THE MATHEMATICS TEXTBOOK AS A STORY:
A NOVEL APPROACH TO THE INTERROGATION OF MATHEMATICS CURRICULUM

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ABSTRACT

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Both the purpose and overarching goal of this dissertation can be summarized with this quote by Buckminster Fuller: “You never change things by fighting the existing reality. To change something, build a new model that makes the existing model obsolete.” That is, to enable substantive positive change in mathematics education, this dissertation builds a new curricular model, tackling the question, “When mathematics textbooks are interpreted as art, what can be learned?” Although unconventional, this approach offers new conceptual tools for teachers and curriculum developers to make sense of the way in which mathematical ideas emerge and develop throughout a curricular sequence and to think anew about mathematics curriculum. Specifically, this work re-conceptualizes a mathematics textbook as a *mathematical story* with mathematical characters, action, setting, moral, and plot. Built from literary theory, especially the frameworks of Bal (2009) and Barthes (1974), the mathematical story framework supports a vision of mathematics curriculum as a complex narrative able to stimulate the imagination and curiosity of students and teachers alike. In particular, the notion of mathematical plot offers a new opportunity to articulate the sequential dynamics affecting a reader’s aesthetic experience, theorized as a tension between questions pursued by a reader and the revelations enabled by the text as the mathematical story unfolds.

Oscillating between the analysis of mathematics textbooks and literary frameworks, the mathematical story constructs were developed and tested. Once stable and consistent, the constructs of mathematical character, action, setting, moral, and plot were carefully defined with

examples from written curriculum. In addition, new characteristics of curriculum made visible with this re-conceptualization were explored and articulated through the analysis of multiple textbooks, focusing attention on what can be learned about the manifestation of mathematical characters and mathematical plots in textbooks. In part, these analyses reveal how a mathematical character, such as the number zero, is introduced and temporally evolves throughout a sequence of curriculum. This interpretation of mathematics textbooks also exposes how the development of a mathematical object involves not only the identification of the character but also the reader's identification with the character. A representation using Barthes' hermeneutic codes is also introduced to describe the mathematical plots of different mathematical stories, enabling the different experiences of reading these stories to be recognized and understood.

As mathematics curriculum broadly affects nearly every aspect of mathematics education (from planning to enacting to assessing), this mathematical story framework supports a renaissance of potential opportunities for mathematics teachers and students. It provides a heuristic for the analysis of math textbooks beyond any specific part (such as a task or a definition) in order to recognize the connective tissue of all the parts and the shape and effect of the whole for a reader. It offers teachers new, yet familiar, language for describing and collaborating on mathematics curriculum such as planned lessons or reflections on enacted lessons, further supporting their curricular design work. In addition, this work offers a conceptual foundation on which designers make important choices regarding the introduction and development of mathematical objects, procedures, and representations. In doing so, this work creates the potential to improve the mathematics curriculum offered to students.

To Laura, Jacob, and Alexander,
with whom my best stories are written

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I owe so many so much that this brief section only scratches the surface. My appreciation for my co-directors of this dissertation, Drs. Glenda Lappan and Nathalie Sinclair, cannot be overstated. Glenda's consistent enthusiasm and confidence in this project buoyed my spirits when challenges appeared insurmountable. Her ability to patiently listen to newly-emerging and unshaped ideas and ask important questions to help me focus my efforts and sharpen my analysis was invaluable. Similarly, Nathalie Sinclair's amazing contributions and support for my project were critically important. Her work in mathematical aesthetic is my constant inspiration and her advice on everything from writing strategies to connections with existing research throughout the project proved remarkably helpful. To call myself lucky to work with these scholars is an understatement.

I am also indebted to many at Michigan State University. In particular, I learned so much through working on the research projects of Dr. Sandra Crespo and Dr. Jack Smith (John P. Smith, III in print). Their important work with teachers and curriculum sets a high bar toward which I will continue to strive. In addition, given the nature of this dissertation, it is not surprising that I also received much support from scholars outside of mathematics education, specifically Dr. Laura Apol and Ann Lawrence. These literary scholars patiently nurtured, identified potential resources, and carefully advanced my project from the beginning. In addition to supporting the project, both regularly offered brilliant insights into my project that I would only come to understand months later.

I was also incredibly lucky to work alongside many graduate students at Michigan State University throughout this project. Particularly, my writing group (Aaron Brakoniecki, Kate

Johnson, and Mari Levin) offered valuable and timely critiques of early drafts and helped me develop confidence in defending my argument. My writing partner, Hanna Figueras, also helped me learn to create reasonable goals and meet them while others (Funda Gonulates, Lorraine Males, and Jill Newton) supported me through discussions about the role and nature of research of mathematics curriculum.

Beyond Michigan State University, there were many others who were instrumental to this project but are too numerous to name. For example, I was lucky to work with many amazing curriculum writers with the CPM Educational Program. These individuals supported my deepening understanding of mathematics curriculum with their incredible insights and challenging proposals. My work with my colleagues at Phillip and Sala Burton Academic High School in San Francisco also reminded me daily of the importance of curriculum *for students*. Last but not least, my high school math students throughout my time at Burton High School regularly gave me reasons to continually strive to be a better mathematics teacher and to develop more meaningful mathematical experiences. In short, this work is for them.

Most of all, I want to acknowledge the critical role of the love and support of my family throughout this process. My partner Laura provided a much-needed sounding board for the emotional and intellectual challenges I faced throughout this process. In addition, our sons, Jacob and Alex, continually reminded me of the intriguing and delightful nature of literature.

PREFACE

This dissertation represents a point in a long path of my development as a mathematics teacher and curriculum scholar. As I reflect on my goals and perspectives for this work, it seems that this aspiration to build and study a mathematical story framework was inevitable. In order to contextualize this work and to help explain why it came to be, I decided to relate a few details about my background coming to this work in this preface.

For as long as I can remember, I have been passionate about mathematics and grew up seeking out mathematical experiences outside of the classroom. This led to my participation of numerous math competitions, "field days", and math clubs over the course of my K-12 experience. I sought out mathematical puzzle books (my favorite section of the bookstore) and remember spending multiple evenings on the same challenging problem or puzzle. From these experiences, I became aware that a problem or puzzle can grab attention, raise larger questions, require me to renegotiate the way I understood a mathematical idea, and invite me to anthropomorphize mathematical objects (to imagine being the object of study). In a very real sense, the numbers and shapes I encountered were friends and enriched my daily experience as much as color or music could.

I continued my study of mathematics in college and decided that I wanted to share this passion with students. I began teaching mathematics right after college in 1989 and was lucky to find a job at an inner-city high school in San Francisco that enabled (and supported) me to engage with students on what could be called "mathematical adventures." These adventures were often sparked by a student's question or an unexpected result, and usually no one knew how the adventure would end until we jointly arrived at it. In class, I aimed not only to help students learn content, but also stimulate their (and my own) interest and passion in the possibilities that lie ahead. It was through working with these students that I recognized that mathematics can be understood in many different ways and learned that the students' experience was equally as important as the learning objective. Early on in my career, I remember working with fellow teachers on curricular questions that I now interpret as trying to recognize (read) the mathematical story of our textbook as well as develop new mathematical stories, such as "but where does this idea go?", "how could I make this more mathematically inspiring?", or "What if I changed the order of [the parts of text, such as lessons or even chapters]?" At this time, the curricular focus was not only on making learning stimulating and enjoyable, but making *mathematics* stimulating and enjoyable. Curriculum choices were made in relation to overall development, and student understanding was viewed in relation to where we were along the broader story.

My changing view and use of mathematics curriculum required more than "following a textbook," and soon I was critiquing the design of my lessons, asking questions such as, "What could help make the focus become important to the student, so that it's the student's question and not my own?" Selfishly, I relished designing experiences that would enable students to gasp in surprise at the mathematics and beg further exploration. Not that this would always happen, but the aim was always there. I wanted students to experience mathematics in a way that showed that it could be exhilarating, surprising, captivating, and full of wonder (much like those emotions described in Hofstadter's (1992) wonderful story of a mathematical adventure). This passion developed into a new career for me, one of writing and editing mathematics textbooks for middle and high school for a non-profit publisher. This valuable experience, which resulted in the creation of seven textbooks, allowed me to work with inspirational fellow teachers and re-imagine what curriculum could offer teachers and students. I began to notice how different parts of a story could affect what came before and what came after. I also became much less interested in replicating existing curricular sequences (assuming, for example, that algebra had to start with a review of arithmetic) and recognized that, though not unbounded, there is great flexibility in the ways mathematical ideas can be sequenced. It was through this work that my own explicit metaphorical connection of stories and curriculum was born.

Between 1995-2005, mathematics education and what it meant to teach mathematics underwent a dramatic change. The curricular questions teachers were asking were often different than those that had shaped most of my teaching work. In conversations with fellow math teachers, it seemed that as a result of newly mandated learning expectations (such as state standards and district curriculum guides), content outcomes were being positioned above (as opposed to along with) student experiences. Although discussions of teaching included pedagogical considerations (particularly with the influence of the NCTM standards), I became concerned that the content sequence, and particularly its potential aesthetic opportunities for students, was rarely part of the discussion. I wondered why many of the teachers I engaged with through professional development (for whom I have a lot of respect) rarely questioned their textbook. I received little reaction from teachers for what I viewed as gross changes to the common mathematical stories found in other texts of the same subject. There was often an ambivalence communicated regarding a choice of sequence, as though some thought, "as long as the content is there, is designed in a way that I prefer, and covers the material before the high-stakes assessment, who cares if chapter 1 starts with X instead of Y?" Even when I passionately tried to convince my fellow teachers that mathematical sequence is worthy of attention, I realized how difficult it was to talk about; there are few ways of conceptualizing and describing the nuances of mathematical changes throughout a lesson, let alone a unit or chapter, particularly aspects on which I had relied so much in my textbook design work.

Given this lack of conceptualization of mathematical progression, it is not surprising that as I started teaching "methods" courses to pre-service teachers, I similarly struggled to find ways to help them to deeply understand the way mathematical ideas in mathematics textbooks grow and build across a unit. Any attempt seemed only to scratch the surface, recognizing only sequence without its *consequence*. As a result, I became passionate about learning how curriculum works and looked for curricular theories that could help fill this void in attention. Although there was evidence that sequences were important, I could find little articulation or analysis of why or how curricular sequences matter. And in much of the analysis of textbooks, curricular elements (such as a definition or a problem) were considered individually as though this independent reading could offer information about the content of the text (consider, for example, work describing the cognitive demand of tasks, which often negotiates the challenge of a task without consideration of what led up to that task).

This dissertation represents my desire to address this challenge. My aim is to support the curricular negotiation and design work of teachers and to improve the mathematical stories with which students are daily confronted. What I bring is passion for and knowledge of mathematics, 17 years of teaching experience with youth who often felt the pursuit of mathematical understanding was not relevant to them, much experience with the fine-grained mathematical decisions of curriculum design, and no assumption of a single mathematical sequence (such as "this" must go before "that"). I aim to offer ways to promote curricular understanding of textbooks, inspire teachers to actively interrogate their textbooks, and to support teachers and other curriculum designers in their inventions of new powerful mathematical stories.

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KEY TO ABBREVIATIONS

References to textbooks are made as follows:

"MPAH" is used to refer to the *My Pals Are Here! Maths* textbook materials

"MPAH1" is used to refer specifically to the grade 2 *My Pals Are Here! Maths* textbook materials

"SFAW" is used to refer to the *Scott Foresman/Addison Wesley Mathematics* textbook materials

"SFAW2" is used to refer specifically to the grade 2 *Scott Foresman/Addison Wesley* textbook materials

"EM" is used to refer to the *Everyday Mathematics* textbook materials

"EM3" is used to refer specifically to the grade 3 *Everyday Mathematics* materials

"Jacobs" is used to refer to the *Mathematics: A Human Endeavor* text by Harold R. Jacobs (1970)

References to parts of textbooks are made as follows:

"WB" is used to refer to the student workbook

"SB" is used to refer to the student text materials

"TB" is used to refer to the teacher materials

Codes for Mathematical Plots in Chapter 5 are abbreviated as follows:

"TH" is used to refer to the thematization code

"PR" is used to refer to the proposal code

"F" is used to refer to the formulated question code

"PA" is used to refer to the partial answer code

"DS" is used to refer to the disclosure code

"EQ" is used to refer to the equivocation code

"JM" is used to refer to the jamming code

"PM" is used to refer to the promise of an answer code

> The concept of "narrative" applies almost directly to mathematics, in that mathematical works – just like many other works of verbal art – tell a story: they have characters, and our information about the characters gradually evolves. (Netz, 2005, p. 262)

CHAPTER 1

Introduction and Literature Review

Due to poor student performance on mathematical assessment and lack of interest in pursuing degrees in higher mathematics, textbooks are often critically analyzed to look for ways to improve their effectiveness. There is current widespread agreement that, in the United States, a vast majority of teachers base mathematics lessons on a set of published materials such as a textbook (Freeman & Porter, 1989; Huntley & Chval, 2010; Kilpatrick, Swafford, & Findell, 2001; Stigler & Hiebert, 1999). For this reason, a common strategy in the U.S. to attempt to change the instruction in the mathematics classroom is the revision of textbooks (Ball & Cohen, 1996; Remillard, 1999; Senk & Thompson, 2003; Willoughby, 2010). Most of these analyses focus attention on the content of texts with an analytic lens that is narrow in focus, such as looking at the cognitive demand of particular tasks or looking at the definitions of certain mathematical objects (e.g., Usiskin, Griffin, Witonsky, & Willmore, 2008). Other types of analyses that provide information about the content of math textbooks include the presence/absence and form of content (e.g., Smith III et al., 2008), the positioning of the students (e.g., Herbel-Eisenmann & Wagner, 2007), and the mathematical approach through which texts organize content, such as whether the concept of solving equations is developed through a functions approach or a balanced expressions approach (e.g., Nie, Cai, & Moyer, 2009).

These lines of research are important. If important mathematical content is missing, or if tasks are found to be unclear, inappropriately trivial, or too challenging, this limits their effectiveness. However, important questions are "How do texts do the work they are fundamentally designed for, that is, sequentially structure and develop mathematical content for a reader?" and "What ramifications in terms of coherence and aesthetic does the sequence of a textbook have for the reader?" With the analytic tools described above, I argue that it is difficult to make sense of how the mathematical content changes and flows throughout a textbook, including how the discrete parts affect each other and work together.

A re-conceptualization of mathematics curriculum in a way that foregrounds how mathematics temporally emerges and changes as one reads[1] sequential passages is the focus of this dissertation. That is, how might a teacher, curriculum developer, or researcher read a mathematics textbook in a way that enables him or her to recognize, critically analyze, and describe its mathematical development[2]? This question is not as simple as it seems. Since mathematically knowledgeable readers are typically confident with their own mathematical interpretations, the individuality of interpretations is sometimes hard to recognize. This makes the analysis of mathematics textbooks unexpectedly challenging, as reading a mathematics textbook for the questions asked above invariably requires interpretation. Thus, a single mathematical task in the middle of a textbook can have a surprising variety of meanings, even to a single reader.

To recognize how content sequence might affect the interpretation of a particular part of a mathematics textbook, it is helpful to consider an example of a textbook with a non-typical mathematical sequence. Although most elementary texts in the United States start the study of

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[1] The conceptualization of "reading" will be discussed in more detail in the theoretical framework in Chapter 2. At this point, however, it is important to note that it is assumed that reading is an active and interactive process as described in Rosenblatt (1988).

[2] Although the words "mathematical development" could potentially refer to multiple aspects of curriculum, such as how a math textbook changes over various publication dates, this dissertation will instead use this term to reference to the emergence and sequential changes of mathematical content as a textbook is read.

mathematics with number and operations, the *Measure Up* curriculum series (Dougherty, 2003), starts with the qualitative comparison of measures instead. Based on the work of Davydov and his colleagues, this focus then transitions into unitization of measures, which eventually leads to experiences that prompt students to develop the concept of number and counting. After this altered sequence, the authors explain that common tasks such as "is 3 < 8 ?" are read differently. Most adult readers might read the statement "3 < 8" as a comparison between two abstract values and determine that the question presents an accurate comparison. However, students in the *Measure Up* program interpret the question differently, as described by its authors:

> At this stage, if asked whether 3 < 8 is a true statement, these children will respond that you have to know what the unit is. As one first grader commented, "If you have three really big units and 8 really small ones, 3 could be greater than 8. But if you're working on a number line, then you know that 3 is less than 8 because all the units are the same." Another first grader noted, when asked to describe what 5 = 5 meant, "It's probably true unless you have a big 5 and a little 5. Like 5 big units and 5 small units, then it isn't true" (Dougherty, 2003, pp. 19–20).

This example reveals the possible effect mathematical sequence can have on mathematical meaning ("What is the meaning of "3"?"). The children's responses provide evidence that in any curriculum, a statement or question must be read in context of what comes before it. They also indicate that reading for the mathematical structure of a textbook goes beyond looking at a table of contents or reading declarations of content standards the authors claim to address. Therefore, how can a portion of text be read in the context of what comes before it and after it? For example, how can teachers read a textbook in order to recognize the potential contributions of curricular content in light of the future development ("What happens if I skip this lesson?" or "Should I introduce that idea at this point?")? Are there other potential sequences of developing mathematical content that can and should also be considered? And in what principled ways can a teacher or curriculum developer compare and contrast mathematical sequences, articulate and critically analyze their affordances and constraints, and identify their mathematical potential?

In addition to the lack of attention to the sequential development of meaning across a passage of a mathematics textbook, the way in which a sequence in mathematics curriculum potentially engages the student reader is often ignored. However, the aesthetic[3] of mathematical sequences is perhaps one of the most important aspects of curriculum to address. For example, lamenting the poor performance of student achievement in both math and English in the U.S., Csikszentmihalyi (1990) argued,

> The chief impediments to learning are not cognitive. It is not that students cannot learn; it is that they do not wish to. If educators invested a fraction of the energy they now spend trying to transmit information in trying to stimulate the students' enjoyment of learning, we could achieve much better results. (p. 115)

Although there have been serious attempts by a few curriculum groups to the contrary, the typical daily progression in most curricula comes across as the directive to the reader, such as "study this mathematical content because you need it to study other related mathematical content coming in a later lesson that you also have little interest in or realize is even coming." Twenty years later and after major investments in curricular reform, textbook authors usually rely on the use of worldly contexts (sending a message similar to, "you should be interested in this because you might own a business some day and need to find the maximum profit") instead of sequences of mathematical content that provoke imagination and curiosity. Typically, ways to make mathematics interesting has been limited to bringing things that interest children into the mathematical development, such as having students decorate symmetric designs or having students study fractions by studying beats on a drum.

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[3] Although the notion of aesthetic will be expanded upon in the theoretical framework, the term here refers to felt response by the reader through reading. Aesthetic refers to both positive and negative experiences.

Sinclair (2001) disagrees with this approach and argues that designing curricula so that aesthetic qualities are derived from domains other than mathematics, "endorses the belief that mathematics itself is an aesthetically sterile domain, or at last one whose potentialities are only realised through engagement with external domains of interest" (p. 25). Drawing from Dewey (1934), who approached aesthetic as an individual's response to an experience rather than an attribute of an object, Sinclair explains that mathematical aesthetic can be viewed as an individual's response to a mathematical experience, such as a sense of fit. With the aim to offer rich, aesthetic experiences for students without denigrating the value of mathematical activity, Sinclair (2001) asks, "Could we reverse the direction of the aesthetic flow, so that it originates in the mathematics?" (p. 25). Sinclair's question is at root a curricular question, as it calls for curricular experiences during which the content engages the child, beyond the medium or context.

Therefore, this dissertation is a first step toward addressing Sinclair's challenge to the field of mathematics education to develop ways to make the mathematical development in curriculum engaging for student readers. By attending to mathematics as a "verbal art," as Netz suggests is possible in his quote, we can learn how curriculum might inspire and transform the reader and how mathematical objects come to be. Reading a mathematical textbook as a narrative that accounts for both parts and whole allows new curricular qualities to become recognizable, such as anticipation, density, and coherence. It also enables the consideration of new questions about texts, such as how some passages of mathematics texts work to instill mathematical curiosity, provoke interest, or stimulate imagination and others not? Math educators could also learn how separate threads of development intertwine or how a particular mathematical idea, such as the meaning of "3," emerges and changes (or not) throughout a sequence. For example, how might deliberate choices of representations of mathematical objects, such as introducing negative numbers in the context of temperature or two-colored tiles (where one side represents +1 and the other represents –1), affect the notion of what negative numbers are, and how do these choices affect later portions of the curriculum? What messages might different sequences of textual forms (such as definitions, examples, problems, exposition, and questions) communicate about the nature of mathematics?

Thus, the broad goal of this dissertation is to complexify the very notion of mathematical development in written curriculum; that is, to develop and analyze strategies for reading math textbooks beyond specific parts (such as a task or a definition) in order to recognize the connective tissue of these parts and the shape of the whole. Specifically, it introduces and analyzes a new conceptualization of mathematics curriculum in order to help teachers, curriculum developers, and researchers come to recognize, characterize, and work with the mathematical development of textbooks. This conceptualization and the ways of reading textbooks that it may enable aims to integrate the curricular parts into a cohesive whole that can be evaluated (by the reader) for its merits, such as for mathematical coherence or aesthetic qualities. This dissertation broadens the understanding of the mathematical design of textbooks by pursuing these general questions:

- What is the nature of mathematical development in curriculum, that is, how the mathematics temporally unfolds in sequence and how the different parts interrelate and contribute to the overall whole?
- What heuristic is suggested by literary theory that might enable math educators (including teachers, teacher educators, curriculum designers) to characterize and interrogate the affordances and constraints of the mathematical development of textbooks?
- How might new ways of reading a textbook enable a reader to critique the sometimes ignored qualities of mathematical development, such as how a portion of curriculum invites a reader to raise questions, builds motivation for the reader to continue, and/or helps the reader to anticipate what will come next?

Studies of Mathematics Textbooks and their Conceptualization

The empirical and theoretical focus on mathematics curriculum, and specifically mathematics textbooks, has emerged in the last 100 years as mathematics educators have studied texts, often to learn what is being (or has been) taught in mathematics classrooms and how it is (or was) taught. In this regard, the mathematical and pedagogical designs of textbooks are used as proxies for the mathematics and pedagogy of enacted curriculum. In this literature review, three focal areas of literature that inform the understanding of mathematics content in textbooks will be discussed: the affordances and limitations of existing content analyses of mathematics textbooks, what has been learned and theorized about the forms and functions of mathematics texts (including textbooks), and existing conceptualizations[4] of mathematics curricula that have been generated through analyses of teacher planning and enacted lessons.

Content analyses of mathematics textbooks. Viewing mathematics textbooks as the mathematical resource for teachers and students, many studies survey and categorize the mathematical content of textbooks to look for explanations of weak student performance on assessments. For example, several of the TIMSS studies (Fuson, Stigler, & Bartsch, 1988; W. H. Schmidt, McKnight, Valverde, Houang, & Wiley, 1997) have compared the proportion of texts devoted to each topic found in U.S. textbooks to those of other countries and documented different grade levels in which topics are introduced and practiced. These studies have revealed that in comparison to slim, focused math textbooks of countries like China and Singapore, many of the texts in the United States follow a "laundry-list approach" (W. H. Schmidt, Wang, &

[4] Note that normally, discussion of theory is found in a theoretical framework. However, since this study is theoretical in nature (that is, the product is theory) as opposed to empirical, and because it concentrates attention on one particular conceptualization of mathematics curriculum, then it is appropriate for the background literature to describe and critique other conceptualizations of mathematics curriculum to help position this conceptualization in the field of mathematics education and curriculum research.

McKnight, 2005, p. 542) of many more topics per grade level. In addition, some mathematical topics, such as the study of decimal fractions, were often found in later grades in U.S. texts. Although these analyses are helpful to shape policy, such as making recommendations of how to change which topics are to be taught at which grades, they do not reveal the nature or quality of how these topics are introduced and developed in the textbooks.

More recently, the Strengthening Tomorrow's Education in Measurement (STEM) Project (e.g., Smith III et al., 2008) analyzed several elementary textbooks from the United States and Singapore in grades K through 4 to build a picture of the nature of the measurement content in textbooks. Using a complex conceptual framework distinguishing between conceptual, procedural, and conventional knowledge elements (the latter involving content which exists largely for communicative purposes, such as the knowledge of standard units), this work documents the largely procedural focus of measurement content in U.S. elementary textbooks. This analysis also contributes a way of qualitatively distinguishing textual forms of content, such as whether the content is found in questions or direct statements, and points to missing concepts which research indicates might be crucial to develop a solid understanding of measurement. However, this framework remains generally neutral on most questions of sequence (should one task be before or after another?) and design (is one set of tasks more coherent than another?). It also analyzes textual elements (problem or sentence of fact) independently from the rest of the development, stripping possibly important contextual clues that can affect the interpretation of the task, which the prompt "Is 3 < 8?" from the *Measure Up* series highlights. In fact, this approach largely treats excerpts of textbooks that are in the same textual form (such as a statement), written for the same audience (such as for the student), focused on the same content (such as comparing lengths of objects) as identical.

Historical analyses of content found in mathematics texts have also contributed understanding about the nature of content in textbooks. One of these, Sinclair (2008), describes the significant influence of Euclid's *Elements* on issues of curricular sequence, such as why very few geometry texts start with the study of three-dimensional space. Sinclair's analysis of textbooks also reveals how changing rationales for the study of geometry, such as utilitarian (i.e., students need to study geometry to be able to live and work) and academic (i.e., geometry helps students develop certain ways of thinking and reasoning), may have impacted content and pedagogic design of textbooks, such as whether or not students are prompted to solve problems or prove theorems. Historical studies, such as Baker et al. (2010), have also raised questions about how the challenging nature of the mathematics found in texts have increased over time, asserting that, "A case can be made that U.S. elementary school mathematics curricula of the late 20th century are both mathematically and cognitively more demanding [than before]" (p. 413). Other studies of more narrow scope, such as Usiskin's analysis of the definitions of quadrilaterals (2008), foreground the different mathematical choices made by authors at various points of time, such as whether or not a trapezoid is defined as a quadrilateral with *at least* one pair of parallel sides or *exactly* one pair of parallel sides.

Instead of focusing on the presence or absence of content, some analyses have instead focused on the different ways textbooks have dealt with topics (Cai, Watanabe, & Lo, 2002; Nie et al., 2009; Son & Senk, 2010). Representative of these, Cai, Lo, and Watanabe (2002) compared the notion of "average" found in some middle school textbooks of the United States with those of China. They discovered that although there are similar goals (i.e., developing a procedure, understanding why that procedure works) in both sets of the texts, the U.S. texts tend to emphasize average *as a representation of data*, while the Chinese texts instead tend to treat average *as a per-unit rate from an equal-sharing scenario*. This analysis highlights the fact that mathematical objects, such as an average, can have multiple natures and interpretations (indicated by the word "as"). These studies demonstrate how tasks that are essentially the same in two different mathematics textbooks may be used for different purposes. These ways of viewing average are each mathematically valid, but result in very different meanings of the mathematical object.

New Questions. Content analyses illuminate ways of understanding the mathematical content in textbooks such as attending to the placement of topics across and within grades, the textual forms of content, the pedagogic goals of the authors, the level of difficulty, and the framing and treatment of content. While a good start, they raise new questions. For example, although U.S. mathematics textbooks very well may contain too many topics at each grade level, as the TIMSS analyses suggest, what principled criteria can help the field decide which topics should go in each grade level? Additionally, STEM'S distinction of the textual form of content found in the textbook does not address how the textual parts of textbooks work together or shape each other. What might be learned when one reads for these connections? How might the interpretation of the content of a part of text (such as a question for the student) change when the textual parts that precede and follow it are taken into account? Finally, the work of Cai and others demonstrates that the mathematical development across a large swath of a textbook can build an image or frame of a mathematical object (such as average *as* a representation of a set of data) that may never itself be explicitly stated anywhere in the text. This reveals that a mathematical textbook can contain content-based messages that might only be recognized by reading a substantial portion of continuous textual passages. However, this analysis stops short of learning how the framing of a mathematical object might change over a textual sequence and how it might affect the interpretation of the parts. When reading a large section of continuous text, what other content-related messages can be read for? How might the way mathematics textbooks shapes meaning be conceptualized?

Textbooks as text. Whereas content analyses of textbooks place focus on the *mathematics* of math textbooks, other research instead places its focus on *text*, considering issues such as syntax, style, intention, rhetoric, and aesthetic of mathematics texts (broadly) and textbooks. This section will first discuss issues of style and argumentation of formal mathematics texts that offer insight into the reading of math textbooks more generally. Then, the focus turns to the form and function of mathematics textbooks in specific. This section will end by discussing research that involves the reading of mathematical textbooks.

Form and functions of math texts. Although this project focuses on advancing understanding of mathematical textbooks, textbooks are a subset of mathematical texts (broadly defined as texts that contain mathematical discussions). The differences between math textbooks and other math texts include audience, purpose, and style (Love & Pimm, 1997). Therefore, recognition of how formal mathematical documents work, such as proofs found in professional mathematical journals, does not alone inform a reading of mathematical textbooks. However, attention to their similarities (e.g., both use abstract symbols) and differences (e.g., their structure) can be instructive.

The writers of formal mathematical texts generally aim to persuade the reader of its credibility and value (Csiszar, 2004). One way to achieve this goal is to hide the origins of the mathematical discoveries, possibly to eliminate any evidence of struggle by the mathematician. Proofs are often "turned around" in final form and generally structured in a *definitions-theorem-proof* order, even though, as persuasively argued by Lakatos (1976), this is opposite to the experience of developing mathematics and arriving at the theorem and definitions presented in the proof. In fact, Leron (1985) argues that the final form works to obscure many of the essential ideas required for understanding the proof and requires the student to decode "the ideas and connections buried beneath the formal code – *the very ideas I suppressed* in coding the proof" (p. 7). In fact, Leron argues, to address the needs of learners, it needs to be supplemented, otherwise the student is "reduced to a step-by-step "execution" of the proof" (p. 9).

Some mathematicians defend this practice as reducing the burden on the reader and as the only way to advance knowledge. Consider Paul Halmos' argument (2003, quoted in Csiszar, 2004, italics added):

> If there were no way to trim, to consolidate, and to rearrange the discovery, *every student would have to recapitulate it*, there would be no advantage to be gained from standing "on the shoulders of giants", and there would never be time to learn something new that the previous generation did not know. (p. 248)

Although the purpose of formal mathematics texts is to convince the reader of its credibility, it may be surprising to learn that even professionally written mathematical proofs found in journals do not spell out every detail. Csiszar (2004) explains that, in reality, these texts are generally written as broad overviews of an argument that require and expect an expert reader to "fill in the gaps" (p. 246). In this way, texts establish and reinforce the author's position within the mathematical community. However, Halmos assumes that "to learn something new," the reader does not need to "recapitulate" the discovery, to construct it for him or herself. Since this contradicts generally accepted theories of how an individual learns, these texts require the reader to have the mathematical competency of the mathematician writer, which hardly treats the reader as a student.

The symbols found in mathematics texts have also received attention. Pimm (1987) notes that since mathematical objects are mental constructs, they must be represented by symbols, and that multiple symbols might refer to the same object (such as $\frac{3}{2}$ and 1.5). Pimm distinguishes four categories of symbols used in mathematics texts to represent mathematical objects (logograms, pictograms, punctuation symbols, and alphabetic symbols) and exposes some of the conventions behind their use. For example, he notes that even though the selection of symbols for variables as in $a+b=cd$ might seem arbitrary, it is probably read with more ease by a mathematician than the statement $! + * = \#\&$. The reader needs to be aware of conventions for naming quantities with alphabetic letters, which dictate that *cd* represents the product of two quantities *c* and *d*. However, because it is unconventional, the use of "#&" in an equation might make its meaning ambiguous to the reader as it could represent a single symbol or the product of two quantities.

In addition to a distinct rhetoric and use of symbols, the analysis of mathematical texts exposes platonic assumptions about the nature of mathematics, especially the view that mathematical objects are fixed external entities (Ernest, 2008a). For example, Pimm (1987) argues that a reader needs to distinguish between object and symbol, and that "... the very real and frequently realized danger is that the symbols themselves, rather than the ideas and processes which they represent, will be taken as the objects of mathematics, *the reality to which the language and notation is pointing and referring*" (p. 159, italics added). Also, through a discourse analysis of mathematical definitions found in both formal mathematics texts and textbooks, Morgan (2005) distinguished between mathematical object and its definition, and explains, "A definition is not identical with the object but is a way of looking at *an independently existing object*" (p. 109, italics added). Although the notion of a "way of looking" helps to make sense of the student responses from the *Measure Up* Program mentioned earlier, this statement assumes that there exists ontologically a single referent for any definition and that when definitions are claimed to be equivalent, they necessarily describe the same object to all readers. However, when different readers read the same mathematical definition, do they necessarily construct the same mental image? Since Reader Response Theory suggests that texts do not mean the same things to different readers due to differences of experience, prior knowledge, and context (Rosenblatt, 1988), it stands to reason that definitions are interpreted in multiple ways.

Form and functions of mathematics textbooks. In contrast to formal mathematical texts, mathematics textbooks are written for multiple audiences (e.g., teachers and students), at least some of whom are in different communities than the authors[5]. This positions the author of the textbook with authority and results in a text that is not necessarily required to develop credibility. This may explain why authors of mathematics textbooks use authoritative voice (see, for example, Herbel-Eisenmann, 2007) and often do not offer any proof of its assertions.

In their analysis of the form and function of mathematics textbooks, Love and Pimm (1996) not only distinguish mathematics textbooks from other math texts (disciplinary texts written by mathematicians for other mathematicians), but also from other forms of pedagogic texts. For example, mathematics textbooks explicitly call upon the reader to respond to questions and problems posed in the text, even though other forms of educative texts (such as gardening manuals) do not often include these features. In addition, a textbook is designed assuming that it will be read and responded to in a linear sequence; that is, part (b) will not only be read after part (a) is read, but it will follow after part (a) is read *and addressed*, such as developing some understanding or producing a response. This is unlike a cookbook, which is not typically designed assuming that before a cook uses one recipe he or she has read and produced all previous recipes in the collection.

[5] Although this is certainly the exception rather than the rule, at least some mathematics textbooks are written by teachers. However, it is assumed that students do not write textbooks.

Recognizing that word problems in textbooks are nearly universal in form and style, Gerofsky (1996, 1999) argues that they constitute a literary genre. Noting that word problems have no "real" referents, the illocutionary force of a word problem can be interpreted as,

> "I am to ignore component 1 and any story elements of this problem, use the math we have just learned to transform components 2 and 3 into the correct arithmetic or algebraic form, solve the problem to find the one correct answer, and then check that answer with the correct answer in the back of the book or turn it in for correction by the teacher, who knows the translation and the answer" (1996, p. 39).

Gerofsky also proposes that the artificial nature of word problems enables an altered reading of a word problem:

> Every year (but it has never happened), Stella (there is no Stella) rents a craft table at a local fun fair (which does not exist). She has a deal for anyone who buys more than one sweater (we know this to be false). She reduces the price of each additional sweater (and there are no sweaters) by 10% of the price of the previous sweater that the person bought (and there are no people, or sweaters, or prices)... (1996, p. 41).

Therefore, at least for word problems, the ontological referents of mathematical textbooks have been challenged; rather than referring to real objects and context, story problems call into being fictitious worlds (Love & Pimm, 1997). This critique is not an attribution of failure, but instead challenges a common assumption that by containing word problems, textbook authors are inviting students to engage in mathematical activity of the world. This ontological quality of textbooks leads Love and Pimm to wonder, "In what ways is this relationship similar to the presumed 'autonomous fictional world' that a novel creates?" (1997, p. 381).

There is also evidence that word problems can provoke the imagination, which might engage the reader in unexpected ways. Pimm (1987) offers an example by author David Roth, who described becoming engaged with the details unmentioned in word problems. For example, after being challenged with a word problem involving a discount on a coat, Roth professed wondering about open questions, such as, "To whom had the haberdasher finally sold the overcoat?" (p. 14). This demonstrates that mathematics textbooks have perlocutionary effect on the reader, and that, though these vary by reader, they can include the raising of questions as well as the stimulation of imagination.

What types of questions might mathematics texts provoke? Besides word problems, how might other parts of the textbook stimulate the imagination? Pursuing questions similar to these, Sinclair (2005) offers the notion of reading mathematics texts as drama, using the imagination to provide the effect of a Greek chorus. To illustrate, Sinclair shares her narrative reading of a proof of the irrationality of $\sqrt{2}$ (see Figure 1.1), offering interpretations such as:

> Well, if $\sqrt{2}$ is rational, I should be able to write is (sic) in a fractional form, as p/q. But if I can write it as p/q, I could also write it as $100p/100q$, and a million other ways, so let me honour my characters by introducing them in their barest form, in reduced form. (narrative reading, para. 1)
>
> I can't get a sense of what that q^2 has to do with anything, so let's write $p^2 = 2q^2$. Aha! Now I'm getting somewhere. This looks nice: nothing is hidden in root signs or in denominators, and p has emerged as the leading actor on the left—the struggle, of p^2/q^2, now has a hero. (narrative reading, para. 2)

Suppose $\sqrt{2}$ is not irrational
Then $\exists\, p, q \in \mathrm{N}$ such that $p/q = \sqrt{2}$ $(p, q) = 1$
So $p^2 = 2q^2$,
Then $2 \mid p^2$,
And $2 \mid p$.
Therefore, $p = 2r, r \in \mathrm{N}$
So $2q^2 = (2r)^2 = 4r^2$ and $q^2 = 2r^2$
Then $2 \mid q^2$,
And $2 \mid q$.
Contradiction, since p and q were supposed to be relatively prime,
$\therefore$ There does not exist p and $q \in \mathrm{N}, (p, q) = 1$ such that $p/q = \sqrt{2}$,
$\therefore \sqrt{2}$ is an irrational number.

Figure 1.1. Sinclair's (2005) proof of the irrationality of $\sqrt{2}$.

Sinclair's interpretation is consistent with Bruner's (1996) notion of narrative thinking, which suggests that individuals construct personal narratives when trying to make sense of an event or idea. In this sense, reading a proof is construed as reading for interactions between parts, looking for what connects each statement together in a coherent string. The reading is drawn to certain objects with certain properties that are revealed over time. Questions arise that are not immediately answered, but all are posed to make sense of the parts in relation to the whole; that is, what each statement tells us in relation to prior statements and how they relate together to address the larger question (is $\sqrt{2}$ irrational or not?).

The reading of mathematical textbooks. Authors of textbooks generally know very little about the individual students who will eventually read their textbooks and are "condemned to plan for faceless people, students shorn of their uniqueness or for all teachers, who become generalized entities often defined in terms of generalized performance roles" (Aoki, 2005, p. 203). Yet the reader of a textbook brings important qualities that affect its interpretation, such as prior knowledge. To distinguish between "the reader" authors write for and the actual reader of a textbook, Weinberg and Wiesner (2010) draw from reader-oriented theory and introduce a framework of three types of reader: the *intended reader* ("the author's image of the reader"), the *implied reader (* "the qualities of the reader needed to correctly interpret the text"), and the e*mpirical reader* ("the qualities of the reader of the text") (p. 3).

According to this theory, the authors of textbooks hold assumptions about the readers of their textbooks, which may or may not describe the actual readers (i.e. the empirical readers). As the textbooks are written, narratives are generated that require certain competencies to interpret that may or may not be part of the repertoire of an empirical reader. Therefore, the text implies a reader, referred to as the implied reader[6].

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[6] Note that according to Weinberg and Wiesner (2010), the definition of an implied reader assumes there exists a "correct interpretation" of any text, a positivist stance that either ignores the multiplicity of meanings possible or privileges one interpretation over others. Yet, this is not

The relationship between the reader and the text is, therefore, a complex one. Studying a text alone reveals nothing about its reader and vice versa. However, despite this independence, it is assumed that changes in the text potentially alter the reading of the text and that changes in the ways of reading the text changes the interpretation of the text. One way to understand this linkage is to consider the qualities of mathematical curriculum that seem linked to student (i.e., reader) responses. For example, although observing how students worked together to make sense of a task including a representation in a dynamic technology environment (called "Microworlds"), Healy and Sinclair (2007) discuss the role of the task design. They observed that when tasks had an unexpected quality, the students produced story-like narratives:

> The microworld activities we have described occurred in the context of carefully constructed tasks, designed with particular learning objectives in mind. Can we identify specific ways in which the design of the tasks and tools impacts upon the emergence of particular story lines and particular mathematical meanings? Reflecting back on our examples, we notice that the tasks are far from neutral in the stories we have presented.... In retrospect, we can see that all the tasks reported in this paper have in common the presentation of something unusual—a step function appearing suddenly in the midst of continuous ones, a turtle that cannot be communicated with, a stick-person who moves mathematically. Built into the tasks, then, was at least one incentive for story-making—*something exceptional that needs explaining*. The computational objects became the characters who, because of their dynamic capabilities, were able to act out the extraordinary event and the storyline provided mathematically orientated interpretations for it. (p. 19)

Unfortunately, most readers (students, particularly) do not encounter mathematics textbooks that often stimulate an explanation of something exceptional. Healy and Sinclair's examination of the narratives that students produce, and thus their "reading" of the task, suggests

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an argument that math textbooks do not require competencies of their readers. The earlier examples of mathematical sentences like $a + b = cd$ and $! + * = \#\&$ offer insight into a readers' need to recognize and understand both mathematical symbols and the conventions of their use. References to "correct interpretations" are likely references to the author's interpretations, which cannot be drawn from the text. Therefore, I propose that a way the implied reader could be defined to eliminate this positivist stance is, "the qualities of the reader needed for being able to obtain *an* interpretation of the text."

a way of confronting an age-old problem in mathematics curriculum; that is, how can we better stimulate imagination and produce interest in mathematical learners?

Conceptualizations of mathematics curriculum. Since teachers regularly make curricular decisions, some mathematics education researchers have used analyses of how teachers make sense of textbooks (and more broadly curriculum) as a basis of conceptualizing the mathematical development of curriculum. In her observation of teachers' interactions with unfamiliar (reform) textbooks, Remillard (1999) connects the potential for changing teacher practice and teacher beliefs with the way they read the textbook. She argues that since reading is interpretive work aimed at sense making, it is through reading that teachers can potentially develop new ways of understanding teaching. Thus, she argues, in order to change enacted curriculum and challenge teacher beliefs, we as teacher educators need to not only change what teachers read but also need to develop new ways for teachers to read. Ben-Peretz (1990) makes this connection explicit, stating that, "the issue becomes one of the interpretative skills needed for a 'reading' of curriculum materials which goes beyond their obvious and explicit meaning. Interpretative skills can be learned and cultivated, leading to an expansion of the repertoire of learning opportunities which teachers offer to their students" (p. 9).

However, what ways of reading to understand the mathematical development of textbooks are currently available for teachers? Clearly one of the central curricular conceptualizations available is the notion of *scope and sequence*. This notion, which often describes not only the content of a course but also the order in which that content is sequenced, is the foundational conceptualization operating behind course syllabi and standards documents. The metaphor of "scope" attends to the range or extent of a course, directing a teacher's focus toward global and large-scale curricular structures (such as "The unit on properties of shapes follows the unit on multiplication" or "The objective of this chapter is for students to gain fluency with adding two-digit integers") instead of fine-grain or daily curricular questions (such as "How is the notion of place value introduced and supported?" or "How does this lesson build from the previous lesson?"). In other words, scope and sequence is a listing of ordered content, much like a table of contents of a textbook, containing little or no attention to connections among topics or ways one topic might affect another.

Perhaps because of these limitations of the notion of scope and sequence, Ball and Cohen (1996) called for educative texts that can in part help the teacher understand how content develops throughout text. They complained that:

> Teachers' guides rarely help teachers to think about the temporal dimensions of curriculum construction. Teachers' guides could, for instance, contribute to teachers' thinking about content and activities appropriate in September as they begin to construct the classroom culture and environment. Teachers' guides could also help teachers to consider ways to relate units during the year... curriculum authors could discuss alternative representations of the ideas and connections among them (p. 7).

Beyond the basic "scope and sequence" construct, three additional conceptualizations are found in mathematics education literature: *curriculum map, learning trajectory,* and *story*. These conceptualizations are metaphorical, mapping relationships and characteristics from other domains outside of education to the notion of mathematics curriculum. In a discussion of the affordance and limitations of metaphors, Sfard (1998) warns that metaphors foreground certain characteristics while shielding from view others. Since metaphors shape the way we think, new metaphors can open up new possibilities and advantages, but can also bring new assumptions. Thus, the choice of metaphor for the mathematical development of curriculum can both limit and enhance the work of teachers. As Sfard explains,

> Because metaphors bring with them certain well-defined expectations as to the possible features of target concepts, the choice of a metaphor is a highly consequential decision.

> Different metaphors may lead to different ways of thinking and to different activities. We may say, therefore, that we live by the metaphors we use. (p. 5)

Since this current project is focused on conceptualizing mathematics curriculum in a way that makes certain aspects (relationships between parts, shape of the whole) salient, each of these metaphorical constructs will be discussed at length and analyzed for their potential benefits and limitations.

Understanding mathematics curriculum as a curriculum map. After observing teachers plan and enact lessons, Remillard (1999) theorized that part of the teacher's curricular work is selecting and ordering of content. To describe the resulting organization of curriculum produced by this work, Remillard used the term *curriculum map*. For the teachers in her study, the construct of curriculum map appears to describe the work of determining the scope and sequence. Remillard even explained that this aspect of curricular work (which she calls an "arena") does not attend to the daily curricular decisions of task selection or development of teachers (which is encompassed in the design and construction arenas, respectively), but rather informs them. Remillard notes that textbooks "offer a curriculum map" that teachers may elect to follow, but also recognizes that teachers may instead develop their own curriculum map.

The metaphor of a "map" for curriculum is not new, and might offer math educators a conceptualization of curriculum that goes beyond scope and sequence. Dewey (1902) used the metaphor of a curricular map to distinguish between the "logical" organization of content as opposed to the everyday "psychological" experiences of the child:

> We may compare the difference between the logical and the psychological to the difference between the notes which an explorer makes in a new country, blazing a trail and finding his way along as best he may, and *the finished map* that is constructed after the country has been thoroughly explored. The two are mutually dependent... the map orders individual experiences, connecting them with one another irrespective of the local and temporal circumstances and accidents of their original discovery. (p. 241, italics added)

Dewey's map metaphor foregrounds the ordering of different concepts and procedures which make up a course, just as a map represents the relative position of different parts of space. When dealing with existing textbooks, this conceptualization might digress into a re-telling of the order of generic content outcomes ("this, and then this"), like scope and sequence, but might also include how parts are related to each other (how the content in chapter 2 might be revisited in chapter 5, for example). In addition, just as a topological map might help one recognize that separate mountains collectively form a mountain range, a teacher forming a curricular map might pay attention to relationships among (or juxtaposition of) different foci of a course. In the end, however, a curriculum map consists of content topics (such as studying graphs of linear equations), and does not reveal important characteristics or qualities of the design that provoke the emergence of meaning for the reader. It may locate content along a sequence, but it gives little information about how it emerged for the reader or its potential impact on the reader. For example, although "the comparison of numbers" might be listed on a curriculum map for the "Is 3<8?" prompt mentioned earlier, this metaphor offers little in understanding the context of this question (i.e., how prior work with measurement impacts the concept of numbers).

Understanding mathematics curriculum as a learning trajectory. A related metaphor that has been offered to help math teachers conceptualize the notion of change throughout curriculum is that of the *learning trajectory* (Breyfogle, Roth McDuffie, & Wohlhuter, 2010). As opposed to the curriculum map metaphor, a learning trajectory focuses on the directedness of intended curriculum, foregrounding the nature of curriculum moments as making progress toward a future goal. As opposed to the map and scope and sequence conceptualizations, which place the focus on the span of content, the learning trajectory focuses on the "instantaneous" decisions of curricular direction (e.g. deciding what to teach next) at various points along the span. This conceptualization is related to that of *hypothetical learning trajectory* (Clements & Sarama, 2004; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Simon & Tzur, 2004), a student-oriented construct which generally describes increasing levels of sophistication students demonstrate as they learn a concept during clinical interviews and teaching experiments. Clements and Sarama (2004) specifically connect the sequence of curricular tasks to this learning trajectory:

> We design tasks that include external objects and actions that mirror the hypothesized mathematical activity of students as closely as possible... These tasks are, of course, sequenced corresponding to the order of the developmental progressions to complete the hypothesized learning trajectory. The main theoretical claim is that such tasks will constitute a particularly efficacious educational program (p. 84).

Although those who write about learning trajectories explain that there is not one but many trajectories through the "conceptual corridor" (Confrey et al., 2009), the metaphor of the vector represents a single path. The metaphor of "trajectory" focuses on direction, much like a vector. However, just as a vector has magnitude, which is not contained in descriptions of its trajectory, the learning trajectory does not necessarily attend to other qualities of unfolding curriculum, such as aesthetic or relational aspects. Said differently, trajectory focuses on the direction, but gives little information about the shape of the whole. A learning trajectory might inform a curriculum designer regarding what content to teach in what order, yet it does inform decisions about how to approach the topic (such as how using a particular task might develop anticipation).

Understanding mathematics curriculum as a story. Although looking for factors that might explain why students in countries such as Japan or Germany routinely outperform those in the U.S. on international mathematical assessments, some studies have analyzed and compared enacted mathematics lessons from classrooms in Japan and Germany with those of the United

States. One of the explanatory factors they explored was the degree to which enacted lessons were mathematically coherent or not (Fernandez, Yoshida, & Stigler, 1992; Stigler & Hiebert, 1999). These studies argue that conceptualizing an enacted lesson as a *story* is a way of identifying connections and understanding how sequential parts relate and build a coherent whole. Stigler and Hiebert explain,

> Imagine the lesson as a story. Well-formed stories consist of a sequence of events that fit together to reach the final conclusion. Ill-formed stories are scattered sets of events that don't seem to connect. As readers know, well-formed stories are easier to comprehend than ill-formed stories. And well-formed stories are like coherent lessons. They offer the students greater opportunities to make sense of what is going on. (p. 61)

The connection of mathematical development to story is not isolated to discussions of lesson coherence, however. As the quote by Netz at the beginning of this chapter indicates, the connection between story (narrative) and mathematics texts has been introduced and considered by mathematicians as well. Indeed, when Lakatos (1976), a philosopher of mathematics, sought to describe the nature of mathematical development, he wrote a play (a particular genre of story). Also, the mathematician Paul Lockhart, in his lament against the current designs of mathematics curriculum, used the metaphor of story to help explain the importance of understanding the context when considering any mathematical property or relationship:

> Mathematical structures, useful or not, are invented and developed within a problem context, and derive their meaning from that context. Sometimes we want one plus one to equal zero (as in so-called 'mod 2' arithmetic) and on the surface of a sphere the angles of a triangle add up to more than 180 degrees. There are no "facts" per se; everything is relative and relational. *It is the story that matters, not just the ending*. (Lockhart, 2009, p. 17, italics added)

Thus, in the context of the mathematical development found in the *Measure Up* program, described earlier, students possibly read "3" and "8" as the results of measuring, interpreting "3" as "3 of some kind of unit." This reading is not "factually wrong;" instead, when read in the context that numbers represent abstract measures, this reading makes sense. Lockhart's metaphor of story offers a way to conceptualize the temporal effects of mathematical development rather than the culminating mathematical statements or "facts."

Like the metaphor of curriculum mapping, viewing curriculum as a story draws attention to the connections among the parts. However, the conceptualization of story also attends to the effect of sequence (such as how would the story change if the murderer were revealed in Chapter 2 instead of Chapter 10?), highlights temporal changes (e.g., Romeo and Juliet meet), and allows one to recognize interdependence (e.g., if Romeo and Juliet don't meet, then they don't fall in love and there is no story). Beyond attention to the directedness toward a target at various points along a path (like the learning trajectory metaphor), the conceptualization of curriculum as a story draws attention to both how mathematical ideas are introduced and changed locally throughout the development, as well as the resulting global curricular form. And just as a story might be viewed by a reader as exciting or boring, the conceptualization of mathematical development as a story can also provide a means for recognizing its aesthetic.

Egan (1988) argues that mathematics curriculum can be organized as stories and claims that this form can help students make meaning of the content. He offers an example of a mathematical story lesson, where students are told a "historical" story (a King's counselor needs to organize marbles in bowls to count the size of an army) and new content (place value) is required to solve a human problem. In this way, Egan proposes to base mathematics content in a fictional story problem, much like those studied by Gerofsky (1996, 1999). However, Egan goes beyond the traditional argument that these motivate because they connect students to the "real world" or how the world *is*. Instead, he argues that by drawing from the historical origins of the mathematical content (what problems the content was developed to solve), mathematical stories may engage students with imagining how the world once *might have been*.

However, Egan's notion of a story still assumes that the characters of the story are humans. What would it mean for a mathematical story that is not about fictional humans, but instead about triangles or systems of equations? How could the content of mathematics textbooks be viewed as a story? Sinclair's (2005) narrative reading of a proof (shown earlier) raises and explores this question and demonstrates a way a story can be read from mathematical text. Despite this example, Sinclair expresses some doubts and raises important questions:

> Now I've substituted the word "text" for "story," a move I am not quite comfortable making, since it is not at all clear that all texts tell stories. In fact, story-telling depends on the text as well as the audience, the reader/writer, as well as the teller (is a story still a story when it is separated from its telling?). I am also not comfortable about substituting text with story since I find it challenging, in mathematics, to identify just what a story is. Must it have a beginning, middle and end? Must it have setting and plot and characters? What kinds of genres are there? (p. 4).

Sinclair identifies many of the challenges raised by the prospect of reading mathematical stories from textbooks. How can mathematical stories be framed in a way to make the metaphorical relationship recognizable and beneficial, and thus potentially tangible and useful for teachers, math educators, and curriculum designers? Unfortunately, before now, this correspondence has not been carefully pursued. If mathematical stories can be fully identified and analyzed, what might tools from literary theory reveal about mathematical development?

Sinclair (2005) takes the metaphor of curriculum as story further, using it to consider how a mathematics textbook might stimulate imagination. Although Sinclair admits that her own association of mathematics text and story is an uncomfortable one, she draws from Barthes to point out that "our experiences of text depend not only on what we read, but on *how we read*" (p. 3, emphasis in original). Thus, she suggests that mathematics texts can produce stories when read in certain ways. Drawing from Barthes' (1974) notion of *writerly texts* (those that invite multiple interpretations and stimulate inquiry and place the reader in the position of writer or producer of meaning) and *readerly texts* (those that are closed to interpretation and do not stimulate the imagination), Sinclair asks the question, "How do we go about creating materials for students that exemplify this kind of narrative, or "writerly" possibility in mathematics?" (p. 9). After comparing math textbooks to those Barthes' describes as readerly, Sinclair challenges our field to imagine math textbooks (or math curriculum broadly) that stimulate imagination, invite a multiplicity of interpretations, and are generative in nature; in short, those that help students/readers produce good stories.

However, asking if math educators can work to produce better mathematical stories assumes that we have a way to recognize and describe these stories. This dissertation builds on the work of Sinclair and others to develop and analyze a conceptualization that enables a reader to recognize, describe, and analyze mathematical development in curriculum broadly through the focus on the mathematical stories found in textbooks. Specifically, it develops and articulates a conceptualization of *mathematical stories* as they play out in textbooks, introduces and demonstrates a heuristic for reading mathematical stories, and analyzes the potential affordances and constraints of this conceptualization when compared to others, such as scope and sequence, curriculum mapping, and curriculum trajectory. The product of this dissertation is a theoretical framework, with illustrations of its use and an analysis of its potential benefits.

Summary of Upcoming Chapters

The next chapter will introduce the theoretical underpinnings of this dissertation, including definitions and assumptions about mathematics texts, narrative, and what it means to read mathematics textbooks. Moreover, this theoretical framework summarizes the literary framework (largely Bal's (2009) narratology framework) that is later used to conceptualize and analyze mathematical stories. In addition, Chapter 2 specifies the contextual details of this project, establishing the questions pursued and the strategies of inquiry applied to develop and test a theoretical framework. These methodological details include an explanation of how texts were selected and a description of the careful process of theory generation.

Chapter 3 draws from Bal's framework to introduce and illustrate the mathematical story framework, an outcome of the theoretical analysis of this dissertation. Using an example from a mathematics textbook, new conceptual tools such as mathematical character, mathematical plot, and mathematical event are defined and discussed.

With the mathematical story framework in place, Chapters 4 and 5 represent an argument that when mathematics texts are read as mathematical stories, new properties of curriculum and new ways to approach content can be recognized and understood. Specifically, in Chapter 4, the manifestation of mathematical objects within written curricula as mathematical characters is analyzed. This analysis attends to both how texts work to help readers identify and build images for mathematical characters, as well as how the text can enable a reader to *identify with* a mathematical character. Then, in Chapter 5, the analytic focus turns to the mathematical plots found in mathematics textbooks. This work introduces an analytic framework from Barthes (1974) to demonstrate a way of recognizing and representing the narrative forces throughout a mathematical sequence.

Finally, Chapter 6 summarizes the dissertation and describes the implications of the mathematical story framework. It connects the framework with other work in mathematics education and poses questions for future inquiry.

"We don't see things as they are, we see things as we are." – Anaïs Nin

CHAPTER 2

Framing the Project

This study draws from a field outside of mathematics education to look anew at the content in mathematics textbooks. In general, it represents an introduction and proof of concept of a metaphorical conceptualization of mathematics curriculum. More specifically, literary theory is used to theorize and test a mathematical story framework to help explain how different story elements and sequences of these elements can offer new understanding of mathematical content in textbooks. As a study of texts and their interpretation, this exploration can be described as a humanities endeavor in which the mathematical textbook is viewed as art, with both structural and aesthetic dimensions. The overall goal of the dissertation is to produce substantive and useful knowledge about mathematics curriculum, particularly new qualities that become visible through this re-conceptualization.

The purpose of this chapter is two-fold: to set out the theoretical assumptions of this inquiry and to describe the strategies[7] of inquiry. The first section of this chapter specifies the underlying theory upon which the metaphorical conceptualization of mathematical stories is supported (in Chapter 3). That is, it frames the work by defining key terms, introducing central tenets, and acknowledging assumptions taken throughout this work. The second section then explains the strategies of inquiry in order to situate and contextualize the products of this dissertation (presented in Chapters 3, 4, and 5). This section describes the scope of this project,

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[7] I recognize that the word "methods" is more commonly used to refer to what I am calling "strategies of inquiry." However, due to the unconventional nature of this project and the role of methods in this analysis, I chose to use the word strategies to acknowledge this difference and to help the reader keep assumptions about the nature of research in mind (whether it be on empirical or theoretical grounds).

addressing what this project is and is not (and cannot be). Moreover, it articulates a process of theory-testing that may be new to many researchers in mathematics education. As humanities research, the claims made in later chapters are not rigorously "proven," in the sense often used in mathematics or social science research; instead, they achieve validity through the strength of argument and the thoroughness of analysis. Thus, this chapter also describes in detail the strategies of inquiry used both for building the mathematical story theoretical framework and for analyzing its affordances, establishing the basis and limitations of claims.

A few comments about the placement of the strategies of inquiry in this chapter may be helpful. Although it may strike a reader as strange to have a theoretical framework found in Chapter 3 presented *after* the methods and strategies of inquiry are detailed in this present chapter, the theoretical nature of this project necessitates this order. The strategies of inquiry for this project are needed to support the production of theory; that is, the mathematical story framework presented in Chapter 3 is an outcome of a process of theoretical analysis detailed in this chapter. Therefore, this dissertation actually has two theoretical frameworks: one in the first section of this current chapter to support the theoretical process of analysis, including assumptions around narrative, text, and reading, and a second in Chapter 3 introducing the mathematical story framework as an outcome of theoretical analysis. The mathematical story framework, in turn, is the underlying framework for the exploration of its usefulness, found in Chapters 4 and 5.

Theoretical Underpinnings

This section starts with a discussion of the nature and assumptions regarding the materials under study: mathematics textbooks. This describes the wide-variety of forms of textbooks and, more importantly, articulates a distinction between what a textbook means and the intended meaning of its author. Following this, the literary lens used to re-conceptualize mathematics textbooks will be introduced and described. Finally, with this literary framework comes an assumption that there is a reader; therefore, the final discussion of this section sets out and complicates assumptions about what it means to read a textbook.

The framing of mathematics textbooks. Although a conceptualization of mathematical story has potential to inform mathematics curriculum broadly, with implications for multiple forms of curricula (e.g., lesson plans, enacted lessons, and written materials), this study narrowly focuses on the interpretation and interrogation of mathematics textbooks, that is, understanding the "textbook curriculum" (Center for the Study of Mathematics Curriculum, n.d.). At the root of this inquiry is a fundamental question, "What is the mathematical development of a textbook" or broader yet, "What is a textbook?" Although the labels *mathematics textbook* and *published mathematics materials* are sometimes used without scrutiny, there are a surprising variety of materials published for school use. Examples include calendars of problems, bound books of puzzles and problems, websites with definitions and examples, dynamic technology applets, boxes of folders each with exposition on a different topic, and bound books with sequentially-designed lessons. Any mixture of these kinds of materials and others may be found in a typical American classroom, but the last of these, the bound text with sequentially-designed lessons, is typically found in nearly every math class in the United States and is the predominant source of math curriculum for the teacher. Since the focus of this inquiry is to understand the nature of the mathematics development of mathematics textbooks as narrative, issues of sequence are central to this analysis. Therefore, only bound texts that have explicit sequences of mathematics content and ancillary materials (e.g., worksheets, transparencies, etc.), directly referred to in the bound materials and designed to be used in classrooms, are referred to in this study as *mathematics textbooks* or *texts*.

Of course, even among mathematics textbooks as defined here, there is great variety in substance, style, and organization (Love & Pimm, 1997). For example, some math texts are organized in chapters of multiple lessons, each designed to be used in a single class period, while others are organized in units of fewer investigations, many of which are designed to take multiple days. In addition, some textbooks contain sections of text that provide worked examples, while other textbooks include reflective activities which ask students to write explanations or descriptions of what they have learned. Still more variance is found in designs of materials directed solely for the student, such as worksheets and homework. Each of these portions of text potentially advances the mathematical content of the text and is included in this analysis to some extent.

As Remillard (1999) suggests, influencing how texts are read and interpreted by teachers and students can influence what happens in mathematics classrooms. However, this dissertation does not (and cannot) provide a description of what may or may not occur in classrooms. It also necessarily cannot describe any student learning that might result from using a mathematics textbook. Instead, through developing and analyzing a new way of reading a math textbook, this project extends the understanding of textbooks beyond collections of activities and conveyors of content. It offers a framework that later can be used to help teachers in their curricular design work (Brown, 2009) and inspire new possibilities for curriculum materials developers.

It should also be noted that this study assumes that the reading of a mathematics textbook does not (and cannot) reveal the intentions of its author. Challenging an assumed link between curricular materials and curricular intention, Ben-Peretz (1990) discusses several examples of how the same curriculum materials might be used to satisfy educative goals in a wide variety of disciplines, such as drama or science. Likewise, she points out that an educational objective, which she reminds us must be interpreted and thus does not have a singular meaning, can result in a wide variety of curriculum materials. She explains that,

> ... it is inappropriate to view any set of curriculum materials purely as an embodiment of writers' intentions. Curriculum materials are more complex and richer in educational potential than can be expressed in any list of preconceived goals or objectives, whether general or specific. (p. 49).

Therefore, this study does not make claims about the intentions of the authors of these textbooks (a worthy study, but one which would require additional sources of information outside the text).

The framing of narrative. The reading of mathematics textbooks as mathematical stories raises the important question *What is a story?* Literary theory has long been focused on addressing this question by examining the nature of literature and its reading (Nodelman & Reimer, 2003). For thousands of years, literary theorists have proposed theoretical structures of analysis that offer ways of understanding, analyzing, and comparing stories. Those who analyze literature pursue answers to questions of the text, such as *How do the different parts of the story work together?*, *How do changes is one part of the story affect other parts of the story, if at all?*

This sub-section introduces Bal's (2009) narratology framework, upon which the mathematical story framework was built, as a way to answer these questions with regard to mathematical textbooks. Bal's framework was selected because it is well recognized[8] and often used by those who study narrative. As a systematic analytic tool that carefully articulates and illustrates narrative, it frames narrative as an interrelated system of layers. The story and fabula

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[8] Indeed, according to Google Scholar, over 1800 research documents cite Bal's narratology framework, which has been translated into three other languages and has three editions. The original (English edition) was published in 1985.

layers present a way to distinguish between the reasoned information of the narrative and how that information is organized and revealed throughout the text, offering a framework that can help differentiate for a reader and analyst the mathematical content learned from mathematics textbooks from the way it is revealed throughout its pages. In addition, with carefully-defined terms within each layer, Bal sets up a comprehensive structure of elements (e.g., character and event), upon which a reading heuristic for mathematical stories can be developed.

The final sub-section will briefly introduce and define specific common literary constructs that will be used in the conceptualization of mathematical stories. These constructs will be revisited in much more detail in Chapter 3, and characters and plot will be theorized even further for the extensive analysis of mathematical stories in Chapters 4 and 5.

Bal's narratology framework. To help understand the complexity of form and function of narrative, Bal (2009) distinguishes between three layers of narrative: (a) the media (including its voice, or narrator) in which the story is told (referred to as *text*), (b) the sequence in which the events as they are encountered and perceived in the text by the reader (referred to as *story*), and (c) the logical sequence of events as constructed by the reader through interpreting the story (referred to as *fabula*). For Bal, an *event* is a transition or change, or as she explains, "the transition from one state to another state, ... a process, an alteration" (2009, p. 189). Bal notes that these three layers are not independent of each other, but that their distinction explains how multiple tellings of the same events differ. Their separation enables analysts to recognize when the same information (the "truths" reconstructed in the fabula by the reader) is organized with different effects. These layers also provide a way to articulate the differences between the information revealed in the story as understood by the reader (the fabula) and how it is temporally revealed (the story).

In short, a reader of a narrative text encounters and interprets a story while reconstructing the logical relations between the events of the story (the reader's fabula). I use the term *reconstructing* because, while the fabula is defined as a construction of a reader[9], so is the story. However, a reader's story is closely linked to the unfolding of the content of the text, and thus typically comes first. In contrast, the fabula is this reader's reworking of the events of the story, using logic. As Bal explains, the fabula is

> ... a theoretical construction, which we can make on the basis of the laws of everyday logic which govern common reality. According to that logic one cannot arrive in a place before one has set out to go there. In a story that [sequence of events] is possible however. (Bal, 2009, p. 79)

Thus, although it may seem on the surface that there is only one sequence of events in a narrative, Bal reveals that there are at least <u>two</u> that can be recognized: the sequence of events which the reader encounters temporally while reading and the chronological sequence deduced by the reader. More specifically, the fabula is the reader's determination of the order, "A happened, then B happened, then C happened, ..." even though this may differ from the order that events A, B, and C are presented in the story. This altered sequence is only one of many possibilities; the story might instead reveal B first, then C, then A for dramatic effect. Thus, Bal suggests, a fabula is a reader's deduction regarding the relationships between events; that is, the reconstruction of the logical chronology of the events of the story.

The story, then, is the manipulation of the fabula, which, Bal explains, "focalizes" (directs the attention of the reader) and otherwise "colours the fabula" (p. 18). That is, a *story* is the way in which the fabula unfolds to the reader. The same constructed relationships between events can be told in many distinct ways, altering its effect. For example, in *Romeo and Juliet*,

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[9] To distinguish these constructs, it can be helpful to link fabula with the notion of *fable*: not in the sense of falsehood, but rather as a creation "made up" (through logic) by the reader.

what effect might changing the media of the story from a play (in verbal form on paper) to that performed in a theater (or even in a movie) have on the audience? Although some aspects of the story may remain the same (for example, the central characters and their roles might not vary between tellings), the attention to the setting or the focus on the action might change greatly. Or, consider the effect when a reader is informed at the beginning that Romeo and Juliet will die at the end (e.g., the written play[10]) compared to a version for which this information is withheld (e.g., the 1996 movie). When a reader knows in advance that the couple will die, he or she may wonder throughout the story, "What will cause them to commit suicide?", whereas without this information, the same reader might instead wonder, "Will they live happily ever after?"

It is important to note that Bal does not suggest that a reader reads/experiences a story and then builds a fabula. Instead, she proposes that a reader logically and simultaneously constructs the fabula throughout the reading and experiencing of the story, though at different points during the sequence, the story or the fabula may be foregrounded. To help represent the interconnected layers attended to by a reader between the layers of story and fabula, I propose the diagram in Figure 2.1.

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[10] From the prologue of *Romeo and Juliet (Shakespeare, n.d.)*:
"Two households, both alike in dignity,
In fair Verona, where we lay our scene,
From ancient grudge break to new mutiny,
Where civil blood makes civil hands unclean.
From forth the fatal loins of these two foes
A pair of star-cross'd lovers take their life;
Whose misadventured piteous overthrows
Do with their death bury their parents' strife.
The fearful passage of their death-mark'd love,
And the continuance of their parents' rage,
Which, but their children's end, nought could remove,
Is now the two hours' traffic of our stage;
The which if you with patient ears attend,
What here shall miss, our toil shall strive to mend."

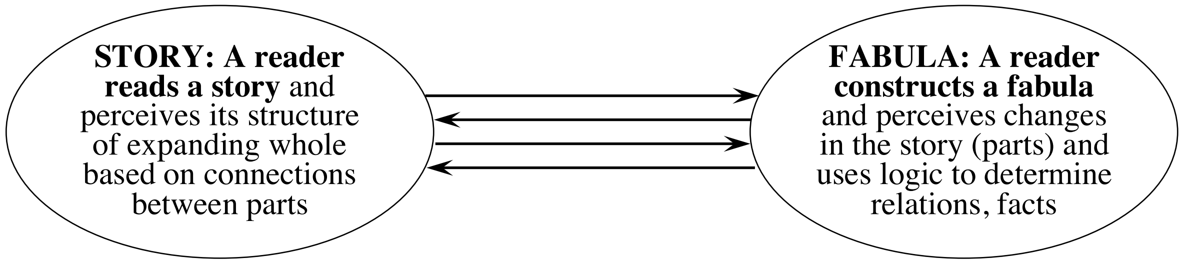

Figure 2.1. The interconnected process between the story and fabula layers by a reader throughout a story.

Building the fabula includes a reader's perception of changes in the story, which helps him or her to identify events (i.e., the story "parts") and deduce the "facts" of the story (e.g., Romeo and Juliet fall in love). Once a reader recognizes an event of the story, then he or she uses logic to determine how that event relates chronologically to the other events so far read in the story. While comparing and connecting events, this reader may perceive an expanding structure of the story "whole," enabling anticipation for what is to come.

Descriptive aspects of literary stories. Bal's (2009) layers of narrative are very useful to support discussions about stories and to articulate the effects that the organization of the fabula in the story has on its reading. Yet merely defining stories as the revelation of the fabula to the reader does not adequately capture the essence or distinguishing elements interwoven throughout the story. Within and between these layers, there are other aspects of stories that are commonly referred to in discussions of stories and help to distinguish each story from others. Therefore, for each layer, Bal isolates and defines numerous elements that help to analyze and describe each narrative layer, some of which are more widely-recognized than others. To narrow the focus of this framework, I have chosen to attend to those narrative elements with which mathematics educators will likely have prior experience (as well as vocabulary). Since the metaphor of mathematical stories offers teachers and other curriculum designers new ways to read, develop, and interrogate mathematics curriculum, focusing on story constructs already familiar to math teachers and other curriculum designers (who may not have an extensive background in literary theory) will arguably offer the most tangible benefits. Therefore, I selected those constructs introduced more than 2300 years ago in Aristotle's *Poetics* (Else, 1967)[11], who proposed that stories (or "tragedies") are *plots* composed of *characters* involved in *action*, placed in various *settings*. Since these four elements (*plot, character, action*, and *setting*) have endured much scrutiny and have been subject to much theoretical exploration, this discussion will briefly define and illustrate these constructs. Finally, additional descriptive elements commonly referenced in discussions of stories will be included, specifically the notion of *moral*[12], the *story unit*, and *frequency* and *repetition*.

Character. In her framework, Bal defines *character* as the "anthropomorphic figures provided with specifying features the narrator tells us about" (p. 112). Bal uses the term anthropomorphic to explain that characters do not need to be human (consider, for example, Baby Bear in *Goldilocks and the Three Bears*), but need to be human-like so that the reader connects human traits with them. The notion of character is embedded in the story layer, since characters and their characteristics are revealed temporally in the story. In addition to changing throughout a story, a character can also effect change and thus act to create an event (such as when Romeo kills himself). Associated with the fabula layer, Bal distinguishes discussions regarding the role of characters in creating events with the term *actor*. Therefore, the character Romeo describes the qualities and information about Romeo built up by a reader through reading the story (such as he is able to fight with a sword, loves Juliet, and is a member of the Montague

[11] Translations of Aristotle's *Poetics* vary greatly. I chose to rely mostly on Else's translation because it is widely cited, although for specific references, I also referred to Epps (1942) for elaboration and clarification.

[12] These literary constructs are further articulated and illustrated in Chapters 3, 4, and 5. These definitions are offered to help the reading of the strategies of inquiry section of this chapter.

family). However, when he fights with Tybalt, Romeo is an actor. Therefore, unlike the common use of the term actor to describe the role of a person playing the part of Romeo in the play, in Bal's framework an actor describes the element in the fabula that corresponds to the character element in the story.

Setting. In contrast, a *setting* can be viewed as the space in which the characters "live" and "are situated" (Bal, 2009). In a literary sense, setting refers to the descriptions of space and location in which the characters and action are placed, such as on the New York subway or in Central Park. Reading for the setting involves imagining and "concretizing" mental pictures enabled by the text (Nodelman & Reimer, 2003). In this way, the reader conjures the experiences and sensory perceptions described in the text. Similar to how character and actor are complementary constructs in the story and fabula layers of Bal's Framework (respectively), setting is found in both layers. In the story, the setting describes the scene of the event, the description and manifestation of space. However, the dimension of location as events transpire through space is its correlate in the fabula layer.

Action. Another element often used to describe what happens in stories is an act (or *action*), which is performed by an actor and results in some form of change (Bal, 2009). Since action is generally needed to create a story (otherwise, nothing happens), actions create markers of change along the temporal experience of reading (*events*). In literary theory, action is closely connected to plot, as plot is often related to the sequence of events demarked by acts (Else, 1967; Nodelman & Reimer, 2003).

Plot. The oldest known definition of plot (by Aristotle) describes it as a sequence of events with recognition that with different sequences come different effects on its reading. In Bal's (2009) framework, this sequence of events is actually what she refers to as story. Instead, Bal theorizes that plot is located neither in the story nor fabula layers, but instead emerges as a result of the reader's negotiation between the two layers while reading the story. This is because the plot unfolds structurally in the story layer (as the reader encounters and interprets the events in sequence), but its effect stems from the reader's logic while he or she connects different parts of the story, reinterprets prior events, and raises questions and anticipates what is to come. Thus, for this study, plot is defined as the tension between what Bal refers to as realization and expectation as the reader constructs the story structure; that is, what is known or believed by the reader and what is desired to be learned by the reader throughout the reading of the story.

Moral. Finally, though not an element of Bal's framework, stories are often pedagogic, a way of teaching that invites the reader/listener to experience a new situation and learn from it (Egan, 1988; King, 2003). Thus, stories are often referred to and described in terms of their overarching messages. These messages are often moralistic, imbued with values and principles distinguishing right from wrong. At times, morals are stated explicitly (as in *Aesop's Fables*), while in other cases the message is gleaned by the reader as a result of reading about the experiences of others. According to Reader Response Theory, the moral read by one reader likely will differ from that by another and stories can have multiple overarching messages to the reader. Therefore, a message (either explicit or implicit) learned by a reader and that may apply to a range of situations outside the story will be referred to as the *moral*[13] of a story.

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[13] I recognize the use of the word "moral" brings with it many historical connotations (e.g., parables in the Bible) that perhaps may not seem appropriate when thinking of a mathematical story. However, the phrase "the moral of the story" is so common that it is likely to have more traction in mapping the notion of literary story to mathematical story than another word (such as "overarching message"). Thus, when there is a common literary word that can be useful in thinking about mathematics curriculum, it has been appropriated in this work. More will be said about this in mathematical story framework in Chapter 3.

The story unit. One aspect of a story that does not focus on the parts of the story but to its overall structure, is what constitutes a story and how it can be isolated as a unit in a larger text. For example, consider how an epic story with much complexity may be composed of multiple embedded stories. How can these stories be identified? Describing what makes a story, Egan (1988) provides good guidance:

Stories are narrative units. That they are units is important. They are distinguishable from other kinds of narratives in that they have particular, clear beginnings and ends. The most basic story begins "Once upon a time" and concludes "they lived happily ever after." "Once upon a time" begins something and "ever after" does not refer to anything in particular except that what began is now ended. "Once upon a time" creates an expectation of a particular kind. We are told that at some particular time and place something happened. This something will involve a conflict or problem of some kind, which the rest of the story will be taken up resolving. (p. 24)

As can be seen by considering popular short stories, not all stories start and end with obvious signals such as, “once upon a time.” Yet, in general, to be considered a story, there is an identifiable start and end with narrative connecting these points. These points could be signaled by the first and last words of the text. However, the fact that embedded stories can be recognized (smaller stories within a larger story) demonstrates that readers can distinguish the start and end of a story without relying on a first or last word. The start, in general, launches a string of events through which a series of questions or curiosity emerge for the reader. The ending, in contrast, signifies a completion, which may be a resolution of “a conflict or problem of some kind” or could be the end of a journey introduced in the story. Thus, a *story unit* is a sequential chain of events that connect the start of inquiry with its resolution. Since plot, as defined previously, is the structured sequence of tensions between withholding information (producing inquiry) and revelation (answering questions) for a reader, then it can be said that a story unit is a portion of narrative that contains a complete plot.

Rhythm and frequency. Bal (2009) points out that an important effect on the reader is a story's *rhythm* (which refers to the proportion of the narrative devoted to different events and how those portions are located throughout the story) and *frequency* (which refers to how events that are similar to the reader are repeated throughout the story). Thus, the rhythm of a story can reveal the areas of emphasis of the story, while the frequency of which the reader is presented with similarity can enable the reader to anticipate what is ahead. For example, when a particular event (such as seeing a daisy in a vase) repeatedly occurs before another event (such as someone's murder), then the reader may pick up the pattern and use it to anticipate that a murder will occur when another daisy in the vase is noted. Bal explains that elaborate descriptions of a setting can slow the pace of a story as it may leave the reader with long portions of narrative that contain no events. Rather than focusing on calculations of the number of words or time it takes to read, Bal recommends examining the patterning of the attention and noting whether "certain episodes are given more attention than others" (p. 99). She explains that the differing amounts of attention to certain parts of the fabula over others can result in that portion of the fabula becoming more developed.

Assumptions about reading. It should be acknowledged that this dissertation applies a structuralist framework to describe and analyze a body of text from mathematics textbooks. Structuralism, a movement that emerged in the middle of the 20th century in France, assumes that a way to understand a phenomenon is to identify and codify its structure. The structuralist movement was later criticized by philosophers partly for its positivistic and ontological stance (that there is a single structure to be recognized that exists independent from the reader/interpreter). Post-structuralist literary theories were developed in part to recognize and account for other ontological assumptions (e.g., that what is thought to be there might not be there and that what is not recognized as there might be there).

Given this criticism, it is reasonable to ask why this dissertation will use a structuralist framework. That is, what might be gained from using a structuralist framework to interpret mathematics texts? As described in the literature review, there is very little conceptualization of structures of mathematics curriculum in a way that accounts for how its parts form a whole. As math educators, we have difficulty answering basic questions about textbooks, such as “In what ways are various mathematical developments the same or different?”, “What can make this mathematical development more interesting?”, “What happens if the sequence is changed?”, and “How does what comes before and after this task affect its interpretation?” This study assumes that an exploration of structure benefits the educational community not only because it offers a new way to understand and describe mathematics curriculum, but also because it enables critique.

Reader Response Theory. To address the positivistic and ontological critiques of structuralism, this study assumes that there is no “true” reading of a mathematical textbook as a mathematical story. That is, it assumes that all readers interpret text with subjectivity. This view is consistent with Reader Response Theory (Rosenblatt, 1988), which explains that any reading not only differs from those of other readers, but also differs from other readings by the same individual. In addition, Barthes (1974), an influential theorist of literature, complicates this view further by arguing that readers of writerly texts (those written with much ambiguity) may produce a multiplicity of meanings. With regard to the work of teachers with mathematics textbooks, other researchers have also recognized the subjectivity of interpretations of text. For example, Remillard's (1999) observation of the curriculum work of teachers suggests that each reading is unique to the teacher since "the meanings the teachers made through reading the text grew out of interactions between their beliefs and elements of the textbook and were situated in the larger context of their teaching" (p. 319).

In addition, it is assumed that text does not "contain" meaning; instead, readers make meaning with the text through a transactional process (Rosenblatt, 1988, 1994). That is, neither the text nor the reader is a determinant factor in the production of meaning. As Rosenblatt explains,

> The reader, we can say, interprets the text. (The reader acts on the text.) Or we can say, the text produces a response in the reader. (The text acts on the reader.) Each of these phrasings, because it implies a single line of action by one separate element on another separate element, distorts the actual reading process. The relation between reader and text is not linear. It is a situation, an event at a particular time and place in which each element conditions the other. (Rosenblatt, 1994, p. 16)

Thus, this dissertation assumes that the readings of textbooks in this study are the readings by a specific reader (i.e., myself). They do not (and cannot) reveal how another reader might interpret the same portion of text, particularly a student. In fact, my previous mathematical and teaching experiences (described in the preface) would suggest that my reading of a mathematics textbook is "atypical." However, the purpose of this dissertation is not to generate particular readings of a mathematics textbook or to describe a typical reading (which is not possible). Rather, the goal is to develop potential new ways to read and understand mathematics curriculum. Similarly, the structure itself is not the goal; instead, I argue that it is the reading for structure that offers new insight.

Forms of reading. Rosenblatt (1994) distinguishes two forms of reading: *efferent* (reading for information, such as trying to read an instruction manual for information on how to set a clock) and *aesthetic* (reading for an experience). This distinction is useful because mathematics texts are often assumed to be read in an efferent manner[14], focused "primarily on what will remain as the residue *after* the reading – the information to be acquired, the logical solution to a problem, the actions to be carried out" (p. 23). However, regardless of how mathematics textbooks are typically read, this dissertation is focused on what can be learned about mathematics curriculum when they are read aesthetically as stories. That is, the readings reported in later chapters and the heuristic developed for reading mathematical stories assumes an aesthetic lens during which the reader takes in and makes (with the text) an experience. As Rosenblatt describes, this type of reading includes paying "attention to the associations, feelings, attitudes, and ideas that these words and their referents arouse within him" (p. 25).

Finally, it is important to note that this study assumes that reading is not a passive process, but instead involves interpretation and activity on the part of a reader as described in Rosenblatt (1994). It assumes a reader uses prior knowledge to make sense of the text and metacognitively monitors his or her understanding of text throughout. Although it might be argued that not all readers engage with text this way, working with text in this way is what is meant in this study by "reading." Therefore, in terms of mathematics textbooks, *reading* includes the aesthetic engagement of the reader as an actor who solves problems, answers questions, and completes tasks.

Summary. Before moving on to the contextual details and strategies of inquiry, a brief summary of the key points from this section may be helpful. For this study, mathematics textbooks are limited to the sequentially-ordered, printed, and bound written mathematics

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[14] Certainly, I know I have read mathematics textbooks for this purpose, and I don't argue with the claim that most mathematics students (particularly those in grades K-12) read for information. Rosenblatt also refers mathematical texts and symbols in examples of texts that are read in an efferent manner: "The mathematician reading his equations, the physicist pondering his formulae, may have no practical purpose in mind, yet their attention is focused on the concepts, the solutions, to be 'carried away' from their reading" (p. 24).

materials designed for classroom use. To support a conceptualization of the content in mathematics textbooks as a form of narrative, the Bal (2009) framework was introduced and described. *Narrative*, then, is defined as a set of inter-related layers (text, story, and fabula), which together represent a comprehensive system enabling the qualities and structure of narrative to be recognized and carefully analyzed. Of course, reading narrative implies that there is a reader reading. This study draws from Rosenblatt's Reader Response Theory (1988) and Barthes' (1974) notion of multiplicity to propose that reading mathematical texts is an active and subjective process. Rather than asserting that texts contain inherent meaning, this study assumes that readers make meaning with the text, a process sometimes referred to as a transaction between reader and text (Rosenblatt, 1994).

Contextual Details and Strategies of Inquiry

Since this project relies on a literary framework and studies texts and their interpretations, this dissertation is best described as humanities research (American Educational Research Association, 2009). It is important to note that the aim of this project is not to determine "the story" told in mathematics textbooks. Instead, the goal is to develop a way of conceptualizing (and thus reading) mathematics curriculum for structure and aesthetic. This dissertation is (a) a proof of concept that mathematical lessons found in textbooks can be read as mathematical stories in a coherent way, and (b) an argument that there is more to be learned about curriculum that emerges when textbooks are read in this way. As such, this work represents a theoretical analysis and is not an empirical study.

According to the reading theory discussed earlier, my reading of mathematics curriculum does not (and cannot) reveal the intentions of the authors nor does it offer a reading of a mathematics student. Therefore, as opposed to the goal of establishing "true" readings (since my reading will necessarily be different than that of other readers), the methods of inquiry for this project focus on enabling (and not hindering) the production of curricular theory. That is, although analytically rigorous, the methods attend to carefully producing explanations of how mathematics curriculum works, developing and testing tools to describe and critique mathematics curriculum, while broadening the potential for improving textbooks. Being a study of text, it is appropriate that these methods include attention on that which comes with the text, namely the *context* of inquiry. This section introduces the questions focused for study, positions the researcher in relation to this work, articulates the details regarding text selection, and presents strategies for reading and accomplishing the goals set in the previous sections.

Overarching questions of inquiry. In light of the goals stated previously, this dissertation broadly seeks to answer the questions: *In terms of mathematics written curriculum, what is a mathematical story? When the mathematical development in curriculum is conceptualized as a mathematical story, what can be learned (relevant to the comparison and characterization of different curricula) about how the mathematical content in mathematics textbooks unfolds as it is sequentially read?* Chapter 3 (which outlines and illustrates the theoretical construct of mathematical story and its elements) directly addresses the first question, while Chapters 4 and 5 address the second question (with respect to mathematical characters and mathematical plots, respectively).

Chapters 4 and 5 also narrow the focus to two important mathematical story constructs in order to support the overall argument that the reading of mathematics textbooks as mathematical stories offers new ways to understand mathematics curriculum. The study of mathematical characters in Chapter 4 was selected because of the central role and importance of mathematical objects and their relationships in mathematics. To focus the inquiry regarding mathematical characters, additional questions of inquiry were developed, specifically: *What can be learned about how mathematical objects develop and change throughout a sequential passage, and what implications might this have for the reading of mathematics curriculum?* That is: (a) *What is (and is not) a mathematical character?,* (b) *What might reading mathematical objects in textbooks as characters within stories enable the reader to notice and learn about the introduction and development of mathematical objects in curriculum?*, (c) *How do the setting and action of the mathematical story contribute, if at all, to the development of a mathematical character?*, and (d) *How can an experienced and knowledgeable reader (i.e., a reader who has learned this content elsewhere) read for character in context of the story and what has come before?*

Additionally, a focus on mathematical plot in Chapter 5 was selected as an opportunity to learn more about the overall structure and aesthetic of mathematical stories found in mathematics textbooks. This work was further focused with the questions: *What can be learned about the aesthetic nature of a curricular sequence?* That is: (a) *How might the structure and aesthetic of the mathematical development be conceptualized as a plot?*, (b) *When mathematics development in textbooks is analyzed for its structure and aesthetic (i.e., its plot), what can be learned about curriculum?*, (c) *How does changing the sequence of the elements of a mathematical story change its plot?*, and (d) *How can an experienced and knowledgeable reader read for the mathematical plot?*

Building and testing a new theoretical framework. The main product of this dissertation is a theory of mathematical sequences in written curriculum (specifically, a theory of mathematical stories with particular attention to mathematical characters and plot). It is assumed that if mathematical stories are thoroughly grounded in literary theory, this construct offers new conceptual tools for math educators to make sense of and critique mathematics curriculum in general, and math textbooks in particular. Therefore, this raises the methodological question of how one uses theory (in this case, a theory of narrative) to build another (i.e., a theory of mathematical stories)?

This theoretical analysis involved the following stages: (1) I sought out and studied what well-accepted literary theorists (e.g., Aristotle, Bal, and Barthes) have written about stories and their constructs (i.e., character, setting, event, plot, and moral), (2) I then performed a close reading (described in more detail later) of mathematics textbook sequences for these metaphorical constructs with respect to the changing and developing mathematical content in written curriculum (e.g., in this sequence, what could be a mathematical character? What could be mathematical setting?), (3) I then returned to literary theory with these interpretations of mathematics written curriculum to check for consistency and identify potential conflicts and challenges, (4) I continued to move back and forth between literary theory and mathematical stories found in textbooks until stability was reached in the interpretations of mathematical story constructs (i.e., consistent interpretations of mathematics content throughout sequences emerged which do not contradict the basic principles of literary constructs according to the theory described earlier), (5) I then read and analyzed multiple mathematical sequences in multiple textbooks for what this conceptualization helps to reveal about mathematics curriculum and identified potential shortcomings of the conceptualization (what it does not reveal).

In addition, it was also important to note the role literary stories (such as Baum's *The Wonderful Wizard of Oz*) played in my analysis. Not only did literary stories help to support and communicate the literary theory on which this study relies, but they also served as the developmental source for the metaphorical mapping. This practice of drawing from literary works to analyze and illustrate particular aspects of literary theory is commonly used in literary frameworks, including both Bal (2009) and Barthes (1974), and is even found in Aristotle's *Poetics*. The use of literary examples in this dissertation not only provided an instructive example in a genre for which literary criticism is most accessible, but also supported the development of a heuristic of reading. That is, when a rich literary example of character is considered, it raises new questions and helps focus the reading of characters in mathematical stories, such as "is there a metaphorical example of this in mathematics textbooks? Does this happen in this lesson? If not, what might it look like and what implications might it have for the reading of mathematical stories?"

To further highlight and strengthen the metaphorical connections between mathematical stories and their literary counterparts, each of the analysis chapters will contain a demonstrative thread of particular aspects of stories with a popular work of fiction, including the *Harry Potter* series (by J.K. Rowling) and *The Wonderful Wizard of Oz* (Baum, 1900).

What brings me to this work and what I bring to this work. The way one reads and the prior knowledge one brings to the reading affect both the meaning of the text and the experience of the reading (Barthes, 1974; Rosenblatt, 1988). As Anaïs Nin articulated so eloquently in the opening quote, we cannot escape who we are and what we bring when we read. The interpretations used during this analysis, then, are not meaningful without some articulation about what I bring to the reading of math textbooks and how I came to this experience. Thus, according to Reader-Oriented Theory (discussed earlier), this sub-section describes the *empirical reader* of this study, "the qualities of the reader of the text" (Weinberg & Wiesner, 2010).

Therefore, to describe the empirical reader of mathematics curriculum for this dissertation, I prefaced this document with my story of how I became intrigued with mathematical stories as a conceptualization of mathematics curriculum and was compelled to develop this theoretical framework. However, I recognize that my background presents what can be viewed as significant challenges for this work. I acknowledge that my extensive classroom experience and curriculum design work (in terms of co-authoring textbooks) makes me far from the typical reader of mathematical texts. I also recognize that my passion for mathematics comes in stark contrast with common student dispositions (although I remain ever hopeful that this can change). Moreover, as a self-described "math geek," I was perhaps more surprised than anyone to find myself adopting a literary framework. Becoming familiar (let alone developing expertise) in literary theory took copious amounts of my time and patience.

Even with these challenges and limitations, my educational and mathematical experience has come with benefits for this work. It afforded me necessary insight about a wide variety of textbooks (diverse in both form and function) which often supported my theoretical testing of story constructs (e.g., If a mathematical character can be read this way for this textbook, what about in that textbook? What other types of textbooks that I know of might challenge this way of reading?). My passion for and experience with teaching enabled me to constantly re-focus my attention on developing a way for this framework to be beneficial for teachers as well. Moreover, though I make no claim that my reading of mathematical stories is similar to those made by students, my teaching experience has convinced me that students nonetheless have aesthetic experiences with mathematics, which I hope to improve and enhance through the development of this framework. My overarching goal and commitment to support student inquiry and curiosity in mathematics curriculum through the development and articulation of this framework prevented me from quitting even when the work seemed daunting and impossible.

The Criteria and Selection of Focal Texts. Much care was used in the selection of the textbooks for this inquiry. As an author of mathematics textbooks, I recognize that I have previously crafted many mathematical stories to which I might unconsciously compare those in this study. Analyzing my own math textbooks, or those that offer similar content, would unnecessarily hinder my aesthetic experience and obscure key parts of the reading heuristic for other math educators. To avoid this conflict, the texts selected for this study address content that was not closely related to content in the textbooks I have co-authored or taught from.

When selecting portions of these mathematics textbooks to analyze, attention was focused at the beginning of the mathematical development. Just as Act III of *Romeo and Juliet* would lose meaning if read without Acts I and II, mathematical stories need to be read and understood in relation to what comes before in the text. For this reason, this study focused on material found at the beginning in the first three chapters. Although not all stories reported in this study started with Lesson 1 of Chapter 1, all later stories (found in Chapter 2 or 3) were read and interpreted in relation to what came before. Therefore, the first three chapters of three focal texts were read and analyzed, allowing some stories of different grain size (chapter length, multiple lesson length, and lesson length) to be considered and used for theory building.

In case different grade levels and different authors offer different types of mathematical stories (which is assumed to be the case), multiple textbook series representing different grade levels were selected. Since this study does not attempt to describe all the mathematical stories found in curriculum, there was no attempt to select texts (or portions of text) that can generally represent mathematics texts. Rather, as a conceptualization study, it is reasonable to assume that a variety of textbook designs would better support the development of a robust mathematical story framework that enables the interpretation of diverse content treatments, as well as developing a reading heuristic that can help educators interrogate a myriad of textbooks. Also, since it is expected that this framework will, in the future, support the work of classroom teachers, the selection of widely-used textbooks, familiar to teachers, is advantageous.

Due to sizable market share and contrasting style, the teacher and student texts of *Everyday Mathematics* (The University of Chicago School Mathematics Project, 2007, "EM") and *Scott Foresman/Addison Wesley* (Charles et al., 2008, "SFAW") were selected to be two of three focal texts. The *Everyday Mathematics* (EM) program is one of the NSF-funded programs developed in response to the recommendations made in the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989). The *Scott Foresman/Addison Wesley Mathematics* (SFAW) textbook series, in contrast, was designed by a large publishing house largely to meet market demands. In addition, a non-U.S. elementary textbook series, *My Pals are Here! Maths* (Kheong et al., 2012, "MPAH") was selected. The teacher and student texts of this series, which (at printing) was widely-used in Singapore, offers the opportunity to learn how this framework might enable the analysis of an even broader set of curricular materials.

Finally, care was taken to choose different grade levels of these texts for analysis so that there would be both similarities in content as well as variety. This allowed questions of how the "same" stories are told in very different ways to be pursued and revealed how the "same" characters actually are quite different when found in different stories. Below is a list of the focal texts that have been selected, along with grade level.

My Pals are Here! Maths (Grade 1, "MPAH1")

Scott Foresman/Addison Wesley Mathematics (Grade 2, "SFAW2")

Everyday Mathematics (Grade 3, "EM3")

In addition to these focal texts, other textbooks (including those for high school) were consulted for stories to provide additional opportunities for theory development and articulation. In particular, a series of lessons found in the textbook *Mathematics: A Human Endeavor* by Harold R. Jacobs (1970) were extensively used in the analysis of mathematical plot.

Portions of the textbooks included in the analysis. Although there is often a lot of information offered in textbooks to support the work of teaching, this analysis was primarily focused on the mathematical stories offered to students of mathematics. Therefore, although the teacher materials were used, the analysis focused on discussions of the activity, discussions, exposition, worked examples, homework, and other forms of curriculum that were intended for students. If the text materials only offered instructions to the teacher (as was usually the case in *Everyday Mathematics*), then these were read for the mathematical development as it was described for students. Since this current study does not (and cannot) attempt to represent the mathematical development that will occur in classrooms[15], then the result of a teacher instruction such as “discuss with the class the patterns that emerge” was instead imagined based on the surrounding context. Even though other readers may have different images of these resulting discussions, this does no harm to the overall goals of this study: to develop a heuristic and understand and describe the nature of mathematical stories as found in textbooks.

It should be noted that the inclusion of homework presents an interesting challenge to the reading of a mathematical story. Many curricular programs design homework that not only focus on the current mathematical topics learned in the lesson, but also review previous ideas and offer repetitive practice. When a lesson was determined to be a complete story (in terms of a story unit defined earlier), then homework that came after the ending of the story was not

[15] An analysis of the mathematical stories of enacted curriculum could certainly follow this study.

included in its analysis. However, when more than one lesson was considered the unit of analysis, the homework was included, at least to the extent that it offered further character development or change the mathematical study under investigation. Unfortunately, including all the homework sometimes made the story appear disconnected (a "threat to coherence" as described by Stigler and Hiebert (1999)). Therefore, since the purpose of homework is usually different than that of class mathematical development, only homework opportunities that directly speak to the mathematical development of the story was included.

Sometimes textbook authors offer optional differentiation recommendations or replacement activities. For example, in the SFAW textbook series, differentiation activities are often provided in boxes that come at the beginning of the teacher materials for the lesson. Whenever optional activities were not clearly marked to have a particular placement within the sequence, they were not included as part of the mathematical story. However, if an optional activity was labeled with where it goes in a sequence, it was considered part of the mathematical story. Finally, if an optional curricular part (such as an activity) was offered as a replacement for another part, then the primary part found in the text was considered[16].

Whenever a textbook for the student was available (as was the case with the MPAH materials), then the student text was the primary source of the mathematical story. However, even when this was the case, the teacher materials were consulted for additional student mathematical curricular elements, such as instructions to the teacher to pose a particular question that was not mentioned in the student text. In addition, any ancillary student materials described in the teacher materials that were designed to be seen and read by all students, including classroom display materials described in the teacher text, were included in the analysis.

[16] However, these alternate activities were also considered to explore the possible effects of changes to the mathematical story when activities are swapped.

Note that the definition of *text* in terms of textbooks is broad and extends beyond printed words on a page. Math textbooks also contain a myriad of other representations, such as photos, drawings, and diagrams. These textual elements often accompany words that refer to them, but they sometimes are also independent on a page. The nonverbal elements fill a variety of purposes (see Kim, 2009 for an extended discussion), including informing, illustrating, appealing, and directing attention of the reader. For the purposes of this analysis, these nonverbal elements were also read and interpreted in the sequence in which they are either referred to in the text (e.g., when a task refers to a diagram) or when the element was encountered on a page from top to bottom. These elements were read for both aesthetic and efferent purposes (Rosenblatt, 1994) and were used in concert with the words to make meaning of the text.

Finally, as described in Love and Pimm (1997), some mathematics textbooks offer "call outs," portions of text that are in the margin. These messages sometimes offer ancillary references to resources or call attention to a particular idea, such as offering a definition of a word in the main body of text. Whenever possible, the sequential text found in the main body of text was considered first, and the call outs were read for their placement in relation to this main body. If there was ambiguity of the sequence (such a reference to a worksheet), then the placement of the call out within the sequence was considered based on its vertical location relative to the other story parts on the page.

Reading Practices. Since one of the goals of this dissertation is to build understanding about the nature of mathematical stories as well as develop a reading heuristic, a careful record of interpretations required the reading and re-reading of the focal texts. This included paying attention to possible connections between textual parts and considering both local details (at the individual task level) and more global structures and qualities that emerge only when considering multiple portions of text. This reading required attention both to the text and to how the attention to the different literary elements helped certain curricular qualities to be recognized. This also required the careful recording of mathematical thinking, work, and reactions as the stories unfolded.

It should be noted that the reading of textbooks in this dissertation does not constitute an expert reading (using my advanced mathematical knowledge to bear) or that of an imaginary novice (e.g., that of some generic student). This is because the reading of text at any point was limited to meanings interpreted from what has come before in the mathematical story. For example, although I could initially interpret the number zero when encountered in a text with a multiplicity of complex meanings, in order to understand the mathematical story of the textbook, I instead closely read this number in its meaning in context. Thus, the readings represent a knowledgeable reading limited to the mathematical story at hand based on a close reading of text. This analysis, therefore, cannot be viewed as a student or teacher interpretation, but instead represents a potential mathematical story from a close reading of text.

The rest of this section describes two layers of reading strategies: one for a first read of a mathematical textbook and another for the subsequent re-readings.

First Reading of Textbook. The focal textbook curricular elements were read and interpreted in the sequence as they appeared in the textbook. To learn about the unfolding nature of mathematics in a math textbook, descriptions of what was mathematically occurring at different temporal "moments" (events) of the development were recorded (consistent with Rosenblatt's (1994) description of reading transaction). Temporal readings were captured in four ways:

- When prompted by the text, the curricular tasks/prompts/questions were completed, answered, solved, and/or read in sequence in a notebook, whenever possible.

- Whenever a question arose or an idea about where the mathematics was heading was stimulated (such as "This makes me anticipate that ..." or "This makes me think of..."), a memo was recorded at that point in the mathematical work.

- A Character Log was generated and referenced for characters that reappeared and continued to be developed.

- Each mathematical story was summarized after its first reading, including initial observations of the interaction of key literary elements.

Each of these practices is elaborated below.

Responding to prompts. Just as analyzing literary stories involves work on the part of the reader to carefully recognize changes and significance of the multiple parts of a narrative, it is assumed that the analysis of mathematical stories similarly requires thorough participation of the reader with the textbook by solving the tasks and answering the questions posed by the author(s). Without thoroughly immersing in the mathematical demands of the text, many parts of the mathematical story may remain hidden from a reader. For example, consider the series of tasks in Figure 2.2 from Jacobs (1970), for which the text prompts readers to draw the path of a ball (always starting at the lower left-hand corner) if it always rebounds at a 45° angle with each side of the table as shown in Problem #6. Since this set of tasks follow a paragraph that introduces questions regarding whether the ball will end in a corner or not and whether it is possible to predict the corner in which the ball will land, a reader might glance at these tasks and assume the mathematical content involved is drawing paths and finding their end corners. However, when solving the tasks, the contrast between the different shapes of the paths, particularly in their contrasting complexity and simplicity, is noticeable, allowing the complexity of the paths (and, by association the number of rebounds) to become a new characterization of mathematical characters (these tables). Without solving the tasks, this characterization, along with possible mathematical questions that could emerge from their forms, would likely be missed.

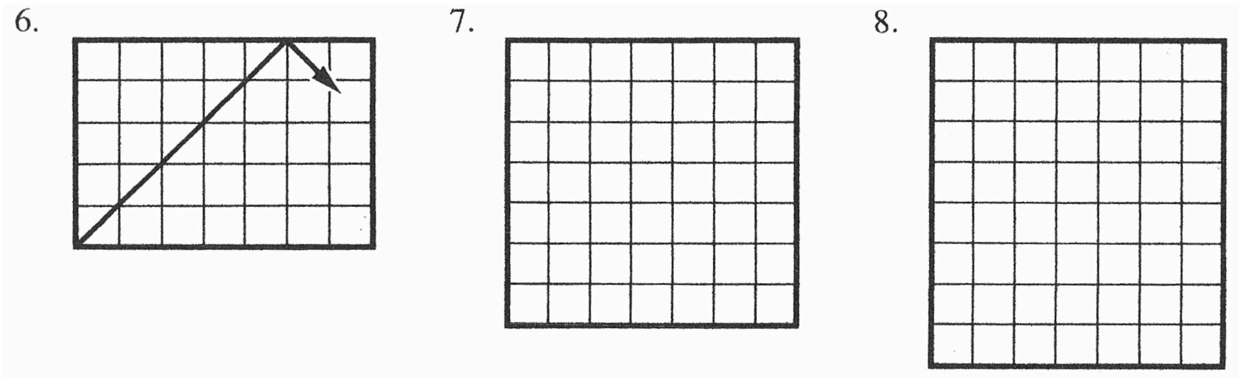

Figure 2.2. Billiards table problems (Jacobs, 1970, p. 5). Reprinted with permission.

More than missing characterizations is in jeopardy. For example, within a sequence of prompts a conflict can arise, providing a reader an opportunity to further construct his or her mathematical fabula (the logical re-construction of the story). One way this can happen is when a strategy is introduced in a textbook that does not apply for a given task or needs adaptation (such as a "counting-on" strategy that instructs a reader to start with the larger of two numbers, and then a subsequent task asks for the sum of 3+3). Another important part of a mathematical story that may be missed is the introduction of a new mathematical character in the solution of a task (such as when asking how many remain after one marble is taken from a group of 1 marble before 0 is introduced).

Thus, in order to recognize the mathematical strategies and concepts as much as possible throughout the textbooks (and thus, have a thorough reading of the unfolding mathematical story), all curricular prompts were answered/solved/completed when possible in a record of analysis to enable later review of strategies used at each point of the mathematical development. If particular strategies were introduced and developed in the text, they were used for subsequent problem-solving. If tools were mentioned (such as the use of graph paper), they were used if available.

Throughout the analysis of mathematics textbooks, there were times when responding to a prompt was not possible because some activities required being in a classroom of students to complete (e.g., when the text prompted me to solve a problem designed by a peer). When the activity was not able to be completed, a memo was instead written in the notebook about the imagined activity, emerging questions (both in terms of ambiguity and in terms of mathematical principles or properties that might emerge from the activity but are not explicitly prompted for), a description of the possible purpose of the activity, and predictions for where a curriculum might be headed based on this activity/prompt (that is, can I predict how this activity might lead to some other mathematical ideas, questions, or activities?).

Questions and anticipations. Plot, as described earlier, represents the tension between what is learned throughout a story and what is desired to be known by the reader. Therefore, reading for mathematical plot involves capturing what I learn throughout the story as well as inquiry that arose for me during the reading. I thus made an effort to record questions, wonderings, and predictions for what lay ahead in the mathematical story through the form of memos. These comments were placed in the analysis record at the point during which the question or anticipation was raised in a contrasting font (e.g., “This makes me think about ...”). These memos became important for noting when the mathematical story structure enabled me to be surprised or when my predictions were accurate. They also enabled me to note when I raised mathematical questions that were never addressed by the mathematical story.

Character Log. To understand the effect of the mathematical story as written in the textbook, the mathematical characters[17] were read as introduced and developed in the text as much as possible. As shown in the previous *Measure Up* example, characters (such as "Ten") in the context of the mathematical story might have very different characteristics than those that might be expected by a knowledgeable math educator. To prevent this conflict, a running log of the central mathematical characters and their characteristics was kept. This electronic log contained the name(s) of the mathematical characters and the characteristics that were noted through the text (such as revealed in a definition or diagram, or that emerged through activity). In addition, this recording noted the setting(s) in which the character was found and any actions that were involved. See Figure 2.3 for a sample log.

Character name(s) and representation(s):			
Lesson and Page	Characteristics	Setting	Actions

Figure 2.3. An example of a Character Log used in the analysis of textbooks.

Importantly, this Character Log supported the reading of text. As a mathematical character repeatedly appears throughout the text, the log was regularly consulted to support the contextual interpretation of the character within the story so far and not from mathematical understanding developed outside the story. When new information about the mathematical characters was revealed later, these new characteristics were added to the log.

[17] The conceptualization of mathematical character will be defined and illustrated in Chapter 3. At this point, is might be helpful to note that mathematical characters are the objects of study in a mathematical story. Therefore, mathematical objects such as numbers (0), triangles, and expressions (2 + 3 = 5) can all be mathematical characters.

Story summary. After reading a mathematical story, solving its problems, jotting all questions and anticipations, and noting central mathematical characters, the residual record represents momentary mathematical points rather than an overall story; that is, after experiencing a walk through the trees, I found I needed to step back and describe the forest. Therefore, for each lesson, a summary of the mathematical story at the lesson level was written. Since different textbooks distinguish different units of content as a lesson, this summary sometimes did not comprise a complete story but instead captured multiple embedded stories or only a part of a larger story. These summaries referred to specific literary elements (character, setting, action, plot, and moral) that through reading emerged as focal.

Strategies of re-readings. Subsequent re-readings focused on fine-tuning theoretical constructs (e.g., what makes this an example of mathematical action?) and developing a reading heuristic (e.g., how could I help another reader recognize this as a mathematical setting?). These later readings also focused on recognizing similarity and differences between mathematical stories, and using these distinguishing qualities to help direct my focus in articulating the framework (that is, what makes these mathematical stories the same? What makes them different?).

Looking Ahead

With these strategies in place, the focal texts were closely read (and re-read) for their story elements as described in this chapter. From this analysis, the mathematical story framework was developed and fine-tuned. Chapter 3 introduces and illustrates this framework, along with introducing a reading heuristic for mathematical stories and their elements.

“It is the function of art to renew our perception. What we are familiar with we cease to see. The writer shakes up the familiar scene, and, as if by magic, we see a new meaning in it.” - Anaïs Nin

CHAPTER 3

Mathematical Story Theoretical Framework

Building upon Bal’s layers of narrative discussed in Chapter 2, this chapter presents a framework that conceptualizes a mathematics textbook as a *mathematical story*. Thus, it repositions a mathematics textbook from an instruction manual or tool to that of art, crafted to offer an experience for a reader (whether good or bad). As Anaïs Nin’s quote above suggests, this repositioning seeks to renew perceptions of familiar curricular forms, to recognize and describe previously hidden qualities. It seeks to shake up conventional ways of seeing textbooks as conveyers of content and aims to invite the imagination of rich new mathematical stories. If a novel can be appreciated for its rich characters or its sudden surprises, then why cannot a mathematics textbook?

The meaning and effect of sequential temporal experiences have been theorized and rigorously studied in terms of novels and short stories alike, but have so far been ignored in regards to mathematics classes. Although it may be unorthodox to consider mathematical objects and activity in these “novel” ways, conceptualizing the unfolding of mathematical content in a textbook as a mathematical story allows new questions to emerge, such as what propels a mathematical story forward? How is the notion of a mathematical object (such as parallelogram) further developed throughout the curriculum? What types of mathematical stories can we find? What happens to the story when the sequence is altered?

Thus, this chapter turns the attention away from literary theory toward a reconceptualization of mathematical stories of mathematics textbooks, as well as their characters, settings, actions, plots, and morals. Specifically, this chapter addresses the questions: *What is (and is not) a mathematical story? How would one read for it?* Building on the narrative theory in Chapter 2, this chapter starts by addressing the three criteria of narrative, introduced by Bal (2009), to justify the interpretation of mathematics textbooks as narrative. In this discussion, Bal's layers of narrative (text, story, and fabula) are defined and illustrated with respect to mathematics curriculum. Next, the chapter discusses each of the story elements discussed in Chapter 2 with regard to mathematical stories (specifically mathematical character, setting, action, plot, and moral). To conclude, a discussion will summarize the framework, clarify what is an is not a mathematical story, and present a heuristic of reading.

The Conceptualization of Mathematics Textbooks as Narrative

In Chapter 2, Bal's (2009) framework of narrative was presented as a system of interrelated layers (text, story, and fabula). Beyond providing the ability to structurally simplify the complex way narratives work, these layers also establish Bal's definition of narrative. Specifically, she articulates that to be considered a narrative, a text ideally meets three conditions (reprinted from Bal, 2009, p. 9):

1. Two types of 'speakers' utter the signs that constitute a narrative text; one does not play a role in the fabula whereas the other does. This difference exists even when the narrator and the actor are one and the same person as, for example, in a narrative related in the first person. The narrator is the same person but at another moment and in another situation than when she originally experienced the events.

2. We can distinguish three layers in a narrative text: the text, the story, and the fabula. Each of these layers can be described.

3. That with which the narrative text is concerned, the 'contents' it conveys to its readers, is a series of connected events caused or experienced by actors presented in a specific manner.

Thus, Bal explains that an essential quality of narratives is, in part, that the three layers can be distinguished and described. This section provides a conceptualization of the metaphorical layers of mathematics textbooks to directly address Bal's condition #2 to support the argument that mathematics textbooks can be recognized and interpreted as narratives. Following this, a section will complete the argument to discuss Conditions #1 and #3 with respect to mathematics textbooks.

To support this discussion, Lesson 1-2 of *Everyday Mathematics* (The University of Chicago School Mathematics Project, 2007) will be used (reprinted in Appendix A). This lesson was chosen because it is found at the beginning of a textbook (and thus, its analysis is not overburdened by what has come before) and offers useful examples of the elements of mathematical stories (i.e., setting, action, character, plot, and moral) under discussion in this chapter. Therefore, this section starts with a brief overview of the textbook lesson referred to throughout the rest of the discussion.

Overview of *Everyday Mathematics* Lesson 1-2. Lesson 1-2 of *Everyday Mathematics* (The University of Chicago School Mathematics Project, 2007, "EM"), follows an introductory lesson focused on extending and completing sequences of whole numbers. It starts with a "Math Message" prompting readers to write both the largest number and smallest number "you can read" (p. 24). These numbers are then shared with the rest of the class and recorded on the board. The teacher is instructed to "review the names of places in a base-ten numeral" by asking children to identify and mark numbers in the tens, ones, and hundreds place.

Next, a Class Number Grid Poster (see Figure 3.1) is used to review patterns and recognize special relationships between numbers in the grid ("Reviewing Number-Grid Patterns" which starts on p. 24). The text then presents a series of 14 prompts for the teacher to ask, such as "What happens to the numbers in a row as you move from left to right?" and "What do the whole numbers in a column have in common?" (p. 24). These questions focus on the way the numbers are organized in the chart, directing attention to how certain numbers (such as those that are adjacent vertically) are numerically related and how this relationship is represented in the numeric symbols used to write the numbers. Following these questions, the teacher is prompted to invite the students to compare some number sequences on a worksheet (completed in the prior lesson) with the number grid.

Class Number Grid

-9	-8	-7	-6	-5	-4	-3	-2	-1	0
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110

Figure 3.1. A Class Number Grid Poster used in Lesson 1-2 of EM3 (The University of Chicago School Mathematics Project, 2007, p. 24). Reprinted with permission from The McGraw-Hill Companies, Inc.

The third portion of the lesson ("Finding Missing Numbers on a Number Grid," which starts on p. 25) extends the notion of a number grid to an arrangement of "any-size consecutive whole numbers" (p. 25). Then, an example is given in the text to show how movement across and down the grid can help relate numbers ("For example, to find 13 more than 577, they can move down one row and then three spaces to the right" (p. 26)). The text prompts readers to then locate four numbers in the grid (587, 554, 580, and 596). This is followed by 12 additional challenges posed by finding numbers more or less than one given on the grid, such as “1 more than 565” and “20 less than 599” (p. 26).

The final activity of this lesson (“Solving Number-Grid Puzzles” which starts on p. 26), challenges the students to fill in missing numbers of various number grids for which only some of the numbers are given. Some show only partial grids, restricting some of the cells of the table to a select group that may only allow for certain numerical relationships. Two of these challenges contain no starting number, enabling a reader to complete the grid based on any desired set of numbers (as long as the numbers are placed with respect to each other in a manner consistent with the entire grid).

Mathematical layers. With the example mathematical story introduced, this section now articulates and distinguishes the metaphorical conceptualization of mathematical text, story, and fabula in order to establish a conceptualization of math textbooks as narrative that is consistent with Bal’s (2009) narrative framework.

Mathematical text. Of the three narratological layers defined by Bal (2009), the one with the most direct correspondence to mathematics textbooks is the *text* layer. Considering the paper-bound textbook as the *text* of a narrative, mathematics textbooks present mathematics content in an ordered sequence containing a variety of verbal (e.g., expository, questions, tasks) and diagrammatic (e.g., illustrations, figures, photos) forms. As described in Chapter 2, this framework assumes no fixed way in which mathematics textbooks are designed (such as including examples or having homework problems presented at the end). However, just as “chapter books” are partitioned into portions of narrative (chapters) that are intentionally sequenced, portions of mathematics textbook are similarly designed in basic units (sometimes referred to as “lessons” or “investigations”) that are aggregated into larger groups (sometimes called “units”, “sections”, or “chapters”) to be read sequentially.

Beyond the form of the text (i.e., the media), however, a literary story is told by a narrator, which Bal describes as a fictitious spokesman who tells the story. Bal frames the narrator as a reader’s construction, not the author, defining narrator as an agent who “relates, who ‘utters’ the signs” of the text (Bal, 2009, p. 9)[18]. Although literary stories can be communicated through a character in first-person narration (when a character tells the story from his or her own point of view), the narrator need not always be identified or named, such as when a story is told in the third person by an external narrator. Bal explains that regardless of the point of view, the voice of the narrator necessarily focalizes the elements of the text, which “colours the story with subjectivity” (p. 8).

As is thoroughly described in Love and Pimm (1997) and Herbel-Eisenmann (2007), mathematics textbooks are usually narrated in the third-person, which means that its directives and voice are given without an acknowledgement to the existence of an author[19]. This means that mathematics textbooks are often communicated with an authority that is to appear objective, even though authorial choices and point of view has colored its presentation. In addition, mathematics textbooks are written with a voice that distances the reader (student, teacher, or both) from the author, further making the claims in the text appear immutable. For example, in the EM lesson, consider the statement “You or a volunteer should record three or four of the largest and smallest numbers on the board” (p. 24). These directives come to a reader (who is

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[18] This is also sometimes referred to as the “implied author.”

[19] There are rare exceptions to this. One is a mathematics textbook called *Life of Fred: Fractions* (S. F. Schmidt, 2010), in which it is written “This is a very silly book. This is a book in which you, the reader, can talk back to me, the author” (p. 15).

assumed to be a teacher) by a narrator who makes no reference to himself or herself. If this sentence had instead started with "I, the author, recommend that you ...", the tone of the text considerably changes.

Therefore, for this framework, the *math text* layer of the narrative interpretation of mathematics textbooks is the media (paper-bound textbooks) and narration (third-person, external) found in textbooks.

Mathematical event and story. Before exploring the metaphorical correspondence of story and fabula with regard to mathematics textbooks, it might first be useful to discuss the basis of their distinction in literary texts: the chronology (sequence in time) of events. As described in Chapter 2, the chronology of a literary story (the ordered timing in which the events are presented to a reader) is not necessarily the same as the chronology of its fabula (the ordered timing of the events of the story as determined by a reader via logic and verbal clues such as "2 weeks ago"). Although a reader may learn of event B after event A while reading a story, he or she may figure out based on verbal clues and logic that B likely occurred before A in reconstructing the fabula (e.g., if B is presented in a flashback). In the analysis of narrative, this distinction is important because changing the sequence of events in a story can change their significance and effect on a reader, even though the fabula does not change. This raises two major questions with regard to the reading of mathematics textbooks as narrative: *What is a mathematical event? How is chronology manifested in a mathematics textbook?*

Assuming that the set of mathematical concepts and images generated by a reader when reading a math textbook changes, then mathematics textbooks offer a reader a chronological layer similar to that of the story layer. That is, a reader encounters and recognizes mathematical ideas[20] (e.g., mathematical objects, relationships, properties, and procedures) in a chronological sequence during his or her reading of a mathematics textbook. Marking points along this sequence are changes and transitions of the mathematical states of these ideas. For example, suppose that when first given the number grid of the EM lesson, a reader recognizes the order of numbers in a row as a sequence of integers (even if he or she does not know that word). After a teacher poses a question to compare how numbers change when moving down one row (a number increases by 10), a new relationship might be recognized. This recognition extends an understanding of the mathematical representation and, therefore, represents a change brought about by the comparison of numbers by this reader. Since literary events are changes throughout a story as the result of action by actors, then a *mathematical event*, likewise, can be conceptualized as a change in the mathematical ideas brought about by the action and experience of actors. Inherent in this definition, then, is that a mathematical event represents a transition from one mathematical state to another, and thus the initial and changed states have a mathematical relationship. For example, if a reader of the EM lesson finds a number that is "13 more than 577" (p. 26) by starting with the cell labeled 577, moving down one row, and moving to the right three to end at 590, then this reader has acted upon the starting value (577) by changing it to 590. This change in mathematical state, which in this case is the value of a number in focus, represents a mathematical event by an actor (the reader).

Metaphorically, then, the *mathematical story* layer attends to the chronological sequence of mathematical events throughout a textbook that is encountered and experienced by a reader. Broadly, these events must be meaningfully connected to help the reader understand "how each

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[20] The phrase *mathematical ideas* will be used throughout the rest of the chapter to refer to the milieu of mathematical concepts and processes encountered by a reader when reading a mathematics textbook.

event develops from what went before and leads to what will happen next" (Nodelman & Reimer, 2003, p. 62). That is, a mathematical story is more than just a sequence of parts; it is a sequence of transitions connecting the parts of the story from one given state (the beginning) to an end goal (the ending). For example, in the EM lesson, a sequence of parts can be summarized as the progression from a reader's generation of very large and small numbers, to a review that large numbers have digits with organization (place value), to the introduction (by the text) of a representation which organizes and relates very large and small numbers to each other (those that are integers), to an exploration of the relationships between the numbers in this representation, and so on. However, to identify the mathematical events of the lesson, and thus recognize the mathematical story, consider the question, "How, when each part is viewed as a transition from the earlier part, are the events connected?" When considering the relationship between very large and very small numbers to the review of place value, then it is reasonable to assume that the place value shows that no matter how large or small, all numbers are written following the same convention. A reader can also note similarities and differences between the numbers (perhaps they all have ones-place digits, but the large values may also have a digit in the hundreds- or thousands-place. With the introduction of the number grid, it can be noted that the grid spatially organizes numbers in a sequence starting from very small numbers to very large numbers (with respect to the mathematical story so far). So the change is the introduction of a new representation that connects these large and small numbers and introduces the sequence of numbers in between. With the new representation in place, the mathematical story continues with a sequence of explorations into the properties of the number grid and how the numbers within it are related to one another. Given its position in the mathematical sequence, this exploration includes finding relationships among numbers in the grid with respect to place value (e.g., adding 1 to a number that does not end in 9 increases the ones digit by 1).

Important to this conceptualization, then, is that a reader (such as a student or a teacher) is a *mathematical actor* of a mathematical story, an agent effecting mathematical change as he or she reads. For example, when a reader performs a mathematical task (e.g., adding 13 to 577 to find 590) or recognizes a mathematical relationship (e.g., determining that a number in the grid is always 10 more than the number directly above it), he or she is an actor effecting change on the mathematical phenomena in focus. In addition, a narrator may also "act" on the mathematical phenomena, when the text presents a sequence of transitions (such as a demonstration of a transformation of an object or an announcement of a new mathematical relationship). Extending the notion of mathematical story to include those mathematical sequences of changes found enacted in classrooms, then additional actors are likely to include the teacher and fellow students. Note that this recognition of the reader as a mathematical actor is special to mathematical stories. In literary stories, an actor is a character (such as Romeo) who acts to change the story (such as drink the poison). Therefore, the character/actor distinction of mathematical stories is more similar to that found in plays, where human actors embody characters and perform actions and where the voice of a narrator may advance the scene from offstage.

Therefore what does a mathematical story look like? As an experience of a reader, it is an accumulation of his or her temporal mathematical experiences and realizations. One way it can look takes its inspiration from Sinclair's (2005) reading of a proof (described in Chapter 1 of this dissertation). For my reading of the EM lesson (which, as explained in Chapter 2, necessarily will differ from that of other readers), this mathematical story starts:

> There are some numbers that are very large and others that are very small in comparison. I wonder what I can learn about these numbers? All are written in a special

way (with place value) that tells me how many hundreds, tens, and ones can generate the value of the number – also, each digit of the number has a name based on its position. Large and small numbers, with others between, can be organized on a number grid – and I see a familiar set of numbers (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) in the second row. The smallest number in the grid is –9 and the largest is 110. I can think of numbers bigger than that! Comparing the rows, I see that each has 10 numbers.

How are these numbers related to each other? I notice that if I move from one number's square (that does not end in 0) to the right, the number increases by 1. Likewise, if I move from one number's square (that does not end in 1) to the left, the number decreases by 1. But what happens if I am at the end of a row? Well, if the number ends in 0 and I want to get to the next number, then I have to move down a row and all the way to the left. The opposite happens when I want to decrease by 1 a number that has a 1 in the units place.

It turns out that besides being in order, the numbers in the grid are also related with those in the same column or row. In fact, the positive numbers in each column have the same units digit, so they form a family of sorts (similar to how family members may share a trait). The whole numbers in the first row are also all negative (except 0 at the end). I can imagine that this grid is really a giant a number line that has been subdivided into groups of ten numbers and stacked atop the next. Another relationship emerges when you move from one number to the number below (or above) it – it increases (or decreases) by 10. There is that number again. I wonder if it is significant?

Moving on, I notice that not only do the numbers change based on how I move about the number grid, but also if I decide how to increase or decrease a given number, I can figure out its new value by moving. I can add 1 by moving 1 to the right (unless I am in the right column, in which case I move to the start of the next row down). So, if I want to add 10, I can move one cell down in the grid. What else can I use this grid to add?

To summarize, the *mathematical story* is defined as an ordered sequence of connected mathematical events encountered by a reader. Inherent in this definition is the need for a reader of a mathematical story to make progress from a beginning state of mathematical ideas toward an ending goal. A reader and narrator are both actors in effecting the mathematical changes.

Mathematical fabula. Although recognizing a chronological story layer of mathematics textbooks allowed a metaphorical correspondent to literary story to be defined, a second chronology with respect to mathematics textbooks corresponding to the literary fabula layer is difficult to identify. According to Bal (2009), a reader re-constructs the literary events beyond the story using logic of chronology. Metaphorically, then, the mathematical fabula should represent a reader's logical re-construction of the mathematics events beyond the text and story layers. Therefore, interpreting mathematical textbooks as narrative raises the critical question: *What is the metaphorical fabula in the reading of mathematics texts?* If mathematics textbooks are to be interpreted as narrative, then Bal's framework suggests that there should be a metaphorical equivalent and without one, the interpretation of mathematics textbooks as narrative is problematic.

Compounding this challenge is the fact that unlike literary stories, the mathematical ideas dealt with in math textbooks (such as the relationship between the values of numbers vertically adjacent in the number grid) are assumed to be timeless[21]. That is, the mathematical objects, properties, and relationships are encountered and interpreted by a reader in the eternal present, independent of time[22]. For example, though a reader may recognize that there are 10 numbers in each row of a number grid and note that each number is 10 more than the number directly above it in the grid, which of these properties necessarily comes first? I argue neither; that is, a number grid could be generated starting from either condition, which then would lead to the manifestation of the other property. Unlike a literary story, which might require a reader to use verbal or situational clues to recognize that the events described in the story must have occurred in a different sequence, a mathematical story makes no such demand on its reader.

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[21] There is also a historical ("genetic") chronology that relates mathematical concepts with respect to time (e.g., how the number "one" was used long before the number "zero"). However, since the historical chronology of mathematical development is rarely included in mathematics textbooks, it is reasonable to assume that this relationship would not be available to a reader without outside resources (and thus, it is not part of this story).

[22] Another interpretation is that the mathematical ideas found in texts refer to an infinite span of time, a Platonic view of a permanent realm of mathematics with existence independent of humans. This framework does not support this view (given the assumed nature of interpretation in the reading of texts) and therefore, frames mathematical claims instead as without reference to time. Although an extended discussion of the epistemological consequences of this distinction is important, it is outside the scope of this project. Therefore, I direct readers instead to a useful discussion of these philosophical stances toward mathematics in Ernest (2008a, 2008b, 2008c).

However, even without a second chronological referent (such as an index to the past or future), a reader can logically relate the mathematical events through the further recognition of mathematical relationships and connections beyond the mathematical story. That is, while reading a mathematical story, a reader may recognize that a later revelation can be used to logically support a prior assertion. For example, a reader may logically determine that the fact that each number in the number grid is 10 more than the number directly above it then can be used to justify that each row must have 10 numbers[23]. Therefore, the re-construction of mathematical events while reading a mathematical story (the *mathematical fabula*) is not based on time, but rather on the logical support for a mathematical concept.

In general, if a mathematical story has events in the sequence of A-B, it is possible that after learning B, a reader figures out that B is a condition for A is true, in deductive terms. Interestingly, then, there is a possible re-sequencing effect when building a mathematical fabula similar to the logical re-sequencing of the chronology of a literary story; that is, through making sense of a mathematical story, a reader may determine that event B comes before event A in a thread of logic in the same way that a condition ("if") precedes the conclusion ("then"). However, I caution that although this example of Aristotelian logic nicely maps the mathematical fabula closer to its literary metaphorical correspondent (in terms of sequence), it unnecessarily limits the possible manifestations of the fabula. Noticing how an algebraic generalization (e.g., $x^2 - y^2 = (x + y)(x - y)$) can justify an arithmetic pattern, perhaps recognized years earlier (e.g., $99 = 10^2 - 1^2 = (10 + 1)(10 - 1)$), is one of numerous possible examples. Logically going beyond a

[23] This is not to say that this particular conclusion will be made by an elementary student (which is not the point of this theoretical work). Instead, this is an argument that readers of a mathematics textbook (which includes students) can make connections and draw relationships between mathematical phenomena beyond the text and story layers, whatever those may be.

story can affect more than just deduction, but instead can include defining, noticing a pattern, predicting, and conjecturing beyond what is present in a mathematical story. In addition, any time in which a reader confronts a mathematical statement that conflicts with a previous statement in a way that requires the logical renegotiation of prior mathematical understanding, the mathematical fabula is involved.

Representing the mathematical fabula can be achieved in a variety of ways, each with a loss of information. However, using the notion of a concept map, the timeless mathematical ideas generated from my reading of the EM mathematical story (provided in the previous sub-section) is given in Figure 3.2.

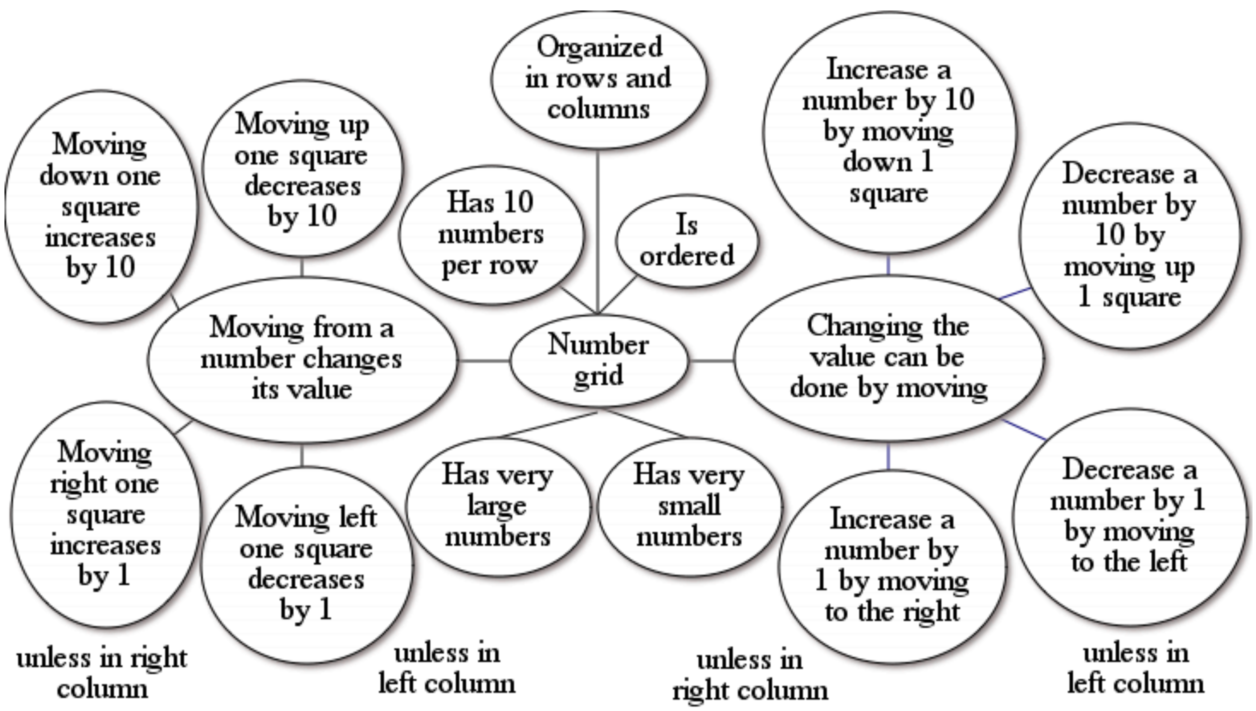

Figure 3.2. Example of a mathematical fabula, representing the mathematical ideas generated by my reading of EM Lesson 1.2 (The University of Chicago School Mathematics Project, 2007).

To summarize, the mathematical fabula is not re-constructed on the grounds of chronological time, but instead using a logic of justification. Since there are many deductive threads that can lead to the same conclusion, then the mathematical fabula is the reorganization of the logic around how certain mathematical ideas support or connect the meaning of other mathematical ideas. For example, if a student who is learning about the addition of whole numbers starts to memorize addition facts (e.g., $2 + 3 = 5$ and $4 + 1 = 5$), these facts may start by being justified through counting collections of discrete objects. Later, when this student learns about the commutative property, then it may be concluded that for every addition fact in which the addends are different, there is another addition fact that is true. Still later, if the mathematical idea of the associative property is introduced (in terms of decomposing and recomposing), then this same student may notice that it supports the notion of addition through regrouping (e.g., if $2 + 3 = 5$ is true, then so is $2 + 2 + 1 = 5$, and therefore, $4 + 1 = 5$).

The recognition and reorganization of the relationships between mathematical ideas (in this case, the addition facts) is similar to rebuilding chronological relationships between temporal events in a literary story. However, the resulting mathematical fabula is atemporal in nature. This conceptualization of a mathematical fabula of curriculum draws upon the goals of the conceptualization of *curricular map*[24] discussed in Chapter 1: a logical construct identifying connections and relationships between elements of mathematical content (including objects, properties, concepts, and procedures).

Since discussions of mathematics curriculum often focus on expected mathematical outcomes (e.g., what is expected to be learned and understood as an outcome of reading the textbook), then the distinction between mathematical story and mathematical fabula becomes even more important. I propose that, in discussions of mathematics textbooks, the primary focus is actually attending to the mathematical fabula layer of curriculum – an atemporal mathematical

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[24] It should be noted that this definition of mathematical fabula avoids the platonic assumptions sometimes found in discussions of mathematics curricula as it is explicitly a result of a reader's construction through reading and is not an external fixed mathematical construct that the reader perceives.

residue beyond the story and text. Objectives like "In this lesson, students learn the Commutative Property" focus on mathematical procedures and concepts yet reveal little about how they are expected to emerge through a reader's experience with the mathematical story. One of the purposes of this theoretical exploration of mathematical stories in mathematics textbooks is to draw attention to the mathematical story layer that has been previously neglected: that of describing the sequential unfolding of mathematical ideas for a reader.

Since the mathematical fabula generally represents the mathematical outcomes of mathematics textbooks for a reader, Bal's distinction also helps to reveal how different mathematical stories might result in different mathematics. For example, consider the illustration of how the students of the *Measure Up* program (discussed in Chapter 1) responded to the inquiry "is $3 < 8$?" It is reasonable to assume that the mathematical fabula constructed by a reader of *Measure Up* would look dramatically different than that of curricula that starts with a focus of number and operations, especially involving the relationship of numbers and measures and how these constructs support (or not) later mathematical constructs (such as generating algebraic expressions).

Summary. Bal's (2009) layered framework offers a distinction between a reader's temporal experience with the emerging content (story) separate from his or her logical re-constructions of the content of the story (fabula), while acknowledging the importance of the selected media. As a metaphorical construct, the *mathematical story framework* then distinguishes between the *text* of a math textbook (e.g., sentences, symbols, diagrams, narrator), the logical mathematical relationships and properties constructed by a reader independent of the sequence in which they are revealed (the *mathematical fabula*), and the manipulation of the fabula that results in a temporal revelation of the mathematical properties and relationships as this reader reads (the *mathematical story*). Thus, Bal's Condition #2 for texts to be classified as narrative (that "We can distinguish three layers in a narrative text: the text, the story, and the fabula. Each of these layers can be described." (Bal, 2009, p. 9)) is met. Consistent with the framing of literary story in Chapter 2, a mathematical story is considered complete if it is a continuous portion of narrative with an identifiable launch of an experience or event (such as the opening of a chapter or lesson or task) and a recognizable point of completion or resolution.

Mathematics textbook as narrative. Returning to Bal's (2009) framework, there are two additional conditions that need to be met in order for a text to be considered a narrative. This section completes the argument by addressing each of these conditions separately.

Speakers of text. Bal's Condition #1 states that ideally, in narrative:

> Two types of 'speakers' utter the signs that constitute a narrative text; one does not play a role in the fabula whereas the other does. This difference exists even when the narrator and the actor are one and the same person as, for example, in a narrative related in the first person. The narrator is the same person but at another moment and in another situation than when she originally experienced the events. (Bal, 2009, p. 9)

Before turning to mathematics textbooks, this discussion first starts with considering this condition within literary narrative. In her second sentence, Bal articulates that two speakers of signs can be the narrator (whom Bal associates with the way the story is told, and thus the text layer) or an actor (whom Bal associates with the agency of events negotiated in the fabula layer), although these can be the same person, as in first-person narratives. Narrators "utter" in the sense that a story is "told" (e.g., Melville's "Call me Ishmael" or Baum's "And so, with Toto trotting along soberly behind her, she started on her journey").

The narrators of mathematics textbooks similarly speak to readers. For example, when a mathematics textbook contains the directive such as "Do your own" (Student Math Journal, 3rd grade, The University of Chicago School Mathematics Project, 2007, p. 3), the narrator speaks directly to a reader and even acknowledges him or her with "your." Other examples include questions (such as "What happens to the numbers in a row as you move from left to right?" and statements (such as ""When you move right, the numbers increase by 1" (on the Math Masters, p. 9)). These types of utterances are commonly found in mathematics textbooks, although different texts will have different ways of addressing readers and will sometimes vary to whom they are addressing (teacher or student). However, narration in mathematics textbooks is not limited to verbal expressions. In presenting the Number Grid, the narrator is also "uttering" an ordered set of signs to communicate relationships between numbers. Even an incomplete number grid, such as the one in Figure 3.3, can be read as the start of a story, such as "There is a part of a number grid with a number 54 in the center. The grid squares above it, below it, and on its left and right were removed, leaving only those diagonal from it. The numbers in these diagonal grid squares are unknown."

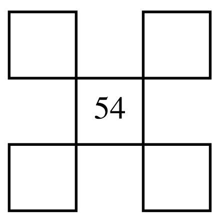

Figure 3.3. A number-grid puzzle adapted from the Student Math Journal of EM3 (2007, p. 3).

Therefore, if a narrator is one speaker of the signs uttered in a narrative text, this raises the question: *What might be another type of speaker of a mathematics text*? As was discussed earlier, actors of mathematics textbooks are those with agency to effect mathematical change (thus, generating a mathematical event) within a mathematical story. When a reader acts with regard to a mathematics textbook (e.g., solving a task, raising or answering a question, making a connection), he or she can be viewed as an actor in progressing the mathematical change of the phenomena of study. As a reader reads, the interpretation of signs (within and beyond the text) can be viewed as an utterance[25]. An example of this is my reading of the number grid in Figure 3.3. As I read this diagram, I utter the signs in the text. However, beyond the text layer, I can continue to act on the grid, such as, "Moving up one row from 54 makes the grid square directly above 44. Then 43 must be to its left in the upper-left corner," and so on. This speaking of the signs of the text are not that different than the narratives produced by students in Healy and Sinclair (2007), although admittedly in that study the students were reading a very different text/media (dynamic geometry software) and were orally narrating their solutions.

With the identification of two types of speakers who utter the signs of a mathematics textbook, Bal's Condition #1 is met.

A series of connected events. Bal's (2009) Condition #3, which states that "That with which the narrative text is concerned, the 'contents' it conveys to its readers, is a series of connected events caused or experienced by actors presented in a specific manner," highlights the fact that a narrative cannot just be a description or a character. That is, <u>something happens</u> (action by actors) which enables the story to make progress (event). For mathematics textbooks, the connection of events was already discussed, but is summarized here to complete the argument that mathematics textbooks can be read as narrative, and thus, as mathematical stories.

It is assumed that the mathematical content in textbooks changes throughout a sequenced portion of text. These transitions occur in mathematical events that are caused or experienced by actors. Nearly everywhere you look in a mathematics textbook, a reader is called upon to act on mathematical objects, whether it is to simplify, rename, compare, transform, or any of the other action verbs sprinkled throughout. In addition, the narrator may also act on mathematical objects

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[25] This is consistent with Sfard's Commognitive Framework (2008), which defines thinking as "individualization of interpersonal communication (the process of communicating between a person and herself, one that does not have to be verbal)" (p. 302).

in the text (such as in a worked example). These actions result in mathematical changes that form mathematical events, which then together forms the content "conveyed to its readers." Therefore, it is argued that the mathematical content of a mathematics textbook is found in a series of connected mathematical events (as defined in this chapter) caused or experienced by actors presented in a specific manner.

Therefore, with all three of Bal's conditions met, my argument that mathematics textbooks can be interpreted as narrative is complete.

Framing Descriptive Aspects of Mathematical Stories

As explained in Chapter 2, stories have additional elements that have, since Aristotle, assisted in their analysis and comparison. These aspects, which includes characters, actions, settings, plot, and morals, address the who, what, how, why, and where of a story. In this section, each of these dimensions is metaphorically theorized and illustrated in relation to mathematical stories. Although mathematical action was defined earlier, it is included here to extend its discussion within this framework with additional illustration.

Mathematical character. As mentioned in Chapter 2, Bal (2009) defines literary characters as "anthropomorphic figures with specifying features the narrator tells us about" (p. 112). Although the use of "figure" as a noun certainly offers specialized meaning in mathematics, its colloquial use has similar meaning, since according to the Oxford English Dictionary means "form, shape," "an attribute of body," "a numerical symbol", "an amount," and "a number." Thus, the notion of character is firmly rooted in the notion of shape (space) and quantity (number and measure), the focus of much of the mathematical investigation in K-12 education. Therefore, metaphorically, it is reasonable to expect that *mathematical characters* are the forms, shapes, and quantities brought into existence (objectified) through reference (e.g., naming, defining, or otherwise drawing attention to it) in the text (similar to the observation made by Love & Pimm, 1997, discussed earlier). That is, the mathematical objects that are the focus of mathematics textbooks, such as the number ("3") or a line segment ($\overline{AB}$), can be interpreted as the characters of a mathematical story.

In terms of mathematics, Bal's reference to anthropomorphism may make recognizing mathematical characters in mathematics textbooks challenging for some readers. However, Bal does not restrict the notion of character only to humans (clearly there are many narratives without humans), and instead emphasizes that part of the way the story layer works is that readers interpret the characters as "human-like" and assign them human characteristics. Analogically, this means that it may be useful to think of mathematical characters as those objects in the mathematical story that can be related to by a reader. For example, as a reader moves from one cell in the number grid to another in the EM lesson, is this movement between objects to other motions between people in his or her life (perhaps, the motion of moving from one classroom down a hall to another?). Also, a reader might associate adding 10 as a convenient move between rows, and possibly view 10 as a hero of the story. That is, how might reader relate the object (such as a geometric figure) to a physical aspect of the world? Examples of anthropomorphizing of mathematical objects, such as the student references to a dancing body in Healy and Sinclair (2007) and "kissing triangles" (Sinclair, 2002), are instructive.

It is important to note that like in literary stories, mathematical characters can have multiple characteristics (which are often called "properties" in mathematics) and can be identified by multiple names. Even after a central mathematical character is formally defined, it is usually further developed with additional characteristics and uses. For example, after a number (such as "3") is introduced into a mathematical story, a reader may learn that it is odd.

Usually much later, this reader will likely learn about another special property of some numbers (including 3), referred to as prime. In terms of a mathematical story, this is referred to as *character development*, and explicit attention to how characters are introduced and developed can offer insight into a critical way that mathematics textbooks work to reveal relationships and properties of characters[26].

In the case of literature, different characters can also be related along different qualities or dimensions (Bal, 2009). For example, it is assumed that Romeo and Juliet are not related in a familial sense (before they are married in the story), but that each is related to a family of other characters. Yet, on a dimension of love, Romeo and Juliet form a relationship that ultimately results in marriage. Likewise, in mathematical stories, different characters might have relationships that depend upon the qualities in focus. For example, the numbers 3, 4, and 7 are related because $3+4=7$. However, when looked at by a different quality, the numbers 3 and 7 are related by oddness, while 4 is even. Just as in fiction, the way the relationships between the mathematical characters are revealed depend on what is the focus of the mathematical story.

The fact that multiple names can refer to the same character can lead to some interesting story developments. Consider the 1982 movie *Tootsie*, where the main character is a male actor (Michael) who creates a female alter-ego (Dorthea) and successfully gets a part in a soap opera (as Dorthea). Meanwhile, an actress in the soap opera (Sandy) knows Michael and Dorthea separately, building a good friendship with Dorthea while despising Michael. At a pivotal moment at the end of the movie, Sandy learns that Michael and Dorthea are just different facades of the same person. Recognizing this aspect of character in literary stories can help to generate new and important questions of analysis in terms of mathematical stories. For example, how

[26] This discussion is extended in Chapter 4.

might a mathematical story (intentionally or unintentionally) similarly focus separately on characters that, on the surface, appear to be different (such as $\frac{1}{2}$ and 0.5) and later reveal them as the same character? How might reading for the mathematical story help a teacher or math educator (who presumably already knows that the multiple labels refer to the same character) recognize why the novice reader might not read these as the same? How might the characteristics of one or more mathematical characters help a reader predict or anticipate the future of the mathematical story or the characteristics of other characters (such as assuming that all prime numbers are odd)?

Identifying the mathematical characters in a mathematics textbook involves reading for the mathematical objects in focus. For example, in terms of the EM lesson, it is useful to ask *What mathematical objects are being talked about? What effect on the mathematical story, if any, would changing these objects have?* As shown in the Math Message and repeatedly throughout the lesson, the notion of number and the relationships between numbers (e.g., larger, smaller, same units digit) is the focus. In some ways, the particular numbers used in the number grid do not have a direct effect on the mathematical story in that relationships between the numbers in the rows and columns could be found whether the number grid started with –9 or with 91. However, regardless of the numbers in the number grid, the mathematical characters 1 and 10 can emerge and remain in focus because the difference between the numbers in the rows and columns would remain the same. Thus, in contrast with a literary character, a mathematical character does not need to be explicitly named in the lesson for it to be part of the mathematical story (for example, if the number grid had not included 1 or 10). Mathematical characters can also be provoked by the text through the solving of tasks or the reader's consideration of questions.

Mathematical setting. In Chapter 2, literary setting was defined as the descriptive space in which the literary characters are located. This raises the question *In a mathematical "world" invoked in mathematics textbooks, in what space do mathematical characters "live"?* In the EM lesson, the mathematical characters (numbers) were found in a number grid: a spatial representation of integers. Similarly, a mathematical story about a triangle might be told in a plane, while one involving a linear function might play out in a table of values. Metaphorically, then, a *mathematical setting* in a mathematical story can be conceptualized as the mathematical representation in which mathematical characters "live." Each of these representations creates a space in which the mathematical characters are found and acted upon.

The choice of setting in a narrative both enables and constrains what is possible in a story and provides clues to a reader about what types of actions to expect (Bal, 2009). For example, a horse-drawn carriage ride is possible in New York City's Central Park but highly unlikely in the subway. However, a setting also can constrain a story by preventing certain characters from interacting (for example, locating a scene inside the house of Montague in *Romeo and Juliet* would prevent the participation of characters of the Capulet family). Turning attention to the EM mathematical story, the setting of the number grid similarly supports particular mathematical actions (such as adding 10 and multiples of 10 quickly) and enables certain characteristics to be noticeable (such as place value). A study of settings in mathematical stories can reveal how representations limit or expose the development of certain aspects of mathematical characters and could help reveal how the changing of setting might offer either new challenges or new insights for a reader. For example, if the setting of the EM mathematical story changed to a number line (instead of a number grid) the number 10 would likely not emerge as a central character.

Note that the notion of context and space used here represents more than the notion of a "real life" word problem and includes all forms of representations in which a mathematical action occurs (defined in the next section) and mathematical characters are found. However, a word problem certainly constructs a setting in which the mathematical characters are set and acted upon. For example, in the problem "José had $76 in his bank account. He withdrew $29. How much money was in his bank account then?" (The University of Chicago School Mathematics Project, 2007, p. 125), a space is constructed in which the character $76 is acted upon along with a fictional actor. This setting limits the potential actions of the characters based on the prior experiences of readers with the setting.

Examining mathematical settings in the conceptualization of mathematical stories may bring to light the way context works to affect the reading. As was argued in Chapter 1, in the case of the *Measure Up* program, the setting of the question "is 3>8?" is called into question when considering the students' responses of "it depends." Without considering the setting in which the mathematical work is placed, a reader is left to construct his or her own setting which in part helps to explain why an adult might find the student responses surprising. A question such as "what is the sum of the angles of a triangle" would evoke to most middle school math teachers (and presumably students) the response "180°." However, as Paul Lockhart's (2009) comment about the importance of mathematical structures as a context (reprinted again below) demonstrates, the setting of the mathematical story heavily influences the meaning taken from a mathematical statement or question:

> Mathematical structures, useful or not, are invented and developed within a problem context, and derive their meaning from that context. Sometimes we want one plus one to equal zero (as in so-called 'mod 2' arithmetic) and on the surface of a sphere the angles of a triangle add up to more than 180 degrees. There are no "facts" per se; everything is relative and relational. *It is the story that matters, not just the ending*. (Lockhart, 2009, p. 17, italics added)

Therefore, a study of setting in relation to the mathematical story helps to situate the mathematical characters and action. If the mathematical story up to a point is set in 2-dimensional space, then it is reasonable to assume that a question about the sum of the angles of a triangle would be interpreted within that setting. Recognizing the influence of setting might help teachers recognize why adding the qualification of "in a plane" when stating that the sum of the angles of a triangle is 180° may not have any impact on a reader, who might question why that needs stating since most everything else in the study of geometry has been constrained to 2-dimensional space.

Setting might also become a conceptual construct that offers opportunities for the development of more interesting stories (more on plot will be said later). Consider the example in Flatland[27] (Abbott, 1884), where a central character (a square) "lives" in a 2-dimensional world. In that story, all shapes are limited in understanding their "reality" (which is all constructed fiction) as 2-dimensional, not aware of the possibility of other settings. When the central character gets propelled into 3-D space, the notion of what it means to be a 2-dimensional shape changes (not just for the character, but arguably also for a reader as well). This dramatic and unexpected shift in setting re-orients everything that came previously in the mathematical story, particularly the noticeable characteristics of the mathematical characters. Thus, changes in settings advance a mathematical story and thus create different *mathematical scenes*. The Flatlands example is also instructive, considering Lockhart's remark noting that the sum of the angles of a triangle on a sphere is not 180°. In a mathematical story, shifting the setting from studying triangles in 2-dimensional space into 3-dimensional space could offer a reader a similar

[27] The book *Flatland* offers a rich example in many ways – particularly in terms of mathematical action, plot, point of view, and certainly thematic message (as it aims at making a larger statement about the hierarchical nature of society at that time). Its setting, however, is perhaps the most focal element of the story (as seen in the title) and it is instructive here in this regard.

experience of surprise and re-orientation. Coming to understand how mathematical settings affect a mathematical story offers a possible window into imagining new mathematical stories and recognizing how some stories work to capture the attention of a reader.

Mathematical action. As described earlier, a *mathematical action* in a mathematical story can be interpreted as a manipulation of a mathematical object, such as a number or shape. Examples of mathematical actions include transformations (such as translating a function on a coordinate plane), decomposition or composition (such as breaking a trapezoidal region into a rectangular and two triangular regions), and operations (such as adding two numbers). What role(s) does the action play in a mathematical story? And how might changing actions change the mathematical story?

Reading for action is generally reading for transition, then, considering the questions *What is happening to the central characters?, What is being done?, What changes?* and *How does it change?* In the case of mathematics curriculum, some actions may be more integral to the mathematical story than others. For example, when learning about finding the area of a complex shape, the decomposition and rearrangement of regions into a new shape is important in changing the temporal story; prior to this act, the region may have been unmeasureable by a reader/actor. Other mathematical actions, however, may not have a major effect on moving the mathematical story forward, such as the repetitious practice of a procedure[28].

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[28] This is not a critique of repetition or practice, but instead allows for a recognition that the mathematical story may not move forward through it. This statement is also not all-inclusive; there are ways that exercises that appear routine can change the mathematical story. For example, in a setting where whole numbers are represented by manipulatives (such as counters) and addition is the merging of sets of counters, a list of exercises for the reader such as $2 + 4$ might invite no new ways of understanding the mathematical characters, setting, or action in the story. However, the inclusion of an exercise, such as $0 + 0$ might spur a reader to reconsider what it means to add (in other words, change the mathematical action).

With regard to the EM lesson, the main focus of the lesson appears to be on action. By asking the questions summarized above, it can be seen that the sequence prompts for actions of increasing sophistication. For example, in the activity entitled "Reviewing Number-Grid Patterns," a reader is asked about the location of several numbers, prompting for reasoning from the numbers that are given in the grid. In most cases, there are multiple ways a reader can do this. However, for increases (and decreases) of 1, a reader can simply move 1 square to the right or left of the starting value. This basic action of moving to the adjacent square to increase (or decrease) a value by 1 enables a reader to find any square's value in a particular row. In fact, the text supports the development of this particular action by asking as its second prompt: "What happens to the numbers in a row as you move from left to right?" (2007, p. 52). The text then complicates this action by considering what happens when starting from the right-most column (or left-most column, in the case of decreasing by1), asking, "How do you get from a number at the end of a row to the next number?" (2007, p. 52) This question requires a new action, namely, moving down a row and back to the left-most column.

Though cumbersome, these two actions can help a reader establish the value of any number of the number grid for this activity. However, a question "What happens if you add 10 to a number?" (2007, p. 53) offers an opportunity of a new action. Counting squares to the right and moving down one row can locate the number that is 10 greater than the starting value. Thus, a reader may notice that this square is directly below the starting value. Since the question does not specify a starting value, it encourages a reader to consider the general situation and describe the overall outcome (that the ending value is directly below the starting value). Although this could become a new action (adding 10 can be done by moving down one row), this is not necessarily the result.

However, the mathematical story again complicates the task, which further encourages a reader to develop new actions. This activity, "Finding Missing Numbers on a Number Grid," removes squares from the grid (see Figure 3.3), thus making it very challenging to use the counting by 1 method. Now confronted with the challenge of finding values in grid squares diagonal with or above a given square, a reader now is presented with an opportunity to use the relationship between adjacent rows to create a new mathematical action. From this analysis, it can be seen that the mathematical sequence encourages the development mathematical actions of increasing sophistication based on what is being added to the starting number.

It should be noted that sometimes the results of mathematical action can also be recognized as a character as well. For example, the expression $x+3$ can be viewed as the action of adding three to a number *x*, but can also be viewed as an object, namely the expression $x+3$ (Sfard, 1991). This ability distinguishes mathematical stories from literary stories. This ambiguity in mathematics offers multiple ways of reading the actions and characters of mathematical stories, and requires careful attention to how the mathematical story frames the object or action. For example, does the setting influence the reading of $x+3$? How does the mathematical story change if $x+3$ is an action versus when it is a character?

Mathematical plot. In addition to character, action, and setting, a story can temporally affect a reader as it unfolds, in ways referred to as its plot. The notion of story is strongly connected to plot, and in some discussions of narrative these words are interchangeable. Although a *story* is defined by Bal (2009) as the manipulation of the fabula, a *plot* refers both to a story's structure connecting beginning to end and its temporal aesthetic, such as provoking suspense, desire, or disinterest for a reader. Due to its complex nature, this section contains an extended discussion and is subdivided into three parts: (a) an elaboration on the tension between a reader's realization and anticipation (introduced in Chapter 2), (b) an introduction of the notion of temporal dynamics[29] and definition of plot, and (c) a conceptualization of mathematical plot.

Bal's tension between expectation and realization. Plot is fundamentally linked to sequence, as changing sequence can have a strong effect on a reader. As Bal explains,

> Playing with sequential ordering is not just a literary convention; it is also a means of drawing attention to certain things, to emphasize, to bring about aesthetic or psychological effects, to show various interpretations of an event, to indicate the subtle difference between expectation and realization, and much else besides. (2009, p. 81)

Bal's (2009) reference to the difference between expectation and realization emphasizes the temporal relationship of a reader to the story and the tension between what is guessed at (anticipated) and what is known (or believed) by this reader at different points in the sequence. Bal explains that as a reader reads a story, he or she looks for clues about what is to come. These clues may be the result of direct statements (such as when the Good Witch of the North tells Dorothy that the Wizard of Oz can use magic and can help her get back to Kansas) or they may be the result of a reader's extension of patterns noticed throughout the story. At various points of the story, facts are "realized" by the reader, meaning they are determined through reasoning or direct means to be true. Realizations by the reader are momentary and can change throughout, but include the reader's determination of the facts of the story. Consider, for example, how the description of the Wizard of Oz at the beginning of the story allows a reader to realize (i.e., come

[29] Since this dissertation is not an effort to develop literary theory and instead seeks to borrow from literary theory a conceptualization of narrative for the purpose of learning more about mathematics curriculum, I have narrowly focused this discussion on conceptualizing plot and how one can read for it. Although entire volumes can be (and are) devoted to the discussion of the plot of a single narrative text, the purpose of this section instead is to ground the notion of mathematical plot in literary theory. Therefore, this section lays the foundation for the argument that if mathematical textbooks are read and interpreted as mathematical stories, then the notion of mathematical plot offers a useful and potential powerful conceptual tool for recognizing both the structure (i.e., its sequence) and aesthetic (i.e., the reader's response) of mathematics curriculum (which is further discussed in Chapter 5).

to know) that the wizard is able to do magic[30] and therefore enables him or her to anticipate that the wizard will be able to help Dorothy go home by the end of the story. When Dorothy and her friends arrive in Oz and learn (i.e., realize) that the wizard is a fraud, this realization violates this expectation and allows for a reader's surprise (and therefore, a reader may feel that the plot has "turned"). Thus, the interaction of a reader's expectation and realization throughout a story explains how a sequence can evoke an aesthetic response because delaying realization offers the possibility of intrigue and wonder, while enabling expectation allows both surprise if an expectation is violated and relief and satisfaction when an anticipated result (such as the confirmation of "who done it") is met.

Therefore, a reader constructs a story as he or she reads, whenever possible renegotiating meaning for prior parts of the story (for example, recognizing by the end of the story that the Munchkins and the Good Witch of the South were misinformed at the beginning of the story) and anticipating what is to come based on what has happened so far in the story. Nodelman and Reimer (2003) explain it this way: "Making predictions ... is part of the basic strategy readers typically apply to all stories: the idea that they are coherent, that everything in them relates in some way to their overall effect" (p. 56). Thus, making sense of the story includes looking for the connections and interrelationships (or lack thereof) between the parts of the story, even when these are not made explicitly. Nodelman and Reimer continue:

> As people read, they constantly imagine reasons for details they learn about, and they either dismiss and forget those reasons or remember and build on them. They formulate questions about what they've read so far and assume the text itself will answer them later. (Nodelman & Reimer, 2003, p. 56)

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[30] From the story: "Oz himself is the Great Wizard," answered the Witch, sinking her voice to a whisper. "He is more powerful than all the rest of us together."

Therefore, Nodelman and Reimer extend Bal's framing to suggest that a reader's response to a text (its aesthetic) is rooted in the making sense of the story, particularly in connecting the parts of the story and asking questions of the text.

Interestingly, although Bal's (2009) framework discusses sequence of events, it stops short of defining the notion of literary plot in relation to the story and fabula layers. Instead, her focus is on describing in general a "narrative system" that can be used to recognize and distinguish elements of the narrative. Thus, her analysis centers on the elements of narrative and not its aesthetic (i.e., a reader's response to it). To define plot in the next section, therefore, I will need to move beyond Bal's framework and connect back with it.

Connecting literary plot with Bal's framework. As was described in the previous section, the Bal (2009) framework well addresses the notion of structure, specifically the layers of narrative, the interrelations of fabula and story, the role of sequence, and a reader's sense of a whole from the interrelation of its parts. However, the in-the-moment aesthetic response by a reader to the parts in relation to the whole (e.g., surprise or anticipation) is neither the story nor the fabula. Brooks (1984)[31] takes major issue with this gap in understanding how the changing forces work on a reader throughout a sequence of narrative, which he calls "temporal dynamics." Instead, he proposes the focus to turn toward how the narratives "work on us" (1984, p. xii). Defining plot as "the temporal dynamics that shape narratives in our reading of them, the play of desire in time that makes us turn pages and strive toward narrative ends" (1984, p. xii), Brooks attends to the compelling nature of narrative (that which "makes us" keep reading). Brooks admits that, in fact, his interest lies in the notion of *plotting*, which he describes as

[31] Although it may appear that Brooks's critique came decades before Bal's framework, they were contemporaries. The original version of Bal's framework was first published in 1985.

> ... the activity of shaping, with the dynamic aspect of narrative – that which makes a plot 'move forward,' and makes us read forward, seeking in the unfolding of the narrative a line of intention and a portent of design that hold the promise of progress toward meaning. (Brooks, 1984, p. xiii)

To understand the forces upon a reader, Brooks' analysis draws from psychoanalytic theories (in particular, Freud's model in *Beyond the Pleasure Principle*), which focus on aspects of a reader's emotional response (particularly, his or her desire) throughout text.

Bal (1986) responds to Brooks' criticism with intrigue, praise, and some complaint. In general, Bal welcomes the extension of focus on the affective. However, in her critique of Brooks (1984), Bal proposes that plot is neither the story nor the fabula, but instead can be viewed as the story's effect on a reader as he or she moves between the story and fabula layers. Using the term *sjuzet*, a term from Russian formalism for "story," Bal asserts, "Plot, then, is neither fabula nor sjuzet, but the work of sjuzet on fabula" (1986, p. 558).

Therefore, although the notion of plot is strongly connected to that of story, for the purposes of this mathematical story framework, *plot* will describe a reader's temporal response(s) to the story (e.g., suspense, desire, or disinterest) as he or she perceives the story's structure and builds the fabula. Thus, it includes a reader's aesthetic response, using Dewey's (1934) notion of aesthetic as an individual's "making of" an experience and not a quality of an object. Although the word "aesthetic" often is attributed to positive aesthetic responses to experiences, such as when someone is "taken in" by the beauty of a particular piece of art, it should be emphasized that this framework explicitly recognizes that not all stories are interesting or compelling. Thus, when a plot's aesthetic is described, both positive and negative responses by a reader are included.

In order to describe the inter-relationships between the theoretical constructs of plot introduced by Bal and Brooks, I propose extending the diagram in Figure 2.1 as shown in

Figure 3.4. Rather than singularly story or fabula, plot is located in a reader's negotiation between these layers of narrative. The plot has both form (the connected structure of the parts making the whole in a given sequence) and effect (a reader's aesthetic reaction). A reader's reaction to the narrative sequence, in part, includes an interaction between expectation and realization, which each can be linked to possible responses by a reader, including anticipation, curiosity, surprise, and satisfaction.

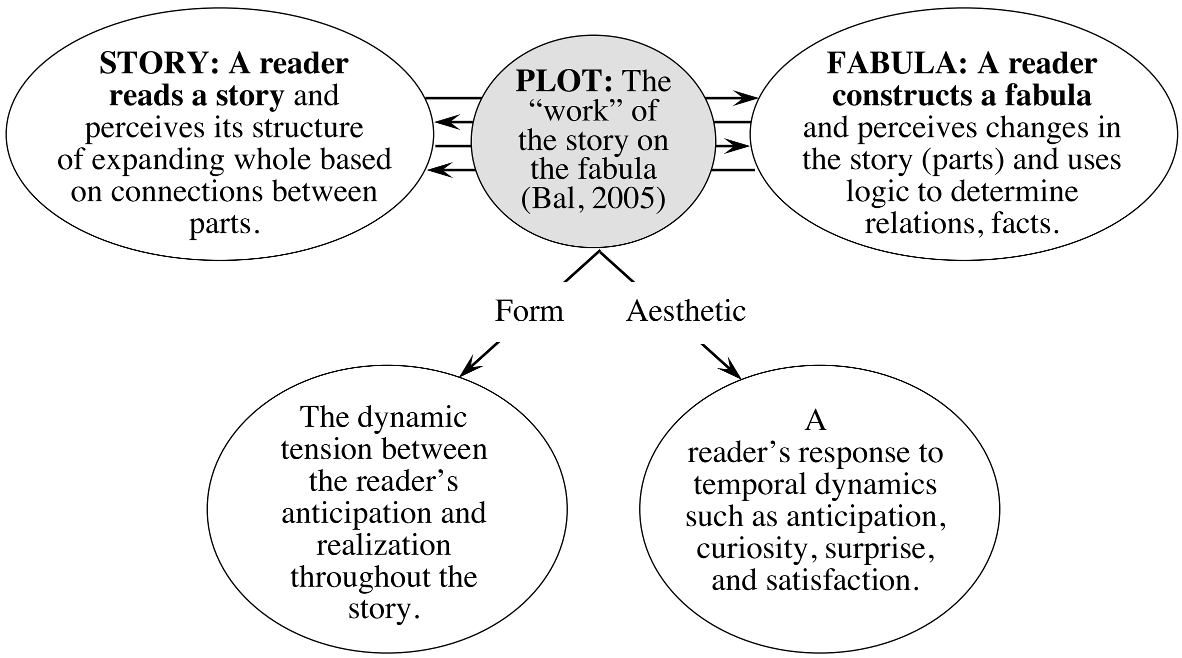

Figure 3.4. A proposed representation of the relationships between plot and Bal's (2009) narratology framework.

Defining mathematical plot. Given this framing of plot, a *mathematical plot* will describe the structure and aesthetic dimensions of a mathematical story, with particular attention to recognizing the dynamic tension between what is known and what is desired to be learned by a reader. Although Chapter 5 expands the notion of mathematical plot, this section discusses the question *What might the temporal dynamics for a reader of a mathematical story look like?*

Since mathematical plot is an aesthetic response, this discussion will first consider the role of aesthetic in mathematical learning (and thus, potentially, the reading of mathematical stories). Specifically in regards of mathematics classrooms, Sinclair (2001) proposes that, "aesthetically-rich learning environments enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26). She notes that part of noticing might rely on perceiving rhythm, a basic element of story according to Bal (2009), introduced in Chapter 2. Sinclair (2001, 2002, 2006, 2009) also identifies multiple aesthetic drives particular to mathematical inquiry, including looking for order and structure (for example, finding a pattern, recognizing symmetry or rhythm), striving for completeness, and aiming for what feels "right" or will "work out" (2006, p. 96). Therefore, in terms of a mathematical story, anticipation may occur when a reader can imagine future mathematical action through the continuation of a pattern or structure, with a vision of closure (such as what a solution might look like) and a way of evaluating the work. This is not to say that all readers will find the same mathematical story aesthetically pleasurable, only that there are mathematical impulses that can drive a reader to (actively) read on.

Thus, the *plot* of a mathematical story is the aesthetic response of a reader[32] as he or she works to perceive structure (and thus, look for order, find patterns, sense rhythm, etc.) and anticipates what is ahead (by wondering, imagining, asking questions). As a tension between the pursuit of mathematical ideas through inquiry and the revelation of information, it is the

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[32] As was discussed in Chapter 2, this analysis makes no claims about *who* would read a mathematics textbook this way, only that it <u>can be read</u> this way. This distinction is important because there are multiple factors (such as the way textbooks are addressed and used in a classroom) that possibly discourage some readers (i.e., students) from recognizing a sequence of mathematics text as a mathematical story. Even though some mathematicians and math educators (e.g., Netz, Thomas, and Sinclair) recognize a possibility of mathematical plot, for example, this does not mean that all or even most readers will. It is important to emphasize that the purpose of this framework is not to explain how students and teachers read mathematics textbooks, it is to re-conceptualize mathematics textbooks in a way that can lay the theoretical foundation for the curricular design work of teachers and open new mathematical possibilities for students.

temporal dynamics of the story, that which encourages a reader to keep reading. Therefore, one way to recognize a mathematical plot is to consider *What mathematical questions are raised (for and by a reader) and when are they answered, if they are?*

In my reading of part of the EM lesson (described above), there were both questions that arose for me by the text (such as "what happens when you add 10 to a number?") and questions generated by me (such as "I wonder what I will learn about these numbers?"). These mathematical questions arose regularly throughout the lesson, and some are answered quickly while others took longer. One question generated by me ("I wonder what I will learn about these numbers?") is an example of a question that remained open throughout the end of the mathematical story. Another question, "I wonder if 10 is significant?", is an example of a question sparked by repetition and sensing structure[33]. These questions represent curiosity raised during my reading of the mathematical story.

The text also raised questions that provoked my curiosity. The initial questions in the Math Message ("Write the largest number you can read. Write the smallest number you can read.") stimulated my imagination and started the story with a potentially large array of mathematical characters, some with very interesting features (especially since my largest number was quite large). However, quickly thereafter, the number grid was introduced and the numbers in focus looked more familiar and, to be frank, boring (i.e., the usual suspects). The imaginative quality of the lesson was gone, not to return again until much later. Namely, after an exploration of moves about the number grid, I found the creative grid shapes in the "Finding Missing Numbers on a Number Grid" activity fun and interesting; they provoked me to imagine how I

[33] Depending on where the mathematical story goes from this point, this question could become the basis of thinking about whether the fact that the difference between these numbers in the grid is related the fact that the rows have 10 numbers. This pursuit would be an example of a reader (me) extending the mathematical fabula.

might design my own missing numbers grid. I was then excited to be able to fill my own interesting number grid at the end of the mathematical story.

In summary, the mathematical plot describes the way in which a mathematical story provokes curiosity and wonder. At its core, it is a description of a reader's inquiry into the subject of the story (mathematical fabula) as it plays out temporally (mathematical story). If a mathematical story can be described as answering questions immediately after they are raised, then the lesson offers little for the reader to look forward to (except being asked more questions). However, mathematical questions that are opened and sustained throughout a mathematical story can offer a reader suspense and wonder and motivate him or her to keep reading (similar to "will Romeo and Juliet live happily ever after?"). More examples of mathematical plots will be illustrated and discussed in Chapter 5.

Mathematical Moral of the Story. In some traditional fairy tales (like Aesop's Fables), stories explicitly note a message, or "moral," that the story is trying to teach. In these cases, the story is positioned as a curricular tool, a device designed for educative use. Similarly, authors of mathematics textbooks sometimes make explicit an educative statement that is the point of the text material (such as a lesson, unit, or investigation). In these cases, the educational objective or the mathematical property or principle could be viewed as the author's intended morals (pedagogical messages).

However, the intended message is not necessarily the same as the message read by a reader. There is a temptation with reading for a moral to assume that there is a single message conveyed through a text ("*the* moral of the story"). Although the author of Aesop's Fables may state a moral at the end of a story, what other messages can a reader take from the story? Ben-Peretz (1990) notes that the same lesson materials may be interpreted differently by different readers and may be used to fulfill different instructional purposes. In addition, critical literary theory has argued that the messages of text in part depend on the questions a reader asks of text, and that reading with different questions results in different meanings and messages (Apol Obbink, 1992).

Since mathematics curriculum has an intentional educative goal, a moral of a mathematical story might still be a useful conceptual construct. Rather than locating a moral with the intention of its author, this mathematical story framework defines a *moral* of a mathematical story as a resulting message or conclusion gleaned by a reader through the reading of the story. Although a reader may not be able to determine the writer's intended message, the morals interpreted by him or her may be explicitly stated in the text or might instead be constructed by the reader from large sections of text. Different readers, then, may (and likely will) take different mathematical messages from the same stories.

It should also be mentioned that a moral is not necessarily the same as the ending of the story; it is a set of experiences together with the ending that forms a message for a reader. In other words, a reader might, at the end of a mathematical story, measure the length of an object (such as the classroom). This measurement is likely not a moral of the story, that is, the point of the lesson or activity. However, it may be across multiple experiences that a reader concludes that when the length of a unit of measure increases, the number of units needed to measure an object decreases. Whether the text states this message explicitly or not, the principle of unit measure compensation could be viewed as a moral of a mathematical story, hence, an educational outcome for a reader. A reader can recognize a moral of a mathematical story by considering, *What is the point of the story?* or *What message can I take from this mathematical story and use to inform other experiences?*

Note that a moral of a mathematical story may not even be mathematical in nature[34]. This conceptualization of mathematics curriculum enables the broad notion of a story moral to include what some theorists have called the *implicit* or *hidden curriculum* (e.g., Eisner, 1979), which are the unstated messages of curriculum. Dewey eloquently made this point, "Perhaps the greatest of all educational fallacies is the notion that a person learns only the particular thing he is studying at the time" (quoted in Eisner, 1979, p. 74). Although the notion of hidden curriculum is often employed in terms of enacted curriculum (experiences in the classroom), a reading of textbooks can also reveal implicit messages about the nature of mathematics.

Summary. Just as studies revealing Poe's poetic style or analyzing the richly described setting do not reveal the story of *The Pit and the Pendulum*, summaries of content found in mathematics textbooks or analyses of their style or voice do not reveal the mathematical stories contained within. Mathematical stories connect a beginning with an ending with a temporal sequence, with characters (mathematical entities that are introduced as objects of study, such as integers), action (changes which advance the story, such as addition), setting (representations in which the characters are found, such as a number line or a coin context), and plot (dynamic tension between what is known and unknown throughout the sequence).

Discussion

This dissertation introduces a new means of interrogating and critiquing mathematics curriculum, potentially offering new ways to improve mathematics curriculum for teachers and students. The conceptualization of mathematical story presented in this chapter, which is metaphorically based on the literary framework of Bal (2009), turns the analytic focus away

[34] The mathematical story *Flatlands* is an excellent example of this, which many interpret as a critique of the hierarchical nature of society in 19th century England.

from curricular outcomes and back toward the way in which a mathematics textbook sequentially reveals information to a reader. The framing of mathematical story provides a new way to use "old" ideas (e.g., familiar literary constructs) to discuss the manifestation and interrelationships of mathematical objects (characters), procedures (action), representations (setting), and mathematical messages (moral). In addition, the conceptualization of mathematical plot introduces a new way to understand the temporal aesthetic experience of reading mathematics textbooks.

Before moving into deeper analyses of mathematical characters and plots in later chapters, this section further clarifies the mathematical story framework by addressing the questions: *Is a word ("story") problem the same thing as a mathematical story (as defined in this chapter)? Are mathematics textbooks a form of narrative? What would not be a mathematical story?* and *How can a mathematically knowledgeable reader read for a mathematical story?*

Contextual word problems. The notion of mathematical stories is a powerful metaphor for the sequential manifestation of mathematical content in a mathematics textbook. However, it should not be confused with another manifestation of story commonly found within mathematics textbooks: contextual word problems, or "story problems." Story problems, which were found in all of focal texts of this study, involve fictional characters in contrived situations. For example, though the EM lesson discussed in this chapter did not contain a story problem, the question "Lara brought 14 candies to school. She gave away 7 during recess. How many candies does she have now?" (Student Math Journal, EM, 3rd grade, 2007, p. 8) appears soon thereafter. This problem is a story in that it relates a situation and has an event (dispersal of candy). However, Lara is not a mathematical entity and the action (giving money) is not a mathematical action. If read as a mathematical story, the quantities of candies can be interpreted as mathematical characters, their subtraction (removal) can be interpreted as mathematical action, and the context at school can be interpreted as the representation. Although this does not make (for me) an interesting mathematical story, interpreting this task in this way does have the required elements of a mathematical story: at least one mathematical event connecting a beginning with an ending (assuming its ending is that Lara has 7 candies). More likely, however, this task and others like it can be considered as part of a larger sequence for how they transition (or not) the mathematical ideas of the lesson.

Therefore, in this example, the word problem becomes an embedded story in an extended mathematical story, and as such, is viewed for how it advances (or not) the mathematical ideas in progress. The temporal aspect of the word problem (that Lara had 14 candies *before* she game some away), which is critical for its classification as a story in literary terms, is less important in relation to mathematical stories. Instead, as a mathematical story, the temporal aspect serves to narrate a mathematical action that is supposedly being performed by different mathematical actors (Lara and friends) than the reader. So, instead of the story prompting a reader to determine "what is 7 less than 14" by acting on the numbers in a number grid, the narrator instead expresses the mathematical event in a new setting with new mathematical actors.

So viewing mathematics textbooks as a broader mathematical story offers new insight into why word problems can be difficult for students. When a majority of the story, as is the case in the EM lesson and subsequent lessons dealing with numbers, takes place in one setting (a number grid), it might be jarring to suddenly have a change in narration, new mathematical actors, and a very different setting. Although mathematical continuity can be recognized by a mathematically knowledgeable reader (i.e., someone who has already constructed the mathematical fabula), it is reasonable to assume that this disruption may obscure the relationships with the rest of the mathematical story. In literary terms, it would be like following the story of *Romeo and Juliet* and suddenly entering a vastly different context in Act 2 (perhaps a space station), during which none of the primary characters can be recognized. Not only might this disruption appear bizarre, it also could halt a reader's sense of mathematical progress and threaten his or her ability to anticipate what is ahead. In contrast, this disruption can be alleviated when a majority of the mathematical story takes place in a single context (such as designing bumper car rinks), as can be found in some mathematics textbooks.

Is a mathematics textbook a narrative? This chapter re-conceptualizes mathematical textbooks as a metaphorical form of narrative by articulating metaphorical correlates to Bal's narrative layers. Despite vast differences in the content of literary stories with that in mathematics textbooks, this chapter establishes a set of mathematical layers that are consistent with Bal's layers in order to recognize the temporal qualities of unfolding mathematical ideas in curriculum. However, it is reasonable to ask *Is this the same as arguing the mathematics textbooks <u>are</u> narrative?*

It is worth mentioning that this question has been raised by others, in particular by Solomon and O'Neill. In Solomon and O'Neill (1998), the authors distinguish two forms of mathematical writing: one that narrates mathematical steps taken by the author (such as "I first factored the polynomial and then I set the factors equal to zero...") and the other that records mathematical claims and logic (such as the proof that $\sqrt{2}$ is irrational, reprinted from Sinclair (2005) in Chapter 1). The recounting of mathematical steps is more similar to literary narrative than mathematical claims, and the researchers argue that the primary distinction is the dimension of time; that is, when recounting chronological mathematical work (with the implicit or explicit "and then I..." connecting the steps), along with the mathematical content comes a reference to time. However, the authors note, as written, mathematical claims have no reference to time. They explain:

> Even if it is embedded in a narrative structure, mathematical argument is itself atemporal — one cannot say for example, 'yesterday $\sqrt{-1}$ was in a certain well-known sense a line perpendicular to the line 1, but today it no longer is' — and it achieves cohesion through logical rather than temporal order. Mathematics is constituted by a logical structure that is not reducible to a temporal sequence of events. (pp. 216-7)

Because of this distinction, Solomon and O'Neill conclude that mathematics text, therefore, is not narrative. Arguing that since "mathematicians operate within a number of non-narrative genres through which mathematical meanings are constituted" (p. 210) the authors emphasize that students must be introduced to non-narrative approaches of communicating mathematical ideas.

In response, Pimm (2006) laments the cleansing of time from mathematics texts, arguing that its form often renders the past invisible. However, he points out some ways the past can be recognized in formal math texts, such as through the reference of past mathematicians in the use of mathematical ideas (e.g., the Pythagorean Theorem) and the temporal nature of diagrams. For example, with regard to a mathematical diagram drawn in The Geometer's Sketchpad, he notes that its temporal creation (and, hence, its story) is retained in its script:

> I can, with the software's help, travel backwards in time. I can revisit the drawing (as it was drawn in the beginning, as it is being drawn now and as it ever shall be so drawn) to hear and see once again the tale of its genesis, a tale that is forever being told. (p. 177)

Importantly, Pimm also points out the temporality of mathematical proof, noting (admittedly briefly) that:

> For a proof to 'work', it must be correctly 'uttered', invoked. For instance, simply shuffle the order of its sentences and it has an efficacy comparable with a similarly-scrambled marriage ceremony: the couple are not, in fact, married, the theorem is not, in fact, proven. (In passing, this is another temporal aspect a proof retains.) (p. 178).

In the theoretical framing of math text (particularly mathematics textbooks) of this dissertation, I have similarly drawn attention to the logical sequence of mathematical ideas. However, since proof (and thus, sentences of proof) are rarely found in K-12 mathematics textbooks, particularly those at the elementary level, the logical connections sequencing the mathematics occurs in a reader's construction of the mathematical fabula. Instead of recognizing the sequence of utterances of a mathematics textbook as logically sequenced and, thus, temporal in the way described by Pimm, this mathematical story framework instead attends to the logical restructuring of the mathematical ideas beyond the sequence in which they are presented in text (the story) in the fabula as a metaphorical analogue to the chronological aspect of narrative.

Because of this metaphorical linkage, this dissertation therefore does not make the claim that mathematics textbooks are narrative, but instead argues that mathematics textbooks can be read as narrative using this metaphorical lens. When reading the mathematics textbook as a mathematical story, a reader, through working to make sense of the elements of the story, re-constructs the mathematical ideas of the text using logical relations.

What would not be a mathematical story? Arguing that mathematics textbooks can be interpreted as mathematical stories is not a claim that all textbooks make for good reading or that all can be understood as a mathematical story when read in this manner. Therefore, this section addresses this issue by considering the conditions of mathematical stories as read in mathematics textbooks to raise and discuss possible curricular "monsters" to this framework (in the sense of Lakatos (1976)). Not only does this help to expand understanding of mathematics curriculum, it also serves to clarify and further define mathematical story.

To be a mathematical story, a reader must be able to recognize a sequence of mathematical events that connects a beginning with an ending. Then it is instructive to consider what if a reader cannot recognize a sequence of mathematical events? This could happen in a mathematics textbook where there is no transition of mathematical ideas. Technically, this could happen with only the introduction of mathematical characters. That is, it seems reasonable to assert that “5” is not a mathematical story, similarly with a longer list of mathematical objects. In this manner, a glossary at the end of a mathematics textbook introduces what can be thought of as the cast of mathematical characters in a fashion similar to that of a playbill or old mystery. However, just as a list of literary characters would leave a reader wondering “what will happen?”, similarly, a glossary fails to offer mathematical events as defined in this dissertation. What is missing from a list of characters is a sequence based on logic[35] and mathematical action; that is, mathematical transformations by an actor.

Suppose there are mathematical events. What other qualities of mathematics textbooks would prevent its interpretation as a mathematical story? One concerns the connections between the events. In other words, are the mathematical ideas introduced in a way in which mathematical ideas continue to make progress and thus are related? Or are they a seemingly random collection of mathematical events that could be ordered in almost any order? To be clear, not every mathematical event must be related to those before and after. Just as there can be major breaks in a literary story (consider, for example, when a story starts out in one country with a set of particular set of characters and then jumps to another seemingly unrelated set of characters on the opposite side of the world), there can be breaks in the mathematical progress (e.g., a development on linear functions might follow a study of probability). As can be seen in the case of the novel, there are usually connected events before and after the discontinuity, which assures a reader that the story is making progress toward a goal of some kind. However, if a

[35] It is assumed that the organization of mathematical glossaries found in mathematics textbooks are organized alphabetically and not based on the logical relations between the entries.

mathematics textbook jumps after every mathematical event with no discernable connection or relationship between, then this can eliminate a sense of progress for a reader. This can render the sequence incomprehensible, with no emerging structure other than randomness. An example of this would be a math program in which content is catalogued on cards in no particular order. This is not to say that these sequences will have no mathematical or curricular value, only that they do not offer a reader a mathematical story as defined here.

Finally, since a mathematical story, as framed here, must connect a beginning with an ending, then it is reasonable to assume that to be recognized as a mathematical story, a reader must perceive an ending or a sense of closure. The need for some closure (and thus, some anticipation of closure) in a mathematical story is more than a reader's need for fulfillment; it attends directly to the manifestation of plot. That is, without a mathematical plot (the tension between what a reader guesses that will emerge and his or her recognition of what has emerged), there is arguably not a mathematical story[36]. This is not to say that all mathematical questions opened for a reader must be answered fully (or even partially) by the end of the story; instead, a sense that a goal of reading has been met is sufficient. For example, it is possible that the closure of a mathematical story is, in fact, a reader's recognition that the answer to the question pursued throughout the mathematical story is unanswerable.

It should be emphasized that having a mathematical plot does not imply that a reader must find a mathematical question and its answer(s) relevant, interesting, or important (and thus, making it a "good" story). Just as a children's story may pursue a question that may not be of interest to an adult (such as how to tie your shoes), similarly, it is assumed that a mathematical

[36] This, perhaps, is another reason why the notions of *story* and *plot* are so interconnected and used interchangeably.

story will not appeal to all readers. Therefore, its appeal is not a quality that distinguishes between what is and is not a mathematical story.

Reading for a mathematical story. Therefore, assuming that mathematics textbooks can be read as narrative (with a mathematical story layer), how might a knowledgeable reader read and recognize a mathematical story? Throughout this chapter, literary theory has been used to raise several questions for the reading of mathematics textbooks. Together, these questions serve as a potential reading heuristic for recognizing mathematical stories and their fabula. For example, asking questions like “what happens in this story (i.e., what are the mathematical events?)” and “How do the mathematical ideas change and transition throughout the story?” can help a reader identify and describe the mathematical progress of a mathematical story. In addition, considering “How, when each part is viewed as a transition from an earlier part, are they connected?” further can help a reader describe the accumulating affects of the sequence; that is, their interdependence. After reading the story, asking, “Overall, what mathematical ideas (including their relationships) were in focus?” can help a reader identify the mathematical fabula of the story and distinguish it from the mathematical story.

Looking Ahead

With this mathematical story framework in place, Chapters 4 and 5 will shift the focus to what can be learned about mathematics textbooks when they are read as a mathematical story, particularly in terms of mathematical characters and mathematical plots.

I remember the second time I finished [teaching with a geometry textbook]. On one of the last days of school we found the equation of a circle using a right triangle. At that point I realized that the whole [geometry] book was about a triangle. The better part of this story was when I shared this realization with my students they all said, "We know." – *Anonymous teacher*

CHAPTER 4

Manifestations of Mathematical Characters

Given that the changing mathematical content within passages of written curriculum can be read as a mathematical story, this chapter begins to identify what can be learned when mathematical objects found in a curricular sequence, such as a number or a symbol for equality, are closely read as characters of a mathematical story. This focus was chosen for multiple reasons. First of all, when defined broadly, characters are an essential component of any story; characters constitute one of the six necessary elements for drama according to Aristotle. Readers pay much attention to characters, both in terms of trying to learn about the characters and in terms of living through the characters. Bal (2009) argues that narrative "thrives on the affective appeal of characters" (p. 112). However, literary characters do not need to have human form to have affective appeal for readers. To the contrary, many children's stories, such as *The Three Little Pigs*, have no human subjects and yet have characters with human characteristics (that is, the pigs talk, act, and express emotions similar to humans). Whether human or not, in all stories, change is manifested either through action with characters (e.g., a fight, a romance) or character development (e.g., a transformation, a new outlook on life). Therefore, it is reasonable to assume that one way of learning how mathematics changes throughout a textbook can be explored through an analysis of the characters of the mathematical story.

The study of mathematical objects and their relations is also important for understanding the mathematics of written curriculum. For example, Sfard's (Sfard, 1991, 2008) recent theorizing of mathematical thinking (which is arguably what a reader does as he or she reads a mathematics textbook) claims that the objectification of mathematical entities is one of the four characteristic features of mathematical discourse. Mathematicians create, define, and carefully study the properties of and processes involving mathematical objects (see, for example, Lakatos, 1976). Without mathematical objects and their relations, it is questionable as to whether mathematical activity would exist[37], as there would be little to "talk about"[38].

Given this central role, this chapter explores how mathematical objects are manifested in mathematics curriculum by asking questions of mathematical stories drawn from literary theory pertaining to literary characters. That is, how can the interpretive lens of mathematical character specifically reveal new ways of understanding mathematics curriculum, and specifically mathematics textbooks? The reading of mathematical characters in mathematical stories as one would read literary characters in literature opens a new unexplored space for observations of mathematics curriculum.

As was discussed in the mathematical story framework in Chapter 3, this approach to reading curriculum assumes that mathematical characters are found within settings and are able to be transformed through action. Even within unimaginative and uninspiring mathematical stories, changes to the mathematical characters can usually be recognized. As a reader moves from one page to the next, the mathematical characters of focus may change or a new property of

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[37] The existence of mathematical activity and mathematical objects is an important ontological question that has been given much attention in philosophy. See, for example, Ernest (2008a, 2008b, 2008c) for an interesting discussion. However, for the purposes of this study, I propose that mathematical activity involves (even if not centrally) the study of mathematical objects and their relationships.

[38] For example, in Sfard's commognition framework of mathematical activity (which she referred to as "mathematizing"), the "objects" of mathematical discourse was described as "what mathematizing is all about" (2008, p. 163).

the mathematical character may be recognized that the reader was not aware of before. However, this chapter presents an argument that recognizing mathematical characters and noting and describing their changes are only part of understanding a mathematical story through the reading of mathematical character.

To support the argument that when read as stories, new qualities of mathematics curriculum can be recognized and described, this chapter will be organized into four main sections: (1) an overview of Bal's (2009) theory regarding story characters in narrative to frame the analysis of mathematical characters found in textbooks, (2) an extensive analysis of elementary mathematics textbook materials for implications regarding the treatment of mathematical objects, (3) a discussion of the implications of this analysis for teaching and learning, and (4) an articulation of the limitations of this conceptualization (i.e., what it does not show). Also, to support the literary discussion in this chapter, a discussion of the character of Professor Severus Snape from the *Harry Potter* series, written by J.K. Rowling, will be woven throughout this chapter as an illustration of various aspects of character in literature[39].

Helping illuminate multiple aspects of character through a literary example is only half of the challenge of this project. Since this analysis is also an argument that this interpretive lens offers the identification of new aspects of mathematics curriculum, the argument must operate on dual planes. Therefore, multiple curricular examples will be used throughout the discussion of mathematical characters to draw further connections between mathematical stories and literary stories and to demonstrate the power of this interpretive lens for mathematics curriculum. To help distinguish these planes, the terms *literary story* and *literature* will refer to stories found in narrative and *literary character* will refer to characters found within literature. Likewise,

[39] This and all other references to *Harry Potter* is to the series of books (7 volumes in all) and not the movies with the same name.

mathematical story and *mathematical character* will refer to their metaphorical counterparts found in curriculum as defined in the mathematical story framework introduced in Chapter 3.

Literary Characters in Stories – Theory and Illustration

The act of reading, which has been defined in this study as an interpretive negotiation between reader and text, brings with it questions of what a reader is reading for. Literary criticism is a field focused in part on making sense of texts by asking questions of texts and reading for particular new ways of understanding them[40]. Therefore, this section will build on the discussion of mathematical character in Chapter 3 to elaborate and illustrate aspects of literary character (described in the story layer in Bal's (2009) narratology framework) in order to build framing questions to pose for mathematical characters in textbooks in the next section. Specifically, this section will discuss theory on the emergence (realization) of literary characters, their changing characterization over time as the story progresses, the role of character in relation to others, and character effects.

The realization of literary characters. To begin with, Bal (2009) points out that part of reading stories is identifying literary characters and building images (or, what she sometimes refers to as "portraits") of them. In essence, this involves bringing characters "to life" (p. 121) by reading literature with questions such as who is this story about and what do I know about him or her? The realization (or creation) of a character, according to Bal, involves the construction of the literary character *by a reader* through the reading of the literary story. That is, once a reader notices that a new character is named or described (that appears different than those the reader is already aware of), he or she can start to seek and collect information about that character and distinguish it from and relate it to other characters. Thus, a reader constructs

[40] This approach to understanding literature becomes particularly clear when the questions asked of literature change, as discussed in Apol Obbink (1992).

an *image* of a literary character (a coherent collection of qualities of the character) throughout the reading of a literary story. The *realization* of a literary character is an ongoing process (in the sense of "coming to be") and emergence of the character's image.

Sometimes trying to understand a character is a driving aspect of a story, a reason a reader wants to keep reading. Bal also explains that readers react to literary characters, referred to as *character effects* (p. 125). A useful example of this is the character Severus Snape (often referred to as "Snape") in the *Harry Potter* series. Early in the first book, Snape is introduced as a professor of potions at the Hogwarts School of Witchcraft and Wizardry. He is repeatedly described with sinister qualities, such as his sneer as he talks or how he liberally assigns punishments to the protagonists (Harry Potter and his friends). Soon, hints of Snape's past involvement with a group called the Death Eaters, founded and led by a notorious villain named Lord Voldemort, further develops Snape's evil traits. However, every now and then contradictory evidence, such as hints that Snape is possibly working against Lord Voldemort as a double-agent, is offered. This evidence, such as the Headmaster's explicit trust in Snape's better nature and Snape's repeated presence at meetings of the Order of Phoenix (an organization aimed at fighting Lord Voldemort), helps to support the generation of questions by the reader about with which group Snape's true allegiance lies.

In terms of the story layer, Bal defines characters as "the anthropomorphic figures provided with specifying features the narrator tells us about" (p. 112). In other words, Bal's notion of character is a literary figure described in the text with particular qualities in which a reader reads into it human traits. Of course, in literary stories, characters are usually representations of people, which, while not real people, can convince a reader that they are real. Bal emphasizes this, stating, "The people with whom literature is concerned are not real people.

They are fabricated creatures made up from fantasy, imitation, memory: paper people, without flesh and blood... the character is not a human being, but it resembles one" (p. 113). The reading of literary stories, therefore, involves constructing and identifying with life-like images of characters described in text. Outside the text, these characters have no material existence; within the realm of the text, they come to existence through the story's reading by a reader. For example, it is an accurate statement to say that within the story of *Harry Potter*, "Severus Snape" exists, even though this character does not exist in the material sense outside the texts.

In addition, Bal explains that qualities of a character "creates an expectation" (p. 125) for a reader that the story may or may not fulfill. Thus, in the case of Snape, a reader may reason that Snape is evil and that the headmaster has been misled, leading that reader to expect more evil-doing by Snape when given an opportunity in the story. This reader's temporary conclusion is independent of whether or not he or she ever learns of Snape's motivation and goals. However, Bal argues that "either way, character features activate the reader" (p. 125) in the sense that they enable a reader to identify, anticipate, and relate to the character.

Bal explains that *characteristics* of characters, those qualities that describe and become part of the description of the character, come from multiple sources. Of course, many are found through explicit descriptive statements made about the character by other characters and the narrator. However, characterization can also be made by a reader through implicit means. For example, Bal notes that characteristics can be drawn through action, noting that, "What a figure does is as important as what he or she thinks, feels, remembers, or looks like" (p. 115). The properties of a character can also be implicitly drawn from the other characters with whom it associates. Thus, what is known about Snape changes as a reader encounters new information about other members of the group called Death Eaters (which, the reader is eventually informed, Snape was once and may still be a member). Similarly, family ties often connote characteristics; one hardly needs to be told that a relative in the Weasley family has red hair!

Even when the character is named after someone from the "real world," such as a story about Harriet Tubman, Bal argues that the character is still a literary creation (although one that a reader likely brings characterizations and assumptions from outside the story). In cases when the character transcends the novel and refers to a person who is either alive or dead, Bal uses the term *referential character*. She explains that the reading of referential characters by a reader may involve a confrontation of the image constructed from the text and this reader's expectation built from prior knowledge of the person. Referential characters may be constructed through non-specific means (such as the reference to the "Prime Minister" in *Harry Potter and the Half-Blood Prince*), which enable a reader to assign qualities to the character that stem from sources outside the text (such as the character as a leader and decision maker, responsible for the safety of the citizens). Bal notes that the use of referential characters (or characteristics) further enables a reader to make predictions about the character.

The temporality of literary characters. One aspect of literary characters that the notion of story helps to recognize is their evolving nature. In literary stories, those characters that are found repeatedly are usually not introduced in final form[41] at their introduction. In fact, a reader usually knows very little about a character at its introduction. Bal (2009) proposes that there are four different principles at work in text to help support the building of the character's image by a reader: "repetition, accumulation, relations to other characters, and transformation" of the

[41] The phrase "final form" is not meant to convey any sense of completeness of a character, but instead refers to the final image of a character produced by a reader by the end of a portion of text. Some characters repeatedly appear in stories, and the reader's image continues to grow throughout. However, other characters make brief appearances (often inconsequential to the story), and are not developed further.

character (p. 127). Thus, she argues, a reader builds an image of a character through successive confrontations with it as the text is read for new information about the character (a process that Bal refers to as “filling in, fleshing out” (p. 127)).

Part of a reader’s construction of a character image involves using logic and searching for coherence to help evaluate if he or she understands what is going on in the story. Therefore, this work of building the image of a character also occurs on the fabula layer, the atemporal “fabrication” constructed by a reader throughout the story as she or he makes sense of the temporal material. Different information about a character, though possibly disparate or contradictory, accumulates over time, enabling a reader to build a convergent image. Bal explains that this negotiation between the story and the fabula is not always smooth and that contradictory characterizations may “confront” a reader. When a contradiction about a character emerges in a literary story, the character can become more complex and can also be the basis of a reader’s inquiry, the means through which a reader raises questions about the character and anticipates answers (resolution).

A character’s temporality is dependent also on the temporality of others, when a reader learns new information about related or associated characters. This is because constructing an image of a literary character involves considering its similarities or differences with other characters and using these comparisons to recognize new potential characteristics of a character. For example, it is Snape’s similarity with other characters (who are explicitly identified in the text as part of the evil group of Death Eaters) that offers potential characterization (such as the possible existence of a “dark mark” on his forearm, a mark that other Death Eaters have) and can help a reader further ask questions of and “flesh out” an image of the character.

An important aspect of a reader's construction of a character's image is that a reader then can draw upon the temporal image (in whatever form it is in) whenever the character is later encountered. Thus, although repetition works to help the reader construct an image, a reader later may read these qualities into the character without explicit mention in the text. For example, since Snape's sneer when he talks to Harry is often mentioned in pivotal scenes, a reader may imagine other dialogue by Snape to Harry with a sneer even when the author does not explicitly mention it. Bal suggests that underlying a reader's creation of a character image is a desire for coherence, explaining that:

When we come across a detailed portrait of a character that has already been mentioned, we are justified in saying that that information – that portrait – 'belongs to' the character, it 'creates' the character, maps it out, builds it up. A certain measure of coherence results. (2009, p. 114)

Although a reader's image of a character changes throughout the text, this does not mean that characters are fixed and permanent throughout a story and that only the reader's growing knowledge of the character changes. Instead, literary characters can also change throughout a story. A character alive at one point in the story may later die, or a character that belongs to one group (such as Snape and the Death Eaters) may change his or her allegiance later in the story. Therefore, part of reading for character is constructing a composite image through repetition, comparisons and relationships with other characters, and noticing possible changes in the qualities of the character and adjusting the image accordingly.

In addition to learning about characters through statements and actions, a reader may make implicit characterizations simply from the character's name. Bal notes how the names of literary characters such as "Miss Marple" and "Snow White" communicate characteristics (such as being unmarried in the case of the former and being innocent in the case of the latter) to a reader. In the case of Severus Snape, other characters often tease him by calling him "Snivellus," further extending his character to include an expectation that he usually complains about his circumstances. In addition, the naming of characters in the *Harry Potter* series can be seen to give status to some characters over others. As Harry talks to his headmaster about "Snape" throughout the books, he is repeatedly reminded, "*Professor* Snape, Harry," a move that not only reminds readers of the status and profession of Snape, but also the position and status of Harry in relation.

The fullness and centrality of character. Although many literary characters, such as Snape, are slowly and richly developed throughout a text, others may only make brief appearances at one point of the story. These supporting characters, such as a member of a crowd or a proprietor of a store at which a scene takes place, may not be further explored in the text throughout the rest of the story and their presence in the story may be only to support the setting, action, or to provide contrast to the central, repeating characters. For these supporting characters, a reader is often left with only the description of the character given at that point. To distinguish rich, fully developed characters that change throughout a story from those that have limited and stable characterization, Bal (2009) relies on Forster's notion of *round* and *flat* characters. However, it should be noted that Bal takes issue with the preference in literary criticism shown for literature with a dominance of round characters. She further argues that in an important way, flat characters also enable complexity by allowing for ambiguity (which can allow mystery to arise), complex relationships, or a series of actions to instead gain the attention of a reader. Therefore, she argues, stories often contain a combination of round and flat characters, and the presence of a flat character might enable a reader's attention to be focused elsewhere.

One aspect of flat characters is that a reader does not need to move beyond the text to further make sense of the character. Although a reader may imagine/create rich thick descriptions of these "flat" characters in the void of text, coherence or enjoyment of the rest of the story does not demand it. In comparison, round characters offer a reader multiple opportunities to build and revise ("flesh out") its image throughout a text. Round characters are often called *central* characters to a story in that the story tends to place central focus on them, which in turn allows them to further develop and to become more complex.

Character effects in literary stories. Although a reader can identify characters (that is, distinguish them from others and build an image), Bal (2009) points out that he or she may also identify *with* a literary character by imagining being the character or relating to it. In this sense, a reader can feel emotion, such as excitement, suspense, or disappointment, during the reading of the events of the story as he or she imagines the character might. A reader may also react to the characters in the story based on his or her personal memories of experiences that are similar to what is happening to the character.

Although readers may remain ambivalent and neutral about a character (particularly one with only a brief appearance in the story), they do not necessarily do so. As Bal puts it, identifying with characters is to "see characters, feel with them and like or dislike them" (p. 113). When a reader identifies with a central character, he or she not only can engage with projected emotions <u>with</u> the character (feeling the emotions the reader imagines the character experiencing), but also <u>for</u> the character (the feelings directed at the character, such as when a reader loves a particular character). Thus, readers can take a stand for (thus, a hero emerges) or against (in the case of a villain), a literary character. Bal offers this as part of the appeal of stories, explaining, "We even go so far as to identify with the character, to cry, to laugh, and to search for or with it, or even against it, when the character is a villain. This is a major attraction of narrative" (Bal, 2009, p. 113). Sometimes, such as in the case of a complex character such as Snape, a reader may oscillate between positive and negative feelings for the character depending on the changing circumstances and revelations throughout the story.

As was mentioned earlier, the repetition of literary characters enables a reader to make predictions. Bal notes that with predictability comes the opportunity of surprise as a "confrontation" of character can lead to suspense (p. 122 and 126). A change in a character, such as when contradictory questions arise about Severus Snape, can stimulate a reader's curiosity because what was initially a two-dimensional "flat" character with a stable list of characteristics then becomes a complex, unpredictable, and surprisingly "round" character. This change can provoke a reader to raise new questions about the character and to wonder how the complexity remained hidden for so long in the story. It also can spur a reader to re-evaluate earlier parts of the story in relation to new information about that character, possibly confronting other prior assumptions or interpretations.

In addition to wonder, reading for character effects also offers other possible aesthetic experiences for a reader, such as surprise (when the predicted quality or behavior of a character later appears to be false) and confusion (when there is a "confrontation" between the reader's image of the character and new contradictory evidence). Bal notes the particular potential of this appeal to emotion when authors use referential characters, such as Santa Claus, in ways that contradict commonly held assumptions about that character (such a when a man dressed as Santa embarks on a murderous rampage in Mario Puzo's *The Godfather*). Interestingly, although anthropomorphizing characters offers a reader potential resources to emotional appeal (which might also repel) of a character, Bal argues that it is also this tendency that may "trap" a reader into human-like assumptions about the character that can later lead to surprise. Bal explains that this happens because readers sometimes place the need to have coherence of the character and its relation to humans above the need to have coherence in the fabula (the abstracted logic of the story).

Framing questions involving characters. To summarize, a character in a literary story is a reader's constructed image of a "paper" person through a repetition of statements and actions representing the character in text. Readers identify characters and their corresponding characteristics through both implicit and explicit means and may identify with the characters through anthropomorphosis, imagining being the characters in the story. Some characters are more developed throughout the story than others, some are recognized by readers from outside the story, and some are more central and pivotal (and thus, consequential) to the story than others.

This discussion of literary characters suggests several questions that a reader can use to learn about the characters in a story. First, to recognize and identify the mathematical characters, the questions "What mathematical character(s) is this story about?", "How does each character come to be?", "What does the story explicitly tell me about each character, either through direct statements or by its name?", "What do I implicitly learn about each character and how do I learn it?", "Do any characters have familiar meaning outside of this mathematical story that affect the reading of the story?" enables a reader to potentially create an initial list of characters along with some of their stable traits. Questions that compare characters, such as "Which characters appear most frequently?", "How are these characters related? How are they different?", "Which characters have consequence on the story (and thus, if replaced with a different set of characteristics would change the story)?" further enable a reader to draw implicit characterizations and to identify central characters to the mathematical story. However, repeatedly asking, "Does this tell me anything more about the mathematical character?" can help a reader recognize ways in which the character is transformed during the story, as well as any qualities of the story that enable the change. Finally, to learn how a character potentially can engage a reader's emotion, questions such as, "Would I want to be this character? Why or why not?", "Can I imagine being this character?", "What does this mathematical character remind me of?", "Did this character surprise me?" and "How do I feel about this character?" may reveal character effects for a reader.

Mathematical Characters in Mathematical Stories

This section now turns the attention away from literature and toward the text found in mathematics textbooks to analyze what can be learned about mathematical stories through the reading for mathematical characters. Using the questions developed from Bal's (2009) framework in the preceding section, this discussion has been similarly organized into four parts that mirror the presentation of literary theory: a consideration of how mathematical characters in mathematical stories are realized, what can be learned about their temporal development, what distinguishes some mathematical characters from others, and the ways mathematical characters can have character effects on a reader. No attempt has been made here to exhaust the subject of mathematical characters. Instead, this chapter seeks to demonstrate that there is a lot to be learned about mathematics curriculum (and textbooks in particular) when the mathematical characters are closely analyzed.

As was stated in Chapter 2, each example presented here is a particular reading (specifically *my* reading of text at a particular time), in order to demonstrate the utility and benefit of the mathematical story framework. At various points of the chapter, another reader may (and likely will) have a different interpretation of a particular textual portion consistent with the mathematical story framework or may recognize an aspect of the mathematical character that is not mentioned here. Rather than a weakness of this framework or the interpretations provided herein, I argue that the variation in interpretations highlights the potential of the mathematical story framework to offer readers insight about the mathematical nature of textbooks.

It bears repeating that this reading does not constitute an expert reading (bringing my advanced mathematical knowledge to bear) or that of an imaginary novice. Instead, the readings presented here represent a knowledgeable reading limited only to the mathematical story at hand based on a close reading of the text. Therefore, interpretations are limited to only those that can be made from the text.

The realization of mathematical characters. Reading mathematical stories with the framing questions involving the identification of mathematical characters ("What mathematical character(s) is this story about?", "How does each character come to be?", "What does the story explicitly tell me about each character, either through direct statements or by its name?", "What do I implicitly learn about each character and how do I learn it?", and "Do any characters have familiar meaning outside of this mathematical story that affect the reading of the story?") offers new insights about how the mathematical characters are "brought to life" (Bal, 2009, p. 121) and play a role in mathematical stories. This section will discuss the realization of mathematical characters using two illustrations from different textbooks: the case of the numbers 1 – 10 (from MPAH1) and the case of addition sentences (from SFAW2). These examples were selected because they both are introduced early in their respective mathematical stories and are then seen in almost every subsequent lesson.

The case of counting numbers (MPAH1). Not surprisingly, mathematical characters found in stories for young elementary students (grades 1-3) consist mostly of whole numbers. Understanding the mathematical story of these elementary texts requires limiting characterizations to those explicit and implicit messages found in the textbooks, as opposed to properties that might be known by a knowledgeable reader. For example, the first lesson (titled "Counting To 10") of MPAH1 introduces counting numbers with a table with ten rows, each that contain four different representations of a counting number (from 1 to 10 in order). See Figure 4.1 for an example similar to two rows from this table. Each row contains a numeral (e.g., "5"), a word (e.g., "five"), a photo of similar or duplicate objects (e.g., photo of 5 identical toy cars), and a photo of a quantity of linking cubes that are not linked together, but instead are stacked in various ways depending on the quantity. The teacher is prompted to have students count the number of items pictured (such as the number of peppers) along with the number blocks, and then associate that quantity with the numeral and word[42].

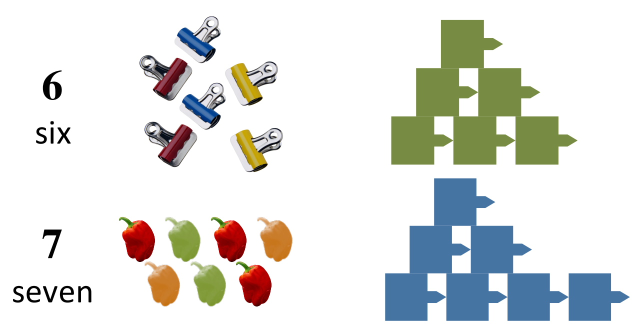

Figure 4.1. Diagram introducing "6" and "7" adapted from MPAH1 (2012). For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

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[42] "Get pupils to look at the pictures in the Pupil's Book and count the number of items with them. Say: '1, 1 bear; 2, 2 flowers; 3, 3 donuts...'" (Teacher materials, MPAH1, p. 7).

So what do I learn about these new characters through this table? Taking a stance that I start this mathematical story knowing nothing about numbers, a few important aspects of these new characters stand out. First, I note that each numeral ("5") is also a word ("five") in a counting sequence. Also, the last number in the counting sequence for a group of objects refers to a quantity of a group of objects. Being tied to a particular quantity of discrete objects (such as 6 clips) distinguishes it from the other characters (such as 7 peppers) because the quantities of objects in those rows differ, so the number 6 is different than the number 7 by one object in the group. However, I notice that each number is tied to a quantity of two different kinds of discrete objects (e.g., peppers and linking cubes), so the number is not tied to a quantity of a particular type of discrete object; that is, the number 6 does not always refer to clips. Also, while the color of the objects being counted as a set can differ (as is the case with the peppers in Figure 4.1), the objects themselves are extremely similar and in some cases copies. This suggests that a number represents a discrete quantity of similar or duplicate objects. Finally, comparing the group of objects in adjacent rows sometimes gives hints of relationships between the numbers in the rows. For example, the linking cubes positioned for the number 7 has the same group of cubes shown for the number 6 with one more cube added to the side. However, not all consecutive rows enable this type of comparison.

For most readers (even those in first grade), the numbers 1 through 10 in MPAH1 are likely to be referential characters. Indeed, the authors rely to some extent on this, as they prompt readers to count the objects in the pictures and connect the quantities with the symbols. If a reader has no knowledge of these numbers, then he or she does not have a sequence with which to count the objects. Even if a counting sequence is all a reader is familiar with, he or she is likely aware that these numbers are well known outside the story. Contrast this with other mathematical characters, such as “quadrilateral” or “decimal,” which, when confronted by a reader in a textbook for the first time, might not be recognized from prior experience.

From a story perspective, these opening pages of MPAH1 do more than just offer initial characterizations of the counting numbers 1 to 10; they set up an expectation that the story from this point on will involve these characters, just as a story might start with “Once upon a time there were three pigs ...”. So far, there is no action implied that works to change these characters, and the settings in which these characters will be later found are only hinted at by the various representations on this opening page (that is, concrete objects may be involved). Since this is so early in the mathematical story, there is not enough information to know how these numbers may be involved, only the expectation to see them again.

This opening of the MPAH1 course represents an example of how mathematical characters are introduced through implicit definition. Although it does not have many of the features of an explicit mathematical definition (with phrases such as, “Let 5 ...”) nor the goal (offering minimal but sufficient criteria to specify the object what the object is and what it is not), this table and the accompanying teacher prompts bring these number characters into being and characterize them (through the counting of groups of objects). Although the text is *explicitly* introducing mathematical characters, many of the characteristics described earlier were read *implicitly* through the noticing of qualities in the diagrams (such as how the pictures show similar objects or comparing the ways different rows organize the grouped objects).

The case of addition sentences (SFAW2). This explicit introduction/implicit characterization can also be found for other types of mathematical objects in other portions of elementary textbooks. For example, in Lesson 1-2 of SFAW2 textbook, the “Learn! Algebra” box, used in this textbook as a way to introduce new vocabulary and demonstrate procedures, a photo shows a young girl holding seven green balloons in one hand and two red balloons in the other. Under a title "Writing Addition Sentences," the text asks, "How many balloons are there in all?" (p. 5) and provides a series of statements aligned so that successive formulations can be compared (see Figure 4.2). This list is followed by the statement, "The sum tells how many there are in all" (p. 5).

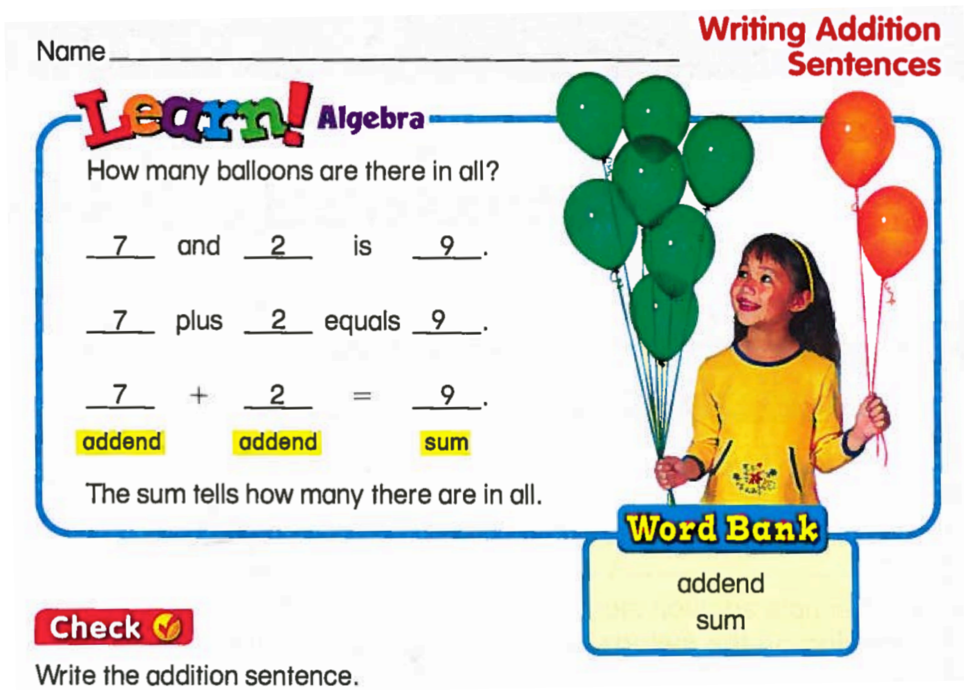

Figure 4.2. The introduction to addition sentences in SFAW2 (2008, p. 5). From SCOTT FORESMAN ADDISON WESLEY MATH MICHIGAN TEACHER EDITION GRADE 2 VOLUME 1 by Charles et al. Copyright © 2008 Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

This introduction to "addition sentence" is also the first time the symbols + and = are found in the SFAW2 mathematical story. Therefore, this segment of text identifies and names a formulation of signs that describe a relationship between mathematical characters (namely, 7, 2, and 9), as well as implicitly introduces the mathematical characters + and =. This relationship was introduced in a previous lesson (e.g., when 7 and 2 are joined together, we get 9), however without an explicit mathematical naming of the relationship. Once the textbook labels these as "addition sentences," it starts to refer to these sets of symbols as mathematical objects in their own right, such as by instructing the reader to "write *the* addition sentence" (p. 5, emphasis added).

What does this textual segment offer me in terms of recognizing an addition sentence and distinguishing it from other mathematical characters? First of all, there is no explicit statement describing or defining an addition sentence, and this ambiguity allows me to choose any or all of the "sentences" in the list as examples of an addition sentence. Thus, the mathematical character "addition sentence" is introduced explicitly, however is characterized almost entirely through implicit means since there are no definitional statements such as "an addition sentence is ..." or "___ is an addition sentence." The title above the box suggests that this textual segment demonstrates the writing of addition sentences, which is further supported by the period after each line, a common punctuation signifying a sentence. The alignment of these addition sentences enable me to interpret the signs "and," "plus," and "+" to be synonymous, and similarly connect the meaning of "is," "equals," and "=." Also, glancing at the picture of the child, I notice that the numbers in the addition sentences match the quantity of balloons in the picture, so I assume that addition sentences describe the quantities in a picture. Since the first number (7) is the number of green balloons on the left and the second number (2) is the number of orange balloons on the right, I conclude that an addition sentence also conveys an order from left to right of groups of objects. In short, I interpret an addition sentence to be an ordered set of symbols (numerals, words (and/is) or other symbols (+/=)) that together describe the joining of discrete quantities of objects represented in a picture.

This example illustrates how, in a mathematical story, a collection of mathematical characters (in this case, numbers and other symbols) can together form a new mathematical character. This occurrence is similar to the naming of a group of characters in a literary story (i.e., how one can refer to the Death Eaters in *Harry Potter*). Once the text names the collection of symbols as "an addition sentence" and asks for "*the* addition sentence," this collection becomes a mathematical character on its own[43]. This naming represents an opportunity for multiplicity of meaning, since the equation $7 + 2 = 9$ can be interpreted as a relationship between mathematical characters or as a mathematical character itself (or both), and this interpretation is largely dependent upon the context in which the group of symbols are found. In the case of my reading, the naming of the collection was pivotal for my recognition of the object, whereas I encountered statements such as "7 and 2 make 9" without being named in the previous lesson with no such recognition.

As was mentioned in the previous section, characteristics of literary characters can emerge through relationships with other characters and are affected by the settings in which the characters are found. The same can be said of mathematical characters. For example, my realization of addition sentences relies on my interpretation of the mathematical characters (the numbers) as groups of concrete and discrete objects. For example, how might the characterization of "addition sentence" change if numbers 7 and 2 are read as locations on a number line? That is, what might "7 and 2" mean then, and how might that meaning affect the reading of "7 + 2 = 9"? In general, certain characteristics of a character are available or noticeable to a reader because of the representational setting. Whereas the order of the whole numbers is apparent on a number line (offering the characteristic of relative size to these characters), when numbers such as 7 and 9 are represented by collections of discrete objects in a

[43] It is also interesting that groups of groups can be objectified. Later in SFAW2 (in Lesson 1-12), addition sentences are grouped with other addition and subtraction sentences to form a "fact family."

photo, the difference in size is not necessarily noticeable. When settings change, character development often follows, as will be discussed in the next section (the temporality of mathematical characters).

In the case of the "addition sentence," the name of the mathematical character also communicates some qualities of character. The other meanings of "sentence" (such as a complete verbal expression with an object and verb) help to convey a sense that an addition sentence represents some form of completeness and informs me both about the object(s) and action(s). The presence of the period at the end of each example of addition sentence in the passage further connects the addition sentences to verbal sentences. Although similar characterization may not occur with the counting numbers 1 through 10, similar characterization may occur with other numbers, such as 102, where the name might help a reader know that 102 is two more than one hundred. Similarly, clues about mathematical characteristics through naming also occur in non-number contexts, such as "triangle" and "variable."

To learn more about how a mathematical character is realized through reading a textbook, a useful question to consider is what is *not* known about the character "addition sentence" at this point? For example, if $\underline{\ 7\ } + \underline{\ 2\ } = \underline{\ 9\ }$ is an "addition sentence," the "addition is joining" lesson before this as well as the contextual question "how many in all?" helps to underscore a meaning that this sentence records an act of joining, so that the left side represents the quantities of two separate sets of discrete items and the right represents the resulting joined quantity. This is different than, for example, a perspective of equality in an equation as a "balance" (see, for example, Sfard, 1994), which metaphorically refers to the relationship between two different quantities (which can be embodied by imagining riding a see-saw with a friend). The process of joining used in SFAW2 instead relates the left quantities and the right quantity as referring to the same objects, only on the left they are two separate sets and on the right they are seen together "in all" as one set. There is no indication (to me) at this point in the text (either contextually or in statements) that there would be any other meaning for equals ("="), other than recording the result of a process of joining. In addition, the meaning of the "sum" (the right-hand term) in this mathematical story is interpreted as the composition of two quantities.

The role of action. It is important to note that these mathematical characters (integers 1 through 10 and the addition sentence $7 + 2 = 9$) are characterized, in part, by the action used to generate or determine them in the mathematical story. In the case of the numbers in MPAH1, the meaning of the numeral and associated word (as a quantity of objects) is interpreted through the act of counting objects in the accompanying photos. The addition sentence in SFAW2 is characterized by the implied act of joining sets of balloons showed in the photo. However, it is important to note that in both of these examples, the naming of the mathematical character before the action (for example, in the title "Writing addition sentences") then raises the question "what is this?" that the procedures are used to answer (in the sense of "What is an addition sentence? If I count the 2 balloons and 7 balloons, I get 9 balloons, so maybe an addition sentence is how I can write down the result of this action").

However, not all mathematical characters are named before they are revealed through action, and the revelation of a new mathematical character through action can offer a surprise for a reader (a delightful surprise, in my case). For example, following the introduction of the counting numbers 1 through 10 in MPAH1, the next textual passage prompts the teacher to "hold 10 unit cubes in your hand. Ask pupils to count the number of cubes aloud. Remove 1 cube and get pupils to count the remaining cubes. Repeat this until there are no cubes left" (teachers guide, p. 6). This activity helps to extend the relationship of consecutive counting numbers in a way that is not revealed in the initial defining table (that is, "6 is one less than 7," as opposed to the initial relationship "7 comes after 6" read as a consequence of the order in the table). This action continues, maintaining the focus of a reader to "what do we call what number is next?" or "what is one less?" until, quite surprisingly, there are no cubes left. This sudden result opens the question "what do we call the quantity when there is nothing to be counted?" since the number zero has not been named or otherwise referred to in the mathematical story prior to that point. As opposed to its naming before the action characterized it (as was done in the case of the numbers 1 – 10 and the addition sentence), zero[44] only emerged and was named after the action enabled it to become part of the mathematical story.

The temporality of mathematical characters. Literary theory suggests that when viewed as elements of a story, mathematical characters can develop throughout a mathematical story and may have (or may not have) fixed characterizations (properties). Throughout textbooks, a mixture of words and diagrams helps readers incrementally build an image of mathematical characters. What can be learned about mathematics curriculum when one reads for this temporality? This section will explore the development of several mathematical characters, asking questions "How are mathematical characters further developed after they are introduced?" and "Does this tell me anything more about the mathematical character?"

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[44] Note that by "zero," I refer to the mathematical character (the whole number less than one) and not any particular symbol used to represent it. This is an important distinction, because technically the symbol "0" has previously appeared in the form of "10," when ten was introduced. However, at that point in the mathematical story, the mathematical character introduced was the quantity of a group of objects, and the numeral 0 was used to write its name. Thus, the numeral 0 was not positioned as a mathematical character of the mathematical story at that point. Much more will be said about zero as a mathematical character in the next section.

To start, this analysis will discuss a temporal reading of a mathematical character (Zero[45]) in MPAH1, which was first introduced through an activity of repeatedly removing cubes from a hand until none were left. There were other central mathematical characters to choose from. For example, Ten (10) is a central character in both MPAH1 and SFAW2. Also, One frequently arises in MPAH1 when sequences of numbers are considered (e.g., noting that 6 is 1 more than 5). Zero was selected because, in addition to its unusual introduction through action, Zero frequently reappeared throughout the first three chapters MPAH1. Its regular appearance, in a variety of settings and in terms of multiple actions, offered many opportunities for character growth throughout the story that highlight its temporality. In addition, unlike Ten and One, it was one of the few characters introduced in the first three chapters of MPAH1 that may not be referential (i.e., well known outside the mathematical story)[46]. After its introduction, Zero is the quantity of what remains when all objects are removed and can also be seen to be less than 1 and the last number in a sequence when counting down[47].

My reading for the characterizations of zero throughout this portion of text was aided with a Character Log (see Appendix B), in which I recorded the instances of this character in

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[45] Since mathematical characters often can be called a variety of names with differing symbols, I have chosen to refer to the mathematical character as "Zero" to emphasize its status as a mathematical character. Although none of the textbooks I have studied have capitalized its name, I hope that by doing so, it will distinguish the references to the mathematical character from references of the symbols used to represent it. Therefore, from this point forward, "0" will refer to the numeral and "zero" will refer to the number-word. The same convention will be used for other numerical mathematical characters, such as Ten, Four, etc.

[46] The analysis of a referential character is additionally complicated because it is assumed that a reader has information about the character from outside the story. This requires the analysis to carefully recognize and articulate what that outside content will be – or, requires the additional reading for how a textual passage relies on prior knowledge that is not supplied by the text.

[47] From my notes: "the character number '0' and the word 'zero' is introduced once no cubes are left. So zero is interpreted as the absence of objects" (MPAH1 reading notes).

sequence along with an interpretation about what I learn about this character through its appearance. This log also included corresponding descriptions of setting and action to enable me to recognize possible relationships between the "side" of the mathematical character I was able to see and the settings and action in which the observation was made. When part of the text revealed something new about Zero, it was given a new line in the log. Although information about characters (characterization) is often repeated in literary stories and this repetition does not necessarily advance what is known about the character, repeated entries in the log allowed me to learn if there were patterns between what is shown about a mathematical character and the corresponding setting and action.

After the discussion of the character development of Zero, implications of character development in textbooks will be discussed, followed by a further elaboration regarding the development of mathematical characters in the mathematical fabula.

The temporal characterization of Zero. After appearing at the beginning of the lesson as a quantity of objects that remains when all objects are removed, Zero does not reappear in either the student or teacher texts until the very end of the lesson (in the workbook material designed to be used in class). In this second appearance, Zero emerges when I am asked how many ties I can find in a picture (there are none). Since this picture is static in nature and there is no evidence that a tie was previously in the picture, I extend the instantaneous[48] characterization of Zero to describe the absence of objects even when there were not starting objects that were removed.

My next encounter with Zero occurs in a series of tasks that provide a set of symbols to be matched (e.g., "0" with "zero," "1" with "one," and so on). These tasks offer no clear

[48] I use "instantaneous" here in a similar way as "instantaneous velocity," or the velocity at a particular moment. In this case, it is a metaphorical strategy to highlight the temporal and changing nature of mathematical characters throughout the story.

reference to quantity of objects present or absent. For example, one task shows a series of train cars linked with each car labeled with a number word (such as "zero") and prompts me to "write in the numbers" (WB, p. 11). When read separately from what came before, it could appear that Zero has no connection to meaning of quantity and instead is simply a set of symbols that are associated with each other. However, unless there is a contradiction, when read as a mathematical story, the prior characterizations carry forward for interpretation. Thus, I read both "0" and "zero" as an absence of objects and the last number in a sequence of counting down.

Following this, a task prompts me to compare Zero with other numbers (such as 2) to determine which is greater or smaller and to "colour the sign with the smaller number" (p. 18, WB). Interestingly, the numerals are printed on what appear to be road signs, obscuring to what quantities they might refer (remember that up to this point, these numbers describe quantities of objects). In the previous cases, Zero referred to a quantity of absence (the number of ties when none are shown, for example) or the number at the end of a countdown sequence. In this case, "0" is on a sign and whatever was absent is left to my imagination.

So far, this series of tasks have enabled an initial image of the mathematical character Zero to emerge. This characterization can be represented with the diagram in Figure 4.3, where the arrows reveal the sequential layers of related characterization (in this case, about the quantity) throughout the story. This character diagram can help recognize that at the next point of the story in which Zero appears, any (or all) of these characterizations may be involved. If not, the character will develop further.

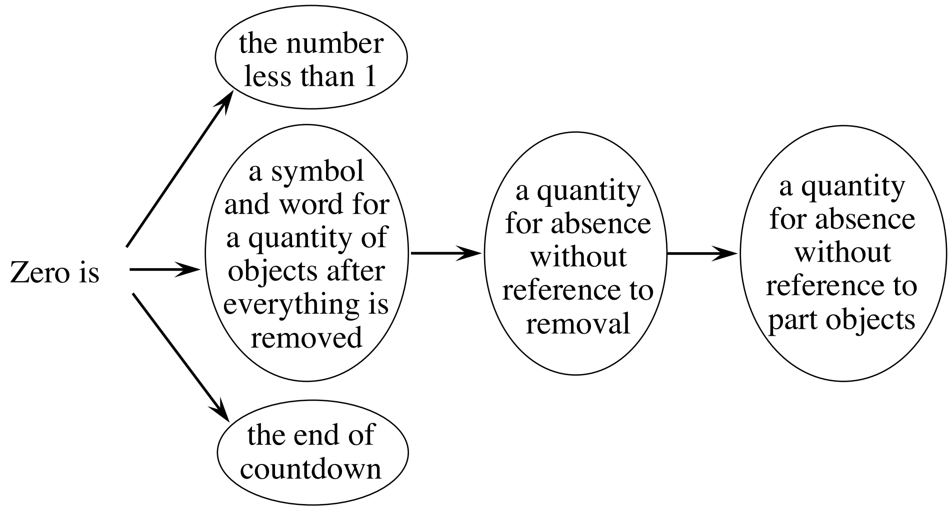

Figure 4.3. Beginning characterization of Zero after the first few encounters in MPAH1 (2012).

The next appearance of Zero provides several sequences of numbers for me to complete, two of which invoke Zero: 4, 3, 2, 1, ___ and ___, 1, 2 (pp. 19 - 20, WB). The first sequence enables me to use the characterization that Zero is the end of a countdown (the number that comes after 1), which was established in the introductory activity with the removal of cubes. The second task, however, which was not directly after the first, requires a new perspective. Seeing the 1 followed by 2 allows me to recognize a counting sequence and yet nothing up to now has required me to start with anything other than one. What could this mean? Counting so far has been associated with discrete objects and the counting activity has always started with 1. Also, the characterizations for Zero developed so far and represented in Figure 4.3, such as "the end of countdown," do not help unless I read the sequence backwards. Therefore, based on the answer, I read a new characterization for Zero, and this enables me to extend my Character Log to note that Zero is the number before 1 in the counting sequence.

The next development of Zero occurs later in Chapter 2 in relation to the decomposition of "number trains" (made of linking cubes, introduced in Lesson 1-2). On page 22 of the student book, a picture of a number train with four cubes is shown, and the text prompts me to "pull it into two groups" with three cubes in one and one single cube in the other. This is the introduction of what this mathematical story calls "number bonds," a relationship between sets of characters. A diagram like the one in Figure 4.4 shows this relationship, where 4 is labeled "whole" and 3 and 1 are labeled "part." The text asks, "How many are there in each group?" and states "3 and 1 make 4." Therefore, based on this example, composition (as in "make") is a result of decomposition. I conclude that if I want to figure out what joins to make a particular number (like 4), I can decompose the number into two groups. This relationship between 4, 3, and 1 is also referred to with the notation "4 — 3 – 1," where the longer dash separates the whole from the parts and the smaller dash separates the parts.

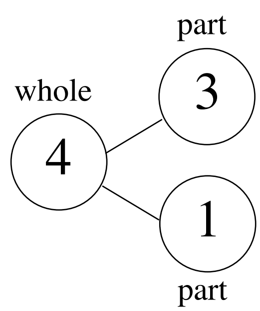

Figure 4.4. A diagram showing a number bond 4 — 3 – 1, adapted from MPAH1 (2012, p. 22).

After this introduction to number bonds, Zero is characterized when the text asks, "What other numbers make 4?" (p. 23, SB). This causes me to look for other ways of "pulling [4] into two groups" (p. 22, SB). The first number bond has a part labeled "0," provoking me to ask new questions: *What is the other part? How can I decompose 4 so that 0 is a part?* The answer in the text ("0 and 4 make 4") reveals that Zero can also be seen as a part of a decomposed number train, and that when Zero is a part, the decomposition is unnecessary and the original number train remains unchanged. Unlike the other numbers, which appear as separate groups of cubes after the trains are pulled apart, there is no physical change when "pulling" 0 from 4. Therefore, Zero is not only a quantity of absence, but, in the case of decomposition, does not change the initial number train.

The character Zero next advances through a change in setting when a “maths balance” is introduced and I am prompted to find “numbers that make 7” (p. 24, SB). A device in a photo contains an arm balanced on a fulcrum with evenly-spaced pins in each direction. Pins are sequentially numbered 1, 2, 3, ... in both directions of the fulcrum. In the student textbook, a photo shows a balanced apparatus with weights hanging on 3 and 4 to the left of the fulcrum and a weight at 7 to the right of the fulcrum to illustrate the number bond 3 – 4 — 7. The teacher is instructed to “display the maths balance” (p. 34, TB) and to “ask pupils to think of other numbers that can make the balance level” (p. 34, TB). Although the text does not show a balance that is unlevel (that is, not balanced), it is assumed the arm can freely pivot about the fulcrum.

Although Zero is not labeled on the balance, its presence is generated through the task “what other numbers make 7?” (p. 24, SB). Since 7 — 3 – 4 is already present in the text (in the example), the weights at 3 and 4 can be moved to other locations that balance with 7. This enables me to generate the options 7 — 1 – 6 and 7 — 2 – 5. What is another option? I consider 7 — 0 – 7, but this presents a challenge as there is not a “0” on the balance. However, the homework for this assignment addresses this situation as it instructs the parent, “Explain to your child that placing the weights at the number 7 on both sides of the balance will give 7 — 7 – 0” (p. 24, SB). Thus, even though the scale has no location labeled “0,” the numbered sequence reading left to right of 3, 2, 1, (fulcrum) 1, 2, 3 suggests that the fulcrum can represent Zero (based on reading this sequence and relating it to counting down). This new setting offers a new implicit characteristic of Zero as a center of balance.

Using the balance, still more characterization of Zero occurs. Since hanging a weight at the fulcrum would have no effect, I notice that whenever Zero is a part, the weights for the other part and the whole must be placed equidistant from the fulcrum to be balanced. In this case, the setting does not highlight an aspect of "absence" of quantity, since a weight at the fulcrum would still have weight (in fact, it would have the same weight that is hung at the other numbers). Instead, the focus of attention in this case is Zero as a relative location on a scale, a center of balance and the only location on the scale at which a weight would have no effect on the balance.

The last development of the characterization of Zero in these first three chapter occurred in a game described on page 31 of the student textbook. In this game, there are two packs of cards, one with cards numbered 1, 2, 3, 0, 1, 2, and 3, and another with cards numbered 1, 2, 3, 4, 5, 6, and 7. One player picks a card from one pack and another player from the other pack. A third player then adds the values shown on the two cards and tells the first two players the sum, which they confirm or challenge. This game, which occurs in Chapter 3, represents the first point in MPAH1 at which Zero can appear as an addend to an addition sentence[49] (introduced earlier in this lesson both an act of joining and as counting on). This opportunity extends the notion of Zero as ineffectual to a value that does not increase the quantity. For example, if I select cards 6 and 0, using a counting on strategy described in the text, I would start with 6 and not count beyond. However I note that because only one of the packs contains 0, the possibility of having 0+0=0 does not occur. Therefore, Zero is still prevented from being a whole or sum (so far), thus avoiding a problematic challenge of the decomposition of nothing.

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[49] That is, even though addition is hinted at during the decomposition activity generating the number bonds, the addition sentence, such as $6 + 2 = 8$, is not introduced until this lesson. Additionally, throughout this lesson, all addends ("parts") are counting numbers until this game.

Further comments on temporality. One aspect of the temporality of mathematical characters is to recognize how the introduction of a character (or the delay of one) affects the mathematical story. For example, imagine if the mathematical story in MPAH1 had delayed the introduction of Zero so that the introduction and study of composition and decomposition of number trains was accomplished without it. This is not so far out of the realm of imagination. This often happens in algebra when solutions of quadratics are studied often for weeks without the introduction of irrational numbers. Without Zero in the mathematical story, learning to add with only counting numbers might result in several changes in the way action is perceived by a reader, such as concluding that "adding makes bigger" or that decomposition always results in breaking into to countable quantities. If Zero's introduction into a story were delayed until after this work with addition and decomposition, contradictions might precipitate in a nontrivial revision of a reader's fabula of the mathematical story.

Based on my reading of the elementary textbooks, each mathematical story on a lesson or multiple lesson level, with rare exceptions, continues the mathematical stories from previous lessons (though perhaps not the lesson that was immediately prior). Although new mathematical characters are periodically introduced (such as the introduction of geometric shapes in Chapter 4 of MPAH1), sequences of mathematics curriculum, when read for mathematical characters, have surprising stability, with new characters introduced "in spurts." Similar to the characters found in multiple episodes of a soap opera, the mathematical characters in a mathematical story often appear and develop throughout multiple lessons of a textbook.

The fullness of mathematical characters. As explained in the literary theory discussion, an interesting feature of literature is the contrast of characters in relation to others. This raises the questions: *What might a full mathematical character be? What can be learned* *about mathematics curriculum if mathematical stories are read for centrality of character*? Drawing from Forster, Bal (2009) explains that round characters are usually psychologically complex. Since mathematical characters do not pretend to be human (unlike Bal's "paper people"), this psychological distinction is not metaphorically helpful. However, drawing from other features of round characters, namely those that develop over the story, offer surprise or provoke a reader's curiosity, a useful metaphoric definition for round mathematical characters can be found. Thus, *round mathematical characters* can be recognized as those that substantially change throughout a sequence, provoke a reader to revisit or change what they know about the character, and stimulate aesthetic responses such as surprise. This is consistent with Bal's description of round literary characters that "undergo a change in the course of the story, and remain capable of surprising the reader" while "flat characters are stable; stereotypical characters that exhibit/contain nothing surprising" (Bal, 2009, p. 115).

Therefore, in MPAH1, Zero is an example of a round mathematical character because its characterization changed substantially throughout the first third of the textbook. In the Character Log, after the initial characterization in Figure 4.3, there were nine additional entries that resulted in a change in characterization throughout these chapters, and there were a total of 19 encounters with Zero throughout these MPAH1 chapters for which there was a different characterization than the previous encounter. Similarly, although a reader at this point in the story (first grade) can interpret Zero as a "preserver" in terms of addition (since adding 0 onto another number preserves the quantity), later in the broader mathematical story, even more dramatic changes are likely to come; with the introduction of multiplication, Zero could be described as a "decimator," wiping out any quantity with which it is multiplied.

With my interpretation of these additional new characterizations of Zero, the diagram in Figure 4.3 is expanded to the one shown in Figure 4.5. This characterization demonstrates how complex a mathematical character can become and shows that once a central mathematical character is introduced, relatively little may be known by a reader. This diagram is temporal in nature. Characterizations were listed in the order (from top to bottom) that they were interpreted in the mathematical story, unless an interpretation was an extension of a previous characterization. When extensions were found (such as when Zero was seen as the number before 1 in a counting up sequence, which was related to it being viewed as the end of a countdown), then this interpretation was placed to the right of the one being extended. To aid in the reference of this diagram, each characterization in the diagram is labeled according to which branch it is located in, and these labels match those in the Character Log.

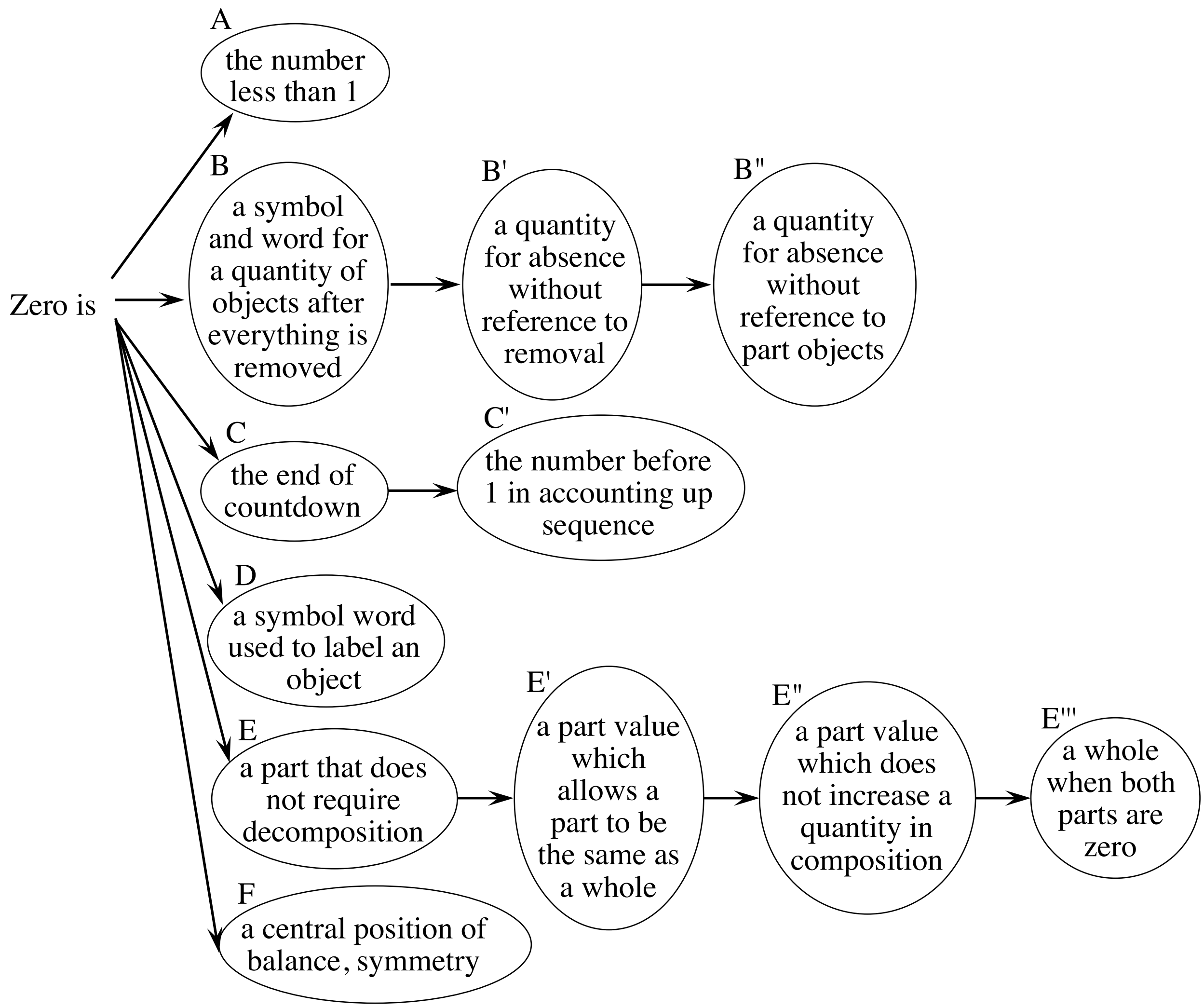

Figure 4.5. Extending the diagram in Figure 4.3 to show the temporal characterization of Zero in MPAH1 (2012) throughout the first three chapters.

However, not all mathematical characters are treated equally in curriculum, and some can be regarded as "flat" when compared to complex characters such as Zero. For example, consider the mathematical character Four, for which the only entries in the first three chapters of MPAH1 occurred during its introduction and in the counting down task. Even though Four makes repeated appearances, as seen in the introduction to the number bond represented in Figure 4.4, it was not critical for any of the mathematical events in the selected chapters (decided by considering the question "What if a different mathematical character were used instead?").

Changing Four to a different mathematical character, such as Five, would be inconsequential since a number train with five cubes can still be decomposed in the same number of ways. In this activity, all that is learned about character Four that was not known before is that it can be decomposed in multiple ways. Although an important character development, this remains the only extension of the character Four throughout the first three chapters of MPAH1. Therefore, in the context of the broader mathematical story across the three chapters, flat mathematical characters such as Four can be viewed as supporting characters that enable the focus on central characters such as One, Ten, and Zero and action (such as joining and decomposing).

Because of the serial nature of mathematical stories throughout textbooks, some mathematical characters that remain as supporting characters for years can later become central characters. For example, Two (which does not get central attention in the grade 1 MPAH1 textbook) could become a central character later in the broad mathematical story when its special properties could be investigated (for example, in the study of prime or even numbers). Considering the mathematical characters found at higher-level mathematics curriculum (consider the unique properties investigated in geometry when the measure of an acute angle in a triangle is 45°, for example), it is apparent that many mathematical characters in elementary schools are just waiting for their turn in the spotlight. This again illuminates the role of setting; certain mathematical characters will have more central roles in particular settings over others. For example, when the attention in curriculum first turns to study the mathematical nature of time in clocks and calendars, some seemingly flat and supporting mathematical characters such as 7, 12, and 60 can become round, central characters.

Character effects in mathematical stories. Bal (2009) writes that readers "see characters, feel with them and like or dislike them" (p. 113). This raises the question: *What can* *be learned about written mathematics curriculum when a textbook is read for the ways a reader might feel for or against a mathematical character?* This section explores this question with the aim to recognize how reading mathematics curriculum as a mathematical story may work to reveal otherwise unrecognized qualities of textbooks. Again, this reading is focused on the questions raised in the literary theory, namely "What if I imagine being the mathematical character?", "What type of experience does a mathematical character offer?", and "How do I feel about this character?" Using my reading of mathematical characters not as substantiation but instead as indications, I identify possible ways mathematical stories might evoke feelings for or against mathematical characters. However, this exploration is not exhaustive, and it is acknowledged that other mathematical stories may have different character effects.

Surprise. A potential confrontation of mathematical character (in the sense described by Bal (2009)) during a mathematical story can evoke wonder and surprise in a reader. For me, the sudden appearance of Zero in MPAH1 after the first activity introducing the counting numbers produced excitement. Its emergence through action, as opposed to how the other numbers were introduced first as symbols that then were connected to action, caused me to view this character as one would a guest that arrived uninvited. While unexpected, this brought for me a pleasant surprise, one that hinted that there were more surprises to come involving Zero. The inclusion of Zero as a decomposed part of a number train "whole" also felt foreign and surprising, sparking a reconsideration of what it means to decompose.

Therefore, how a mathematical character is contrasted with others in the story can affect the feelings about the character, although each reader's feelings towards the character can differ. Although it was introduced with other numbers, Zero became an interesting mathematical character because the mathematical story in MPAH1 continually, though subtly, revealed how

Zero had many characteristics that set it apart from all the other mathematical characters. For example, other numbers affected quantities while adding, yet Zero did not. Zero (along with Ten) was also a boundary to counting in MPAH1, the end goal of counting down from any of the other numbers. Zero regularly offered additional challenge (in comparison to other numbers) because it represents a quantity that is neither countable nor visible. Ironically, the one facet that became predictable about the character Zero was that it was going to be different and surprising.

It is also important to note how the placement of the removal task enabled my surprise in the introduction of Zero. This removal task followed an introduction of the mathematical characters One through Ten (described earlier). Entering the removal task, I assumed that I already was aware of the mathematical characters involved in this portion of the lesson and that this portion of the story was now focused on relations between them through action (i.e., the action of taking 1 away). Until the quantity of "nothing" emerged through the action of removing cubes, there was no hint that a new mathematical character would emerge.

Throughout my reading of text, I noticed that when a mathematical character consistently led to no surprise throughout an extended mathematical story (i.e., multiple chapters of a textbook), it became predictable and uninteresting to me as a reader. An example of this would be the number Six, for which I cannot find one interesting thing to say as the result of the different mathematical stories analyzed throughout this project. Although Six has interesting qualities that I know of from outside this analysis (it is, after all, perfect), the most information about this number in MPAH1 was that it is a symbol to represent the quantity of six discrete objects, is 1 less than 7, and is 1 more than 5. Since my characterization of Six was consistently reinforced and was never challenged throughout the sequence, my encounters with problems involving Six did not generate any expectation of surprise or excitement. This is not to say that there might not be a future point in the extended mathematical story for which I could imagine that Six would lead to surprise; instead, there is nothing from this portion of the story in MPAH1 that leads me to expect or predict that this will happen.

Identifying with a mathematical character. When reading the mathematical stories for how they might use mathematical characters, I recognized that I drew upon my own relationships in order to interpret relationships between mathematical characters. That is, forming analogous connections between mathematical characters and their relationships and people with whom a reader is familiar can allow him or her to identify with a mathematical character. An example of this occurs in the mathematical story in Lesson 2-2 of SFAW2 involving doubles. This mathematical story opens a direction to the teacher for a "warm up" to:

> Review doubles to 12. Call out the doubles combinations with sums to 12 (1 + 1, 2 + 2, 3 + 3, and so on, up to and including 6 + 6). Have children, in unison, call out the sums. Practice them in sequence and then out of sequence. (p. 45)

During my reading of this mathematical story, I imagined calling out the doubles in the warm-up as an exercise in adding. Immediately, the patterns in the addends revealed that these addition problems were special. There is also a rhythm to the reading of these "doubles combinations," and the resulting sums form a sequence that is reasonably familiar (2, 4, 6, 8, ...). Thus, this activity immediately positions certain numbers (2, 4, 6, etc.) as numbers of interest for this lesson and names them as "doubles."

The next portion of the story, the "Learn!" box (see Figure 4.6, with its corresponding teacher notes), states that "doubles facts are addition facts with two addends that are the same" and provides an example of doubles fact (7 + 7 = 14). Following this box, the text prompts the teacher to ask, “Is 17 a double? Why or why not?”, for the first time raising the possibility that not all numbers have this quality, highlighting the special quality of being a double[50].


Learn!

Doubles facts are addition facts with two addends that are the same.

7 + 7 = 14

Word Bank
doubles fact
addition fact
addend

Check

Draw to show the double.
Then write the addition sentence.

2 Teach

Learn!

Have children look at the two sets of stars. Review the word *double* and the terms addition fact and doubles fact.

Ongoing Assessment

Talk About It

- **Is 17 a double? Why or why not?** *(No, because 17 cannot be divided into two equal parts)*
- **How can you tell whether or not a number is the sum of two equal addends?** *(If it is, then it can be divided, or separated, into two equal parts)*

Figure 4.6. The “Learn!” sections of Lesson 2-2 in SFAW2 (p. 45). From SCOTT FORESMAN ADDISON WESLEY MATH MICHIGAN TEACHER EDITION GRADE 2 VOLUME 1 by Charles et al. Copyright © 2008 Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

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50 Since the mathematical story up to this point has only introduced whole numbers, I assumed that “doubles” refer to even whole numbers.

Therefore, this mathematical story starts by introducing a special quality of mathematical characters and then prompts the reader to establish that this quality is only associated with certain mathematical characters and not others. This not only raises the possibility that not all numbers are doubles but also calls to question how one can know if a number is a double or not? Although the notion of a doubles fact seemingly addresses this (by focusing on when the addends are the same), this attention is on a quality of an addition fact, not on the sum alone. Therefore, when given any sum, such as 17, determining if it is a double is not only a matter of checking whether the addends are "the same." Instead, it is a matter of whether or not such an addition sentence is possible. During my reading of this mathematical story, this question of whether 17 is a double prompted me to consider whether an addition sentence ___ + ___ = 17, and the best I could do was get two addends that were 1 apart (that is, 8 + 9).

Therefore, 2, 4, 6, and other doubles were members of an exclusive club of which 17 was not a member. Positioning the mathematical character of 17 against the doubles enabled me to draw from my own experiences of being excluded from a group, which caused me to sympathize with 17. Although it may seem unusual to identify with a number in this manner, this identification with one character helped to further develop the characterizations of others (now exclusive members of a group). And while 17 had not previously received any special attention earlier in the mathematical story, its new role in the mathematical story led to the development of curiosity and interest. That is, since the number 17 was now identified as a mathematical character that was not part of the doubles club, then this caused me to wonder *To what club might it belong? Are there other mathematical characters that are left out of the doubles club?* The raising of these types of questions can be productive in that they can lead to the recognition of an equally sizable group (the odd numbers) that are not doubles.

Mathematical hero. My last example of mathematical character effects involves how a mathematical story can position a mathematical character in a sequence to "save the day," much as a superhero might in a different genre. Sinclair's (2005) reading of the proof that $\sqrt{2}$ is irrational, described in Chapter 1, demonstrates how a number can be positioned as a hero within a story. Similarly, mathematical stories in textbooks offer this opportunity. For example, in Lesson 3-1 of EM3, four students are initially selected to walk heel-to-toe along a side of the classroom while counting steps. The measurement of each student's counts are recorded in a chart on the board using language of approximation, such as "*about, almost, between ___ and ___*, and *a little less than*" (p. 171). The text then prompts the teacher to ask, "Can anyone suggest a unit name for each measurement?" raising the need for a unit name for a measurement as well as the possibility that different measurements might use different units. So at this point in the mathematical story, there is an attribute of the classroom being measured (a length of wall in the classroom) and there are now four (potentially) different measures for that quantity using four different units of length. The focus on unit names (such as "Carmen-shoes", p. 171) explicitly introduces these four units as mathematical characters of this story.

To better make sense of this portion of the mathematical story, it is important to take into account what has occurred before this point in the story. Earlier in the course, lengths were found using a ruler in Lesson 1-4. In that lesson, the teacher is prompted to demonstrate how to measure the length of line segments with a ruler to the nearest inch and centimeter. Therefore, early on it is suggested that linear segments can have multiple measures (using inches and centimeters) and there is no sign of contradiction or conflict. Previously, no attention was given to why measurements with different units result in different counts of units for the same object's attribute. Therefore, it is assumed in Lesson 3-1 that having multiple measures is expected and that these measures can all be associated with the length of a classroom. Since the next prompt is for the teacher to also measure the length using the same procedure, the introduction to yet another measure for this length is not interpreted as problematic.

However, the mathematical story hints at a conflict when text then asks, “Why weren’t all the measurements the same?” (p. 172, TB) and prompts the teacher to “point out the need for agreeing on a ‘class shoe’ length that everyone in the class can use” (p. 172, TB). Although the “need” for the agreement in this passage is vague, clues for its motivation are offered in an accompanying image of a page from the Student Reference Book. The text states, “The problem with using body measures is that they are different for different people” (p. 133, SRB) and a picture shows two students measuring an object with their thumbs. The child to the left states, “It’s 1 thumb wide,” to which the child on the right responds, “No! It’s 2 thumbs wide.” The phrasing of “the problem with” and the child’s response of “No!” indicate that there is a conflict when multiple measurements (mathematical characters) are found for the same object and that this problem could spark disagreements between measurers (actors). This conflict then is resolved in the text with the introduction of the standard unit:

> Using **standard units** of length solves this problem. The standard units never change. They are the same for everyone. If two people measure the same object using standard units, their measurements will be the same or almost the same. (p. 133, SRB)

Following this introduction, a new picture of the children actors is presented indicating that the disagreement is resolved, where the left child states, “it’s 2 inches long” and the right states, “That’s right! It’s 2 inches long” (p. 133, SRB). Therefore, the introduction in this lesson of the inch unit as a standard unit that eliminates the conflict positions it as a hero of the mathematical story. The text then continues by prompting the teacher to find a standard unit of length that represents the class (through a process of averaging) and then this "standard unit" representing a "class shoe" is used to measure the length of the classroom wall.

Implications of Reading for Mathematical Characters

Using literary theory as a way to interrogate the manifestation of mathematical objects within written mathematics curriculum, this chapter argues that the conceptualization of mathematical sequences as stories offers new ways of understanding mathematics curriculum and to implications for learning and teaching of mathematics in general. Therefore, this section offers a discussion of the potential challenges and benefits to learning and teaching mathematics that is relevant to mathematical characters and identifies connections with other areas of research in math education. These remarks are organized to first address issues related to student learning, followed by comments directed to how this analysis supports mathematics teaching.

Implications for student learning. Reading textbooks for mathematical characters can enable analysts to gain insight into the curriculum's role in supporting (as well as possibly inhibiting) student learning of mathematics. This section will start by discussing how reading for mathematical characters in textbooks may support the development of diverse concept images (e.g., Tall & Vinner, 1981). Then, attention will shift to discuss possible challenges mathematical stories pose for students, such as relating and distinguishing mathematical characters and revisiting and revising character images. Finally, this section will end with a discussion of how a mathematical character's image grows or becomes constrained throughout the mathematical story.

Supporting the development of concept images. Several researchers who study mathematical thinking and learning have offered frameworks to describe the different ways individuals may structure meaning. For example, Tall, Vinner, Dreyfus, and others have proposed that mathematical knowledge is composed of a web of “concept images” of mathematical concepts (such as a limit or function) to describe the phenomenon observed when individuals who appear familiar with formal mathematical definitions draw upon other ways of thinking about the mathematical object at different times and contexts (Rasslan & Vinner, 1995; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Though the researchers expected to observe individuals using formal definitions of mathematical actions and objects such as limits and functions to reason mathematically, their data instead revealed that in different contexts, individuals often drew from other informal notions. In this framing of mathematical thinking, “concept images” differ by individual and include both a mental picture as well as its associated properties (Even, 1993). For example, for Zero, a mental image might be an image of an empty plate (found in a task in MPAH1) or a sense of ineffectuality (in the sense that it does not change an addend when counting on). Vinner and his colleagues have demonstrated that some individuals hold conflicting concept images (of which they may not be aware), namely differing ways of viewing a mathematical idea that are logically at odds. This indicates that concept images are separate interpretations of a mathematical concept (such as a way of describing a mathematical object) that do not necessarily have relation to one another.

I propose that the diagram of characterizations read of 0 in MPAH1 (in Figure 4.5) can be viewed as a potential map of concept images made by a reader (in this case, me) of relations and characteristics of a mathematical concept (the concept of zero). The separate branches of the character map can be viewed as sets of images that do not have close relation. Reader Response Theory not only suggests that different readers of the same story will construct different characterization mappings (collectively, Bal’s “character image”), it argues that this is likely (Rosenblatt, 1988). However, whatever the particular characterizations that emerges for a reader, this way of reading for the development of a central mathematical character enables the recognition of a diverse set of ways that zero appears in the text, how it is involved in action (such as decomposition), and how it is realized through different settings (such as the maths scale and the absence of a tie in a picture). Since zero was never explicitly defined in MPAH1, these characterizations (e.g., a quantity of absence, as a part in an addition sentence) were generated implicitly, which suggests that concept images can be built for a mathematical object that is never explicitly defined.

The study of the emerging development of a mathematical character (such as Zero) throughout the beginning of MPAH1 helps to explain how the development of multiple concept images is supported by mathematics curriculum. That these different ways of characterizing Zero emerged through my reading of different types of tasks set in different settings with different actions also supports Tall and Vinner's observation that different types of questions on surveys would call upon different images – just as the reading of later portions of the literary story call upon different prior characterizations of a literary character realized earlier. As each new characterization was recognized, it was usually used again later in the mathematical story for interpretation of some parts of text and not for other parts of text. For example, when I was confronted in MPAH1 with the possibility of having cards in the game that would prompt me to add a counting number (such as 6) and 0, my characterizations of Zero up to that point (as a quantity of objects after everything is removed) was not helpful. However, interpreting Zero as an ineffectual part for decomposition was useful for recognizing the lack of change when adding (counting on) zero with 6. Therefore, at different points in the sequence of mathematics text, different characterizations of Zero were the basis for interpreting the text.

Challenges and support for reading for the mathematical story. Considering the mathematical development of a textbook as a story raises new potential sources of challenge for the student. When the same name, such as 0, is used in very different ways (such as a representation of absence when everything is removed versus a lack of action in decomposition), it is up to the reader to decide if these characters are the same or merely have the same name. Since different people can have the same name, is it unreasonable for a reader to assume that the same label cannot reference two different mathematical characters? Consider the sign "x" and the various ways it is used in algebra as an unknown value or as a general representation of a set of values. When encountering a character in a story (whether literary or not), what enables a reader to decide if they refer to the same or different objects?

Throughout my reading of MPAH1, I assumed that each encounter with Zero referred to the same mathematical character. However, this is not to say that all readers will read with this assumption. Indeed, distinguishing two different characters with the same name might be one way a reader deals with contradictory characterizations. The possibility of parsing a mathematical character into multiple characters with the same name could explain why students who encounter the same mathematical characters in different stories do not recognize them to be the same. For example, students who easily recognize within an algebra course the object $t^2 = 64$ as an equation which is true for only certain values of t might stare at the same object in a physics course without any recognition or connection. Although a reader of any of the *Harry Potter* texts would probably recognize that the label "Ron" refers to a friend of Harry's, they are not likely to assume that a "Ron" encountered in a different story refers to the same character. Yet, in literary stories, some characters do make unexpected appearances (such as how Sherlock

Holmes shows up in stories not written by Arthur Canon Doyle). What enables a reader to recognize and decide when a name for a character represents the same set of characterizations?

Reading with meanings generated outside the story also has implications for the reading and understanding of referential mathematical characters. Although reading mathematical stories for how mathematical objects are manifested within a sequence requires attention to how the character image develops through the text, Rosenblatt and others argue that all readers draw from prior knowledge. Therefore, the notion of referential characters in mathematical stories helps explain why students bring colloquial or everyday meanings of words such as "equals" or "number" (e.g., Hersh, 1997). Not only is this expected, but this reading suggests that mathematics textbooks appear to rely on referential mathematics characters at times, such as expecting students in the first lesson of MPAH1 to be able to count objects with the numbers 1, 2, 3 that were being introduced. Similarly, names of characters can bring associated properties that sometimes mislead a reader. Consider, for example, the milliliter, which a reader might connect with "a millionth of a liter," instead interpreting the suffix "milli" to represent "a thousandth of." Reading mathematics curriculum in a textbook as an evolving mathematical story can help a reader (such as a teacher, researcher) identify portions of curriculum (or adjust if necessary) that might help a reader to recognize how the meanings in "plain English" is different in "math lingo" (Hersh, 1997, p. 48).

Just as later revelations about round literary characters, such as Severus Snape, enable a reader to reinterpret earlier parts of a literary story with a new perspective, later characterizations of mathematical characters enable a reader to revisit prior portions of the mathematics textbook for reinterpretation. For example, the use of the balance to think about Zero enabled me to reinterpret the prior task with the decomposition of a cube train. In the case of the cubes, Zero required the original train to not change through decomposition, but I did not see Zero as a part for which the other part and the whole were equal since there were never two separate quantities to compare. The balance helped to see that when Zero is one of the parts, then the other part is (separately) equal to the whole. After reinterpreting the cube decomposition, I could "see" 4 — 4 – 0 as two separate cube trains that are of equal length, where one train had been put into two groups where one group had zero cubes.

This temporal change in sophistication of understanding of the relationship between parts and whole, exposed by the close reading of mathematical characters throughout a mathematical story, also helps to highlight a potential challenge faced by expert readers when analyzing mathematical textbooks. That is, when reading a portion of text that calls upon knowledge of a particular mathematical object, experts can choose any one or more of an elaborate set of ways of viewing that object as a basis for interpretation. Although reading with complex concept images at hand enables help experts recognize potential mathematical conflicts and contradictions presented in the textual materials, it can also blind the expert reader to the increasing complexity of the concept throughout a sequence. The consequence is that experts (e.g., teachers and researchers) might expect novice readers (i.e., students) to read with complex characterizations of mathematical objects early in the mathematical story when certain properties may only be revealed later, if at all. Reading mathematics curriculum as a temporal mathematical story can help teachers look for (and adjust if necessary) later parts of the mathematical story that enable expected characterizations to emerge.

The constraining and expanding potential of a mathematical character. Bal (2009) explains that beyond what is known about a literary character is what can be imagined about it. As an image of a mathematical character is developed, some characterizations are refinements of existing characterizations while others open up new areas of potential development. For example, in a literary sense, learning of a new character named “Chris” might engage a reader’s imagination in ways that conjures up a variety of possibilities about that character. At its beginning, however, its character image may not include information about the gender of the character. That is not to say that a reader might not guess or imagine the character as female or male. However, in cases of ambiguity, readers are left with an array of imagined possibilities. Then, if later a pronoun, such as “she,” is used in the text to refer to the character, this narrows the array of imagined possibilities while increasing a reader’s image of the character. Further elaborations may additionally narrow the set of imagined possibilities. Thus, there is an inverse relationship between the complexity of a mathematical character and the extent of the imagined possibilities for the character. As a character continues to evolve with more complexity, each new characterization can further reduce its ambiguity, narrowing the array of imagined possibilities for that character.

Reading mathematics textbooks as a mathematical story helps to recognize how this phenomenon can occur throughout a curricular mathematical sequence. For example, when Zero was introduced in MPAH1, it was interpreted as the quantity for the absence of objects after cubes were removed. With this start, little is known, leaving much to be imagined. For example, I could imagine the symbol “0” as an icon for the empty hand, encompassing nothing. When new settings or actions are encountered later in the story without reference to Zero, a reader can liberally make assumptions. For example, if the mathematical story shifts to division and Zero is not encountered as a divisor, a reader may imagine or assume a range of possibilities for what might result. Once explored, however, the elaboration of Zero has a constraining effect, enabling the reader to construct boundaries of what may or may not be recognized as Zero. These boundaries become more and more stable as characterizations are repeated throughout the sequence. However, rather than being demarcated strictly by the limits of formal mathematical definitions, as is commonly assumed, a reader's emerging image of a mathematical character instead becomes bounded by his or her limits of imagination and reasoning.

In addition to constraint of a character image, a reader's inquiry and anticipation for a mathematical character may expand its array of possibilities. The introduction of mathematical characters at the beginning of a mathematical story helps to orient a reader to look for these characters in the upcoming events of the story. It might work to focus the attention of the students – such as "This story opens with an set of numbers that can be found when ever a number is added to itself. These are called the "doubles." Notice them. This story is going to be about how these are special. Let's see what there is to learn about this group of numbers." This type of introduction early in a mathematical story also offers an opportunity for a reader to imagine how the story might advance, anticipating new characteristics that may result. A reader might also raise questions about the mathematical character before learning too much about it, which further expands possibilities for characterization.

How reading for mathematical characters can support teaching. One possible benefit of the interpretation of mathematics curriculum as a mathematical story is the empowerment it offers teachers to recognize how mathematical characters are positioned within a development (Is it a central character? Supporting?). When a teacher sees a potential benefit for making a character more central, he or she could purposefully adjust the mathematical story to draw more attention to it. This could be done, for example, by explicitly noting the surprising arrival of 0 ("Taking that cube away gave us a number we haven't seen yet") or asking students to imagine how they might find Zero in later parts of the mathematical story ("What do you predict will happen to 0?").

Reading mathematical stories for character effects, such as the surprise introduction of Zero offers and the alienation of 17 from the set of doubles, also offers a teacher opportunities to imagine ways to play on these effects during the enacted lesson. As "storytellers," teachers choose to highlight (or deemphasize) their own temporal reactions to the mathematical story (such as adding "Wow! Who knew that 0 would pop up like that?" or "Poor 17! How sad."). Beyond adding dramatic effect, it is possible that doing so can help students recognize the aesthetic of mathematical work and view it as emblematic, that is, to come to expect to be personally moved by mathematical developments in stories. This, in turn, may shift the typical student mathematical experience closer to that of mathematicians[51].

In addition to capitalizing on aesthetic opportunities throughout a lesson, reading for mathematical characters can enable a teacher to recognize how the settings chosen for the various scenes of a mathematical story are key to revealing certain characterizations. This offers a teacher (and curriculum materials designer) new criteria on which to decide which representations to use. For example, does the teacher aim to emphasize Zero as a quantity or as a location (the fulcrum) on a balance? The role of associated actions is also available for scrutiny by a teacher. For example, through a close reading of the mathematical sequence in MPAH1 regarding number bonds, a teacher might notice that the initial representation of parts and whole

[51] The study of the role of aesthetic in the work of mathematicians is well described in Sinclair (2001, 2002, 2004, 2006) and Burton (1999). A helpful illustration of this is the emotional response described by Hofstadter (1992) when he realized that all of the five "most special of special points" relating triangles and circles were collinear with the exception of the incenter: "Poor little neglected incenter" (p. 1). This reaction, along with a dissatisfaction with its exclusion spurred his lengthy investigation into why the incenter was not included, leading to the discovery of other points of concurrency that are collinear with the incenter.

is brought about through decomposition (and not composition). This recognition enables a teacher to think about the impact of starting from decomposition and consider questions such as, "how does decomposition help think about composition? How does this story transition to be one about adding together?"

Finally, the reading for mathematical characters can enable teachers to predict and recognize why students look at mathematical objects as they do. The heuristic of reading for the development of explicit and implicit characterizations allows certain characteristics of a mathematical character (such as Zero) to be recognized that otherwise might not be recognized by a knowledgeable reader. For example, in its abstract form, the number 2 is not a description of quantity of objects; it is a quantity regardless of any reference to worldly objects. However, when looking at the introduction of the number 2 in MPAH1 through the lens of what is revealed about this character and how this number carries through the story (as the number of marbles to count on, for example on page 28, SB), it can be seen that 2 is a mathematical character firmly rooted in the concrete world.

Looking Ahead

This chapter has focused narrowly on mathematical stories for what can be learned about the manifestation of mathematical objects. The next chapter, focusing on mathematical plot, will look for ways to read a sequence as a whole as a way to learn how the parts (such as the mathematical characters) interrelate and together make the whole.

"Remember: Plot is no more than footprints left in the snow after your characters have run by on their way to incredible destinations. Plot is observed after the fact rather than before. It cannot precede action. It is the chart that remains when an action is through. That is all Plot ever should be. It is human desire let run, running, and reaching a goal. It cannot be mechanical. It can only be dynamic. So, stand aside, forget targets, let the characters, your fingers, body, blood, and heart do." – Ray Bradbury, *Zen in the Art of Writing*

CHAPTER 5

Mathematical Plot: The Structure and Effects of Mathematical Stories

Chapter 4 demonstrated how reading mathematical sequences for mathematical objects not only can enable a reader to recognize how an object comes to be for the reader temporally throughout the text, but also how he or she can identify with it. It explained how a reader can develop an image of a character throughout a story and at the same time reacts (in various, independent ways) to that evolving image. For example, after activities focused on finding doubles, the prompt "Is 17 a double?" offered surprise because it unexpectedly shifted my attention from finding doubles through adding to evaluating whether a number is a double. This quality (which can be called *doubleness*) had not, by that point, ever been suggested to me as important or special, and therefore, the question about determining if a number has this quality was unanticipated. This prompt's effect on my reading, namely its emotional provocation, is because of its position in the sequence. Since the prompt followed the generation of numerous doubles, it provoked my empathy since it is excluded from a large group of other characters in focus (the doubles). If this prompt had started the mathematical story, then a different response, such as wonder, would likely have been triggered (for example, thinking *17 is different than all of these doubles. I wonder what other ways 17 is special?*). Then again, if this prompt were instead asked after a sequence of similar questions, such as "Is 6 a double? Is 3 a double? Is 10 a double? Is 17 a double?" it might have been interpreted as an invitation to practice a procedure, possibly evoking disinterest or annoyance.

In broad terms, this chapter analyzes the ways mathematical sequences such as these in textbooks compel a reader to desire, in the sense used by Bradbury in the opening quote, to keep reading while shaping his or her experience with the mathematics. It responds to Sinclair's (2001) challenge (introduced in Chapter 1) to the field of mathematics education to learn how mathematics textbooks might inspire the reader's curiosity and imagination. Specifically, this chapter presents a proof of concept that mathematical sequences in texts can be read for the ways the mathematical changes throughout a sequence might affect a reader, as well as an argument that this lens offers a new window into the structure and affordances of mathematics curriculum more generally. In the way that a novel can engross a reader and compel him or her to keep reading, this theoretical analysis seeks to support the future creation of mathematics curriculum that is designed to provoke a reader's intrigue and wonder in the mathematical phenomena and generate interest to embark on new mathematical terrain.

However, the focus of this chapter is not limited to understanding the effects of particular moments of the mathematics sequence (such as the prompt "Is 17 a double?"), but also addresses how collections of these moments in an ordered sequence work on a reader. To learn more about the aesthetic of parts of a mathematical story in relation to the whole (a continuous sequence in which the part is embedded), this chapter focuses on the conceptualization of *mathematical plot*. Althought the word *plot* has historically been used to refer to multiple aspects of narrative, this analysis will focus on its use to describe how a reader may experience the story as he or she temporally encounters the multiple moments that make up a mathematical story. The attention to the consequences and benefits of sequence in mathematics curriculum builds a theoretical foundation upon which other mathematical sequences can be critiqued and improved.

Rationale

I chose to focus on mathematical plot for multiple reasons. First, I argue that the notion of mathematical plot in curriculum highlights an otherwise hidden aspect of mathematics curriculum. Although other aspects of mathematical stories directly correspond to a single mathematical phenomenon (for example, mathematical characters can be mapped to the reader's identification of mathematical objects and the mathematical setting can be identified as the representation), the conceptualization of a mathematical plot surprisingly has no single corresponding mathematical correlate. In contrast, plot involves both logic (making sense of what is happening in a story) and aesthetic (a reader's response to the story, such as surprise), mathematical qualities that are often considered separately. In analyzing a story's plot, a reader grasps and attends to both its logic and aesthetic as co-dependent forces. That is, the logical sense a reader constructs and seeks while reading a mathematical sequence alters his or her aesthetic experience, and vice versa.

In addition, the construct of mathematical plots offers mathematics educators (including curriculum designers and researchers) a new construct for the critique of mathematics curriculum. Examining mathematical stories can highlight possible aesthetic implications of sequences for a student reader of the textbook and can provide a teacher with a heuristic to evaluate possible affordances and constraints of sequential changes when designing lessons. Thus, mathematical plot is a conceptual tool that can assist in the development of mathematical sequences that potentially offer new curricular materials, and thus new possible aesthetic experiences for students.

Background

The suggestion that reading mathematical sequences as narrative can offer new insight about the structure and effect of mathematical texts, while atypical, has already been made by a small group of mathematicians. For example, Thomas (2002; 2007) compares mathematical proofs to narratives and argues that mathematicians "postulate some things to talk about and the relations that they are to have among themselves... this is analogous to telling a new story about characters already known from their appearance in the local mythology" (2002, p. 44). He also describes the similar effects on the reading of stories and mathematical proofs, claiming that both have an "implicit question," specifically "how will it turn out?" Thomas points out that, similar to narratives, mathematical texts can also have dramatic tension, and he connects "gripping" narrative consequence (the effects of action) with mathematical implications based on logic. Thus, Thomas uses story as an analogy to highlight the potential temporal response to mathematical proof by a reader.

A historian of mathematics, Raviel Netz, specifically focuses on the aesthetic effects of mathematics sequences in historical mathematical texts. To compare the structural form and the aesthetic affordances of mathematical sequences of the texts of Euclid and Archimedes, Netz (2005) relies on a metaphorical interpretation of mathematical text as narrative. By contrasting his reading of two historical texts with distinctly different aesthetic qualities, Netz points out that mathematical sequences are not pre-determined but instead are the result of authorial choices by the mathematician. As Netz explains,

> The author may chose (sic) to reveal as much or as little of the plot as he or she pleases; he or she may structure this information in many possible sequences. Such choices may possess aesthetic value, and in this way mathematical texts may possess an aesthetic dimension. (2005, p. 261)

To illustrate the role of these choices, Netz explains that the text in Euclid's *Elements* builds from "the absolute nothingness of the foundations of geometry" to achieve a "capstone" of the Pythagorean theorem. At the beginning, Euclid offers the reader (in this case, Netz) few clues about the direction the mathematical sequence may be headed. Instead, Euclid's sequence foregrounds an incremental progression, focusing the reader's attention more on how each part relates (logically) to the part before. In contrast, Archimedes' *Sphere and Cylinder* sets up a reader's expectation of a major accomplishment at its beginning (a derivation involving the volume and surface area of a sphere), but then immediately shifts its focus to mathematical conclusions involving other, seemingly unrelated two-dimensional shapes. Netz explains that Archimedes' structuring of this sequence enabled him (the reader) to sense "a brilliant master-stroke" (p. 257) and argues that removing the initial declaration of the goal would have destroyed this effect. The ability for a mathematical sequence to provoke surprise or suspense, which Netz connects with the tension between what he refers to as "freedom" (the ability of the mathematician to choose how what to argue in the line of mathematical reasoning and when to argue it) and "necessity" (the logical requirements of the argument), rests in the mathematical sequence.

To analyze structure, Netz (2005) uses an analogy of poetry to describe the rhythmic patterns of mathematical sequences. He explains,

> For while the rhythm of long and short syllables is not itself a marked feature of Greek mathematical texts, the texts are marked by other rhythmic patterns, which are of clear aesthetic significance. The rhythmic pattern of verse represents the fact that verse is built from clearly defined units – lines – that participate in larger-scale structures – stanzas – and possess an internal structure – feet. Greek mathematical texts – perhaps more than any other prose style – are similarly built from clearly defined units, which allow a similar structural analysis. This is especially true of proofs which ... possess the strong syntagmatic structure of the "since – therefore" sequence. This sequence works on *assertions*, and combines them into *arguments*. (p. 265)

To compare the structural deductive chains of reasoning in Euclid's and Archimedes' texts, Netz (1999, 2005) proposes structural diagrams (see Figure 5.1) which show how assertions (numbered and placed at the bottom of triangles) lead to conclusions (numbered and placed at the top of a triangle), which become assertions for further conclusions, and so on. These patterns reflect a "style" of a mathematical deduction used by authors, and Netz argues that each style has aesthetic effect on a reader. For example, by maintaining an evenly-paced structure of mathematical results, Netz argues that "Euclid does not aim to startle, in a quick stroke, but to impress, in a stately progression" (p. 259). In contrast, Archimedes' structure unexpectedly calls upon the reader to revisit earlier results (e.g., theorem 4 comes back to help produce 15), thereby obscuring the direction of where the sequence is headed and setting up the potential for a reader's surprise.

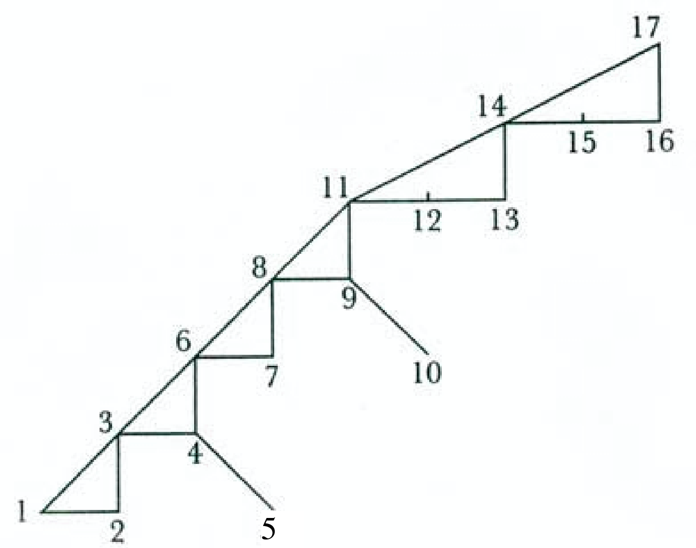
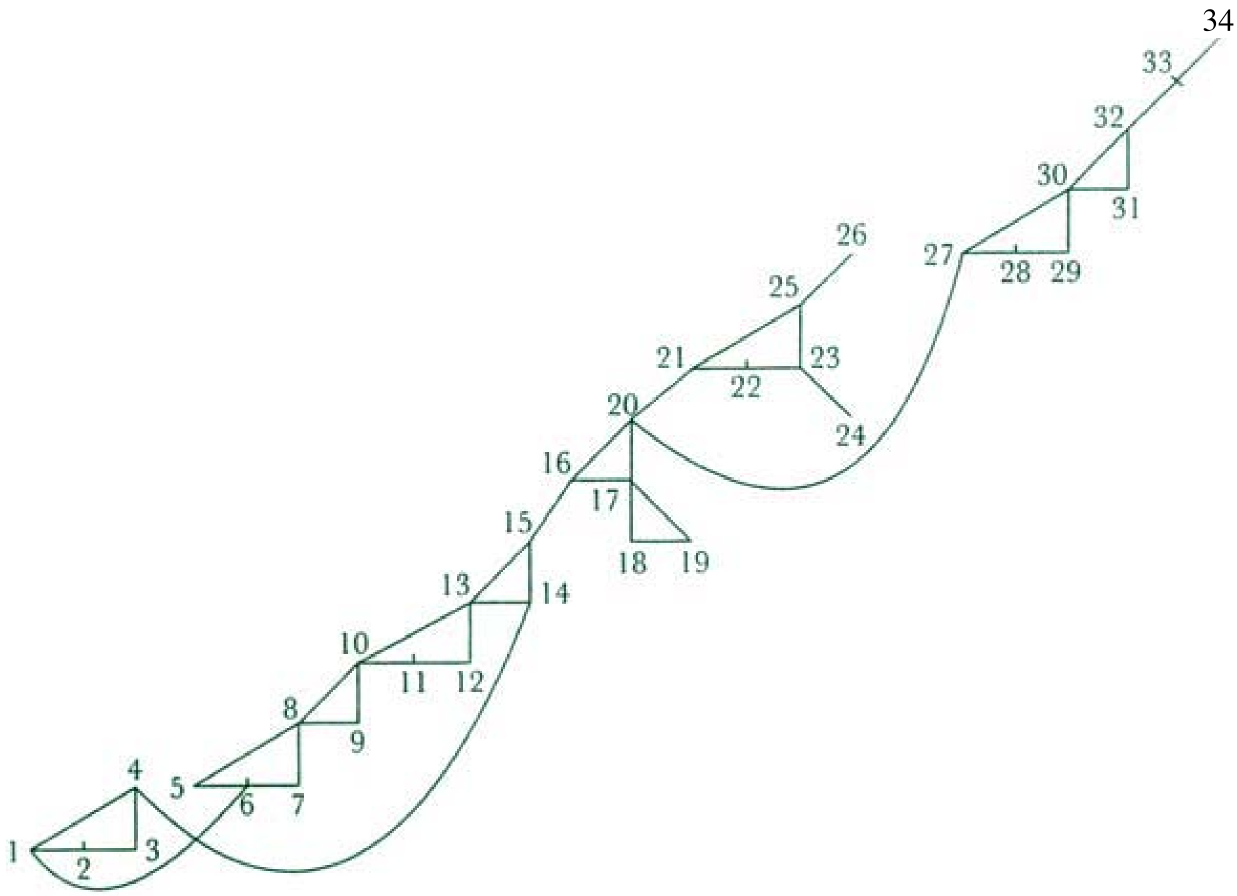


Figure 5.1. Netz's structures of mathematical deductive arguments in Book 1 of Euclid's *Elements* (top) and at the start of Archimedes' *Method* (bottom) (2005, pp. 266–7). With kind permission of Springer Science and Business Media.

Figure 5.1 (cont'd)



Given that Netz' diagrams in Figure 5.1 help represent his interpretation of the sequential structure of mathematical argument, it is reasonable to ask do his diagrams represent the mathematical plots of these texts? These diagrams describe how each part of the text logically relates to the others and together form a whole. They also enable a window into a reader's potential surprise, or lack there of, as the reader of Archimedes' *Method* would be unlikely to anticipate how conclusion #4 will be used again with #13 to form the result in conclusion #15. However, what is not captured is the way a reader is compelled (or repelled) by the mathematical story, provoking what a reader wants to know at any given point in the story. That is, does a reader even want to learn anything as he or she reads the story? And so, what? By narrowing his attention to the sequential logical thread of mathematical conclusions throughout a text, Netz's representation offers his fabula of each text: the logical interrelationships he made between assertions and conclusions interpreted from the text.

Though benefitted by the work of Thomas and Netz, this chapter turns its focus away from professional and historical mathematics texts toward what is arguably a very different genre of mathematics text: current K-12 mathematics textbooks. In addition, it draws from well-accepted literary theory involving the structure and aesthetic of stories to support the conceptualization of mathematical plot and how it can be analyzed. To learn how mathematical plots are manifested in written curriculum, this chapter extends Bal's distinction of story and fabula layers with Barthes' (1974) framework, designed to analyze narrative text for its writerly opportunities. Descriptions of the process of reading for plot will also be taken from Nodelman and Reimer's (2003) survey of theories involving children's literature. This chapter also uses a popular literary example, *The Wonderful Wizard of Oz* (Baum, 1900)[52], to articulate and demonstrate theoretical aspects of plot.

To start, the literary theory regarding events, sequence, and plot presented in Chapter 3 will be revisited to introduce a methodology for the exploration of mathematical plots found in textbooks. Barthes' (1984) analytic framework will be introduced and connected to Bal's layers, extended to define a unit of plot (referred to as *story arc*), and a coding of a literary plot using Barthes' analytic tools will be offered. Barthes' framework was selected because it is well-accepted in literary theory and because it provides useful analytic tools. Should other scholars who study mathematics stories choose to use different literary framings of plot to analyze and produce new understandings of the aesthetic affordances of mathematics curriculum, this only strengthens the overall argument of this chapter (and dissertation) that the conceptualization of

[52] Note that this is in reference to the 1900 text and not the popular 1939 movie. Although most events of the story between the two media forms are similar, there are some key differences.

mathematical sequences as stories is beneficial and widens the array of tools that can offer curricular insight and judgment.

Following this, a reading of a mathematical plot of a special mathematical story will be carefully analyzed to demonstrate what it offers one to learn about written mathematics curriculum. After this elaborated example, a contrasting mathematical plot found in an elementary textbook will be presented and discussed to highlight some of the affordances of this conceptualization of mathematical sequence, such as the role of specific features of a lesson (e.g., worked examples) in a mathematical story and the possible benefits or loss when the sequence is changed. Finally, the implications this framework has for students and teachers, as well as its limitations, will be discussed.

An Analytic Framework for Plot

Experiencing plot is not necessarily the same act as reading for plot. That is, a reader can sense a structure and respond aesthetically without consciously recognizing the temporal dynamics in play. Since the goal of this chapter is about what can be learned about mathematics curriculum when plot is deliberately read for, this sub-section will introduce how one can read for these temporal dynamics of a story. Since plot has both structure and aesthetic, heuristic strategies to read a story for plot must attend to both.

Analyzing a literary plot. Plot was defined in Chapter 3 as neither the story nor fabula layers of narrative (Bal, 2009), but instead as the "temporal dynamics" (Brooks, 1984) on a reader as he or she moves between these layers with anticipation and realization. Nodelman and Reimer (2003) explain how this tension works, explaining that "good" stories (in the view of the particular reader) support the reader "to want answers, then gives them enough to tantalize them, and holds back enough to keep them reading" (p. 64). Nodelman and Reimer (2003) go on to connect Bal's tension of anticipation and realization with Brook's focus on the plot's effect on a reader's pleasure and desire for answers, explaining that "good plots almost always provide a two-fold pleasure – first, the pleasure of incompleteness, the tension of delaying and anticipating completion; the second, the pleasure of the completion" (p. 65). Succinctly, if the plot of a story captivates a reader, he or she might respond to the story's temporal dynamics in multiple ways, including curiosity and wonder (the raising of questions), desire (to want to know), surprise (when anticipation is violated), and satisfaction (when closure is reached).

In her critique of Brooks, Bal (1986) draws from another literary theorist (Barthes) to articulate the notion of plot, explaining that it is "the interaction of Barthes's proairetic and hermeneutic codes, that of the sequence of happenings through time and that of enigma and answer" (p. 558). Briefly, Barthes' (1974) analytic framework proposes five codes to describe the "voices" woven throughout the text, which mark a reader's negotiation and pleasure of narrative. Barthes' theory is that a reader writes the text as he or she reads and that some texts are more "writerly" than "readerly," meaning that they invite multiple ways of interpreting and building meaning and require much more work by a reader. Barthes' codes (along with their abbreviated forms and description of voice) are hermeneutic (HER, the "Voice of Truth"), proairetic (ACT, the "Voice of Empirics"), connotation (SEM, the "Voice of the Person"), cultural (REF, the "Voice of the Sciences"), and symbolic (SYM, the "Voice of the Symbol").

Although all of Barthes' (1974) codes can be used to describe a reader's "making" of literature, Bal's (1986) notion of plot draws specific attention to both the hermeneutic and the proairetic codes because, together, these can help one read for plot. The first, the hermeneutic code, is a way to track the questions raised and pursued by a reader temporally throughout the story. Barthes describes this code as the "voice of truth" because it marks a reader's pursuit of "truth" within the story. Once a reader raises a question, it is the drive to answer that question (and, thus, find the "truth") that compels the reader to keep reading. In addition, Barthes' proairetic code marks the sequence of actions or changes throughout the story. Thus, the process of moving from question to answer is sequential, and since changing the sequence of story actions potentially changes the how the questions drive a reader forward, the HER and ACT codes are interlinked and together can represent the plot of a story.

Thus, Barthes' (1974) hermeneutic code provides a way for a reader to recognize the changing anticipation and emerging realizations throughout the story, specifically how inquiry is inspired, held, and brought to a close. The ACT codes mark the events (i.e., changes) throughout the story and the HER codes track a readers' questions of the story throughout (e.g., How will Dorothy get home?). These two codes work together, because changes in the sequence of events can (and often do) change the questions throughout the sequence as well as how long they are held and how the reader makes progress throughout the story. Once a reader formulates a question, he or she looks for clues that help point toward an answer.

It is important to note Barthes' (1974) notion of *lexia*, a textual fragment that can be described with one, two, or three codes. He explains that while reading, a reader continually searches fragments of text for meaning, and that these textual fragments are somewhat arbitrary. In short, a lexia is a portion of text for which there is something to note, in which some meaning that has been established for the reader. For Barthes' purpose of recognizing the ways a literary story works, the arbitrariness of lexias underscores how different readers will create different meaning from text. Thus, Barthes' argument is not that his codes will enable a reliable methodological structure that will enable the same coding(s) by all readers. Instead, these codes help a reader distinguish and describe the questions driving him or her during the emergence of meaning throughout a story, and therefore are possible motivational forces for other readers.

Since this analysis limits its attention to plot and is not taking advantage of Barthes' other analytic codes, the appropriate lexia of text for analysis is that fragment which is identified as an event of the story. Since the hermeneutic marks progress toward an answer, it is reasonable to assume that an event marks the change in that progress. That is, as events occur in throughout a story, new questions open for the reader and new information is realized. The sequenced events (the ACT codes) create temporal markers of the story parts during which the reader pursues questions of the story. Therefore, reading for plot requires the identification of the events of the story. As described in Chapters 2 and 3, to be considered an event, the story must advance. Action by characters (such as Dorothy throwing water on the witch, thereby killing her) and changes of setting (such as when Dorothy and her friends arrive in Oz) are examples of changes that form events, whereas learning that Dorothy's dog is named Toto is not. Analytically, these events can be identified by continually asking of the story "Has the story progressed?" and "What, if anything, has changed?"

Barthes also offers ways to describe the way a reader makes progress or how this progress is frustrated while reading a story. To further describe a reader's transition from question to answer, Barthes offers ten sub-codes of the hermeneutic code: (1) *thematization*, the setting up of the subject of one or more questions of the story; (2) *proposal*, an opening of the possibility of mystery; (3) *formulation* of the question, the raising of a question (by the reader or the text) to be pursued throughout the story; (4) *promise* of an answer (or the request of an answer explicitly by a character in the text), an indication to the reader that an answer will eventually be found later in the story; (5) *snare*, an attempt in the text to mislead the reader;

(6) *equivocation*, an ambiguity that contains both snare and truth; (7) *jamming*, a suggestion by the text that the question is unanswerable; (8) *suspended answer*, a delay of answering; (9) *partial answer* interpreted or generated by a reader; and (10) *disclosure*, the explicit closing of a question in the text by a reader. The relationship of Barthes' analytic framework, with his five major codes and the hermeneutic sub-codes, is shown in the diagram in Figure 5.2. Since coding an event with a hermeneutic sub-code is tantamount to identifying where the reader is in the pursuit of questions, this can be done by considering "What do I want to know?", "What have I learned?", and "What am I expecting?"

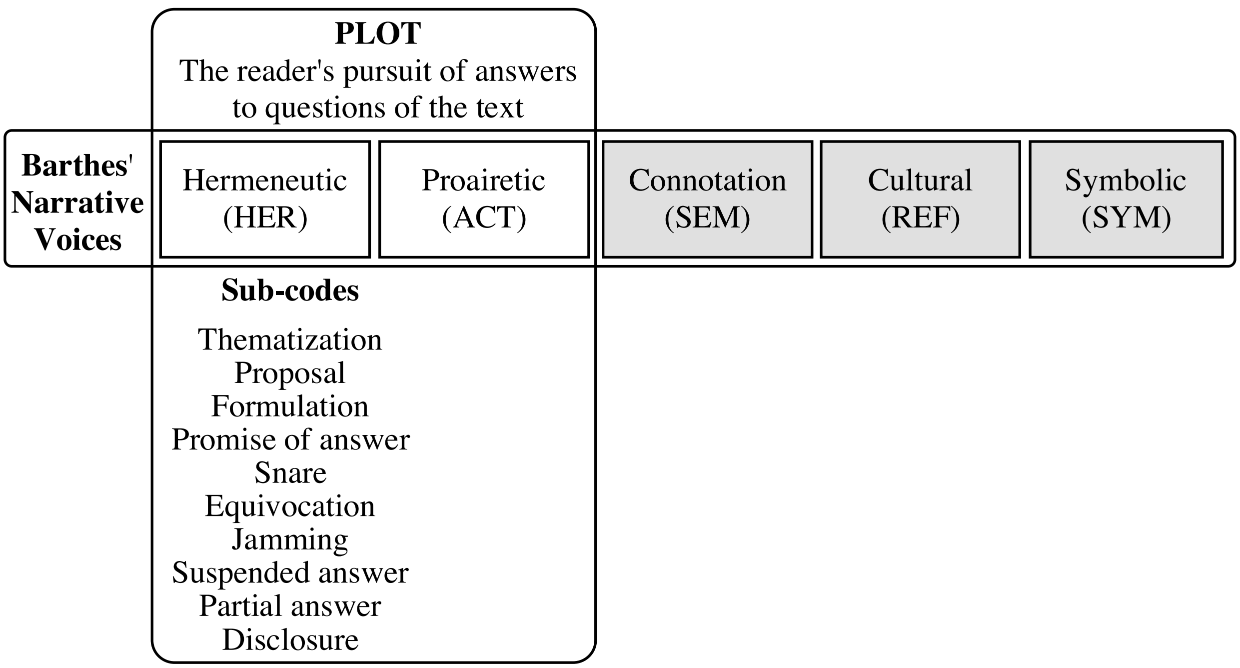


Figure 5.2. A diagram representing Barthes' (1974) analytic framework for narrative.

Thus, Barthes' hermeneutic sub-codes help to identify and describe how the story hinders or helps a reader to make progress on answering a question. They offer a way to describe Brook's "temporal dynamics," the in-the-moment forces in the story affecting the reader. Put together, the sub-codes help to explain how the story plays on both the anticipation and realization of reader and how these forces can draw a reader forward through the text or halt his or her progress. For example, *thematization*, *proposal*, *formulation of the question*, and the *promise of an answer* can mark the generation of a reader's curiosity and wonder, his or her desire to come to realize (know) something more. During the pursuit of an answer, a reader may anticipate possible realizations, which can be further supported and sustained if he or she recognizes a *promise of an answer* in the text. When a reader encounters/writes a *partial answer*, he or she might realize some aspect(s) of the answer and anticipate its disclosure.

In contrast, the *equivocation*, *snare*, *jamming*, *suspended answer*, and *disclosure* sub-codes all note parts of the story that potentially arrest a reader's anticipation and foreground realization. For example, both *equivocation* and *snare* potentially lead a reader to make a false realization, which can cause him or her to believe (in the moment) that the answer has been found. When this happens, the reader is offered nothing further to anticipate and the groundwork is laid for surprise later if the reader realizes he or she has been misled. A reader may halt anticipation for disclosure when the text suggests that the question is not answerable (*jamming*) or when the text changes the subject and the reader can find no progress toward an answer (*suspended answer*). Finally, a reader's sense of final completion at the question's *disclosure* marks the final transition from anticipation to realization.

Story Arcs. Barthes' (1974) analytic framework, particularly the hermeneutic and proairetic codes, also provides a strategy for a reader to identify portions of a story for which a complete (or incomplete) path from question to answer is experienced. Since these units of text do not necessarily make up the entire story and may vary in length, it is reasonable to assume that the way a reader asks and answers questions throughout the reading of a story describes the tension between Bal's anticipation (wondering about the future story) and realization (coming to know what happens), and thus describe the overall plot of the story. The sequenced transition from a reader's question to its answer (if one is found) will be referred to as a *story arc*[53]. Story arcs may include all of the aspects described by Barthes' hermeneutic sub-codes or only some, and Barthes points out that the sub-codes may also repeat and be found in any order. However, because of the way story arc is defined in this study, each will have a formulation of a question (whether it is raised by the text explicitly or not). Note that a story arc may be left open at the end of the story, leaving a reader with questions. However, for the sake of avoiding unproductive analysis, story arcs will be limited to cases where the reader makes progress toward an answer of a question through the reading of the story (i.e., there is at least one other sub-code than the formulation).

A Coded Example of Literary Plot. As Bal (2005) suggests, Barthes's ACT and HER codes can together be used to analyze the plot of a literary story. This section will illustrate how these codes together present my reading of the plot of the popular children's story *The Wonderful Wizard of Oz* by L. Frank Baum (1900). However, before presenting this illustration, a few methodological clarifications should be made. First, the act of coding multiple progressions from question to answer (that is, multiple story arcs) requires a re-reading of the text. For example, because of the nature of a *snare* (a case when a reader is misdirected toward an answer that later is "proved" false), it can usually not be identified "in the moment" during a first reading of the story[54] and instead requires the reader to encounter evidence of its falsehood later in the story. Likewise, a *proposal* of a mystery may be a hint that a reader may miss during the first read, but in hindsight can be recognized as laying the seeds for a later formulation during a

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[53] This unit of text is sometimes referred to as "story line." However, I have decided to use the term *story arc* since the reference of a curved arc at the opening and closing of a question-to-answer can draw attention to the opening and closing of the portion of the story.

[54] In fact, when a snare is noted by a reader on the first read, this represents a case when the story is very predictable, and thus, offers little surprise for the reader.

re-read. Therefore, this method of coding must occur after the first read, and even better, after multiple readings of the text. This strategy is consistent with Barthes' theory. Arguing that questions can be formed upon re-reading (an activity he explicitly supports), Barthes does not limit the identification of questions to an initial reading of the text. His notion of plurality of texts includes mysteries that exist even after the text is read and re-read; the opportunity and invitation of questioning and wondering.

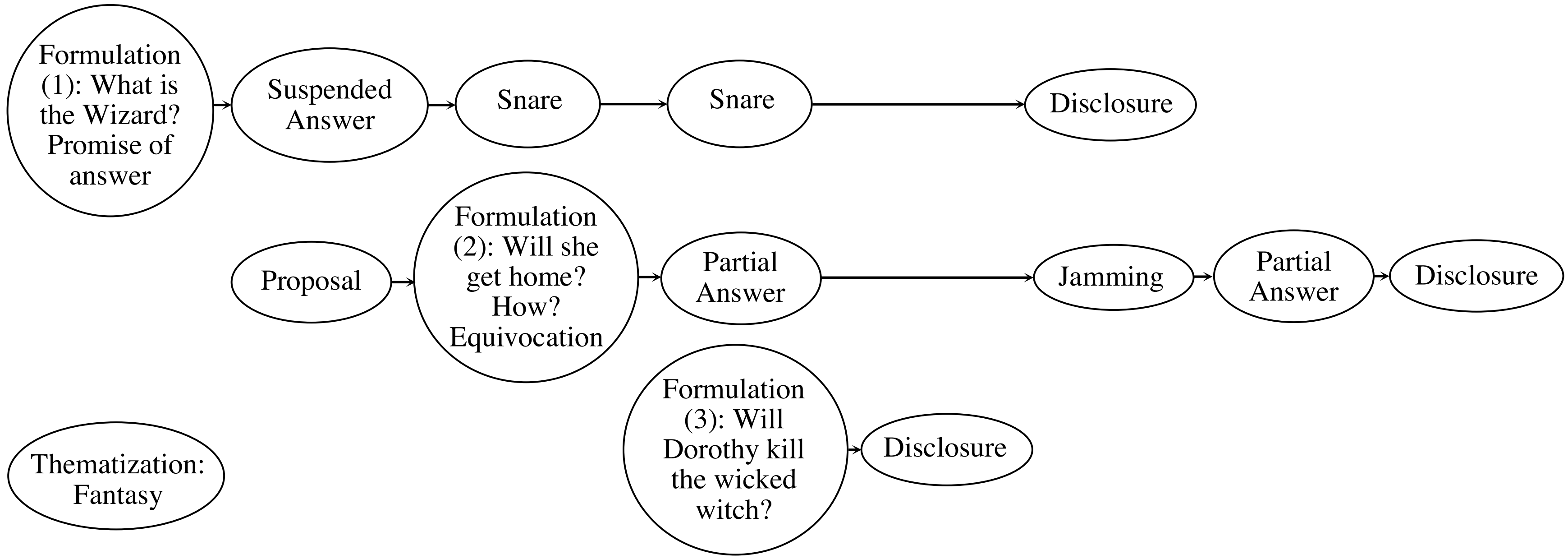
Secondly, for the purposes of illustrating a literary plot, this example only focuses on some of the events and questions involving Dorothy. Therefore, this discussion is far from an exhaustive representation of the plot of *The Wonderful Wizard of Oz* (1900). Several key events of the story that add much value and meaning to the story, such as the house killing the Wicked Witch of the East and the introduction of the Lion, Tin Man, and Scarecrow, were intentionally omitted for the sake of brevity.

Finally, for the purposes of illustration at the story level, the grain size of what counts as an event (and, thus, a ACT code) for this analysis is quite large since only seven main events are coded (making a total of eight ACT codes when the title is included). Thus, what is captured in an event in the analysis because it represents a change (e.g., when Dorothy and her friends battle and kill the Wicked Witch of the West) may represent multiple pages and sometimes an entire chapter of the story. Therefore, although several other events could be coded throughout the story, only those that pertain to these selected story arcs were included for this illustration and discussion. Each hermeneutic (HER) sub-code (What am I trying to find out?) is paired with a corresponding event (ACT) code (What is happening at that point in the story?), and the sequence of these coding pairs represent the structure of the story arc. Then, the multiple story arcs together represent the plot of the overall story. Since multiple story arcs may be developed simultaneously, the arcs will be labeled HER (to represent their hermeneutic transitions) with a number. Thus, once the first formulated question is recognized, all progress on that question will be labeled HER1. In contrast, ACT codes will be numbered sequentially from the beginning (starting with the title). To distinguish this use of numbering, the pound sign (#) will be used for ACT codes, indicating it is a numbering sequence that represents a relative position in a sequence: ACT#1, ACT#2, ACT#3, and so on.

In *The Wonderful Wizard of Oz*, one story arc immediately opens from the title with its reference of a "wizard" and an unfamiliar name "Oz." This (ACT#1, the story begins) sets up the theme (HER: *thematization*) for the story as one of fantasy and opens for me a initial question I expect to be answered at some point in the story, namely, "What is the Wizard of Oz?" (HER1: *formulation, promise of answer*). However, since the story then opens with Dorothy in Kansas and no mention of wizards, this question is not immediately addressed (HER1: *suspended answer*). Instead, I'm introduced to a young girl named Dorothy and her dog Toto, who live on a very isolated farm in Kansas with her Aunt and Uncle. A cyclone threatens and lifts the house with Dorothy and Toto in it (ACT#2, Dorothy moves). When Dorothy awakens, she finds herself in a new land full of color and merriment and meets the munchkins and the Good Witch of the North, opening the possibility for an enigma, namely how Dorothy will get home (ACT#3, new setting and characters). When Dorothy then asks how she can return home, this explicitly raises the related questions "Will Dorothy and Toto get home? How?" (HER2: *formulation*). The witch tells Dorothy to go to "the City of Emeralds" via a dangerous road "paved with yellow brick" to visit Oz, a great and good wizard, misleading both Dorothy and the reader as well as opening ambiguity for how Dorothy might get home (HER1: *snare*, HER2: *equivocation*). Dorothy and her dog then embark on a long journey and eventually arrive in

Emerald City (ACT#4, new setting) and encounters the fear and trepidation of "the Great and Terrible Oz" by its residents. When Dorothy meets Oz, he continues the ruse of his magical abilities, telling her "If you wish me to use my magic power to send you home again you must do something for me first. Kill the Wicked Witch of the West" (p. 128) (HER1: *snare*). Not only does this continue to mislead Dorothy and the reader, it raises new questions, which is "Will Dorothy kill the wicked witch? How?" (HER3: *formulation*). Dorothy and her friends commence on a journey to find the witch (ACT#5, change in scene, new goal), and after much struggle and attacks, succeeded accidentally when throwing water at the witch in anger (HER3: *disclosure*). Upon Dorothy's return to the Wizard of Oz (ACT#6, return to the Wizard), she learns that he is a fraud and has no magical powers, which appears to eliminate any chance of returning to Kansas (HER1: *disclosure*, HER2: *jamming*). Dorothy meets the Good Witch of the North (ACT#7, new character), who tells her "All you have to do is to knock the heels together three times and command the shoes to carry you wherever you wish to go" (p. 257) (HER2: *partial answer*). Dorothy follows her instructions and returns to Kansas (ACT#8, returns home) (HER2: *disclosure*).

A diagram that represents these interwoven story arcs is introduced in Figure 5.3.



Sequence of Events (ACT codes)

#1 Title: The Wonderful Wizard of Oz	#2 A cyclone moves Dorothy from Kansas to a new land far away.	#3 Dorothy asks how to get home, and is told to follow the yellow-brick road. She embarks on her journey.	#4 Dorothy arrives in the Emerald City and meets the Wizard who presents a job (killing the wicked witch) to get home.	#5 Dorothy and friends kill the wicked witch and return to the Wizard.	#6 Dorothy discovers that the Wizard is a fraud.	#7 The Good Witch of the North appears, tells Dorothy how to get home.	#8 Dorothy returns back to Kansas.

Figure 5.3. Overlapping story arcs from *The Wonderful Wizard of Oz* (Baum, 1900).

This diagram, which for brevity only describes a few story arcs in *The Wonderful Wizard of Oz*, highlights several general aspects of plot. For example, a story can have many open questions under pursuit throughout its text. The questions formulated by a reader (in this case, me) can occur at any time throughout the text, and are sometimes the result of questions explicitly asked in the text (such as when Dorothy explicitly asks how she can get home[55]) and other time are implicitly raised (such as the reference to the Wizard in the title). These story arcs weave together throughout the sequential events, at times foregrounding progress on one question, and later foregrounding another. Some formulated questions are answered relatively quickly (such as wondering, when Dorothy hesitates to kill the Wicked Witch, whether she will actually go through with it, in which the next chapter answers this question), while others make slow progress or are even delayed or extended through obfuscation. The story arcs, which appear separate, are actually critical to one another. For example, some events can close one question (such as when the Wizard is revealed to be an "ordinary" man from Omaha) and simultaneously jamming another (since this revelation then creates doubt on if and how Dorothy will get home).

Summary. Of Barthes' (1974) five codes to describe a reader's experience with the text, the hermeneutic (HER) and proairetic (ACT) codes together mark the interaction between the sequence of action and the reader's pursuit of inquiry, and thus represent plot (Bal, 1986). A portion of the story through which a reader transitions from question to answer will be referred to as a story arc, and a reader's progress (or lack thereof) toward an answer can be represented and described by pairing the hermeneutic progress (Barthes' HER codes) with the change in the story that enabled it (ACT).

[55] Dorothy asks the Munchkins, "I am anxious to get back to my aunt and uncle, for I am sure they will worry about me. Can you help me find my way?" (Baum, 1900, p. 25).

Readers demand and pursue questions of stories, particularly regarding its logic and purpose. This involves looking backward (i.e., renegotiating meaning through logically building the fabula) and forwards (i.e., with anticipation of what is to come through structural perception of the evolving whole story) throughout the temporal reading experience (Nodelman & Reimer, 2003). A reader's negotiation of the fabula and story layers throughout a story involves considering, "Why is this part of the story?", "How does this connect with what comes before?", "How does it change what I thought of what came before?", and "What do I think will come next?" The relational connections (whether explicitly stated or not) between the events of the story are crucial for a reader to develop anticipation, as there can be little hope for a reader of anticipating what comes next if the parts of the story appear completely unrelated. Heuristic questions that can help a reader identify events (ACT) are "Has the story progressed?" and "What, if anything, has changed?", while questions that help track progress on enigmas of the story include "What do I want to know?", "What have I learned?", and "What am I expecting?".

Mathematical Plots in Written Curriculum

This section will analyze how reading a mathematics lesson for its mathematical plot can reveal both the form and effect of a sequence of mathematical events in a textbook, forming a mathematical story that may (or may not) hold a reader's interest. Similar to how, in literature, an *event* describes of change within a story, in a mathematical story, a *mathematical event* describes a mathematical change or advance in the story (e.g., the introduction of a mathematical character, a change in setting, or the development of a new mathematical relationship). When a mathematical sequence is read as a story, its *mathematical plot* describes the temporal dynamics that shapes the reader's experience associated with the tension between what is known and what is desired to be known by the reader. As a structure, a mathematical plot represents a sequence of mathematical events perceived by the reader as he or she that marks the progress of inquiry. By pairing Barthes' (1974) hermeneutic sub-codes with corresponding sequential event codes, the changing anticipation and realization of the reader will be represented, describing how a story temporally connects a reader's wonder, curiosity, and desire to know how the story ends with his or her surprise and/or satisfaction with its ending. These sub-codes, along with more detail with their possible manifestations in mathematics text, are presented in Appendix C.

It may be helpful to note that this analysis will not (and cannot) offer all there is to say on the notion of mathematical plot. It presents an argument that the metaphorical conceptualization of mathematical plots (a) rests on a stable theoretical foundation and (b) allows for the identification of new and useful aspects of written mathematics curriculum that were previously hidden from analysis. In short, it is a proof of concept of an interpretive framework that offers a window into the structure of mathematics curriculum. As with any interpretive framework, interpretations will vary among different readers; however, as opposed to being a threat of analysis, multiple interpretations instead are additional evidence that new relational qualities can be recognized and described when mathematics curriculum is read as mathematical stories.

This argument will be organized into three main parts. First, the mathematical plots of two lessons will be presented. One of these, using a lesson of a textbook selected because of its intention to motivate a reader, will be carefully elaborated to demonstrate the way Barthes' codes can be used to analyze and map the progression of multiple story arcs. The mathematical plot of the other lesson will be, in comparison, briefly described in order to offer a contrasting case. The second part will then discuss the qualities of mathematics curriculum that appear through this conceptualization, through the contrasting of the two mathematical plots. Then, the last section will articulate particular ways that mathematical plots differ from literary plot, thus "speaking back" to the literary framework upon which this work rests. In addition, I will identify implications this conceptualization of written mathematics curriculum for students and teachers.

Choosing Examples of Mathematical Plot. To help demonstrate the analytic reading of a mathematical plot, two curricular lessons were selected based on their ability to illustrate aesthetic and structural qualities that the mathematical plot framework foregrounds. First, Lesson 1 from *Mathematics: A Human Endeavor* (Jacobs, 1970), was selected from a text specifically designed to interest students in mathematics in order to demonstrate the aesthetic potential of curriculum to highlight what the reading for mathematical plot enables (as an existence proof of sorts). This textbook, subtitled "A Textbook for Those Who Think They Don't Like the Subject," is highly regarded by both textbooks designers and mathematicians. In its foreword, popular mathematical writer Martin Gardner critiques the math texts of the day for their incomprehensibility and dullness and praises Jacobs for his effort to build interest in mathematics through an appeal to its beautiful structure. Gardner highlights this in perhaps the highest compliment that can be given to a mathematics textbook writer:

> Mr. Jacobs is a pedagogue of a different breed. It takes only a few glances at his book to see that he enjoys mathematics hugely, that he appreciates its incredible variety and structural beauty. Above all, he knows how to present mathematics in a way most likely to provide an average student with what is fashionable to call "motivation." It is unfortunate that there are not many more textbooks having this kind of approach, with the author's sense of humor and with his insight into the minds of the students whom our educational system is supposed to teach. (Jacobs, 1970, p. x).

Since this textbook was designed specifically to interest readers in mathematics and to provide new aesthetic opportunities, it is reasonable to assume it offers potential candidates for rich and interesting mathematical plots designed to attract and hold a reader's attention.

Second, a lesson from *Scott Foresman/Addison Wesley* (SFAW2) was selected because on the surface it shares many of the same hermeneutic qualities as the Jacobs lesson. However, when the mathematical plot of the SFAW2 lesson is analyzed, dramatic differences between it and the Jacobs' lesson emerge. These similarities and differences highlight the affordances of the analytic framework and raise important curricular questions regarding mathematical textbooks.

A Reading and Analysis of the Mathematical Plot of Jacobs Lesson 1. To organize this discussion, an overview of the sequential parts of the lesson is first described. Following this, a thorough coding of its mathematical plot is offered to demonstrate how the literary codes can be used to describe the overlapping transitions between mathematical questions and answers. Finally, some of the qualities of form and aesthetic of this lesson is highlighted.

This analysis was generated after more than three readings. The first reading was as a first-time reader unfamiliar with the text and content, during which I thoroughly read all words, attempted every problem, recorded observations, and made note of any questions that occurred to me throughout the sequence. During my second reading, I focused on identifying the key parts of the lesson and identified the changes throughout (i.e., mathematical events). During my third and subsequent readings, I organized a table of the sequenced events, described what change in the mathematical story they represented, and coded each event with HER sub-codes to describe the progress of the plot, if any was perceived. Since each read is important, they are interwoven throughout the analysis provided in this section. This strategy of weaving multiple levels of reading together in the presentation of this analysis is consistent with how Barthes (1974) presents his analysis of Honore' de Balzac's *Sarrasine*, which weaves together multiple voices, including both the emerging realization of the unfolding story along with the descriptive analytic coding of a later read.

The manner in which the written curriculum is woven throughout the analysis is also consistent with Barthes (1974). The written text material is provided in Appendix D for the reader to read (and, I expect, enjoy!), uninterrupted and unhampered by my analysis. In addition, to keep the analysis "close" to the text, similar sequential events are grouped and reprinted prior to its analysis, following Barthes' example.

The overview of the mathematical story of Jacobs Lesson 1. Lesson 1 (Jacobs, 1970, pp. 2–7, reprinted in Appendix D) opens the course with an investigation of the path a billiard ball would make on differently-sized tables. It starts with a discussion about how expert billiards players can not only control the path of a ball, but can also envision its path. The author explains that the path of the ball can be modeled with mathematics, noting "mathematicians like the game of billiards because the paths of the balls can be precisely calculated by mathematical methods" (p. 2). The text provides diagrams of two differently-sized tables that sequentially demonstrates the path of a ball when shot from the lower left corner at a 45° angle with the sides of the table. From this, the author raises (but does not answer) several questions about the path of the ball for the reader (me) to consider and invites me to ask still more questions. Then, the text offers two sets of exercises that are recommended to the teacher that all students attempt. Set I provides a sequence of eight tables and instructs me to copy and complete the path of the ball on graph paper. These tables are followed by four questions that invite me to review the group of paths so far generated and make observations and conjectures. Set II then prompts me to study a related group of tables (all with a height of 6 units) to learn how changing one dimension changes the path of the ball. This is followed by a sequence of five questions, asking me to reflect on the results, find patterns, and make predictions about the behavior of the ball on still other tables. The author also offers a third set of exercises, but it is described by the author in the note to the teachers as optional and is "intended to challenge the students and it is not expected that everyone will be able to do them" (p. xiv). This set of exercises repeats the exercise found in Set II with a height of 7 units.

A reading of the mathematical plot with coding.

The Path of a Billiard Ball

AN expert billiard player's ability to control the path of a ball seems almost miraculous. Mathematicians like the game of billiards because the paths of the balls can be precisely calculated by mathematical methods. Lewis Carroll, the author of *Alice in Wonderland* and a mathematics teacher at Oxford University as well, liked to play billiards. He even invented a version of the game to be played on a circular table!

A skilled billiard player can picture the path of a ball before he hits it. The path is determined by how the ball is hit, by the *shape* of the table, and by the positions of the other balls. The ordinary billiard table is about twice as long as it is wide (approximately

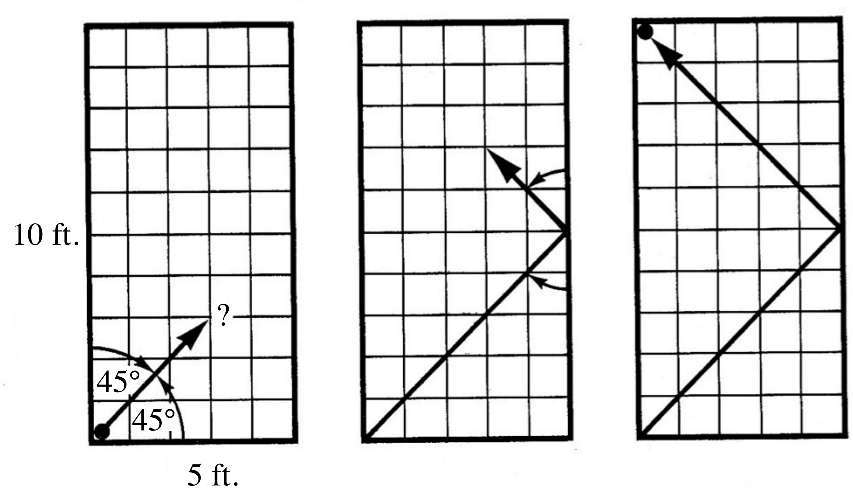


Figure 5.4. Part 1 of Lesson 1 of Jacobs (1970, p. 2). Reprinted with permission.

The title of the lesson (ACT#1: Title) sets the theme of the mathematical story (HER: *Thematization*), triggering anticipation that the story will be focused on the path of a ball on a billiard table. The first paragraph introduces the possibility that the path of a billiard ball on a table can be controlled by an expert and even calculated by a mathematician (ACT#2: Introducing the role of mathematics as modeling the path of a ball) (HER: *Proposal:* Pay attention to shape of the table, that may matter).

> 10 feet by 5 feet). Suppose a ball is hit from one corner so that it travels at a 45° angle with the sides of the table.* If it is the only ball on the table, where will it go? The first diagram (on the facing page) shows us the direction of the ball as it is hit from the corner. The second diagram shows that the ball hits the midpoint of the longer side of the table. (A billiard table, unlike a pool table, does not have any pockets.) When the ball strikes the cushion, it rebounds from it at the same angle. The angles of hitting and rebounding have been marked with curved lines to show that they are equal. The third diagram shows that the ball goes to the corner at the upper left, and we will assume that the ball always stops when it comes to a corner.

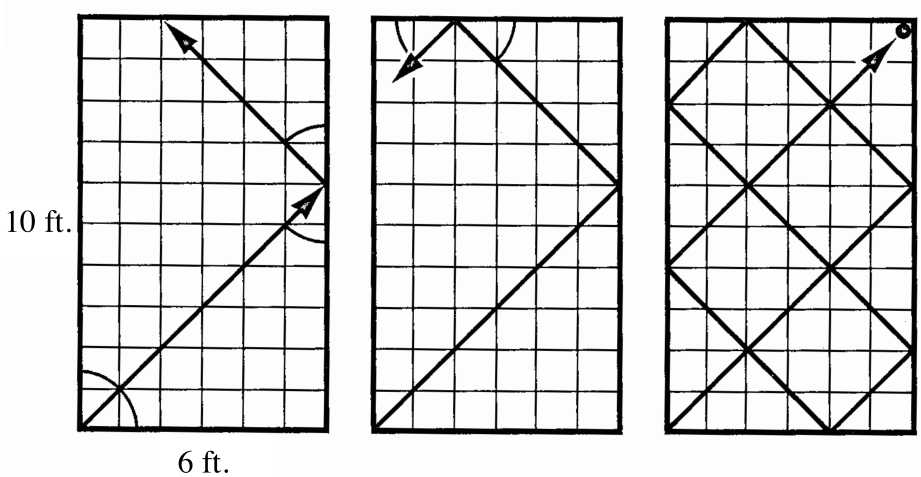
Figure 5.5. Part 2 of Lesson 1 of Jacobs (1970, p. 3). Reprinted with permission.

This paragraph introduces the notion of the shape of the table, which confirms my suspicion that shape may matter. The author then describes an "ordinary billiard table" that is "about twice as long as it is wide" and shows some diagrams of tables that I assume to be typical. The author then proposes a hypothetical situation, "Suppose a ball is hit from one corner so that it travels at a 45° angle with the sides of the table. If it is the only ball on the table, where will it go?" (pp. 2-3)[56]. Since the tables were presented in the text before I encounter this question, I now realize that I have seen the "answer" before the question was even posed. However, this does not ruin the effect of the question for me. Instead, it makes me realize that there was a mathematical question behind that diagram; that is, the diagrams can take on new meaning when

[56] The text also has a footnote with a directive to read an appendix (with some background on angles and angle measures) if I am confused. Since the text separates this mathematical content from the mathematical story, I did not include it in this analysis.

viewed as an answer to a question. Revisiting the diagram, I now pay attention to the path of the ball and notice that the diagram shows the starting path of a ball with an arrow. At the end of the arrow, where I assume the path of the ball would continue, is a question mark also emphasizing the question "where will it go?" A sequence of diagrams shows that the ball would end up in the upper right corner (ACT#3: Introduction of table and path of ball) (HER1: *Formulation:* Where will this ball travel? *Disclosure:* It rebounds off the right side of the table and ends up in the upper-left corner).

Let's try this again on a table with a different shape. Suppose the table is 6 feet by 10 feet.



Again the ball is hit from the lower left-hand corner at a 45° angle with the sides. This time, after the first rebound, it misses the other corner and hits the top side as shown in the second diagram. It rebounds from the side in a new direction so that the angles of hitting and rebounding are again equal. When we follow the path of the ball on the rest of its journey, we see that it rebounds several times more before finally coming to rest in a corner. This time, however, it is the corner in the upper right.

Figure 5.6. Part 3 of Lesson 1 of Jacobs (1970, p. 3). Reprinted with permission.

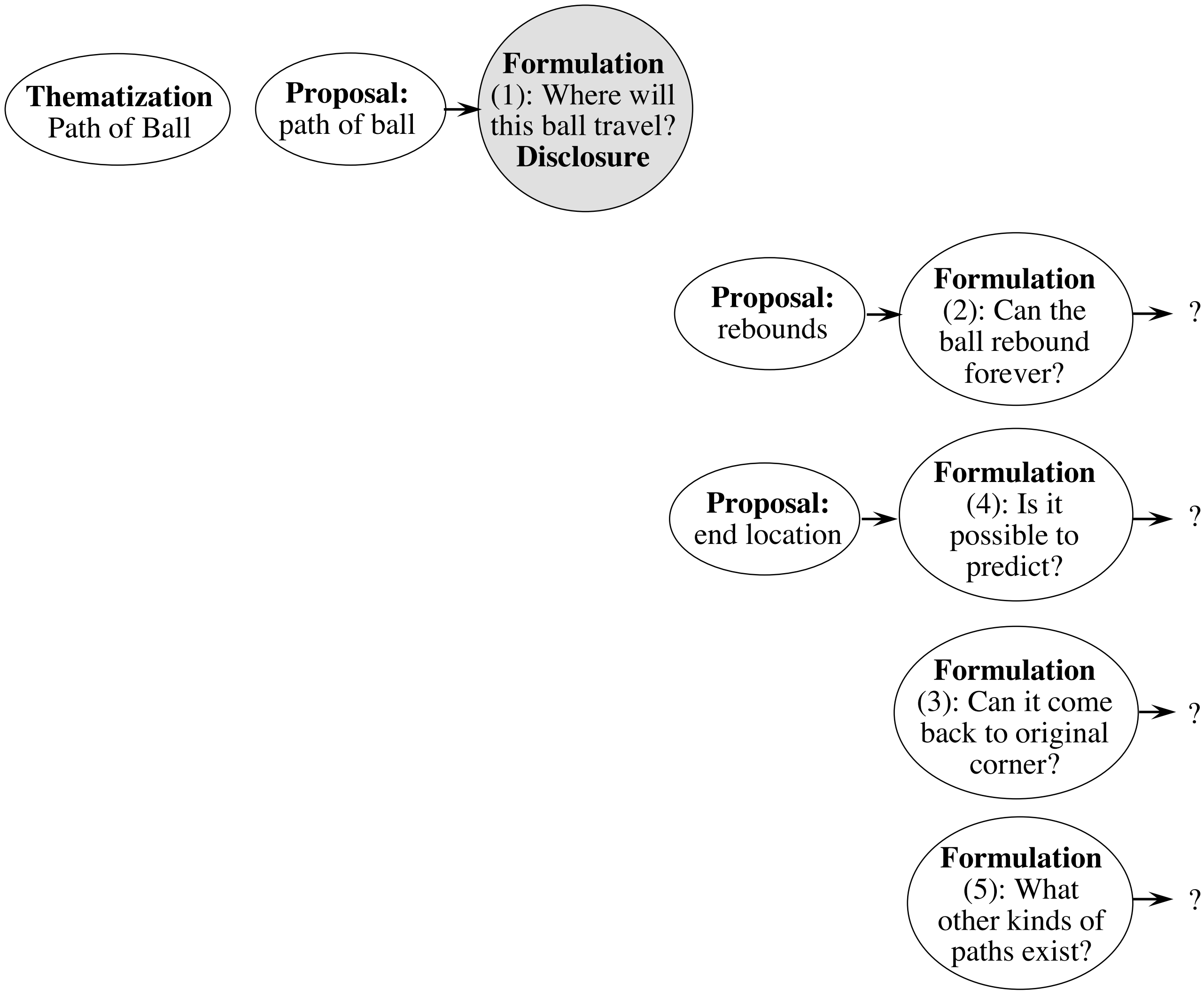
A second table with "a different shape" (6 ft. by 10 ft.) is proposed and solved (ACT#4: New table). So the text changes only one dimension and now I see a lot more zigzags and the ball ends up in the upper right corner. I note the lack of a compelling pattern such as the same ending spot or the same appearance of path. The author describes the multiple rebounds and sequential diagrams, again showing the path. The author's phrase "it rebounds several times more before finally coming to rest in a corner" (p. 3), emphasizes the higher number of rebounds (HER: *Proposal*: Pay attention to the number of rebounds, that is important). The author also draws attention to the fact that this ball ended in a different pocket (with the phrase "This time, however, it is the corner in upper right" (p. 3) (HER: *Proposal:* Pay attention to the end location of the ball, it may be important).

> These two tables suggest all kinds of questions about tables of other shapes. Will the ball always end up in one of the table's corners, or, ignoring friction, could it go on rebounding from the walls forever? Can it ever come back to the original corner? If the ball ends up in a corner, and you know the length and width of the table, is it possible to predict which corner without drawing a diagram? You can probably think of other questions as well. We are faced with quite a puzzle. A 20th century American mathematician has said:
>
> "Puzzles are made of the things that the mathematician, no less than the child, plays with, and dreams and wonders about, for they are made of the things and circumstances of the world he lives in."*

Figure 5.7. Part 4 of Lesson 1 of Jacobs (1970, pp. 3–4). Reprinted with permission.

At the end of this portion of text, the author refers to the two tables and explicitly poses several new questions about the possible paths of a ball (ACT#5: New questions to think about). Because it is not identified as an exercise, I assume it does not demand an immediate answer but instead offers mathematical questions to keep in mind. Therefore, I anticipate that this mathematical story will later address these questions (HER2: *Formulation*: "Will the ball always end up in one of the table's corners?" (p. 3); HER3: *Formulation:* "Can it ever come back to the original corner?" (p. 3); HER4: *Formulation:* "If the ball ends up in a corner, and you know the length and width of the table, is it possible to predict which corner without drawing a diagram?" (pp. 3-4)). Then, in a move rarely found in other textbooks, the author suggests that the reader is able to pose more questions, stating, "You can probably think of other questions as well. We are faced with quite a puzzle" (p. 4). I pause here, feeling as though the author is daring me to wonder, to push new boundaries of what might be known about the paths of the ball. Looking at the different paths resulting in the two given examples, I wonder, "What other kinds of paths are there?" (HER5: *Formulation*).

So, throughout the first third of the lesson, five questions have been raised of which only one has been answered. Each of the proposals can be linked to one of the eventual questions raised by the text. For example, the proposal that there might be interesting things to learn about the rebounds of a billiard ball can be connected to the formulated question "Can the ball rebound forever?" asked later in the text. Using a diagram similar to that in Figure 5.3, the codes for this opening of the lesson can be organized as shown in the diagram in Figure 5.8. This diagram, which shows the mathematical sequence from top-to-bottom and uses shading to denote the end of a story arc, shows the start of several possible story arcs. Note that F4 (the fourth formulated question) is above F3 because its proposal was encountered before F3.



Sequence of Events (ACT codes)

#1 Title: The Path of a Billiard Ball	#2 Intro paragraph, proposal of math model	#3 First worked example	#4 Second worked example	#5 Paragraph 5, new math questions posed

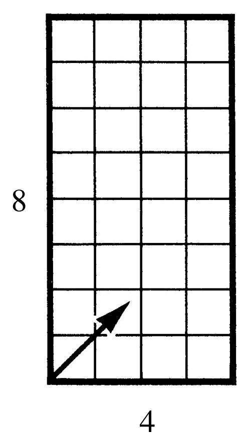
Figure 5.8. The mathematical plot of the beginning of Lesson 1 of Jacobs (1970).

EXERCISES

Set I

On graph paper (4 units per inch is convenient), make a diagram of each of the following tables. Use the *same number of units of length and width* as shown, and write the dimensions along the sides, as has been done for table 1. Now continue drawing the path of each ball as far as it can go. (We will assume that the ball comes to a stop when it reaches a corner.) Always start the path from the lower left-hand corner and notice that since the ball is always hit at a 45° angle with each side of the table, it always moves 1 unit up or down for 1 unit left or right. If the ball ends up in a corner, mark the corner with a large dot. (Please do not write in this book. You *won't* learn anything extra by doing so, and you *will* spoil the problems for the next student who uses it.)

1.



2.

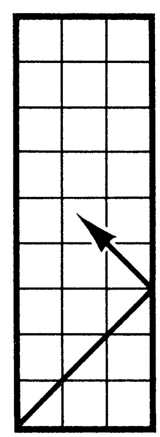


Figure 5.9. Part 5 of Lesson 1 of Jacobs (1970, p. 4). Reprinted with permission.

The text next offers a series of eight exercises in Set I (Jacobs, 1970, pp. 4-5) followed by four follow-up questions. I am instructed to copy the table onto graph paper and to "continue drawing the path of each ball as far as it can go" (Jacobs, 1970, p. 4). Problem #1 (ACT#6: New table) provides a 4x8 table, which results in a single rebound like the 5x10 table and ends up in the upper left corner (HER6: *Formulation*: Where does this ball end up? *Partial Answer*). The next table, which is 3x9 (ACT#7: New table), shows the first rebound (HER7: *Formulation*: Where does this ball end up?). After extending this path, I notice that it also zigzags up but ends in the top right corner, which I confirm with the answer in the back of the book (HER7: *Disclosure*). So, looking back to connect this result with earlier parts of the story, I now notice that the 5x10, 4x8, and 3x9 tables all result in a zigzag path up the table to a corner which are different than the table in example 2 (HER5: *Partial answer*).

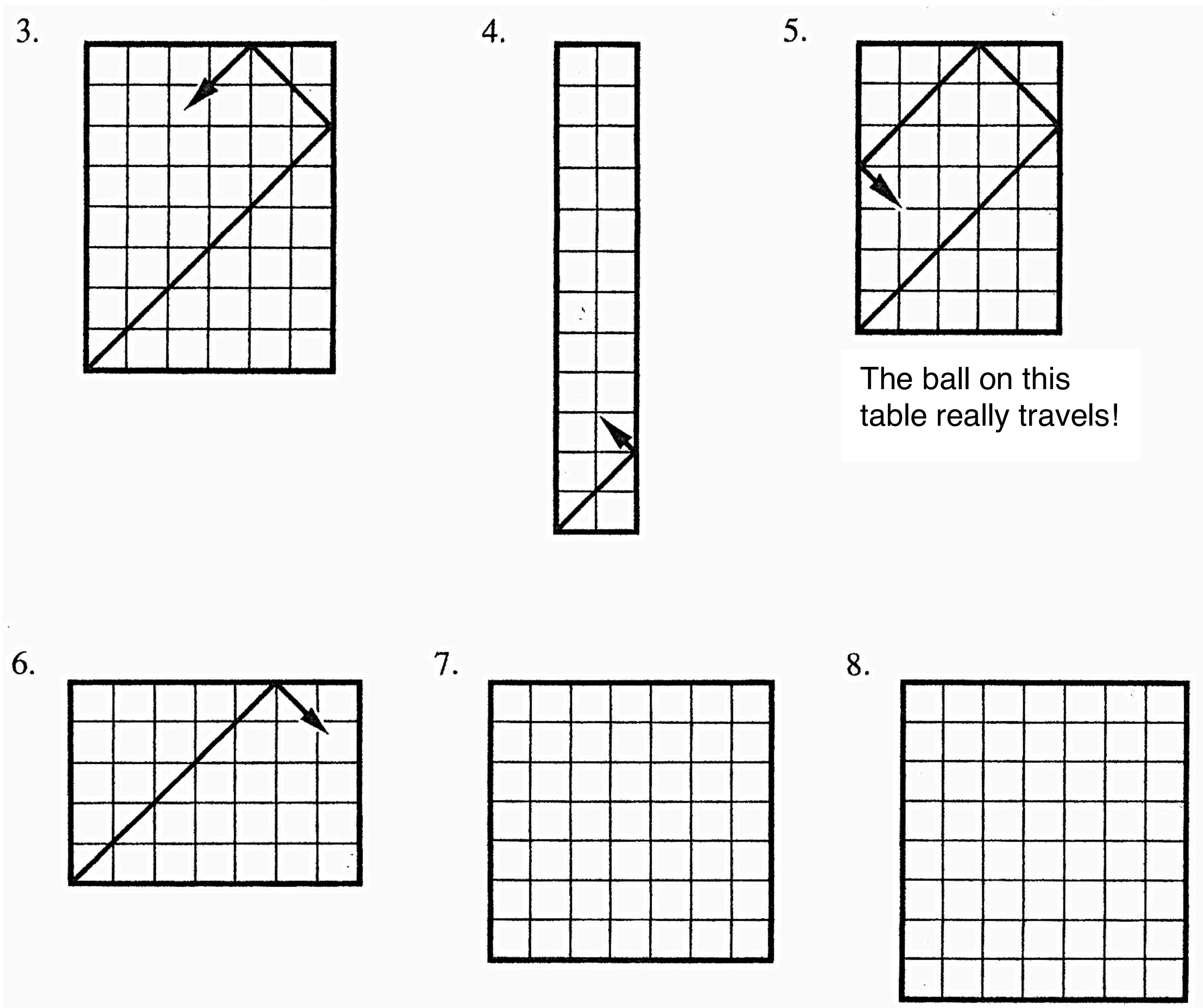


Figure 5.10. Part 6 of Lesson 1 of Jacobs (1970, p. 5). Reprinted with permission.

Next, a 6x8 table in Problem #3 (ACT#8: New table) is shown with the first two rebounds (HER8: *Formulation*: Where does this ball end up?). I draw the path and learn that this ball ends in the upper-left corner (HER8: *Partial answer*). Instead of a zigzag pattern, this table had more of a more complex, crisscross pattern like the path found in example 2 (HER5: *Partial answer*). Wondering why this problem was here in the sequence, its difference from the earlier tables made me notice how some paths are more complex than others (*Proposal*: Pay attention to complexity). Problem #4 (ACT #9: new table) provides a 2x12 table (HER9: *Formulation*: Where does this ball end up?). The path zigzags up the table, provoking the question, "Why are some paths more complex than others?" (HER10: *Formulation*).

Problem #5 shows a 5x7 table (ACT#10: New table) with the path shown with the first three rebounds. Below the table, it says "The ball on this table really travels!" (Jacobs, 1970, p. 5). Drawing the path (HER11: *Formulation*: Where does this ball end up?), it seems to take a lot longer to complete than prior tables. After 10 rebounds, the ball finally ends in the upper right corner. I confirm my answer with the answer key in the back of the book (HER11: *Disclosure*). With a crisscross pattern similar to the table from Problem #3, this table is the most complex so far. I notice that the ball travels over every square unit of the table, unlike those in example 2 and Problem #3 (HER5: *Partial answer*). The next table (Problem #6, 7x5) (ACT#11: New table, HER12: *Formulation*: Where does this ball end up?) results in a similar path, a tight crisscross pattern ending in the upper-right corner (HER12: *Partial answer*). The familiarity of this path provokes me to wonder *Is this table related to the table in Problem #5?* (HER13: *Formulation*). After comparing the tables, I see the tables have the same dimensions and that one is just rotated 90° (HER13: *Partial answer*). Like viewing a billiard table from a different location, the fact that it has the same dimensions convinces me that they must have the same paths.

Problem #7 offers a surprise (ACT#12: New table, HER14: *Formulation*: Where does this ball end up?). After a mixture of different paths of varying complexities and rebound patterns, the path on this 7x7 table is shockingly simple in that it has no rebounds (HER14: *Partial answer*). I immediately recognize this as a new type of path (HER5: *Partial answer*) and realize that with square tables, all complexity disappears (HER10: *Partial answer*). Although the 7x8 table in problem #8 appears "squarish" (ACT#13: New table, HER15: *Formulation*: Where does this ball end up?), it surprisingly results in another tight crisscross pattern (HER15: *Partial answer*).

With this new part of the mathematical story, the diagram in Figure 5.8 can be expanded (see Figure 5.11). For space considerations, abbreviations are used (F for formulated questions, PR for proposals, PA for partial answers, DS for disclosures, EQ for equivocations) and codes are placed on the same horizontal line if the prior story arc was opened and closed during a single event, except when an additional code occurred that was unrelated to prior codes (such as in ACT#8, when a proposal was also coded). Again, the shaded bubbles indicate story arcs that appear to me to end at by this point.

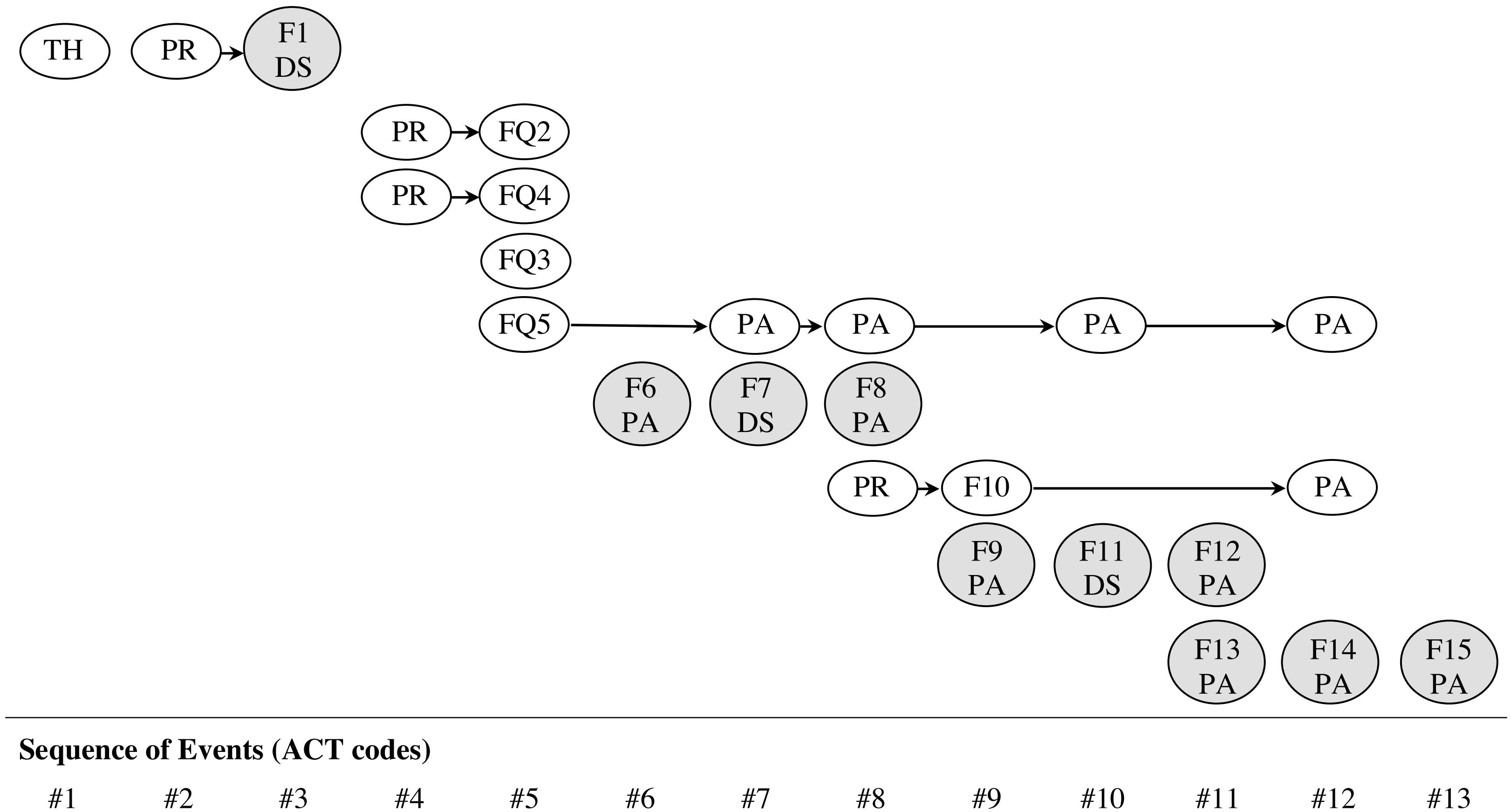


Figure 5.11. The mathematical plot of the middle portion of Lesson 1 of Jacobs (1970).

Note that questions HER2, HER3, and HER4, raised in the first five events, have not yet been revisited by the end of ACT#13, and at this point in the story it is unclear if they ever will. When compared with the diagram in Figure 5.8, a new mathematical plot pattern appears during the Set I table exercises (events ACT#6 through ACT#13). Rather than opening up questions that remain open, each event contains one (or more) question(s) that is asked and answered either by the reader or by the reader and the author within the single event (which will be referred to as *momentary questions*). The momentary questions appear in sets of three, at the end of which an extended code branches the diagram onto a new line. HER5 (regarding the various kinds of paths the ball can take) is revisited regularly and always on the second event of the set, which helped me keep interest in moving forward throughout each set of three. Notice that with the exception of events ACT#6 and ACT#13, all of the events so far have multiple codes, which indicates that at most points of the story, progress was made on multiple mathematical questions.

9. On which table does the ball have the simplest path? Can you explain why?
10. What do you notice about the paths on tables 5 and 6? Can you explain?
11. Do you think the ball will always end up in a corner?
12. If the ball starts from the lower left-hand corner, do you think it can end up in any of the four corners?

Figure 5.12. Part 7 of Lesson 1 of Jacobs (1970, p. 5). Reprinted with permission.

After the exercises with table diagrams, Problem #9 (ACT#14: Focus changed to simple paths) provokes me to return to the square and think about why the square is unique in this effect. I notice that whenever a table is a square, the path of the ball will lie on the diagonal and the ball will travel directly to the upper-right corner. This makes me realize that there is a type of tables I can predict at which corner the ball will land (HER4: *Partial answer*). Problem #10

(ACT#15: Focus changed to comparison of tables) asks what I notice about the tables in Problems #5 and #6, to which I restate my earlier observation (HER13: *Repeat partial answer*). Problem #11 changes the focus (ACT#16) to whether the ball will always end up in a corner. At this point, I notice that all the paths so far have ended in a corner and I wonder if a path could ever rebound forever. I try to draw a few new tables to construct one that would not end in a corner and do not get far. I write "I think so, I tried to find one that didn't but couldn't. Not sure why" (HER2: *Partial answer*). Problem #12 asks whether I think the ball "can end up in any of the four corners?", a question I hadn't considered before this (ACT#17: Change in focus to corners, HER16: *Formulation*). I notice that the ball has only landed in the upper-left and upper-right corners so far, so I temporarily conclude that the ball only lands in the upper corners (HER16: *Equivocation*). After starting at a few of the completed tables, I realize that the ball cannot return to its starting position because at some point it will have to reverse its path which is contrary to the rules of rebounding (HER3: *Partial answer*).

Increasing the diagram to include these additional events, the overlapping story arcs of this mathematical story so far can be seen in Figure 5.13, using F for formulated questions, PR for proposals, PA for partial answers, DS for disclosures, EQ for equivocations. Note that because F13 ended up being revisited (in ACT#15), the diagram was adjusted to show this by lowering F14 and F15.

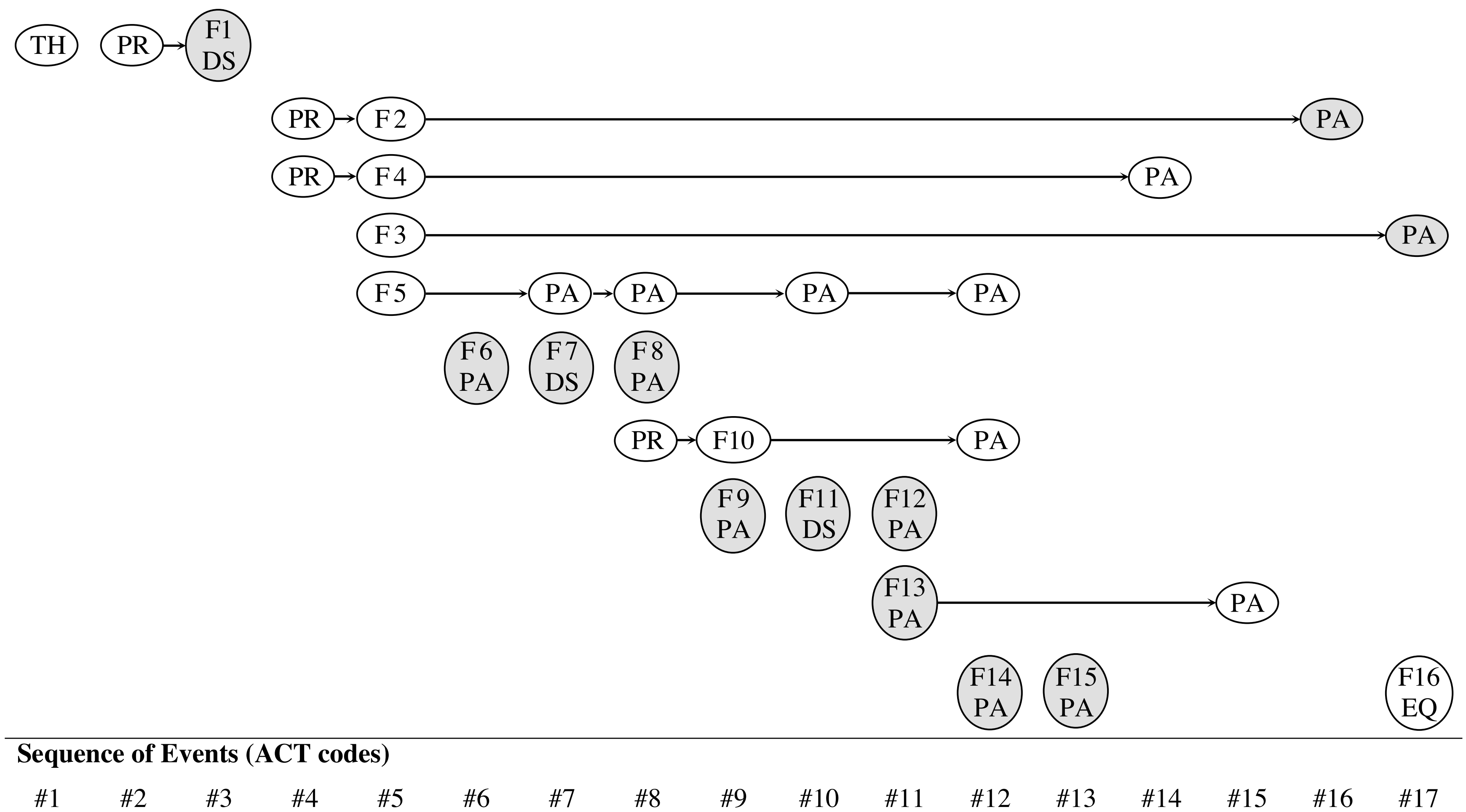


Figure 5.13. Coding of events for first two-thirds of Lesson 1 of Jacobs (1970).

Some observations about the mathematical plot so far are appropriate before moving on to the next part of the story. The final portion of Exercise Set I directed me to revisit and make progress on questions F2, F3, and F4, left open at the start of the lesson. Thus, this revisit helps me connect this part of the mathematical story with the opening discussion before the exercises. Like the overarching question "How will Dorothy get home?", these questions become the framing questions of the mathematical story so far, and looking back with hindsight, I can recognize that almost all the sequential parts appear to be in the service of answering them. Another, F10 (*What makes some paths more complicated than others?*), which was proposed in ACT#8 and raised in ACT#9, has made little progress. Therefore, looking ahead, I anticipate making sense of the complexity of paths; that is, learning what makes some paths more complicated than others.

Set II

The paths we have drawn so far are wildly unpredictable. A slight change in the shape of the table can make a tremendous difference in where the ball will go. Compare your drawings of the last two tables. Table 8 is the same width as table 7 and only one unit longer, yet the paths are entirely different.

Figure 5.14. Part 8 of Lesson 1 of Jacobs (1970, pp. 5–6). Reprinted with permission.

With the opening of Set II (ACT#18: New set of exercises), rather than helping me make sense of the complexity, the author unexpectedly throws me a curve. The author writes "the paths we have drawn so far are wildly unpredictable" (Jacobs, 1970, p. 5) and points out that even just changing the dimensions of a table slightly changes the path considerably. This paragraph, two-thirds of the way into the story, gives me momentary pause and causes me to question whether there is a discernable pattern after all (HER4). My investment in drawing many tables and paths and answering questions is not for naught, is it? I had figured out a way to predict the path on one type of table (squares), but I had to agree with the author that the other results did not make sense. The reassurance at the end of Set I, where I felt progress on finding patterns throughout this sequence now is jeopardized (HER4: *Jamming*).

The shape of the table determines the path of the ball in some way that is not yet clear. What determines the *shape of the table?* Two things: its length and its width. These dimensions can change, or vary, from one table to the next and are called *variables*. The path of the ball, then, is determined by the shape of the table which, in turn, is determined by two variables.

Figure 5.15. Part 9 of Lesson 1 of Jacobs (1970, p. 6). Reprinted with permission.

This next paragraph (ACT#19: focus on pattern) offers me some hope, however. Stating, "The shape of the table determines the path of the ball in some way that is not yet clear" (Jacobs, 1970, p. 6), the use of the word *yet* at this point communicates to me that although so far a pattern has not emerged, there is one (HER4: *Promise of answer*). Furthermore, it seems that the author expects that I may not see the pattern yet, but that will at some point in the future be remedied. My worries raised in the preceding paragraph dissipate and I'm left with a sense of optimism that eventually I will find a pattern and be able to predict the path of a ball from the table.

It would be simpler if there were only one variable. Let's keep the length of the table the same (we will hold it *constant*), vary the width, and see what happens.

1-6. Draw a set of six billiard tables with lengths of 6 units and with widths of 1, 2, 3, 4, 5, and 6 units. Start the ball from the lower left-hand corner as before and mark where the ball ends up with a large dot.

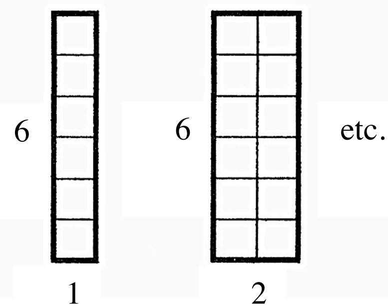


Figure 5.16. Part 10 of Lesson 1 of Jacobs (1970, p. 6). Reprinted with permission.

Next, the text points out that it would be simpler (and perhaps make a pattern more evident) "if there were only one variable. Let's keep the length of the table the same... vary the width, and see what happens" (Jacobs, 1970, p. 6) (ACT#20: Analysis of a related set of tables). It prompts me to focus on the tables with height 6 units. I draw out all of the tables and complete their paths. I notice that the tables with widths 1, 2, and 3 units all follow the zigzag pattern I noted earlier. The table with width 4 units surprises me because the path ended up in the bottom right corner, causing me to revisit my prediction that the ball would only land in the top corners (HER16: *Partial answer*).

7. Does the result for any of these tables surprise you? Which one and why?
8. What are the two dimensions (length and width) of the table with the *simplest* path?
9. What are the dimensions of the table with the most *complicated* path?
10. If a giant billiard table had a length of 100 feet, what width should it have for the ball to travel the simplest possible path?
11. What width should it have for a very complicated path?

Figure 5.17. Part 11 of Lesson 1 of Jacobs (1970, p. 6). Reprinted with permission.

Following the drawing of these tables, the text provides four follow-up questions. Problem 7 (ACT#21: Focus on surprise) prompts me to revisit and repeat my observation from ACT#20 (HER16: *Repeat partial answer*), since I already noted my surprise about the 4x6 table. However, this question reveals an intentional authorial hand in the choice of tables provided in Set I as I suspect that the author expected me to be surprised.

Problem 8 (ACT#22: Change of focus) asks which of the tables resulted in the simplest path, prompting me to revisit my prior observation about the square and answer that square paths are the simplest paths (HER5: *Repeat partial answer*). This feels like a delay in my progress. However, Problem 9 (ACT#23: Change of focus) prompts me to consider which table resulted in the most complicated path[57]. Comparing the paths, I notice that the 5x6 table is more complex than the others, resulting in a tight crisscross pattern. I remember that the author had emphasized that the 7x8 table was very different than the 7x7 table earlier, and noticed that both complicated path had dimensions that differed by one unit (HER10: *Partial answer*, HER5: *Partial answer*).

---

[57] Note that the properties *most complicated* and *simplest* paths are not defined in the textbook. I interpreted *most complicated* to be a path that has the most tightly overlapping rebounds, passing through every square unit on the table at least once. I interpreted the *simplest* path as the path with the fewest rebounds.

At the end, problems #10 and #11 (ACT#24: New tables) ask me to predict the width of a table with length 100 units that would result in the simplest or the most complicated path (HER17: *Formulation*: *What width makes a simple path? What width makes a complicated path?*). At this point, I am confident that the square (100x100) has the simplest path. However, since it is not feasible to draw a table to scale that is 100x99 or 100x101, I have to rely on my observation for Problem #9 and predict that a width of 99 units (or 101 units) would have a most complicated path (HER17: *Partial answer*).

Adding these final codes for Set II to our diagram, the entire mathematical plot from beginning to end of this lesson can be seen in Figure 5.18, using F for formulated questions, PR for proposals, PA for partial answers, DS for disclosures, JM for jamming, PM for promise of answer, and EQ for equivocations. In all, there were 24 events identified, including the title. There were 17 formulated questions, of which six were overarching questions (F2, F3, F4, F5, F10, and F16). By the end of Set II, all but one of the overarching questions appear answered to some extent. In addition, the events following the explosion of formulated questions in ACT#5 can be seen to be very productive. After the initial overarching questions are introduced in ACT#5, only five of the remaining 19 events did not introduce or make progress on an overarching question (ACT#6, ACT#11, ACT#13, ACT#21, and ACT#24).

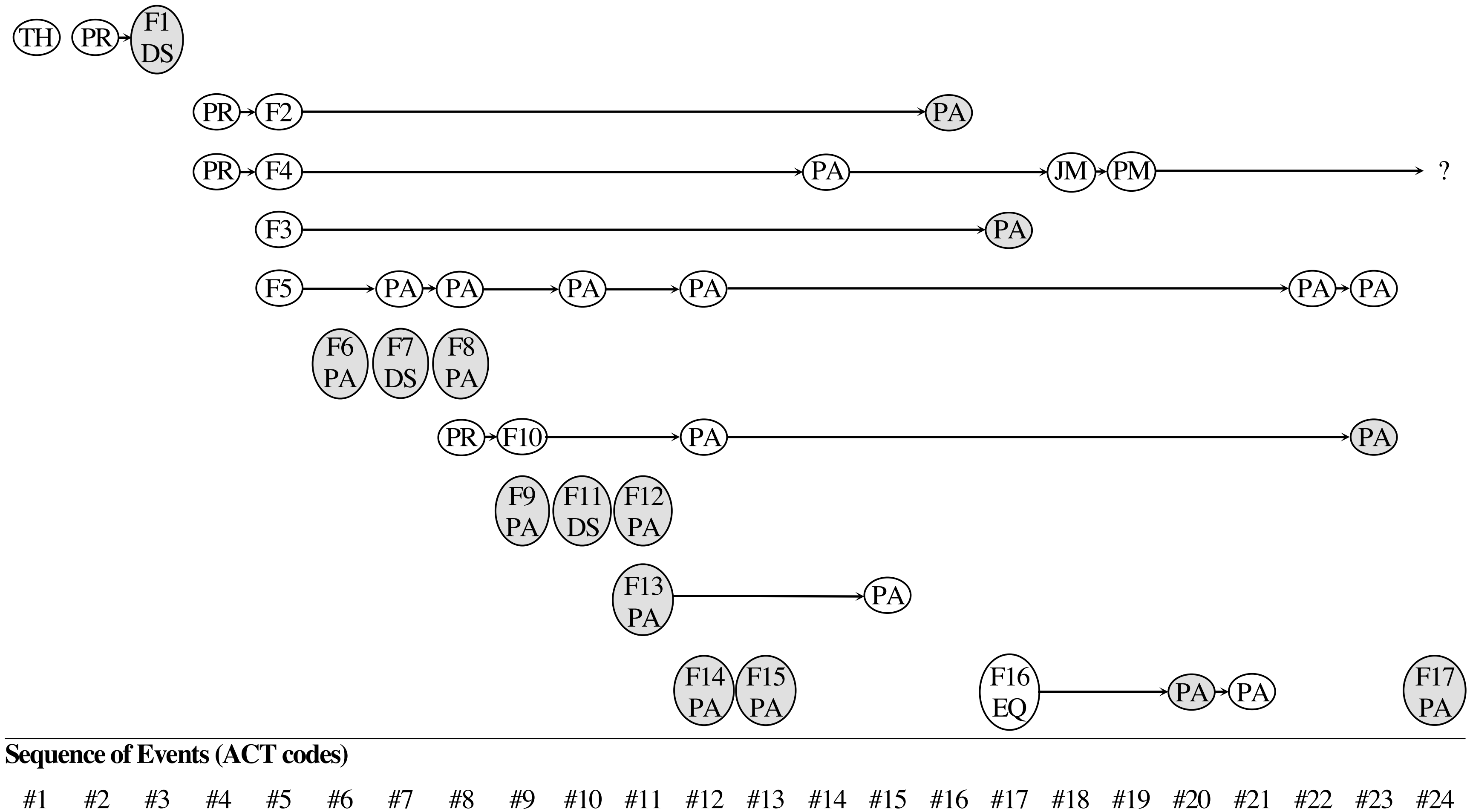


Figure 5.18. The mathematical plot of Lesson 1 of Jacobs (1970).

The structure and aesthetic of the mathematical plot. Looking back on this elaborated analysis (which is summarized in Appendix E), several comments about the mathematical plot can be made. For instance, it can be seen how a structure of a mathematics lesson emerges as it is read and understood as the simultaneous pursuit of multiple mathematical overarching questions with sequences of momentary questions. The diagrams in Figures 5.8, 5.11, 5.13, and 5.18 each represent a different moment of the reading, and the changes in structure between them demonstrate how a mathematical story changes in time. For example, in the first portion of the mathematical story (Figure 5.8), the main aspect of the mathematical plot was to open several overarching questions that would mostly be addressed throughout the rest of the mathematical story. In contrast, the second portion contained a series of momentary questions (Problems #3 through #8), each with a similar goal (to draw the path of the ball and find out where it ends), which by their unpredictable variation provoked a new overarching question (Figure 5.11).

By the end of the mathematical story, all but one of the overarching questions are resolved in some way. Surprisingly, however, F4 (*Is it possible to predict the end corner?*) was not satisfied by the end of this lesson, even after an explicit question on page 4, a sense of progress with predicting paths for square tables, temporary jamming at the beginning of Set II, and a promise of an answer. The lack of disclosure for this question is not a minor triviality. I was so interested in this question by the end of this lesson that I continued to work through the optional Set III exercises, hoping to learn the answer. When no answer emerged for me, I proceeded to the next lesson, where resolution was finally achieved[58]. I argue that its unresolved status at the end of the lesson supported my anticipation for what was to come, and provided me with a strong sense of satisfaction when it was eventually resolved.

[58] I choose not to spoil the ending for the reader, but instead recommend finding a copy of this textbook and enjoying the rest of this mathematical story.

The prospect that overarching questions can span multiple mathematical stories highlights that, although this lesson can be viewed as a mathematical story on its own, it can also be viewed as part of a longer, more elaborate mathematical story. Much like how a question in a literary story (such as "How will Dorothy get home?") can give a reader something to anticipate beyond a single chapter and can remain unanswered for a long time, an unanswered question at the end of a mathematics lesson can stimulate his or her interest and enable a reader to look forward to the future parts of the mathematical story.

Although withholding an answer can raise a reader's interest, maintaining the interest of the reader throughout an extended overarching question might take more than time. The jamming move in ACT#18 and ACT#19 was not only surprising (which raised my interest in its own right), it was a critical reminder that the question remained open after all the others from the beginning had been resolved. Had the author not re-engaged me with this question in these later events, I may have assumed the question was dropped and forgotten about it altogether, losing its motivational force. In addition, the author followed the jamming with a critical signal that I interpreted as a promise of an answer, which encouraged me to continue to anticipate its answer moving forward.

In addition to the surprise through the equivocation and jamming, this lesson also offered a third form of surprise. Whether intentional or not, the sequence of the tables provided in Set I attracted my attention to different types of ways the path can zigzag about the table and also set up my anticipation for more rebounding paths. This is not to say that I could not have asserted that a square would have the simplest path from the beginning. If I had questioned (either by myself or with the help of the text) whether a path existed that would have no rebounds, I'm sure I would have figured it out. However, nothing in the text provoked me to think about a possible table for which the ball would have no rebounds, and so I assumed that the tables would have rebounds and would differ by how the ball rebounded. Therefore, the encounter with the square in Problem 7 conflicted with my anticipation for more rebounding paths, generating my surprise and appreciation.

Finally, the mathematical plot of the Jacobs lesson contained multiple instances when answers (those generated by a reader) were prompted for even after the corresponding questions were considered answered. This occurred, for example, in ACT#21, when the text asked me about any surprises that I had in the prior work. Since I had already noted my surprise and answered the F16 in ACT#20, this later prompt required a revisit to a question that felt answered already. Another example occurred in ACT#22, when the text asked me which table was the simplest and why, prompting me to repeat my reasoning generated earlier about why the square must have the simplest path in ACT#17. These moments in the story were the only points when a feeling of progress felt impeded. Throughout the rest of the story, a considerable momentum of progress was palpable.

Another example of a mathematical plot. As shown in Figure 5.18, the mathematical plot of Jacobs' lesson can be understood to involve multiple enduring mathematical questions, which were pursued simultaneously through a majority of the mathematical story. However, the special structure of this mathematical plot is made even more salient when contrasted with the mathematical plot of another lesson. Using the same careful reading and coding method demonstrated with the Jacobs' lesson, the mathematical plot of Lesson 1-8 of SFAW2 (Charles et al., 2008) was analyzed in order to contrast Jacobs' lesson with a more contemporary (and typical) lesson example. What follows is an extended overview of the lesson, followed by a representation of its mathematical plot. Since the purpose of this example is not to exemplify a methodological analysis, but instead is to offer a contrasting mathematical plot form to highlight some of the affordances of this conceptual lens, the extensive details of the coding of this lesson have been omitted. Therefore, to help the reader make sense of the mathematical plot, I have provided a more extensive overview.

The overview of the mathematical story of SFAW2 Lesson 1-8. SFAW2's Lesson 1-8, which follows a lesson involving acting out story problems, deciding if they should be solved with addition or subtraction, and writing the corresponding number sentence, is titled *Adding in Any Order*. It opens with an "Investigating the Concept" activity that prompts me and a partner to each place colored cubes on a side of a part-part mat (see the example diagram in Figure 5.19, which was provided in the textbook in an earlier lesson). My partner and I are then asked to build addition sentences by reading the mat from left-to-right from opposite sides, resulting in two different number sentences, such as $4+6=10$ and $6+4=10$.

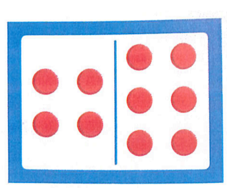


Figure 5.19. A part-part mat diagram provided in SFAW2 (Charles et al., 2008, p. 3A). From SCOTT FORESMAN ADDISON WESLEY MATH MICHIGAN TEACHER EDITION GRADE 2 VOLUME 1 by Charles et al. Copyright © 2008 Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

I'm told that these two equations are called "related facts." I am then asked to compare these addition sentences, and I notice that they both have the same numbers, the addends are in different order, and the sums are the same. Following this, I'm asked whether all addition sentences with different addends have a related fact, which I conjecture that they do (since any type of number sentence represented on a part-part mat can be read from opposite sides).

Then, in the boxed section entitled "Learn," containing a worked example, a picture of a child is shown wondering what happens to the sum when the order of the addends is changed (see Figure 5.20). Next to this, the text answers by stating "the sums are the same when two numbers are added in a different order" (p. 23). Then $3+5=8$ and $5+3=8$ are represented by joined cube trains (using contrasting colors for the addends) and with addition sentences. Again, a name for this relationship is introduced, but this time they are called "related addition facts" and a Word Bank below indicates that "related fact" is a new vocabulary term. Because of the introduction of the word "addition," I wonder if there will be other types of related facts, like related subtraction facts[59]?

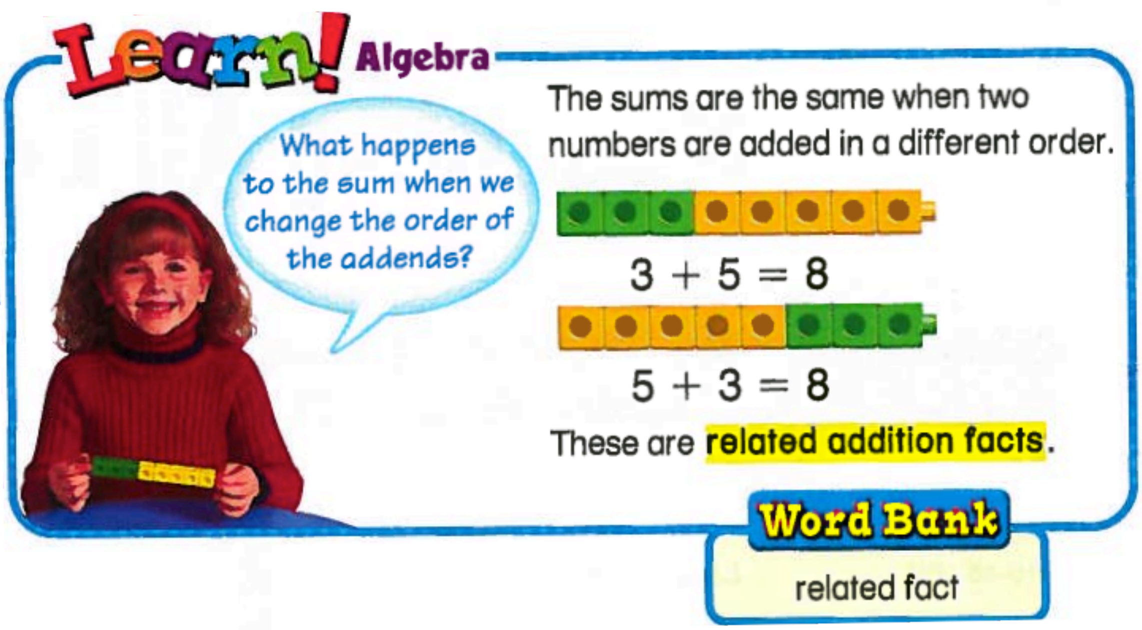


Figure 5.20. A mathematical question and worked example found in Lesson 1-8 of SFAW2 (Charles et al., 2008, p. 23). From SCOTT FORESMAN ADDISON WESLEY MATH MICHIGAN TEACHER EDITION GRADE 2 VOLUME 1 by Charles et al. Copyright © 2008 Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

[59] In my reading, this question made no progress during this lesson. However, in a later lesson, Lesson 1-10, it was partially addressed. Although never referred to as a related subtraction sentence, the relationship between addition and subtraction sentences is named a "fact family."

Under this box, another worked example (Problem #1, with the equations $6+1=7$ and $1+6=7$) enables me to count the like-colored parts of the joined cube train and notice how these form the given number sentences from left-to-right. The text then provides three exercises for me to complete, one (Problem #2, with two cube trains with 4 green cubes followed by 6 yellow cubes, and vice versa) that is very similar to the worked example in that the cube trains are presented horizontally and the numbers I am to fill in are represented by line segments. For this, I count the colored cubes and fill in number sentences. I then do the same for the vertical cube trains in Problems #3 and #4 ($3+2=5$ and $1+3=4$, respectively). With the exception of the change from horizontal to vertical format, these questions feel repetitive, for all I am doing is counting the cubes and writing the numbers in given lines.

At the bottom of the page, the portion of text entitled "Think About It" asks me whether a cube train with 3 yellow and 3 red cubes (from left-to-right) train has a related addition fact (see Figure 5.21). I notice that I could switch the colors in the cube train so that the 3 red cubes are on the left, and that this would result in a cube train that was noticeably different. Therefore, I answer that this train does have a related addition fact. However, the answer for the problem in the teacher text states that this cube train does not have a related addition fact. This extends my understanding about an addition sentence, specifically, that it does not have reference to its original referents. In addition, this tells me that related number facts must have unequal addends, a requirement that was not previously mentioned.

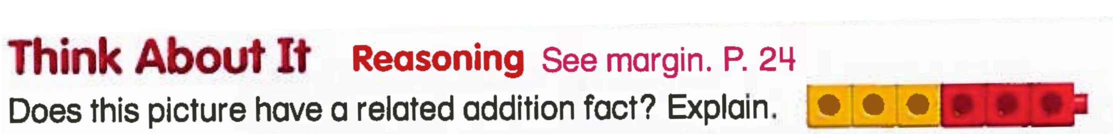


Figure 5.21. The Think About It question from Lesson 1-8 in SFAW2 (Charles et al., 2008, p. 23). From SCOTT FORESMAN ADDISON WESLEY MATH MICHIGAN TEACHER EDITION GRADE 2 VOLUME 1 by Charles et al. Copyright © 2008 Pearson Education, Inc. or its affiliates. Used by permission. All Rights Reserved.

On the next page, I encounter some horizontal and vertical sums without contextual referents and write the related addition fact. Again, a worked example demonstrates what I am to do. Without cubes to count, I look at the worked example and notice that the addition related fact uses the same symbols, but this time with the addends switched. I replicate this process throughout these exercises. During this work, I am reminded (by the teacher) that, "changing the order of the numbers when they are adding does not change the sum in any way" (p. 24). It is also noted that just having one of the same addends is not enough to make two addition sentences related addition facts. At the end of the lesson, I am prompted to write a math story based on a picture (describe the picture). Finally, in my journal, I am prompted to copy and write a related addition fact for each of three different addition sentences without contexts ($5+2=7$, $9+0=9$, and $4+8=12$).

The mathematical plot of SFAW2 Lesson 1-8. In the analysis, 18 sequential events were identified. A total of 16 formulated questions were posed and answered. One additional question was posed (about related subtraction sentences), but since this question saw no progress in this lesson, it was dropped for analysis. Of these 16 formulated questions, the first three were overarching questions, raised in this order: *What will be added?*, *What makes the equations related?*, *What happens to the sum when we change the order of the addends?* These are followed by 13 momentary questions[60]. Using the same diagramming process as used for Jacobs' lesson, a diagram for the mathematical plot of SFAW2 Lesson 1-8 showing the sequential development throughout the mathematical story is offered in Figure 5.22.

[60] Technically, F4 was proposed in a prior event (the lesson title). However, since the question was formulated and answered in the same event, it is still classified as a momentary question.

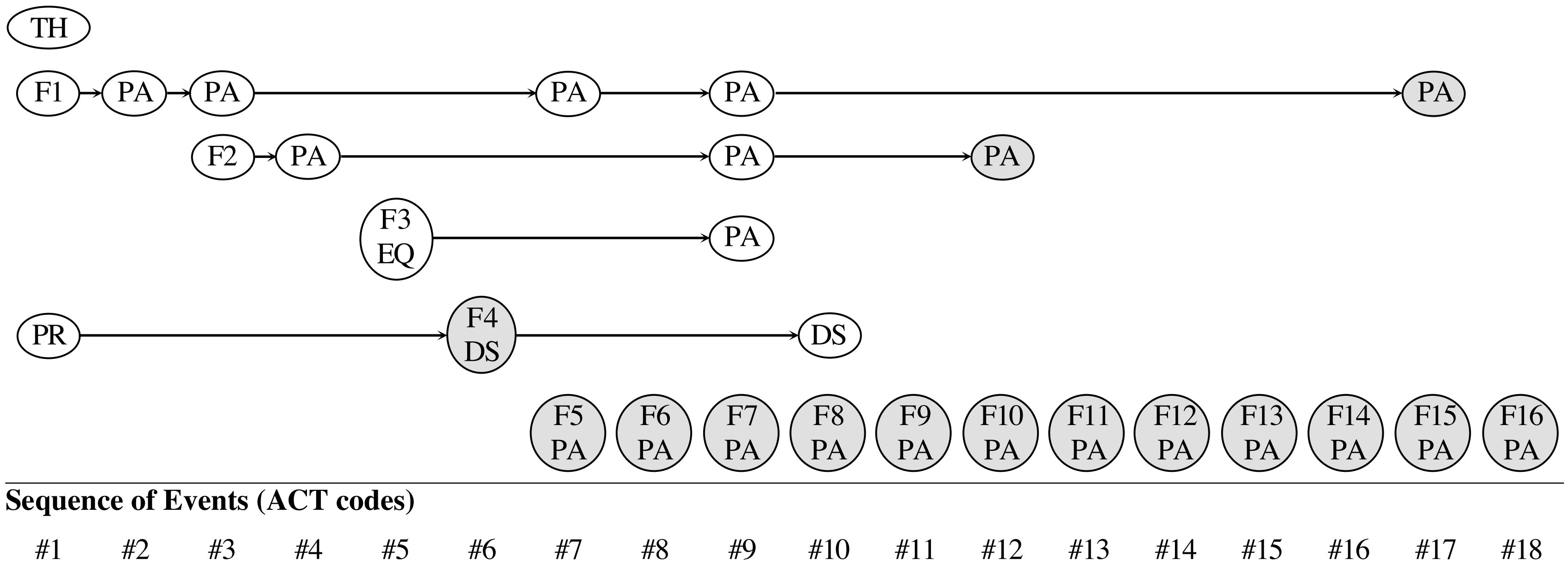


Figure 5.22. Coding of SFAW2 Lesson 1-8, using F for formulated questions, PR for proposals, PA for partial answers, DS for disclosures, and EQ for equivocations.

Unlike the Jacobs lesson, for which all the story arcs involved the path of a ball on various billiard tables, this lesson involves two unrelated pursuits. One of these foci starts with the title "Adding in Any Order," which raises for me the formulated question F1 "What will be added?" and focuses on the properties of the mathematical characters being added (e.g., numbers or concrete objects). This inquiry advances when the tasks or text draws my attention to what is being added. In contrast, the two other overarching questions, F2: *What makes the equations related?* and F3: *Do you think there is a related fact for every addition sentence with two different addends? Why?*, as well as all remaining momentary questions, focus on related addition facts. These latter questions are arguably the focus of the lesson, as indicated by the title *Adding in Any Order*. Therefore, the overarching question that, at first glance, appears to hold the mathematical plot together across the lesson does not even address the main content of the lesson. In *The Wonderful Wizard of Oz*, this would be similar to having an overarching question that was not central or important to the main work of the story, such as "What type of breed of dog is Toto?" Although a question such as this might be of interest to the reader, its answer is likely inconsequential for Dorothy's adventure in Oz and how she will get back home.

Thus, the diagram in Figure 5.22 can be reorganized so that these two foci are distinguished (see Figure 5.23). Grouping the story arcs for F1 and F15 together and separating them from the other story arcs enables the recognition of the shifts in attention. The mathematical plot codes above the dashed line represent those that could be called "off topic," and thus weaker than those below the dashed line. I should emphasize, again, that this separation of topics is based on my reading and feeling as though there are multiple foci in this lesson. It helps to show that in ACT#17, I felt a question come "out of the blue" that seemed unrelated (in terms of content) with what came immediately before and after.

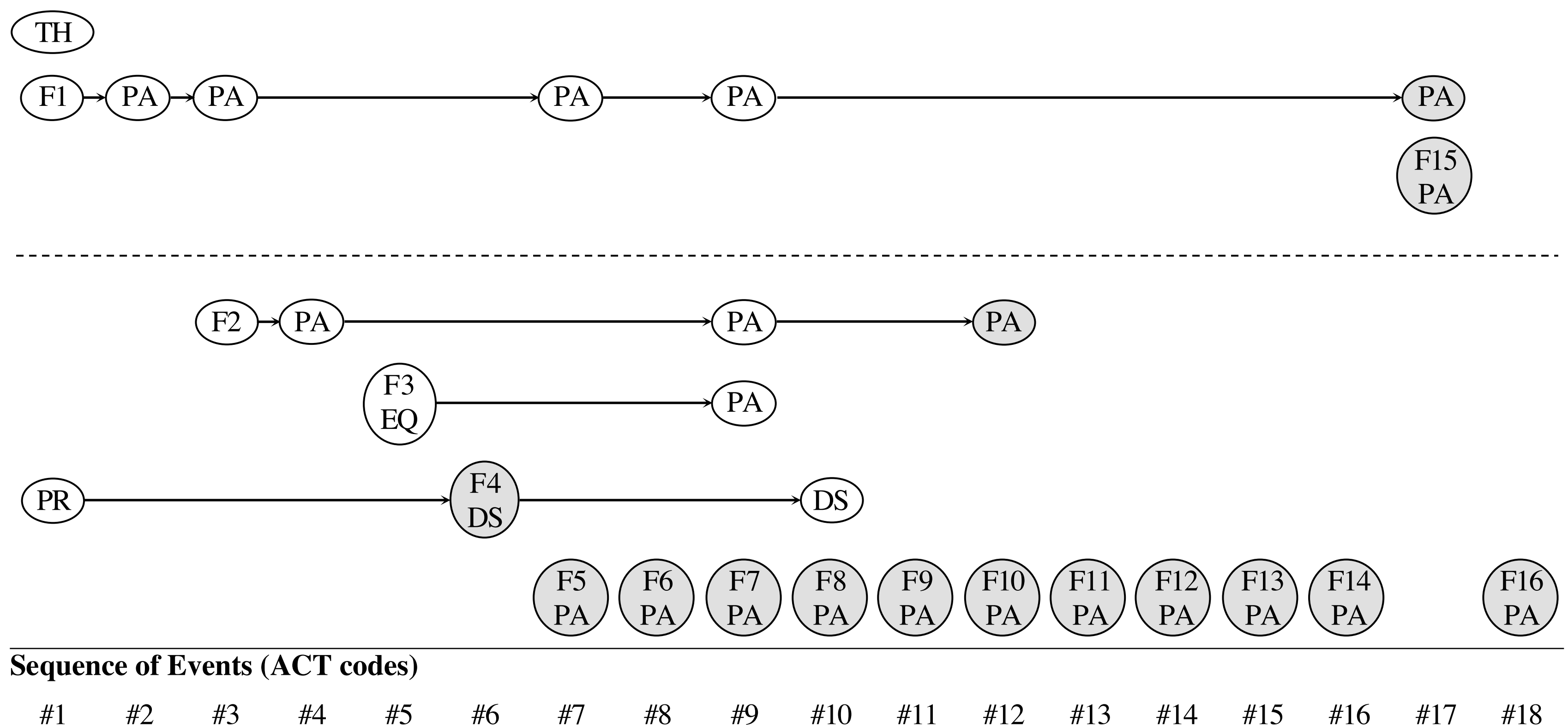


Figure 5.23. My reading of the mathematical plot for SFAW2 Lesson 1-8 with story arcs grouped by related focus.

One of the most appealing parts of this lesson was encountering the photo of a child thinking a mathematical question: “What happens to the sum when we change the order of the addends?” (p. 23, F4). In fact, this photo was the reason this lesson was marked for further analysis during my first read. It was the first, and nearly only[61], explicit suggestion during my reading of the first three chapters of the SFAW2 textbook that a child can (and should) ask a mathematical question. At first glance, the question seemed like a possible overarching question central to the lesson (which, by its title, was focused on the order of addends). And yet, after analyzing the mathematical plot, no further progress was made on this question. Therefore, this question did not end up being a driving force of the mathematical development of this lesson.

Qualities of Written Curriculum Evident through Reading for Mathematical Plot

The examples of mathematical plots discussed in the previous section help to reveal some of the structural (i.e., the way the sequence of events builds to make a coherent whole) and aesthetic (i.e., the reader’s response) differences which can emerge while reading mathematical stories. However, in addition to demonstrating a method of analysis, this chapter seeks to argue that a general understanding of written mathematics curriculum can be gained if textbooks are read and understood in this way. Therefore, moving across these particular examples, this section highlights some of the curricular insight afforded by this conceptualization, particularly with regard to structure and aesthetic.

Although no two readers will have the same experience, and thus there is no single mathematical plot offered by a mathematical sequence, this framework supports a deeper understanding of the mathematics curriculum in use in classrooms today and offers a potential

[61] The only other occurrence in the core section (i.e., not the optional materials) of Chapters 1- 3 was found in the Learn box of Lesson 3-4, focused on making an organized list. In this lesson, a photo of a student is shown with a thought bubble “What two groups of ten make 100?”

heuristic for identifying new and interesting mathematical stories in the future. This section divides the discussion regarding what can be learned about mathematics curriculum when read for its mathematical plot in two parts; the first part will introduce some of the structural and aesthetic forms of mathematical stories as found through the contrast of these two existing plots described in the previous section. The second section then turns the attention to the sequence in a mathematical story both to highlight the necessity of the duality of the proairetic and hermeneutic codes and to demonstrate the affordance of the mathematical plot framework in changing sequences.

Structural Differences of the Mathematical Plots and their Effects. Although many observations about the structure of mathematical plots have been offered so far, the contrast of my readings of the mathematical plots offers new insight into the dynamic forces on a reader of mathematical stories. This section elaborates the roles and uses of momentary questions, worked examples, and equivocation, and introduces the notion of density of mathematical inquiry in a mathematical story.

Differences in the use and effects of momentary questions. One of the striking differences between the two analyzed lessons was the role and sequential placement of the momentary questions (those that are asked and answered during the same event). On the surface, both lessons have a string of tasks for a reader to answer and these exercises are encountered after one or more worked examples. However, when considering the mathematical plot of both mathematical stories, there were some important differences in the role of the momentary questions for the reader. For example, in the SFAW2 lesson, once momentary questions start in ACT#6, no additional overarching questions are introduced to the mathematical story, signaling that these exercises generated no curiosity. In addition, nine of the 13 momentary questions occur alone in an event, which indicates that during most of these exercises, I made no mathematical progress on the overarching questions. Therefore, in the SFAW2 lesson, these questions were repetitive (e.g., the variety did not spark any curiosity) and offered me very little opportunity for progress[62] of the story.

In contrast, one new overarching story arc began in Jacobs' lesson after the occurrence of the first of a string of consecutive momentary question (F10), and the momentary questions, while occurring consecutively (with the exception of F17), sometimes occur in events with multiple formulated questions. Also, in the Jacobs lesson, only three of the 10 momentary questions involving exercises occur alone in an event (F6, F15, and F17), indicating that for most of these exercises, I made progress on one or more of the overarching questions of the mathematical story. Therefore, it may therefore be no surprise that as I advanced through the Jacobs lesson, I felt that the exercises had something to offer me (e.g., there was potential for closure of one or more of the overarching questions), which increased my desire to move forward through the exercises.

Differences in the effects of equivocation. The cause and effect of the equivocations found in my reading of in both lessons also highlights differences in the mathematical plots. In the Jacobs lesson, an equivocation occurred when I was prompted to consider in which corners the ball could end up. Since the ball ended in the top corner for all of the exercises in Set I, I conjectured in ACT#17 that the ball would only end in one of the top corners. In hindsight, the text never lied to me; however, by restricting my view of tables for which the path ended in the bottom-right corner, I was given the opportunity to answer the question incorrectly. This

[62] I want to stress that this does not mean there are not other forms of progress that might be made through repeated exercises, such as increased fluency or a decrease in errors. However, in this framework, unless these questions help the reader generate a question or make progress on an overarching question of the mathematical story, then the mathematical plot is not advanced.

ambiguity set up my surprise later, and the author's explicit mention of this surprise caused me to perceive an intentional effort to mislead me[63]. Rather than feeling duped, however, it helped me recognize that I had accepted a property of paths based on examples without question. I also enjoyed the surprise later, as it provoked my curiosity about what other interesting properties of paths might surprise me in future parts of the story (which became realized in the next lesson).

In contrast, the equivocation in the SFAW2 lesson occurred when, early-on, I interpreted what made related facts different in a way that was later obviously not what the authors expected based on an answer to the Think About It in ACT#5. Since no definition of the term "related fact" was given at the time of its introduction, the correction via an answer in the text surprised me. However, I also felt as though I was expected to know that even though the colors of the cube train can change order, the addition sentences are not considered different and therefore there is no related fact when the addends are equal. As opposed to provoking anticipation of future parts of the mathematical story as Jacobs' equivocation had done, this equivocation resulted in my looking backward, renegotiating earlier parts of the mathematical story, and raising doubts about what else I thought I had understood.

Differences in the use and effects of worked examples. Comparing the mathematical plots of these two lessons also reveals differing roles of the worked examples in both lessons. In the SFAW2 lesson, the worked examples occurred (in ACT#6 and ACT#8) after all of the overarching questions were introduced and partially answered (including the case of equivocation, where the answer was later recognized to be incorrect) and before the string of exercises resulting in the momentary questions. The delay in the worked examples allowed the

[63] It is important to note that I cannot make any claims about what the author intended, only what I perceived the intentions to be. Therefore, I felt an orchestrated effort, whether or not the author actually intended to do so.

earlier parts of the mathematical story (e.g., building related facts with the part-part mat) to feel more open, in the sense that the answers to the questions in the Investigating the Concept activity had no predetermined quality. During this portion of the lesson, I was able to appreciate how the result of writing equations from both sides of the part-part mat (from left-to-right) was not revealed until I actually imagined completing the activity (since I had no partner). Therefore, when I entered the activity and drew out a part-part mat with counters in each part, I was unaware at that point that I would get related addition facts and, although I did not have any clue about what to anticipate at that early point, I also did not feel the story was constrained. However, after the worked examples, the SFAW2 mathematical story felt very constrained. After multiple exercises that did not provoke any new overarching questions, I started to only anticipate more exercises similar to the worked examples.

In contrast, the worked examples in the Jacob lesson occurred before all of the overarching questions. They demonstrated the behavior of the ball and helped to define the notion of the "path" as a static object to be reasoned about by the reader. The sequential diagrams in these examples helped me recognize their temporal quality, the unfolding progression of the path of the ball. The role of these examples for my reading was to allow me to perceive the path of the ball both as a dynamic process and as a final object (the final diagram of the completed path). The opportunity to perceive the paths as objects might not have been the case if I were left to generate these paths myself. For example, for each of these tables, I am able to mentally follow the path about the table to learn where the ball ends up. Without a trace of the path, I might have instead focused on the dynamic quality of the path, perhaps noting at the end only their ending points. This change would probably jeopardize the generation of F5, yet likely would have allowed a different focus (such as a focus on how many rebounds, since the more rebounds, the more mentally taxing to visualize).

Therefore, for the Jacobs lesson, the worked examples did more than establish and demonstrate important properties of rebounding. It also enabled me to see the path of the ball as an elaborate design of zigzagging paths, which helped me wonder about the possible kinds of paths (F5). Rather than ending my curiosity and the anticipation of new developments in the story, these worked examples instead provoked curiosity. In fact, the author explicitly points to the worked examples as the basis for raising new questions and proposes that the reader do likewise.

Differences in the density of questions. Finally, one of the most stunning differences of the mathematical stories revealed through this analysis was the difference in the number of the story arcs that overlapped throughout my readings of the lessons. At any given time throughout the middle half of the Jacobs lesson (between ACT#6 and ACT#17), I was pursuing between five and six mathematical questions (including both momentary and overarching), all of which were related to the path of balls on billiard tables. For example, in ACT#12, I was pursuing F2, F3, F4, F5, F10, and F14[64]. This is starkly different than my reading of the SFAW2 lesson, during which I focused on at most four questions, only three of which were related to the commutativity of addition. This degree of complexity of the mathematical story, which requires the reader to be able to maintain and pursue questions for longer periods of time throughout numerous changes, can be thought of the mathematical plot's density. The greater the density, the more the reader is challenged to consider and pursue answers, and the mathematical story can be thought of as thick with mystery (as in *the plot thickens*).

[64] Note that although F13 was also later revisited in ACT#15, it was not counted in ACT#12 because it was answered in ACT#11.

It can be noted that, although these mathematical stories had different degrees of density for me as a reader, both lessons had a similar pattern of increasing and then decreasing density across the lesson. The number of questions that I entertained throughout each sequence (not counting those that were already answered at any point) display, when graphed, a similar shape of rapid increasing, followed by leveling off and then slowly decreasing. These graphs are shown in Figure 5.24.

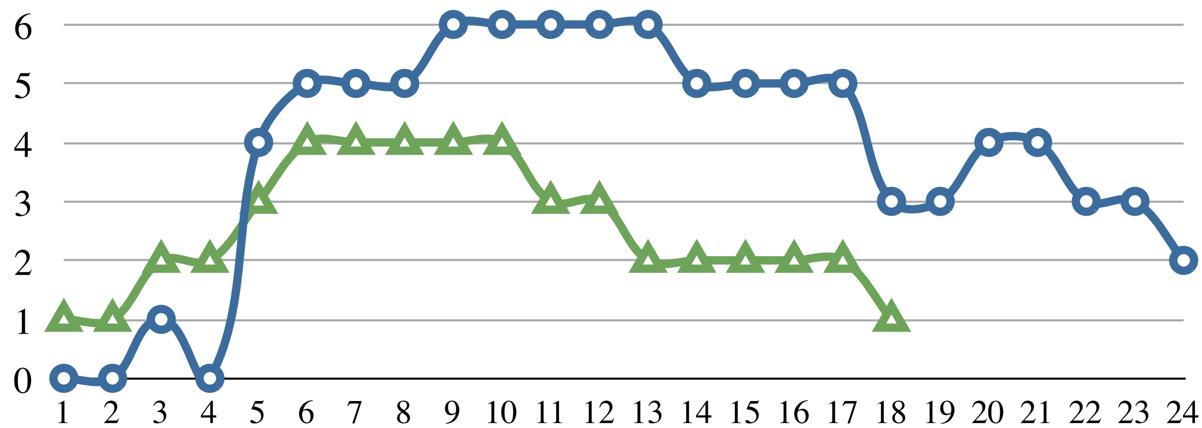


Figure 5.24. The number of mathematical questions (vertical axis) pursued across events (horizontal axis) for both the Jacobs (with circles) and SFAW2 (with triangles) lessons.

One other important distinction in relation to density between these two lessons can be noted when comparing the rate of change of density from event to event. In SFAW2, the density changed from one event to another by at most one question, meaning one mathematical question was either added to or removed from pursuit. For example, in ACT#4, two questions were under pursuit (F1 and F2), while in ACT#5, three questions were pursued (F3 was added and neither F1 or F2 were answered). However, the Jacobs lesson had two points in the mathematical story when the change was greater than one. The greatest was the relative explosion of questions opened at the beginning when Jacobs presents the reader three overarching questions and stimulates the reader (me) to generate another. At this point, I went from considering zero questions in ACT#4 to four overarching questions in ACT#5. Another large change near the end of the lesson between ACT#17 and ACT#18, where the density dropped by two questions, considerably lightening the conceptual load.

Effects of the sequence of events. Another way the conceptualization of mathematical plots can help increase an understanding of written curriculum is through considering different hypothetical sequences of events. Although it is tempting to focus much attention to Barthes' hermeneutic code (with the sub-codes used in this analysis) to understand the forces on the reader, it is important to keep in mind that Brook's referred to the *temporal* dynamics of plot, the dynamics experienced by a reader in real-time while reading the story. Barthes' proairetic code, which marks the sequence of events (ACT), although not time-specific (e.g., each ACT does not represent a length of time), orders these events to help the analyst understand how multiple story arcs overlap or not. A minor change of the sequence of events in a literary story (such as whether Juliet wakes up before or after Romeo kills himself) can make a surprisingly important change in the effect of the story and, in many cases, can change the story altogether.

The same can be recognized about the sequence of events in mathematical stories. Therefore, similar to Netz's observation of the choice of the mathematician to logically sequence the assertions and conclusions in a mathematics text, an author of a mathematical story (such as in a mathematics textbook or in a planned lesson) chooses a sequence of events of the fabula to create the story. For example, what if the lesson had opened with the photo of the student wondering "What happens to the sum when we change the order of the addends?" (p. 23) and then had proceeded on to the part-part mat activity of the Investigating the Concept? I argue that, at least for me, a couple of things might have changed about my reading of that section. First, the focus on reading the part-part mat might have provoked me to read for the effects of order of the addends. It also may have helped me notice that the sum cannot change because the total quantity represented by both addends together on the mat is literally the same. That is, when read from opposite sides simultaneously, the total quantity of the counters remains unaffected. Therefore, the presence of this overarching question in the early events of this mathematical story likely would have changed my focus during this activity (which instead was answering "what do I notice" in ACT#3 and became about looking for differences and commonalities). In addition, this change in sequence might have enabled me to start the Investigating the Concept (the introduction of the part-part mat), anticipation and purpose. As the lesson currently is written, I did not have any anticipation of why I was reading a part-part mat and what it might reveal.

Discussion and Implications

Mathematical plot is a conceptualization of mathematical sequences that makes note of a reader's temporal experience and sense of structure as he or she reads. It helps to capture how a reader uses parts (events) of a mathematical development to make sense of a whole in much the same way that he or she can make sense of a literary story. Adapting a construct developed for literary stories, this framework exposes and articulates some of the ways mathematical sequences can affect us (as readers), shaping our experiences with mathematics.

Yet the work connecting mathematical plots with literary plots is only beginning. Through the analysis of mathematics curriculum, important ways they relate, as well as important contrasts and distinctions, can now be recognized. This section, therefore, focuses on strengthening the framework by returning to the Bal and Barthes literary frameworks to make additional observations about the theoretical framework of mathematical plot. It describes some ways mathematical plots differ from literary plots and proposes changes that may help to better describe the form and effect of mathematical sequences. It also discusses the potential for this conceptualization to help imagine new mathematical stories. Finally, this section will complete the argument regarding the potential affordances of mathematical plots by discussing the additional implications it holds for students and teachers.

Revisiting Barthes' framework. The coding of mathematical plots with Barthes' framework revealed several characteristics about the emergence of questions during the mathematical stories and my temporal pursuit of their answers. The structures of the two lessons are appropriately distinguishable as they represent my different mathematical experiences. Whereas the Jacobs lesson continually felt rich and fulfilling, the SFAW2 lesson held my interest for at the beginning and then felt "flat," tedious, with little progress.

However, the Barthes framework offers other benefits as well. Particularly, this framework offers the ability for the analyst to tease apart the mathematical development of the different threads of reasoning throughout a mathematical story. For example, reading across the HER10 story arc in the coding table, I can recognize that I started to notice the difference in the complexity of paths in ACT#8 and then formulated F10 ("What makes some paths more complicated than others?") in ACT#9. Then, after a surprising square table was introduced in ACT#12, I realized that square tables would never require the ball to rebound and therefore would offer no complexity. However, at this point, I still did not know what might make some tables more complex than others except that when the side lengths are not equal, the resulting path is not complex. Then in ACT#23, the prompt focused my attention on comparing the tables for which had the most complicated path. I realized at that point that tables with orthogonal dimensions that differed by one unit (e.g., 5x6 and 7x8 units) resulted in the most complex paths, and I started to feel like I had a potential answer. Recognizing a single line of reasoning across a mathematical story such as this is difficult in the milieu of exercises and worked examples found in a textbook. In addition, this framework also allows a reader to characterize how and when particular mathematical ideas (e.g., the complexity of paths) emerge, develop, are frustrated, delayed, and are (hopefully) eventually resolved.

Even with all that Barthes's framework potentially reveals about mathematical sequences, some important aspects of the experience of reading mathematics curriculum are not easily captured in either the diagram or the coding table. For example, as was pointed out earlier, not all equivocations result in the same effect on the reading of a sequence. Where one can result in delight, another might provoke shame for not having interpreted a meaning correctly. Therefore, merely showing the equivocation code followed by a set of partial answers does not help to represent its aesthetic effect on the reader. Yet, the role of equivocation in mathematics curriculum has important ramifications for the learning of mathematics. Although it could be argued that learning mathematics is about *removing* ambiguity and making precise, I posit that since all of reading involves interpretation, ambiguity is impossible to remove from mathematics curriculum even in cases when the authors painstakingly define every mathematical term and prove every theorem. When viewed as a mathematical story, we can recognize how a reader reads into each story and builds connections that are unwritten between parts, no matter how much is (un)written. With the Jacobs mathematical story, I found evidence of a storyteller who not only is aware of the equivocation in mathematics curriculum, but who took advantage of it to delight students (e.g., the question "Did any of the tables surprise you?").

Highlighting another distinction that is not apparent when looking at Figure 5.18, some of the formulated questions pursued throughout my reading were explicitly stated in the text while others were raised by me independently of the text, such as when I asked F5 "What types of paths are there?" in the Jacobs lesson. Barthes' (1974) framework does not distinguish between questions asked in the text and those not (which I will refer to as *explicit* and *implicit*, respectively), since he proposes that when a question is printed in the text, the reader must still "write" it for it to become part of the plot. That is, in order for a question to become a motivational force for the reader, he or she needs to notice it[65], interpret it, and desire an answer (this agency will be referred to as "adopting" a question). Therefore, a reader's adoption of an explicit question is not essentially different than his or her raising of an implicit question; what is different is its stimulation.

For the benefit of mathematical stories in written curriculum, however, I argue that the difference between explicit and implicit mathematical questions is an important distinction. If it is important for a particular mathematical question to be considered and answered by the reader to make productive sense of future parts of the story (such as "How can we predict where the ball will end up?"), then it needs explicit formulation in the text. Although explicitly stating the question does not guarantee that a reader will adopt the question, doing so at least offers the reader the opportunity to negotiate the question in case he or she does not formulate it from other means (such as the variation of the end position of the ball).

This analysis also provides evidence that bypassing the need for a reader's question may also have serious consequences for the reader. When a mathematical question, the answer of which gives purpose and meaning of future events in the mathematical story, is never explicitly stated, it is possible a reader may not recognize the significance of a future explicit statement of its answer. For example, when I encountered the first Jacobs worked example (a series of diagrams showing the path of a ball), I focused on the way the ball rebounded. Only after I read the question did I realize that it also showed me where the ball ended up. Without the question

[65] A reader could ignore the question in text or miss it all together, resulting in it not becoming part of the mathematical plot.

being asked, the answer had a very different meaning for me. In addition, encountering an answer without a question offers a very different experience for me. Rather than being a mathematical accomplishment of a line of reasoning, the answer may instead appear on the surface as just another statement of fact for the reader to accept; rather than a sense of satisfaction and completion, it brought ambivalence.

Finally, it might be useful to add an additional code to further distinguish partial answers and disclosures when they are first encountered in a sequence from those that are repeats. In Barthes' analysis, he makes no distinction, likely because he would argue that even when encountering identical text referring to the same answer, its meaning(s) can change[66]. In mathematics curriculum, a situation in which I am prompted to answer the same question a second time, such as when I was prompted to describe the table that had the simplest path after I had already formulated and answered this question earlier, has a qualitatively different effect than when the question is first entertained. Even though the text does not explicitly ask for a repeat of an answer, it nonetheless provoked me to restate a prior realization. The need to ask these questions in the text turned out, for me, to be unnecessary. These moments felt like a step backwards or a halt in progress rather than a movement forward in the mathematical story.

Some might conclude that when a question is repeated so that a reader may feel a lack of progress, it should be removed. However, I do not necessarily recommend that these prompts be excluded from the mathematical story for several reasons. First of all, these three repeated codes highlight the fact that the unfolding mathematical story enabled me to *anticipate* the direction of

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[66] This is not to suggest that literature does not return to the same questions repeatedly. For example, during a mystery, a reader might realize ("conclude") the answer to an overarching questions (who done it?) using deduction before the text explicitly states the answer. Thus, when the reader encounters the detective's solution, it may be a repeat of the answer the reader already suspected.

the mathematical changes by sensing an overall structure to the momentary parts. Second, without asking the questions, another reader who instead anticipated other aspects of the mathematical story (or none at all) might not have considered these questions. And lastly, because of their inclusion, the author displays his or her intention of what is revealed in the mathematical story, thus helping to focus the reader's attention to an important (to the author) aspect of the content development without needing to say "look at this, this is important." Therefore, being able to tag these experiences with a new code (such as *repeat partial answer*) would allow for their distinction and allow analysts to learn more about their manifestation in written curriculum.

Imagining new mathematics curriculum. This mathematical plot framework can also help a designer imagine possible new mathematical stories unlike those analyzed in this chapter or possibly ever seen before. As discussed earlier, the diagrams used to organize the temporal transitions throughout the events can highlight structural differences between mathematical plots in terms of emergence of questions, density of story arcs, and resolution. When different structures of mathematical plots are considered, like those provided in Figure 5.25, interesting curricular challenges and questions emerge, such as which mathematical questions are suitable to span an entire lesson and which are not? Is the design in (a) possible or do overarching questions eventually emerge? What type of implications does a design such as (b) have for students and teachers? What other mathematical plot structures might be explored?

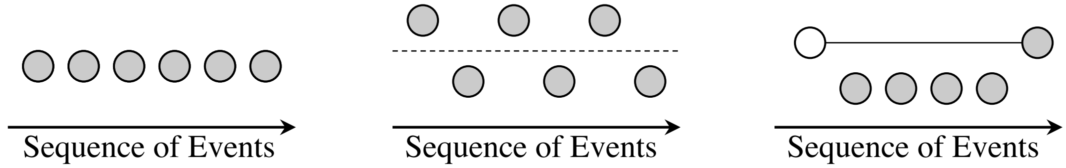


Figure 5.25. Thought experiment regarding the density of mathematical stories.

The structural aspect of density also offers some possibly generative thought experiments in terms of mathematics curriculum. Considering, for example, what might be said about mathematical stories such as the contrasting cases represented in the density diagrams in Figure 5.26 might be productive for mathematics teachers seeking to understand the potential effects of different mathematical sequences for students. Seeking opportunities, or lack thereof, of potential opening and closing mathematical questions (including both momentary and overarching) throughout a lesson can provide curriculum designers (including teachers) a new heuristic tool.

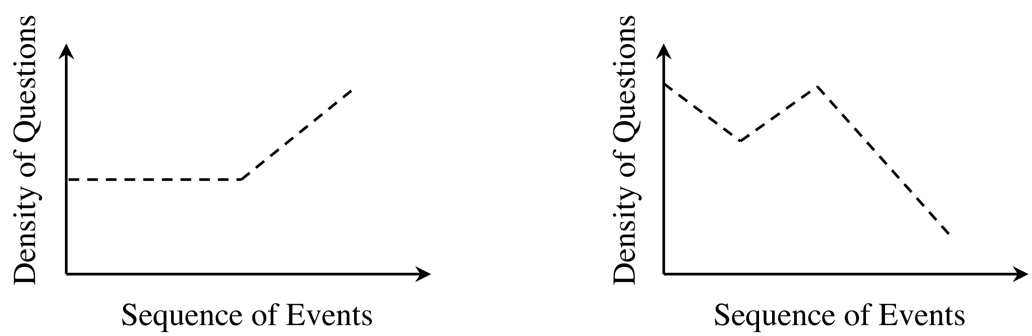


Figure 5.26. Imagined mathematical stories with different density patterns.

The absence of two of Barthes' sub-codes from the lessons analyzed in this chapter (*suspended answer* and *snare*) offer an opportunity to mentally extend the possibilities for mathematics curriculum. The suspended answer describes when a question is raised and then, for whatever reason, the story changes the subject to an unrelated topic. This could be seen when the question about the Wizard of Oz was raised from the title and yet the story starts in Kansas with no mention of a wizard or Oz. The Jacobs lesson did not shift my attention away from the topic of the path of balls on a table. Even though some of the overarching questions opened at the beginning of the story remained unaddressed for numerous events, the fact that the focus remained on the paths of balls on billiards tables maintained my expectation that I may find an answer at any moment. The SFAW2 lesson also did not get a suspended answer code even though I felt that the mathematical story abruptly changed topic in ACT#17. This was because the overarching question at that point (What will be added?) is an open question with no proper closure. Since each event focused on adding, it was possible to be addressed again at any point. However, mathematical stories that do suspend an answer are not difficult to imagine and the reader may be aware of examples of this currently in existence. A mathematics lesson that shifts between different unrelated content throughout a lesson or across lessons in a disconnected fashion would likely abound with suspended answer codes.

Likewise, a snare is a falsehood, an explicit misdirection, that leads a reader astray and diverts him or her from the (intended, and later to be developed) answer of an overarching question. Although it is likely controversial to suggest that mathematics textbooks lie to readers, is this really so hard to imagine? What may feel like an ambiguity to one reader may come across as misdirection to another. Although the epistemological differences are beyond the scope of this discussion, I wish to point out here that a snare can easily describe those moments at which a mathematics textbook explicitly states a mathematical idea that is momentarily true, but later is not true after more mathematical concepts are developed. This can be seen in a case when a text explicitly states that "adding makes bigger" in a portion of a mathematical story when only counting numbers are involved. Once zero and negative numbers are introduced, this statement can be recognized as false. Rather than simply lamenting this practice, however, I assert that these types of snares are sometimes unavoidable due to the incremental nature of mathematics curriculum. The mathematical plot framework instead offers an opportunity to think both about how snares can be dealt with as well as potential aesthetic implications. Just like how a reader of *The Wonderful Wizard of Oz* may be surprised and concerned to learn at the end that the Wizard has no magic and will not be able to send Dorothy home, *snares* provide opportunities for readers, as a result of an unexpected turn of events, to revisit assumptions and rewrite prior understandings. How might mathematics curriculum be designed to similarly captivate and delight readers while appropriately requiring that they challenge and adjust earlier naïve understandings?

The rhythm of mathematics curriculum. One aspect of the structure of mathematical plot that can be noticed with the diagrams of story arcs, such as in Figure 5.18, is a rhythm of opening and closing questions that changes throughout the mathematical story. As described in the opening section of this chapter, Netz (2005) compared the felt rhythm of ancient mathematics texts with the stanzas of a poem. However, with this mathematical plot framework focused on enduring inquiry, a musical analogy might be more appropriate. The overarching questions are similar to sustained notes spanning a melody. In contrast, the momentary questions appear (and last) like staccato notes in music, quickly punctuating the story forward. With this analogy, the sudden introduction of four formulated questions in ACT#4 of Jacobs' lesson is the striking of a chord, which then undulates with the addition of new notes here and there throughout the remainder of the story. Barthes (1974) similarly uses an analogy of a musical score to describe the multitude of voices throughout narrative, comparing the hermeneutic voices throughout as "what sings, what flows smoothly, what moves by accidentals, arabesques, and controlled ritardandos through an intelligible progression (like the melody often given the woodwinds)" (p. 29). This juxtaposition of short and long "notes" helps to emphasize the coordination and appropriateness of both elements in a story. If all notes are long, for example, one might think of a dirge. If all notes are fleeting, however, the story may flit by like Rimsky-Korsakov's *Flight of the Bumblebee*. These different rhythms could be studied to learn more about the success of some mathematics lessons. That is, recognizing that there will always be individual differences, it still may be useful to learn if is there a particular type of rhythm of mathematical plots that generally attract and hold students' attention.

Revisiting Bal's tensions. Bal's (2009) notion of the tension of anticipation and realization can be seen in the mathematical plot diagrams such as Figure 5.18, where a single mathematical question can be seen to remain open through multiple attempts at answering. Realization, which is the emergence of understanding by a reader, is experienced when a reader answers a formulated question (regardless if the answer is explicitly stated in the text). As seen with the two example mathematical stories, the realization of an answer in a mathematical story can occur within a single event (as with a momentary question), but sometimes an answer takes many events to be "fully" realized[67]. Consider, for example, the formulated question F5 in the Jacobs lesson regarding what types of paths exist on a billiards table. At first, I realized that some paths zigzag upward while others rebound along the top wall. Then I encountered a path with no rebounds, and noticed that some paths are more tightly wound about the table than others. Note, however, that I never sensed a final resolution to this question. By the end, had I encountered and identified all the possible types of paths? The inductive approach of this lesson withholds this answer from me directly, requiring me to decide what is sufficient to satisfy my curiosity. This uncertainty of resolution allowed my interest in this formulated question to be maintained throughout the mathematical story.

In contrast, consider the possible effect on my anticipation and realization if the author had instead made a chart at the beginning introducing and naming the types of paths with a sample or two diagrammed for the reader. With this approach, the author removes all tension by

[67] The notion of the reader's epistemological role in the closing of questions will be discussed in a later sub-section.

supporting the realization immediately, possibly before I raised the question. Later, if I ever formulated this question, it may instead be for recall, as in, "what types of tables were there again?", prompting me to return back to the list. When considering this alternative, Jacobs' sequence of events, particularly the way the sequence of tables enable glimpses of types of paths to emerge slowly throughout the Set I, can be appreciated for provoking and maintaining my interest. Bal (2009) explains that when the reader already is set up with the answer at the beginning of a literary story, this evokes a sense of predestination, explaining,

> Nothing can be done, we can only watch the progression towards the final result, in the hope that next time we recognize the writing on the wall. This type robs the narrative of suspense, at least a certain kind of suspense. The suspense generated by the question 'How is it going to end?' disappears; we already know how it is going to end. (p. 93)

Bal's quote above describes my experience during the SFAW2 lesson, when (in ACT#6) the formulated question posed in the photo of the student is answered immediately (see Figure 5.20). The subsequent exercises offered me no anticipation of surprise or expanded insight, only practice with adding and switching the order of addends. Later, in ACT#10, when the answer is restated, the only surprise was the fact that it was repeated, since to me the question was resolved earlier in the mathematical story.

Conceptualizing mathematical plot turns. Of course, if the SFAW2 exercises had offered me new insight, I would have been delighted and surprised. In general, surprise is evoked when what a reader anticipates in the story (based on hints in the text) is violated. For example, for a formulated question, a reader may anticipate one or more possible answers, which may have varying degrees of specificity. If the text's eventual "reveal" is inconsistent with his or her expected outcome, especially in opposition to, then surprise is almost certainly triggered. This was the case when I anticipated that all the paths would end in one of the top corners during the Jacobs lesson, only to learn in ACT#20 that the ball can end in the lower right corner. In this sense, my temporal anticipation was sustained through multiple paths that were consistent with my expectation.

Unfortunately, the mathematical plot diagram in Figure 5.18 only offer glimpses and hints of my anticipation throughout the story. In the case of the equivocation, this code identifies points in the story when I expected one answer to be true and then learned later it was false. However, other anticipations, such as the expectation that all tables would have rebounds only to encounter the square later, are not visible at all in the diagram. If anticipation is viewed as a way that the reader projects forward future experiences in the story (such as predicting how the story ends), this projection can be described at any moment with a linear extension of story arc based on what the reader anticipates at that point. A series of diagrams in Figure 5.27 might offer a way to conceptualize this temporal anticipation and violation, explaining why some might say the plot has "turned." In Figure 5.27(a), a reader in ACT#1 formulates a question and makes predictions about its possible outcomes (answers), represented by a dashed line extended beyond the question. After the question is formulated, this prediction may later be supported (in a sense, fortified) by the text, such when the Jacobs lesson offered me additional evidence that the ball only ends in the top corners. This can be represented with a partial answer in ACT#2 in Figure 5.27(b), in line with the reader's prediction since it is consistent. When this reader encounters an unexpected result, however, he or she may now sense a change in direction of the story, shown in Figure 5.27(c).

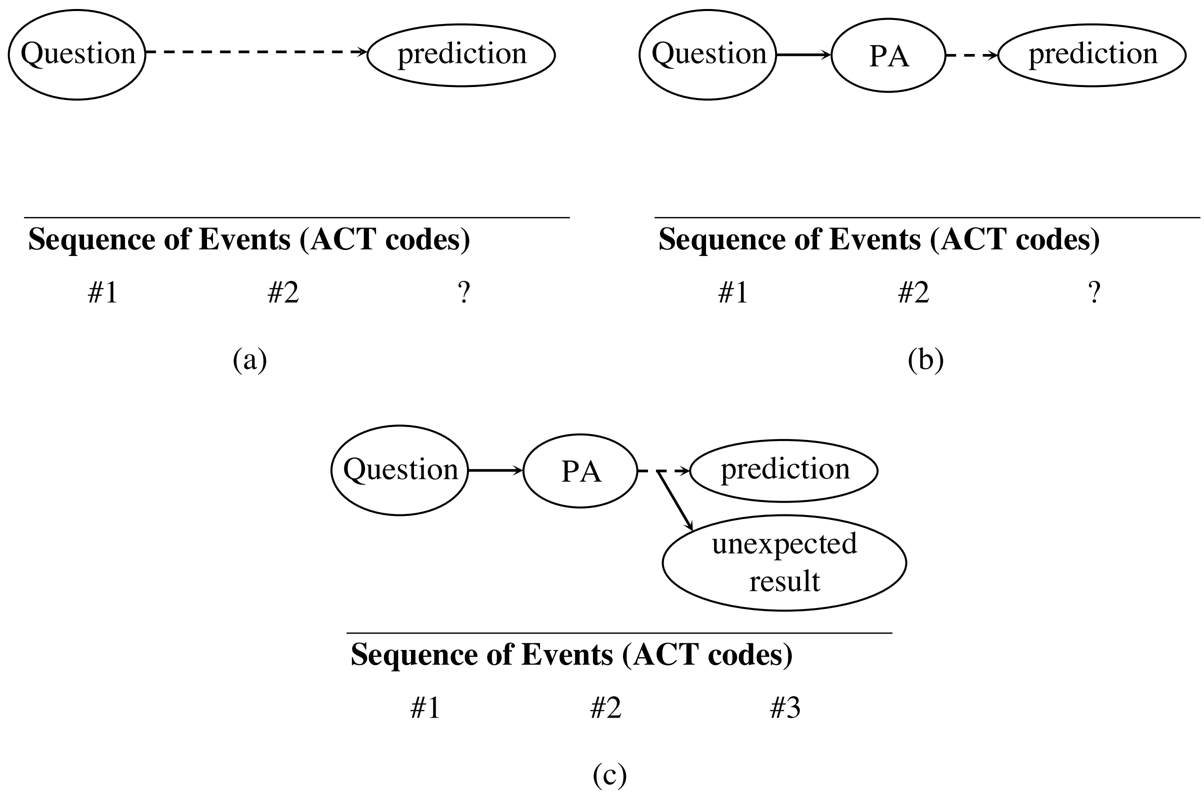


Figure 5.27. A proposal for representing surprising changes of mathematical plot.

It may appear that this framing of a plot turn may only explain the aesthetic of a story during the first read by a reader. Instead, the recognition of multiple possible outcomes might also increase the enjoyment on subsequent reads. For example, perhaps readers that have read a story previously can become captivated because the reader enjoys reliving the emotions and desires of the story. Sinclair (2005) connects this phenomenon to the enjoyment mathematicians sometimes express with reproving a theorem or the reactions of students when re-solving a task they have solved before, noting that this changes the compelling question driving the reading: "The dramatic tension shifts; it's no longer about 'what's going to happen'? or 'Is it true?' The shift is a meta-cognitive one; you're interested in how you know what you know" (p. 13).

How a reader's epistemology may affect the effect of a mathematical plot. Bal (2009) suggests that different stories can be categorized by the type of suspense experienced by the reader and proposes that this effect can be described by what is known by the reader and the characters of the story. Briefly, Bal asserts that four types of suspenseful stories can be recognized and described based on the relationship between the knowledge of the reader and that (presumed) of the characters. When a major question is raised in a literary story, such as “How will Dorothy get home?”, Bal describes four possible combinations of knowledge status between the reader and characters: (1) neither the reader nor the characters know the answer, which sets a reader up for a mysterious adventure to reach the answer; (2) the reader knows the answer but the characters presumably do not know, which often involves a threat and builds suspense for whether the characters will learn the answer in time; (3) the characters know the answer but the reader does not, which instead has the effect of a secret that a reader may want to learn; and (4) both the reader and the characters both know the answer, which eliminates suspense altogether. These combinations are represented in Figure 5.28.

		Characters	
		Do not know answer	Know answer
Reader	Does not know answer	mystery	secret
	Knows answer	threat	no suspense

Figure 5.28. Effects of literary plot depending on the knowledge status of the reader and characters (Bal, 2009).

In a mathematical story, it is not useful to attribute knowledge to mathematical characters. That is, what might it mean for a mathematical character such as a 4x6 billiard table or an addition sentence like $3 + 4 = 7$ to know something? Thus, unless knowledge is attributed to mathematical objects, Bal's framework of suspense does not at first appear informative to mathematical plots. However, when considering the epistemological stance of the reader toward mathematics, that is, what it means to know mathematics, Bal's framework regarding suspense can offer insight into the possible effects of mathematical stories. For example, one stance a reader might take is that mathematical texts (i.e., the content in them) mirror reality and mathematical knowledge is what is known about a "platonic realm of ideal mathematical objects" (Ernest, 2008a, p. 2). When encountering an overarching question, a reader with these views (that mathematics is a fixed body of existing truths to be discovered) might experience the mathematical plot of a mathematical story as uncovering a secret withheld by the universe.

On the other hand, another epistemological position identified by Ernest is that of mathematical knowledge "as social and cultural knowledge that is publicly shared, both within the mathematical community and more widely as well" (Ernest, 2008a, p. 3). A reader with this stance might instead take the stance that since mathematics is a shared human creation, then there is no presumed or predefined answer. This stance could help the mathematical plot to feel like a mystery or adventure as the reader (with the help of the mathematical characters) works towards an answer. So, even though the notion of the knowledge of mathematical characters in this way has a different basis than Bal's notion of knowledge of literary characters, future work could study whether the juxtaposition of the reader's knowledge with his or her epistemological stance toward mathematics has a similar effect on the type of suspense generated.

Additional implications for students. The notion of mathematical plots in written curriculum challenges common assumptions about the role of written curriculum in mathematics teaching and learning. For example, rather than viewing the end of a mathematics lesson as a closing of questions, the example with Jacobs lesson demonstrated the aesthetic benefits that came when questions remained open at the end. How might we go beyond a typical announcement at the end of the lesson ("Tomorrow, we'll...") so that the student instead is left with unanswered questions that make him or her want to return for the next installment?

Stepping backwards to view a mathematical plot as a whole structured experience, rather than looking at the particular coded moments, some additional comments can be made about the density of mathematical plots. For example, the greater the density, the greater the inquiry by the reader, which intensifies a reader's felt need for resolution. Although the sequence of events is certainly not determined by some fixed intervals of time and therefore is not a constant unit of measure, each event represents a noticeable change to the analyst. Therefore, the change in the number of questions under pursuit as the number of events increases represents a widening or narrowing of a mathematical story. For example, when this rate is negative, a reader is answering more questions than opening new questions, which reduces the load on the reader and makes the lesson feel narrowly focused. A constant rate, which can occur throughout momentary questions, indicates points where a mathematical story may feel "flat."

While reading a mathematical story, there are times when a reader formulates an implicit question that is not followed up in the story. These questions, like when I wondered whether there would be any related subtraction facts in the SFAW2 lesson, appear to be dropped and irrelevant at that point in the story. However, when what is considered as the mathematical story is broadened to include multiple related lessons, later progress on these types of earlier dropped questions sometimes may be recognized, and at that later point, the reader may connect this progress with his or her anticipation. With what Bal (2009) refers to as *hints*, stories (including mathematical stories) can subtly support a reader's anticipation to know where the story is headed. This anticipation not only enables a reader to look forward to more of the story, but also potentially offers satisfaction and gratification when the anticipated outcome is met. What are the potential benefits for students when they successfully anticipate future parts of the story? What types of hints do students pick up on?

If mathematical plots were designed to encourage students to anticipate with wonder, perhaps more students would want to continue further into the broader mathematical story. Perhaps no one better describes the role of anticipation in learning than Wong (2007), who, drawing from Dewey (1934), explains,

> Anticipation is the tension in the dramatic line that connects the "what if" to "what is." The excitement of sensing an opening to a possible world and the irresistible urge to move into the world best describes the motivation of a student who suddenly sits bolt upright in class and exclaims, "I have an idea! What if . . . " Anticipation is what transforms an ordinary occurrence into an event saturated with significance and moving forward with dramatic energy. Whether the learner is engaged in reading a story, watching a film, or conducting scientific inquiry, anticipation is what moves us to the edge of our seat so that we may see better and be better prepared for what we might see. (p. 208)

Wong's tension of "what if" and "what is" renames Bal's tension between anticipation (e.g., what the reader guesses or expects) and realization (e.g., what comes to be true, according to the reader). Wong's notion of a sense of an opening to a possible world wonderfully describes Barthes' *proposal*, a sense of the possibility of an enigma. Wong's irresistible urge to move and his "moving forward with dramatic energy" describe the temporal dynamics that can motivate the reader to formulate (or adopt) a question and pursue its answer. In addition, Wong aptly highlights the role of significance and the preparation for what is to come, in how a question can help a reader recognize the significance of an answer. Anticipation offers a reader a sense of connectedness with a story, a feeling about being in the groove and perceiving hints of what is to come. If mathematics students can successfully anticipate forward into the mathematical story, that is, accurately construct ideas about where the mathematical story is headed, it is possible they might be more confident in asking more questions, gaining a sense of trust that he or she can ask productive questions in anticipation of textual support for disclosure. Although the gap between being led in a mathematical story through curriculum and leading your own mathematical adventure as a mathematician might is immense (see Hofstadter, 1992 for a good example), fostering mathematical anticipation might at least partially offer a bridge.

Reading a mathematical lesson to identify the overarching mathematical questions throughout directly addresses an age-old question often heard by students wondering "what is the point of learning this?" Besides the important cultural and political aspects of this question (e.g., is it to raise test scores?), on a day-by-day level, the notion of mathematical plot brings to the foreground the purposes a reader might hold for continuing to read a mathematical story. As was found in my reading of the SFAW2 mathematical plot, some events of the story continued even once all the overarching questions were closed, which fairly raises the question of why are they there? Are they part of the mathematical story? Are they instead something else? Designing curriculum with the mathematical plot in mind can help designers (including teachers) attend to mathematical questions that support readers to recognize the purpose of parts of the story (such as jumping from table to table in Jacobs' lesson) throughout mathematics curriculum (both in written and enacted forms), which can potentially lead to mathematical stories that hold the interest of students.

My reading for mathematical plot also points out how, within a mathematical story, what it means for a question to be answered is not always clear even when the reader has advanced mathematical knowledge. This challenges the notion of mathematics curriculum as building certainty. For example, in the case of an equivocation, a question that at one point appears answered turned out to be the seed for later surprise. Had the lesson ended after ACT#17, I might have continued to assume my conjecture was correct without question. The decision for what counts as "closed" will vary by reader and relates to the standards of evidence used by the reader to be convinced that the partial answer they have generated is "realized."

How to determine something is correct may also rest upon genre knowledge. For example, in a mystery book, the reader familiar with this genre likely expects the questions about who perpetrated the crime to be answered. Even if there is overwhelming evidence that one particular character has "done it," an avid mystery reader is likely to continue to hold some amount of skepticism that the character actually is guilty until the detective explicitly reveals the solution to the mystery. This is because enough mysteries contain "turns" at the end of the plot that suddenly reveal to a reader his or her flawed logic or because additional information is later supplied. Similarly, in each math text, a reader may also build up a sense of what it means for a question to be closed, depending on the question and how it is found through reading.

Additional implications for teachers. The reading for mathematical plot also holds special opportunities for mathematics teachers. Beyond the analysis of mathematical stories as they are encountered in written textbooks as was done in this chapter (what can be called "static"), recognizing mathematical plots supports the curricular decisions regarding sequence made by a teacher planning a lesson as well as the dynamic decisions in the moment during a lesson. Considering how changing the sequence of events may affect the mathematical plot can help a teacher (or other curriculum designer) notice when an opportunity of surprise or satisfaction might be lessened or eliminated. For example, the unpredictable changes between simple and complex paths of Jacobs' lesson (e.g., between the tables in Problems 3, 4, and 5) helped draw my attention to the differences of complexity and supported my generation of a new overarching question. Recognizing this role of sequence of the tables could help a designer recognize a potential loss of a reader's curiosity when considering other possible sequences (such as when a teacher decides to make a worksheet using problems from a textbook, or when a teacher chooses to only select some of the tables). This is not to say that this focus on complexity or its generative quality could not be achieved with another sequence, only that this juxtaposition made the change in complexity salient and had a role in generating inquiry. Reading for the mathematical plot might help reveal a potential loss in this generative quality if the tables were instead, for example, grouped by similar paths. Also, simply shifting the 4x6 table from Set II to Set I in the Jacobs lesson would have eliminated the equivocation in ACT#17 (regarding the possible corners at which the ball could end up), which would have eliminated my surprise in ACT#20.

Therefore, with the conceptualization of mathematical plot comes a useful curricular heuristic for designers of mathematics curriculum. Asking critical questions of mathematics curriculum, such as "How does the form (the story arcs) and effect of the mathematical story change if this event is moved earlier or later?" or "What overarching question(s) might help to motivate or give purpose to this string of exercises?" can potentially create generative and motivating opportunities that otherwise might have been missed. For example, when the exercises in the SFAW2 Lesson 1-8 are examined for answers they might work toward, and therefore what questions might be sustained through that work, new curricular opportunities for mathematical coherence emerge. Since most of the lesson was focused on the commutative property, questions such as "Does changing the order of the addends always leave the sum unchanged?" or "When might it help to change the order of the addends?" may positively affect the role of the repetitive practice problems by enabling the reader to gain more than automaticity.

Finally, reading for mathematical plots enables a teacher to recognize the complexity of the mathematical questions in play. This density of mathematical stories can easily be overlooked when mathematics curriculum is not read as a mathematical story. For example, on the surface, a single task such as problem #7 (the square billiard table) in the textbook appears to challenge the reader with only one question, namely, where does the ball end up? However, when reading a mathematics textbook for how these different moments work together to open and close questions across a sequence, it becomes apparent that this question is only one of many that might be "worked on" by a reader.

"You never change things by fighting the existing reality. To change something, build a new model that makes the existing model obsolete." — Buckminster Fuller

CHAPTER 6

Concluding Remarks

What happens when a mathematics textbook is viewed as a story? What can be learned about the manifestation of content in the textbook? These questions largely guided this project and helped to develop the framework and analyses within. However, they do little to describe the underlying purpose for these questions in the first place: to reframe mathematics textbooks in a way that generates new curricular possibilities for students, teachers, and other mathematics educators (including textbook authors). Beyond a theoretical exercise, this endeavor aims to offer a new response to a very old problem, described eloquently by a pre-service teacher after observing a tenth grade mathematics class:

> I don't know how they stand it. I couldn't stand it, and I wasn't even there all day. The lessons are so monotonous it makes you crazy. They just sit there and sort of listen or take notes or respond to the same leading, empty questions. I just thought with the *Standards* and all the new technology and everything that it would be better than when I was in school. But it's not. And yet, it absolutely has to be. *We can't just keep doing the same old thing*. (Allen-Fuller, Robinson, & Robinson, 2010, p. 231)

In other words, this dissertation could be described as an endeavor to suggest how "we," as a mathematics community, can avoid "doing the same old thing." It is an attempt to turn at least some of the attention away from much of the dominant focus on meeting standards, and toward a vision of mathematics curriculum as a complex narrative able to stimulate imagination and curiosity in readers. This was important two years ago when I started this project, and is even more important now due to the recent adoption of the Common Core State Standards in Mathematics ("CCSSM", 2010). The adoption of a single mathematical sequence for grades K-

12 by well over 40 states of the U.S. (granted, for multiple, complex social and educative reasons), has greatly limited the curricular potential in that it has essentially committed a majority of the country to the same mathematical story. Alternate sequences, such as those envisioned by innovative curricula such as the *Measure Up* Program (discussed in Chapter 1), are now not available for a majority of the children in the U. S. regardless of potential and documented educative advantages for students. Since this adopted "national" mathematical sequence largely parallels the historically dominant content sequence from elementary through high school mathematics, this policy has largely cemented in place "the same old thing."

Or has it? Is this new policy the death knell of thinking anew regarding mathematics curriculum? Taking Buckminster Fuller's vision of changing existing reality to heart (see opening quote), this mathematical story framework, as a new model, offers me hope of changing this existing views of curriculum, inspiring provocative educative opportunities for students, teachers, and curriculum designers. To help explain how, this concluding chapter begins with a review of the nature of this project, followed by a summary of the outcomes of this work (particularly with regard to its implications for existing teachers, pre-service teachers, and curriculum materials designers). After this, I propose additional ways the mathematical story framework can continue to evolve and learn from literary theory and pose questions for future inquiry to learn more about how this conceptualization of mathematics curriculum can benefit teachers and students.

The Evolving Nature of this Project

Due to the unusual nature of this dissertation, I wish to situate its outcomes by first describing the project as I eventually came to understand it. In short, this dissertation represents a theoretical argument. In Chapters 1 and 2, I argued that the study of and work with mathematics textbooks could benefit from literary theory. Then, in Chapter 3, I argued, through the careful definition of metaphorical constructs based on well-accepted narrative theory, that mathematics textbooks could be read as mathematical stories. Following this, in Chapters 4 and 5, I completed the argument that this theoretical framing of mathematics textbooks is both reasonable and useful.

By viewing mathematics textbooks as *art* with both structure and aesthetic dimensions, this study is an example of humanities research. Therefore, to study the nature and effect of mathematics textbooks, I chose to draw from outside the field of mathematics (and science in general) to that of literary theory. This move was warranted because for thousands of years, literary theory has been focused on answering challenging questions of narrative, trying to understand and describe how and why narrative does what it does. Within this field, Bal's (2009) narratological framework is particularly well-respected for how it clearly defines and articulates the elements of narrative that contribute to its form and function. In a complex field where terms like "story" and "plot" are sometimes used in conflicting ways, Bal offered clarity.

Using Bal's framework, however, introduced numerous challenges for me. This immersion into literature and literary theory, for me, was like learning a new language. In response, I began to read stories. Although I could generally be described as a casual and "picky" reader before, I became immersed in both children's stories and adolescent literature, reading more literature during my dissertation than at any other time of my life. The mixture of texts (across age groups and media) enabled me to continually challenge and refine my evolving understanding of narrative as I started to recognize particular "literary moves" that reminded me of "mathematical moves" I had used in my work as a textbook author. It was through the intersection of exploring Bal's (and, later, Barthes') framework, engaging with stories, and reading and interpreting mathematics textbooks that the constructs in this dissertation emerged.

In addition to becoming fluent in literary constructs, during this time I also learned about the complex roles of an analysis of text in this work. I began to understand, for example, that as a study, Barthes' (1974) reading of *Sarrasine* was not positioned as the interpretation, but instead was the means through which Barthes accomplished three goals: (a) the introduction and description of his theoretical constructs, (b) the demonstration of their use as analytic tools for studying other narrative, and (c) the substantiation of his argument that these theoretical constructs are both reasonable and useful. This revelation profoundly changed my understanding of the role of analysis in literary theory; the analysis of text itself is not empirical data but instead is the source for the germination of new ideas and constructs. It was then that I realized that as a study of text, the knowledge produced in this dissertation would not be any particular reading (which is highly subjective and differs by the individual), but instead would offer potential insight into what can be learned about mathematical stories through this way of reading.

Finally, I also studied the form in which literary theory was communicated. I noticed how each of the theorists (including Bal, Brooks, and Barthes) centered their arguments with a demonstrative novel or short story, allowing the selected examples to instruct and support the interpretation of the theoretical constructs being proposed. These examples obviously restricted what was able to be discussed, and thus I started to pay careful attention to the choices of mathematical stories used in the analyses presented in Chapters 3, 4, and 5.

Summary of Outcomes

As a result, in Chapter 3, I introduced the main outcome of this dissertation: a new conceptual framework that enables the recognition and description of how mathematical content is manifested in textbooks. Using a metaphorical interpretation of Bal's (2009) narrative layers, I reframed paper-bound mathematics textbooks as *mathematics text*, a sequence of mathematical events as encountered by the reader as a *mathematical story*, and the reader's reconstruction of the mathematical ideas across and beyond the story as the *mathematical fabula*. These definitions relied on many other theoretical constructs, particularly that of *mathematical event*, which was defined as a transition in mathematical ideas from one state to another by an actor. With this new framing, I argued, mathematics textbooks can meet Bal's criteria for narrative and thus can be interpreted as narrative. This clarification then allowed an articulation and illustration of other metaphorical mathematical constructs to follow, particularly mathematical characters, action, setting, moral and plot.

Through the testing of this emerging mathematical story framework, particularly with respect to mathematical characters and plot, Chapters 4 and 5 were developed. These chapters constitute a "proof of concept" that new knowledge about mathematics textbooks can be learned through the use of this narrative lens. For example, the study of mathematical characters in mathematical stories in Chapter 4 revealed that mathematical objects are introduced not only through direct statements (such as a definition), but also sometimes through mathematical actions. It demonstrated the nuanced changes in the meaning of mathematical objects as found embedded within a curricular sequence, as shown with the emerging complexity of Zero in *My Pals are Here! Maths* (Kheong et al., 2012). Through the close examination of my own anthropomorphism of mathematical objects in these mathematical stories, I noted different types of opportunities in mathematics textbooks for identifying with mathematical characters, such as empathizing with 17 in a lesson from *Scott Foresman/Addison Wesley* (Charles et al., 2008).

The analysis of mathematical plot in Chapter 5 was also fruitful and enabled me to illustrate how its framing can help to make sense of the structural and temporal forces of reading mathematics text. Barthes' (1974) set of sub-codes, which were used to mark mathematical transitions from question to answer within the mathematics textbooks, helped me to articulate felt differences in the reading of different lessons, such as the density of inquiry, the ignition of curiosity, the generation of anticipation, and the turn of surprise. For example, when examining how the events of these mathematical stories were structurally interdependent (that is, informing and affecting one another), I discovered how the sequence of tasks (such as the juxtaposition of billiard tables with complex and simple paths) can support the anticipation of mathematical inquiry, and thus, enabled me to predict the upcoming direction of the mathematical story. Also, I recognized how the acknowledgement of an equivocation within the mathematical story could set up later pleasurable surprise, which in turn strengthened my interest in pursuing answers to raised questions. Another example is the emerging revelation of the different roles of momentary questions within the mathematical story; in one story, they interacted to provoke my curiosity and help me make progress toward resolving inquiry, while in another, they offered me little progress and decreased my expectation of any advancement of the mathematical story.

Beyond the critique of existing mathematics textbooks, these chapters support a new vision of what mathematics curriculum can be: an enticing mathematical story that provokes curiosity and a reader's desire for more. This re-imagining of the potential for mathematics curriculum offers particular benefits for the curricular design work of teachers. With this framework comes a heuristic of reading mathematics stories that can enable a mathematics teacher to critique his or her lesson plans and textbook to recognize the possible aesthetic forces of the unfolding of mathematical ideas in a given sequence. In doing so, it supports a teacher's reconsideration of the sequence within and across lessons, offering ways to recognize the potential benefits (or disadvantages) of certain changes. The metaphorical mapping of curricular elements with narrative equivalents (e.g., mathematical objects can be interpreted as mathematical characters) provides teachers new (yet familiar) language for describing and collaborating on mathematics curriculum, further supporting their curricular design work.

However, the potential benefit for teachers reaches far beyond gross (or even minor) changes of sequence. Just as a fiction writer carefully chooses the moment in a story to introduce a character or to reveal "who done it," so too might a designer[68] of mathematics curriculum carefully consider when in a sequence to introduce mathematical objects or reveal important properties or relationships. Similarly, just as placing a setting of a story in New York City can offer particular affordances and constraints in a story (compared to Boise, Idaho), so too is the decision to set the study of integers on a number line versus two-colored tiles. Therefore, a contribution of this dissertation is to offer a conceptual foundation on which designers make mathematical choices regarding objects and representations. Finally, this new language offers new ways for teachers to identify and describe meaningful mathematical connections across large portions of textbook materials, the kind described by the anonymous teacher who recalled a memorable curricular realization, presented earlier in Chapter 4 and reprinted here:

> I remember the second time I finished [teaching with a geometry textbook]. On one of the last days of school we found the equation of a circle using a right triangle. At that point I realized that the whole [geometry] book was about a triangle. The better part of this story was when I shared this realization with my students they all said, "We know."

The mathematical story framework also offers an opportunity for schools of education to support the development of curriculum design by pre-service teachers (and, by extension, new

[68] Although curriculum designer has often been used in math education to refer to a textbook author, in this work, it includes the curricular work of a teacher in the spirit of Brown (2009).

teachers as they enter the classroom). In addition to helping to complexify simple models of curriculum, this new conceptualization of mathematics curriculum offers a view of how curricular moments within a sequence (e.g., a mathematical statement, the asking of a question) can interact with one another, affecting the temporal meaning and requiring renegotiation of earlier moments and future anticipations. Thus, it can help new teachers recognize the potential effects of curricular decisions in the moment (such as how choosing to have a student present a particular solution could change the way in which a later task is interpreted). It can also allow the re-conceptualization of lesson planning beyond a list of activities and actions to a carefully crafted story that connects beginning with ending.

Finally, it almost goes without saying that this framework also provides new opportunities for those who design curricular materials to be used in mathematics classrooms. The analytic tools defined and demonstrated in Chapters 3, 4, and 5 offer a method for authors to recognize the manifestations of mathematics within materials, illuminating the possible effects of curricular choices. For example, upon the recognition of the central role of a mathematical character in an extended mathematical story (such as Zero in *My Pals are Here! Maths*), a materials designer can make different decisions about how this character appears (or does not) throughout the story. This framework also foregrounds the interaction of the mathematical story elements (particularly the settings, actions, and characters), which enables a designer to recognize the limiting and expanding potential of each within the story (such as how the selection of a balance scale or use of counters changes the characterization of Zero). Finally, the conceptualization of mathematical stories potentially enables curriculum to be designed with opportunities for readers (students and teachers) to spontaneously inquire, anticipate mathematical development, and, most importantly, to want to know how each episode ends.

Future Inquiry

With this framework in place, a potentially rich set of research questions can now be studied. Particularly, the heuristic questions for reading mathematical stories and their elements (drawn from literary theory and proposed in Chapters 3, 4, and 5) need testing and refinement with both teachers and pre-service teachers alike. That is, *which questions support the recognition of the mathematical changes throughout a sequence by teachers? What new questions enable teachers to critique their mathematical stories (both in terms of mathematics textbooks and lesson plans)? Once teachers can attend to the emerging temporal mathematical changes throughout a planned lesson, what, if any, differences can be seen in their lesson plans? Also, does the introduction of the mathematical story framework allow teachers to notice and/or describe additional mathematical and curricular qualities of their textbooks that otherwise were not visible? Finally, when teachers (both those who have learned of the framework and those who have not) are given written mathematics curriculum materials and asked to describe ways in which they might improve the lesson, is there a difference? If so, along what dimensions?*

Although this dissertation develops a theoretical foundation for future studies in mathematical stories, there are additional ways that this framework can benefit from literary theory. That is, using the same method of directing questions of text from literary theory toward mathematical stories, what might be learned? One direction of exploration that can inform our understanding of mathematics textbooks and mathematics curriculum in general is to identify how some texts (including movies) push design boundaries and to turn that into new questions regarding the boundaries of mathematics curriculum. For example, the movie *Momento* is well known as a story that completely reversed the sequence of the fabula in the story. In other words, the last chronological event that occurred in the fictional series of events was revealed first, then the second-to-last, and so on. *What would be its equivalent in terms of a mathematical story? It is possible, and if not, why not? What might its benefits (and/or consequences) be?*

Another example of literature that contradicts some basic assumptions about narrative (such as a fixed sequence) is the text *Meanwhile* (Shiga, 2010). This book offers an interesting story structured so that the reader is confronted with numerous choices throughout the narrative, each of which not only change the story (the sequence of events) but also the fabula (because the events also change depending on the choices the reader makes). What might a similarly structured mathematics textbook look like? Is it possible, and if not, why not? What might its benefits (and/or consequences) be? With the increasing presence of technology in mathematics classrooms, with hyperlinks that naturally offer the opportunity for reader-designed sequences, the exploration of this question is increasingly relevant[69].

Once possible genres of mathematical stories emerge and are articulated, new questions can be explored, such as If we look at the genres of mathematical stories found within a textbook, what type of variation is found?, What might be the role of variation of mathematical story genres?, Do certain genres lead to improvements in learning or retention by students?, and Are there forms or genres that are noticeably missing and could lead to the exploration of new mathematical stories? What mathematical stories are "told" in our classrooms?

As described earlier, one of the potential benefits of this framework is in the work with pre-service teachers, for whom there is currently little to offer by way of conceptualizing mathematical changes in a curricular sequence. To address this need, reading comprehension activities developed to help learners interrogate literature may be adapted for use with pre-service teachers reading for mathematical stories. For example, in order to help readers

[69] Sinclair (2005) offers an interesting discussion and example of using hyperlinks to allow for student choice in the progression through mathematical tasks.

understand the significance of authorial choices, Nodelman and Reimer (2003) recommend the consideration of alternatives, such as "How would the events of a story seem different if we heard about them in a different order or if they were told from the point of view of a different character?" (p. 45). They also suggest that the readers "can write alternative versions of texts and then compare them with the original" (p. 45). What impact might these types of activities offer to mathematics teachers who are trying to understand the mathematical development of their textbooks? Future studies could examine whether the conceptualization of mathematical stories leads to more coherence in mathematical lesson planning of pre-service teachers, or whether pre-service teachers are able to better recognize and articulate the mathematical similarity and differences between two different sequences of the same curricular tasks.

Not to be forgotten, I am hopeful that the introduction of this framework will positively impact the mathematics curriculum that students confront daily. From the point of view of a student, what is the mathematical story? What story elements do they pay attention to? Although these questions are well outside the scope of this dissertation, the notion of mathematical stories could eventually lead to potentially useful line of research examining how the plot may influence (or not) student learning and motivation. For example, which mathematical stories do students describe in aesthetically glowing terms (e.g., thrilling, interesting, surprising, gripping), and what qualities of those mathematical stories may contribute to that reaction? What is the relationship (if any) between student-identified aesthetically desirable mathematical stories enacted in classes and student achievement?

Other important questions involving students include how can the use of equivocation be better understood so that other curriculum designers can similarly delight and interest more students? And what other narrative codes within a mathematical sequence offer potential understanding of the affordance of a lesson for curriculum designers? I propose that future work on the mathematical plot of mathematics curriculum seek to distinguish and understand the varying qualities of narrative moves (such as proposal, equivocation, snare, and jamming) in mathematics curriculum, particularly those that appear important in terms of mathematical meaning or experience of the student reader.

Concluding Thoughts

As I look back on my reasons for embarking on this work (described in the Preface), I am struck by the optimism in the goal of developing and analyzing a conceptual tool that can help recognize and advance possible improvements in mathematics curriculum. Instead of theorizing about selective parts of mathematics curriculum, such as an examination of definitions or the treatment of a particular content strand, I focused on the entire structure as a dynamic system of parts that interact with each other to support the reader's meaning making. In particular, I hoped that this new conceptualization of mathematics textbooks could help open up new possible ways to imagine and design mathematics curriculum.

As curriculum (broadly) affects nearly every aspect of schooling (from planning to enacting to assessing), this framework supports a renaissance in mathematics curriculum in which curriculum design takes into account both logic and aesthetic. It renders "obsolete" existing views about curriculum in research, such as whether a portion of a mathematics textbook (e.g., a definition or a task) can be understood and analyzed independently without regard to what came before. And it seeks to change reality for students, who are often condemned to mathematical stories that "do the same old thing."

APPENDICES

APPENDIX A

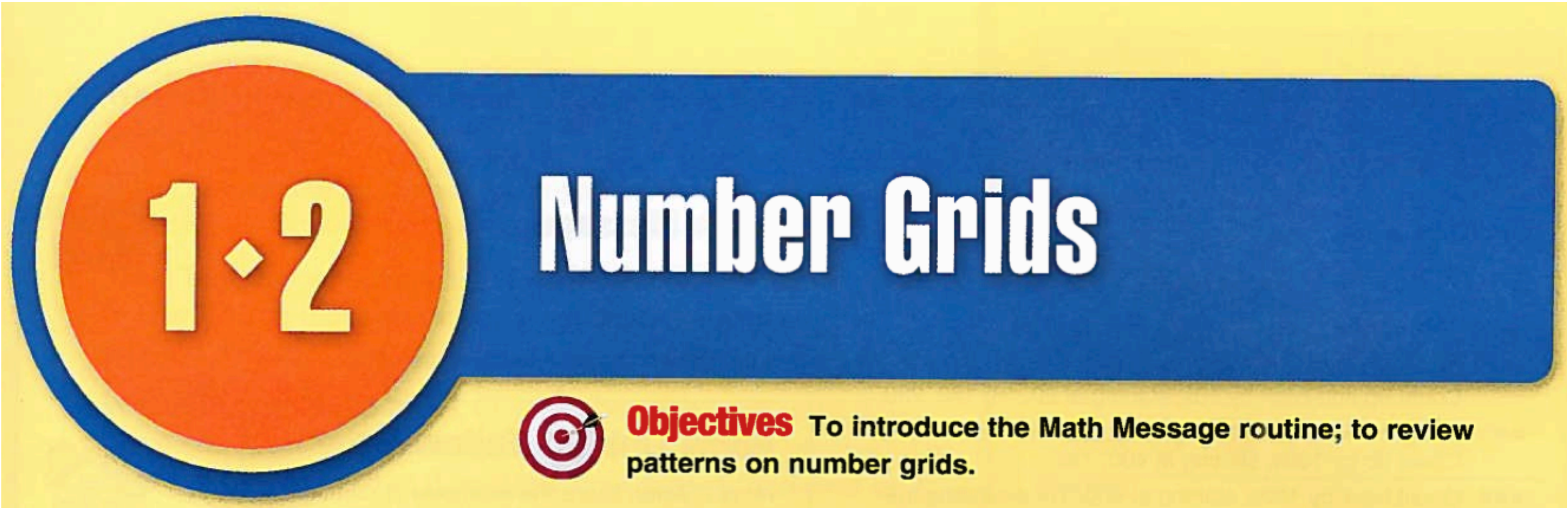
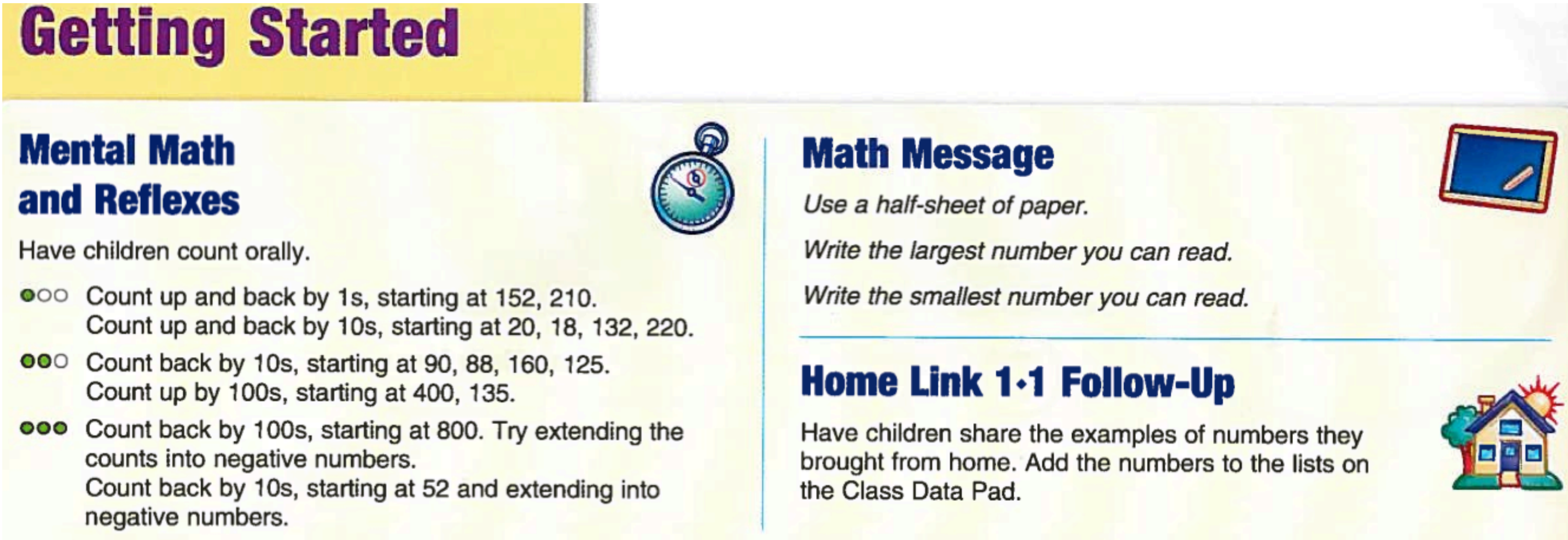


Figure A.1. Top portion with lesson objective of page 23 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enable the reading of small print, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.



Getting Started

Mental Math and Reflexes

Have children count orally.

●○○ Count up and back by 1s, starting at 152, 210.
Count up and back by 10s, starting at 20, 18, 132, 220.

●●○ Count back by 10s, starting at 90, 88, 160, 125.
Count up by 100s, starting at 400, 135.

●●● Count back by 100s, starting at 800. Try extending the counts into negative numbers.
Count back by 10s, starting at 52 and extending into negative numbers.

Math Message

Use a half-sheet of paper.

Write the largest number you can read.

Write the smallest number you can read.

Home Link 1•1 Follow-Up

Have children share the examples of numbers they brought from home. Add the numbers to the lists on the Class Data Pad.

Figure A.2. Top portion of page 24 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enable the reading of small print, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

Some children may benefit from doing the **Readiness** activity before you begin Part 1 of the lesson. See the Readiness activity in Part 3 for details.

NOTE Some children may record fractions, decimals less than one, or negative numbers as the smallest numbers they know. This is to be encouraged, but do not expect most children to be familiar with these concepts at this time.

Class Number Grid

1 Teaching the Lesson

Math Message Follow-Up

WHOLE-CLASS DISCUSSION

Have children read their largest and smallest numbers. You or a volunteer should record three or four of the largest and smallest numbers on the board. Ask someone to read the largest and the smallest numbers recorded on the board.

Review the names of places in a base-ten numeral. For example, ask children to circle the tens place in the largest number on the board, to underline the hundreds place, and to put an X through the thousands place.

Figure A.3. Middle portion of page 24 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enable the reading of small print, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

Class Number Grid

-9	-8	-7	-6	-5	-4	-3	-2	-1	0
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110

The Class Number Grid Poster used throughout third grade.

▶ Reviewing Number-Grid Patterns

WHOLE-CLASS DISCUSSION

(*Math Journal 1*, p. 1)

Use the Class Number Grid Poster to review number-grid patterns.

- How many numbers are in each row? 10
- What happens to the numbers in a row as you move from left to right? The numbers increase by 1.
- What happens to the numbers in a row as you move from right to left? The numbers decrease by 1.
- How do you get from a number at the end of a row to the next number? Move to the first number in the next row
- What do the whole numbers in a column have in common? They end in the same digit.
- Describe the numbers in the first row. They are negative except for 0. Point out that the number grid is like a number line stacked up in groups of 10.

Figure A.4. Bottom portion of page 24 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enable the reading of small print, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

- What happens to a number when you move to the number below it? It increases by 10.
- What happens to a number when you move to the number above it? It decreases by 10.
- What happens when you add 1 to a number that does not end in 9 or 0? You move one space to the right. How does the number change? The ones digit is increased by 1.
- What happens if you add 1 to a number that ends in 9? You move one space to the right. How does the number change? The ones digit is changed to 0, and the tens digit is increased by 1. If the tens digit is 9, it is changed to 0 and the hundreds digit is increased by 1.
- What happens when you add 1 to a number that ends in 0? You go to the first number in the next row. How does the number change? The ones digit becomes 1 and the tens and hundreds digits stay the same.
- What happens if you add 10 to a number? You move to the number below it. If you add 100? When adding 100, increase the digit in the hundreds place by 1.
- If we decided to continue the number grid, what would be the next number? 111 How do you know? One more than 110 is 111. Ten more than 101 is 111.
- If we decided to begin the number grid before −9 what would be the next number? −10 How do you know? One less than −9 is −10. Ten less than 0 is −10.

Invite children to describe similarities and differences between the number sequences on *Math Journal 1,* page 1 and the class number grid.

Figure A.5. Top portion of page 25 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

Student Page

Date Time

LESSON 1·1

Number Sequences

Complete the number sequences.

Unit

1. 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, ...

2. 918, 919, 920, 921, 922, 923, 924, 925, 926, ...

3. 1,415; 1,416; 1,417; 1,418; 1,419; 1,420; 1,421; ...

4. 301, 311, 321, 331, 341, 351, 361, ...

5. 4,316; 4,326; 4,336; 4,346; 4,356; 4,366; ...

Try This

6. 7,628; 7,728; 7,828; 7,928; 8,028; 8,128

***Math Journal 1*, p. 1**

Figure A.6. Math Journal image at the top of page 25 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

▶ Finding Missing Numbers on a Number Grid

WHOLE-CLASS ACTIVITY

(*Math Journal 1*, p. 3)

Remind children that a **number grid** can contain any-size consecutive whole numbers—that is, numbers in the hundreds, thousands, and beyond. The number grid on journal page 3 shows numbers in the 5 and 6 hundreds.

Have children write numbers on the journal page grid as suggested. Encourage them to describe how to use the number-grid patterns to find the correct spaces for the numbers.

Student Page

For the contents of this text box, see Figures A.9 and A.10.

Figure A.7. Bottom portion of page 25 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

NOTE The translation of new mathematical vocabulary may present difficulty for children who are learning English. Several different English words or phrases may translate into a single word in another language. For example, *less* and *minus* translate into the same word in some languages. English language learners may hear *1 less than 570* and think it means *1 minus 570.* Write these problems, their symbolic representations, and their answers on the board.

For example, to find 13 more than 577, they can move down one row and then three spaces to the right.

Enter the following numbers in your number grid:

587, 554, 580, 596

- 1 more than 565 566
- 1 less than 570 569
- 10 more than 562 572
- 10 less than 593 583
- 4 more than 544 548
- 7 less than 574 567
- 13 more than 577 590
- 15 less than 593 578
- 20 more than 585 605
- 20 less than 599 579
- 23 more than 585 608
- 26 less than 620 594

Figure A.8. Top portion of page 26 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

Student Page

Date Time

LESSON 1·2 **Number-Grid Puzzles**

1. Follow your teacher's directions.

541			544				548		550
551		553	554		556			559	
	562			565	566	567		569	570
	572		574			577	578	579	580
581		583		585		587	588	589	590
		593	594		596			599	
	602			605	606		608		
			614						620

Figure A.9. Top portion of Math Journal image on page 25 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

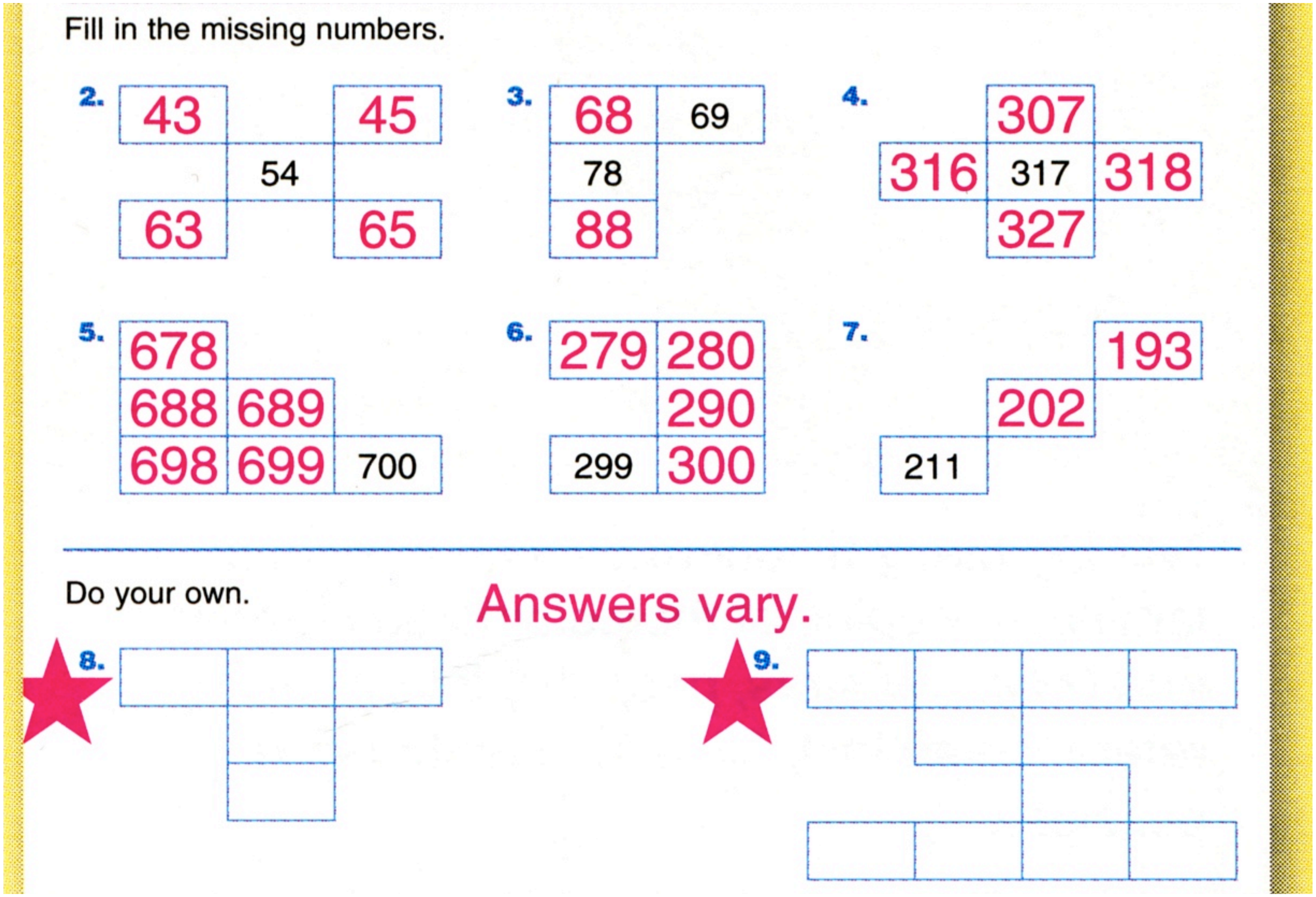


Figure A.10. Bottom portion of Math Journal image on page 25 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

Ongoing Assessment: Informing Instruction

Watch for children who have difficulty ordering numbers greater than 110. Suggest that if the beginning digits of the number are the same, they can order the numbers based on the last two digits. For example, to find the number that comes after 574, ask what number comes after 74. Write 75 in the next space on the grid. Now look at the hundreds place and write a 5 (5 hundred) in front of the 75. 575 comes after 574.

Solving Number-Grid Puzzles

INDEPENDENT ACTIVITY

(*Math Journal 1,* p. 3)

Fill in the **number-grid puzzle** in Exercise 2 together. Have children complete the rest of the page on their own. When most children have completed Exercises 3–9, have them share answers and solution strategies. Ask a few volunteers to write the puzzles they made up on the board for the class to solve.

Ongoing Assessment: Recognizing Student Achievement

Journal page 3 Problems 8 and 9

Use **journal page 3, Problems 8 and 9** to assess children's progress toward describing and extending numerical patterns. Children are making adequate progress if they are able to complete the number-grid puzzles with 2-digit numbers. Some children may use 3-, 4-, and 5-digit numbers; still others may use negative numbers.

[Patterns, Functions, and Algebra Goal 1]

Figure A.11. Excerpt of page 26 of Lesson 1-2 of *Everyday Mathematics* Teacher Lesson Guide Vol. 1 Grade 3 (The University of Chicago School Mathematics Project, 2007). Copyright © The McGraw-Hill Companies, Inc. *Note*: To enhance readability, the materials across these pages have been separated and enlarged. To learn how these parts fit together across multiple pages of the *Everyday Mathematics* Teacher Guide, please consult the teacher guide directly.

APPENDIX B

Table B.1. *Character Log for Zero (MPAH1)*

SB = Student book
TB = Teacher book
WB = Student Workbook (in class work)

Character name(s) and representation(s): "Zero," 0, "zero"				
Ref	Page (Text)	Note, characteristics	Setting	Actions
A, B, C	8 (SB), 6 (TB)	Zero is introduced as the quantity of objects (cubes, bugs) that remains when all are removed, also is the amount less than 1, the "end" or stopping number of counting backwards	Physical (cubes in hand) Pictorial (bugs in book)	Removing or decreasing quantity, finding the quantity
B'	9 (WB)	Zero represents a lack of objects (even when something is not removed)	Pictorial (picture of snowman)	Finding quantity
D	10, 11 (WB)	Zero is a label for an object without reference to quantity	Symbols in text	Matching symbol and word
B"	18 (WB)	Zero is a quantity that can be compared to others in terms of size without accompanying objects that can be counted	Symbols in text	Comparing values
A, C, C'	19, 20 (WB)	Zero is a term in a sequence of numbers, the number after 1 (when counting down), the number before 1 (when counting up)	Symbols in text	Completing a sequence
B", D	23 (WB)	Zero is a symbol label on object – not representing quantity, but instead as a name	Pictorial (picture of farm with numbered eggs)	Comparing values

Table B.1 (cont'd)

E	23 (SB), 33 (TB)	Zero is a symbol we write for a decomposed part when the starting train is not pulled apart.	Physical (cubes in hand) Pictorial (cube train in book)	Decomposing "pulling apart"
F, E'	24 (SB), 34 (TB)	Zero is a balance point, a fulcrum on a balance scale. Rather than representing nothing, 0 is a position. When 0 is a part, the other part equals the whole.	Balance scale	Placing weights on a scale so that the arm is level.
B'	27 (SB), 37 (TB), 25 (WB), 26 (WB), 27 (WB)	Zero is an absence of objects (nothing)	Pictorial (cup raised to show no beads under it)	Finding quantities
B", D	31 (WB)	Zero is a symbol label on object – not representing quantity of what it labels, but now representing a value to be added to another value.	Pictorial (pictures labeled with numbers)	Adding values
B'	32 (WB)	Zero is an absence of objects	Pictorial (picture of flower without butterflies)	Finding quantities
E	34 (WB)	Zero is an amount that requires no change of the whole in decomposition (analogue to not decomposing a cube train)	Pictorial (cubes in text)	Decomposing
B", D	39 (WB)	Zero is a symbol label on object – not representing quantity, but instead as a name	Pictorial (picture of fish with number labels)	Comparing values

Table B.1 (cont'd)

E'	40 (WB)	Zero is a symbol for a part in a number bond, lets a part match the whole	Symbols in text (number bond diagram with numerals)	Finding parts to add up to whole
E''	31 (SB), 51 (TB)	Zero is a symbol that does not increase a quantity in composition	Symbols on cards	Adding (composition or counting on)
B'	37 (SB), 57 (TB)	Zero is a representation of a plate with no apples (that apples are moving onto). So now, Zero is not just absence of objects, it is the absence of particular objects that are getting attention (because there is the plate)	Pictorial	Representing addition story with a number bond and addition sentence
D, E''	48 (SB), 63 (TB), 49 (SB), 63 (TB)	Zero is a label of a value on a butterfly that can be added to another number on the same butterfly.	Pictorial	Adding values, matching parts and whole
E'''	57 (SB), 67 (TB)	Zero is a symbol representing a part OR a whole (first time it is a whole)	Symbols in text	Adding values
E', G	58 (SB), 68 (TB)	Zero is a number for a part when a given part matches the whole, also not a quantity when decomposing sweets	Symbols in text, Pictorial (10 candies that are shared between two girls)	Adding values, decomposing

APPENDIX C

Table C.1. *Hermeneutic Sub-Codes (Adapted from Barthes, 1974)*

Sub-Code	Description
(1) *thematization*, the setting up of the subject of the question framing the story arc	This sets the theme for the content around which the story questions will be asked and answered. For example, if the title of the lesson is about parallel lines, it can orient the reader that the mystery may involve parallel lines. The title of the lesson is often a source of thematization, but that does not mean that formulated questions that are not related to the title are not allowed. The thematization may arise in other ways too, such as in minor headings or a change in subject throughout the different sections of the lesson.
(2) *proposal*, an opening of the possibility of mystery	This sub-code identifies story material that piqued the reader's interest and supported his or her later raising of (or recognition of) one or more formulated questions (see next sub-code).
(3) *formulation* of the question, where the text provokes a question by the reader	This sub-code is used to mark points throughout the story at which the reader is provoked to raise a question of the text or to adopt a question by the text. The question may be explicitly stated in the text, may be generated as a response to a statement of the text, or may instead draw from different parts of the text.
(4) *promise* or request of an answer, an indication to the reader of an eventual answer	This sub-code is used to mark points in a story where the text indicates that a question will be answered. However, this is not just because the text asks the reader a question, as texts can open a question that it never answers. An example is a statement such as, "When parallel lines are cut by a transversal, there is a special relationship among the angles. Let's investigate" (Serra, 2008, p. 126). The claim that a relationship exists and the prompt for an investigation can be interpreted as an indication that later the relationship of the angles will eventually be revealed.
(5) *snare*, an attempt to mislead the reader	This sub-code indicates moments in the story when a reader can later identify upon reflection that he or she was explicitly misled. Unlike equivocation (see next sub-code), this code reflects what can be later be recognized as "a lie" by the narrator, such as when Dorothy (and, thus, the reader) is told that Oz is "a great wizard" who can help her get home in *The Wonderful Wizard of Oz* (Baum, 1900).

Table C.1. (cont'd)

Sub-Code	Description
(6) *equivocation*, an ambiguity that contains both snare and truth	This sub-code indicates moments in the story when a reader can later identify (upon reflection) that he or she was misled through ambiguity. Unlike a snare, the text contains no overt lie and is technically accurate.
(7) *jamming*, the suggestion that the question is unanswerable	This sub-code is used to mark when the text suggests that a question cannot (or will not) be answered. It is the opposite of the sub-code *promise*. In a mathematics textbook, this would include a statement by the author that a question raised in the text will not be addressed due to the advanced nature of the content.
(8) *suspended answer*, the delay of the answer	This sub-code is used to mark when the text moves away from the pursuit of a question to an unrelated topic, focusing on a different, unrelated question, which causes the reader to notice that the question is dropped for the time being. For example, a novel might change scenes and characters but later go back and answer questions raised in an earlier episode.
(9) *partial answer* on the part of the reader	This sub-code is used to mark moments during which a reader can later identify (upon reflection) that he or she partially answered a question (through a statement by a text or through analytic means) but that the final answer was not yet found.
(10) *disclosure* of the answer by the text for the reader	This sub-code is used to mark the moments of a story during which a formulated question is explicitly answered in the text. However, the text can explicitly answer the question but in subtle ways. For example, the text may pose the problem for the reader to answer in the text and then later refer to the results of the problem and state the answer (but not as an answer, but instead as a "remember this" or "you may have noticed that..."). An answer may also be stated in an answer key at the end of the textbook.

APPENDIX D

Lesson 1

The Path of a Billiard Ball

AN expert billiard player's ability to control the path of a ball seems almost miraculous. Mathematicians like the game of billiards because the paths of the balls can be precisely calculated by mathematical methods. Lewis Carroll, the author of *Alice in Wonderland* and a mathematics teacher at Oxford University as well, liked to play billiards. He even invented a version of the game to be played on a circular table!

A skilled billiard player can picture the path of a ball before he hits it. The path is determined by how the ball is hit, by the *shape* of the table, and by the positions of the other balls. The ordinary billiard table is about twice as long as it is wide (approximately

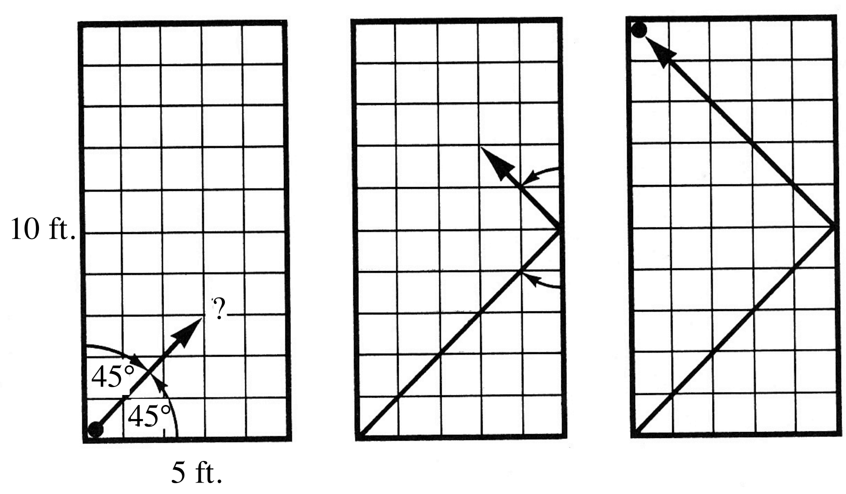
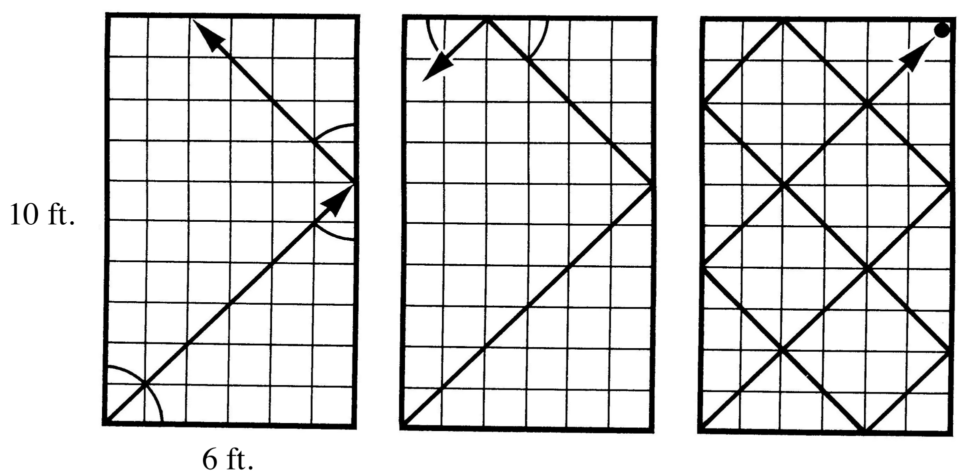


Figure D.1. Excerpt of Lesson 1 of Jacobs (1970, p. 2). Reprinted with permission.

10 feet by 5 feet). Suppose a ball is hit from one corner so that it travels at a 45° angle with the sides of the table.* If it is the only ball on the table, where will it go? The first diagram (on the facing page) shows us the direction of the ball as it is hit from the corner. The second diagram shows that the ball hits the midpoint of the longer side of the table. (A billiard table, unlike a pool table, does not have any pockets.) When the ball strikes the cushion, it rebounds from it at the same angle. The angles of hitting and rebounding have been marked with curved lines to show that they are equal. The third diagram shows that the ball goes to the corner at the upper left, and we will assume that the ball always stops when it comes to a corner.

Let's try this again on a table with a different shape. Suppose the table is 6 feet by 10 feet.



Again the ball is hit from the lower left-hand corner at a 45° angle with the sides. This time, after the first rebound, it misses the other corner and hits the top side as shown in the second diagram. It rebounds from the side in a new direction so that the angles of hitting and rebounding are again equal. When we follow the path of the ball on the rest of its journey, we see that it rebounds several times more before finally coming to rest in a corner. This time, however, it is the corner in the upper right.

These two tables suggest all kinds of questions about tables of other shapes. Will the ball always end up in one of the table's corners, or, ignoring friction, could it go on rebounding from the walls forever? Can it ever come back to the original corner? If

*If this isn't clear to you, look on page 494.

Figure D.2. Excerpt of Lesson 1 of Jacobs (1970, p. 3). Reprinted with permission.

the ball ends up in a corner, and you know the length and width of the table, is it possible to predict which corner without drawing a diagram? You can probably think of other questions as well. We are faced with quite a puzzle. A 20th century American mathematician has said:

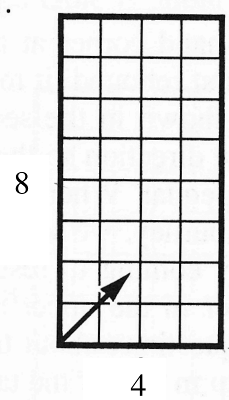
"Puzzles are made of the things that the mathematician, no less than the child, plays with, and dreams and wonders about, for they are made of the things and circumstances of the world he lives in."*

EXERCISES

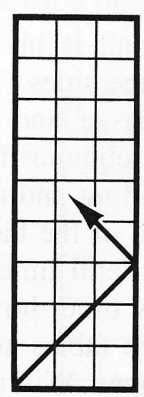
Set I

On graph paper (4 units per inch is convenient), make a diagram of each of the following tables. Use the *same number of units of length and width* as shown, and write the dimensions along the sides, as has been done for table 1. Now continue drawing the path of each ball as far as it can go. (We will assume that the ball comes to a stop when it reaches a corner.) Always start the path from the lower left-hand corner and notice that since the ball is always hit at a 45° angle with each side of the table, it always moves 1 unit up or down for 1 unit left or right. If the ball ends up in a corner, mark the corner with a large dot. (Please do not write in this book. You *won't* learn anything extra by doing so, and you *will* spoil the problems for the next student who uses it.)

1.



2.



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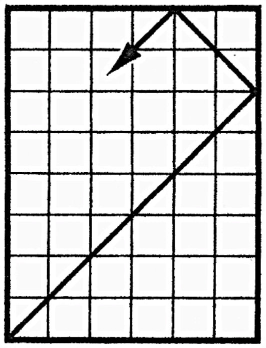
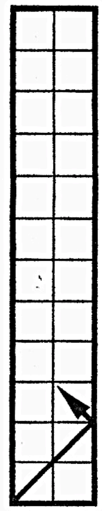
*Edward Kasner, in his book, *Mathematics and the Imagination,* Simon and Schuster, 1940, pp. 188–189.

Figure D.3. Excerpt of Lesson 1 of Jacobs (1970, p. 4). Reprinted with permission.

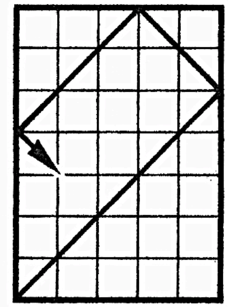
3.



4.

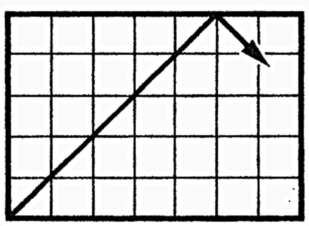


5.

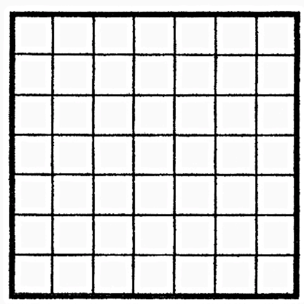


The ball on this table really travels!

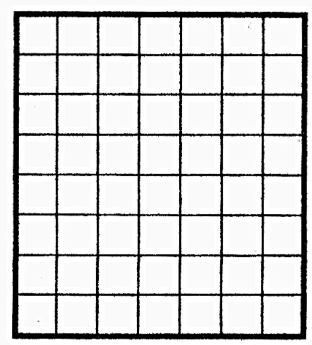
6.



7.



8.



9. On which table does the ball have the simplest path? Can you explain why?
10. What do you notice about the paths on tables 5 and 6? Can you explain?
11. Do you think the ball will always end up in a corner?
12. If the ball starts from the lower left-hand corner, do you think it can end up in any of the four corners?

Set II

The paths we have drawn so far are wildly unpredictable. A slight change in the shape of the table can make a tremendous difference in where the ball will go. Compare your drawings of

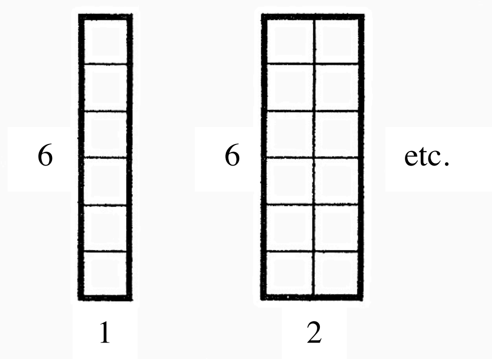
Figure D.4. Excerpt of Lesson 1 of Jacobs (1970, p. 5). Reprinted with permission.

the last two tables. Table 8 is the same width as table 7 and only one unit longer, yet the paths are entirely different.

The shape of the table determines the path of the ball in some way that is not yet clear. What determines the *shape of the table?* Two things: its length and its width. These dimensions can change, or vary, from one table to the next and are called *variables*. The path of the ball, then, is determined by the shape of the table which, in turn, is determined by two variables.

It would be simpler if there were only one variable. Let's keep the length of the table the same (we will hold it *constant*), vary the width, and see what happens.

1-6. Draw a set of six billiard tables with lengths of 6 units and with widths of 1, 2, 3, 4, 5, and 6 units. Start the ball from the lower left-hand corner as before and mark where the ball ends up with a large dot.



7. Does the result for any of these tables surprise you? Which one and why?

8. What are the two dimensions (length and width) of the table with the *simplest* path?

9. What are the dimensions of the table with the most *complicated* path?

10. If a giant billiard table had a length of 100 feet, what width should it have for the ball to travel the simplest possible path?

11. What width should it have for a very complicated path?

Figure D.5. Excerpt of Lesson 1 of Jacobs (1970, p. 6). Reprinted with permission.

APPENDIX E

Table E.1. *Plot Coding for Jacobs Lesson 1*

ACT	Event: What changed?	Description	HER Sub-code: Details
#1	Title:Theme set	"The Path of a Billiard Ball" (p. 2)	**Thematization:** I expect this mathematical story to be about the path of a ball on a billiard table.
#2	Introductory Paragraph: A mathematical model is proposed to understand the path of balls.	"... the paths of the balls can be precisely calculated by mathematical methods" and "a skilled billiard player can picture the path of a ball before he hits it" (p. 2).	**Proposal:** Interest in how the ball behaves is piqued. A ball's path can be calculated using math and pictured by a skilled player.
#3	Introduction of table and path of ball.	First example (5ft. by 10ft.) is proposed and solved. ("Suppose a ball is hit from [the lower left] corner so that it travels at a 45° angle with the sides of the table. If it is the only ball on the table, where will it go?" (p. 3)). Diagrams show rebound and end corner (upper-left).	**HER1: Formulation (Implicit):** *Where will this ball travel?* **Disclosure:** Answer is given (Draw a diagram, ball rebounds with same 45° angle).
#4	Second worked example: Another mathematical character is introduced and analyzed for the path of the ball.	Second example (6 ft. by 10 ft.) is proposed and solved. ("Let's try this again on a table with a different shape. Suppose the table is 6 feet by 10 feet" (p. 3). Diagrams show rebound and end corner (upper-right).	**Proposal:** Pay attention to the number of rebounds, that is important. **Proposal**: Pay attention to the end location of the ball, it may be important.

Table E.1 (cont'd)

ACT	Event: What changed?	Description	HER Sub-code: Details
#5	Paragraph #5: New mathematical questions posed	The author notes that the "two tables suggest all kinds of questions about tables of other shapes" (p. 3).	**HER2: Formulation:** "Will the ball always end up in one of the table's corners?" (p. 3) **HER3: Formulation:** "Can it ever come back to the original corner?" (p. 3) **HER4: Formulation:** "If the ball ends up in a corner, and you know the length and width of the table, is it possible to predict which corner without drawing a diagram?" (pp. 3-4) **HER5: Formulation:** *What other kinds of paths are there?*
#6	Set I Problem #1: A new math character is introduced and the path is analyzed.	4x8 table is given with path started, I am prompted to identify the ending corner (upper left corner).	**HER6: Formulation:** *Where does this ball end up?* **HER6: Partial answer** (reader generates answer)
#7	Set I Problem #2: A new math character is introduced and the path is analyzed.	3x9 table is given with path started. I notice that the ball zigzags up to the upper right corner like the 4x8 and 5x10 tables.	**HER7: Formulation:** *Where does this ball end up?* **HER7: Disclosure** (reader generates answer, text provides answer in the back) **HER5: Partial answer**: some tables result in a zigzag path up the table to the corner.

Table E.1 (cont'd)

ACT	Event: What changed?	Description	HER Sub-code: Details
#8	Set I Problems #3: A new math character is introduced and the path is analyzed.	6x8 table is given with path started, shows rebound off top. I notice this path had many more rebounds than tables #1 and #2.	**HER8: Formulation:** *Where does this ball end up?* **HER8: Partial answer** (reader generates answer) **HER5: Partial answer**: some tables result in a crisscross path up and down until reaching a corner. **Proposal:** Some paths are complex, others simple.
#9	Set I Problem #4: A new math character is introduced and the path is analyzed.	2x12 table is given with the start of its path. The path ends up pretty simple (only zigzagging up to top).	**HER9: Formulation:** *Where does this ball end up?* **HER9: Partial answer** (reader generates answer) **HER10: Formulation:** *What makes some paths more complicated than others?*
#10	Set I Problem #5: A new math character is introduced and the path is analyzed.	5x7 table is given with path started. I notice that this path resulted in a lot of rebounds and the path seemed to zigzag all over the table.	**HER11: Formulation:** *Where does this ball end up?* **HER11: Disclosure** (reader generates answer, text provides answer in the back) **HER5: Partial answer**
#11	Set I Problem #6: A new math character is introduced and the path is analyzed. It is determined that it is related to the mathematical character in problem #5.	7x5 table is given with path started, rebound off top first is given, I notice that the table can be wider (horizontally) than it is tall (vertically). This zigzags a lot too, feels the same as #5. I notice that it had the same table turned sideways, so the path should be the same.	**HER12: Formulation:** *Where does this ball end up?* **HER12: Partial answer** (reader generates answer) **HER13: Formulation:** *Is this related to table in problem #5?* **HER13: Partial answer** (reader generates answer)

Table E.1 (cont'd)

ACT	Event: What changed?	Description	HER Sub-code: Details
#12	Set I Problem #7: A math character is introduced with an interesting quality (no rebounds) and the path is analyzed.	7x7 table is given without a path started. I am surprised that this table has no rebounds following after two tables with many rebounds.	**HER14: Formulation:** *Where does this ball end up?* **HER14: Partial answer** (reader generates answer) **HER5: Partial answer:** Some paths have no rebounds. **HER10: Partial answer:** Square tables are not complex.
#13	Set I Problem #8: A new math character is introduced and the path is analyzed.	7x8 table is given without a path started.	**HER15: Formulation:** *Where does this ball end up?* **HER15: Partial answer** (reader generates answer)
#14	Set I Problem #9: Entire group of paths are compared for simplicity.	"On which table does the ball have the simplest path? Can you explain why?" (p. 5). I noticed that #7 is a square, and that the 45° path will take me to the upper-right corner.	**HER4: Partial answer:** When it is a square, I can predict that the ball will end up in the upper right corner.
#15	Set I Problem #10: Tables are compared and related.	"What do you notice about the paths on tables 5 and 6? Can you explain?" (p. 5). I already answered this before in problem #6 and this had me revisit my answer.	**HER13: Repeat partial answer** (reader generates answer)
#16	Set I Problem #11: Possibility of an infinite path is considered.	"Do you think the ball will always end up in a corner?" (p. 5). I try to draw a few tables to see if I could find one that would never land in a corner but cannot.	**HER2: Partial answer:** So far, all of them have, so I conjecture that the ball will always end up in a corner.

Table E.1 (cont'd)

ACT	Event: What changed?	Description	HER Sub-code: Details
#17	Set I Problem #12: Entire group of paths are compared for which end corners at which the ball ended.	"If the ball starts from the lower left-hand corner, do you think it can end up in any of the four corners?" (p. 5). I notice that the ball never seems to end up in the lower pockets. I wonder about the starting point – to end up there, the ball has to reverse its path somewhere, which I do not believe is possible given how it rebounds.	**HER16: Formulation:** Can the ball travel to any of the other three corners? **HER16: Equivocation:** The paths provided so far end up in the upper pockets only. **HER3: Partial answer:** Based on my work so far, I do not think the ball can ever return to its starting corner.
#18	Set II Introduction, paragraph 1: Refocusing on predicting path.	The paths are unpredictable. Even tables that are very close in size (7x7 and 7x8) result in vastly different paths.	**HER4: Jamming:** The path of the ball is unpredictable.
#19	Set II Introduction, paragraph 2: Focus on "shape" and dimensions of the table	The path is determined by shape in a way that "is not yet clear" (p. 6). I assume it will become clear later.	**HER4: Promise of answer:** The pattern is not *yet* evident, but I can expect it to become evident later.
#20	Set II Probs #1-6: Set of related mathematical characters (a "family" with height of 6) are analyzed and compared for similarities and differences.	The problem prompts me to find where the ball will end up for each of the tables that have a height of 6 for widths 1, 2, 3, ..., 6. I draw the tables and paths and record the number of rebounds and the end pocket for each. I do not see any patterns, but am surprised that a 4x6 table ends up in the lower right-hand pocket. I also note that the 6x6 confirmed my earlier conjecture (that a square will require no rebounds and will end up in the upper right corner).	**HER16: Partial answer:** I change my conjecture to state that a ball can travel to three of the four corners, depending on the shape of the table.

Table E.1 (cont'd)

ACT	Event: What changed?	Description	HER Sub-code: Details
#21	Set II Problem 7: Focus on surprising result	The text asks, "Does the result for any of these tables surprise you? Which one and why?" (p. 6). I repeat my earlier response that I was surprised the ball can end up in the lower right-hand corner.	**HER16: Repeat partial answer**
#22	Set II Problem 8: Focus on simplest path.	What are the dimensions of the table with the simplest path? I repeat my earlier note that it was again a square. There is no simpler path, since it has no rebounds.	**HER5: Repeat partial answer**
#23	Set II Problem 9: Focus on complicated path	"What are the dimensions of the table with the most *complicated* path?" (p. 6). I note that of this set, the 5x6 table has the most complicated path, like the 7x8 path I saw earlier. I wonder if complexity has something to do with having consecutive values for side lengths.	**HER10: Partial answer** **HER5: Partial answer**
#24	Set II Problem 10 and 11: Focus on simplest path.	Problem #10 asks for the width of the table with length 100 feet that would have the simplest path. Based on prior work, I think that a square would have the simplest path (width of 100 feet) and a with of 100+1 or 100-1 would have the most complicated path.	**HER17: Formulation:** *What width makes a simple path? What width makes a complicated path?* **HER17: Partial answer** (reader generates answer)

REFERENCES

REFERENCES

Abbott, E. A. (1884). *Flatland: A romance of many dimensions*. London: Dover Publications.

Allen-Fuller, K., Robinson, M., & Robinson, E. (2010). Curriculum as a change agent: High schools that rise to the challenge and what they stand to gain. In B. J. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics Curriculum: Issues, Trends, and Future Directions* (pp. 231–246). Reston, VA: National Council for Teachers of Mathematics.

American Educational Research Association. (2009). *Standards for Humanities-Oriented Research in AERA Publications*. Retrieved from http://www.aera.net/publications/Default.aspx?menu_id=32&id=1850

Apol Obbink, L. (1992). Feminist theory in the classroom: Choices, questions, voices. *The English Journal*, *81*(7), 38–43.

Aristotle. (1942). *The Poetics of Aristotle*. (P. H. Epps, Trans.). Chapel Hill: University of North Carolina Press.

Baker, D., Knipe, H., Collins, J., Leon, J., Cummings, E., Blair, C., & Gamson, D. (2010). One hundred years of elementary school mathematics in the United States: A content analysis and cognitive assessment of textbooks from 1900 to 2000. *Journal for Research in Mathematics Education*, *41*(4), 383–423.

Bal, M. (1986). Tell-tale theories. *Poetics Today*, *7*(3), 555–564. doi:10.2307/1772511

Bal, M. (2009). *Narratology: Introduction to the theory of narrative*. (C. Van Boheemen, Trans.) (3rd ed.). Toronto: University of Toronto Press.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is – or might be – the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, *25*, 6–8, 14.

Barthes, R. (1974). *S/Z*. (R. Miller, Trans.). New York: Macmillan.

Baum, L. F. (1900). *The Wonderful Wizard of Oz*. Chicago, IL: George M. Hill Company.

Ben-Peretz, M. (1990). *The teacher-curriculum encounter: Freeing teachers from the tyranny of texts*. Albany, NY: SUNY Press.

Breyfogle, M. L., Roth McDuffie, A., & Wohlhuter, K. A. (2010). Developing curricular reasoning for grades pre-K-12 mathematics instruction. In B. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics Curriculum : Issues, Trends, and Future Directions*. Reston VA: National Council of Teachers of Mathematics.

Brooks, P. (1984). *Reading for the plot: Design and intention in narrative*. London: Oxford Press.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction* (pp. 17–36). New York: Routledge.

Bruner, J. S. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, *37*, 121–143.

Cai, J., Watanabe, T., & Lo, J. J. (2002). Intended treatments of arithmetic average in US and Asian school mathematics textbooks. *School Science and Mathematics*, *102*(8), 391–404.

CCSSO, & NGA. (2010). *Common Core State Standards for Mathematics*. Joint effort of the Council of Chief State School Officers (CCSSO) and the National Governers Association (NGA). Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Center for the Study of Mathematics Curriculum. (n.d.). Curriculum Research Framework. *Curriculum Research Framework*. Framework. Retrieved December 6, 2010, from http://csmc.missouri.edu/research_framework.php

Charles, R., Crown, W., & Fennell, F. (2008). *Mathematics*. Glenview, IL: Scott-Foresman/Addison Wesley.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, *6*(2), 81–89. doi:10.1207/s15327833mtl0602_1

Confrey, J., Maloney, A. P., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In *33rd Conference of the International Group for the Psychology of Mathematics Education*. Thessaloniki, Greece.

Csikszentmihalyi, M. (1990). Literacy and intrinsic motivation. *Daedalus*, *119*(2), 115–140.

Csiszar, A. (2004). Stylizing rigor: Or, why mathematicians write so well. *Configurations*, *11*(2), 239–268.

Dewey, J. (1902). The child and the curriculum. In L. A. Hickman & T. M. Alexander (Eds.), *The Essential Dewey* (Vol. 1, pp. 236–245). Chicago, Illinois: Indiana University Press.

Dewey, J. (1934). *Art as experience*. New York: Penguin Group.

Dougherty, B. J. (2003). Voyaging from theory to practice in learning: A view from Hawaii. In *International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17–32). Honolulu, HI: International Group for the Psychology of Mathematics Education.

Egan, K. (1988). *Teaching as story telling: An alternative approach to teaching and the curriculum*. Chicago, IL: Routledge.

Eisner, E. W. (1979). The three curricula that all schools teach. In *The Educational Imagination: On the Design and Evaluation of School Programs* (pp. 74–92). New York, NY: Macmillan.

Else, G. F. (1967). *Aristotle: Poetics*. Ann Arbor, MI: Univ of Michigan Press.

Ernest, P. (2008a). Towards a semiotics of mathematical text (part 1). *For the Learning of Mathematics*, *28*(1), 2–8.

Ernest, P. (2008b). Towards a semiotics of mathematical text (Part 2). *For the Learning of Mathematics*, *28*(2), 39–47.

Ernest, P. (2008c). Towards a semiotics of mathematical text (Part 3). *For the Learning of Mathematics*, *28*(3), 42–49.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*, 94–116.

Fernandez, C., Yoshida, M., & Stigler, J. W. (1992). Learning mathematics from classroom instruction: On relating lessons to pupils' interpretations. *The Journal of the Learning Sciences*, *2*(4), 333–365.

Freeman, D. J., & Porter, A. C. (1989). Do textbooks dictate the content of mathematics instruction in elementary schools? *American Educational Research Journal*, *26*(3), 403–421.

Fuson, K. C., Stigler, J. W., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, Taiwan, and the United States. *Journal for Research in Mathematics Education*, *19*(5), 449–456.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics*, *16*(2), 36–45.

Gerofsky, S. (1999). Genre analysis as a way of understanding pedagogy in mathematics education. *For the Learning of Mathematics*, *19*(3), 36–46.

Healy, L., & Sinclair, N. (2007). If this is our mathematics, what are our stories? *International Journal of Computers for Mathematical Learning*, *12*, 3–21.

Herbel-Eisenmann, B. (2007). From intended curriculum to written curriculum: Examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, *38*, 344–369.

Herbel-Eisenmann, B., & Wagner, D. (2007). A framework for uncovering the way a textbook may position the mathematics learner. *For the Learning of Mathematics*, *27*(2), 8–14.

Hersh, R. (1997). Math lingo vs. plain English: Double entendre. *The American Mathematical Monthly*, *104*(1), 48–51. doi:10.2307/2974822

Hofstadter, D. (1992). From Euler to Ulam: Discovery and dissection of a geometric gem, (Preprint version available from author).

Huntley, M. A., & Chval, K. (2010). Teachers' perspectives on fidelity of implementation to textbooks. In B. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics Curriculum : Issues, Trends, and Future Directions*. Reston VA: National Council of Teachers of Mathematics.

Jacobs, H. R. (1970). *Mathematics: A human endeavor*. San Francisco, CA: W. H. Freeman and Co.

Kheong, F. H., Ramakrishnan, C., & Wah, B. L. P. (2012). *My pals are here! Maths* (2nd ed., Vol. 1A). Singapore: Marshall Cavendish.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kim, R. Y. (2009). *Text + book = Textbook? Development of a conceptual framework for non-textual elements in middle school mathematics textbooks* (Doctoral dissertation). Michigan State University, United States -- Michigan. Retrieved from http://search.proquest.com.proxy1.cl.msu.edu/dissertations/docview/304949605/abstract/1381F9AB7826ADAF007/1?accountid=12598

King, T. (2003). *The truth about stories: A native narrative*. Toronto: House of Anansi Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Leron, U. (1985). Heuristic presentations: The role of structuring. *For the Learning of Mathematics*, *5*(3), 7–13.

Lockhart, P. (2009). *A mathematician's lament*. New York: Bellevue Literary Press.

Love, E., & Pimm, D. (1997). "This is so": A text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (Vols. 1-2, Vol. 1, pp. 371–409). Netherlands: Kluwer Academic Publishers.

Morgan, C. (2005). Word, definitions and concepts in discourses of mathematics, teaching and learning. *Language and Education*, *19*(2), 102–116.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge, UK: Cambridge University Press.

Netz, R. (2005). The aesthetics of mathematics: A study. In P. Mancosu, K. F. Jørgensen, & S. A. Pedersen (Eds.), *Visualization, Explanation and Reasoning Styles in Mathematics* (pp. 251–293). Netherlands: Springer.

Nie, B., Cai, J., & Moyer, J. C. (2009). How a standards-based mathematics curriculum differs from a traditional curriculum: With a focus on intended treatments of the ideas of variable. *ZDM*, *41*(6), 777–792.

Nodelman, P., & Reimer, M. (2003). *The pleasures of children's literature* (3rd ed.). Boston: Allyn and Bacon.

Pimm, D. (1987). *Speaking mathematically: Communication in the mathematics classroom*. New York: Routledge & Kegan Paul.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity* (pp. 160–190). New York, NY: Springer.

Rasslan, S., & Vinner, S. (1995). The graphical, the algebraic and their relation: The notion of slope. In *Proceeding of the 19th Conference of the International Group for the Psychology of Mathematics Edcuation* (Vol. 2, pp. 264–271). Recife, Brazil.

Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry*, *29*(3), 315–342. doi:10.1111/0362-6784.00130

Rosenblatt, L. M. (1988). *Writing and reading: The transactional theory* (Technical Report No. 416). University of Illinois at Urbana-Champaign: Reading Research and Education Center.

Rosenblatt, L. M. (1994). *The reader, the text, the poem: The transactional theory of the literary work*. Carbondale and Edwardsville: Southern Illinois University Press.

Schmidt, S. F. (2010). *Life of Fred: Fractions*. Reno, NV: Polka Dot Publishing.

Schmidt, W. H., McKnight, C., Valverde, G. A., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims: A cross-national investigation of curricular intentions in school mathematics*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*, 525–559.

Senk, S. L., & Thompson, D. R. (2003). *Standards-based school mathematics curricula: What are they? What do students learn?* Mahwah, N.J.: Lawrence Erlbaum Associates.

Serra, M. (2008). *Discovering geometry*. Emeryville, CA: Key Curriculum Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44–55.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13.

Sfard, A. (2008). *Thinking as communicating : Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Shiga, J. (2010). *Meanwhile: Pick any path. 3,856 story possibilities*. New York, NY: Abrams Books.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91. doi:10.1207/s15327833mtl0602_2

Sinclair, N. (2001). The aesthetic IS relevant. *For the Learning of Mathematics*, *21*(1), 25–32.

Sinclair, N. (2002). The kissing triangles: The aesthetics of mathematical discovery. *International Journal of Computers for Mathematical Learning*, *7*(1), 45–63. doi:10.1023/A:1016021912539

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*(3), 261–284.

Sinclair, N. (2005). Chorus, colour, and contrariness in school mathematics. *THEN: Journal*, *1*(1). Retrieved from http://thenjournal.org/feature/80/

Sinclair, N. (2006). The aesthetic sensibilities of mathematicians. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity* (pp. 87–104). New York, NY: Springer. Retrieved from http://philmat.oxfordjournals.org/content/16/1/149.short

Sinclair, N. (2008). *The history of the geometry curriculum in the United States*. Charlotte, NC: Information Age Publishing.

Sinclair, N. (2009). Aesthetics as a liberating force in mathematics education? *ZDM*, *41*(1), 45–60. doi:10.1007/s11858-008-0132-x

Smith III, J. P., Dietiker, L., Lee, K., Males, L., Figueras, H., Mosier, A., … Sisman, G. (2008). Framing the analysis of written measurement curricula. In *Annual Meeting of the American Educational Research Association*. New York, NY.

Solomon, Y., & O'Neill, J. (1998). Mathematics and narrative. *Language and Education*, *12*(3), 210–221. doi:10.1080/09500789808666749

Son, J. W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, *74*(2), 117–142.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Simon and Schuster.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

The University of Chicago School Mathematics Project. (2007). *Everyday mathematics*. Chicago, IL: Wright Group/McGraw-Hill.

Thomas, R. (2007). The comparison of mathematics with narrative. *Perspectives On Mathematical Practices*, 43–59.

Thomas, R. S. D. (2002). Mathematics and narrative. (M. Senechal, Ed.)*The Mathematical Intelligencer*, *24*(3), 43–46. doi:10.1007/BF03024731

Usiskin, Z., Griffin, J., Witonsky, D., & Willmore, E. (2008). *The classification of quadrilaterals: A study in definition*. Charlotte, NC: Information Age Publishing Inc.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of functions. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Weinberg, A., & Wiesner, E. (2010). Understanding mathematics textbooks through reader-oriented theory. *Educational Studies in Mathematics*, 1–15. doi:10.1007/s10649-010-9264-3

Willoughby, S. (2010). Reflections on five decades of curriculum controversies. In B. J. Reys, R. E. Reys, & R. Rubenstein (Eds.), *Mathematics Curriculum: Issues, Trends, and Future Directions* (pp. 77–85). Reston, VA: National Council for Teachers of Mathematics.

Wong, D. (2007). Beyond control and rationality: Dewey, aesthetics, motivation, and educative experiences. *Teachers College Record*, *109*(1), 192–220.