SUMS OF DISTINCT DIVISORS OF RATIONAL INTEGERS AND SUMS OF DISTINCT DIVISORS OF QUADRATIC INTEGERS

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B. M. Stewart
Major professor

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Bernard Jacobson

candidate for the degree of

Doctor of Philosophy

Final examination, June 1, 1956, 3:00 P.M., Physics-Mathematics Building

Dissertation: Sums of Distinct Divisors of Rational Integers and Sums of Distinct Divisors of Quadratic Integers

Outline of Studies

Undergraduate Studies, Western Reserve University 1948-1951
Major Subject: Mathematics
Minor Subject: Chemistry

Graduate Studies, Michigan State University 1951-1956
Major Subject: Mathematics (algebra)
Minor Subjects: Mathematics (analysis, topology, applied mathematics)

Biographical Items

Born, April 7, 1928, Cleveland, Ohio

Military Service, United States Army 1946-1947

Experience: Graduate Assistant, 1951-1955
Instructor, 1955-1956
Michigan State University

Member of Phi Society, Pi Mu Epsilon, Sigma Xi, Mathematical Association of America, American Mathematical Society

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BY

BERNARD JACOBSON

A THESIS

Submitted to the School of Graduate Studies of Michigan State University of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

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AN ABSTRACT

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ABSTRACT

In a recent paper B.M. Stewart [6] discussed sums of distinct positive divisors of rational integers. In this dissertation these results are generalized. Let $\alpha(M)$ be the number of positive integers n which can be written in the form $n = \sum d$, where the d are distinct positive or negative divisors of M. The author has proved that $\alpha(M) = \sigma(M)$ if and only if n is of the form $n = 2^b 3^c \prod_{i=1}^{k} p_i^{a_i}$ where b and c are not both zero, $3 < p_1 < p_2 \cdots < p_k$, $p_1 \leqq 2\sigma(2^b 3^c) + 1$ and $p_{j+1} \leqq 2\sigma(2^b 3^c \prod_{i=1}^{j} p_i^{a_i}) + 1$ for $j = 1, 2, \cdots, k-1$. The function $\alpha(M)/\sigma(M)$ is everywhere dense on the interval 0 to 1.

In the quadratic fields $x + y\sqrt{2}$ and $x + y\sqrt{5}$ every integer in the field can be written as a finite sum of distinct units, the algorithm produced depending upon the representation of each integer of the field as a lattice point in the plane. In any real quadratic field there exist infinitely many integers n, having the property that every integer in the field can be written as a finite sum of distinct divisors of n. Explicitly if $a + b\sqrt{m}$ is the unit of smallest absolute value for

which $a > 0$ and $b > 0$, then any integer $2^{t+1}\sqrt{m}$ where $2^t > a$ satisfies this condition. The proof again depends upon the representation of each integer of the field as a lattice point in the plane.

For the imaginary quadratic fields the set $A(m)$ is defined where an integer n belongs to $A(m)$ if and only if there exists a rational integer n' such that every integer of the form $x + y\sqrt{m}$ where $-n' \leqq x \leqq n'$ and $-n' \leqq y \leqq n'$ and no other integer can be represented as a sum of distinct divisors of n. It is shown that for $m = -2$ numbers of the form $R\sqrt{-2}$ belong to $A(-2)$ where $R = \prod p_i^{a_i}$, $p_i \equiv 2$, 5 or 7 mod 8 and $\alpha(R) = \sigma(R)$. When $m = -1$, numbers of the form $R = \prod p_i^{a_i}$ belong to $A(-1)$, where $p_i \equiv 3 \bmod 4$ and $\alpha(R) = \sigma(R)$. When $m < -2$ and $m \not\equiv 1 \bmod 4$, integers of the form $2^t\sqrt{m}$ belong to $A(m)$. If $m < -2$ and $m \equiv 1 \bmod 4$, then the set $A(m)$ is empty.

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INTRODUCTION

In a recent paper B.M. Stewart [6] discussed sums of distinct positive divisors of rational integers. In this dissertation these results are generalized and extended. In Chapter I these results are extended to sums of distinct divisors of rational integers where the divisors may be positive or negative. In Chapters II through IV sums of distinct divisors of integers in the real quadratic fields are discussed. Sums of distinct divisors of integers in the imaginary quadratic fields are investigated in Chapters V and VI.

In each case a maximal set **A** or **A**(m) is defined and the problem is to find which integers of a given field belong to this set. For several fields a complete characterization of the integers belonging to this set is given. For the remaining fields we show that the set **A**(m) is not empty by exhibiting infinitely many integers which do belong to **A**(m).

CHAPTER I

THE RATIONAL FIELD

For a given rational positive integer n let $\alpha(n)$ be the number of positive integers which can be written as the sum of distinct divisors of n. The divisors may be positive or negative. Let A be the set of all integers for which $\alpha(n) = \sigma(n)$.

LEMMA 1. If $(n,2) = 1$, $(n,3) = 1$ and $n \neq 1$, then n does not belong to A.

Lemma 1 is true because $\sigma(n)-3$ cannot be written as a sum of distinct divisors of n.

LEMMA 2. If $n=2^t$, then n belongs to A.

LEMMA 3. If $n = 3^t$, then n belongs to A.

PROOF: The lemma is true for $t = 1$. Assume that the lemma is true for $t = k-1$. Let $t = k$. Every divisor of 3^{k-1} is a divisor of 3^k. By the induction hypothesis every integer between 1 and $\sigma(3^{k-1}) = \frac{3^k-1}{2}$ can be written as a sum of distinct divisors of 3^{k-1} and thus as divisors of 3^k. Every integer from $\frac{3^k-1}{2} + 1 = \frac{3^k+1}{2} = 3^k-\frac{(3^k-1)}{2}$ to 3^k can be written as 3^k minus a sum of distinct divisors of 3^{k-1}. Every integer from 3^k to $\sigma(3^k) = \frac{3^{k+1}-1}{2} = \frac{3\cdot 3^k-1}{2} = 3^k+ \frac{3^k-1}{2} = 3^k + \sigma(3^{k-1})$ can be written as 3^k plus a sum of distinct divisors of 3^{k-1}. Thus by induction n belongs to A.

LEMMA 4. If n belongs to A and p is an odd prime with $(n,p) = 1$, then np belongs to A if and only if $\sigma(n) \geq \frac{p-1}{2}$.

PROOF: Suppose that n belongs to A and $\sigma(n) < \frac{p-1}{2}$. The numbers $\frac{p-1}{2}$ and $\frac{p+1}{2}$ cannot be represented as sums of distinct divisors of np since the largest number which can be represented without using p is $\sigma(n) < \frac{p-1}{2}$. The smallest number which can be represented, using p, is $p-\sigma(n) > p- \frac{p-1}{2} = \frac{p+1}{2}$. Thus the condition is necessary. We now prove that the condition is sufficient. Suppose that n belongs to A and that $\sigma(n) \geq \frac{p-1}{2}$. Let r be any integer such that $0 \leq r \leq \sigma(n)$. Every integer between rp and $rp + \sigma(n)$ can be written as $p\sum_{d/n} d + \sum_{d'/n} d'$. Now let r be any integer such that $0 \leq r+1 \leq \sigma(n)$. Every integer from $rp + \frac{p-1}{2} +1 = (r+1)p - (\frac{p-1}{2})$ to $(r+1)p$ can be written as $p\sum_{d/n} d - \sum_{d'/n} d'$. Thus np belongs to A.

LEMMA 5. If n belongs to A and $(n,p) = 1$, then np^t belongs to A if and only if $\sigma(n) \geq \frac{p-1}{2}$.

PROOF: The condition is necessary in order to represent $\frac{p-1}{2}$ and $\frac{p+1}{2}$. Suppose that $\sigma(n) \geq \frac{p-1}{2}$. The lemma was proved true for $t = 1$ in Lemma 4. Let us assume that the lemma is true for $t = k-1$ and prove it true for $t = k$ by induction. Every divisor of np^{k-1} is a divisor of np^k. Thus by the induction hypothesis every integer from 1 to $\sigma(np^{k-1})$ can be written as a sum of distinct divisors of np^k. Every integer from $p^k - \sigma(np^{k-1})$ to p^k can be written as $p^k - \sum_{d/np^{k-1}} d$. However we have that

$$\sigma(np^{k-1}) + 1 = 1 + \sigma(n)\sigma(p^{k-1}) \geq 1 + \frac{(p-1)}{2}\frac{(p^k-1)}{(p-1)} = \frac{p^k+1}{2}$$

$$= p^k - \frac{(p^k-1)}{2} = p^k - \frac{(p-1)}{2}\frac{(p^k-1)}{(p-1)} \geq p^k - \sigma(n)\sigma(p^{k-1}) =$$

$p^k - \sigma(np^{k-1})$. Thus every integer up to $p^k - \sigma(np^{k-1})$ can be represented. Let r be any integer such that $0 \leq r \leq \sigma(n)$. Every integer from rp^k to $rp^k + \sigma(np^{k-1})$ can be written as $p^k \sum_{d/n} d + \sum_{d'/np^{k-1}} d'$. Now let $r+1 \leq \sigma(n)$. Every integer from $(r+1)p^k - \sigma(np^{k-1})$ to $(r+1)p^k$ can be written as $p^k \sum_{d/n} d - \sum_{d'/np^{k-1}} d'$. This proves the lemma since $rp^k + \sigma(np^{k-1}) + 1 \geq (r+1)p^k - \sigma(np^{k-1})$.

LEMMA 6. Let $n = \prod_{i=1}^{k} p_i^{t_i}$, where $p_i < p_j$ for $i<j$. If n does not belong to A, then nq^s does not belong to A for all primes $q > p_k$ and all $s > 0$.

PROOF: Let $n_j = \prod_{i=0}^{j} p_i$ for $j \leq k$. Let $p_0 = n_0 = 1$.

Since n does not belong to A there must be a smallest integer j such that n_j does belong to A and n_{j+1} does not belong to A. By Lemmas 2, 3, and 5, $\sigma(n_j) < (p_{j+1} - 1)/2$. Let R be the sum of all divisors of n greater than or equal to p_{j+1}. We will show that $R - (p_{j+1} - 1)/2$ cannot be represented as a sum of distinct divisors of n. If all the divisors of n which are greater than or equal to p_{j+1} are used positively, the smallest integer which can be represented is $R - \sigma(n_j) > R - (p_{j+1} - 1)/2$. If any divisor $d \geq p_{j+1}$ is not used positively, the largest number which can be written as a sum of distinct divisors of n is $R - d + \sigma(n_j) < R - p_{j+1} + (p_{j+1} - 1)/2 < R - (p_{j+1} - 1)/2$.

As a direct result of Lemmas 1 through 6, we can now state the following theorems:

THEOREM 1. An even integer belongs to A if and only if it has one of the two following factorizations as a product of primes:

i) $n = 2^t$, for all $t \geq 0$.

ii) $n = 2^t \prod_{i=1}^{k} p_i^{t_i}$ with $p_r < p_s$ for $r < s$; $t \geq 1$, $t_i \geq 1$ and $\frac{p_j - 1}{2} \leq \sigma\left[\frac{n}{\prod_{i=j}^{k} p_i^{t_i}}\right]$ for $1 \leq j \leq k$.

THEOREM 2. An odd integer belongs to A if and only if it has one of the two following factorizations as a product of primes:

i) $n = 3^t$ for all $t \geq 0$.

ii) $n = 3^t \prod_{i=1}^{k} p_i^{t_i}$ with $3 < p_r < p_s$ for $r < s$; $t \geq 1$

$t_i \geq 1$ and $\frac{p_j - 1}{2} \leq \sigma\left[\frac{n}{\prod_{i=j}^{k} p_i^{t_i}}\right]$ for $1 \leq j \leq k$.

THEOREM 3. There exist arbitrarily large square free integers in A.

PROOF: Let $n_k = \prod_{i=1}^{k} p_i$ be the product of the first k primes in their natural order. For every k, n_k belongs to A since $p_j \leq n_{j-1} + 1 \leq \sigma(n_{j-1}) + 1 < 2\sigma(n_{j-1}) + 1$ for all j.

LEMMA 7. If p is a prime such that $p > 2\sigma(T) + 1$, then $\alpha(pT) = 2\alpha(T)[\alpha(T) + 1]$.

PROOF: Every positive sum of distinct divisors of pT can be written as $n = n_1 p + n_2$ where n_1 is zero or any positive number which can be written as a sum of distinct divisors of T and n_2 is zero or any positive or negative integer which can be written as a sum of distinct divisors of T. The maximum value for n_2 is $\sigma(T)$. Since $n_1 p + \sigma(T) < n_1 p + \frac{(p-1)}{2} = (n_1+1)p - p + \frac{p-1}{2} =$

$(n_1+1)p - \frac{(p+1)}{2} < (n_1+1)p - \frac{(p-1)}{2} < (n_1+1)p - \sigma(T)$,

there is no overlapping. We have $[\alpha(T)+1]$ choices for n_1 and $2\alpha(T) + 1$ choices for n_2 except when n_1 is zero.

If n_1 is zero, then n_2 must be positive. Thus we have $\alpha(pT) = [\alpha(T)+1][2\alpha(T)+1] - [\alpha(T)+1] = 2\alpha(T)[\alpha(T)+1]$.

DEFINITION: $s(M) = \frac{\alpha(M)}{\sigma(M)}$.

THEOREM 4. $s(M)$ is everywhere dense on the interval 0 to 1.

PROOF: Given any x and any y such that $0 < x < y \leqq 1$, we seek an integer R such that $x < s(R) < y$. Let $R = pM$ where M belongs to A and $p > 2\sigma(M) + 1$. By Lemma 7 $s(R) = s(pM) = \frac{2\sigma(M)[\sigma(M)+1]}{(p+1)\sigma(M)} = \frac{2[\sigma(M)+1]}{p+1}$. We seek an integer M and a prime p satisfying the above conditions and such that $\frac{1}{y} < \frac{p+1}{2[\sigma(M)+1]} < \frac{1}{x}$. Let $u = \frac{1}{y}$, $u(1+\varepsilon) = \frac{1}{x}$, and $v = 2u[\sigma(M)+1]$. We know by Theorems 1 and 2 that we can find an M belonging to A which is arbitrarily large so that $v = 2u[\sigma(M)+1]$ can be made arbitrarily large. By the Cahen-Stieltjes theorem [1] we know that for sufficiently large v there exists a prime p such that $v - 1 < p < (v-1)(1+\varepsilon) = v(1+\varepsilon) - 1 - \varepsilon < v(1+\varepsilon) - 1$. We now have $(v-1) < p < v(1+\varepsilon) - 1$.

$$v < p + 1 < v(1+\varepsilon),$$

$$2u[\sigma(M)+1] < p + 1 < 2u[\sigma(M)+1][1+\varepsilon],$$

$$\frac{1}{y} < \frac{p+1}{2[\sigma(M)+1]} < \frac{1}{x} .$$

CHAPTER II

SPECIAL QUADRATIC FIELDS

In this chapter and in the chapter following this we will focus our attention upon the integers in the quadratic field $R(m) = x + y\sqrt{m}$, where m is square free, greater than zero and not congruent to one modulo four.

The domain of integers $D(m)$ of $R(m)$ are the numbers of the form $x + y\sqrt{m}$ where x and y are rational integers. The units are those numbers of $D(m)$ where $x^2 - my^2 = \pm 1$. Each domain has an infinite number of units and a basic unit such that if $u_1 = a_1 + b_1\sqrt{m}$ is the unit of smallest absolute value for which both x and y are positive, then every unit in the domain can be written $\pm u_1^n$ for $n = 0, \pm 1, \pm 2, \cdots$.

Let $u_j = u_1^j$ for $j = 0, 1, 2, \cdots$. In this way we may divide the units into one set $S_0 = \{+1, -1\}$ of two units and infinitely many sets $S_i = \left\{u_i, -u_i, \frac{1}{u_i}, \frac{-1}{u_i}\right\}$ of four units. Every unit belongs to one and only one set.

DEFINITION: $A(m)$ is the set of integers $\{n\}$ in the field $R(m)$ which have the following property: n belongs to $A(m)$ if and only if every integer in the domain $D(m)$ can be written as a finite sum of distinct divisors of n.

DEFINITION: $A'(m)$ is the set of integers $\{n\}$ in the field $R(m)$ which have the following property: n belongs to $A'(m)$ if and only if every integer in the domain $D(m)$ can be written as a finite sum of distinct associates of n.

We will show that for all values of m the set $A(m)$ is not empty and that for $m > 2$ the set $A'(m)$ is empty.

$m = 2$

In this field $u_1 = 1 + \sqrt{2}$, $u_2 = 3 + 2\sqrt{2}, \cdots$. Let a_i be the rational part of u_i and b_i be the coefficient of $\sqrt{2}$ in u_i. Let $B_k = \sum_{i=1}^{k} b_i$.

Since the integers of $D(2)$ are given as $x + y\sqrt{2}$ where x and y are both rational integers we may consider any integer P of $D(m)$ as a lattice point P in the plane. We will fill the plane with concentric squares whose centers are the origin and whose diagonals are the coordinate axes. The length of the diagonal of the k square is $4B_k$. It will be shown that every integer lying in the k square can be represented as a sum of distinct units belonging to the first k sets $S_0, S_1, \cdots, S_k$.

LEMMA 1. $a_{k+1} = a_k + 2b_k$ and $b_{k+1} = a_k + b_k$.

PROOF: $a_{k+1} + b_{k+1}\sqrt{2} = u_{k+1} = u_k(1+\sqrt{2}) = (a_k + b_k\sqrt{2})(1+\sqrt{2}) = (a_k + 2b_k) + (a_k + b_k)\sqrt{2}$. Therefore $a_{k-1} = a_k + 2b_k$ and $b_{k+1} = a_k + b_k$.

LEMMA 2. $a_k + b_k \leqq 4B_{k-1}$ for all $k \geqq 3$.

PROOF: The lemma is true for $k = 3$ and $k = 4$. We will assume that the lemma is true for $3 \leqq k \leqq n$ and prove the lemma true by induction. Let $k = n + 1$. Using Lemma 1 we have: $a_{n+1} + b_{n+1} = 2a_n + 3b_n = 5a_{n-1} + 7b_{n-1} = a_{n-1} + b_{n-1} + 4a_{n-1} + 6b_{n-1} = a_{n-1} + b_{n-1} + 4b_n - 4b_{n-1} + 6b_{n-1} = a_{n-1} + b_{n-1} + 2b_{n-1} + 4b_n \leqq a_{n-1} + b_{n-1} + 4(b_{n-1} + b_n)$. Using the induction hypothesis we find that the extreme right hand side of the equation is less than or equal to $4B_{n-2} + 4(b_{n-1} + b_n) = 4B_n$.

LEMMA 3. $a_k \geqq b_k$.

PROOF: $a_k = a_{k-1} + 2b_{k-1} \geqq a_{k-1} + b_{k-1} = b_k$.

LEMMA 4. $a_k + b_k \geqq 2B_{k-1}$.

PROOF: The lemma is true for $k = 1$. Assume that the lemma is true for $k = n$. $a_{n+1} + b_{n+1} = a_n + 2b_n + a_n + b_n$

By the induction hypothesis the right hand side of the equation is greater than or equal to $2b_n + 2B_{n-1} = 2B_n$.

THEOREM 1. The integer 1 belongs to A(2).

PROOF: By inspection it can be seen that every integer in the first three squares can be represented as a sum of distinct units in the sets S_0, S_1, S_2 and S_3. Let us assume that every integer in the first k-1 squares can be written as a sum of distinct units in the first k sets of units, $S_0, S_1, \cdots, S_{k-1}$ and prove the theorem by induction. Consider any point inside the k square. Since the square and all sets of units are symmetric with respect to both axes, there is no loss in generality by assuming that the point lies in the first quadrant, on the positive x axis or on the positive y axis. If the point lies inside the k-1 square a required representation is assured by the induction hypothesis. Therefore consider a point with rectangular coordinates (x,y) which lies in the first quadrant, in the k square but not in the k-1 square.

Thus we have: $2B_{k-1} < x + y \leqq 2B_k$,

$$y \geqq 0, \qquad x \geqq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies above the line $y - x = 2(b_k - B_{k-1})$. In Case 2 the point lies below the line $y-x=2(B_{k-1}-a_k)$. In Case 3 the point lies on or between these two lines. Case 3 actually exists because Lemma 4 implies that $b_k - B_{k-1} \geqq B_{k-1} - a_k$.

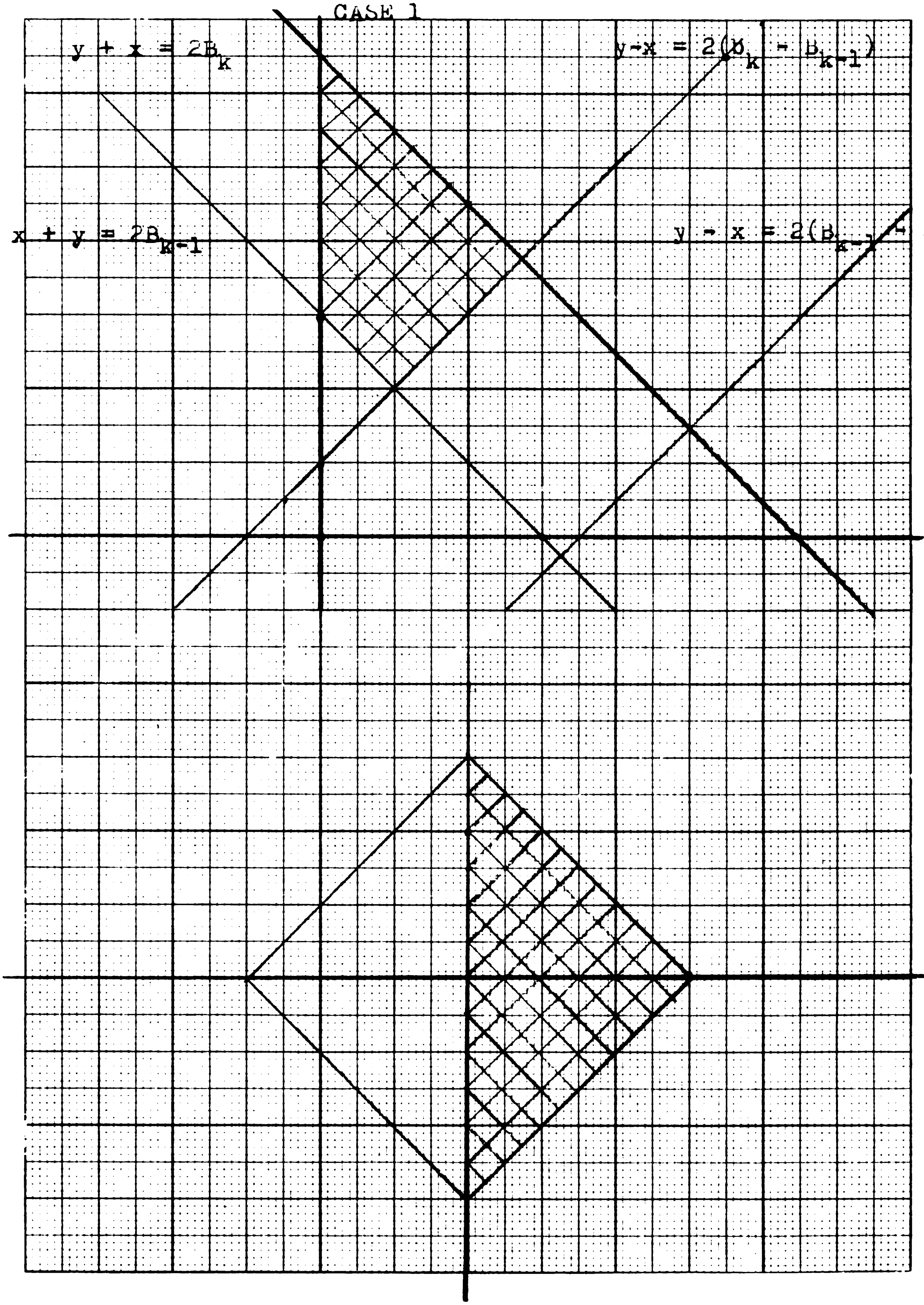
Case 1. $2B_{k-1} < x + y \leqq 2B_k$,

$$y - x > 2(b_k - B_{k-1}),$$

$$y \geqq 0, \qquad x \geqq 0.$$

If we subtract $2b_k$ from y, we obtain a point P' with

CASE 1



rectangular coordinates $x' = x$ and $y' = y - 2b_k$. Thus $x' + y' \leqq 2B_k - 2b_k = 2B_{k-1}$, $y' - x' \geqq 2(b_k - B_{k-1}) - 2b_k = -2B_{k-1}$, $x' \geqq 0$. Thus the point P' lies inside the k-1 square and by the induction hypothesis can be represented as a sum σ_{k-1} of distinct units in the first k sets of units. $P' = \sigma_{k-1}$. $P' = P - 2b_k\sqrt{2}$. Thus $P = \sigma_{k-1} + 2b_k\sqrt{2} = \sigma_{k-1} + (a_k + b_k\sqrt{2}) + (-a_k + b_k\sqrt{2})$ and the theorem is proved in Case 1.

Case 2. $2B_{k-1} \leqq x + y \leqq 2B_k$,

$$y - x \leqq 2(B_{k-1} - a_k),$$

$$y \geqq 0, \qquad x \geqq 0.$$

If we subtract $2a_k$ from x, we obtain a point P' with coordinates $x' = x - 2a_k$ and $y' = y$. Thus we have:

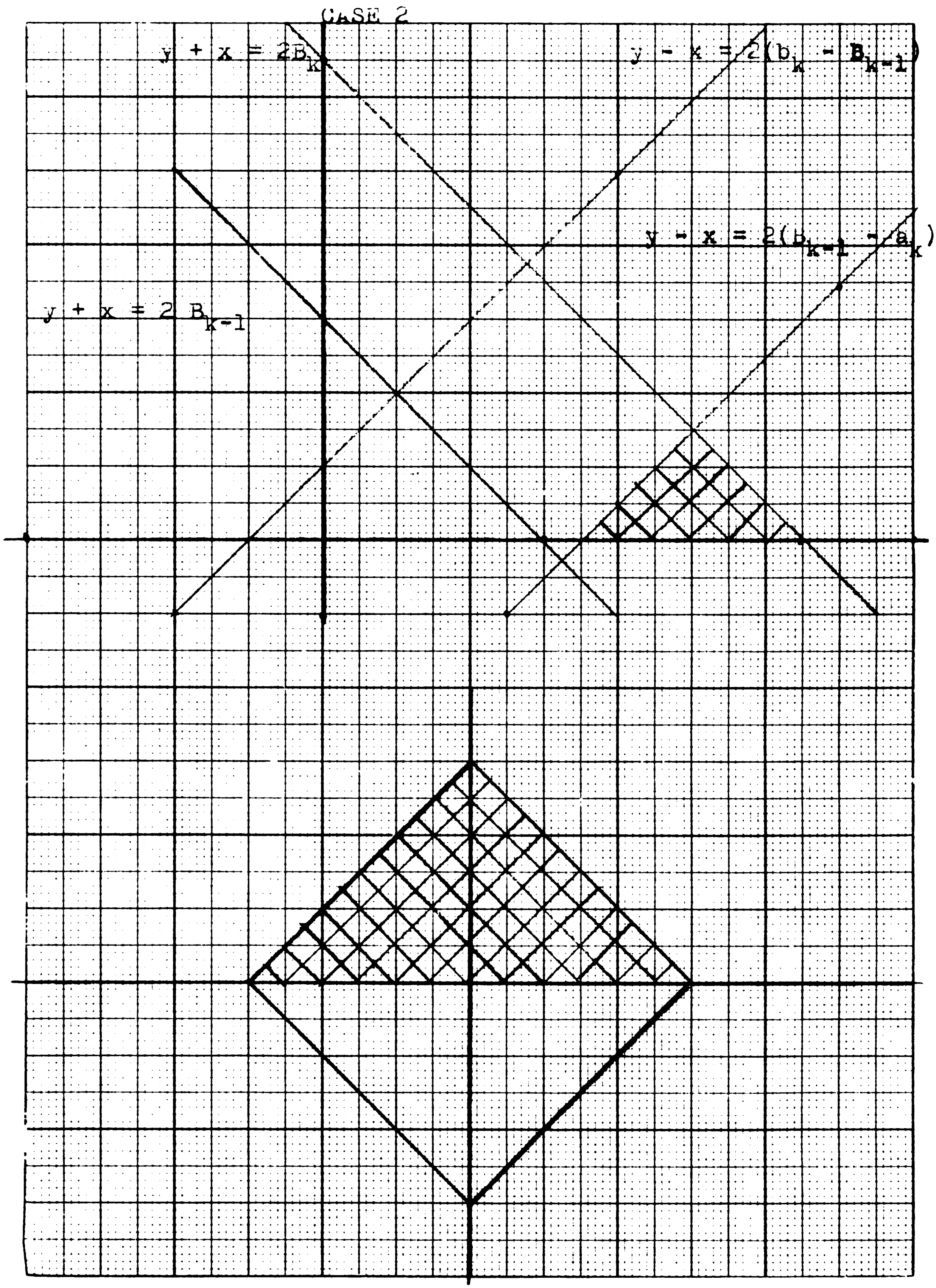
$$x' + y' \leqq 2B_k - 2a_k \leqq 2B_k - 2b_k = 2B_{k-1},$$

$$y' - x' \leqq 2(B_{k-1} - a_k) + 2a_k = 2B_{k-1},$$

$y' \geqq 0$. Thus the point P' lies inside the k-1 square and can be represented as a sum σ_{k-1} of distinct units in the first k sets $S_0, \cdots, S_{k-1}$. $P' = \sigma_{k-1}$. $P' = P - 2a_k$. Thus $P = \sigma_{k-1} + 2a_k = \sigma_{k-1} + (a_k + b_k\sqrt{2}) + (a_k - b_k\sqrt{2})$. Thus the theorem is proved for Case 2.

Case 3. $2B_{k-1} \leqq x + y \leqq 2B_k$,

$$2(b_k - B_{k-1}) \geqq y - x \geqq 2(B_{k-1} - a_k).$$


CASE 2
y + x = 2B_k
y - x = 2(b_k - B_{k-1})
y - x = 2(B_{k-1} - a_k)
y + x = 2 B_{k-1}

If we subtract a_k from x and b_k from y we obtain a point P' whose coordinates are $x' = x - a_k$ and $y' = y - b_k$. Using lemma 2 we obtain the following inequalities:

$x' + y' \geqq 2B_{k-1} - a_k - b_k \geqq 2B_{k-1} - 4B_{k-1} = -2B_{k-1}$,

$x' + y' \leqq 2B_k - a_k - b_k \leqq 2B_k - 2b_k = 2B_{k-1}$,

$y' - x' \geqq 2B_{k-1} - 2a_k - b_k + a_k \geqq 2B_{k-1} - a_k - b_k \geqq -2B_{k-1}$,

$y' - x' \leqq 2b_k - 2B_{k-1} - b_k + a_k = a_k + b_k - 2B_{k-1} \leqq$

$4B_{k-1} - 2B_{k-1} = 2B_{k-1}$.

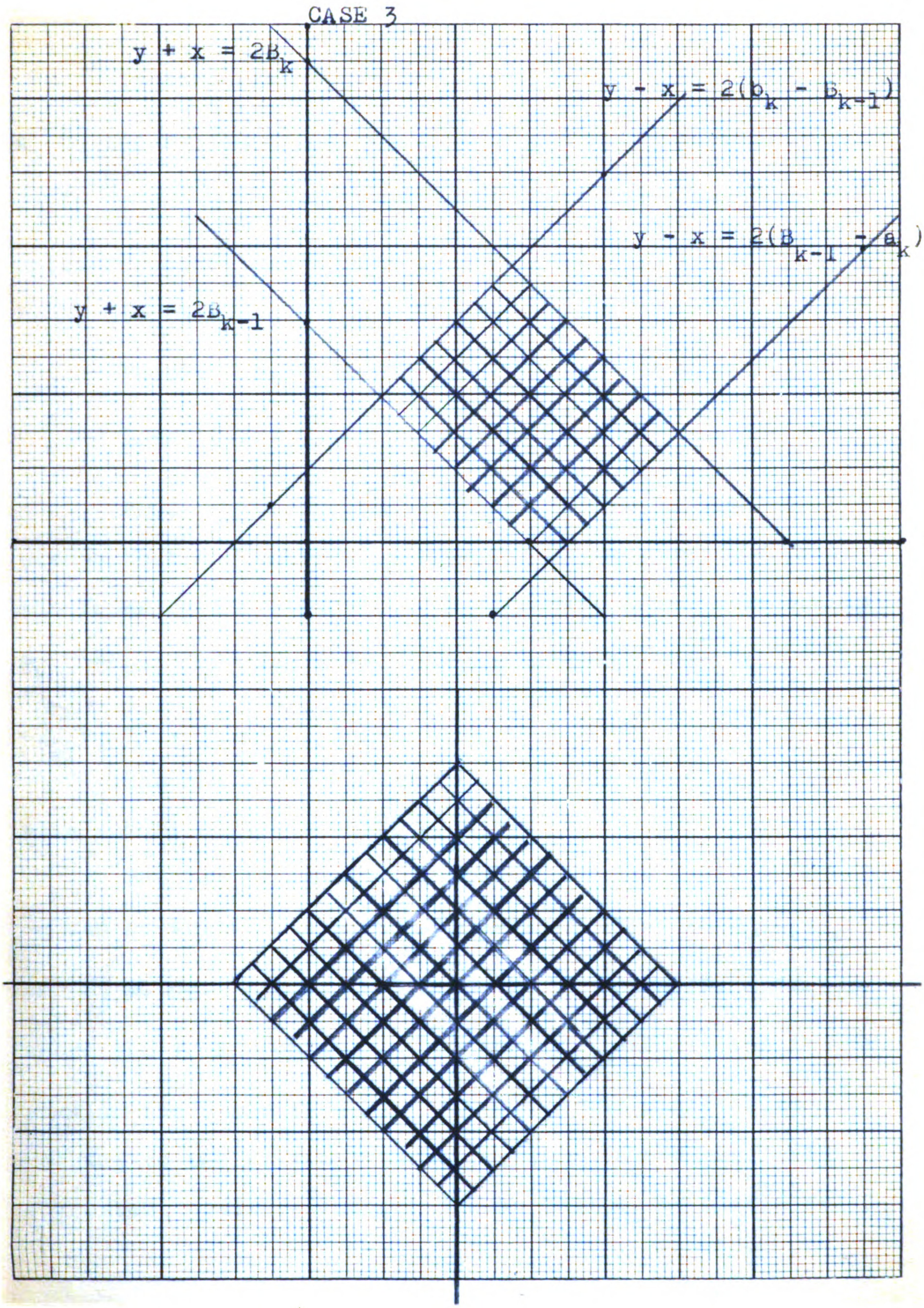
Thus the point P' lies inside the k-1 square and can be represented as a sum σ_{k-1} of distinct units in the first k sets of units. $P' = \sigma_{k-1}$. $P' = P - (a_k + b_k\sqrt{2})$. Thus $P = \sigma_{k-1} + (a_k + b_k\sqrt{2})$ and the theorem is proved.

COROLLARY 1. Every integer in D(2) is in A(2).

THEOREM 2. The only integers in A'(2) are units.

PROOF: It was proved in theorem 1 that the units belong to A'(2). Let X be any integer which is not a unit and let Y be any integer which can be represented as a sum of distinct associates of X. Using norms we obtain from $Y = \sum X_i = \sum X u_i = X \sum u_i$ the following equation:

$N(Y) = N(X \sum u_i) = N(X) N(\sum u_i)$. If the norm of X does not divide the norm of Y, the required representation is not possible.


CASE 3
y + x = 2B_k
y - x = 2(b_k - B_{k-1})
y - x = 2(B_{k-1} - a_k)
y + x = 2B_{k-1}

$m = 3$

In this field $u_1 = 2 + \sqrt{3}$, $u_2 = 7 + 4\sqrt{3}, \cdots$. Let a_i be the rational part of u_i, b_i be the coefficient of $\sqrt{3}$ in u_i and $B_k = \sum_{i=1}^{k} b_i$. We will first prove that the units do not belong to A(3).

LEMMA 1. $2b_k > a_k$ for all $k > 1$.

PROOF: $2b_k = 2a_{k-1} + 4b_{k-1} > 2a_{k-1} + a_k - 2a_{k-1} = a_k$.

LEMMA 2. $a_k > 2\sum_{i=1}^{k-1} a_i + 3$ for $k > 2$.

PROOF: The theorem is true for $k = 3$. Let us assume that the theorem is true for $2 < k \leqq n-1$ and prove the lemma by induction. Let $k=n$. Using Lemma 1 we obtain $a_n = 2a_{n-1} + 3b_{n-1} > 2a_{n-1} + a_{n-1} > 2a_{n-1} + 2\sum_{i=1}^{n-2} a_i + 3 = 2\sum_{i=1}^{n-1} a_i + 3$.

THEOREM 1. The numbers 2 and -2 cannot be written as a sum of distinct units of D(3).

PROOF: We can see by inspection that neither 2 nor -2 can be written as a sum of distinct units in S_0, S_1, S_2 and S_3 alone. Assume that 2 can be represented, $2 = \sum v_i$ where the v_i are units. If v_i and $-v_i^{-1}$ both appear in the summand their sum is a multiple of $\sqrt{3}$ and contributes nothing to the rational part. Therefore there must exist a greatest positive integer j such that v_i is in the set S_j,

v_i appears in the summand, and neither $-v_i$ nor $-v_i^{-1}$ appear in the summand. Lemma 2 tells us that $j \leqq 2$ and the theorem is proved.

We will now show that the prime $p = (1 + \sqrt{3})$ belongs to $A(3)$. Let $c_{2t} + d_{2t}\sqrt{3} = a_t + b_t\sqrt{3}$ and $c_{2t+1} + d_{2t+1}\sqrt{m} = pu_1^{\ t}$. We now separate the divisors of p into the sets $V_k = [(c_k+d_k\sqrt{3}),(c_k-d_k\sqrt{3}),(-c_k+d_k\sqrt{3}),(-c_k-d_k\sqrt{3})]$. Every divisor of p belongs to exactly one of these sets. Each set contains exactly one divisor $c_k + d_k\sqrt{3}$, with both c_k and d_k non negative. Let $D_k = \sum_{i=1}^{k} d_i$.

We will make use of the following equations:

$c_k = 2c_{k-2} + 3d_{k-2}$ and $d_k = c_{k-2} + 2d_{k-2}$ for all $k > 1$;

$c_k = c_{k-1} + 3d_{k-1}$ and $d_k = c_{k-1} + d_{k-1}$ when k is odd;

$c_k = \frac{1}{2}[c_{k-1} + 3d_{k-1}]$ and $d_k = \frac{1}{2}[c_{k-1} + d_{k-1}]$ for k even.

LEMMA 3. $c_k + d_k \leqq 4D_{k-1}$.

PROOF: The lemma is true for $k = 2$ and $k = 3$. Let us assume that the lemma is true for $k \leqq n$ and prove that the lemma is true for $k = n + 1$. We have

$c_{n+1} + d_{n+1} = 3c_{n-1} + 5d_{n-1} = c_{n-1} + d_{n-1} + 4d_{n-1} + 2c_{n-1}$.

Noting that $2c_{n-1} < 4d_n$, and making use of the induction hypothesis we see that the right hand side of the equation is $\leqq 4D_{n-2} + 4d_{n-1} + 4d_n = 4D_n$ and the lemma is proved.

LEMMA 4. $c_k + d_k > 2D_{k-1}$.

PROOF: The lemma is true for $k = 2$ and $k = 3$. Let us assume that the lemma is true for $k \leq n$ and prove the lemma by induction. Let $k = n + 1$.

$c_{n+1} + d_{n+1} = c_{n-1} + d_{n-1} + 2c_{n-1} + 4d_{n-1} > 2D_{n-2} + 2d_{n-1}$.

THEOREM 2. $1 + \sqrt{3}$ belongs to $A(3)$.

PROOF: We construct concentric squares with center at the origin and with diagonal lengths equal to $4D_k$. Because both the sets of divisors and the squares are symmetric with respect to both coordinate axes there is no loss in generality by considering only those points in the first quadrant, on the positive x axis or on the positive y axis. By inspection we can see that every integer in the first square can be written as a sum of distinct divisors in the sets V_0 and V_1. In order to prove the theorem by induction we assume that every point in the first k - 1 squares can be represented as a sum of distinct divisors taken only from the first k sets $V_0, \cdots, V_{k-1}$. Let P be any point with coordinates (x,y) which lies inside or on the boundaries of the k square. If P lies in the k - 1 square, the theorem is proved. If P does not lie in the k - 1 square, we have the following inequalities:

$$2D_{k-1} < x + y \leq 2D_k,$$

$$x \geq 0, \qquad y \geq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies on or above the line $y - x = 2d_k - 2D_{k-1}$. In Case 2 the point lies on or below the line $y - x = -2c_k + 2D_{k-1}$ and in Case 3 the point lies between these lines. Lemma 4 assures us that Case 3 actually exists.

Case 1. $2D_{k-1} < x + y \leqq 2D_k$,

$y - x \geqq 2d_k - 2D_{k-1}$,

$x \geqq 0$.

If we subtract $2d_k$ from y we obtain a point P' with rectangular coordinates $y' = y - 2d_k$ and $x' = x$.

$x' + y' \leqq 2D_k - 2d_k = 2D_{k-1}$,

$y' - x' \geqq 2d_k - 2D_{k-1} - 2d_k = -2D_{k-1}$,

$x' \geqq 0$.

The resulting point P' lies inside the $k - 1$ square and can be represented as a sum σ_{k-1} of distinct divisors in the first k sets $V_0, \cdots, V_{k-1}$. $P' = P - 2d_k$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2d_k = \sigma_{k-1} + (c_k + d_k\sqrt{3}) + (-c_k + d_k\sqrt{3})$. This completes Case 1.

Case 2. $2D_{k-1} < y + x \leqq 2D_k$,

$y - x \leqq -2c_k + 2D_{k-1}$,

$y \geqq 0$.

If we subtract $2c_k$ from x we obtain a point P' with rectangular coordinates $y' = y$ and $x' = x - 2c_k$.

$y' + x' \leqq 2D_k - 2c_k < 2D_k - 2d_k = 2D_{k-1}$,

$y' - x' \leqq -2c_k + 2D_{k-1} + 2c_k = 2D_{k-1}$,

$y' \geqq 0$.

The resulting point P' lies inside the k-1 square and can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V_0, \cdots, V_{k-1}$. $P' = P - 2c_k$. $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2c_k = \sigma_{k-1} + (c_k + d_k\sqrt{3}) + (c_k - d_k\sqrt{3})$. This completes Case 2.

Case 3. $2D_{k-1} < y + x \leqq 2D_k$,

$$-2c_k + 2D_{k-1} < y - x < 2d_k - 2D_{k-1}.$$

If we subtract d_k from y and c_k from x we obtain a point P' with coordinates $y' = y - d_k$ and $x' = x - c_k$. $y' + x' \leqq 2D_k - c_k - d_k < 2D_k - 2d_k = 2D_{k-1}$. Applying Lemma 3 to each of the following inequalities we obtain:

$y' + x' > 2D_{k-1} - c_k - d_k \geqq 2D_{k-1} - 4D_{k-1} = -2D_{k-1}$,

$y' - x' < 2d_k - 2D_{k-1} + c_k - d_k = c_k + d_k - 2D_{k-1} \leqq 4D_{k-1} - 2D_{k-1} = 2D_{k-1}$,

$y' - x' > -2c_k + 2D_{k-1} + c_k - d_k = -c_k - d_k + 2D_{k-1} \geqq -4D_{k-1} + 2D_{k-1} = -2D_{k-1}$.

The resulting point P' lies inside the k-1 square and can be represented as a sum σ_{k-1} of distinct divisors in the first k sets $V_0, V_1, \cdots, V_{k-1}$. $P' = P - c_k - d_k\sqrt{3}$. $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + c_k + d_k\sqrt{3}$. This completes Case 3 and the theorem is proved.

COROLLARY. If $x + y\sqrt{3}$ is an integer in $D(3)$ with $x \equiv y$ modulo 2, then $x + y\sqrt{3}$ belongs to $A(3)$.

PROOF: $\dfrac{x + y\sqrt{3}}{1 + \sqrt{3}} = \dfrac{(x - 3y) + (y - x)\sqrt{3}}{-2}$. If we have $x \equiv y$ modulo 2, then the right hand side is an integer and $x + y\sqrt{3}$ is divisible by $1 + \sqrt{3}$. It then follows from Theorem 2 that $x + y\sqrt{3}$ belongs to $A(3)$.

$m = 6$

In this field $u_1 = 5 + 2\sqrt{6}$, $u_2 = 49 + 20\sqrt{6}, \cdots$. Let b_i be the coefficient of $\sqrt{6}$ in u_i and a_i be the rational part of u_i. We will first prove that the units do not belong to $A(6)$.

LEMMA 1. 2 divides b_k.

PROOF: The lemma is true for $k = 1$. Let us assume that the lemma is true for $k = n-1$ and prove the lemma by induction. $b_n = 5b_{n-1} + 2a_{n-1}$. By the induction hypothesis 2 divides b_{n-1} and thus divides the right hand side of the equation. Therefore 2 divides b_n.

THEOREM 1. The units do not belong to $A(6)$.

PROOF: Lemma 1 tells us that in every sum of units the coefficient of $\sqrt{6}$ is even. $x + y\sqrt{6}$ can not be represented as a sum of units when y is odd.

We will now show that the integer 2 belongs to $A(6)$. Let us separate the divisors of 2 into sets V_i as follows:

$$V_0 = [1, -1, 2, -2]$$
$$V_1 = [2+\sqrt{6}, 2-\sqrt{6}, -2-\sqrt{6}, -2+\sqrt{6}]$$
$$V_2 = [u_1, u_1^{-1}, -u_1, -u_1^{-1}]$$
$$V_3 = [2u_1, 2u_1^{-1}, -2u_1, -2u_1^{-1}]$$
$$V_4 = [(2+\sqrt{6})u_1 = 22+9\sqrt{6}, 22-9\sqrt{6}, -22+9\sqrt{6}, -22-9\sqrt{6}]$$
$$\vdots$$

In every set V_k $(k>0)$ there exists exactly one divisor $c_k + d_k\sqrt{6}$ with both c_k and d_k positive. The divisors in the set V_k are $c_k + d_k\sqrt{6}$, $c_k - d_k\sqrt{6}$, $-c_k - d_k\sqrt{6}$ and $-c_k + d_k\sqrt{6}$. Let $D_k = \sum_{i=1}^{k} d_i$. We will make use of the following equations:

$c_k = 2c_{k-1}$ and $d_k = 2d_{k-1}$ when $k \equiv 0 \bmod 3$, and $k \neq 0$,

$c_k = c_{k-1} + 3d_{k-1}$ and $d_k = \frac{c_{k-1}}{2} + d_{k-1}$ if $k \equiv 1$ or 2 modulo 3. In either case $2d_k < c_k$ and $3d_k > c_k$ for $k > 1$.

LEMMA 2. $c_k + 2d_k < 8D_{k-1}$.

PROOF: The lemma is true for $k = 3$. Let us assume that the lemma is true for $k = n-1$ and prove the lemma true by induction. Let $k=n$. If $k = n \equiv 0 \bmod 3$, $c_n + 2d_n = 2c_{n-1} + 4d_{n-1} < c_{n-1} + 7d_{n-1} = c_{n-1} + 2d_{n-1} + 5d_{n-1} < 8D_{n-2} + 5d_{n-1} < 8D_{n-1}$. If $k = n \equiv 1$ or $2 \bmod 3$, $c_n + 2d_n = 2c_{n-1} + 5d_{n-1} < c_{n-1} + 8d_{n-1} = c_{n-1} + 2d_{n-1} + 6d_{n-1} < 8D_{n-2} + 6d_{n-1} < 8D_{n-1}$.

LEMMA 3. $c_k + 2d_k > 4D_{k-1}$.

PROOF: The lemma is true for $k = 2$. Let us assume that the lemma is true for $k = n-1$ and prove the lemma true by induction. Let $k = n$. If $k = n \equiv 0 \bmod 3$, $c_n + 2d_n = 2c_{n-1} + 4d_{n-1} > c_{n-1} + 6d_{n-1} = c_{n-1} + 2d_{n-1} + 4d_{n-1} > 4D_{n-2} + 4d_{n-1} = 4D_{n-1}$. If $k = n \equiv 1$ or $2 \bmod 3$, $c_n + 2d_n = 2c_{n-1} + 5d_{n-1} > c_{n-1} + 7d_{n-1} = c_{n-1} + 2d_{n-1} + 5d_{n-1} > 4D_{n-2} + 5d_{n-1} > 4D_{n-1}$.

THEOREM 2. 2 belongs to $A(6)$.

PROOF: We construct concentric diamonds with centers at the origin, whose sides are given by the equations $2y + x = \pm 4D_k$ and $2y - x = \pm 4D_k$. Because the sets of divisors and the diamonds are both symmetric to both coordinate axes there is no loss in generality by considering only those points lying in the first quadrant, on the positive x axis or on the positive y axis. By inspection we can see that every point in the first two diamonds can be represented as a sum of distinct divisors in the sets V_0, V_1 and V_2. In order to prove the theorem by induction we assume that every point in the first k-1 diamonds can be represented as a sum of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. Let P be any point with coordinates (x,y) which lies inside or on the boundaries of the k diamond. If P lies in the k-1 diamond, the theorem is proved. If P does not lie in the k-1 diamond, we have the following inequalities:

$$4D_{k-1} < 2y + x \leqq 4D_k,$$

$$x \geqq 0, \qquad y \geqq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies on or above the line $2y - x = 4d_k - 4D_{k-1}$. In Case 2 the point lies on or below the line $2y - x = -c_k + 4D_{k-1}$. In Case 3 the point lies between these lines. Lemma 3 assures us that Case 3 actually exists.

Case 1. $4D_{k-1} < 2y + x \leqq 4D_k,$

$$2y - x \geqq 4d_k - 4D_{k-1},$$

$$x \geqq 0.$$

If we subtract $2d_k$ from y we obtain a point P' with rectangular coordinates $y' = y - 2d_k$ and $x' = x$.

$$2y' + x' \leqq 4D_k - 4d_k = 4D_{k-1}, \qquad x' \geqq 0,$$

$$2y' - x' \geqq 4d_k - 4D_{k-1} - 4d_k = -4D_{k-1}.$$

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. $P' = P - 2d_k\sqrt{6}$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2d_k\sqrt{6} = \sigma_{k-1} + (c_k + d_k\sqrt{6}) + (-c_k + d_k\sqrt{6})$. This completes Case 1.

Case 2. $4D_{k-1} < 2y + x \leqq 4D_k$,

$$2y - x \leqq -2c_k + 4D_{k-1},$$

$$y \geqq 0.$$

If we subtract $2c_k$ from x we obtain a point P' with rectangular coordinates $y' = y$ and $x' = x - 2c_k$.

$$2y' + x' \leqq 4D_k - 2c_k < 4D_k - 4d_k = 4D_{k-1},$$

$$2y' - x' \leqq -2c_k + 4D_{k-1} + 2c_k = 4D_{k-1},$$

$$y' \geqq 0.$$

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. $P' = P - 2c_k$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2c_k = \sigma_{k-1} + (c_k + d_k\sqrt{6}) + (c_k - d_k\sqrt{6})$. This completes Case 2.

Case 3. $4D_{k-1} < 2y + x \leqq 4D_k$,

$$-2c_k + 4D_{k-1} < 2y - x < 4d_k - 4D_{k-1}.$$

If we subtract c_k from x and d_k from y we obtain a point P' with coordinates $y' = y - d_k$ and $x' = x - c_k$.

$2y' + x' \leqq 4D_k - 2d_k - c_k < 4D_k - 4d_k = 4D_{k-1}$.

Applying Lemma 2 we also obtain the following inequalities:

$2y' + x' > 4D_{k-1} - 2d_k - c_k > -4D_{k-1}$,

$2y' - x' < 4d_k - 4D_{k-1} - 2d_k + c_k = c_k + 2d_k - 4D_{k-1} < 4D_{k-1}$,

$2y' - x' > -2c_k + 4D_{k-1} - 2d_k + c_k = 4D_{k-1} - c_k - 2d_k > 4D_{k-1}$.

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. $P' = P - c_k - d_k\sqrt{6}$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + c_k + d_k\sqrt{6}$. This completes Case 3 and the theorem is proved.

COROLLARY. If $x \equiv y \equiv 0$ modulo 2, then the integer $x + y\sqrt{6}$ belongs to A(6).

$m = 7$

In this field $u_1 = 8 + 3\sqrt{7}$, $u_2 = 127 + 48\sqrt{7} \cdots$. Let a_i be the rational part of u_i and b_i be the coefficient of $\sqrt{7}$ in u_i. We will first prove that the units do not belong to A(7).

LEMMA 1. 3 divides b_k for all k.

PROOF: The lemma is true for $k = 1$. Let us assume that the lemma is true for $k = n-1$ and prove that the lemma is true by induction. When $k = n$, we have $b_n = 8b_{n-1} + 3a_{n-1}$. 3 divides b_{n-1} and thus 3 divides the right hand side of the equation. Therefore 3 divides b_n.

THEOREM 1. The units do not belong to A(7).

PROOF: In every sum of units the coefficient of $\sqrt{7}$ is divisible by three and thus $x + y\sqrt{7}$ cannot be represented if $y \equiv 1$ or 2 modulo 3.

We will now show that the integer 2 belongs to A(7). Let us separate the divisors of 2 into the sets V_k as follows:

$V_0 = [1, -1, 2, -2]$

$V_1 = [3 + \sqrt{7}, 3 - \sqrt{7}, -3 - \sqrt{7}, -3 + \sqrt{7}]$

$V_2 = [u_1, -u_1, u_1^{-1}, -u_1^{-1}]$

$V_3 = [2u_1, -2u_1, 2u_1^{-1}, -2u_1^{-1}]$

$V_4 = [(3+\sqrt{7})u_1 = 45 + 11\sqrt{7}, 45-11\sqrt{7}, -45-11\sqrt{7}, -45+11\sqrt{7}]$

$\cdots$

In every set V_k $(k>0)$ there exists exactly one divisor $c_k + d_k\sqrt{7}$ with both c_k and d_k positive. The divisors in the set V_k are $c_k + d_k\sqrt{7}$, $c_k - d_k\sqrt{7}$, $-c_k - d_k\sqrt{7}$ and $-c_k + d_k\sqrt{7}$. Let $D_k = \sum_{i=1}^{k} d_i$. We will make use of the following equations:

$c_k = 2c_{k-1}$ and $d_k = 2d_{k-1}$ when $k \equiv 0$ modulo 3,

$c_k = (3c_{k-1} + 7d_{k-1})/2$ and $d_k=(c_{k-1}+ 3d_{k-1})/2$ if $k \not\equiv 0$ mod 3.

LEMMA 2. $5d_k < 2c_k \leqq 6d_k$.

PROOF: The lemma is true for $k = 1$. Let us assume that the lemma is true for $k = n-1$ and prove that the lemma is true for $k = n$. When $k = n \equiv 0$ mod 3 we have:

$5d_n = 10d_{n-1} < 4c_{n-1} = 2c_n = 4c_{n-1} \leqq 12d_{n-1} = 6d_n$.

When $k = n \equiv 1$ or 2 modulo 3 we have:

$5d_n = (5c_{n-1} + 15d_{n-1})/2 < (6c_{n-1} + 13d_{n-1})/2 < 2c_n =$

$3c_{n-1} + 7d_{n-1} < 3c_{n-1} + 9d_{n-1} = 6d_n$.

LEMMA 3. $2c_k + 5d_k < 20D_{k-1}$ for $k > 2$.

PROOF: The lemma is true for $k = 3$. Let us assume that the lemma is true for $k = n-1$ and prove that the lemma is true by induction. For the case $k = n$ and $n \equiv 0$ mod 3 we apply Lemma 2 and the induction hypothesis to obtain $2c_n + 5d_n = 4c_{n-1} + 10d_{n-1} =$

$2c_{n-1} + 5d_{n-1} + 2c_{n-1} + 5d_{n-1} < 20D_{n-2} + 11d_{n-1} < 20D_{n-1}$.

When $k = n \equiv 1$ or 2 modulo 3, we apply Lemma 2 and the induction hypothesis to obtain:

$2c_n + 5d_n = (11c_{n-1} + 29d_{n-1})/2 \leqq (4c_{n-1} + 50d_{n-1})/2 =$

$2c_{n-1} + 5d_{n-1} + 20d_{n-1} < 20D_{n-2} + 20d_{n-1} = 20D_{n-1}$.

LEMMA 4. $2c_k + 5d_k > 10D_{k-1}$.

PROOF: The lemma is true for $k = 2$. Let us assume that the lemma is true for $k = n-1$ and prove that the lemma is true by induction. For the case $k = n \equiv 0$ modulo 3, we apply the induction hypothesis and Lemma 2 to obtain $2c_n + 5d_n = 4c_{n-1} + 10d_{n-1} > 10D_{n-2} + 10d_{n-1} = 10D_{n-1}$. When $k = n \not\equiv 0$ modulo 3, we apply Lemma 2 and the induction hypothesis to obtain:

$2c_n + 5d_n = (11c_{n-1} + 29d_{n-1})/2 = 2c_{n-1} + 5d_{n-1} +$

$(7c_{n-1} + 19d_{n-1})/2 > 10D_{n-2} + 10d_{n-1} = 10D_{n-1}$.

THEOREM 2. The integer two belongs to $A(7)$.

PROOF: We construct concentric diamonds with centers at the origin, whose sides are given by the equations $5y + 2x = \pm 10D_k$ and $5y - 2x = \pm 10D_k$. Since the sets of divisors and the diamonds are symmetric to both coordinate axes there is no loss in generality in considering only those points lying in the first quadrant, on the positive x axis or on the positive y axis. By inspection we can see that every point in the first two diamonds can be represented as a sum of distinct divisors in the sets V_0, V_1 and V_2. In order to prove the theorem by induction we assume that every point in the first k-1 diamonds can be represented as a sum of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. Let P be any point with coordinates (x,y) which lies inside or on the boundaries of the k diamond. If P lies in the k-1 diamond, the theorem is proved. If P does not lie in the k-1 diamond, we have the following inequalities:

$$10D_{k-1} < 5y + 2x \leqq 10D_k,$$

$$x \geqq 0, \qquad y \geqq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies on or above the straight line $5y - 2x = 10d_k - 10D_{k-1}$. In Case 2 the point lies on or below the line $5y - 2x = -4c_k + 10D_{k-1}$. In Case 3 the point lies between the two lines. Lemma 4 assures us that Case 3 actually exists.

Case 1. $10D_{k-1} < 5y + 2x \leqq 10D_k$,

$$5y - 2x \geqq 10d_k - 10D_{k-1},$$

$$x \geqq 0.$$

If we subtract $2d_k$ from y we obtain a point P' with rectangular coordinates $y' = y - 2d_k$ and $x' = x$.

$$5y' + 2x' \leqq 10D_k - 10d_k = 10D_{k-1}, \qquad x' \geqq 0,$$

$$5y' - 2x' \geqq 10d_k - 10D_{k-1} - 10d_k = -10D_{k-1}.$$

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors V_0, V_1, $\cdots$,V_{k-1}. $P' = P - 2d_k\sqrt{7}$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2d_k\sqrt{7} = \sigma_{k-1} + (c_k + d_k\sqrt{7}) + (-c_k + d_k\sqrt{7})$. This completes Case 1.

Case 2. $10D_{k-1} < 5y + 2x \leqq 10D_k$,

$5y - 2x \leqq -4c_k + 10D_{k-1}$,

$y \geqq 0$.

If we subtract $2c_k$ from x, we obtain a point P' with rectangular coordinates $y' = y$ and $x' = x - 2c_k$. Applying Lemma 2 we obtain:

$5y' + 2x' \leqq 10D_k - 4c_k < 10D_k - 10d_k = 10D_{k-1}$,

$5y' - 2x' \leqq -4c_k + 10D_{k-1} + 4c_k = 10D_{k-1}$,

$y' \geqq 0$.

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors V_0, V_1, $\cdots$,V_{k-1}. $P' = P - 2c_k$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + 2c_k = \sigma_{k-1} + (c_k + d_k\sqrt{7}) + (c_k - d_k\sqrt{7})$.

This completes Case 2.

Case 3. $10D_{k-1} < 5y + 2x \leqq 10D_k$,

$$-4c_k + 10D_{k-1} < 5y + 2x < 10d_k - 10D_{k-1}.$$

If we subtract c_k from x and d_k from y we obtain a point P' with rectangular coordinates $y' = y - d_k$ and $x' = x - c_k$. Applying Lemma 2 we obtain:

$$5y' + 2x' \leqq 10D_k - 5d_k - 2c_k < 10D_k - 10d_k = 10D_{k-1}.$$

Applying Lemma 3 to the following, we obtain:

$$5y' + 2x' > 10D_{k-1} - 5d_k - 2c_k > 10D_{k-1} - 20D_{k-1} = -10D_{k-1},$$

$$5y' - 2x' < 10d_k - 10D_{k-1} - 5d_k + 2c_k = 2c_k + 5d_k - 10D_{k-1}$$
$$< 20D_{k-1} - 10D_{k-1} = 10D_{k-1},$$

$$5y' - 2x' > -4c_k + 10D_{k-1} + 2c_k - 5d_k = 10D_{k-1} - 2c_k - 5d_k$$
$$> 10D_{k-1} - 20D_{k-1} = -10D_{k-1}.$$

The resulting point P' lies inside the k-1 diamond and thus can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V_0, V_1, \cdots, V_{k-1}$. $P' = P - c_k - d_k\sqrt{7}$ and $P' = \sigma_{k-1}$. Solving for P we obtain $P = \sigma_{k-1} + c_k + d_k\sqrt{7}$. This completes Case 3 and the theorem is proved.

COROLLARY. If $x \equiv y \equiv 0$ modulo 2, then the integer $x + y\sqrt{7}$ belongs to A(7).

CHAPTER III

REAL QUADRATIC FIELDS
$m > 7$ and $m \equiv 2$ or $3 \bmod 4$

In this chapter we will focus our attention upon the integers in the quadratic fields $R(m)$ where m is square free, greater than seven and not congruent to one modulo four. We will show that if the norm of u_1 $[N(u_1)]$ equals -1, then the number $T = 2^t\sqrt{m}$ belongs to $A(m)$ and if $N(u_1) = +1$, then the number $T = 2^{t+1}\sqrt{m}$ belongs to $A(m)$ where t is chosen so that $2^t \leqq a_2/a_1 < 2^{t+1}$.

Let us separate the divisors of T into the sets $V_0, V_1, \cdots$ where $V_0 = S_0$ and $V_k = 2^r S_{h+1}$ where $h \geqq 0$, $0 \leqq r \leqq t$ and $k = h(t+1) + r + 1$. In every set V_k there exists exactly one divisor $c_k + d_k\sqrt{m}$ with both c_k and d_k non-negative. A divisor of T belongs to at most one set. We will make use of the fact that $a_k = a_1a_{k-1} + b_1b_{k-1}m$ and $b_k = a_1b_{k-1} + b_1a_{k-1}$. We will first prove a series of lemmas for the fields where $N(u_1) = -1$.

LEMMA 1. $2^t > a_1 + 1/2a_1$.

PROOF: $2^t > a_2/2a_1 = (a_1^2+b_1^2m)/2a_1 = (2a_1^2+1)/2a_1$.

LEMMA 2. $b_1/a_1 > b_k/a_k$ for $k > 1$.

PROOF: Since $a_1^2 - b_1^2m = -1$ and $b_{k-1} > 0$, we have $0 > (a_1^2 - b_1^2m)b_{k-1}$.

$a_1b_1a_{k-1} + b_1^2b_{k-1}m > a_1b_1a_{k-1} + a_1^2b_{k-1}.$

$b_1(a_1a_{k-1} + b_1b_{k-1}m) > a_1(b_1a_{k-1} + a_1b_{k-1}).$

$b_1a_k > a_1b_k.$

LEMMA 3. $a_k/b_k < 3a_1/b_1$ for all k.

PROOF: $0 < b_{k-1}(a_1^2-b_1^2m+2a_1^2) + 2a_1b_1a_{k-1}.$

$a_1b_1a_{k-1} + b_1^2b_{k-1}m < 3a_1^2b_{k-1} + 3a_1b_1a_{k-1}.$

$b_1(a_1a_{k-1} + b_1b_{k-1}m) < 3a_1(a_1b_{k-1} + b_1a_{k-1}).$

$b_1a_k \quad < 3a_1b_k.$

LEMMA 4. $(2b_1a_1-b_1)a_{k-1}/a_1 < 3(2^{t+1}-1)b_{k-1}$ for $k>1$.

PROOF: By Lemma 3 we know that $b_1a_{k-1} < 3a_1b_{k-1}$. Since $2a_1 - 1 > 0$, we may multiply both sides by $2a_1-1$, and obtain $(2a_1b_1-b_1)a_{k-1}< 3(2a_1^2-a_1)b_{k-1}< 3(2a_1^2-a_1+1)b_{k-1}.$ Since $a_1 > 0$, we may divide both sides by a_1 and obtain $(2b_1a_1-b_1)a_{k-1}/a_1 < 3(2a_1+\frac{1}{a_1} -1)b_{k-1} < 3(2^{t+1} -1)b_{k-1}.$

LEMMA 5. $b_k + \frac{b_1}{a_1}a_k \leq 4(2^{t+1} -1)B_{k-1}$ when $k > 1$.

PROOF: If $k = 2$ we have that $b_k + \frac{b_1a_k}{a_1} =$

$2a_1b_1 + \frac{b_1}{a_1}(2a_1^2 + 1) = 4b_1(a_1 + \frac{1}{4a_1}) < 4(2^t)b_1 < 4(2^{t+1}-1)b_1.$

Let us assume that the lemma is true for $k \leq n-1$ and prove the lemma by induction. Let $k = n$.

For $k = n$ we have that $b_n + \frac{b_1}{a_1}a_n$ is equal to

$a_1b_{n-1} + b_1a_{n-1} + \frac{b_1}{a_1}(a_1a_{n-1} + b_1b_{n-1}m) = b_{n-1}(a_1+\frac{b_1^2m}{a_1}) + 2b_1a_{n-1} = b_{n-1}(2a_1+\frac{1}{a_1}) + 2b_1a_{n-1}$. By Lemma 1 the right hand side is less than $2^{t+1}b_{n-1} + 2b_1a_{n-1} = (2^{t+1}-1)b_{n-1} + b_{n-1} + \frac{b_1}{a_1}a_{n-1} + a_{n-1}(2a_1b_1-b_1)/a_1$. Using Lemma 4 and the induction hypothesis we find that the right hand side is $< (2^{t+1}-1)b_{n-1} + 4(2^{t+1}-1)B_{n-2} + 3(2^{t+1}-1)b_{n-1}$. The latter is exactly $4(2^{t+1}-1)B_{n-1}$ which completes the proof.

LEMMA 6. $2^rb_k + \frac{b_1}{a_1}2^ra_k \leq 4(2^{t+1}-1)B_{k-1} + 4(2^r-1)b_k$.

PROOF: If $k = 1$ and $r = 1$ the lemma is true. When $r = 1$ and $k > 1$, we use Lemmas 3 and 5 to see that $2b_k + \frac{2b_1}{a_1}a_k < 4(2^{t+1}-1)B_{k-1} + b_k + 3b_k = 4(2^{t+1}-1)B_{k-1} + 4b_k$. Thus the lemma is true for all values of k when $r =1$. Let us assume that the lemma is true for $r \leq n-1$ and prove the lemma by induction. Using Lemma 3 and the induction hypothesis: $2^nb_k + 2^n\frac{b_1}{a_1}a_k = 2(2^{n-1}b_k + 2^{n-1}\frac{b_1}{a_1}a_k) \leq 4(2^{t+1} - 1)B_{k-1} + 4(2^{n-1} -1)b_k + 2^{n-1}b_k + 2^{n-1}(3b_k) = 4(2^{t+1} - 1)B_{k-1} + 4(2^n - 1)b_k$. This completes the proof of the lemma.

Let $D_k = \sum_{i=o}^{k} d_i$ and $s = \frac{b_1}{a_1}$. We can now summarize the results of Lemmas 1 through 6 as follows:

LEMMA 7. $d_k + sc_k \leqq 4D_{k-1}$ and $s > d_k/c_k$ for $k > 1$.

We now turn our attention to the fields in which $N(u_1) = +1$ and prove the analogous lemmas. We will let $s = (\sqrt{m})^{-1}$ and note that $s > b_k/a_k$ for all k

LEMMA 8. $2^t > a_1 - 1/2a_1$.

PROOF: $2^t > a_2/2a_1 = (a_1^2 + b_1^2 m)/2a_1 = (2a_1^2 - 1)/2a_1$.

LEMMA 9. $sa_k < \frac{3}{2}b_k$ for $k > 0$.

PROOF: $a_k^2 - b_k^2 m = 1 < \frac{5}{4}b_k^2 m$.

$a_k^2 < \frac{9}{4}b_k^2 m$.

$a_k < \frac{3}{2}b_k\sqrt{m}$.

LEMMA 10. $2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3})$.

PROOF: Since $m \geqq 10$ we have that $a_1^2 = 1 + mb_1^2 > 10$ which implies that $a_1 > 3$ and $0 < 2a_1^2 - a_1 - 4$. Thus

$4a_1^2 - 2a_1 < 6a_1^2 - 3a_1 - 4$.

$2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3})$.

LEMMA 11. $b_{k-1}/a_1 + (2a_1 - 1)sa_{k-1} < 3(2^{t+1} - 1)b_{k-1}$ for $k>1$.

PROOF: Making use of Lemmas 9 and 10, we have:

Let $D_k = \sum_{i=o}^{k} d_i$ and $s = \frac{b_1}{a_1}$. We can now summarize the results of Lemmas 1 through 6 as follows:

LEMMA 7. $d_k + sc_k \leqq 4D_{k-1}$ and $s > d_k/c_k$ for $k > 1$.

We now turn our attention to the fields in which $N(u_1) = +1$ and prove the analogous lemmas. We will let $s = (\sqrt{m})^{-1}$ and note that $s > b_k/a_k$ for all k

LEMMA 8. $2^t > a_1 - 1/2a_1$.

PROOF: $2^t > a_2/2a_1 = (a_1^2+b_1^2m)/2a_1 = (2a_1^2-1)/2a_1$.

LEMMA 9. $sa_k < \frac{3}{2}b_k$ for $k > 0$.

PROOF: $a_k^2 - b_k^2m = 1 < \frac{5}{4}b_k^2m$.

$a_k^2 < \frac{9}{4}b_k^2m$.

$a_k < \frac{3}{2}b_k\sqrt{m}$.

LEMMA 10. $2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3})$.

PROOF: Since $m \geq 10$ we have that $a_1^2 = 1 + mb_1^2 > 10$ which implies that $a_1 > 3$ and $0 < 2a_1^2 - a_1 - 4$. Thus

$4a_1^2 - 2a_1 < 6a_1^2 - 3a_1 - 4$.

$2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3})$.

LEMMA 11. $b_{k-1}/a_1 + (2a_1-1)sa_{k-1} < 3(2^{t+1}-1)b_{k-1}$ for $k>1$.

PROOF: Making use of Lemmas 9 and 10, we have:

$s(2a_1^2-a_1)a_{k-1} < 9/4(2a_1^2-a_1-\frac{4}{3})b_{k-1} < 3(2a_1^2-a_1-\frac{4}{3})b_{k-1}$. If we divide both sides of the inequality by a_1, we obtain $s(2a_1-1)a_{k-1} < 3(2a_1-1-\frac{1}{a_1})b_{k-1} - \frac{1}{a_1}b_{k-1}$. Using Lemma 8 we have $b_{k-1}/a_1 + s(2a_1 - 1)a_{k-1} < 3(2a_1 - \frac{1}{a_1} - 1)b_{k-1} < 3(2^{t+1} - 1)b_{k-1}$.

LEMMA 12. $b_k + sa_k \leq 4(2^{t+1} - 1)b_{k-1}$ for $k > 1$.

PROOF: When $k = 2$ we have that $b_2 + sa_2 = 2a_1b_1 + s(a_1^2+b_1^2m) = 2a_1b_1 + s(1+2b_1^2m) = 2a_1b_1 + 2b_1^2\sqrt{m} + s = 2a_1b_1(1+\frac{b_1\sqrt{m}}{a_1}) + s < 4a_1b_1 + s = 4b_1(a_1-\frac{1}{2a_1}) + \frac{2b_1}{a_1} + s$. Applying lemma 8 we see that the right hand side is $< 4b_1(2^t) + 2b_1/a_1 + s < 4b_1(2^t) + 3s < 4b_1(2^{t+1} - 1)$. This proves the lemma for $k = 2$. Let us assume that the lemma is true for $k \leq n-1$ and prove the lemma by induction. $b_n + sa_n = a_1b_{n-1} + b_1a_{n-1} + s(a_1a_{n-1}+b_1b_{n-1}m) = b_{n-1}(a_1+b_1\sqrt{m}) + (b_1+sa_1)a_{n-1} < 2a_1b_{n-1} + 2sa_1a_{n-1}$. By Lemma 8 the right hand side is less than $(2^{t+1}+\frac{1}{a_1})b_{n-1} + 2sa_1a_{n-1} = (2^{t+1}-1)b_{n-1} + (1+\frac{1}{a_1})b_{n-1} + sa_{n-1} + s(2a_1-1)a_{n-1} = (2^{t+1}-1)b_{n-1} + b_{n-1} + sa_{n-1} + b_{n-1}/a_1 + (2a_1-1)sa_{n-1}$. Making use of Lemma 11 and the induction hypothesis we find that the right hand side is less than $(2^{t+1} - 1)b_{n-1} + 4(2^{t+1} - 1)B_{n-2} + 3(2^{t+1}-1)b_{n-1}$ $= 4(2^{t+1} -1)B_{n-1}$ which proves the lemma.

LEMMA 13. $2^r b_k + 2^r sa_k \leq 4(2^{t+1}-1)B_{k-1} + 4(2^r-1)b_k$ for all r when $k > 1$ and $r > 1$ when $k = 1$.

PROOF: Lemma 9 tells us that $sa_1 < b_1$ which implies that $4b_1 + 4sa_1 < 4\cdot 3b_1$ and the lemma is proved for $k =1$ and $r = 2$. If $k > 1$ and $r = 1$, then we have $2b_k + 2sa_k < 4(2^{t+1}-1)B_{k-1} + b_k + \frac{3}{2}b_k < 4(2^{t+1}-1)B_{k-1} + 4b_k$. Let us assume that the lemma is true for $r = n-1$ and prove the lemma by induction. Using the induction hypothesis and Lemma 9 we have $2^n b_k + 2^n sa_k = 2(2^{n-1}b_k+2^{n-1}sa_k) <$ $4(2^{t+1}-1)B_{k-1} + 4(2^{n-1}-1)b_k + 2^{n-1}b_k + \frac{3}{2}(2^{n-1})b_k <$ $4(2^{t+1}-1)B_{k-1} + 4(2^{n-1}-1)b_k + 4(2^{n-1})b_k =$ $4(2^{t+1}-1)B_{k-1} + 4(2^n - 1)b_k$. Thus the lemma is proved.

We can now summarize the results of Lemmas 8 to 13 as follows:

LEMMA 14. $d_k + sc_k \leq 4D_{k-1}$, and $s > d_k/c_k$.

The first part of the lemma is true only for $k \geq 3$.

We will now proceed to show that the units do not belong to A(m) and T does belong to A(m). It is obvious that if $b_1 \neq 1$, the units do not belong to A, for $b_1 | b_k$

implies that the coefficient of $\sqrt{m}$ in any sum of units is congruent to zero modulo b_1. We make use of the following lemma to show that the units do not belong to A(m) if $b_1 = 1$.

LEMMA 15. $a_k > 2\sum_{i=1}^{k-1} a_i + 3$ for $k > 1$.

PROOF: $a_1^2 = \pm 1 + m \geqq m - 1 \geqq 9$ implies that $a_1 \geqq 3$. If $k = 2$, then $a_2 = a_1^2 + b_1^2 m > 2a_1 + 3$. Let us assume that the lemma is true for $k = n-1$ and prove the lemma by induction. $a_n = a_1 a_{n-1} + b_{n-1} m > 3a_{n-1} > 2a_{n-1} + 2\sum_{1}^{n-2} a_i + 3$ $= 2\sum_{i=1}^{n-1} a_i + 3$.

THEOREM 1. The units do not belong to A(m).

PROOF: We have shown that when $b_1 \neq 1$, the theorem is true. By Lemma 15 no unit in S_j for $j > 1$ can be used to represent the number two. It can be seen by inspection that the number two cannot be represented using only units in S_0 and S_1. Therefor the number two cannot be written as a finite sum of distinct units.

THEOREM 2. The number T belongs to the set A(m).

PROOF: We may consider every integer P of D(m) as a lattice point P in the plane with rectangular coordinates x and y. We will fill the plane with concentric diamonds with centers at the origin and whose diagonals are the coordinate axes. The y intercepts of the sides of the k diamond are $2D_k$ and $-2D_k$ and the x intercepts are $2D_k/s$ and $-2D_k/s$. For every point in the k diamond we have:

$$-2D_k \leqq y + sx \leqq 2D_k,$$

$$-2D_k \leqq y - sx \leqq 2D_k.$$

Let V_{-1} be the set $1,2,4,\cdots,\frac{T}{\sqrt{m}},\sqrt{m},2\sqrt{m},\cdots,T$. It will be shown that every integer lying in the k diamond can be represented as a sum of distinct divisors in the first k+1 sets $V_{-1},V_1,V_2,\cdots,V_k$. Since the divisors in V_j and the diamonds are symmetric with respect to both coordinate axes there is no loss of generality by considering only points in the first quadrant on the positive x axis and on the positive y axis. If $N(u_1) = -1$, then we have $2^t > a_1$ and $2^{t+1} - 1 > 2a_1 - 1 > \frac{2b_1}{s} - 1 > 2b_1 - 1.$

Thus every integer in the first diamond can be written as a sum σ_1 of distinct divisors in the sets V_{-1} and V_1. If $N(u_1) = +1$, we have $2^{t+2} - 1 > 4a_1 - 1 > \frac{4b_1}{s} - 1 >$ $4b_1 - 1$ and $2a_1 + 2^{t+2} - 1 > 6a_1 - 1 > \frac{6b_1}{s} - 1.$ Also $2b_1 + 2^{t+2} - 1 > 6b_1 - 1.$ Thus every integer in the second diamond can be represented as a sum σ_2 of distinct divisors in the sets V_{-1},V_1 and V_2.

Let us assume that every integer $P' = x' + y'\sqrt{m}$ in the first k-1 diamonds can be represented as a sum of distinct divisors σ_{k-1} of T without using any divisors in the sets V_j for $j > k-1$ and prove the theorem by induction. Let P be any point (x,y) in the k diamond. If P lies in the $k-1$ diamond, then the theorem is proved. If P is not in the $k-1$ diamond we have $2D_{k-1} < y + sx \leqq 2D_k$, $x \geqq 0$ and $y \geqq 0$. We divide the possibilities into three cases. In Case 1 the point lies on or above the line $y - sx = 2d_k - 2D_{k-1}$. In Case 2 the point lies on or below the line $y - sx = -2sc_k + 2D_{k-1}$. In Case 3 the point lies between these two lines.

Case 1. $2D_{k-1} < y + sx \leqq 2D_k$,

$$y - sx \geqq 2d_k - 2D_{k-1},$$

$$x \geqq 0, \qquad y \geqq 0.$$

If we subtract $2d_k$ from y we obtain a point P' with rectangular coordinates $x' = x$ and $y' = y - 2d_k$.

$$y' + sx' \leqq 2D_k - 2d_k = 2D_{k-1},$$

$$y' - sx' \geqq 2d_k - 2D_{k-1} - 2d_k = -2D_{k-1}.$$

Since $x' = x \geqq 0$, the point P' lies in the $k-1$ diamond and $P' = \sigma_{k-1}$. Since $P' = P - 2d_k\sqrt{m}$, we have $P = \sigma_{k-1} + 2d_k\sqrt{m} = \sigma_{k-1} + (c_k + d_k\sqrt{m}) + (-c_k + d_k\sqrt{m})$ and the theorem is proved for Case 1.

Case 2. $2D_{k-1} < y + sx \leqq 2D_k$,

$$y - sx \leqq 2D_{k-1} - 2sc_k,$$

$$y \geqq 0, \qquad x \geqq 0.$$

If we subtract $2c_k$ from x we obtain a point P' with rectangular coordinates $x' = x - 2c_k$ and $y' = y$.

$$y' + sx' \leqq 2D_k - 2sc_k < 2D_k - 2d_k = 2D_{k-1},$$

$$y' - sx' \leqq 2D_{k-1} - 2sc_k + 2sc_k = 2D_{k-1}.$$

Since $y' = y \geqq 0$, the point P' lies in the k-1 diamond and $P' = \sigma_{k-1}$. From $P' = P - 2c_k$ we obtain $P = \sigma_{k-1} + 2c_k = \sigma_{k-1} + (c_k + d_k\sqrt{m}) + (c_k - d_k\sqrt{m})$ and the theorem is proved in Case 2.

Case 3. $2D_{k-1} < y + sx \leqq 2D_k$,

$$-2sc_k + 2D_{k-1} < y - sx < 2d_k - 2D_{k-1}.$$

If we subtract d_k from y and c_k from x we obtain a point P' with coordinates $x' = x - c_k$ and $y' = y - d_k$. Making repeated use of Lemmas 7 and 14, we obtain:

$$y' + sx' \leqq 2D_k - d_k - sc_k < 2D_k - d_k - d_k = 2D_{k-1},$$

$$y' + sx' > 2D_{k-1} - d_k - sc_k \geqq 2D_{k-1} - 4D_{k-1} = -2D_{k-1},$$

$$y' - sx' < 2d_k - 2D_{k-1} - d_k + sc_k \leqq -2D_{k-1} + 4D_{k-1} = 2D_{k-1},$$

$$y' - sx' > -2sc_k + 2D_{k-1} - d_k + sc_k \geqq -2D_{k-1}.$$

Thus the point P' lies inside the k-1 diamond and $P' = \sigma_{k-1}$. Since $P' = P - c_k - d_k\sqrt{m}$, we have that $P = \sigma_{k-1} + (c_k + d_k\sqrt{m})$ and the theorem is proved in Case 3. This completes the proof of the theorem.

CHAPTER IV

REAL QUADRATIC FIELDS

$m \equiv 1$ modulo 4

In this chapter we will focus our attention upon the integers in the quadratic fields $R(m)$, where m is square free, greater than zero and congruent to one modulo four.

The domain of integers $D(m)$ of $R(m)$ is the set of numbers of the form $x + y\sqrt{m}$, where x and y are both rational integers or halves of odd rational integers. The units are those numbers of $D(m)$ where $x^2 - my^2 = \pm 1$. Each field has an infinite number of units and one basic unit u_1 such that if $u_1 = a_1 + b_1\sqrt{m}$ is the unit of smallest absolute value for which both x and y are positive, then every unit in the field can be written $\pm u_1^n$ for $n = 0, \pm 1, \pm 2, \cdots$.

Let $u_j = u_1^j$ for $j = 0, 1, 2, \cdots$. In this way we may divide the units into one set $S_0 = [+1, -1]$ of two units and infinitely many sets $S_i = [u_i, -u_i, -u_i^{-1}, u_i^{-1}]$ of four units each. Every unit belongs to exactly one set.

DEFINITION: $A(m)$ is the set of integers $\{n\}$ in the field $R(m)$ which have the following property; n belongs to $A(m)$ if and only if every integer in the domain $D(m)$ can be written as a sum of distinct divisors of n.

We will show that for all m the set $A(m)$ is not empty and that for $m = 5$ every integer belongs to A -`.

$$m = 5$$

In this field $u_1 = (1 + \sqrt{5})/2$, $u_2 = (3 + \sqrt{5})/2$, $u_3 = 2 + \sqrt{5}$, $\cdots$. Let a_i be the rational part of u_i and b_i be the coefficient of $\sqrt{5}$ in u_i. Also let $B_k = \sum_{i=1}^{k} b_i$.

The integers of $D(5)$ may be considered to be those points in the plane for which x and y are rational integers or halves of odd rational integers. We will fill the plane with concentric squares with centers at the origin, whose diagonals are the coordinate axes and whose sides are given by the equations $x + y = \pm 2B_k$ and $y - x = \pm 2B_k$. It will be shown that every integer lying in the k square can be represented as a sum of distinct units belonging to the first $k + 1$ sets $S_0, \cdots, S_k$. We will make use of the following equations:

$a_k = (a_{k-1} + 5b_{k-1})/2$ and $b_k = (a_{k-1} + b_{k-1})/2$.

LEMMA 1. $a_k + b_k \leqq 4B_{k-1}$.

PROOF: The lemma is true for $k = 1$. Let us assume that the lemma is true for $k = n-1$ and prove the lemma true by induction. Let $k = n$.

$$a_n + b_n = a_{n-1} + 3b_{n-1} = a_{n-1} + b_{n-1} + 2b_{n-1} <$$

$$4B_{n-2} + 2b_{n-1} < 4B_{n-1}.$$

LEMMA 2. $a_k + b_k > 2B_{k-1}$.

PROOF: The lemma is true for $k = 2$. Let us assume that the lemma is true for $k = n-1$ and prove that the lemma is true by induction. Let $k = n$.

$$a_n + b_n = a_{n-1} + 3b_{n-1} = a_{n-1} + b_{n-1} + 2b_{n-1}$$

$$> 2B_{n-2} + 2b_{n-1} = 2B_{n-1}.$$

THEOREM 1. The units belong to A(5).

PROOF: We can see by inspection that every integer in the first square can be represented as a sum of distinct units in S_0 and S_1. Let us assume that every integer in the first k-1 squares can be written as a sum of distinct units taken from the first k sets of units, $S_0, S_1, \cdots, S_{k-1}$ and prove the theorem by induction. Consider any point inside the k square. Since the square and the sets S_k are symmetric with respect to both axes there is no loss in generality in considering only points in the first quadrant, on the positive x axis and on the positive y axis. If the point lies inside the k-1 square, the required representation is assured by the induction hypothesis. Therefore we may consider only points in the first quadrant, in the k square and not in the k-1 square.

Thus we have: $2B_{k-1} < x + y \leqq 2B_k$,

$$y \geqq 0, \qquad x \geqq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies on or above the line $y - x = 2b_k - 2B_{k-1}$. In Case 2 the point lies on or below the line $y-x = -2a_k + 2B_{k-1}$. In Case 3 the point lies between these two lines. Lemma 2 tells us that Case 3 actually exists.

Case 1. $2B_{k-1} < x + y \leqq 2B_k$,

$$y - x \geqq 2b_k - 2B_{k-1},$$

$$x \geqq 0.$$

If we subtract $2b_k$ from y we obtain a point P' with rectangular coordinates $x' = x$ and $y' = y - 2b_k$.

$$x' + y' \leqq 2B_k - 2b_k = 2B_{k-1},$$

$$y' - x' \geqq 2b_k - 2B_{k-1} - 2b_k = -2B_{k-1},$$

$$x' \geqq 0.$$

The resulting point P' lies in the k-1 square and thus by the induction hypothesis it can be represented as a sum σ_{k-1} of distinct units in the first k sets of units $S_0, S_1, \cdots, S_{k-1}$. $P' = \sigma_{k-1}$ and $P' = P - 2b_k\sqrt{5}$. Thus $P = \sigma_{k-1} + 2b_k\sqrt{5} = \sigma_{k-1} + (a_k + b_k\sqrt{5}) + (-a_k + b_k\sqrt{5})$. This completes Case 1.

Case 2. $2B_{k-1} < x + y \leqq 2B_k$,

$$y - x \leqq -2a_k + 2B_{k-1},$$

$$y \geqq 0.$$

If we subtract $2a_k$ from x we obtain a point P' with rectangular coordinates $x' = x - 2a_k$ and $y' = y$.

$$x' + y' \leqq 2B_k - 2a_k < 2B_k - 2b_k = 2B_{k-1},$$

$$y' - x' \leqq -2a_k + 2B_{k-1} + 2a_k = 2B_{k-1},$$

$$y' \geqq 0.$$

The resulting point P' lies in the k-1 square and thus by the induction hypothesis it can be represented as a sum σ_{k-1} of distinct units in the first k sets of units $S_0, S_1, \cdots, S_{k-1}$. $P' = \sigma_{k-1}$ and $P' = P - 2a_k$. Therefore $P = \sigma_{k-1} + 2a_k = \sigma_{k-1} + (a_k + b_k\sqrt{5}) + (a_k - b_k\sqrt{5})$. This completes Case 2.

Case 3. $2B_{k-1} < x + y \leqq 2B_k$,

$$2B_{k-1} - 2a_k < y - x < 2b_k - 2B_{k-1}.$$

If we subtract a_k from x and b_k from y, we obtain a point P' with coordinates $x' = x - a_k$ and $y' = y - b_k$.

$$x' + y' \leqq 2B_k - b_k - a_k < 2B_k - 2b_k = 2B_{k-1}.$$

Applying Lemma 1 we obtain the following inequalities:

$$x' + y' > 2B_{k-1} - a_k - b_k \geqq -2B_{k-1},$$

$$y' - x' < 2b_k - 2B_{k-1} - b_k + a_k = a_k + b_k - 2B_{k-1} \leqq 2B_{k-1},$$

$$y' - x' > 2B_{k-1} - 2a_k - b_k + a_k = 2B_{k-1} - a_k - b_k \geqq -2B_{k-1}.$$

The resulting point P' lies in the k-1 square and thus by the induction hypothesis can be represented as a sum σ_{k-1} of distinct units in the first k sets of units $S_0, S_1, \cdots, S_{k-1}$. $P' = \sigma_{k-1}$ and $P' = P - a_k - b_k\sqrt{5}$. Solving for P we obtain $P = \sigma_{k-1} + a_k + b_k\sqrt{5}$. This completes Case 3 and the theorem is proved.

$m > 5$

In this section we will show that if $N(u_1) = -1$, then the number $T = 2^{t-1}(1+\sqrt{m})$ belongs to A(m) and if $N(u_1) = +1$, then the number $T = 2^t(1+\sqrt{m})$ belongs to A(m), where t is chosen so that $2^t \leqq a_2/a_1 < 2^{t+1}$. In either case 2^t divides T.

Let us separate the divisors of T into the sets $V_0 = S_0$, and $V_k = 2^r S_{h+1}$ where $k = h(t + 1) + r + 1$, $h \geqq 0$, and $0 \leqq r \leqq t$. In every set V_k there exists exactly one divisor $c_k + d_k\sqrt{m}$ with both c_k and d_k non-negative. A divisor of T belongs to at most one set. We will make use of the fact that $a_k = a_1a_{k-1} + b_1b_{k-1}m$. and $b_k = b_1a_{k-1} + a_1b_{k-1}$. We will first prove a series of lemmas for the fields in which $N(u_1) = -1$.

LEMMA 3. $2^t > a_1 + 1/2a_1$.

PROOF: $2^t > a_2/2a_1 = (a_1^2 + b_1^2m)/2a_1 = (2a_1^2 + 1)/2a_1$.

LEMMA 4. $b_1/a_1 > b_k/a_k$ for $k > 1$.

PROOF: Since $a_1^2 - b_1^2m = -1$ and $b_{k-1} > 0$, we have

$0 > (a_1^2 - b_1^2m)b_{k-1}$.

$a_1b_1a_{k-1} + b_1^2b_{k-1}m > a_1b_1a_{k-1} + a_1^2b_{k-1}$.

$b_1(a_1a_{k-1} + b_1b_{k-1}m) > a_1(b_1a_{k-1} + a_1b_{k-1})$.

$b_1a_k > a_1b_k$.

LEMMA 5. $a_k/b_k < 3a_1/b_1$ for all $k > 1$.

PROOF: $0 < b_{k-1}(a_1^2 - b_1^2m + 2a_1^2) + 2a_1b_1a_{k-1}$.

$a_1b_1a_{k-1} + b_1^2b_{k-1}m < 3a_1^2b_{k-1} + 3a_1b_1a_{k-1}$.

$b_1(a_1a_{k-1} + b_1b_{k-1}m) < 3a_1(a_1b_{k-1} + b_1a_{k-1})$.

$b_1a_{k-1} < 3a_1b_{k-1}$.

LEMMA 6. $(2b_1a_1-b_1)a_{k-1}/a_1 < 3(2^{t+1}-1)b_{k-1}$ for $k>1$.

PROOF: From Lemma 5 we have $b_1a_{k-1} < 3a_1b_{k-1}$. Because $(2a_1 - 1)/a_1 > 0$, we may multiply both sides of the above inequality by $(2a_1 - 1)/a_1$ and obtain the following.

$$(2a_1b_1-b_1)a_{k-1}/a_1 < 3(2a_1 + \frac{1}{a_1} - 1)b_{k-1} < 3(2^{t+1} - 1)b_{k-1}.$$

LEMMA 7. $b_k + b_1a_k/a_1 \leqq 4(2^{t+1} - 1)B_{k-1}$ for $k > 1$.

PROOF: When $k = 2$ we have that $b_k + b_1a_k/a_1 =$

$$2a_1b_1 + b_1(2a_1^2+1)/a_1 = 4b_1(a_1 + \frac{1}{4a_1}) < 4\cdot2^tb_1 < 4(2^{t+1}-1)b_1.$$

Let us assume that the lemma is true for $k \leqq n - 1$ and prove the lemma true by induction. Let $k = n$.

$$b_n + b_1 a_n/a_1 = a_1 b_{n-1} + b_1 a_{n-1} + b_1(a_1 a_{n-1} + b_1 b_{n-1} m)/a_1$$

$$= b_{n-1}\left(a_1 + \frac{b_1^2 m}{a_1}\right) + 2b_1 a_{n-1} = b_{n-1}\left(2a_1 + \frac{1}{a_1}\right) + 2b_1 a_{n-1}.$$

By Lemma 3 the right hand side is less than

$2^{t+1} b_{n-1} + 2b_1 a_{n-1} = (2^{t+1}-1)b_{n-1} + b_{n-1} + b_1 a_{n-1}/a_1 +$

$a_{n-1}(2a_1 b_1 - b_1)/a_1$. Applying Lemma 6 and the induction hypothesis we find that the right hand side is less than

$(2^{t+1} - 1)b_{n-1} + 4(2^{t+1} - 1)B_{n-2} + 3(2^{t+1} - 1)b_{n-1}$.

LEMMA 8. $2^r b_k + 2^r b_1 a_k/a_1 \leqq 4(2^{t+1}-1)B_{k-1} + 4(2^r-1)b_k$.

PROOF: The lemma is true when $k = 2$ and $r = 1$. When $r = 1$ and $k > 2$, we apply Lemmas 5 and 7 to obtain

$$2b_k + 2b_1 a_k/a_1 < 4(2^{t+1} - 1)B_{k-1} + b_k + 3b_k$$

$$= 4(2^{t+1} -1)B_{k-1} + 4b_k.$$

Thus the lemma is true for all values of $k > 1$ when $r = 1$. Let us assume that the lemma is true for $r \leqq n - 1$ and prove the lemma true by induction. Let $r = n$. Applying Lemma 5 and the induction hypothesis we obtain

$2^n b_k + 2^n b_1 a_k/a_1 = 2(2^{n-1} b_k + 2^{n-1} b_1 a_k/a_1) \leqq$

$4(2^{t+1} - 1)B_{k-1} + 4(2^{n-1} - 1)b_k + 2^{n-1} b_k + 2^{n-1}(3b_k) =$

$4(2^{t+1} - 1)B_{k-1} + 4(2^n - 1)b_k$. This completes the proof.

Let $D_k = \sum_{i=1}^{k} d_i$ and $s = b_1/a_1$. We can now summarize the results of Lemmas 3 through 8 as follows:

LEMMA 9. $d_k + sc_k \leqq 4D_{k-1}$ and $s > d_k/c_k$ for $k > 1$.

We now turn our attention to the fields where $N(u_1) = +1$ and prove the analogous lemmas. We will let $s = (\sqrt{m})^{-1}$ and note that $s > b_k/a_k$ for all k.

LEMMA 10. $2^t > a_1 - 1/2a_1$.

PROOF: $2^t > a_2/2a_1 = (a_1^2 + b_1^2 m)2a_1 = (2a_1^2 - 1)/2a_1$.

LEMMA 11. $sa_k < 3b_k/2$ for $k \geqq 1$.

PROOF: $a_k^2 - b_k^2 m = 1 < 5b_k^2 m/4$.

$$a_k^2 < 9b_k^2 m/4.$$

$$a_k < 3b_k^2\sqrt{m}/2.$$

LEMMA 12. $2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3})$.

PROOF: For $m \geqq 13$, $a_1^2 = 1 + mb_1^2 > 4$ implies $a_1 > 2$ which in turn implies $0 < 2a_1^2 - a_1 - 4$. Therefore

$$4a_1^2 - 2a_1 < 6a_1^2 - 3a_1 - 4.$$

$$2a_1^2 - a_1 < \frac{3}{2}(2a_1^2 - a_1 - \frac{4}{3}).$$

LEMMA 13. $b_{k-1}/a_1 + (2a_1 - 1)sa_{k-1} < 3(2^{t+1} - 1)b_{k-1}$.

PROOF: Applying Lemmas 11 and 12, we obtain:

$$s(2a_1^2 - a_1)a_{k-1} < \frac{9}{4}(2a_1^2 - a_1 - \frac{4}{3})b_{k-1} < 3(2a_1^2 - a_1 - \frac{4}{3})b_{k-1}.$$

Dividing both sides of the inequality by a_1, we obtain:

$$s(2a_1 - 1)a_{k-1} < 3(2a_1 - 1 - \frac{1}{a_1})b_{k-1} - b_{k-1}/a_1.$$

Applying Lemma 10 we obtain:

$$\frac{b_{k-1}}{a_1} + s(2a_1 -1)a_{k-1} < 3(2a_1 - \frac{1}{a_1} -1)b_{k-1} < 3(2^{t+1} -1)b_{k-1}.$$

LEMMA 14. $b_k + sa_k \leqq 4(2^{t+1} - 1)B_{k-1}$ for $k > 1$.

PROOF: When $k = 2$, we have that $b_2 + sa_2 =$
$2a_1b_1 + s(a_1^2 + b_1^2m) = 2a_1b_1 + s(1 + 2b_1^2m) = 2a_1b_1 + 2b_1^2\sqrt{m} + s.$
$= 2a_1b_1(1 + \frac{b_1}{a_1s}) + s < 4a_1b_1 + s = 4b_1(a_1 - \frac{1}{2a_1}) + \frac{2b_1}{a_1} + s.$

Applying Lemma 10 we see that the right hand side is <
$4b_1(2^t) + 2b_1/a_1 + s < 4b_1(2^t) + 3s < 4b_1(2^{t+1} - 1).$

This proves the lemma for $k = 2$. Let us assume that the lemma is true for $k \leqq n-1$ and prove the lemma by induction.
$b_n + sa_n = a_1b_{n-1} + b_1a_{n-1} + s(a_1a_{n-1} + b_1b_{n-1}m) =$
$b_{n-1}(a_1 + b_1\sqrt{m}) + (b_1 + sa_1)a_{n-1} < 2a_1b_{n-1} + 2sa_1a_{n-1}.$

Applying Lemma 10, the right hand side is less than
$(2^{t+1} + \frac{1}{a_1})b_{n-1} + 2sa_1a_{n-1} = (2^{t+1} - 1)b_{n-1} + (1 + \frac{1}{a_1})b_{n-1} +$
$sa_{n-1} + s(2a_1 - 1)a_{n-1} = (2^{t+1} - 1)b_{n-1} + b_{n-1} + sa_{n-1} +$
$b_{n-1}/a_1 + (2a_1 - 1)sa_{n-1}$. Applying Lemma 13 and the induction hypothesis, the right hand side is less than
$(2^{t+1} - 1)b_{n-1} + 4(2^{t+1} - 1)B_{n-2} + 3(2^{t+1} - 1)b_{n-1} =$
$4(2^{t+1} - 1)B_{n-1}$. This completes the proof of the lemma.

LEMMA 15. $2^rb_k + 2^rsa_k \leqq 4(2^{t+1}-1)B_{k-1} + 4(2^r-1)b_k$ for all r when $k > 1$ and $r > 1$ when $k = 1$.

PROOF: Lemma 11 tells us that $sa_1 < b_1$ which implies that $4b_1 + 4sa_1 < 4(3b_1)$ and the lemma is proved for $k = 1$ and $r = 2$. If $k > 1$ and $r = 1$, then we have $2b_k + 2sa_k < 4(2^{t+1} - 1)B_{k-1} + b_k + \frac{3}{2}b_k < 4(2^{t+1}-1)B_{k-1} + 4b_k$. Let us assume that the lemma is true for $r = n-1$ and prove the lemma by induction. Let $r = n$. Applying Lemma 11 and the induction hypothesis we obtain:

$$2^n b_k + 2^n sa_k = 2(2^{n-1}b_k + 2^{n-1}sa_k) <$$
$$4(2^{t+1} - 1)B_{k-1} + 4(2^{n-1} - 1)b_k + 2^{n-1}b_k + \tfrac{3}{2}(2^{n-1})b_k.$$
$$< 4(2^{t+1} - 1)B_{k-1} + 4(2^{n-1} - 1)b_k + 4(2^{n-1})b_k =$$
$$4(2^{t+1} - 1)B_{k-1} + 4(2^n - 1)b_k.$$

We can now summarize Lemmas 10 to 15 as follows:

LEMMA 16. $d_k + sc_k \leqq 4D_{k-1}$ and $s > d_k/c_k$ for $k \geqq 3$.

We will now proceed to show that the units do not belong to $A(m)$ and that the integer T does belong to $A(m)$. It is obvious that if $b_1 \neq 1/2$, then the units do not belong to $A(m)$. We make use of the following lemma to show that the units do not belong to $A(m)$ when $b_1 = 1/2$

LEMMA 17. $a_k > 2\sum_{i=1}^{k-1} a_i + 3$ when $k > 1$, $m \geqq 21$ and $b_1 = 1/2$.

PROOF: $a_1^2 = b_1^2 m \pm 1 \geq m/4 - 1 > 4$ implies $a_1 > 2$. $a_2 = a_1^2 + b_1^2 m > 2a_1 + 4 > 2a_1 + 3$. Therefore the lemma is true for $k = 2$. Let us assume that the lemma is true for $k \leq n-1$ and prove the lemma true by induction. Let $k = n$. For $m \geq 21$ we have the following inequalities:

$$4 - m < -4 < -4/mb_{n-1}^2.$$

$$4mb_{n-1}^2 - m^2 b_{n-1}^2 < -4.$$

$$4a_{n-1}^2 - b_{n-1}^2 m^2 = 4a_{n-1}^2 - 4mb_{n-1}^2 + 4mb_{n-1}^2 - m^2 b_{n-1}^2 < 4-4 = 0.$$

$$a_{n-1} < (b_{n-1}m)/2.$$

$$a_n = a_1 a_{n-1} + b_1 b_{n-1} m > 2a_{n-1} + (b_{n-1}m)/2 > 2a_{n-1} + a_{n-1}.$$

Applying the induction hypothesis to the right hand side we obtain: $a_n > 2a_{n-1} + 2\sum_{i=1}^{n-2} a_i + 3 = 2\sum_{i=1}^{n-1} a_i + 3$.

THEOREM 2. The units do not belong to $A(m)$.

PROOF: If $b_1 = 1/2$ and $m \geq 21$, then Lemma 17 tells us that no unit in S_j for $j > 1$ can be used to represent the number two as a sum of distinct units. It can be seen by inspection that the number two cannot be written as a sum of distinct units using only units in S_0 and S_1. When $k = 13$ we have that $a_k > 2\sum_{i=1}^{k-1} a_i + 6$ for $k > 3$. Thus no unit in S_j for $j > 3$ can be used in the representation of the number five. By inspection it can be seen that the number five cannot be represented using only units in S_0, S_1, S_2 and S_3. This completes the proof.

THEOREM 3. The number T belongs to A(m).

PROOF: We may consider every integer P of D(m) as a point P in the plane with rectangular coordinates x and y where x and y are either integers or halves of odd integers. We will fill the plane with concentric diamonds whose diagonals are the coordinate axes. The y intercepts of the sides of the k diamond are $\pm 2D_k$ and the x intercepts are $\pm 2D_k/s$. For every point in the k diamond we have:

$$-2D_k \leqq y + sx \leqq 2D_k,$$

$$-2D_k \leqq y - sx \leqq 2D_k.$$

Let V be the set of all divisors of T which are not contained in V_j for some $j > 0$. It will be shown that every integer lying in the k diamond can be written as a sum of distinct divisors of T contained in the first k + 1 sets V, V_1, V_2,$\cdots$,V_k. The divisors in V_j and the diamonds are symmetric with respect to both coordinate axes. Thus there is no loss in generality by considering only points in the first quadrant, on the positive x axis and on the positive y axis.

Let us first consider a representation for every integer in the first diamond in the case where $N(u_1) = -1$. Because $2^{t+1} - 1 > 2a_1 - 1 > 2b_1/s - 1 > 2b_1 - 1$, every rational integer between 0 and $2b_1$ and between 0 and $2b_1/s$ can be written as a sum of distinct divisors of 2^t. Thus every integer in the first diamond can be written in the form: $\sum_r 2^r(1 + \sqrt{m})/2 \pm \sum_z 2^z$.

Let us now consider a representation for every integer in the second diamond in the case where $N(u_1) = +1$.

$$2^{t+2} - 1 > 4a_1 - 1 > 4b_1/s - 1 > 4b_1 - 1.$$

$$2a_1 + 2^{t+2} - 1 > 6a_1 - 1 > 6b_1/s - 1.$$

$$2b_1 + 2^{t+2} - 1 > 6b_1 - 1.$$

Applying the above inequalities we see that every integer in the second diamond can be written either as $2^r(1 + \sqrt{m})/2 \pm 2^z$ or as $(2^r + 2b_1)(1 + \sqrt{m})/2 \pm 2^z \pm 2a_1$.

Therefore when $N(u_1) = -1$, every integer in the first diamond can be written as a sum σ_1 of distinct divisors in the sets V and V_1. When $N(u_1) = +1$, every integer in the second diamond can be written as a sum of distinct divisors in the sets V, V_1 and V_2.

Let us assume that every integer $P' = x' + y'\sqrt{m}$ in the first k-1 diamonds can be represented as a sum σ_{k-1} of distinct divisors of T without using any divisors in the sets V_j for $j > k-1$, and prove the theorem by induction. Let P be any point (x,y) in the k diamond where x and y are either rational integers or both halves of odd rational integers. If P lies in the k-1 diamond, then the theorem is proved. If P does not lie in the k-1 diamond, then we have the following inequalities:

$$2D_{k-1} < y + sx \leqq 2D_k,$$

$$x \geqq 0, \qquad y \geqq 0.$$

We divide the possibilities into three cases. In Case 1 the point lies on or above the straight line

$$y - sx = 2d_k - 2D_{k-1}.$$

In Case 2 the point lies on or below the straight line

$$y - sx = -2sc_k + 2D_{k-1}.$$

In Case 3 the point lies between these two lines.

Case 1. $2D_{k-1} < y + sx \leqq 2D_k,$

$$y - sx \geqq 2d_k - 2D_{k-1},$$

$$x \geqq 0, \qquad y \geqq 0.$$

If we subtract $2d_k$ from y we obtain a point P' with rectangular coordinates $x' = x$ and $y' = y - 2d_k$.

$$y' + sx' \leqq 2D_k - 2d_k = 2D_{k-1},$$

$$y' - sx' \geqq 2d_k - 2D_{k-1} - 2d_k = -2D_{k-1},$$

$$x' = x \geqq 0.$$

The resulting point P' lies in the k-1 diamond and thus can be written as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V, V_1, \cdots, V_{k-1}$. $P' = \sigma_{k-1}$. $P' = P - 2d_k\sqrt{m}$. Solving for P we obtain the following:

$$P = \sigma_{k-1} + 2d_k\sqrt{m} = \sigma_{k-1} + (c_k + d_k\sqrt{m}) + (-c_k + d_k\sqrt{m}).$$

This completes the proof of the theorem for Case 1.

Case 2. $2D_{k-1} < y + sx \leqq 2D_k$,

$$y - sx \geqq 2D_{k-1} - 2sc_k,$$

$$y \geqq 0, \qquad x \geqq 0$$

If we subtract $2c_k$ from x,we obtain a point P' whose rectangular coordinates are $x' = x - 2c_k$ and $y' = y$.

$$y' + sx' \leqq 2D_k - 2sc_k < 2D_k - 2d_k = 2D_{k-1},$$

$$y' - sx' \geqq 2D_{k-1} - 2sc_k + 2sc_k = 2D_{k-1},$$

$$y' = y \geqq 0.$$

The resulting point P' lies in the k-1 diamond and thus it can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V, V_1, \cdots, V_{k-1}$. $P' = \sigma_{k-1}$. $P' = P - 2c_k$. Solving for P we obtain:

$$P = \sigma_{k-1} + 2c_k = \sigma_{k-1} + (c_k + d_k\sqrt{m}) + (c_k - d_k\sqrt{m}).$$

Thus the point P can be represented as a sum of distinct divisors taken from the sets $V, V_1, \cdots, V_k$. This completes the proof of the theorem for Case 2.

Case 3. $2D_{k-1} < y + sx \leqq 2D_k$,

$$-2sc_k + 2D_{k-1} < y - sx < 2d_k - 2D_{k-1}.$$

If we subtract d_k from y and c_k from x, we obtain a point P' whose rectangular coordinates are $x' = x - c_k$ and $y' = y - d_k$.

Applying Lemmas 9 and 13, we obtain:

$y' + sx' \leqq 2D_k - d_k - sc_k < 2D_k - 2d_k = 2D_{k-1}$,

$y' + sx' > 2D_{k-1} - d_k - sc_k \geqq 2D_{k-1} - 4D_{k-1} = -2D_{k-1}$,

$y' - sx' < 2d_k - 2D_{k-1} - d_k + sc_k \leqq 4D_{k-1} - 2D_{k-1} = 2D_{k-1}$,

$y' - sx' > -2sc_k + 2D_{k-1} - d_k + sc_k = 2D_{k-1} - d_k - sc_k$

$\geqq 2D_{k-1} - 4D_{k-1} = -2D_{k-1}$.

The resulting point P' lies inside the k-1 diamond and thus it can be represented as a sum σ_{k-1} of distinct divisors in the first k sets of divisors $V, V_1, \cdots, V_{k-1}$. $P' = \sigma_{k-1}$. $P' = P - c_k - d_k\sqrt{m}$. Solving for P we obtain: $P = \sigma_{k-1} + c_k + d_k\sqrt{m}$. Thus the point P can be represented as a sum of distinct divisors taken from the sets $V, \cdots, V_k$. This completes Case 3 and the theorem is proved.

CHAPTER V

THE GAUSSIAN FIELD

In this chapter we will focus our attention upon the domain of integers G in the Gaussian field. The integers in G are numbers of the form $x + iy$, where x and y are rational integers and $i = \sqrt{-1}$. The units of G are the numbers $+1$, -1, i and $-i$. Because we have only four units in G it is impossible to represent every integer in G as a sum of distinct divisors of any one integer n of G. However we can define a set A of integers in G whose properties are analogous to the properties of the rational integers belonging to the set A defined in chapter one.

DEFINITION: A is the set of all integers $\{n\}$ in G which have the following properties: n belongs to A if and only if there exists a rational integer n' such that every integer $x + iy$ in G satisfying the inequalities

$$-n' \leqq x \leqq +n' \qquad -n' \leqq y \leqq +n'$$

and no other integer in G, can be represented as a sum of distinct divisors of n.

If we represent every Gaussian integer as a lattice point in the plane we can see that geometrically we have a situation analogous to the situation in Chapter I. In Chapter I we represented every integer on the real line between $-a$ and $+a$ as a sum of distinct divisor of a rational integer. Here we are going to represent as a sum of distinct divisors of a Gaussian integer every lattice point inside a square whose center is at the origin and whose sides are given by the equations, $y = \pm n'$ and $x = \pm n'$.

LEMMA 1. If n belongs to A and d divides n, then d is either real or pure imaginary.

PROOF: Let n be a Gaussian integer belonging to A. Let $d_j = a_j + ib_j$ be the divisors of n. If $\sum a_j = n'$, every divisor lying in the first or fourth quadrants or on the positive x axis must be included in the summand. If $\sum b_j = n'$, every divisor lying in the first or second quadrants or on the positive y axis must be included in the summand. Thus the integer $n' + in'$ cannot be represented unless every divisor lying in the first quadrant is used twice.

THEOREM 1. A Gaussian integer n belongs to A if and only if n is an associate of a rational integer having one of the following factorizations as a product of primes:

i) $n = 3^t$ for all $t \geq 0$.

ii) $n = 3^t \prod_{j=1}^{k} p_j^{t_j}$ with $3 < p_r < p_s$ for $r < s$, $t \geq 1$

$t_i \geq 1$, $p_j \equiv 3 \bmod 4$ and $\frac{p_i - 1}{2} \leq \sigma\left[\frac{n}{\prod_{j=i}^{k} p_j^{t_j}}\right]$ for $i = 1, 2, \cdots, k$.

PROOF: By Lemma 1, n must be a rational integer and all divisors of n must be rational integers or the associates of rational integers. If we factor n as a product of primes in the Gaussian field, each prime factor must be a rational prime which is also a prime in G. Thus if n belongs to A, it has the form i) or ii). If n has one of these forms, then Theorem 2 of Chapter I tells us that every rational integer between $-\sigma(n)$ and $+\sigma(n)$ can be represented as a sum $\sum d_j$ of distinct rational divisors of n. Every number of the form iy for $-\sigma(n) < y < +\sigma(n)$ can be represented as $i\sum d_k$, where the d_k are distinct rational divisors of n. If we let $\sigma(n) = n'$, every number in the square can be written as $\sum d_j + i\sum d_k = \sum d_j + \sum id_k$.

THEOREM 2. There exist arbitrarily large square free rational integers in A.

PROOF: Let $n_1 = 3$, $n_2 = 21$ and $n_k = \prod_{j=1}^{k} p_j$ be the product of the first k rational primes which are congruent to 3 modulo 4. Either $n_k + 2$ or $n_k + 4$ is congruent to 3 modulo 4 and therefore divisible by a prime q which is congruent to 3 modulo 4. However $(4,p_j) = 1$ implies that $(p_j,q) = 1$. Thus for $k > 1$ we have:

$p_{k+1} \leqq q \leqq n_k + 4 = n_k + 3 + 1 < \sigma(n_k) < 2\sigma(n_k) + 1.$

When $k = 1$ we have, $7 < 8 + 1 = 2\sigma(3) + 1$. Thus n_k belongs to A for all k.

CHAPTER VI

IMAGINARY QUADRATIC FIELDS

$m \not\equiv 1$ modulo 4

In this chapter we will focus our attention upon the integers in the quadratic fields R(m), where m is square free, less than -1 and congruent to two or three modulo four.

The domain of integers D(m) in R(m) is the set of numbers of the form $x + y\sqrt{m}$, where x and y are both rational integers. The only units in D(m) are ± 1.

DEFINITION: A(m) is the set of all integers $\{n\}$ in D(m) which have the following properties: n belongs to A(m) if and only if there exists a rational integer n' such that every integer $x + y\sqrt{m}$ in D(m) satisfying the inequalities $-n' \leq x \leq n'$ and $-n' \leq y \leq n'$ and no other integers in D(m) can be represented as a sum of distinct divisors of n.

We shall identify with every integer $x + y\sqrt{m}$ a point $P = (x,y)$ in the plane. An integer n in D(m) belongs to A(m) if and only if every point in a square, whose center is the origin and whose sides are parallel to the coordinate axes, and no point outside of this square can be represented as a sum of distinct divisors of n.

By using exactly the same argument as that used in

Lemma 1 of Chapter V we can prove the following lemma.

LEMMA 1. If n belongs to $A(m)$ and d divides n, then either $d = x$ or $d = y\sqrt{m}$, where x and y are rational integers.

$\underline{m = -2}$

Every integer in $D(-2)$ can be factored uniquely as a product of primes in $D(-2)$. Lemma 1 tells us that if n belongs to $A(-2)$, then $n = 2^t \prod_{i=1}^{k} p_i\sqrt{m}$, where each p_i is a rational prime which is also a prime in $D(-2)$. Therefore $p_i \equiv 5$ or 7 modulo 8. Let $n_j = 2^t \prod_{i=1}^{j} p_i$ for $1 \leqq j \leqq k$. The largest positive rational integer which can be written as a sum of distinct divisors of n is $\sigma(n_k)$. In order to represent every rational integer between $-\sigma(n_k)$ and $+\sigma(n_k)$ as a sum of distinct divisors of n, n_k must satisfy Theorem 1 of Chapter I. We can now prove the following theorem.

THEOREM 1. n belongs to $A(-2)$ if and only if $n = n_k\sqrt{m}$, where n_k satisfies Theorem 1 of Chapter I and $p_i \equiv 5$ or 7 modulo 8.

PROOF: If n belongs to $A(-2)$, Lemma 1 tells us that n_k must have the form $2^t\prod p_i$, where $p_i \equiv 5$ or 7 modulo 8. In order to represent every rational integer between $-\sigma(n_k)$ and $+\sigma(n_k)$, n_k must satisfy Theorem 1 of

Chapter I. In order to represent every integer of the form $y\sqrt{-2}$ for $-\sigma(n_k) \leqq y \leqq +\sigma(n_k)$ we must have $n = n_k\sqrt{-2}$

Conversely if $n = n_k\sqrt{-2}$, every integer in the square can be written as $\sum_{d/n_k} d + \left(\sum_{d'/n_k} d'\right)\sqrt{-2}$. Because the maximum value for each sum is $\sigma(n_k)$, no integer outside the square can be represented as a sum of distinct divisors of n.

THEOREM 2. $A(-2)$ contains integers n with arbitrarily large square free n_k.

PROOF: Let $n_k = 2\prod_{i=1}^{k} p_i$, where $p_1 = 5$, $p_2 = 7$ and $\prod_{i=1}^{k} p_i$ is the product of the first k primes which are congruent to 5 or 7 modulo 8. In order to prove the theorem we must show that n_k satisfies Theorem 1 of Chapter I for all k. If $k = 1$ or $k = 2$, n_k satisfies this condition. For $k > 2$ we have that at least one of the numbers $n_k/2 + 2$, $n_k/2 + 4$, $n_k/2 + 6$ or $n_k/2 + 8$ must be congruent to 5 modulo eight and therefore must be divisible by a prime (q) congruent to 5 or 7 modulo 8. Because $(q,2) = (q,4) = (q,6) = (q,8) = 1$, we have $(q,n_k) = 1$. Therefore $p_{k+1} \leqq q \leqq n_k/2 + 8 < 2\sigma(n_k) + 1$ and n_k satisfies Theorem 1 of Chapter I for all k.

$m < -2$

In this section we will consider the quadratic fields $R(m)$, where m is square free, less than -2 and not congruent to 1 modulo 4. This implies that m is congruent to 2, 3, 6 or 7 modulo 8. Let $h = -m$. h is congruent to 1, 2, 5 or 6 modulo 8.

THEOREM 3. If $a + b\sqrt{m}$ divides $2^r\sqrt{m}$, then either a or b must be zero.

PROOF: Let us suppose that $a + b\sqrt{m}$ divides $2^r\sqrt{m}$ and that neither a nor b are zero. This implies that $a^2 - mb^2$ divides $2^{2r}m$ or that $a^2 + hb^2$ divides $2^{2r}h$, where a, b, h and r are rational integers and r and h are positive.

$$\begin{aligned} &a^2 + hb^2 = 2^t h', \\ &0 \leqq t \leqq 2r \;, \quad h' \mid h. \end{aligned} \tag{1}$$

We may rewrite equation (1) as

$$h'(a/h')^2 + (h/h')b^2 = 2^t, \tag{2}$$

where a/h' and b are non-zero rational integers. The left hand side of equation (2) is greater than $h' + h/h'$ $> h' + 4/h' \geqq 4$. Thus $t \geqq 3$.

If either a or b is an even rational integer, then the other must also be even and equation (1) reduces to

$$(a^2/4) + h(b^2/4) = 2^{t-2}h', \tag{3}$$

where $a^2/4$ and $b^2/4$ are rational integers. If either

$a^2/4$ or $b^2/4$ are even rational integers, we may continue our method of descent and obtain $(a/4)^2 + h(b/4)^2 = 2^{t-4}h'$ where $a^2/16$ and $b^2/16$ are rational integers. Continuing in this way we obtain

$$(a^2/2^s) + h(b^2/2^s) = 2^{t-s}h' \qquad (4)$$

where either $t - s < 3$ or both $(a^2/2^s)$ and $(b^2/2^s)$ are odd rational integers. Because of the results following equation (2), we cannot have $t - s < 3$. Therefore a solution to equation (1) with a and b even rational integers implies a solution to equation (1) with a and b odd rational integers and $t \geqq 3$.

When a and b are both odd rational integers we have

$$a^2 + hb^2 \equiv h + 1 \equiv 2,\ 3,\ 6 \text{ or } 7 \bmod 8.$$

Therefore there exist no solutions in non-zero rational integers to equation (1) and thus the theorem is proved.

THEOREM 4. $T = 2^t\sqrt{m}$ belongs to $A(m)$ for all t.

PROOF: Theorem 3 tells us that the only divisors of T are powers of 2 and $\sqrt{m}$ times powers of 2. Every rational integer between $-\sigma(2^t)$ and $+\sigma(2^t)$ and no other rational integer can be written as a sum of distinct divisors of T. Thus every integer $x + y\sqrt{m}$ in $D(m)$ where

$$-\sigma(2^t) \leqq x \leqq +\sigma(2^t) \quad \text{and} \quad -\sigma(2^t) \leqq y \leqq +\sigma(2^t)$$

can be written as $\sum_{d/2^t} d + \sqrt{m}\sum_{d'/2^t} d'$. No other integer can be so represented.

CONCLUSION

In this dissertation the problem of characterizing the sets A and A(m) has been completely solved for the rational field and the fields where $m = 2$, $m = 5$, $m = -1$ and $m = -2$. In every other quadratic field except where m is negative and congruent to one modulo four, there exist infinitely many integers which do belong to A(m) and infinitely many integers which do not belong to A(m). However in these fields we do not have a complete characterization of the integers which do belong to A(m). One of the many difficulties which arises is that if $a + b\sqrt{m}$ divides a given integer, then $a - b\sqrt{m}$ need not be a divisor of this given integer. Thus it is impossible to separate the divisors into sets which are symmetric to both the x and y axes.

When m is negative and congruent to one modulo four the set A(m) is empty. This arises from the fact that integers of the form $x + y\sqrt{m}$, where x and y are halves of odd integers cannot be represented using only divisors of the form d_i and $d_i\sqrt{m}$ where the d_i are rational integers.

Another area for further investigation is in the study of the behavior of sums of distinct divisors of integers in fields of higher degree.

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