THE DISTRIBUTION OF THE NUMBER OF COMPONENTS OF A RANDOM MAPPING FUNCTION

THESIS FOR THE DEGREE OF PH. D.

MICHIGAN STATE UNIVERSITY

JAY ERNEST FOLKERT

1955

This is to certify that the

thesis entitled

The Distribution of the Number of Components
of a Random Mapping Function

presented by

Jay Ernest Folkert

has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Mathematics

Major professor

Date July 25, 1955

THE DISTRIBUTION OF THE NUMBER OF COMPONENTS OF A RANDOM MAPPING FUNCTION

By

JAY ERNEST FOLKERT

A THESIS

Submitted to the School for Advanced Graduate Studies of Michigan State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1955

ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to Dr. Leo Katz who supervised the work involved in the solution of the problem. His advice, encouragement and patience have been most important throughout the author's entire program of study.

The writer deeply appreciates the financial support of the Office of Naval Research.

THE DISTRIBUTION OF THE NUMBER OF COMPONENTS OF A RANDOM MAPPING FUNCTION

By

Jay Ernest Folkert

AN ABSTRACT

Submitted to the School for Advanced Graduate Studies of Michigan State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

Year 1955

Approved Leo Katz

JAY ERNEST FOLKERT ABSTRACT

From the collection, F, of functions, f, which map the set, Ω, of N elements into Ω, the general subclass, $G_{\{r_i\}}$, is selected. This class is composed of functions such that for each $f \in G_{\{r_i\}}$ the point $x_i \in \Omega$ has r_i images in Ω, $i = 1, 2, \ldots, N$. A probability space is constructed by taking $G_{\{r_i\}}$ and attaching equal probability to each $f \in G_{\{r_i\}}$. A random mapping of Ω into Ω relative to $G_{\{r_i\}}$ is defined as corresponding to the selection of an f from $G_{\{r_i\}}$ with uniform probability.

A subset ω of Ω is a <u>component</u> of the function if and only if it is a minimal, non - null subset such that $f(\omega) \subset \omega$ and $f^{-1}(\omega) \subset \omega$. Every mapping function, $f \in G_{\{r_i\}}$ decomposes Ω into a number of disjoint components. Therefore, to each f in $G_{\{r_i\}}$ there corresponds a number, c, which is the number of components induced in Ω by this f. If f is selected at random from $G_{\{r_i\}}$ then c is a random variable. The problem is to find the probability distribution of c.

The method used is one in which auxiliary sums, S_μ, $\mu = 1, 2, \ldots, N$ are defined. These sums are accounted for by two approaches. By the first approach a formula in terms of N and r_i is derived for computation of S_μ, $\mu = 1, 2, \ldots, N$. By the second approach, S_μ is found in terms

of the probability, P_c, of exactly c components, c = 1, 2, ..., N. A matrix solution of this expression for P_c in terms of S_μ is given. The result is the exact probability distribution of the number of components.

Particular cases of general mapping are considered as specializations of the mapping under $G_{\{r_i\}}$. These are the subclass, G_r, of functions under which each element of Ω has the same number, r, of images and the subclass, G_1, of functions under which each element of Ω has only one image.

Hollow mapping in the sense that no point is permitted to map into itself is also considered. For this case, the subclasses, $H_{\{r_i\}}$, H_r, and H_1 are defined in a manner parallel to the general case and the exact probability distribution is found for each.

Numerical examples are included to illustrate each of the cases considered. The amount of computation involved in these suggests the need for an approximation to the exact distribution. Therefore a binomial approximation is developed. The results of the exact and approximate distributions for these examples are included in tabular form for reasons of comparison.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF TABLES

1. INTRODUCTION

<u>1:1. Basic considerations</u>. This thesis is concerned with the collection, F, of transformations or functions, f, on the points of a set, Ω, of N elements into the set Ω. The most general kind of a function takes a point of Ω into an arbitrary number of points of the set. Under such a function the point x_1 maps into r_1 points of Ω, x_2 maps into r_2 points of Ω, etc.

A subset ω of Ω is a <u>component</u> of the transformation or function if and only if it is a minimal, non-null subset such that points of ω map only into points of ω and every point which maps into a point of ω is itself a point of ω. Symbolically, the subset ω is a component of the function if it is a minimal, non-null subset $\ni f(\omega) \subset \omega$ and $f^{-1}(\omega) \subset \omega$. It is essential that ω be minimal.

Every mapping function, $f \varepsilon F$, decomposes Ω into a number of disjoint components. For example, if f were a cyclic permutation taking x_i into x_{i+1}, mod N, there would be only one component. On the other hand, if f were the identity transformation, there would be N components. Consequently, depending on the function involved, there could be 1, 2, ..., or N components induced in Ω by a function, f.

1:2. Statement of the problem and the plan of attack.

A suitably chosen subset $\mathfrak{F}$ of the finite collection, F, of functions is considered. A probability space is constructed by taking $\mathfrak{F}$ and attaching equal probability to each $f \in \mathfrak{F}$. A random mapping of Ω into Ω relative to $\mathfrak{F}$ is defined as corresponding to the selection of an $f \in \mathfrak{F}$ with uniform probability. To each $f \in \mathfrak{F}$ there corresponds a number, c, which is the number of components induced in Ω by this function. if f is selected at random from $\mathfrak{F}$, then c is a random variable. The probability distribution of c for various choices of $\mathfrak{F}$ is the core of this thesis.

There are two main parts in this dissertation. Part I is the general case in which a point may map into itself. For any fixed set, $\{r_i\}$, $i = 1, 2, \ldots, N$, of values, there exists a subclass, $G_{\{r_i\}} \subset F$, of functions such that for each $f \in G_{\{r_i\}}$ the points $x_1, x_2, \ldots, x_N$ of Ω have $r_1, r_2, \ldots, r_N$ images respectively. Of special interest is the case where $r_i = r$ for all i. $(0 < r \leq N)$. This introduces the subclass, $G_r \subset F$, of functions under which each point of Ω has the same number, r, of images. Moreover, the special case where $r = 1$ yields the subclass G_1 of functions under which each point has only one image.

Part II is the hollow case and deals with the subclass $H_{\{r_i\}} \subset F$ of completely nonidentical functions under which no point is permitted to map into itself. As in the general case, the special cases H_r and H_1 are considered.

These are actually subsets of the corresponding G sets and are defined in a manner parallel to the subsets $G_{\{r_i\}}$, G_r and G_1.

Starting with the general case, the procedure is to consider the subset $\mathcal{F}$ to be the subset $G_{\{r_i\}}$ with r_1, r_2, ..., r_N images in Ω for the points x_1, x_2, ..., x_N respectively for each $f \in G_{\{r_i\}}$. Using matrix representation, which will be described in section 2:1, calculation formulas for auxiliary sums, S_μ, $\mu = 1, 2, \ldots, N$, are found in terms of N and the number of images, r_i. This is followed by a derivation of the same auxiliary sums by a different approach. By this second method S_μ is found in terms of the probability, P_c, that exactly c components are induced in Ω by f chosen at random from $G_{\{r_i\}}$. Then, by matrix algebra, the last derivation is solved for P_c. The result is that the exact probability distribution of the number of components under the random mapping $G_{\{r_i\}}$ is found in terms of S_μ. Since the S_μ, $\mu = 1, 2, \ldots, N$, can be calculated by the first set of formulas, the distribution of c can be computed.

The results for the collection, $G_{\{r_i\}}$ of functions are easily adapted to give similar results for the classes G_r and G_1 by making proper changes in the values of r_i in the formulas.

The hollow case is treated in a manner parallel to the general case in as much as the subsets $H_{\{r_i\}}$, H_r and H_1 are considered in turn. Moreover, the results of the

general case can be adapted to this case by making proper changes in the limits of summation and proper substitutions in the formulas for the auxiliary sums to account for the fact that no point may map into itself.

The numerical examples which are included for each of the above cases indicate that considerable computation is involved in finding the exact distribution. The examples which are presented are relatively trivial. In order to minimize the work involved in more complicated cases a binomial approximation to each distribution is derived and illustrated by numerical examples.

1:3. Previous work. If $\mathcal{T}$ is restricted to the class of permutations of N elements, components become cycles. Under a permutation any point of the finite set Ω maps into a second, the second maps into a third, etc. until at some stage a point maps into the first member of the sequence. At this stage a cycle is formed. Gontcharoff [1] has given several moment generating functions for the number of cycles of the elements of a permutation group of N elements. His results of the asymptotic behavior of the distribution of the number of components are repeated in different forms by others. Feller [2] reports that if c is the number of cycles formed, the distribution of c is asymptotically normal with mean equal to log N and standard deviation equal to $(\log N)^{1/2}$. More recently Greenwood [3]

showed by different methods that for large N, the mean of the number of cycles is approximately equal to log N + C, where C is Euler's constant, and that the variance is approximately equal to $\log N + C - \frac{\pi^2}{6}$.

The case where $\mathfrak{F}$ is the class of single - valued mappings on N elements was considered in part by Metropolis and Ulam [4] . (This is the class which is called G_1 in section 3:3.) They defined the study of a random function as the study of the probability distribution over the Ω^{Ω} sample points of f(x) on Ω into Ω. Using the word tree instead of component, they proposed the question of the expected number of components under random mapping. They suggested that the answer is no doubt of the order of log N and base their surmise on the result which was previously obtained for cycles.

Kruskal [5] describes the structure of a component as consisting of a cycle together with a number of trees rooted in the elements of the cycle. This is consistent with the definition used in this thesis. He, too, considered the class of single - valued mappings and in particular dealt with the question proposed by Metropolis and Ulam. He showed that the expected number, E, of components under a random mapping function is

$$E = \sum_{m=1}^{N} \frac{N!}{(N-m)!\,mN^{m}} \quad . \tag{1:1}$$

Moreover, for large N this reduces to the following asymptotic result,

(1:2) $E \sim {}^1/_2 (\log 2N + C)$, where C is Euler's constant.

This dissertation will present the entire distribution for multiple - valued functions as well as single - valued functions. Thus its results extend well beyond those given by Kruskal.

2. THE GENERAL CASE

If no restriction is placed on the mapping of Ω into Ω, it is conceivable that a point could map into itself. We shall refer to this type of mapping as the general case. The collection, F, of all functions on Ω into Ω is admittedly broad for it does not specify the number of images of the different points of Ω. Therefore the subclass, $G_{\{r_i\}}$, is considered. This class consists of those functions, f, under which each point $x_i \in \Omega$ has r_i images in Ω, $i = 1, 2, \ldots, N$. It is assumed that $r_1, r_2, \ldots, r_N$ are known. Further, for each $f \in G_{\{r_i\}}$ the mapping of Ω into Ω is unique. Equal probability is assigned to each f in $G_{\{r_i\}}$. Since to each f in $G_{\{r_i\}}$ there corresponds a number, c, which is the number of components induced in Ω by this f, c becomes a random variable if f is selected at random from $G_{\{r_i\}}$.

2:1. The calculation formula for the auxiliary sum, S_μ. In order to begin the derivation of this formula it is necessary to introduce some notation.

(2:1) Let $\mathcal{I} = \{i\}$, $i = 1, 2, \ldots, N$ be the set of indices of the points $x_i \in \Omega$.

(2:2) Let $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ be a μ-part partition

of ϑ into subsets such that:

(a) α_j is non - empty for each $j \leq \mu$

(b) $\alpha_j \cdot \alpha_k = 0$ for $j \neq k$

(c) $\alpha_1 + \alpha_2 + \dots + \alpha_\mu = \vartheta$.

(2:3) Let ω_{α_j} be that subset of elements of Ω with indices belonging to α_j, $j = 1, 2, \dots, \mu$.

(2:4) Let $E(\alpha_1, \alpha_2, \dots, \alpha_\mu)$ be the event when Ω is divided into subsets in accordance with (2:2) and (2:3), ω_{α_j} $(j = 1, 2, \dots, \mu)$ has the properties:

(a) $f(\omega_{\alpha_j}) \subset \omega_{\alpha_j}$

(b) $f^{-1}(\omega_{\alpha_j}) \subset \omega_{\alpha_j}$

It is emphasized that none of the subsets ω_{α_1}, $\omega_{\alpha_2}, \dots, \omega_{\alpha_\mu}$ need be minimal subsets having the properties (a) and (b) of (2:4). Therefore, $E(\alpha_1, \alpha_2, \dots, \alpha_\mu)$ is not equivalent with decomposition of Ω into components.

(2:5) Let $p(\alpha_1, \alpha_2, \dots, \alpha_\mu)$ be the probability of the occurrence of $E(\alpha_1, \alpha_2, \dots, \alpha_\mu)$ on the condition that f is a random selection from $G_{\{r_i\}}$.

(2:6) Let $S_\mu = \sum_{(\alpha_1, \alpha_2, \dots, \alpha_\mu)} p(\alpha_1, \alpha_2, \dots, \alpha_\mu)$, $\mu = 1, 2, \dots, N$, where the sum is over all possible choices of $(\alpha_1, \alpha_2, \dots, \alpha_\mu)$ in accordance with (2:2) for a fixed value of μ.

(2:7) Let k_j = the number of indices in α_j, $j = 1, 2, \ldots, \mu$. This implies that there are k_1 elements in ω_{α_1}, k_2 elements in ω_{α_2}, etc. Consistent with (2:2) $k_j > 0$ and $k_1 + k_2 + \ldots + k_\mu = N$

The indicated summation in (2:6) can now be considered in two parts. By theorem 3, section 4, chapter 2 of Feller [2.], for any fixed set of values $(k_1, k_2, \ldots, k_\mu)$, there are $\binom{N}{k_1, k_2, \ldots, k_\mu}$ ways in which the N indices of ϑ can be divided into μ groups of which the first contains k_1 indices, the second contains k_2 indices, etc. Here $\binom{N}{k_1, k_2, \ldots, k_\mu}$ is the multinomial coefficient and equals $\frac{N!}{k_1!\, k_2! \ldots k_\mu!}$. However, $(k_1, k_2, \ldots, k_\mu)$ can take on different values and still satisfy the conditions of (2:7). For each choice of $(k_1, k_2, \ldots, k_\mu)$ the different choices of $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ must be considered. For convenience, therefore, it is possible to write (2:6) as a double sum,

$$\text{(2:6')} \quad S_\mu = \sum_{(k_1, k_2, \ldots, k_\mu)} \; \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}^{(k)} p_{(\alpha_1, \alpha_2, \ldots \alpha_\mu)}$$

The symbol $\sum\limits_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}^{(k)}$ means that for a fixed set of values $(k_1, k_2, \ldots, k_\mu)$ the sum is taken over all possible choices of $(\alpha_1, \alpha_2 \ldots., \alpha_\mu)$ in accordance with (2:2).

To find a workable expression for S_μ, it is necessary to evaluate $p(\alpha_1, \alpha_2, \ldots., \alpha_\mu)$. In order to do this, each $f \in G\{r_i\}$ is represented by an N by N matrix, $B^f = (b^f_{ij})$ where b^f_{ij} is either zero or unity. This representation is based on an approach suggested for sociometric data by Forsyth and Katz [6] and by Katz [7]. It was used in a similar way by Katz and Powell [8]. Each of the rows and columns of B^f is representative of a point of Ω. The elements of each row denote the mapping of the point of Ω which this row represents. For example, the i th row of B^f describes the mapping of $x_i \in \Omega$. If b^f_{iq} is unity, then x_i maps, under f, into $x_q \in \Omega$. If b^f_{iq} is zero, then x_i does not map into x_q. Since the functions $f \in G\{r_i\}$ have been assigned equal probability, the r_i ones in row x_i may with equal probability appear in this row in any of $\binom{N}{r_i}$ ways, $i = 1, 2, \ldots., N$.

The event that B^f has the form

(2:3) $$B^f = \begin{pmatrix} B_{11} & & & 0 \\ & B_{22} & & \\ & & \ddots & \\ 0 & & & B_{\mu\mu} \end{pmatrix}$$

is equivalent to the event, $E_{\alpha_1, \alpha_2, \ldots, \alpha_\mu}$. In (2:8) O stands for an array of zeros, B_{jj} is a k_j by k_j principal minor of B^f, $j = 1, 2, \ldots, \mu$. This form is equivalent to $E(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ because the k_1 rows of B_{11} can represent the elements of ω_{α_1}, those of B_{22} can represent the elements of ω_{α_2}, etc. Since unity can appear only in the positions indicated in (2:8), the properties, $f(\omega_{\alpha_j}) \subset \omega_{\alpha_j}$ and $f^{-1}(\omega_{\alpha_j}) \subset \omega_{\alpha_j}$, $j = 1, 2, \ldots, \mu$ are obviously satisfied.

The procedure for the evaluation of $p(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ is as follows. For a fixed set of values $(k_1, k_2, \ldots, k_\mu)$ which satisfy (2:7), one of the $\binom{N}{k_1, k_2, \ldots, k_\mu}$ choices of elements from Ω to represent the subsets ω_{α_1}, ω_{α_2}, ..., ω_{α_μ} is made. By interchanging rows and corresponding columns, B^f is arranged that the rows and columns representing the k_1 points of ω_{α_1} appear in the first k_1 positions of B^f, those representing the k_2 points of ω_{α_2} next, etc. For purposes of identification the rows of B^f are then labeled from top to bottom: $x_{1_1}, x_{1_2}, \ldots x_{1k_1}; x_{2_1}, x_{2_2}, \ldots, x_{2k_2}; \ldots\ldots\ldots\ldots\ldots; x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu k_\mu}$. The corresponding number of ones in these rows is denoted by: $r_{1_1}, r_{1_2}, \ldots,$

r_{1k_1}; r_{2_1}, r_{2_2}, ..., r_{2k_2}; ......; r_{μ_1}, r_{μ_2}, ..., $r_{\mu_{k_\mu}}$.

The following auxiliary notation is now introduced:

(2:9) p_{i_j} = probability that the r_{i_j} ones in the jth row of B_{ii} are in the k_i columns of B_{ii} as indicated in (2:8), $i = 1, 2, \ldots, \mu$ and $j = 1, 2, \ldots, k_i$.

In row x_{i_j}, the r_{i_j} ones could appear in any of the N positions in the row if no restrictions were placed on B^f. Therefore there are $\binom{N}{r_{i_j}}$ ways in which they could be arranged in this row. However, if these r_{i_j} ones are to appear only in the k_i columns of B^f to satisfy (2:8), then there are only $\binom{k_i}{r_{i_j}}$ ways in which they could be arranged. The result is

$$(2:10) \quad p_{i_j} = \frac{\binom{k_i}{r_{i_j}}}{\binom{N}{r_{i_j}}}, \quad i = 1, 2, \ldots, \mu \text{ and } j = 1, 2, \ldots, k_i.$$

In order to have B^f take form (2:8), the events for which the probabilities are given in (2:10) must all occur at the same time. Since these events are all independent, the probability of their simultaneous occurrence is the product of these probabilities.

Therefore,

$$(2:11) \quad P(\alpha_1, \alpha_2, \ldots, \alpha_\mu) = \prod_{i=1}^{\mu} \prod_{j=1}^{k_i} \frac{\binom{k_i}{r_{i_j}}}{\binom{N}{r_{i_j}}}.$$

For a fixed set of values of $(k_1, k_2, \ldots, k_\mu)$ the above procedure would be the same for each of the

$\binom{N}{k_1,\ k_2,\ \ldots.,\ k_\mu}$ choices of elements from Ω to form the subsets $\omega_{\alpha_1},\ \omega_{\alpha_2},\ \ldots,\ \omega_{\alpha_\mu}$. Since (2:11) is a general result, it is the expression which can be used in (2:6') to give

$$(2{:}12)\quad S_\mu = \sum_{(k_1,\ k_2,\ \ldots.,\ k_\mu)} \ \sum_{(\alpha_1,\ \alpha_2,\ \ldots.,\ \alpha_\mu)}^{(k)} \prod_{i=1}^{\mu} \prod_{j=1}^{k_i} \frac{\binom{k_i}{r_{i_j}}}{\binom{N}{r_{i_j}}},$$

$$\mu = 1,\ 2,\ \ldots.,\ N.$$

The above derivation together with the limits on $(k_1,\ k_2,\ \ldots.,\ k_\mu)$ which are given in (2:7) and the fact that the factors of the denominators of (2:12) come from each of the N rows of B^f constitutes a proof of

THEOREM 1. <u>If N is the number of elements in Ω, if r_ν is the number of images of the point $x_\nu \in \Omega$, $\nu = 1,\ 2,\ \ldots.,\ N$ and if $(\alpha_1,\ \alpha_2,\ \ldots.,\ \alpha_\mu)$ is defined by (2:2), the value of the auxiliary sum, S_μ, is given by</u>

$$(2{:}12')\quad S_\mu = \frac{1}{\prod_{i=1}^{N}\binom{N}{r_i}} \sum_{\substack{k_1,k_2,\ldots,k_\mu>0 \\ k_1+k_2+\ldots.+k_\mu = N}} \ \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_\mu)}^{(k)} \prod_{i=1}^{\mu} \prod_{j=1}^{k_i} \binom{k_i}{r_{i_j}},$$

$$\mu = 1,\ 2,\ \ldots.,\ N.$$

Formula (2:12') provides the means for computing S_μ, $\mu = 1,\ 2,\ \ldots.,\ N$, from the initial conditions (i.e., with N given and r_i, $i = 1,\ 2,\ \ldots.,\ N$ known, the auxiliary sums can be calculated). The values of S_μ will be used to find the exact probability distribution.

2:2. The exact probability distribution of the number of components. In order to find the exact distribution of the number of components, it is necessary to obtain an expression involving the probability that an $f \in G_{\{r_1\}}$ selected at random induces exactly c components in Ω . This is done by finding a formula for S_μ , $\mu = 1, 2, \ldots, N$ by a different method.

Before deriving this expression for S_μ , we prove:

LEMMA 1. If $M(c, \mu)$ = the number of ways in which c components can be distributed into μ subsets with none of them being empty and if $\mathcal{S}_c^\mu$ are Stirling numbers of the second kind[1], then

$$M(c, \mu) = \mu! \, \mathcal{S}_c^\mu \tag{2:13}$$

PROOF. The distribution of one or more components into a subset implies that all the elements of the components are elements of the subset and that the total number of elements in the components equals the number in the subset. No regard is given to arrangement of components within the subset.

Whitworth [10] in proposition XXII proves that the number of ways in which c different things can be distributed

---

[1]Stirling numbers of the second kind are defined by Jordon [9] as: $\mathcal{S}_n^m = \left[\frac{\Delta^m x^n}{m!}\right]_{x=0}$, where $\Delta^m x^n = (x+m)^n - m(x+m-1)^n + \frac{m(m-1)}{2!}(x+m-2)^n - \ldots\ldots$

into μ parcels (without blank lots) is c! times the coefficient of x^c in the expansion of $(e^x - 1)^\mu$. He describes "different things" as those which, for purposes of the problem, are not identical and he defines "parcels" as an unarranged class. Therefore, it is consistent to allow "things" to be considered "components" and "parcels" to be "subsets". Thus M (c,μ) equals c! multiplied by the coefficient of x^c in the expansion of $(e^x - 1)^\mu$.

Jordan [9] in section 71, formula (5) shows that

(2:14) $$(e^x - 1)^\mu = \sum_{c=\mu}^{\infty} \frac{\mu!}{c!} \mathcal{S}_c^\mu x^c .$$

Therefore,

(2:13) $M(c,\mu) = \mu!\, \mathcal{S}_c^\mu$ and the proof of lemma 1 is complete.

It is now possible to prove

THEOREM 2. <u>If P_c is the probability of exactly c components and if $\mathcal{S}_c^\mu$ are Stirling numbers of the second kind, the value of S_μ is given by</u>:

(2:15) $$S_\mu = \mu! \sum_{c=\mu}^{N} P_c \mathcal{S}_c^\mu , \quad \mu = 1, 2, \ldots, N.$$

PROOF. In accordance with the hypothesis of lemma 1, the event $E(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ can be described by a condition which is equivalent to those given in (2:4).

(2:16) $E(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ = the event that when Ω

is divided into μ subsets, (ω_{α_1}, ω_{α_2}, ..., ω_{α_μ}), in accordance with (2:2) and (2:3), it is true that the c components induced in Ω by f can be distributed into these μ subsets with none of them being empty for all $c \geq \mu$.

S_μ is the sum of the probabilities of the occurrence of $E(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$. If there are a fixed number of components, each of the ways of distributing these components into the μ subsets of Ω constitutes an occurrence of $E(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$. Consequently,

$$(2:17) \qquad S_\mu = \sum_{c = \mu}^{N} P_c \cdot M(c, \mu), \quad \mu = 1, 2, \ldots, N.$$

Using lemma 1 with (2:17), theorem 2 is proved.

The next step in deriving the exact probability distribution is to solve (2:15) for P_c.

THEOREM 3. <u>If S_μ, $\mu = 1, 2, \ldots, N$ is given by (2:12') and if</u>

$$(2:18) \qquad W_\mu = \frac{S_\mu}{\mu!}$$

<u>and if S_μ^c are Stirling numbers of the first kind</u>[2], <u>then</u>

---

[2] Stirling numbers of the first kind are defined by Jordan [9] as $S_n^m = \left[\frac{1}{m!} D^m (x)_n\right]_{x = 0}$, where $D^m (x)_n$ is the m th derivative of: $x(x-1)(x-2) \ldots\ldots (x-n+1)$.

(2:19) $$P_c = \sum_{\mu=c}^{N} W_\mu S_\mu^c \quad , \ c = 1, 2, \ldots\ldots, N.$$

PROOF. Using (2:18), (2:15) becomes

(2:20) $$W_\mu = \sum_{c=\mu}^{N} P_c \mathcal{S}_c^\mu , \ \mu = 1, 2, \ldots\ldots, N.$$

In matrix notation (2:20) is written:

(2:20') $W = P\mathcal{S}$, where

(2:21) $$W = (W_1 \ W_2 \ W_3 \ \ldots\ldots \ W_N) ,$$

(2:22) $$P = (P_1 \ P_2 \ P_3 \ \ldots\ldots \ P_N) \text{ and}$$

(2:23) $$\mathcal{S} = \begin{pmatrix} \mathcal{S}_1^1 & 0 & 0 & \ldots\ldots & 0 \\ \mathcal{S}_2^1 & \mathcal{S}_2^2 & 0 & \ldots\ldots & 0 \\ \mathcal{S}_3^1 & \mathcal{S}_3^2 & \mathcal{S}_3^3 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \mathcal{S}_N^1 & \mathcal{S}_N^2 & \mathcal{S}_N^3 & \ldots\ldots & \mathcal{S}_N^N \end{pmatrix} .$$

Since $\mathcal{S}_n^\mu = 0$ for all $\mu > n$, $\mathcal{S}$ is a triangular matrix. Moreover, $\mathcal{S}_n^n = 1$ for all n. Therefore, $|\mathcal{S}| = 1$, $\mathcal{S}$ is non-singular and has an inverse $\mathcal{S}^{-1}$, which is also a triangular matrix of the same form. From (2:20'),

(2:24) $$P = W\mathcal{S}^{-1} .$$

Using Jordan [9], it will be shown that:

$$(2:25)\quad \mathcal{S}^{-1} = S = \begin{pmatrix} S_1^1 & 0 & 0 & \cdots\cdots & 0 \\ S_2^1 & S_2^2 & 0 & \cdots\cdots & 0 \\ S_3^1 & S_3^2 & S_3^3 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ S_N^1 & S_N^2 & S_N^3 & & S_N^N \end{pmatrix},$$

where S_μ^c are Stirling numbers of the first kind. Jordan shows

$$(2:26)\qquad \sum_{i=m}^{n} \mathcal{S}_n^i \, S_i^m = \delta_n^m ,$$

where δ_n^m is the Kronecker delta. This implies

$$(2:27)\qquad S = I \text{ and } S = \mathcal{S}^{-1} .$$

Now (2:24) becomes

$$(2:24')\qquad P = W\,S,$$

which in non - matrix notation is (2:19) and theorem 3 is proved.

Formula (2:19) is analogous to the formula (3.1) for the combination of events of chapter 4 of Feller [2.] The method of inclusion and exclusion used by Feller is different and does not work in the proof of (2:19). Katz [11.] employed the method used in this thesis. It involves setting

up a system of equations involving the desired quantity. (In our case the desired quantity was P_c.) Then this system is solved so that this desired quantity is expressed explicitly in terms of other calculable quantities.

A short table of values of Stirling numbers of the first kind from Jordan [9.] is included in table 1 so they can be used in the computation of numerical examples.

TABLE 1

STIRLING NUMBERS OF THE FIRST KIND - S_μ^c

μ \ c	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	-1	1	0	0	0	0	0	0
3	2	-3	1	0	0	0	0	0
4	-6	11	-6	1	0	0	0	0
5	24	-50	35	-10	1	0	0	0
6	-120	274	-225	85	-15	1	0	0
7	720	-1764	1624	-735	175	-21	1	0
8	-5040	13068	-13132	6769	-1960	322	-28	1

These numbers are related by a recurrence formula:

$S_{\mu+1}^c = S_\mu^{c-1} - \mu S_\mu^c$ with $S_1^j = \delta_1^j$ and $S_\mu^0 = 0$. Thus, the table may be extended indefinitely.

<u>2:3 A numerical example</u>. Consider $G_{\{r_i\}}$ defined by

$N = 6$ and $r_i = (1, 1, 2, 2, 3, 3)$. The values of the auxiliary sums by (2:12') are as follows.

If $\mu = 1$, $k_1 = N$ and $S_1 = 1$.

Since $\binom{k}{r} \doteq 0$ when $r > k$,

$$S_2 = \frac{1}{\binom{6}{1}^2\binom{6}{2}^2\binom{6}{3}^2}\left\{4\binom{1}{1}\binom{5}{1}\binom{5}{2}^2\binom{5}{3}^2 + 2\left[\binom{2}{1}^2\binom{4}{2}^2\binom{4}{3}^2\right.\right.$$

$$\left.\left. + 4\binom{2}{1}\binom{2}{2}\binom{4}{1}\binom{4}{2}\binom{4}{3}^2 + \binom{2}{2}^2\binom{4}{1}^2\binom{4}{3}^2\right] + 20\binom{3}{1}^2\binom{3}{2}^2\binom{3}{3}^2\right\}$$

$$S_2 = \frac{212,884}{3.24 \times 10^6} \doteq .065,705$$

$$S_3 = \frac{1}{3.24 \times 10^6}\left\{3\left[2\binom{1}{1}^2\binom{4}{2}^2\binom{4}{3}^2\right] + 6\left[4\binom{1}{1}\binom{2}{1}\binom{2}{2}\binom{3}{2}\binom{3}{3}^2\right.\right.$$

$$\left.\left. + 2\binom{1}{1}\binom{2}{2}^2\binom{1}{1}\binom{3}{3}^2\right]\right\}$$

$$S_3 = \frac{3636}{3.24 \times 10^6} \doteq .001,122$$

S_4, S_5 and S_6 all vanish because it is impossible to choose the k's so that all $\binom{k_i}{r_i} > 0$ in a given product.

Using (2:18) and (2:19) with table 1, the exact probability distribution of the number of components is:

$$P_1 = \frac{1,567,385}{1.62 \times 10^6} \doteq .967,522$$

$$P_2 = \frac{52,312}{1.62 \times 10^6} \doteq .032,291$$

$$P_3 = \frac{303}{1.62 \times 10^6} \doteq .000,187$$

$$P_4 = P_5 = P_6 = 0$$

3. PARTICULAR CASES OF GENERAL MAPPING

3:1. The class G_r. A special case of $G_{\{r_i\}}$ of some importance in applications is the class G_r. Under each $f \in G_r$, each point of Ω has the same number, r, of images. Any situation where each element under consideration maps into the same number of elements is of this type. While this case was implicitly covered in chapter 2, it is interesting to note what effect this particular kind of mapping will have on the computation formulas of the probability distribution.

Since G_r is a special case of $G_{\{r_i\}}$, the results of chapter 2 apply and can be modified to fit the situation that $r_i = r$. The modification of the formula for S_μ, $\mu = 1, 2, \ldots, N$, is threefold. First, with $r_i = r$ the subscripts on r can be removed. Second, the factors obtained from B_{ii}, $i = 1, \ldots, \mu$, are all equal and the double product of (2:12') can be changed to a single product with exponents. Third, for a fixed set of values, $(k_1, k_2, \ldots, k_\mu)$, the $\binom{N}{k_1, k_2, \ldots, k_\mu}$ choices of elements from Ω to form the subsets $(\omega_{\alpha_1}, \omega_{\alpha_2}, \ldots, \omega_{\alpha_\mu})$ will all give the same value for $p(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ because the number of images of each point of Ω is the same. Therefore, the indicated summation, $\sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}^{(k)}$ can be replaced by the

factor, $\binom{N}{k_1, k_2, \ldots, k_\mu}$. The result is

COROLLARY 1. <u>If N is the number of points in Ω and r is the number of images of each point, then it follows from theorem 1 that</u>

$$(3:1) \qquad S_\mu = \frac{1}{\binom{N}{r}^N} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 0 \\ k_1 + k_2 + \ldots + k_\mu = N}} \binom{N}{k_1, k_2, \ldots, k_\mu} \prod_{i=1}^{\mu} \binom{k_i}{r}^{k_i},$$

$$\mu = 1, 2, \ldots, N.$$

REMARK 1. For computational purposes (3:1) can be written as:

$$(3:1') \qquad S_\mu = \frac{1}{\left[N^{(r)}\right]^N} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 0 \\ k_1 + k_2 + \ldots + k_\mu = N}} \binom{N}{k_1, k_2, \ldots, k_\mu} \prod_{i=1}^{\mu} \left[k_i^{(r)}\right]^{k_i}, \mu = 1, 2, \ldots, N,$$

where $N^{(r)} = N(N-1) \ldots\ldots (N-r+1)$

Since the expression for P_c in (2:10) does not depend directly on the value of r but rather on the values of S_μ, $\mu = 1, 2, \ldots, N$, the probability distribution for this case can be computed as for the class $G_{\{r_i\}}$.

<u>3:2. A numerical example.</u> Consider G_r to be defined by $N = 6$ and $r = 2$. The values of the auxiliary sums by (3:1') are

$$S_1 = 1$$

$$S_2 = \frac{1}{(30)^6}\left\{2\binom{6}{2,4}(12)^4(2)^2 + \binom{6}{3,\ 3}(6)^6\right\} = \frac{44}{9375}$$

$$S_3 = \frac{1}{(30)^6}\binom{6}{2,\ 2,\ 2}(2)^2 = \frac{2}{253,125}$$

Since it is impossible to choose the k's so that all $\binom{k_i}{r} > 0$ in a given product, S_4, S_5, and S_6 all vanish.

The exact probability distribution as computed by (2:19) is:

$$P_1 = \frac{757,595}{759,375} \doteq .997,656$$

$$P_2 = \frac{1,779}{759,375} \doteq .002,343$$

$$P_3 = \frac{1}{759,375} \doteq .000,001$$

$$P_4 = P_5 = P_6 = 0$$

<u>3:3. The class, G_1.</u> The class of functions, f, under which each point of Ω has only one image in Ω is called G_1. This is the class of mappings considered by Metropolis and Ulam [4.] and Kruskal [5.]. Since $G_1 \subseteq G_r$, the results contained in (3:1) apply in this case. However, because of prior interest in this class of functions, it is appropriate to show explicitly the expressions which apply to this particular case.

Since each point of Ω has only one image in Ω, r = 1. The result which follows from (3:1) is given in

COROLLARY 2. If N is the number of elements in Ω and each point in Ω has only one image, then

$$(3:2) \quad S_\mu = \frac{1}{N^N} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 0 \\ k_1 + k_2 + \ldots + k_\mu = N}} \binom{N}{k_1, k_2, \ldots, k_\mu} \prod_{i=1}^{\mu} (k_i)^{k_i}, \quad \mu = 1, 2, \ldots, N.$$

Recently tables of the binomial probability distribution [12.] have been published. An alternative form of (3:2) for which these tables are useful is given in

THEOREM 4. If S_μ, $\mu = 1, 2, \ldots, N$, is given by (3:2) and if $b(k; n, p) = \binom{n}{k} p^k q^{n-k}$ is the binomial probability distribution, then

$$(3:3) \quad S_\mu = \sum_{\substack{k_1, k_2, \ldots, k_\mu > 0 \\ k_1 + k_2 + \ldots + k_\mu = N}} b\left(k_1; N, \frac{k_1}{N}\right) b\left(k_2; N - k_1, \frac{k_2}{N-k_1}\right) \cdots\cdots b\left(k_{\mu-1}; N - k_1 - \cdots - k_{\mu-2}, \frac{k_{\mu-1}}{N - k_1 - \cdots - k_{\mu-2}}\right),$$

$\mu = 1, 2, \ldots, N.$

PROOF: $\binom{N}{k_1,\ k_2,\ \ldots,\ k_\mu} = \dfrac{N!}{k_1!\ k_2!\ \ldots\ k_\mu!}$

$$= \frac{N!}{k_1!\ (N-k_1)!} \cdot \frac{(N-k_1)!}{k_2!\ (N-k_1-k_2)!} \cdot \ldots \cdot \frac{(N-k_1-\ldots-k_{\mu-2})!}{(N-k_1-\ldots-k_{\mu-1})!\ k_{\mu-1}!}$$

$$= \binom{N}{k_1}\binom{N-k_1}{k_2} \ldots \binom{N-k_1-k_2-\ldots-k_{\mu-2}}{k_{\mu-1}}.$$

$$\frac{\prod_{i=1}^{\mu} (k_i)^{k_i}}{N^N} = \left(\frac{k_1}{N}\right)^{k_1} \cdot \left(\frac{k_2}{N}\right)^{k_2} \cdot \ldots \cdot \left(\frac{k_\mu}{N}\right)^{k_\mu},$$ with

$k_1,\ k_2,\ \ldots,\ k_\mu > 0$ and $k_\mu = N - k_1 - k_2 - \ldots - k_{\mu-1}$.

From these facts, (3:2) can be written as a "telescoping product",

$$(3{:}2') \quad S_\mu = \sum_{\substack{k_1, k_2, \ldots, k_\mu > 0 \\ k_1+k_2+\ldots+k_\mu = N}} \left\{\binom{N}{k_1}\left(\frac{k_1}{N}\right)^{k_1}\left(\frac{N-k_1}{N}\right)^{N-k_1}\right\}$$

$$\left\{\binom{N-k_1}{k_2}\left(\frac{k_2}{N-k_1}\right)^{k_2}\left(\frac{N-k_1-k_2}{N-k_1}\right)^{N-k_1-k_2}\right\}\ldots$$

$$\ldots\ldots \left\{\binom{N-k_1-\ldots-k_{\mu-2}}{k_{\mu-1}}\left(\frac{k_{\mu-1}}{N-k_1-\ldots-k_{\mu-2}}\right)^{k_{\mu-1}}\right.$$

$$\left.\left(\frac{N-k_1\ \ldots-k_{\mu-1}}{N-k_1-\ldots-k_{\mu-2}}\right)^{N-k_1-\ldots-k_{\mu-1}}\right\}, \mu = 1,\ 2,\ \ldots.\ N.$$

By the hypothesis of theorem 4, (3:3) follows immediately from (3:2') and the theorem is proved.

Again (2:19) can be used to compute the exact probability distribution because this formula is not directly dependent on the value of r but rather on the value of S_μ which must be computed by either (3:2) or 3:3).

3:4. A numerical example. Consider G_1 to be defined by N = 8 and r = 1. The values of the auxiliary sums as computed by (3:3) are:

$$S_1 = 1$$

$$S_2 = b(1;8,1/8) + b(2;8,1/4) + b(3;8,3/8) + b(4;8,1/2) + b(5;8,5/8) + b(6;8,3/4) + b(7;8,7/8).$$

Since $\binom{N}{k} = \binom{N}{N-k}$, it follows from the hypothesis of theorem 4 that $\binom{N}{k}\left(\frac{k}{N}\right)^k \left(\frac{N-k}{N}\right)^{N-k} = \binom{N}{N-k}\left(\frac{N-k}{N}\right)^{N-k}\left(\frac{k}{N}\right)^k$

and as a result,

(48) $b(k;N,\frac{k}{N}) = b(N-k;N,\frac{N-k}{N})$ and S_2 becomes:

$$S_2 = 2b(1;8,1/8) + 2b(2;8,1/4) + 2b(3;8,3/8) + b(4;8,1/2) .$$

Using tables of the binomial distribution [12.] for the values which it contains and computation by means of the hypothesis of theorem 4 for other values:

$$S_2 = .785,391,8 + .622,924,8 + .563,263,8 + .273,437,6$$

$$S_2 = 2.245,018,0$$

$$\begin{aligned} S_3 = {} & b(1;8,1/8)\left[2b(1;7,1/7) + 2b(2;7,2/7) + 2b(3;7,3/7)\right] \\ & + b(2;8,1/4)\left[2b(1;6,1/6) + 2b(2;6,1/3) + b(3;6,1/2)\right] \\ & + b(3;8,3/8)\left[2b(1;5,1/5) + 2b(2;5,2/5)\right] \\ & + b(4;8,1/2)\left[2b(1;4,1/4) + b(2;4,1/2)\right] \\ & + b(5;8,5/8)\left[2b(1;3,1/3)\right] + b(6;8,3/4)b(1;2,1/2) \end{aligned}$$

$$S_3 = .792{,}514{,}8 + .552{,}749{,}6 + .425{,}376{,}8 + 333{,}252{,}1$$
$$+ .250{,}339{,}4 + .155{,}731{,}2$$

$$S_3 = 2.509{,}963{,}9$$

$$S_4 = b(1;8,1/8)b(1;7,1/7)[2b(1;6,1/6) + 2b(2;6,1/3)$$
$$+ b(3;6,1/2)] + b(1;8,1/8)b(2;7,2/7)[2b(1;5,1/5)$$
$$+ 2b(2;5,2/5)] + b(1;8,1/8)b(3;7,3/7)[2b(1;4,1/4)$$
$$+ b(2;4,1/2)] + b(1;8,1/8)b(4;7,4/7)[2b(1;3,1/3)]$$
$$+ b(1;8,1/8)b(5;7,5/7)b(1;2,1/2)$$
$$+ b(2;8,1/4)b(1;6,1/6)[2b(1;5,1/5) + 2b(2;5,2/5)]$$
$$+ b(2;8,1/4)b(2;6,1/3)\ 2b(1;4,1/4) + b(2;4,1/2)$$
$$+ b(2;8,1/4)b(3;6,1/2)2b(1;3,1/3)$$
$$+ b(2;8,1/4)b(4;6,2/3)b(1;2,1/2)$$
$$+ b(3;8,3/8)b(1;5,1/5)[2b(1;4,1/4) + b(2;4,1/2)]$$
$$+ b(3;8,3/8)b(2;5,2/5)\ 2b(1;3,1/3)$$
$$+ b(3;8,3/8)b(3;5,3/5)b(1;2,1/2)$$
$$+ b(4;8,1/2)b(1;4,1/4)\ 2b(1;3,1/3)$$
$$+ b(4;8,1/2)b(2;4,1/2)b(1;2,1/2)$$
$$+ b(5;8,5/8)b(1;3,1/3)b(1;2,1/2)$$

$$S_4 = .276{,}374{,}8 + .189{,}056{,}3 + .140{,}590{,}7 + .102{,}539{,}1$$
$$+ .102{,}539{,}1 + .062{,}584{,}8 + .189{,}056{,}3$$
$$+ .124{,}969{,}5 + .086{,}517{,}3 + .086{,}517{,}3 + .140{,}590{,}7$$
$$+ .048{,}666{,}0 + .051{,}269{,}6 + 062{,}584{,}9$$

$$S_4 = 1.715{,}120{,}9$$

$$S_5 = b(1;8,1/8)b(1;7,1/7)\Big\{b(1;6,1/6)\big[2b(1;5,1/5)$$
$$+ 2b(2;5,2/5)\big] + b(2;6,1/3)\big[2b(1;4,1/4) + b(2;4,1/2)\big]$$
$$+ b(3;6,1/2)2b(1;3,1/3) + b(4;6,2/3)b(1;2,1/2)\Big\}$$
$$+ b(1;8,1/8)b(2;7,2/7)\Big\{b(1;5,1/5)\big[2b(1;4,1/4)$$
$$+ b(2;4,1/2)\big] + b(2;5,2/5)2b(1;3,1/3)$$
$$+ b(3;5,3/5)b(1;2,1/2)\Big\} + b(1;8,1/8)b(3;7,3/7)$$
$$\Big\{b(1;4,1/4)2b(1;3,1/3) + b(2;4,1/2)b(1;2,1/2)$$
$$+ b(1;3,1/3)b(1;2,1/2)\Big\}$$
$$+ b(2;8,1/4)b(1;6,1/6)\Big\{b(1;5,1/5)\big[2b(1;4,1/4)$$
$$+ b(2;4,1/2)\big] + b(2;5,2/5)\big[2b(1;3,1/3 + b(1;2,1/2)\big]\Big\}$$
$$+ b(2;8,1/4)b(2;6,1/3)\Big\{b(1;4,1/4)2b(1;3,1/3)$$
$$+ b(2;4,1/2)b(1;2,1/2)\Big\}$$
$$+ b(2;8,1/4)b(3;6,1/2)b(1;3,1/3)b(1;2,1/2)$$
$$+ b(3;8,3/8)b(1;5,1/5)\Big\{b(1;4,1/4)2b(1;3,1/3)$$
$$+ b(2;4,1/2)b(1;2,1/2)\Big\}$$
$$+ b(3;8,3/8)b(2;5,2/5)b(1;3,1/3)b(1;2,1/2)$$
$$+ b(4;8,1/2)b(1;4,1/4)b(1;3,1/3)b(1;2,1/2)$$

$$S_5 = .094,528,2 + .062,484,7 + .043,258,7 + .025,634,8$$
$$+ .062,484,7 + .038,452,1 + .021,629,3 + .043,258,7$$
$$+ .021,629,3 + .025,634,8 + .062,484,7 + .038,452,1$$
$$+ .021,629,3 + .038,452,1 + .019,226,0 + .021,629,3$$
$$+ .043,258,7 + .021,629,3 + .021,629,3 + .025,634,8$$

$$S_5 = .753,020,9$$

$$S_6 = b(1;8,1/8)b(1;7,1/7)b(1;6,1/6)\Big\{b(1;5,1/5)\big[2b(1;4,1/4)$$
$$+ b(2;4,1/2)\big] + b(2;5,2/5)\big[2b(1;3,1/3) + b(1;2,1/2)\big]\Big\}$$
$$+ b(1;8,1/8)b(1;7,1/7)b(2;6,1/3)\Big\{b(1;4,1/4)2b(1;3\,1/3)$$

$+ b(2;4,1/2)b(1;2,1/2)\}$

$+ b(1;8,1/8)b(1;7,1/7)b(3;6,1/2)b(1;3,1/3)b(1;2,1/2)$

$+ b(1;8,1/8)b(2;7,2/7)b(1;5,1/5)b(1;4,1/4)2b(1;3,1/3)$

$+ b(1;8,1/8)b(2;7,2/7)b(1;5,1/5)b(2;4,1/2)b(1;2,1/2)$

$+ b(1;8,1/8)b(2;7,2/7)b(2;5,2/5)b(1;3,1/3)b(1;2,1/2)$

$+ b(1;8,1/8)b(3;7,3/7)b(1;4,1/4)b(1;3,1/3)b(1;2,1/2)$

$+ b(2;8,1/4)b(1;6,1/6)b(1;5,1/5)b(1;4,1/4)2b(1;3,1/3)$

$+ b(2;8,1/4)b(1;6,1/6)b(1;5,1/5)b(2;4,1/2)b(1;2,1/2)$

$+ b(2;8,1/4)b(1;6,1/6)b(2;5,2/5)b(1;3,1/3)b(1;2,1/2)$

$+ b(3;8,3/8)b(1;5,1/5)b(1;4,1/4)b(1;3,1/3)b(1;2,1/2)$

$S_6 = .061,283,1 + .028,839,1 + .010,814,7 + .019,226,1$

$+ .009,613,0 + .009,613,0 + .010,814,7 + .019,226,1$

$+ .009,613,0 + .009,613,0 + .009,613,0 + .010,814,7$

$S_6 = .209,093,3$

$S_7 = b(1;8,1/8)b(1;7,1/7)b(1;6,1/6)b(1;5,1/5)\{b(1;4,1/4)$

$2b(1;3,1/3) + b(2;4,1/2)b(1;2,1/2)\} + b(1;8,1/8)$

$b(1;7,1/7)b(1;6,1/6)b(2;5,2/5)b(1;3,1/3)b(1;2,1/2)$

$+ b(1;8,1/8)b(1;7,1/7)b(2;6,1/3)b(1;4,1/4)$

$b(1;3,1/3)b(1;2,1/2) + b(1;8,1/8)b(2;7,2/7)$

$b(1;5,1/5)b(1;4,1/4)b(1;3,1/3)b(1;2,1/2)$

$+ b(2;8,1/4)b(1;6,1/6)b(1;5,1/5)b(1;4,1/4)$

$b(1;3,1/3)b(1;2,1/2)$

$S_7 = .014,419,6 + .004,806,5 + .004,806,5 + .004,806,5$

$+ .004,806,5$

$S_7 = .033,645,6$

$$S_8 = b(1;8,{}^1/8)b(1;7,{}^1/7)b(1;6,{}^1/6)b(1;5,{}^1/5)$$
$$b(1;4,{}^1/4)b(1;3,{}^1/3)b(1;2,{}^1/2)$$

$$S_8 = .002,403,3$$

Using these results together with table 1 and (2:19), the exact probability distribution is:

$$P_1 = .405,628$$
$$P_2 = .408,440$$
$$P_3 = .153,895$$
$$P_4 = .028,893$$
$$P_5 = .002,970$$
$$P_6 = .000,169$$
$$P_7 = ,000,005$$
$$P_8 = .000,000,06$$

4. THE HOLLOW CASE

4:1. Preliminaries. If the mapping of Ω into Ω is restricted so that no point is permitted to map into itself, the mapping is called hollow. The subclass of functions which represents this type is called $H_{\{r_i\}}$. Analogous to $G_{\{r_i\}}$, $H_{\{r_i\}}$ is composed of functions, f, under which each $x_i \in \Omega$ has r_i, $i = 1, 2, \ldots, N$, images in Ω.

Hollow mapping is of special interest in the field of social psychology. In sociometric tests an individual chooses the individuals in a group with whom he wishes to be associated. In some cases a variety in the number of choices made by an individual is permitted. In other cases, all individuals must make an equal number of choices. In still other instances only the prime choice is made. If each of the N individuals making the choices is considered to correspond to a point x_i and if his choices for associates correspond to the r_i images of x_i, $i = 1, 2, \ldots, N$, then a hollow mapping situation exists provided no individual is permitted to choose to be associated with himself.

The number of choices permitted in different instances gives rise to different subsets of hollow mappings.

The situation where there is a variety in the number of answers by different individuals in the group is covered by the class, $H_{\{r_i\}}$. If H_r and H_1 are subsets of functions defined for the hollow case as G_r and G_1 were for the general case, then H_r covers the situation where each individual in the group chooses the same number of associates and H_1 covers the case where each individual makes only the one best choice. H_r and H_1 are considered in chapter 5.

4:2. The auxiliary sums and the exact probability distribution of the number of components. Since $H_{\{r_i\}} \subset G_{\{r_i\}}$, the results of chapter 2 can be adapted to the hollow case. To make the event, $E_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}$, possible under hollow mapping there must be at least two elements in the subset ω_{α_j}, $j = 1, 2, \ldots, \mu$. This means the k_j, $j = 1, 2, \ldots, \mu$, must be greater than one. The matrix representation, $B^f = (b^f_{ij})$ is different in that $b^f_{jj} \equiv 0$ for all $j = 1, 2, \ldots, N$. Consequently, the required form of B^f, which is equivalent to the event, $E_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}$ given in (2:8), is modified in that the main diagonal elements of the principal minors, B_{ii}, $i = 1, 2, \ldots, \mu$, are all zeros. Therefore, the r_{i_j} ones in the j th row of B_{ii}, $i = 1, 2, \ldots, \mu$ and $j = 1, 2, \ldots, k_i$ may appear only in any of the remaining $(k_i - 1)$ positions if B^f is to have the form equivalent to (2:8). At the same time if no restrictions are placed on

the form of B^f, the ones in any row could appear in any of $(N - 1)$ positions. Since $k_j > 1$, $j = 1, 2, \ldots, \mu$, there could be at most $\left[\frac{N}{2}\right]$ subsets formed from the index set, $\mathcal{J}$, where $\left[\frac{N}{2}\right]$ is the largest integer in the quotient $\frac{N}{2}$. Moreover, there could be at most $\left[\frac{N}{2}\right]$ components induced in Ω by a function, f.

Using the above facts, formulas for the auxiliary sums, S_μ, $\mu = 1, 2, \ldots, \left[\frac{N}{2}\right]$, and for the probability, P_c, of exactly c components, $c = 1, 2, \ldots, \left[\frac{N}{2}\right]$, must be modified in order to make them valid for the hollow case. The theorems which were proved for the general case are now listed with proper modifications for the hollow case. They are numbered with primes to show the correspondence between the cases.

THEOREM 1'. <u>If N is the number of elements in Ω and r_ν is the number of images of the point $x_\nu \in \Omega$, $\nu = 1, 2, \ldots, N$, and if $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ is defined by (2:2) then the value of S_μ is given by</u>

$$(4:1)\quad S = \frac{1}{\prod_{i=1}^{N} \binom{N-1}{r_i}} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 1 \\ k_1 + k_2 + \ldots + k_\mu = N}} \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_\mu)}^{(k)} \prod_{i=1}^{\mu} \prod_{j=1}^{k_i} \binom{k_i - 1}{r_{ij}}, \quad \mu = 1, 2, \ldots, \left[\frac{N}{2}\right].$$

THEOREM 2'. If P_c is the probability of exactly c components and $\mathcal{S}_c^{\mu}$ are Stirling numbers of the second kind, then

$$(4{:}2)\quad S_\mu = \mu! \sum_{c=\mu}^{\left[\frac{N}{2}\right]} P_c \mathcal{S}_c^{\mu}, \ \mu = 1, 2, \ldots, \left[\tfrac{N}{2}\right].$$

THEOREM 3'. If $W_\mu = \frac{S_\mu}{\mu!}$, $\mu = 1, 2, \ldots, \frac{N}{2}$, given by (4:1), and if S_μ^c are Stirling numbers of the first kind, then

$$(4{:}3)\quad P_c = \sum_{\mu=c}^{\left[\frac{N}{2}\right]} W_\mu S_\mu^c, \ c = 1, 2, \ldots, \left[\tfrac{N}{2}\right].$$

4:3. A numerical example. Consider $H_{\{r_1\}}$ defined by $N = 6$ and $r_1 = (1, 1, 2, 2, 3, 3)$. The values of the auxiliary sums are, by (4:1):

$$S_1 = 1$$

$$S_2 = \frac{1}{\binom{5}{1}^2\binom{5}{2}^2\binom{5}{3}^2}\left\{2\binom{1}{1}\binom{3}{2}^2\binom{3}{3}^2\right\} = \frac{18}{2.5 \times 10^5}$$

$$S_2 = .000{,}072$$

S_3, S_4, S_5, and S_6 all vanish.

By (4:3), the exact probability distribution for this example is:

$$P_1 = .999,964$$

$$P_2 = .000,036$$

$$P_3 = P_4 = P_5 = P_6 = 0 .$$

5. PARTICULAR CASES OF HOLLOW MAPPING

5:1. The class H_r. A special case of $H_{\{r_i\}}$ for the hollow case is the class H_r which is defined as G_r was for the general case. (That is each point of Ω has the same number, r, of images in Ω.) Although this class was mentioned in section 4:1 and implicitly considered in section 4:2, for completeness, the exact formulas for computation are shown. The corollaries are numbered with primes to show the parallelism between the results for G_r in section 3:1 and those for H_r. The reasons for the modifications are covered in section 3:1.

COROLLARY 1'. If N is the number of points in Ω and r is the number of images of each point, then it follows from theorem 1' that

$$(5:1) \qquad S_\mu = \frac{1}{\binom{N-1}{r}^N} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 1 \\ k_1+k_2+\cdots+k_\mu=N}} \binom{N}{k_1, k_2, \ldots, k_\mu} \prod_{i=1}^{\mu} \binom{k_i-1}{r}^{k_i},$$

$$\mu = 1, 2, \ldots, \left[\frac{N}{2}\right].$$

REMARK 1'. For purposes of computation (5:1) can be written

$$(5:1') \quad S_\mu = \frac{1}{\left[(N-1)^{(r)}\right]^N} \sum_{\substack{k_1,k_2,\ldots,k_\mu \geq 1 \\ k_1+k_2+\ldots+k_\mu = N}} \binom{N}{k_1,k_2,\ldots,k_\mu} \prod_{i=1}^{\mu} \left[(k_{i-1})^{(r)}\right]^{k_i}, \quad \mu = 1,2,\ldots,\left[\frac{N}{2}\right].$$

Since the formula for P_c in (4:3) does not depend directly on r but rather on the values of $S_\mu, \mu = 1, 2, \ldots \left[\frac{N}{2}\right]$, the exact probability distribution can be computed by using this formula.

5:2 A numerical example. Consider H_r defined by $N = 6$ and $r = 2$. The values of the auxiliary sums by (5:1') are:

$$S_1 = 1$$

$$S_2 = \frac{1}{(20)^6} \binom{6}{3, 3} (2)^6 = \frac{20}{10^6} = .000,020$$

$$S_3 = 0$$

The resulting exact probability distribution by use of (4:3) is:

$$P_1 = .999,990$$

$$P_2 = .000,010$$

$$P_3 = 0$$

5:3. The class H_1. As mentioned in section 4:1,

the subclass H_1 is defined for the hollow case as G_1 was for the general case. To complete the parallelism between the hollow and the general case, the formulas which apply specifically to random mapping under functions from H_1 are given as they were for the class H_r.

COROLLARY 2'. <u>If N is the number of elements in Ω and each point in Ω has only one image, then</u>

$$(5:2) \quad S_\mu = \frac{1}{(N-1)^N} \sum_{\substack{k_1, k_2, \ldots k_\mu > 1 \\ k_1+k_2+\ldots+k_\mu = N}} \binom{N}{k_1, k_2, \ldots, k_\mu} \prod_{i=1}^{\mu} (k_i-1)^{k_i},$$

$$\mu = 1, 2, \ldots, \left[\frac{N}{2}\right].$$

In order to make the tables of the binomial probability distribution [12.] useful an alternative form of (5:2) is presented in

THEOREM 4'. <u>If S_μ, $\mu = 1, 2, \ldots, \left[\frac{N}{2}\right]$ is given by (5:2) and if $b(k;n,p) = \binom{n}{k} p^k q^{n-k}$ is the binomial distribution, then</u>

$$(5:3) \quad S_\mu = \frac{(N-\mu)^N}{(N-1)^N} \sum_{\substack{k_1, k_2, \ldots, k_\mu > 1 \\ k_1+k_2+\ldots+k_{\mu-1} \le (N-1)}} b\left(k_1; N, \frac{k_1-1}{N-\mu}\right)$$

$$b\left(k_2; N-k_1, \frac{k_2-1}{N-k_1-\mu+1}\right) \cdots b\left(k_{\mu-1}; N-k_1-\ldots-k_{\mu-2}, \frac{k_{\mu-1}-1}{N-k_1-\ldots-k_{\mu-2}-2}\right),$$

$$\mu = 1, 2, \ldots, \left[\frac{N}{2}\right].$$

PROOF:

$$\binom{N}{k_1, k_2, \ldots, k_\mu} = \binom{N}{k_1}\binom{N-k_1}{k_2} \cdots\cdots \binom{N-k_1-\ldots-k_{\mu-2}}{k_{\mu-1}}$$

$$k_1, k_2, \ldots\ldots, k_\mu > 1$$

$$(k_\mu - 1) = (N-k_1-k_2-\ldots\ldots-k_{\mu-1} - 1)$$

Therefore, (5:2) can be written as a "telescoping product",

$$(5{:}2') \quad S_\mu = \frac{(N-\mu)^N}{(N-1)^N} \sum_{\substack{k_1, k_2, \ldots, k_{\mu-1} \geq 1 \\ k_1+k_2+\ldots+k_{\mu-1} \leq N-1}} \left\{ \binom{N}{k_1} \left(\frac{k_1 - 1}{N-\mu}\right)^{k_1} \left(\frac{N-k_1-\mu+1}{N-\mu}\right)^{N-k_1} \right\}$$

$$\left\{ \binom{N-k_1}{k_2} \left(\frac{k_2-1}{N-k_1-\mu+1}\right)^{k_2} \left(\frac{N-k_1-k_2-\mu+2}{N-k_1-\mu+1}\right)^{N-k_1-k_2} \right\} \cdots\cdots$$

$$\cdots\cdots \left\{ \binom{N-k_1-\ldots-k_{\mu-2}}{k_{\mu-1}} \left(\frac{k_{\mu-1}-1}{N-k_1-\ldots-k_{\mu-2}-2}\right)^{k_{\mu-1}} \left(\frac{N-k_1-\ldots-k_{\mu-1}-1}{N-k_1-\ldots-k_{\mu-2}-2}\right)^{N-k_1-\ldots-k_{\mu-1}} \right\}$$

$$\mu = 1, 2, \ldots\ldots, \left[\frac{N}{2}\right].$$

Using the notation of the hypothesis of theorem 4', (5:3) follows and the theorem is proved.

The formula which is used to compute the exact distribution after S_μ, $\mu = 1, 2, \ldots, \left[\frac{N}{2}\right]$ has been computed by (5:2) or (5:3) is (4:3) because it does not depend on the value of r but on the value of S_μ.

5:4 A numerical example. Consider H_1 defined by $N = 8$ and $r = 1$. The values of the auxiliary sums as found by use of (5:3) are

$$S_1 = 1$$

$$S_2 = \frac{(6)^8}{(7)^8}\left[2b(2;8,{}^1/6) + 2b(3;8,{}^1/3) + b(4;8,{}^1/2)\right]$$

$$S_2 = .390,607,4$$

$$S_3 = \frac{(5)^8}{(7)^8}\Big[2b(2;8,{}^1/5)b(2;6,{}^1/4) + b(2;8,{}^1/5)b(3;6,{}^1/2)$$
$$+ 2b(3;8,{}^2/5)b(2;5,{}^1/3) + b(4;8,{}^3/5)b(2;4,{}^1/2)\Big]$$

$$S_3 = .036,354,0$$

$$S_4 = \frac{(4)^8}{(7)^8}\left[b(2;8,{}^1/4)b(2;6,{}^1/3)b(2;4,{}^1/2)\right]$$

$$S_4 = .000,437,1$$

S_5, S_6, S_7, and S_8 all vanish.

By (4:3), the exact distribution is:

$$P_1 = .816,705$$

$$P_2 = .177,327$$

$$P_3 = .005,950$$

$$P_4 = .000,018$$

$$P_5 = P_6 = P_7 = P_8 = 0$$

6. A BINOMIAL APPROXIMATION OF THE DISTRIBUTION OF THE NUMBER OF COMPONENTS

6:1. Introductory considerations. Since the exact distribution is known, an approximate distribution is useful only if it is more easily computed. The above numerical examples, although restricted to relatively trivial cases show that considerable work is involved in the computation of the exact distribution. Therefore, an approximation is worthy of investigation.

Because the distribution of the number of components is discrete and since it is conceivable, in some applications, that the set Ω will be composed of a relatively small number of elements, it seems feasible to use a distribution of the discrete type for the approximation. A binomial approximation is therefore found and the results have proved to be rather good.

6:2. Derivation of an approximate probability distribution. The binomial distribution has only two parameters, N and p. With N fixed, only p needs to be estimated. Since the mean of the binomial distribution equals N times p, an approximation of p could be obtained by equating the expected value of the random variable, c , which is the number of components in Ω , with the expected value of c if it were distributed as the binomial distribution. However, the expected value of the number, c , of components is not easily obtained in terms of one or more of the S_μ, $\mu = 1, 2, \ldots, N$. Kruskal [5.] gave the result for the special case where the mapping was single-valued. Since this seems to be quite difficult in the general case, it is convenient to consider a related variable, n^{c-1}, where n is a positive integer. For such a variable the following theorem is proved.

THEOREM 5: <u>If c is the number of components in Ω, n is a positive integer and S_μ the auxiliary sums given by (2:12'), then the expected value of n^{c-1} is given by</u>

$$E\left(n^{c-1}\right) = \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} \frac{S_\mu}{\mu}. \tag{6:1}$$

PROOF: Consider the arbitrary quantity, A, defined as:

$$A = \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} \frac{S_\mu}{\mu}$$

From (2:15): $S_\mu = \mu! \sum_{j=\mu}^{n} P_j \mathscr{S}_j^{\mu}$.

Therefore, $A = \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} (\mu-1)! \left[\sum_{j=\mu}^{N} P_j \mathscr{S}_j^{\mu} \right].$

The coefficient of P_c in $A = \sum_{\mu=1}^{c} \binom{n-1}{\mu-1} (\mu-1)! \mathscr{S}_c^{\mu}.$

By changing the notation slightly, the coefficient of

P_c in $A = \frac{1}{n} \sum_{\mu=1}^{c} n^{(\mu)} \mathscr{S}_c^{\mu}.$

But Jordan [9], section 58, formula 2, shows

$$n^c = \sum_{\mu=1}^{c} \mathscr{S}_c^{\mu} (n)_{\mu} \text{ , where } (n)_{\mu} = n(n-1)....(n-\mu+1) = n^{(\mu)}.$$

Therefore, the coefficient of P_c in $A = n^{c-1}$; and by the definition of the expected value, $A = E\ (n^{c-1})$ and (6.1) follows. This completes the proof of theorm 5.

Using (6:1) the results for a few integers are:

(a) $E\ (1^{c-1}) = 1$

(b) $E\ (2^{c-1}) = 1 + \frac{S_2}{2}$

(c) $E\ (3^{c-1}) = 1 + S_2 + \frac{S_3}{3}$

Since $E(1^{c-1})$ gives a trivial result and since $E\ (2^{c-1})$ is obviously the simplest expected value to compute, it is convenient to use this to find a binomial approximation of the distribution of the number of components.

In order to find the parameter p which will determine the binomial distribution for fixed N, it is necessary to prove the following theorem.

THEOREM 6. <u>If (c-1) is a random variable distributed binomially with parameters (N-1) and p, then</u>

$$(6:2) \qquad E(n^{c-1}) = \left[1 + (n-1)p\right]^{N-1}$$

PROOF: By hypothesis: $b(c-1;\, N-1,\, p) = \binom{N-1}{c-1} p^{c-1} q^{N-c}$

Therefore, $$E\,(n^{c-1}) = \sum_{c=1}^{N} n^{c-1} \binom{N-1}{c-1} p^{c-1} q^{N-c}.$$

This can be rewritten:

$$E\,(n^{c-1}) = \sum_{c=1}^{N} \binom{N-1}{c-1} (np)^{c-1} q^{N-c} = (q + np)^{N-1}$$

Since q = 1 - p, (6:2) follows and the proof of theorem 6 is complete.

An estimate of p which will determine the binomial approximation of the distribution of the number of components is now possible by equating the two values of $E(2^{c-1})$ obtained from theorems 5 and 6. This is not an estimate in the statistical sense. No sampling is involved. It is simply an approximation which results from equating the expected value of 2^{c-1} if c is the number of components in Ω with the expected value of 2^{c-1} if (c-1) has binomial distribution with parameters (N-1) and p.

Using the results in (6:1) and (6:2) with n = 2 and equating the expected values of 2^{c-1}, the result for the estimate of p is:

$$(6:3) \qquad p = \left[1 + \frac{S_2}{2}\right]^{\frac{1}{N-1}} - 1.$$

Using (6:3) and denoting the approximate probability of c components by Q_c , the formula is

$$(6:4) \quad Q_c = \binom{N-1}{c-1} p^{c-1} q^{N-c} \ , \ c = 1, 2, \ldots., N.$$

It is noted that this binomial approximation of the distribution of the number of components can be determined by finding only the value of S_2. For the general case formula (2:12'), (3:1') or (3:3) can be used, depending on the values of r. After S_2 has been found, p is found by (6:3) and the approximate distribution is found by (6:4).

For the hollow case, the same procedure may be used and the same formulas apply with the exception that N is replaced by $\left[\frac{N}{2}\right]$ throughout. For completeness the formulas are listed with primes as they apply to the hollow case.

$$(6:3') \qquad p = \left[1 + \frac{S_2}{2}\right]^{\frac{1}{\left[\frac{N}{2}\right] - 1}} - 1$$

$$(6{:}4') \qquad Q_c = \binom{\left[\frac{N}{2}\right] - 1}{c-1} p^{c-1} q^{\left[\frac{N}{2}\right] - c}, \quad c = 1,\ 2,\ \ldots.,\ \left[\frac{N}{2}\right].$$

6:3 Numerical examples. In connection with the exact distribution discussed previously six numerical examples were presented to illustrate each of the classes of random mapping functions. These same examples are now presented so that the approximate distribution of the number of components can be compared with the exact distribution. Accordingly, an estimate of p is found for each example and the approximate probabilities are shown together with the exact probabilities, (found earlier) for the general and hollow cases in tables 2 and 3 respectively.

If $G_{\{r_i\}}$ is defined by $N = 6$ and $r = (1, 1, 2, 2, 3, 3)$, then, from section 2:3, $S_2 = .065,705$ and by (6:3) $p = .006,486$. The values of Q_c computed by (6:4) are shown in the first section of table 2.

If G_r is defined by $N = 6$ and $r = 2$, then, from section 3:2, $S_2 = \frac{44}{9375}$ and, by (6:4), $p = .000,467$. The values of Q_c computed by (6:4) are given in the second section of table 2.

If G_1 is defined by $N = 8$ and $r = 1$, from section 3:4, $S_2 = 2.245,018$ and therefore $p = .113,506,6$. Again the values of Q_c are given in the third section of table 2.

If $H_{\{r_1\}}$ is defined by N = 6 and r = (1, 1, 2, 2, 3, 3), then S_2 = .000,072 by section 4:3. Using (6:3'), p = .000,018. The values of Q_c are computed by (6:4') and given in the first section of table 3.

If H_r is defined by N = 6 and r = 2, by section 5:2, S_2 = .000,020 and by (6:3'), p = .000,005. The values of Q_c as computed by (6:4') are given in the second section of table 3.

If H_1 is defined by N = 8 and r = 1, by section 5:4, S_2 = .390,607,4 and by (6:3'), p = .061,270,5. The values of Q_c are given in the third section of table 3.

The agreement between the tabulated values is reasonably good. In the hollow case, there is virtually no difference between the exact and approximate values when N= 6, r_1 = (1, 1, 2, 2, 3, 3) and when N = 6, r = 2. In the general case, agreement is to at least the third decimal place for these examples. When N = 8 and r = 1, there is variation but there is definite agreement in the pattern of the distributions.

For larger values of N the computation of the exact distribution is cumbersome. Thus, comparison becomes difficult. Katz [13.] has shown that the exact probability of one component when N = 20 and r = 1 is .264,68. The approximate probability given by (6:4) for this set of values is: Q_1 = .295,227. This indicates very little change in accuracy in the probability of one component for larger values of N.

TABLE 2

COMPARISON OF THE EXACT AND APPROXIMATE DISTRIBUTIONS OF THE NUMBER OF COMPONENTS IN THE GENERAL CASE

	$N = 6$, $r_1 = (1,1,2,2,3,3)$		$N = 6$, $r = 2$		$N = 8$, $r = 1$	
C	P_c	Q_c	P_c	Q_c	P_c	Q_c
1	.967,522	.967,988	.997,656	.997,667	.405,628	.430,258
2	.032,291	.031,597	.002,343	.002,331	.408,440	.385,631
3	.000,187	.000412	.000,001	.000,002	.153,895	.148,129
4	—	.000,003	—	—	.028,893	.031,611
5	—	—	—	—	.002,970	.004,047
6	—	—	—	—	.000,169	.000,311
7	—	—	—	—	.000,005	.000,013
8	—	—	—	—	.000,000	.000,000

TABLE 3

COMPARISON OF THE EXACT AND APPROXIMATE DISTRIBUTIONS OF
THE NUMBER OF COMPONENTS IN THE HOLLOW CASE

	$N = 6,\ r_i = (1,1,2,2,3,3)$		$N = 6,\ r = 2$		$N = 8,\ r = 1$	
C	P_c	Q_c	P_c	Q_c	P_c	Q_c
1	.999,964	.999,964	.999,990	.999,990	.816,705	.827,221
2	.000,036	.000,036	.000,010	.000,010	.177,327	.161,977
3	—	.000,000	—	.000,000	.005,950	.010,572
4	—	—	—	—	.000,018	.000,230

7. SUMMARY

After Kruskal [5.] had solved the problem of the expected number of components for a single-valued random mapping function, the question of the probability distribution of the number of components was a logical next step. Moreover, the question of what would happen if the mapping were multiple-valued seemed worthy of consideration. By a method somewhat analogous to that used by Feller [2.] for the combination of events but more nearly like that used by Katz [11.] , the exact probability distribution of the number of components of a multiple-valued random mapping function was found. Results for particular mappings which restricted in various ways the number of images of each point became special cases of the general solution.

Hollow mapping in the sense that no point was permitted to map into itself was considered because of the interest in this type in application to social situations. The probability distribution for this case followed very readily from the general solution.

Numerical examples, which were included as illustrations, revealed that the amount of computation increases enormously with increase in N. The binomial approximation, which was presented, does minimize the work but sacrifices some of the accuracy.

Rubin and Sitgreaves [14.] , in a paper made available after the main part of the problem which is considered in this thesis was completed, showed some results which are related to the problem. Dealing only with single-valued functions (corresponding to the class G_1 in the thesis) they have found the distribution of the number of components by a completely different method. Thus, for single-valued functions their result overlaps the result for the class of functions, G_1, presented here. They also considered other topics dealing with size and composition of components. Questions concerning the size and composition of components formed under multiple-valued random mapping functions remain unanswered.

It is hoped that the results obtained here will be useful in applications. In social situations, divisions into groups are bound to occur. Whether these divisions follow essentially the theoretical distribution or whether they deviate significantly so that they must be accounted for on the basis of age, prejudice, etc. rather than on chance remains as part of the problem of the application of the results presented in this thesis.

REFERENCES

1. V. Gontcharoff, "Du domaine d'analyse combinatore," Bulletin de l' Academie des Sciences U.R.S.S. Serie Mathematique, Vol. 8, No. 1 (1944), pp. 45-48.

2. W. Feller, An Introduction to Probability Theory and Its Applications, John Wiley and Sons, 1950.

3. R. E. Greenwood, "The number of cycles associated with the elements of a permutation group". The American Mathematical Monthly, Vol. 60, No. 6 (1953) pp. 407-409.

4. N. Metropolis and S. Ulam, "A property of randomness of an arithmetical function", The American Mathematical Monthly, Vol. 60, No. 4 (1953) pp. 252-253.

5. M. D. Kruskal, "The expected number of components under a random mapping function", The American Mathematical Monthly, Vol. 61, No. 6 (1954) pp. 392-397.

6. E. Forsyth and L. Katz, "A matrix approach to the analysis of sociometric data: preliminary report", Sociometry, Vol. IX No. 4 (1946) pp. 340-347.

7. L. Katz, "On the matric analysis of sociometric data", Sociometry, Vol X, No. 3 (1947) pp. 233-240.

8. L. Katz and J. H. Powell, "The number of locally restricted directed graphs", Proceedings of the American Mathematical Society, Vol. 5, No. 4 (1954) pp. 621-626.

9. C. Jordan, Calculus of Finite Differences, Chelsea Publishing Co., 1947.

10. W. A. Whitworth, Choice and Chance, G. E. Stechert and Co., 1925.

11. L. Katz, "The distribution of the number of isolates in a group", Institute of Statistics, University of North Carolina, Mimeo. Series 36, (1950).

12. U. S. Department of Commerce, Tables of the Binomial Probability Distribution, National Bureau of Standards, AMS6, U. S. Gov't. Printing Office (1950).

13. L. Katz, "Probability of indecomposability of a random mapping function", The Annals of Mathematical Statistics (in press).

14. H. Rubin and R. Sitgreaves, "Probability distributions related to random transformations of a finite set", Applied Mathematics and Statistics Laboratory, Stanford University, Technical Report No. 19A (1954).