THE CONTINUOUS-TIME PRINCIPAL-AGENT PROBLEM WITH MORAL HAZARD AND RECURSIVE PREFERENCES.

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ABSTRACT

THE CONTINUOUS-TIME PRINCIPAL-AGENT PROBLEM WITH MORAL HAZARD AND RECURSIVE PREFERENCES.

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The thesis presents a solution to the principal-agent problem with moral hazard in a continuous-time Brownian filtration with recursive preferences, and pay over the contract's lifetime. Recursive preferences are essentially as tractable as time-additive utility because the agency problem induces recursivity in the principal's utility even in the time-additive case. Furthermore, recursive preferences allow more flexible modeling of risk aversion. The thesis develops various results on Backward Stochastic Differential Equation (BSDE), Functional Ito Calculus, an extension of Kuhn-Tucker Theorem and a maximum principle for multi-dimensional BSDEs. These concepts in conjunction with the theoery of gradient and supergradient density are used to derive a first order condition for the principal-agent problem.

Various examples have been worked out with closed form solutions. The thesis also presents applications of Functional Ito Calculus in Finance. Various other problems of Financial Economics such as Pareto Optimality, Altruism, direct utility for wealth are solved using the technique developed under recursive preferences. The theory developed will be very useful in further development of BSDE applications and Functional Ito Calculus in financial mathematics.

DEDICATION

To my family

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Introduction

1.1 Thesis overview

The thesis deals with the three problems that belong to the intersection between stochastic analysis and financial economics. The problems are:

1. Continuous-time Principal-Agent Problem with moral hazard under recursive preferences.
2. General maximization principle with application to problems that arise in financial economics.
3. A Functional version of Itô formula.

The problems will now be described in greater detail together with the statistical/mathematical techniques that are being used in solving them.

1.2 Principal-Agent Problem

In economics, the principal-agent (owner-employee) problem deals with the difficulties that arise when a principal hires an agent to pursue the principal's interest. This problem is found in most owner/employee relationships. The most common example is when the owner (shareholders or principal) of a corporation hires an executive such as a Chief Executive Officer (CEO or agent). The term 'principal' and 'agent' is frequently used in the economics literature. The specific problem that is the focus here arises when a principal gets into a contract to compensate the agent for performing certain work that is useful to the principal and agent's effort is noncontractible, i.e. the agent's effort is not part of the contract. In the basic version of the problem, the agent chooses an effort scheme in order to maximize his/her utility. The principal, in turn, offers a compensation contract for the agent in order to induce the agent to perform his/her duty in a way that will maximize the principal's own utility. The principal moves first by offering compensation package to the agent, who in turn selects the optimal effort scheme from his/her point of view.

The above problem has a moral hazard component, in the sense that the agent's effort is not part of the contract and the principal does not always observe the agent's effort. So as explained above the principal has to choose carefully the compensation package that is being offered to the agent. The goal is that the effort scheme chosen by the agent in order to maximize his/her utility will also maximize the utility of principal. Another complication in the selection of the compensation package arises because the agent may have employment opportunities elsewhere, so the agent's initial utility must exceed some fixed amount.

In the continuous-time Brownian version, first examined in [27], the impact of effort choice is typically modeled by replacing the underlying Gaussian probability measure by an equivalent one. That is, the agent's efforts change the probability measure, which by Girsanov's theorem is equivalent to a change in the drift of the driving Brownian motion. This is a convenient way to model, for example, the impact of effort on the growth rate of a cash flow process.

In this work, both the principal and the agent are using a type of utility known as generalized recursive utility. This class of utility is defined as the solution of Backward Stochastic Differential Equation (BSDE). The advantage of this class of utility is that it allows to break the link between risk aversion and intertemporal substitution. Preferences are defined recursively over current consumption and future consumptions. It is well known that a forward-backward structure is inherent to the principal-agent problem and so the use of recursive preferences is both flexible and natural. Previous work has considered only additive utility, which is well known to arbitrarily link intertemporal substitution and risk aversion (see, for example, [19]). Yet time-additivity offers essentially no advantage in tractability because agent optimality induces recursivity to the principal's preferences even in the additive case. The generalized recursive utility class was introduced in [34] with the goal of unifying the stochastic differential utility (SDU) formulation of [13], and the multiple-prior formulation of [5]. Unlike the additive class, the recursive class allows distinct treatment of aversion to variability in consumption across states and across time. Furthermore, the class can accommodate source-dependent (domestic versus foreign, for example) risk aversion, differences in agent and principal beliefs, as well as first-order risk aversion (which imposes a higher penalty for small levels of risk) in addition to the standard second-order risk aversion.[1] Also, [47] shows that SDU, a special case of the recursive class, includes the robust control formulations of [2], [24], and [37].

Some of the statistical/mathematical techniques used to solve this problem are also used

[1] See [44] and [48]. We consider only second-order risk aversion in this paper, but extensions to the first-order case (modeled by kinks in the aggregator) can be handled along the lines of [44].

in the next problem, which will be discussed briefly in the next section.

1.3 General Maximization Problem

Here, the topic of interest is the development of general maximization principle under recursive preferences. The maximization principle is developed for the optimization of a linear combination of recursive utilities for several agents. The theory can be applied to solve various economic problems such as Pareto optimality, Altruism, etc. One of the applications in this thesis of the maximum principle is in solving the principal-agent problem described in Chapter 4. To introduce the maximization problem an overview of Translation-Invariant (TI) preferences is presented. After developing the thoery, it is applied in detail to the TI case.

The study of generalized recursive utility is based on the theory of BSDE. The theory of stochastic calculus is the basis for the theory of BSDE. More specifically, the method of proving existence and uniqueness of solutions of BSDE is a combination of the martingale representation theorem and fixed-point theorem. Other techniques that are being used in the maximization problem are utility gradient and dynamic programming theory. This technique is an extremely useful method as it provides a way of determining first order conditions for optimality. The main result is based on an extension of Kuhn-Tucker theorem, a Lagrange multiplier type of result, that is being used here for an optimization problem in infinite dimesion setup with restrictions formulated as inequalities.

1.4 Functional Itô Formula

The Functional Itô formula is an extension of the classical Itô formula for functionals that are defined on the history of the process rather than the current value of the process. This result was developed in [16] for the continuous case and was extended in [6] to the general semimartingale case. The thesis develops a new and much simpler proof for the general case. The proof of the theorem uses an original setup to the problem, which makes it easy to link functional Itô formula with the classical Itô formula. Some applications to optimal control and portfolio theory are presented.

2

Functional Itô Calculus

2.1 Introduction

Itô's stochastic calculus has been used to analyze random phenomenon and has lead to new fields in stochastic processes. There are many applications of Itô's stochastic calculus in various fields of science such as probability, statistics, mathematics etc. The traditional Itô formula applies to functions of the current value of a semimartingale. But in many applications, such as mathematical finance and statistics, it is natural to consider functionals of the entire path of a semimartingale. The Itô formula was extended recently by [16] to the case of functionals of paths of continuous semimartingales.

This work was motivated by the talk of Bruno Dupire at 6th World Congress of the Bachelier Finance Society (Toronto, June, 2010) which described several applications of the functional Itô formula. Among the applications was an explicit expression for the integrand in the martingale representation theorem, and a formula for the running maximum of a continuous semimartingale.

The setting in [16] uses somewhat exotic concepts such as vector bundles and some new types of derivatives. We modify Dupire's setup to a standard one: The vector bundle is replaced by the usual right continuous left limit space of functions, and Dupire's derivatives are replaced by standard directional derivatives in infinite-dimension spaces. With this setup the functional Itô formula comes as a simple and natural extension of the standard Itô formula. Furthermore, this work provide a simple proof of the extension to the more general case of cadlag semimartingales (which allows for jumps in the stochastic processes). The proof for the cadlag case in [6] uses the analytic approach developed by [20]. However the proof here is much shorter and uses traditional and basic concepts. A statement of the result in the special case of (continuous) Brownian semimartingales appears in [7].

2.2 Notation and Definitions

Consider the usual stochastic base $(\Omega, \{\mathcal{F}_t, 0 \le t \le T\}, \mathcal{F}, P)$. Let $X_t(\omega) \in \mathbb{R}^d,\ 0 \le t \le T$, denote an adapted RCLL semimartingale. Let $D\left([0,T], \mathbb{R}^d\right)$ be the space of RCLL $\mathbb{R}^d$-valued functions on $[0,T]$ equipped with the supremum metric. Let $F : D\left([0,T], \mathbb{R}^d\right) \to \mathbb{R}$ be a functional. Define

$$Y_t(s) = \{X(s \wedge t),\ 0 \le s \le T\};$$

that is, Y_t represents the history of the process, X up to time t. Observe that $(Y_t)_{0 \le t \le T}$ is adapted to the $\{\mathcal{F}_t\}$ filtration, even though Y_t is defined on the interval $[0,T]$.

The following notation will be used throughout:

$$1_{[t,T]}(s) = \begin{cases} 0 & \text{if} \quad 0 \le s < t, \\ 1 & \text{if} \quad t \le s \le T. \end{cases}$$

Also, $\{e_i\}_{i=1,..d}$ will denote the canonical basis in $\mathbb{R}^d$ (that is, e_i is a length-d vector with a one in the ith position and zeros elsewhere). Then the following definition of the directional derivatives of the functional F will be used.

Definition 2.1. *For $x \in D\left([0,T],\mathbb{R}^d\right)$ we denote by $D_i^1 F(x;[t,T])$ the (directional) derivative of F at x in the direction of the $\mathbb{R}^d$-valued process $1_{[t,T]}e_i$. Namely,*

$$D_i^1 F(x;[t,T]) = \lim_{h\to 0} \frac{F(x + h1_{[t,T]}e_i) - F(x)}{h}.$$

Similarly we denote the second-order derivative in the direction $1_{[t,T]}e_i$ and $1_{[t,T]}e_j$ by

$$D_{ij}^2 F(x;[t,T]) = \lim_{h\to 0} \frac{D_j^1 F(x + h1_{[t,T]}e_i;[t,T]) - D_j^1 F(x;[t,T])}{h}$$

Note that time can be taken as one of the components in $D\left([0,T],\mathbb{R}^d\right)$. In general the functional itself is not necessarily dependent on time (see Section 2.5 below). However, the derivatives defined above are functions of $(x,t) \in D\left([0,T],\mathbb{R}^d\right) \times [0,T]$. To consider continuity of the derivatives we define a distance on $D\left([0,T],\mathbb{R}^d\right) \times [0,T]$ by $\widetilde{d}\,[(x,t),(y,s)] = \|x-y\|_\infty + |t-s|$, where $\|x-y\|_\infty = \sup_{0\le t\le T} \|x(t)-y(t)\|$. Define $(x_n,t_n) \underset{\widetilde{d}}{\to} (x,t)$ if $\widetilde{d}\,[(x_n,t_n),(x,t)] \underset{n\to\infty}{\to} 0$.

Let

$$D^1 F(x;[t,T]) = \left(D_i^1 F(x;[t,T]);\ i = 1,...d\right),$$
$$D^2 F(x;[t,T]) = \left(D_{ij}^2 F(x;[t,T]);\ i = 1,...d;\ j = 1....d\right).$$

2.3 Functional Itô Formula

Assume throughout the following continuity condition on the functional.

Condition 2.1. *The functional* $F : D\left([0,T],\mathbb{R}^d\right) \to \mathbb{R}$ *is continuous and* $D^k F(x;[t,T])$, $k = 1,2$, *exist and are continuous in* x *and* t. *All continuity relates to the metric* $\widetilde{d}$. *(In other words, if* $\widetilde{d}\,[(x_n,t_n),(x,t)] \to 0$ *then* $F(x_n) \to F(x)$ *and* $D^k F(x_n;[t_n,T]) \to D^k F(x;[t,T])$, $k = 1,2$.*)*

Next, the proof for the functional Itô formula for the case of continuous semimartingales is given. Formally this follows from the RCLL case which will be proved later, but the proof of the continuous case shows clearly some of the ideas being using.

Theorem 2.1 (Functional Itô formula for continuous semimartingales)**.** *Let* X_t *be a* d-*dimensional continuous semimartingale and* $F : D\left([0,T],\mathbb{R}^d\right) \to \mathbb{R}$ *a functional that satisfies Condition* (2.1). *Then*

$$F(Y_t) = F(Y_0) + \sum_{i=1}^{d}\int_0^t D_i^1 F(Y_s;[s,T])dX_s^i + \frac{1}{2}\sum_{i,j\le d}\int_0^t D_{ij}^2 F(Y_s;[s,T])d\left[X_s^i, X_s^j\right]. \tag{2.1}$$

Proof. Consider partitions $\left\{t_i^n, 0 \le i \le n, 1 \le n \le \infty\right\}$ of $[0,T]$ such that mesh of the partition goes to 0 as $n \to \infty$, and define a sequence of approximations for Y_t as

$$App^n(Y_t) = \sum_{k=0}^{n-1} Y_t(t_k^n)1_{[t_k^n,t_{k+1}^n)}, \quad \text{for each } t \in [0,T].$$

For t restricted to $[t_{k-1}^n, t_k^n]$ we have

$$App^n(Y_t) = App^n(Y_{t_{k-1}^n}) + \left(X_t - X_{t_{k-1}^n}\right)1_{\left[t_k^n,T\right]}, \quad t \in [t_{k-1}^n, t_k^n]. \tag{2.2}$$

If $t \in [t_{k-1}^n, t_k^n]$, then $F(App^n(Y_t)) = f(X_t, A)$ for some function $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$, where the random variable $A \in \mathcal{F}_{t_{k-1}^n}$. It is easy to confirm

$$\nabla_x f(X_t) = D^1 F(App^n(Y_t); [t_k^n, T]),$$
$$\nabla_{xx} f(X_t) = D^2 F(App^n(Y_t); [t_k^n, T]), \quad \text{if } t \in [t_{k-1}^n, t_k^n].$$

The classical Itô formula for functions implies that (2.1) holds in $[t_{k-1}^n, t_k^n]$, hence for $t \in [0, T]$. Defining $t_n(t) = \min\{t_k^n : t_k^n \geq t\}$, the formula becomes

$$F(App^n(Y_t)) = F(App^n(Y_0)) + \sum_{i=1}^{d} \int_0^t D_i^1 F(App^n(Y_s); [t_n(s), T]) dX_s^i \tag{2.3}$$
$$+ \frac{1}{2} \sum_{i,j \leq d} \int_0^t D_{ij}^2 F(App^n(Y_s); [t_n(s), T]) d\left[X_s^i, X_s^j\right].$$

The goal is now to complete the proof of the theorem by letting $n \to \infty$ in (2.3), relying on the dominated convergence theorem for stochastic integrals, which is quoted for the reader's convenience. □

Theorem 2.2 (Dominated Convergence Theorem for Stochastic Integrals)**.** *Let X be a semimartingale. If (K^n) is a sequence of predictable processes converging to zero pointwise a.s. and if there exists a locally bounded predictable process K such that $|K^n| \leq K$, for all n, then $\int_0^{\cdot} K^n(s) dX_s \to O(p)$.*

Proof. See Theorem 4.31, Chapter IV in [29]. □

Remark 2.1. If X is a continuous semimartingale, then K^n and K can be taken as progressively measurable in the above theorem.

The following following lemma will be used for the proof of theorem.

Lemma 2.1. *Let* $t_n : [0,T] \to [0,T]$ *be a sequence of functions with* $t_n \geq I$*, where* $I(t) = t$*. Let* $F : D\left([0,T], \mathbb{R}^d\right) \to \mathbb{R}$*. Assume* $\|x_n - x\|_\infty + \|t_n - I\|_\infty \underset{n\to\infty}{\to} 0$*, where* $x_n, x \in D\left([0,T], \mathbb{R}^d\right)$*. Then, under the continuity assumptions of Condition 2.1,*

$$\sup_{0\leq t\leq T}\left\{\left\|D^iF(x_n;[t_n(t),T]) - D^iF(x;[t,T])\right\|\right\} \underset{n\to\infty}{\to} 0, \quad i = 1, 2.$$

Proof. If not then $\exists\varepsilon > 0$, a sequence of integers $n_k \uparrow \infty$, and a sequence $s_k \in [0,T]$ such that

$$\left\|D^iF(x_{n_k};[t_{n_k}(s_k),T]) - D^iF(x;[s_k,T])\right\| \geq \varepsilon. \tag{2.4}$$

By moving to a subsequence we can assume without loss of generality that $s_k \to s^*$, for some $s^* \in [0,T]$. Because t_n converges uniformly to I, we also get $t_{n_k}(s_k) \underset{k\to\infty}{\to} s^*$. Our continuity assumption implies $D^iF(x_{n_k};[t_{n_k}(s_k),T]) \underset{k\to\infty}{\to} D^iF(x;[s^*,T])$ as well as $D^iF(x;[s_k,T]) \underset{k\to\infty}{\to} D^iF(x;[s^*,T])$. This contradicts (2.4). ■

Next, the proof of the functional Itô formula is continued. Because Y_t has a continuous path it follows that $(App^n(Y_t), t_n(t)) \underset{\tilde{d}}{\to} (Y_t, t)$, and by our continuity assumptions, we obtain, for each $t \in [0,T]$:

$$F(App^n(Y_t)) \underset{n\to\infty}{\to} F(Y_t) \quad \text{and}$$

$$D^kF(App^n(Y_t);[t_n(t),T]) \underset{n\to\infty}{\to} D^kF(Y_t;[t,T]), \quad k = 1, 2.$$

Observe that for each $x \in D\left([0,T], \mathbb{R}^d\right)$, it follows from our assumptions that $D^iF(x;[t,T])$ is continuous in t, and hence it is bounded on $[0,T]$. Therefore,

$$\sup_{0\leq t\leq T}\left\{\left\|D^iF(Y_t;[t,T])\right\|\right\} < \infty, \ a.s., \quad i = 1, 2.$$

Also, $D^iF(App^n(Y_t);[t_n(t),T])$ is bounded on $[0,T]$ a.s. for each n. By Lemma 2.1

conclude that

$$\sup_{n\geq 1, 0\leq t\leq T}\left\{\left\|D^iF(App^n(Y_t);[t_n(t),T])\right\|\right\}<\infty,\ a.s.,\quad i=1,2.$$

Use Theorem 2.2 for each of the two stochastic integrals in (2.3). For the first and second integrals, respectively, take

$$K^{n,1}(t)\equiv\sum_{i\leq d}\left\{D_i^1F(App^n(Y_t);[t_n(t),T])-D_i^1F(Y_t;[t,T])\right\},$$
$$K^{n,2}(t)\equiv\sum_{i,j\leq d}\left\{D_{ij}^2F(App^n(Y_t);[t_n(t),T])-D_{ij}^2F(Y_t;[t,T])\right\},$$

with the locally bounded processes $K^{(1)}(t)$ and $K^{(2)}(t)$, respectively, where

$$K^{(i)}(t)=\sup_{n\geq 1, 0\leq s\leq t}\left\{\left\|D^iF(App^n(Y_s);[t_n(s),T])\right\|+\left\|D^iF(Y_s;[s,T]\right\|\right\},\quad i=1,2.$$

□

Remark 2.2. By adding two more steps to the above proof, one can produce a proof without relying on the classical Itô formula (obviously the classical Itô formula is a corollary of the theorem by taking $F(Y_t)=f(X(t))$). The two steps are:

1. Showing that if (2.1) holds for F and G, then (2.1) holds for $F\cdot G$ (integration by parts formula).
2. Showing (2.1) in the case where $F(Y_t)=f(Y_t(t_1),...,Y_t(t_n))$ by using polynomial approximation (Weirstrass theorem) and stopping times.

Because those steps appear in the proof of the classical Itô formula, we can conclude that the Functional Itô formula is a natural extension of Itô formula with one more step.

The next step is to prove the functional Itô formula for RCLL semimartingales.

Theorem 2.3 (Functional Itô Formula for RCLL Semimartingales)**.** *Let X_t be a d-dimensional RCLL semimartingale and $F : D\left([0,T],\mathbb{R}^d\right) \to \mathbb{R}$, a functional that satisfies Condition* (2.1)*. Then*

$$F(Y_t) = F(Y_0) + \sum_{i=1}^{d} \int_0^t D_i^1 F(Y_{s-};[s,T])dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t D_{ij}^2 F(Y_{s-};[s,T])d[X_s^{i,c}, X_s^{j,c}]$$

$$+ \sum_{s\le t}\left[F(Y_s) - F(Y_{s-}) - \sum_{i=1}^{d} D_i^1 F(Y_{s-};[s,T])\Delta X_s^i\right], \tag{2.5}$$

where X_t^c is the continuous part of the semimartingale X_t, $\Delta X_t = X_t - X_{t-}$, and

$$Y_{t-}(u) = \begin{cases} X(u) & \textit{if} \quad u < t, \\ X(t-) & \textit{if} \quad t \le u \le T. \end{cases}$$

Proof. Define $t_k^n = \frac{k}{n}T$, $k = 1, ..., n$, and again let $t_n(t) = \inf\left\{t_k^n : t_k^n > t\right\}$. For a fixed n, define a sequence of stopping times

$$\alpha_0^n = 0, \quad \alpha_{k+1}^n = \inf\left\{t > \alpha_k^n : \left|X(t) - X(\alpha_k^n)\right| > \frac{1}{n}\right\} \wedge t_n(\alpha_k^n) \wedge T.$$

We remark that the above sequence of stopping times can be viewed as a union of the fixed stopping times $\left\{t_k^n\right\}$ and $\left\{\tau_k^n\right\}$, where $\tau_0^n = 0$, $\tau_{k+1}^n = \inf\left\{t > \tau_k^n : \left|X(t) - X(\alpha_k^n)\right| > \frac{1}{n}\right\} \wedge T$. Since $X(t)$ is a RCLL semimartingale, we have $\alpha_{k+1}^n > \alpha_k^n$, a.s. on the event $\left\{\alpha_k^n < T\right\}$. Also there exists a $k < \infty$ such that $\alpha_k^n = T$. Next, define an approximation of Y_t as follows. For $t \in [\alpha_k^n, \alpha_{k+1}^n)$,

$$App^n(Y_t) = \left[\sum_{l=0}^{k-1} X(\alpha_l^n)\cdot 1_{[\alpha_l^n,\alpha_{l+1}^n)}\right] + X(\alpha_k^n)\cdot 1_{[\alpha_k^n, t_n(t))} + X(t)\cdot 1_{[t_n(t),T]} \tag{2.6}$$

(The bracketed term on the right side of (2.6) is 0 if $k = 0$.) Observe that $App^n(Y_t)$ as defined in Equation (2.6) is adapted. If $t \in [\alpha_k^n, \alpha_{k+1}^n)$ then $F(App^n(Y_t)) = f(X(t), A)$ for some function $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$, where the random variable $A \in \mathcal{F}_{\alpha_k^n}$. It is easy to confirm that

$$\begin{aligned}\nabla_x f\left(X_{t-}\right) &= D^1 F(App^n(Y_{t-}); [t_n(t), T]),\\ \nabla_{xx} f\left(X_{t-}\right) &= D^2 F(App^n(Y_{t-}); [t_n(t), T]), \quad \text{if } t \in [\alpha_k^n, \alpha_{k+1}^n).\end{aligned}$$

Apply the classical Itô formula for RCLL semimartingales to get (2.5) for $F(App^n(Y_t))$ on the interval $[\alpha_k^n, \alpha_{k+1}^n)$, and hence for $[0, T]$. The formula becomes

$$\begin{aligned}F(App^n(Y_t)) = F(App^n(Y_0)) &+ \sum_{i=1}^{d} \int_0^t D_i^1 F(App^n(Y_{s-}); [t_n(s), T]) dX_s^i + \qquad (2.7)\\ &\frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{ij}^2 F(App^n(Y_{s-}); [t_n(s), T]) d[X_s^{i,c}, X_s^{j,c}] +\\ &\sum_{0 \le s < t} \left[\begin{array}{c} F(App^n(Y_s)) - F(App^n(Y_{s-})) \\ -\sum_{i=1}^{d} D_i^1 F(App^n(Y_{s-}); [t_n(s), T]) \Delta X_s^i \end{array} \right].\end{aligned}$$

The goal is now to complete the proof of the theorem by letting $n \to \infty$ in (2.7) relying on Theorem 2.2. First we show $\|App^n(Y_t)(u) - Y_t(u)\|_\infty \underset{n\to\infty}{\to} 0$. Indeed, we estimate, for $u \in [0, T]$,

$$\left|App^n(Y_t)(u) - Y_t(u)\right| = \begin{cases} \left|X(\alpha_k^n) - X(u)\right| < 2/n, & \text{if} \quad \alpha_k^n \le u < \alpha_{k+1}^n \le t, \\ \left|X(\alpha_k^n) - X(u)\right| < 2/n, & \text{if} \quad \alpha_k^n \le u \le t < \alpha_{k+1}^n, \\ \left|X(\alpha_k^n) - X(t)\right| < 2/n, & \text{if} \quad \alpha_k^n \le t < u < \alpha_{k+1}^n, \\ |X(t) - X(t)| = 0, & \text{if} \quad t_n(t) \le u \le T. \end{cases}$$

It follows that $(App^n(Y_t), t_n(t)) \underset{\widetilde{d}}{\to} (Y_t, t)$ and $(App^n(Y_{t-}), t_n(t)) \underset{\widetilde{d}}{\to} (Y_{t-}, t)$.

By Condition (2.1), for each $t \in [0, T]$,

$$F(App^n(Y_t)) \underset{n\to\infty}{\to} F(Y_t), \quad F(App^n(Y_{t-})) \underset{n\to\infty}{\to} F(Y_{t-}),$$
$$D^k F(App^n(Y_{t-}); [t_n(t), T]) \underset{n\to\infty}{\to} D^k F(Y_{t-}; [t, T]), \quad k = 1, 2.$$

Next, we establish the local boundedness needed in Theorem 2.2. Exactly as in the proof of Theorem 2.1, Condition (2.1) implies

$$\sup_{0\le t\le T} \left\{ \left\| D^i F(Y_{t-}; [t, T]) \right\| \right\} < \infty, \ a.s., \quad i = 1, 2,$$

and by using Lemma 2.1 we can conclude that

$$\sup_{n\ge 1, 0\le t\le T} \left\{ \left\| D^i F(App^n(Y_{t-}); [t_n(t), T]) \right\| \right\} < \infty, \ a.s., \quad i = 1, 2.$$

We use Theorem 2.2 for each of the two stochastic integrals in (2.7). For the first and second integrals, respectively, take

$$K^{n,1}(t) \equiv \sum_{i\le d} \left(D_i^1 F(App^n(Y_{t-}); [t_n(t), T]) - D_i^1 F(Y_{t-}; [t, T]) \right),$$
$$K^{n,2}(t) \equiv \sum_{i,j\le d} \left(D_{ij}^2 F(App^n(Y_{t-}); [t_n(t), T]) - D_{ij}^2 F(Y_{t-}; [t, T]) \right),$$

with the locally bounded processes $K^{(1)}(t)$ and $K^{(2)}(t)$, respectively, where

$$K^{(i)}(t) = \sup_{n\ge 1, 0\le s\le t} \left\{ \left\| D^i F(App^n(Y_{s-}); [t_n(s), T]) \right\| + \left\| D^i F(Y_{s-}; [s, T] \right\| \right\},$$
$$i = 1, 2.$$

So we can conclude the following convergence in probability:

$$\sum_{i=1}^{d}\int_0^t D_i^1 F(App^n(Y_{s-});[t_n(s),T])dX_s^i$$
$$+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t D_{ij}^2 F(App^n(Y_{s-});[t_n(s),T])d[X_s^{i,c},X_s^{j,c}]$$
$$\underset{n\to\infty}{\to}\sum_{i=1}^{d}\int_0^t D_i^1 F(Y_{s-};[s,T])dX_s^i+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t D_{ij}^2 F(Y_{s-};[s,T])d[X_s^{i,c},X_s^{j,c}].$$

The last step is to deal with the sum involving jumps in formula (2.7). Let

$$g_n(s)=F(App^n(Y_s))-F(App^n(Y_{s-}))-\sum_{i=1}^{d}D_i^1 F(App^n(Y_{s-});[t_n(s),T])\Delta X_s^i,$$
$$g(s)=F(Y_s)-F(Y_{s-})-\sum_{i=1}^{d}D_i^1 F(Y_{s-};[s,T])\Delta X_s^i,\quad s\in[0,T].$$

Observe that $g_n(s)\underset{n\to\infty}{\to}g(s),\ s\in[0,T]$. So to prove that $\sum_{0\le s<T}g_n(s)\underset{n\to\infty}{\to}\sum_{0\le s<T}g(s)$, a.s., we need to find $G(s)\ge\sup_n|g_n(s)|$ so that $\sum_{0\le s<T}G(s)<\infty$, a.s. By the Taylor's theorem applied to $F(App^n(Y_s))$ we get that $\exists\ r_n(s)\in[0,1]$ such that

$$|g_n(s)|\le\sum_{i,j=1}^{d}\left|D_{ij}^2 F(App^n(Y_{s-})+r_n(s)\cdot 1_{[t_n(s),T]}\cdot\Delta X_s;[t_n(s),T])\right|\left|\Delta X_s^i\cdot\Delta X_s^j\right|$$
$$\le\sup_{\substack{n\ge1,\ 0\le s\le T\\ r\in[0,1],\ 1\le i,j\le d}}\left\{\left|D_{ij}^2F\left(\begin{array}{c}App^n(Y_{s-})\\+r\cdot 1_{[t_n(s),T]}\cdot\Delta X_s;[t_n(s),T]\end{array}\right)\right|\cdot\sum_{i,j=1}^{d}\left|\Delta X_s^i\cdot\Delta X_s^j\right|\right\}$$
$$\equiv G(s).$$

Because $\sum_{0 \le s < T} \sum_{i,j=1}^{d} \left| \Delta X_s^i \cdot \Delta X_s^j \right| \le d \cdot \sum_{i=1}^{d} [X^i, X^i]_T < \infty$, a.s., we will get that $\sum_{0 \le s < T} G(s) < \infty$ if we can show that

$$H \equiv \sup_{\substack{n \ge 1,\ 0 \le s \le T \\ r \in [0,1],\ 1 \le i,j \le d}} \left| \begin{array}{c} D_{ij}^2 F(App^n(Y_{s-}) \\ +r \cdot 1_{[t_n(s),T]} \cdot \Delta X_s; [t_n(s), T]) \end{array} \right| < \infty, \text{ a.s.}$$

If not then there exist subsequences $n_k \to \infty,\ [0,T] \ni s_k \uparrow s^*$(or $s_k \downarrow s^*$ or $s_k = s^*$), $[0,1] \ni r_k \to r^*, 1 \le i,j \le d$ so that

$$\left| D_{ij}^2 F(App^{n_k}(Y_{s_k -}) + r_k \cdot 1_{[t_{n_k}(s_k),T]} \cdot \Delta X_{s_k}; [t_{n_k}(s_k), T]) \right| \to \infty. \tag{2.8}$$

However,

$$\begin{aligned} &\left| D_{ij}^2 F(App^{n_k}(Y_{s_k -}) + r_k \cdot 1_{[t_{n_k}(s_k),T]} \cdot \Delta X_{s_k}; [t_{n_k}(s_k), T]) \right| \\ &\to \begin{cases} \left| D_{ij}^2 F(Y_{s^*-}; [s^*, T]) \right| & \text{if } s_k \uparrow s^*, \\ \left| D_{ij}^2 F(Y_{s^*}; [s^*, T]) \right| & \text{if } s_k \downarrow s^*, \\ \left| D_{ij}^2 F(Y_{s^*-} + r^* \cdot 1_{[s^*,T]} \cdot \Delta X_{s^*}; [s^*, T]) \right| & \text{if } s_k = s^*. \end{cases} \end{aligned} \tag{2.9}$$

Because Condition 2.1 implies that $D^2 F(x; [t,T])$ is bounded on $[0,T]$ for each $x \in D\left([0,T], \mathbb{R}^d\right)$, we conclude that (2.9) contradicts (2.8). Therefore H is finite a.s. $\square$

2.4 Applications

2.4.1 Optimal Control

Let $(E, \mathcal{B}(E))$ be a *mark space*, where E is either Euclidean space, with $\mathcal{B}(E)$ denoting the Borel σ-algebra, or a discrete space, with $\mathcal{B}(E)$ denoting the set of all subsets of E. Let uncertainty be driven by a one-dimensional Brownian motion B and an independent sequence $\{(T_n, J_n)\}$ of random jump times and E-valued random jump values, respectively, such that $T_{n+1} > T_n$, a.s. and $\lim_{n\to\infty} T_n = \infty$, a.s. Associated with the sequence $\{(T_n, J_n)\}$ is the counting random measure $p : \Omega \times \mathcal{B}([0,T]) \otimes \mathcal{B}(E) \to \{1, 2, \dots\}$, defined as

$$p([0,t], S) = \sum_{n=1}^{\infty} 1\{T_n \leq t,\ J_n \in S\}, \quad t \leq T, \quad S \in \mathcal{B}(E),$$

which denotes the number of jumps by time t whose values fall within the (Borel measurable) set S. The random measure p is known as an integer-valued point process.

The compensator of $p(\omega, dt \times dz)$ is assumed to be of the form $h(\omega, t, dz)\, dt$, for an *intensity kernel* h. The corresponding compensated random measure is defined as

$$\widehat{p}(\omega, dt \times dz) = p(\omega, dt \times dz) - h(\omega, t, dz)\, dt.$$

We can interpret $h(\omega, t, S)$ as the time $t-$ conditional per-unit-time probability of a jump whose magnitude falls in set S.

Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^n)$ (that is, the set of $\mathbb{R}^n$-valued predictable processes) denote the set of admissible controls. For any control $\alpha \in \mathcal{A}$, consider the $\mathbb{R}$-valued state variable semimartingale process X_t^{α} satisfying (recall that Y_{t-} represents the history of X just before time t)

$$dX_t^\alpha = \mu\left(t, Y_{t-}^\alpha, \alpha_t\right) dt + \sigma\left(t, Y_{t-}^\alpha, \alpha_t\right) dB_t + \int_E \upsilon\left(t, Y_{t-}^\alpha, \alpha_t, z\right) \widehat{p}\,(dt \times dz)\,, \tag{2.10}$$

with $\mu, \sigma : \Omega \times [0,T] \times D\,[0,T] \times \mathbb{R} \to \mathbb{R}$ and $\upsilon : \Omega \times [0,T] \times D\,[0,T] \times \mathbb{R} \times E \to \mathbb{R}$. We assume that μ, σ, υ satisfy appropriate conditions that guarantee unique solution for equation (2.10). (See Theorem 7 in [39] for a precise formulation of the conditions.)

We assume $E\left(\int_0^T \left|f(s, Y_s, \alpha_s)\right|\, ds + \left|g(Y_T)\right|\right) < \infty$, for all $\alpha \in \mathcal{A}$ and define for any time-t starting history Y_t the value function

$$V(t, Y_t) = \sup_{\alpha \in \mathcal{A}} E_t\left(\int_t^T f(s, Y_s, \alpha_s)\, ds + g\left(Y_T\right)\right), \quad t \in [0,T]\,, \tag{2.11}$$

where the dynamics of X for $s \geq t$ are as in (2.10), though we henceforth omit the superscript α from Y.

The agent's problem is to choose the control process, α that maximizes the time-0 value function:

$$V(0, Y_0) = \sup_{\alpha \in \mathcal{A}} V^\alpha(0, Y_0) = \sup_{\alpha \in \mathcal{A}} E\left(\int_0^T f(s, Y_s, \alpha_s)\, ds + g\left(Y_T\right)\right).$$

The following informal argument motivates the Bellman equation (2.15) below. Given any history Y_t of the process X_t, the Bellman principle in our setting is

$$V(t, Y_t) = \sup_{\alpha \in \mathcal{A}} E_t\left(\int_t^{t+\Delta t} f(s, Y_s, \alpha_s)\, ds + V(t + \Delta t, Y_{t+\Delta t})\right), \tag{2.12}$$

where Δt satisfies $t + \Delta t \leq T$. Furthermore, functional Itô's formula (Theorem 2.3) implies (we assume that V satisfies Condition 2.1)

$$V(t+\Delta t, Y_{t+\Delta t}) - V(t, Y_t) = \int_t^{t+\Delta t} \mathcal{D}^{\alpha_s} V(s, Y_s) ds + \int_t^{t+\Delta t} dM_s^{\alpha_s}, \quad (2.13)$$

where the drift operator $\mathcal{D}^{\alpha_t}$ and local martingale $M_t^{\alpha_t}$ are defined by[1]

$$\begin{gathered} \mathcal{D}^{\alpha_t} V(t, Y_t) = D_t V(t, Y_t) + D^1 V(t, Y_{t-}; [t, T]) \mu(t, Y_t, \alpha_t) + \\ \frac{1}{2} D^2 V(t, Y_{t-}; [t, T]) \sigma^2(t, Y_t, \alpha_t) + \\ \int_E \left\{ \begin{array}{c} V(t, Y_{t-} + 1_{[t,T]} \upsilon(t, Y_{t-}, \alpha_t, z)) - V(t, Y_{t-}) \\ -D^1 V(t, Y_{t-}; [t, T]) \upsilon(t, Y_{t-}, \alpha_t, z) \end{array} \right\} h(t, dz), \end{gathered}$$

and

$$\begin{aligned} dM_t^{\alpha_t} = {} & D^1 V(t, Y_{t-}; [t, T]) \sigma(t, Y_{t-}, \alpha_t) dB_t + \\ & \int_E \left\{ V(t, Y_{t-} + 1_{[t,T]} \upsilon(t, Y_{t-}, \alpha_t, z)) - V(t, Y_{t-}) \right\} \hat{p}(dt \times dz). \end{aligned}$$

We assume that for all $\alpha \in \mathcal{A}$, the local martingale $M_t^{\alpha_t}$ is a martingale (namely, we assume $\left\{ M_\tau^{\alpha_\tau} : \tau \in [0, T],\ \tau \text{ stopping time} \right\}$ is uniformly integrable). Then we get from (2.13)

$$E_t \left(V(t+\Delta t, Y_{t+\Delta t}) - V(t, Y_t) \right) = E_t \left(\int_t^{t+\Delta t} \mathcal{D}^{a_s} V(s, Y_s) ds \right). \quad (2.14)$$

Comparing equation (2.12) and (2.14) we get by dividing by Δt and letting $\Delta t \downarrow 0$ that,

[1] In what follows, we use the notation $D_t V(t, Y_t) = D_t^1 V(t, Y_{t-}; [t, T]),\ D^i V(t, Y_{t-}; [t, T]) = D_Y^i V(t, Y_{t-}; [t, T])$ for $i = 1, 2$.

for any history Y_t,

$$0 \geq f(t, Y_{t,}^{\alpha}\alpha_t) + \mathcal{D}^{\alpha_t} V(t, Y_t^{\alpha}), \quad t \in [0,T], \quad \text{all } \alpha \in \mathcal{A} \tag{2.15}$$
$$0 = f(t, Y_{t,}^{\alpha^*}\alpha_t^*) + \mathcal{D}^{\alpha_t^*} V(t, Y_{t,}^{\alpha^*}), \quad t \in [0,T],$$

where $\alpha^* \in \mathcal{A}$ satisfies $V(t, Y_t) = E_t\left(\int_t^{t+\Delta t} f(s, Y_{s,}\alpha_s^*)\, ds + V(t+\Delta t, Y_{t+\Delta t})\right)$.

The only change in the following standard verification Lemma is in the drift term $\mathcal{D}^{\alpha_t}$.

Lemma 2.2 (Verification). *Let $V : [0,T] \times D[0,T] \to \mathbb{R}$ satisfy the terminal condition $V(T,y) = g(y)$ for all $y \in D[0,T]$. Assume Condition 2.1 holds for V and there exists a control $\alpha^* \in \mathcal{A}$ for which (2.15) holds. Then α^* is optimal.*

Proof. Consider any admissible process $\alpha \in \mathcal{A}$. By (2.15) we have

$$\mathcal{D}^{\alpha_t} V(t, Y_t^{\alpha}) = -f(t, Y_{t,}^{\alpha}\alpha_t) - p_t$$

for some nonnegative process p (which is zero when $\alpha = \alpha^*$). Therefore

$$\begin{aligned} g\left(Y_T^{\alpha}\right) - V(0, Y_0) &= \int_0^T \mathcal{D}^{\alpha_s} V(s, Y_s^{\alpha}) ds + \int_0^T dM_s^{\alpha_s} \\ &= -\int_0^T \left\{ f(s, Y_s^{\alpha}, \alpha_s) + p_s \right\} ds + \int_0^T dM_s^{\alpha_s}. \end{aligned}$$

Taking expectations we have

$$\begin{aligned} V(0, Y_0) &= E\left(\int_0^T f(s, Y_s^{\alpha^*}, \alpha_s^*) ds + g\left(Y_T^{\alpha^*}\right)\right) \\ &\geq E\left(\int_0^T f(s, Y_s^{\alpha}, \alpha_s) ds + g\left(Y_T^{\alpha}\right)\right) \end{aligned}$$

for all $\alpha \in \mathcal{A}$. □

2.4.1.1 Example - Optimal Portfolio

Let B denote d-dimensional Brownian motion, and X a wealth process for an agent trading in $n \geq d$ financial securities. The control process $\phi \in \mathcal{P}(\mathbb{R}^n)$ represents the dollar investments in the risky assets. The budget equation is

$$dX_t = \left(X_t r + \phi_t' \mu^R\right) dt + \phi_t' \left(\sigma_t^R dB_t + \int_E \upsilon^R(t,z)\, \hat{p}(dt \times dz)\right), \quad X_0 = x,$$

where $x \in \mathbb{R}$ is initial wealth, $r \in \mathbb{R}_+$ is the riskless rate, $\mu^R \in \mathbb{R}^n$ the excess (above the riskless rate) instantaneous expected returns of the risky assets, $\sigma^R \in \mathbb{R}^{n\times d}$ the return volatility corresponding to Brownian noise, and $\upsilon^R(t,z) \in \mathbb{R}^n$ the sensitivity of returns to a jump of size z, $z \in E$. We assume σ^R is full rank and

$$\int_0^t \left(\left|\phi_s' \mu^R\right| + \phi_s' \sigma^{R\prime} \sigma^R \phi_s + \int_E \left|\phi_s' \upsilon^R(s,z)\right| h(s,dz)\right) ds < \infty \quad \text{a.s.} \quad \text{for all } t < T.$$

To rule out doubling-type strategies we assume that $E\left[\sup_{t\in[0,T]} \left|X_t^-\right|^2\right] < \infty$, where $X_t^- = \max\{0, -X_t\}$.

The agent is assumed to maximize the expected value of some functional $u : D[0,T] \to \mathbb{R}$ of the wealth process history:

$$\max_{\phi\in\mathcal{P}(\mathbb{R}^n)} Eu\left(Y_T\right).$$

We assume u satisfies Condition 2.1 as well as

(i) (monotonicity) $D^1 u(y;[t,T]) > 0$ for all $y \in D([0,T])$ and $t \in [0,T]$.

(ii) (concavity)

$$u(\lambda y_1 + (1-\lambda)y_2) \geq \lambda u(y_1) + (1-\lambda)u(y_2), \quad \text{for all } y_1, y_2 \in D([0,T]),\ \lambda \in [0,1].$$

The time-t value function given history Y_t is $V\left(t, Y_t\right) = \max_{\phi \in \mathcal{P}(\mathbb{R}^n)} E_t u\left(Y_T\right)$.

Lemma 2.3. *Assumption (i) implies* $D^1 V(t, Y_{t-}; [t,T]) > 0, \quad t \in [0,T]$. *Assumption (ii) implies* $V(t, Y_t)$ *is concave in the direction* $1_{[t,T]}$ *for all* $t \in [0,T]$. *That is,*

$$V(t, Y_t + \{\lambda h + (1-\lambda)k\}\, 1_{[t,T]}) \geq$$
$$\lambda V(t, Y_t + h 1_{[t,T]}) + (1-\lambda) V(t, Y_t + k 1_{[t,T]}), \quad \textit{for all } h, k \in \mathbb{R},\ \lambda \in [0,1].$$

Proof. To prove the first result, observe that if $z(0) = 0$ and $z(t)$ is smooth and increasing, i.e. $\frac{dz_t}{dt} \geq 0$, then $u(y+z) \geq u(y),\ y \in D\left([0,T]\right)$. Indeed, by using the Functional Itô formula we get $u(y+z) - u(y) = \int_0^T D^1 u(Y_{t-} + Z_t; [t,T]) \frac{dz_t}{dt} \cdot dt \geq 0$, where $Z_t(s) = z(s \wedge t)$, and $Y_{t-}(s) = y\left(s\right)$ if $s < t$ and $Y_{t-}(s) = y\left(t-\right)$ if $s \geq t$. Let $Y_T^{h,\phi}$ denote the time-T wealth history from investing the additional h dollars at t in the money-market account, and note that $r > 0$ implies $Y_T^{h,\phi}(s) = Y_T^{\phi}\left(s\right) + h \cdot e^{r(s-t)} \cdot 1_{[t,T]}\left(s\right) \geq Y_T^{\phi}\left(s\right) + h \cdot 1_{[t,T]}\left(s\right)$. So we conclude that $u\left(Y_T^{h,\phi}\right) \geq u\left(Y_T^{\phi} + h \cdot 1_{[t,T]}\right)$ and consequently $V(t, Y_t^h) \geq E_t(u(Y_T^{h,\phi})) \geq E_t(u\left(Y_T^{\phi} + h \cdot 1_{[t,T]}\right))$ where the first inequality follows because $Y_T^{h,\phi}$ is generally a suboptimal policy. We calculate

$$\begin{aligned} D^1 V(t, Y_{t-}; [t,T]) &= \lim_{h \downarrow 0} \frac{V(t, Y_{t-}^h) - V(t, Y_{t-})}{h} \\ &\geq \lim_{h \downarrow 0} \frac{E_t\left\{u\left(Y_T^{\phi} + h \cdot 1_{[t,T]}\right) - u\left(Y_T^{\phi}\right)\right\}}{h} \\ &\geq E_t\left(\lim_{h \downarrow 0} \frac{u\left(Y_T^{\phi} + h \cdot 1_{[t,T]}\right) - u(Y_T)}{h}\right) \\ &= E_t\left(D^1 u\left(Y_T^{\phi}; [t,T]\right)\right) > 0, \end{aligned}$$

where the second inequality follows from Fatou's lemma. For the proof of the second part, let ϕ_h and ϕ_k denote the optimal portfolio processes (after time t) for $V(t, Y_t^h)$ and

$V(t, Y_t^k)$ for any $h, k \in \mathbb{R}$. Letting $\lambda \in [0,1]$ then

$$\begin{aligned}\lambda V(Y_t^h) + (1-\lambda)V(Y_t^k) &= E_t\left\{\lambda u\left(Y_T^{h,\phi_h}\right) + (1-\lambda)u\left(Y_T^{k,\phi_h}\right)\right\}\\ &\le E_t\left\{u\left(\lambda Y_T^{h,\phi_h} + (1-\lambda)Y_T^{k,\phi_h}\right)\right\}\\ &\le V(t, \lambda Y_t^h + (1-\lambda)Y_t^k),\end{aligned}$$

where the first inequality follows from the concavity of u, and the last follows because the wealth history $\lambda Y_T^{h,\phi_h} + (1-\lambda)Y_T^{k,\phi_h}$ is feasible (with the portfolio $\lambda\phi_h + (1-\lambda)\phi_k$) but not necessarily optimal for $Y_t + \{\lambda h + (1-\lambda)k\}\, 1_{[t,T]}$. □

The concavity of V in the direction $1_{[t,T]}$ gives $D^2V(t, Y_{t-}; [t,T]) \le 0$, but we will assume the inequality is strict: $D^2V(t, Y_{t-}; [t,T]) < 0$. Lemma 2.2 implies that if ϕ satisfies

$$\begin{aligned}0 = \max_{\phi_t} \quad & D_tV(t, Y_t) + D^1V(t, Y_{t-}; [t,T])\left(X_t r_t + \phi_t' \mu_t^R\right) +\\ & \frac{1}{2}D^2V(t, Y_{t-}; [t,T])\phi_t' \sigma^R \sigma^{R\prime} \phi_t +\\ & \int_E \left\{V(t, Y_{t-} + 1_{[t,T]}\phi_t' \upsilon^R(t,z)) - V(t, Y_{t-})\right\} h(t, dz), \quad t \in [0,T],\end{aligned}$$

then ϕ is optimal. The first-order condition

$$\begin{aligned}0 = & D^1V(t, Y_{t-}; [t,T])\mu_t^R + D^2V(t, Y_{t-}; [t,T])\phi_t' \sigma^R \sigma^{R\prime} +\\ & \upsilon^R(t,z) \int_E D^1V(t, Y_{t-} + 1_{[t,T]}\phi_t' \upsilon^R(t,z))h(t, dz)\end{aligned}$$

implies that the optimal portfolio $\phi_t = \phi(t, Y_t)$ solves the implicit equation

$$\phi_t = R(t, Y_t)\left(\sigma^R \sigma^{R\prime}\right)^{-1} \left(\begin{array}{c} \mu^R + \\ \upsilon^R(t,z)\int_E \frac{D^1 V(t, Y_{t-} + 1_{[t,T]}\phi_t' \upsilon^R(t,z))}{D^1 V(t, Y_{t-};[t,T])} h(t, dz) \end{array} \right),$$
$$\text{where } R(t, Y_t) = -\frac{D^1 V(t, Y_{t-};[t,T])}{D^2 V\left(t, Y_{t-};[t,T]\right)}.$$

In the absence of jumps ($\upsilon^R = 0$), the agent invests in a mean-variance efficient portfolio as in the standard model in which u is a concave function of terminal wealth only, but in our case the scale factor $R(t, Y_t)$ depends both on time and the history of the wealth process. The presence of jumps distorts the investments. Supposing for simplicity that $\sigma^R = I$ and jumps are Poisson distributed with intensity λ (that is, $E = \{1\}$ and $\int_E h(t, dz) = \lambda$) then (recall $D^1 V$ is positive)

$$\phi(t, Y_t) = R(t, Y_t)\left(\mu^R + \upsilon^R(t,1)\frac{D^1 V(t, Y_{t-} + 1_{[t,T]}\phi_t' \upsilon^R(t,1))}{D^1 V(t, Y_{t-};[t,T])}\lambda\right).$$

The impact on the optimal investment of a positive potential return jump for stock i, $\upsilon_i^R(t,1) > 0$, is to increase the optimal investment in that stock relative to stock j with no jump component ($\upsilon_j^R(t,1) = 0$). Conversely, the impact of a negative jump, $\upsilon_i^R(t,z) < 0$, is the opposite. The effect of the jump increases with the intensity of the jumps.

2.4.2 Equilibrium Consumption and Risk Premia

A common problem in finance and economics is to derive the interest rate and risk premium process under which a given consumption process is optimal for an agent trading in financial markets. For example, in a single-agent economy with a given endowment process, the interest rate and risk premium processes such that the endowment is the optimum consumption process corresponds to a pure-trade asset market equilibrium.

Let $\mathcal{L}(\mathbb{R})$ denote the set of $\mathbb{R}$-valued adapted processes, and $\mathcal{C} \subset \mathcal{L}(\mathbb{R})$ the set of admissible consumption processes.[2] For any $c \in \mathcal{C}$ we let C_t, $0 \leq t \leq T$ denote the history of the consumption up to time t : $C_t(s) = \{c(s \wedge t), \ 0 \leq s \leq T\}$. The utility derived from any consumption process is given by a utility function $U : \mathcal{C} \to \mathbb{R}$ is given by

$$U(c) = Eu\left(C_T\right).$$

for some monotonic and concave functional $u : D[0,T] \to \mathbb{R}$. In contrast to the example in Section 2.4.1.1, utility is derived from consumption, which is financed through trading via the budget equation

$$dX_t = \left(X_t r_t - c_t + \phi_t' \mu_t^R\right) dt + \phi_t' \sigma_t^R dB_t, \quad X_0 = x,$$

where $x \in \mathbb{R}$ is initial wealth and B is d-dimensional Brownian motion. The instantaneously riskless rate $r \in \mathcal{L}(\mathbb{R})$, excess instantaneous expected returns $\mu^R \in \mathcal{L}(\mathbb{R}^n)$, and return diffusion $\sigma^R \in \mathcal{L}\left(\mathbb{R}^{n \times n}\right)$ are allowed to be stochastic processes. For simplicity, we assume $n = d$ and that σ_t^R is invertible for all t (therefore markets are complete). Consider a consumption process given by

$$dc_t = \mu^c(t, C_t)\, dt + \sigma^c(t, C_t)\, dB_t, \tag{2.16}$$

where $\mu^c : [0,T] \times D[0,T] \to \mathbb{R}$ and $\sigma^c : [0,T] \times D[0,T] \to \mathbb{R}^d$. We suppose the existence of a square integrable process $\pi \in \mathcal{L}\left(\mathbb{R}_{++}\right)$ (the *utility gradient density* at c) which satisfies

$$\lim_{\alpha \downarrow 0} \frac{U(c + \lambda h) - U(c)}{\lambda} = E \int_0^T \pi_s h_s ds, \text{ for all } h \text{ such that } c + \lambda h \in \mathcal{C} \text{ for some } \lambda > 0.$$

[2] Typically $\mathcal{C} = \mathcal{L}(\mathbb{R})$ or $\mathcal{C} = \mathcal{L}\left(\mathbb{R}_{++}\right)$.

(An example is provided below) Finally, suppose π is some functional of the history of the consumption process:

$$\pi_t = F\left(t, C_t\right), \quad t \in [0, T], \tag{2.17}$$

where F is some smooth strictly positive functional $F : [0, T] \times D\,[0, T] \to \mathbb{R}_{++}$. Then assuming no constraints on trading, if c in (2.16) is the optimal consumption process, then π is also a state-price density process (see, for example, [12]) satisfying

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t' dB_t,$$

where η is the market price of risk process defined by $\eta_t = \left(\sigma_t^R\right)^{-1} \mu_t^R$. Applying functional Itô's lemma to (2.17) yields the interest rate and market price of risk process:

$$\begin{aligned}
r_t &= -\frac{\begin{pmatrix} D_t F\left(t, C_t; [t, T]\right) + D^1 F\left(t, C_t; [t, T]\right) \mu^c\left(t, C_t\right) \\ +\frac{1}{2} D^2 F\left(t, C_t; [t, T]\right) \left\|\sigma^c\left(t, C_t\right)\right\|^2 \end{pmatrix}}{F\left(t, C_t\right)}, \\
\eta_t &= -\frac{D^1 F\left(t, C_t; [t, T]\right)}{F\left(t, C_t\right)} \sigma^c\left(t, C_t\right).
\end{aligned}$$

The following example illustrates a utility function class and corresponding utility gradient density expression.

Example 2.1. The generalized recursive utility process U (see [34]) is part of the solution (U, Σ) to the backward stochastic differential equation (BSDE)

$$dU_t = -G\left(t, c_t, U_t, \Sigma_t\right) dt + \Sigma_t' dB_t, \quad U_T = 0, \tag{2.18}$$

where $G : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is assumed differentiable, and concave and Lipschitz continuous in (c_t, U_t, Σ_t). Then [38] show that a unique solution exists. The utility of consumption process $c \in \mathcal{C}$ is the initial value U_0 of the solution. Suppose for some smooth functional $H : [0,T] \times D\,[0,T] \to \mathbb{R}$ satisfies

$$
\begin{aligned}
0 &= D^1 H\left(t, C_t; [t,T]\right) \mu^c\left(t, C_t\right) + \frac{1}{2} D^2 H\left(t, C_t; [t,T]\right) \left\|\sigma^c\left(t, C_t\right)\right\|^2 \\
&\quad + G\left(t, c_t, H\left(t, C_t\right), D^1 H\left(t, C_t; [t,T]\right) \sigma^c\left(t, C_t\right)\right) \\
0 &= H\left(T, C_T\right).
\end{aligned}
$$

Then $\left(H\left(t, C_t\right), D^1 H\left(t, C_t; [t,T]\right) \sigma^c\left(t, C_t\right)\right)$ solves the BSDE (2.18). Using the abbreviation $G\left(t\right) = G\left(t, c_t, H\left(t, C_t\right), D^1 H\left(t, C_t; [t,T]\right) \sigma^c\left(t, C_t\right)\right)$ the utility gradient density at c is the solution of the SDE

$$
\begin{aligned}
\pi_t &= \mathcal{E}_t G_c\left(t\right), \text{ where} \\
\frac{d\mathcal{E}_t}{\mathcal{E}_t} &= G_U\left(t\right) dt + G_\Sigma\left(t\right)' dB_t, \quad \mathcal{E}_0 = 1.
\end{aligned}
$$

2.5 Comparison with Dupire's setup

The following concepts are used in Dupire's work for the one-dimensional ($d = 1$) case.

The state space of the process of interest is taken as the vector bundle $\Lambda = \bigcup_{t\in[0,T]} \Lambda_t$, where $\Lambda_t = D([0,t], \mathbb{R})$. Let $\widetilde{X}(t) \in \mathbb{R},\ 0 \le t \le T$. Derivatives of a functional are based on the following perturbations of $x_t = \left\{\widetilde{X}(s),\ 0 \le s \le t\right\} \in \Lambda_t$:

$$
\begin{aligned}
x_t^h(s) &= x_t(s) + h 1_{\{s=t\}}, \quad s \le t, \\
x_{t,h}(s) &= x_t(s) 1_{\{s<t\}} + x_t(t) 1_{\{t \le s \le t+h\}}, \quad s \le t+h \le T.
\end{aligned}
$$

The distance in space Λ is defined, for $x, y \in \Lambda$, where $x_t \in \Lambda_t$ and $y_s \in \Lambda_s$, as

$$
d_\Lambda(x_t, y_s) = \left\|x_{t,s-t} - y_s\right\|_\infty + s - t, \quad s \ge t.
$$

For any functional $f : \Lambda \to \mathbb{R}$, $x_t \in \Lambda_t$, the vertical first and second derivatives are defined as

$$\Delta_x f(x_t) = \lim_{h \to 0} \frac{f(x_t^h) - f(x_t)}{h}, \qquad \Delta_{xx} f(x_t) = \lim_{h \to 0} \frac{\Delta_x f(x_t^h) - \Delta_x f(x_t)}{h},$$

and the horizontal derivative is defined as

$$\Delta_t f(x_t) = \lim_{h \to 0+} \frac{f(x_{t,h}) - f(x_t)}{h}.$$

Using the above definitions of derivatives and examining the Taylor expansion, Dupire derived the functional Itô formula for the continuous case.

To see the correspondence between Dupire's setup and ours, we map the functional $f : \Lambda \to \mathbb{R}$ to a functional $F : D([0,T], \mathbb{R}^2) \to \mathbb{R}$ by defining for each RCLL $(\widetilde{X}(t), Z(t)) \in \mathbb{R}^2$:

$$F(x, z) = f(x_{|Z(T)| \wedge T}).$$

In particular if we take $Z(s) = t \wedge s$, $0 \le s \le T$ with $t \in [0,T]$, then $F(x,z) = f(x_t)$. It follows that for $X(t) = (\widetilde{X}(t), t) \in \mathbb{R}^2$ we get $F(Y_t) = f(x_t)$, where, consistent with our earlier notation, $Y_t(s) = X(t \wedge s)$. We also get $f(x_t^h) = F(Y_t + h1_{[t,T]}e_1)$ and $f(x_{t,h}) = F(\widetilde{X}(t \wedge \cdot), (t+h) \wedge \cdot) = F(Y_t + h1_{[t,T]}e_2)$. We conclude that

$$\Delta_x f(x_t) = D_1^1(Y_t; [t,T]), \qquad \Delta_t f(x_t) = D_2^1(Y_t; [t,T]).$$

Finally observe that if $f(x_{t,h}) = f(x_t)\ \forall\ 0 \le t \le T,\ 0 \le h \le T - t$, then F above can be considered as a functional on $D([0,T], \mathbb{R})$. This will be the case with examples like the maximum functional, $F(x) = \max\limits_{0 \le t \le T} \widetilde{X}(t)$ or δ-modulus of continuity, $F(x) = \max\limits_{|t-s| < \delta} \left| \widetilde{X}(t) - \widetilde{X}(s) \right|$.

3

Some preliminary concepts from Optimization and Financial Economics

3.1 Introduction to Optimization under constraints

This section deals with the methods used in solving constrained optimization problems. Here, the main goal is to prove variations of Kuhn-Tucker theorem which is a Langrage multiplier type of result used by us in infinite dimensional setting. This cover problems where there are multiple constraints that may be non-binding, namely contraints which are formulated as inequalities.

The Kuhn-Tucker conditions are simply the first-order conditions for a constrained optimization problem. Linear programming is a special case covered by the Kuhn-Tucker theory.

In this section we will be proving variations of Kuhn-Tucker theorem. The method of the proof is similar to the proof of Kuhn-Tucker theorem in [36](See theorem 1 in Section 9.4). We will start by proving the theorem for general normed spaces and then specialize to collection of processes. We will need the following definitions for the theorem.

Definition 3.1. *Let B and Z be normed spaces, $X \subseteq B$, $G : X \to Z$ and $f : X \to \mathbb{R}$.*

a) X is called "extended convex" if $\forall\ x_1, x_2 \in X$ there is $\delta = \delta(x_1, x_2) > 0$ so that

$$\alpha x_1 + (1-\alpha)x_2 \in X \quad \text{for all} -\delta \leq \alpha \leq 1+\delta.$$

b) Denote $H_x \equiv \{h \in B : x + h \in X\},\ x \in X$. Let X be extended convex and let $x \in X$, $h \in H_x$. We denote the Gateaux derivative *of G in the direction h as*

$$\delta G(x,h) = \lim_{\alpha \to 0} \frac{G(x+\alpha h) - G(x)}{\alpha}.$$

Observe that since X is extended convex, we don't have to restrict α to be positive in the definition of $\delta G(x,h)$. If $\delta G(x,h)$ exists for all $h \in H_x$ we say that G is Gateaux differentiable at x.

c) The functional $\delta f(x, \cdot) : H_x \to \mathbb{R}$ will be called the supergradient *for f at x if*

$$f(x+h) - f(x) \leq \delta f(x,h), \qquad \forall h \in H_x.$$

d) Let Z contain a convex cone A; namely, $\alpha x + \beta y \in A,$ for each $\alpha, \beta > 0,\ x, y \in A$. Also we denote $x \geq y$ (respectively $x > y$) if $x - y \in A$ (respectively $x - y \in$interior(A)) and A is a positive cone. The point $x_0 \in X$ is said to be a regular point *of $\{G(x) \leq 0\}$ if $G(x_0) \leq 0$ and there is $h \in H_{x_0}$ so that $G(x_0) + \delta G(x_0, h) < 0$, where $\delta G(x_0, h)$ is* *the Gateaux derivative of G at x_0. Finally, we denote $z^* \geq 0$ for any $z^* \in Z^*$(Z^* is the dual space of Z) if $z^*[x] \geq 0$ for each $x \in A$.*

Remark 3.1. (a) H_x is convex (respectively extended convex) if X is convex (respectively extended convex), but it isn't necessarily a linear subspace of B. For a mapping $T : H_x \to Y$, where Y is a vector space, to be linear simply means

$$T(ah_1 + bh_2) = aT(h_1) + bT(h_2), \quad \text{for all } a, b \in \mathbb{R},\ h_1, h_2, ah_1 + bh_2 \in H_x.$$

If $\delta G(x, \cdot)$ (respectively $\delta f(x, \cdot)$) is linear in this sense we say that G has a linear Gateaux derivative (respectively linear supergradient) at x.

(b) We can define the supergradient of G in the same way as we defined for f, and we can define $x_0 \in X$ to be a *regular point* in the same way as in Definition 3.1(d) with the understanding that $\delta G(x_0, h)$ stands for the supergradient of G.

Next we will define a weaker concept than extended convex that will be useful in certain cases (see the concept of extended convex for processes in Definition 3.3)

Definition 3.2. *We will say that X, a subset of a vector space B, is "weakly extended convex with respect to a collection of functions $F = \{f : B \to \mathbb{R}^d\}$" if $f(X)$ is extended convex in $\mathbb{R}^d$ for each $f \in F$.*

Example 3.1 (Extended Convex)**.** Let B be a vector space whose elements are all the real-valued functions defined on the set A (say). Let $C = \{f \in B : f(x) > 0,\ x \in A\}$. Obviously $C \subset B$ is convex. A simple condition for C to be extended convex is:

$$0 < \inf_{x \in A}\{f(x)\} \leq \sup_{x \in A}\{f(x)\} < \infty, \quad \text{for all } f \in C.$$

3.1.1 Kuhn-Tucker Theorem

Theorem 3.1 (Kuhn-Tucker)**.** *Let B be a normed space, Z a normed space that contains a positive cone P with non-empty interior, $X \subset B$ extended convex, $G : X \to Z$ and $f : X \to \mathbb{R}$. Let $x_0 \in X$ satisfy $G(x_0) \leq 0$ and $f(x_0) = \min_{\{x \in X : G(x) \leq 0\}} \{f(x)\}$. Assume:*

(i) G has a linear Gateaux derivative and f has a linear supergradient at x_0.

(ii) x_0 is a regular point of $\{G(x) \leq 0\}$.

Then there is $z^* \in Z^*,\ z^* \geq 0$ such that

$$\delta f(x_0, h) + z^*[\delta G(x_0, h)] = 0,\ h \in H_{x_0},$$
$$z^*[G(x_0)] = 0.$$

Proof. In the space $W = \mathbb{R} \times Z$, define the sets

$$A = \{(r, z) : r \geq \delta f(x_0; h),\ z \geq G(x_0) + \delta G(x_0; h) \text{ for some } h \in H_{x_0}\},$$
$$D = \{(r, z) : r \leq 0,\ z \leq 0\}.$$

The set D is obviously convex. Letting (r_1, z_1) and $(r_2, z_2) \in A$, and using the fact that the supergradient of f and the Gateaux derivative of G are linear, we have $(\lambda r_1 + (1 - \lambda)r_2, \lambda z_1 + (1 - \lambda)z_2) \in A$ where $0 \leq \lambda \leq 1$. Therefore A is also convex. The set D contains interior points because P does. If $(r, z) \in A$, with $r < 0$ and $z < 0$, then there exists $h \in H_{x_0}$ such that

$$\delta f(x_0; h) < 0, \quad G(x_0) + \delta G(x_0; h) < 0.$$

The point $G(x_0) + \delta G(x_0; h)$ is the center of some sphere of radius ρ contained in the negative cone in Z. Then for $0 < \alpha < 1$ the point $\alpha(G(x_0) + \delta G(x_0; h))$ is the center of an open sphere of radius $\alpha\rho$ contained in negative cone; hence so is the point $(1-\alpha)G(x_0)+\alpha(G(x_0)+\delta G(x_0;h)) = G(x_0)+\alpha\delta G(x_0;h)$. By the definition of the Gateaux derivative we have

$$\| G(x_0+\alpha h) - G(x_0) - \alpha\delta G(x_0;h) \| = o(\alpha).$$

Therefore $G(x_0+\alpha h) < 0$ for sufficiently small α. By definition of supergradient for f we have $f(x_0+h) - f(x_0) \leq \delta f(x_0;h)$. Then the supposition $\delta f(x_0;h) < 0$ implies $f(x_0+h) < f(x_0)$. This contradicts the optimality of x_0; therefore A contains no interior points of D.

By Theorem 3, Section 5.12 (from [36]), there is a hyperplane separating A and D. Hence there are r_0, z^*, δ such that

$$r_0 r + z^*[z] \geq \gamma \text{ for all } (r,z) \in A,$$

$$r_0 r + z^*[z] \leq \gamma \text{ for all } (r,z) \in D.$$

Because $(0,0)$ belongs to both A (choose $h=0$) and D, we have $\gamma = 0$. It follows at once that $r_0 \geq 0$ and $z^* \geq 0$ (otherwise, you can choose $(r,z) \in D$ with one component 0 and the other negative, resulting in a contradiction). Furthermore, $r_0 > 0$ because of the existence of $h \in H_{x_0}$ such that $G(x_0)+\delta G(x_0;h) < 0$ (by regularity of x_0). By scaling we can assume without loss of generality that $r_0 = 1$. From the separation property, we have for all $h \in H_{x_0}$

$$\delta f(x_0;h) + z^*[G(x_0)+\delta G(x_0;h)] \geq 0$$

(because trivially $(\delta f(x_0;h),\ [G(x_0)+\delta G(x_0;h)]) \in A$). Setting $h=0$ gives $z^*[G(x_0)] \geq 0$ but $G(x_0) \leq 0$, $z^* \geq 0$ implies $z^*[G(x_0)] \leq 0$ and hence $z^*[G(x_0)] = 0$. We conclude that

$$\delta f(x_0;h) + z^*[\delta G(x_0,h)] \geq 0, \quad \forall h \in H_{x_0}. \tag{3.1}$$

For any $h \in H_{x_0}$, extended convexity of X implies that there exists $\gamma > 0$ such that $x_0 \pm \gamma h \in X$; that is, $\pm\gamma h \in H_{x_0}$. By linearity of the supergradient and Gateaux derivative with respect to h, we have

$$\pm\gamma \left\{\delta f(x_0;h) + z^*[\delta G(x_0,h)]\right\} = \delta f(x_0;\pm\gamma h) + z^*[\delta G(x_0,\pm\gamma h)].$$

The last equation together with equation 3.1 gives

$$\delta f(x_0;h) + z^*[\delta G(x_0,h)] = 0.$$

□

Corollary 3.1. *Using a proof similar to that of Theorem 3.1, we can replace assumption (i) of that theorem by one of the following assumptions*

a) G has a linear Gateaux derivative and f has a linear Gateaux derivative at x_0.

b) G has a linear supergradient and f has a linear Gateaux derivative at x_0.

c) G has a linear supergradient and f has a linear supergradient at x_0.

Next we will prove below a corollary to Theorem 3.1 for processes. We will start with a variation of the definition of extended convex which is appropriate to a collection of stochastic processes.

Definition 3.3. *A collection of vector-valued stochastic processes X will be called extended convex if for all $x_1, x_2 \in X$ there is a process $\delta = \delta(\omega, t; x_1, x_2) > 0$ such that for each process $\alpha = \alpha(\omega, t)$ that satisfies $-\delta \leq \alpha \leq 1 + \delta$ we have*

$$\alpha x_1 + (1-\alpha)x_2 \in X.$$

Observe that the concept just defined is essentially a special case of weakly extended convex (see Definition 3.2) with respect to the functionals $\Omega \times [0, T]$. Namely, each (ω, t) represents a functional given by: $x \to x(\omega, t)$, for each process $x \in X$.

Example 3.2. To show that $\mathcal{L}_2\left(\mathbb{R}^n_{++}\right)$ is extended convex, for any $x_1, x_2 \in \mathcal{L}_2\left(\mathbb{R}^n_{++}\right)$ we let $\delta\left(t\right) = \min\left(x_1\left(t\right), x_2\left(t\right)\right) / \max\left(x_1\left(t\right), x_2\left(t\right)\right)$.

We next define the concepts of *supergradient density* and *gradient density* for real-valued functionals defined on a collection of processes.

Definition 3.4. *Let $X \subseteq \mathcal{L}_2\left(\mathbb{R}^n\right)$ be an extended convex subset of processes. Let $\nu : X \to \mathbb{R}$ be a functional. For any $x_0 \in X$,*

(a) The process $\pi \in \mathcal{L}_2\left(\mathbb{R}^n\right)$ is a supergradient density *of ν at x_0 if*

$$\nu\left(x_0 + h\right) - \nu\left(x_0\right) \leq \left(\pi \left| h\right.\right) \quad \text{for all } h \in H_{x_0}.$$

(b) $\pi \in \mathcal{L}_2\left(\mathbb{R}^n\right)$ is a gradient density *of ν at x_0 if*

$$\left(\pi \left| h\right.\right) = \lim_{\alpha \downarrow 0} \frac{\nu\left(x_0 + \alpha h\right) - \nu\left(x_0\right)}{\alpha} \quad \text{for all } h \in H_{x_0}.$$

Corollary 3.2. *Let $X \subseteq \mathcal{L}_2\left(\mathbb{R}^n\right)$ be an extended convex set of processes. Let $G : X \to \mathbb{R}^m$ and $f : X \to \mathbb{R}$. Let $x_0 \in X$ satisfy $G(x_0) \leq 0$ and $f(x_0) = \min_{\{x \in X : G(x) \leq 0\}} \{f(x)\}$. Assume:*

(i) G has a gradient density, and f has a supergradient density or a gradient density at x_0.

(ii) x_0 is a regular point of $\{G(x) \leq 0\}$.

Then there is a $z^* \in \mathbb{R}^m_+$ *such that*

$$\delta f(x_0, h) + z^{*\prime} \left[\delta G(x_0, h)\right] = 0, \quad \textit{all } h \in H_{x_0},$$
$$z^{*\prime} \left[G(x_0)\right] = 0.$$

Proof. Recall that all the processes that we deal with are assumed to be progressively measurable. We will prove the corollary under the assumption that f has a supergradient density at x_0. The proof goes exactly as Theorem 3.1 through equation (3.1). Fix some $h \in H_{x_0}$. Because X is extended convex there exists a $\gamma \in \mathcal{L}\left(\mathbb{R}_{++}\right)$ such that $x_0 \pm \gamma h \in X$. For each $\varepsilon > 0$, define $A(\varepsilon) = \{(\omega, t) : \gamma(\omega, t) > \varepsilon\}$. Because 1_A is a progressively measurable process and $\varepsilon 1_{A(\varepsilon)} < \gamma$, we have $x_0 \pm \varepsilon h 1_{A(\varepsilon)} \in X$. As in the proof of Theorem 3.1 we get (from equation (3.1))

$$\delta f(x_0; h 1_{A(\varepsilon)}) + z^{*\prime} \left[\delta G(x_0, h 1_{A(\varepsilon)})\right] = 0$$

Since gradient and supergradient densities are continuous in the increment and $h 1_{A(\varepsilon)} \to h$ in $\mathcal{L}_2\left(\mathbb{R}^n\right)$ as $\varepsilon \to 0$, we have

$$\delta f(x_0; h) + z^{*\prime} \left[\delta G(x_0, h)\right] = 0.$$

The proof is similar when we assume that f has gradient density at x_0. $\square$

3.2 Generalized Recursive Utility

In this section, we define recursive utility. We also present computations of supergradient density and gradient density for recursive utility. Continuous-time recursive utility was first defined in [13]. The generalized recursive utility class was introduced in [34] to unify the stochastic differential utility (SDU) formulation of [13], and the multiple-prior formulation of [5]. Generalized recursive utility is defined as a solution to a general BSDE.

3.2.1 Recursive utility and BSDEs

All uncertainty is generated by d-dimensional standard Brownian motion B over the finite time horizon $[0,T]$, supported by a probability space $(\Omega,\mathcal{F},P)$. All processes appearing in this paper are assumed to be progressively measurable with respect to the augmented filtration $\{\mathcal{F}_t : t\in[0,T]\}$ generated by B. For any subset $V\subseteq\mathbb{R}^n$ (respectively $V\subseteq\mathbb{R}^{n\times m}$), let $\mathcal{L}(V)$ denote the set of V-valued progressively measurable processes, and, for any $p\geq 1$, denote

$$\mathcal{L}_p(V)=\left\{x\in\mathcal{L}(V):\ E\left[\int_0^T\|x_t\|^p\,dt+\|x_T\|^p\right]<\infty\right\},$$

$$\mathcal{S}_p(V)=\left\{x\in\mathcal{L}(V):E\left[\operatorname*{esssup}_{t\in[0,T]}\|x_t\|^p\right]<\infty\right\},$$

where $\|x_t\|^2=x_t'x_t$ (respectively trace$(x_t'x_t)$). It is well known that $\mathcal{L}_2(V)$ with $V\subseteq\mathbb{R}^n$ is an inner product space with inner product defined by

$$(x\,|y)=E\left[\int_0^T x_t'y_t dt+x_T'y_T\right],\quad x,y\in\mathcal{L}_2(V).$$

The qualification "almost surely" is omitted throughout.

We have $N\geq 1$ agents. The set of *consumption plans* is an extended convex set $\mathcal{C}\subseteq\mathcal{L}_2\left(\mathbb{R}^k\right)$, $k\leq N$ (see Definition 3.3 in Appendix B for a precise formulation of extended convex set of processes). For any $c\in\mathcal{C}$, we interpret c_t, $t<T$, as a length-k vector of consumption rates, and c_T as a vector of lump-sum terminal consumption. For any $c\in\mathcal{C}$ and bounded $b\in\mathcal{L}(\mathbb{R}^k)$ we assume that $\tilde{c}\in\mathcal{C}$ where $\tilde{c}_t^i=\max\left(c_t^i+b_t^i,c_t^i/2\right)$, $i=1,\ldots,k$, $t\in[0,T]$.

Remark 3.2. An example of a set $\mathcal{C}$ that satisfies the required assumptions is the following:

$$\mathcal{C} = \{c(t,\omega) \in A_{t,\omega},\ 0 \le t \le T,\ \omega \in \Omega\} \cap \mathcal{L}_2\left(\mathbb{R}^k\right),$$

where $A_{t,\omega}$ is a convex and open set in $\mathbb{R}^k$ containing $\mathbb{R}^k_{++}$.

For any $c \in \mathcal{C}$ we define the $\mathbb{R}^N$-valued utility process Z as part of the pair $(Z, \Sigma) \in \mathcal{L}_2\left(\mathbb{R}^N\right) \times \mathcal{L}_2\left(\mathbb{R}^{N\times d}\right)$ that solves the BSDE

$$dZ_t(c) = -\Phi(t, c_t, Z_t, \Sigma_t)\, dt + \Sigma_t dB_t, \quad Z_T = \Phi\left(T, c_T\right). \tag{3.2}$$

The function $\Phi : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \mathbb{R}^N$ is called the *aggregator* and is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{k+N+N\times d})$ measurable, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0,T]$. The terminal-utility aggregator does not depend on $\left(Z_T, \Sigma_T\right)$. We let $\Phi^i(t)$ and Σ^i_t denote the i^{th} row of $\Phi(t)$ and $\Sigma(t)$, respectively. Let $\Phi_c \in \mathbb{R}^{N\times k}$ denote the matrix with typical element $\Phi^{ij}_c = \partial\Phi^i/\partial c^j$; $\Phi_{Z^j} \in \mathbb{R}^N$ the vector with typical element $\Phi^i_{Z^j} = \partial\Phi^i/\partial Z^j$; and $\Phi_{\Sigma^m} \in \mathbb{R}^{N\times d}$ the matrix with typical element $\Phi^{ij}_{\Sigma^m} = \partial\Phi^i/\partial\Sigma^{mj}$.

Initial BSDE existence and uniquness results, based on the type of Lipschitz continuity condition assumptions were first obtained in [38]. Based on the same assumptions [17] gave an better proof. BSDE theory has been further developed in [23], [32], [4] and others(see [31]). We will be using Lipschitz-assumptions for existence and uniqueness in this section but in later chapters, we will assume weaker conditions like quadratic or concave aggregators.

Condition 3.1. *a)* $\Phi^i(t,\cdot),\ i = 1,\dots,N$ *has continuous and uniformly bounded derivatives w.r.t.* c, Z, Σ.

b) $\Phi^i_c(t) > 0$ *for all* $t \in [0,T]$ *and* $i = 1,\dots,N$.

c) $\Phi(t,0,0,0) \in \mathcal{L}_2\left(\mathbb{R}^N\right)$.

d) $E(\left\|\Phi(T, c_T)\right\|^2) < \infty$.

Remark 3.3. Under Condition 3.1, there exists a unique pair $(Z_t, \Sigma_t) \in \mathcal{L}_2\left(\mathbb{R}^N\right) \times \mathcal{L}_2\left(\mathbb{R}^{N\times d}\right)$ which solves (3.2) for each $c \in \mathcal{C}$.

3.2.2 Utility supergradient and gradient density calculation

Consider a solution (Z, Σ) to (3.2) for $c \in \mathcal{C}$. Define the adjoint process ε, with some initial value $\varepsilon_0 \in \mathbb{R}_+^N \setminus \{0\}$, as the solution to the SDE (see the notation for derivatives in Section 3.2.1)

$$d\varepsilon_t^i = \sum_{j=1}^{N} \varepsilon_t^j \Phi_{Z^i}^j \left(t, c_t, Z_t, \Sigma_t\right) dt + \sum_{j=1}^{N} \varepsilon_t^j \Phi_{\Sigma^i}^j \left(t, c_t, Z_t, \Sigma_t\right)' dB_t, \quad i = 1, \ldots, N; \tag{3.3}$$

which can be written more compactly as

$$d\varepsilon_t^i = \varepsilon_t' \Phi_{Z^i} \left(t, c_t, Z_t, \Sigma_t\right) dt + \varepsilon_t' \Phi_{\Sigma^i} \left(t, c_t, Z_t, \Sigma_t\right) dB_t, \quad i = 1, \ldots, N.$$

Observe that condition 3.1(a) imply existence and uniqueness of the adjoint process ε.

We will be proving two lemmas which will involve the calculation of super gradient and gradient density.

Lemma 3.1. *Suppose $c \in \mathcal{C}$ and (Z, Σ) solve the BSDE (3.2) and ε solves the SDE (3.3) for some $\varepsilon_0 \in \mathbb{R}^N$. Assume $\Phi^i(t, \cdot)$ is a concave function, $\Phi_{Z^i}^j(t) \geq 0$ and $\Phi_{\Sigma^i}^j(t) = 0$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$ and $t \in [0, T]$. Then for all h such that $c + h \in \mathcal{C}$ we have*

$$\varepsilon_0' \left\{Z_0\left(c+h\right) - Z_0\left(c\right)\right\} \leq E\left(\int_0^T \varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t) h_t dt + \varepsilon_T' \Phi_c(T, c_T) h_T\right).$$

We will first start by proving the positivity of ε_t, solution to SDE (3.3).

Lemma 3.2. *Assume $\Phi_{Z^i}^j(t) \geq 0$ and $\Phi_{\Sigma^i}^j(t) = 0$ for all $t \in [0, T]$, $i \neq j$, $i, j \in$*

$\{1,2,\ldots,N\}$. *Then* ε_t^i, *the solution of equation* (3.3), *satisfies* $\varepsilon_t^i \geq 0$, *if* $\varepsilon_0^i \geq 0$, $i = 1,\ldots,N$, $0 \leq t \leq T$.

Proof. Consider the SDE

$$d\eta_t^i = \eta_t^i \Phi_{Z^i}^i \left(t, c_t, Z_t, \Sigma_t\right) dt + \eta_t^i \Phi_{\Sigma^i}^i \left(t, c_t, Z_t, \Sigma_t\right)' dB_t, \quad i = 1,\ldots,N.$$

Under Condition 3.1 and the assumptions for the lemma, we can easily check that the conditions of Theorem 1.1 of [21] are satisfied. It follows from their theorem that $\varepsilon_0^i \geq \eta_0^i$ implies $\varepsilon_t^i \geq \eta_t^i$, $i = 1,\ldots,N$. Therefore, by selecting $\varepsilon_0^i = \eta_0^i \geq 0$,we get

$$\varepsilon_t^i \geq \eta_t^i \geq 0, \quad t \in [0,T], \quad i = 1,\ldots,N.$$

□

Using the nonnegativity of ε, we now prove Lemma 3.1. Let $c, c+h \in \mathcal{C}$, and let $(Z(c), \Sigma(c))$ and $(Z(c+h), \Sigma(c+h))$ denote the solutions to equation (3.2). Define $\Delta = Z(c+h) - Z(c)$ and $\delta = \Sigma(c+h) - \Sigma(c)$. Using integration by parts we have

$$d\left(\varepsilon_t' \Delta_t\right) = -\varepsilon_t' \left\{ \begin{array}{c} \Phi_Z\left(t, c_t + h_t, Z_t + \Delta_t, \Sigma_t + \delta_t\right) - \\ \Phi\left(t, c_t, Z_t, \Sigma_t\right) - \Phi_Z(t) Z_t - \left(\Phi_\Sigma(t), \Sigma_t\right) \end{array} \right\} dt + M_t$$

for some local martingale M, where we use the abbreviation for derivatives $\Phi_i(t) = \Phi_i\left(t, c_t, Z_t, \Sigma_t\right)$ for $i \in \{c, Z, \Sigma\}$. By concavity,

$$\Phi\left(t, c_t + h_t, Z_t + \Delta_t, \Sigma_t + \delta_t\right) - \Phi\left(t, c_t, Z_t, \Sigma_t\right) - \Phi_c(t) h_t - \Phi_Z(t) Z_t - \left(\Phi_\Sigma(t), \Sigma_t\right) \leq 0.$$

Let $\{\tau_n : n = 1,2,\ldots\}$ be an increasing sequence of stopping times such that $\tau_n \to T$, a.s., and M stopped at τ_n, i.e., $M_{t\wedge\tau_n}$, is a martingale. Integrating and taking expectation we get (note that the ε is nonnegative by Lemma 3.2)

$$\varepsilon_0' \Delta_0 \leq E\left(\int_0^{\tau_n} \varepsilon_t' \Phi_c\left(t\right) h_t dt + \varepsilon_{\tau_n}' \Delta_{\tau_n}\right).$$

Because $\Phi_Z\left(t\right)$ and $\Phi_\Sigma\left(t\right)$ are uniformly bounded we get $E\left[\sup_{t\leq T}\|\varepsilon_t\|\right]^2 < \infty$. Using a similar argument as in Proposition 2.1 from [17] we get $E\left[\sup_{t\leq T}\|Z_t\|\right]^2 < \infty$. Therefore $E\left[\left(\sup_{t\leq T}\|\varepsilon_t\|\right)\cdot\left(\sup_{t\leq T}\|Z_t\|\right)\right] < \infty$. Letting $n \to \infty$ and interchanging limit and expectation we get

$$\varepsilon_0' \Delta_0 \leq E\left(\int_0^{T} \varepsilon_t' \Phi_c\left(t\right) h_t dt + \varepsilon_T' \Phi_c\left(T\right) h_T\right)$$

using also the concavity of $Z^i(T) = \Phi^i(T, c_T)$ in c_T for each i.

Lemma 3.3. *Suppose $c \in \mathcal{C}$ and (Z, Σ) solve the BSDE* (3.2). *For any initial value $\varepsilon_0 \in \mathbb{R}^N$, the utility gradient density of $\varepsilon_0' Z_0$ at c is given by*

$$\pi_t = \varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t), \quad t < T, \quad \pi_T = \varepsilon_T' \Phi_c(T, c_T),$$

where ε satisfies the SDE (3.3).

We start by defining the following notation. Given $x \in \mathbb{R}^{n^2\times d}$ and $y \in \mathbb{R}^n$, we define the product between x and y as

$$(x, y) = \left\{\text{trace}\left(x'y\right); \, i = 1, \ldots, N\right\}.$$

Proposition 3.1 (linear BSDE). *Let $\beta \in \mathcal{L}\left(\mathbb{R}^{n\times n}\right)$ and $\gamma \in \mathcal{L}\left(\mathbb{R}^d\right)$ both be uniformly bounded, $\varphi \in \mathcal{L}_2\left(\mathbb{R}^n\right)$, and $E(\|\xi\|^2) < \infty$. Then the linear BSDE*

$$-dY_t = \left(\varphi_t + \beta_t Y_t + Z_t \gamma_t\right) dt - Z_t dB_t, \quad Y_T = \xi, \tag{3.4}$$

has a unique solution (Y,Z) *in* $\mathcal{L}_2(\mathbb{R}^n)\times\mathcal{L}_2\left(\mathbb{R}^{n\times d}\right)$*. Furthermore* Y_t *satisfies*

$$\Gamma_t' Y_t = E_t\left(\Gamma_T'\xi + \int_t^T \Gamma_s'\varphi_s ds\right), \tag{3.5}$$

where Γ_t *is the* $\mathbb{R}^n$*-dimensional adjoint process defined by the forward linear SDE*

$$d\Gamma_t = \beta_t\Gamma_t dt + \Gamma_t\gamma_t' dB_t, \quad \Gamma_0 = a \in \mathbb{R}^n. \tag{3.6}$$

Proof. Since β and γ are bounded processes, the linear aggregator is uniformly Lipschitz, and therefore there exists a unique solution to BSDE 3.4. Applying Ito's lemma we get

$$\begin{aligned} d\left(\Gamma_t' Y_t + \int_0^t \Gamma_s'\varphi_s ds\right) &= (d\Gamma_t)' Y_t + \Gamma_t' dY_t + (d\Gamma_t)' dY_t \\ &= (\Gamma_t\gamma_t' dB_t)' Y_t + \Gamma_t' Z_t dB_t, \end{aligned}$$

which is a local martingale. Theorem 2.1 of [17] implies $Y \in \mathcal{L}_2(\mathbb{R}^n)$. By a similar argument as in Proposition 2.1 from [17] we have $\sup_{s\le T}\|Y_s\|$ and $\sup_{s\le T}\|\Gamma_s\|$ are square-integrable random variable and $\left(\sup_{s\le T}\|Y_s\|\right)\cdot\left(\sup_{s\le T}\|\Gamma_s\|\right)$ is an integrable random variable. Therefore the local martingale is a martingale, and so integrating and taking conditional expectation with respect to $\mathcal{F}_t$ we get (3.5). □

Remark 3.4. For every t and $1 \le i \le n$ there exists an initial value $a \in \mathbb{R}^n$ such that $\Gamma_t = \boldsymbol{e}_i$ and therefore

$$Y_t^i = E_t\left(\Gamma_T'\xi + \int_t^T \Gamma_s'\varphi_s ds\right).$$

This will follow from the representation of solution of the SDE (3.6) as $\Gamma_t = A_t a,$ where A_t is a nonsingular $n\times n$ matrix.

Next we present a version of Proposition 3.1 that will be useful in the proof of Lemma

3.3.

Proposition 3.2. *Let $\beta \in \mathcal{L}\left(\mathbb{R}^{n\times n}\right)$ and $\gamma \in \mathcal{L}\left(\mathbb{R}^{n^2\times d}\right)$ both be uniformly bounded, $\varphi \in \mathcal{L}_2\left(\mathbb{R}^n\right)$, and $E(\|\xi\|^2) < \infty$. Then the BSDE*

$$-dY_t = \left(\varphi_t + \beta_t Y_t + (\gamma_t, Z_t)\right)dt - Z_t dB_t, \quad Y_T = \xi,$$

has a unique solution (Y, Z) in $\mathcal{L}_2\left(\mathbb{R}^n\right) \times \mathcal{L}_2\left(\mathbb{R}^{n\times d}\right)$. Furthermore Y_t satisfies

$$\Gamma_t' Y_t = E_t\left(\Gamma_T' \xi + \int_t^T \Gamma_s' \varphi_s ds\right),$$

where, for any $a \in \mathbb{R}^n$, Γ_t is the $\mathbb{R}^n$-dimensional adjoint process defined by the forward linear SDE

$$d\Gamma_t^i = \sum_{j=1}^n \Gamma_t^j \beta_t^{ji} dt + \sum_{j=1}^n \Gamma_t^j \Pi_i^j(t) dB_t, \quad \Gamma_0 = a,$$

and where $\Pi_i(t)$ is the $n \times d$ matrix defined by concatenating the i^{th} row from each $n \times d$ block in γ_t (consisting of n blocks).

Proof. The proof is similar to Proposition 3.1. □

Proof of Lemma 3.3. Let h be a process such that $c + h \in \mathcal{C}$. Since $\mathcal{C}$ is (extended) convex, we have $c + \alpha h \in \mathcal{C}$ for any constant $\alpha \in [0, 1]$. Let $(Z^\alpha, \Sigma^\alpha)$ be the solution to the BSDE (3.2) corresponding to $c + \alpha h$. By the results on BSDEs in [3], the derivative $(\partial Z, \partial\Sigma)$ of $(Z^\alpha, \Sigma^\alpha)$ with respect to α is given by the solution of following BSDE:

$$-d\partial Z_t = (\Phi_c(t)h_t + \Phi_Z(t)\partial Z_t + (\Phi_\Sigma(t), \partial\Sigma_t))dt - (\partial\Sigma_t)dB_t, \quad \partial Z_T = \Phi_c(T, c_T)h_T. \tag{3.7}$$

To get the exact form of $(\partial Z, \partial\Sigma)$, we use the adjoint process $\varepsilon_t \in \mathbb{R}^N$ presented as a solution of the equation (3.3). Observe that due to Condition 3.1 we have existence and uniqueness of ε. By Proposition 3.2 (Observe that in [3] we have $E\left[\operatorname{esssup}_{t\in[0,T]} \|Z_t\|^2\right] < \infty$

and equation (3.3) is a linear SDE), the solution $(\partial Z, \partial\Sigma)$ of equation (3.7) is given by

$$\varepsilon_t' \partial Z_t = E_t \left(\int_t^T \varepsilon_s' \Phi_c(s, c_s, Z_s, \Sigma_s) h_s + \varepsilon_T' \Phi_c(T, c_T) h_T \right), \quad t \in [0, T] .$$

□

4

General Maximization Principle

4.1 Introduction

We study a class of optimization problems involving linked recursive preferences in a continuous-time Brownian setting. Such links can arise when preferences depend directly on the level or volatility of wealth, in principal-agent (optimal compensation) problems with moral hazard, and when agents are altruistic in the sense that utility is affected by the utility and risk levels of other agents. We characterize the necessary first-order conditions, which are also sufficient under additional conditions ensuring concavity. We also examine applications to optimal consumption and portfolio choice, and applications to Pareto optimal allocations.

The optimization problems we study all reduce to maximizing a linear combination of a multidimensional BSDE system. This system was proposed in [17] as an extension of [13]'s stochastic differential utility (SDU). [34] show, in the single-agent (one-dimensional) case, that this recursive specification allows considerable flexibility in separately modeling risk aversion and intertemporal substitution, and unifies many preference classes including SDU and multiple-prior formulations ([5]; [2]; [37]). We show that the multidimensional analog can be used to model altruism and direct dependence of utility on wealth, as well as the links in utility induced by moral hazard in principal/agent problems (see [35]).

Another contribution of our paper is that we define a multidimensional extension of the translation-invariant (TI) class of BSDEs, introduced by [44] as an extension of time-additive exponential utility. We show that the solution to this class of BSDEs can simplify to the solution of a single unlinked BSDE and a system of pure forward equations. Furthermore the solution method simplifies and easily generalizes, and the conditions for sufficiency are relaxed compared to the general case. The simplification of the solution for this class is illustrated in Example 4.9 which solves the optimal consumption/portfolio problem with homothetic preferences and direct utility for wealth.

Our solution method is based on an extension of the utility-gradient approach originating in [8] and [30] for additive utilities, and extended by [46] and [15] to recursive preferences. Our general optimization result (Theorem 4.1 below) can be viewed as the natural multidimensional extension of Theorem 4.2 of [18]. They derive a maximum principle for the optimal consumption/portfolio problem of a single agent with recursive utility and nonlinear wealth dynamics. They formulate their problem in terms of BSDEs for utility and wealth and obtain first-order conditions (FOCs) in terms of two adjoint processes, which represent utility and wealth gradient densities. We consider a general system of linked BSDEs and obtain FOCs in terms of a system of linked adjoint processes.

4.2 General Maximization Principle

4.2.1 Optimization Problem

We will use the same setting as Chapter 3. The definition of utility and the condition 3.1 from Chapter 3 will be assumed in this chapter. We have N agents and their preferences follow generalized recursive utility.

Fixing some nonzero weights $\beta \in \mathbb{R}_{+}^{N}$, the problem is

$$\max_{c \in \mathcal{C}} \beta' Z_0(c) \text{ subject to } Z_0(c) \geq K \tag{4.1}$$

(i.e., $Z_0^i(c) \geq K^i$, $i = 1, \ldots, N$), where $K^i \in \mathbb{R} \cup -\infty$ (to allow for void constraints).

Next we present the maximum principle for multidimensional BSDEs, which is our solution to (4.1).

Theorem 4.1 (Maximum Principle)**.** *Suppose $c \in \mathcal{C}$ and (Z, Σ) solve the BSDE (3.2) and ε solves the SDE (3.3).*

a) (Necessity) If c solves the problem (4.1) then there is some $\kappa \in \mathbb{R}_{+}^{N}$ such that

$$\varepsilon_0 = \beta + \kappa, \quad \varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t) = 0, \quad t \in [0, T], \tag{4.2}$$
$$\kappa' \{Z_0(c) - K\} = 0, \quad Z_0(c) \geq K.$$

b) (Sufficiency) Assume $\Phi^i(t, \cdot)$ is a concave function, $\Phi^j_{Z^i}(t) \geq 0$ and $\Phi^j_{\Sigma^i}(t) = 0$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$, and $t \in [0, T]$. If (4.2) holds then c is optimal.

Proof. a) We will use Corollary 3.2 from Chapter 3 with $-G = Z_0(c) - K$, and $-f = \beta' Z_0(c)$ to prove the necessity part. By Definition 3.1(d) in Chapter 3, c is a regular point of $\{c \in \mathcal{C} : G(c) \leq 0\}$ if $G(c) + \delta G(c; h) < 0$ for some h such that $c + h \in \mathcal{C}$, and where $\delta G(c; h)$ is the Gateaux derivative of G at c in the direction $h \in \mathcal{C}$. By Lemma 3.3, we see that in our case the condition for a regular point is satisfied if for every initial value

$\varepsilon_0 = \boldsymbol{e}_i$, $i = 1, \ldots, N$, there exists an h with $c + h \in \mathcal{C}$ and

$$E\left(\int_0^T \varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t) h_t dt + \varepsilon_T' \Phi_c(T, c_T) h_T\right) > 0. \tag{4.3}$$

Obviously since $\varepsilon_0 = \boldsymbol{e}_i$, it follows that the solution ε of the SDE (3.3) is not identically zero. From the assumed properties of $\mathcal{C}$ it follows that $c + h \in \mathcal{C}$ for h defined by

$$h_t^i = 1 \wedge \left(-c_t^i/2 \vee \left\{\varepsilon_t' \Phi_{c^i}(t, c_t, Z_t, \Sigma_t)\right\}\right), \quad i = 1, \ldots, k, \quad t \in [0, T]. \tag{4.4}$$

It is easy to confirm (4.3) for the h defined above. By Corollary 3.2, if c is optimal then there exists a $\kappa \in \mathbb{R}_+^N$ such that the Gateaux derivative of $\beta' Z_0(c) + \kappa' \{Z_0(c) - K\}$ in the direction of h is 0 for all h such that $c + h \in \mathcal{C}$ and $\kappa' \{Z_0(c) - K\} = 0$. Using Lemma 3.3 to compute the Gateaux derivative, or utility gradient density, for $(\beta + \kappa)' Z_0(c) - \kappa' K$ we therefore get

$$E\left(\int_0^T \varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t) h_t dt + \varepsilon_T' \Phi_c(T, c_T) h_T\right) = 0, \quad t \in [0, T],$$

$$\forall h \text{ such that } c + h \in \mathcal{C},$$

where ε solves the SDE (3.3) with initial value $\varepsilon_0 = \beta + \kappa$.

Because the above statement is true $\forall h$ such that $c+h \in \mathcal{C}$, we have $\varepsilon_t' \Phi_c(t, c_t, Z_t, \Sigma_t) = 0$, $t \in [0, T]$ (otherwise we get a contradiction using h defined as in (4.4)). This completes the necessity proof.

b) Lemma 3.1, $\varepsilon_0 = \beta + \kappa$ and (4.2) together imply

$$\beta' \{Z_0(c + h) - Z_0(c)\} \leq \kappa' \{Z_0(c) - Z_0(c + h)\} = \kappa' \{K - Z_0(c + h)\} \tag{4.5}$$

(the equality follows from complementary slackness).

It follows immediately that $\beta' Z_0(c + h) > \beta' Z_0(c)$ implies $\kappa' \{K - Z_0(c + h)\} >$

0, and therefore a violation of at least one constraint. $\square$

We now use Theorem 4.1 to sketch a characterization of the optimum as the solution to an forward-backward stochastic differential equation (FBSDE) system. Furthermore, a reduction in dimensionality is attained, which can be useful for small N.

Assume that ε^i is strictly positive for some $i \in \{1, 2, \ldots, N\}$; by relabeling we obtain strict positivity of ε^1. Define λ_t by

$$\lambda_t = \left(1, \frac{\varepsilon_t^2}{\varepsilon_t^1}, \ldots, \frac{\varepsilon_t^N}{\varepsilon_t^1}\right)', \quad t \in [0, T]. \tag{4.6}$$

The FOCs (4.2) imply $\lambda_t' \Phi_c(t, c_t, Z_t, \Sigma_t) = 0$. Assume that we can invert the FOCs to solve for consumption. That is, there exists a function $\varphi : \Omega \times [0, T] \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \to \mathbb{R}^k$ satisfying $\lambda_t' \Phi_c(t, \varphi(t, \lambda_t, Z_t, \Sigma_t), Z_t, \Sigma_t) = 0$. Applying Ito's lemma to compute the dynamics of λ^i, we can express the FOCs (4.2) as a FBSDE system for (Z, Σ, λ):

$$\begin{aligned}
dZ_t(c) &= -\Phi(t, c_t, Z_t, \Sigma_t)\, dt + \Sigma_t dB_t, \quad Z_T = \Phi\left(T, c_T\right), \\
d\lambda_t^i &= \lambda_t' \left\{\Phi_{Z^i}(t) - \lambda_t^i \Phi_{Z^1}(t) + \left(\lambda_t^i \Phi_{\Sigma^1}(t)\, \Phi_{\Sigma^1}'(t) - \Phi_{\Sigma^i}(t)\, \Phi_{\Sigma^1}'(t)\right) \lambda_t\right\} dt \\
&\quad + \lambda_t' \left\{\Phi_{\Sigma^i}(t) - \lambda_t^i \Phi_{\Sigma^1}(t)\right\} dB_t, \\
\lambda_0^i &= \left(\beta^i + \kappa^i\right) / \left(\beta^1 + \kappa^1\right), \quad i = 2, \ldots, N, \\
c_t &= \varphi(t, \lambda_t, Z_t, \Sigma_t), \\
0 &= \kappa' \{Z_0(c) - K\}, \quad Z_0(c) \geq K, \quad \kappa \geq 0.
\end{aligned} \tag{4.7}$$

4.3 Translation Invariant (TI) BSDEs

In this section we examine an aggregator form which leads to a particularly tractable solution. We show that when $k = N - 1$ the solution reduces to solving a single unlinked backward equation, followed by a system of $N-1$ forward SDEs. Furthermore, we use a dynamic-programming argument that relaxes the sufficiency conditions of Theorem 4.1.

We assume throughout this section that $\mathcal{C}$ satisfies, in addition to the previously assumed properties, the following: $c+v \in \mathcal{C}$ for all $c \in \mathcal{C}$ and $v \in \mathbb{R}^k$. We also assume a TI aggregator, defined as follows:

Definition 4.1. *A TI aggregator takes the form*

$$\Phi(t, c, Z, \Sigma) = \psi(t, Mc - Z, \Sigma), \quad \Phi(T, c) = Mc + \zeta,$$

for some $\psi : \Omega \times [0, T] \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \to \mathbb{R}^N$, where $M \in \mathbb{R}^{N \times k}$ is assumed to satisfy rank$(M) = k$, and $\zeta = \left(\zeta^1, \ldots, \zeta^N\right)' \in \mathcal{F}_T$ is assumed to satisfy $E\|\zeta\|^2 < \infty$.

We can interpret ζ as supplemental lump-sum terminal consumption in addition to terminal component of the control c (say, a lump-sum endowment; an intermediate endowment can be included via the ω argument of ψ).

A key property of the TI class is the (easily verified) quasilinear property

$$Z_t(c+\alpha) = Z_t(c) + M\alpha, \quad t \in [0, T], \quad \text{for all } \alpha \in \mathbb{R}^k. \tag{4.8}$$

Special cases of the TI class are common in finance. Example 4.9 shows that homothetic preferences and the standard present value operator (that is, the budget equation considered as a BSDE) are both within the TI class after transforming to logs. The following example shows that additive exponential preferences are also in the TI class.

Example 4.1. Suppose

$$\psi^i(t, Mc - Z, \Sigma) = -\exp\left(-a^{i\prime}(Mc_t - Z_t)\right) - \frac{1}{2}\Sigma_t^{i\prime}\Sigma_t^i, \quad i = 1, \ldots, N,$$

where $a^i = \left(a_1^i, \ldots a_N^i\right)' \in \mathbb{R}^N$ and we normalize with $a_i^i = 1$ for $i = 1, \ldots, N$. Then the ordinally equivalent utility $z_t^i = -\exp\left(-Z_t^i\right)$ satisfies (under sufficient integrability)

$$z_t^i = -E_t \left\{ \int_t^T \exp\left(-a^{i\prime} M c_s\right) \prod_{j \neq i} \left| z_s^j \right|^{a_j^i} ds + \exp\left(-\left(M^i c_T + \zeta^i\right)\right) \right\}.$$

Agent i's utility is increasing (decreasing) in agent j's utility if $a_j^i < 0$ ($a_j^i > 0$). If $a^i = \mathbf{e}_i$ then agent i's utility is standard additive exponential.

We show below that the matrix M plays a key role in determining the properties of the solution. A specification that arises in principal-agent problems and Pareto efficiency problems is given in the following example.

Example 4.2. Suppose each agent's aggregator depends only their own consumption and utility level, and total consumption is given by some stochastic process $C \in \mathcal{L}(\mathbb{R})$. Defining $c_t = \left(c_t^1, \ldots, c_t^{N-1}\right)'$ as the first $N-1$ agents' consumption (and so $k = N-1$), then $c_t^N = C_t - \mathbf{1}' c_t$, where $\mathbf{1}$ denotes a vector of ones. Then M is obtained by stacking a $\mathbb{R}^{(N-1)\times(N-1)}$ identity matrix I_{N-1} on top of an $\mathbb{R}^{(N-1)}$-valued row vector of -1s:

$$M = \begin{pmatrix} I_{N-1} \\ -\mathbf{1}' \end{pmatrix} \tag{4.9}$$

The agents' aggregators in this simple case take the form

$$\begin{aligned} dZ_t^i &= -f^i\left(t, c_t^i - Z_t^i, \Sigma_t\right) dt + \Sigma_t^{i\prime} dB_t, \quad Z_T^i = c_T^i + \zeta^i, \quad i = 1, \ldots, N-1, \\ dZ_t^N &= -f^N\left(t, C_t - \mathbf{1}' c_t - Z_t^N, \Sigma_t\right) dt + \Sigma_t^{N\prime} dB_t, \quad Z_T^N = C_t - \mathbf{1}' c_T + \zeta^N, \end{aligned}$$

for some $f^i : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{N\times d} \to \mathbb{R}$. For example, in a class of principal-agent problems with a single principal (let agent-N represent the principal here) and agents $1, \ldots, N-1$, the principal's problem is to choose the pay processes c to maximize the utility of the principal's (i.e., $\beta = \boldsymbol{e}_N$) cash-flow process $C - \mathbf{1}'c$, which is what remains of C

after the agents are paid, subject to the "participation constraint" $Z_0^i \geq K^i$, $i = 1, \ldots, N-1$ (and $K^N = -\infty$). The agents' cumulative actions/effort changes the measure, which adds a dependence of each aggregator on the other agents' utility diffusion processes.

The key to the tractability of the TI class is the next lemma which shows that the adjoint processes ε must lie in the null space of M'. When $k = N - 1$, this implies that λ defined in (4.6) is a constant vector.

Lemma 4.1. *In the TI class, at the optimum c we have*

$$\varepsilon_t' M = 0, \quad t \in [0, T], \tag{4.10}$$

where ε_t satisfies (3.3) with $\varepsilon_0 = \beta + \kappa$.

Proof. On the one hand, letting $\varepsilon_0 = \beta + \kappa$, Theorem 4.1 implies

$$\lim_{\alpha \downarrow 0} \frac{\varepsilon_t' \{Z_t(c + \alpha v) - Z_0(c)\}}{\alpha} = 0.$$

On the other hand, by quasilinearity (4.8), the left-hand side equals $\varepsilon_t' M v$. Therefore $\varepsilon_t' M v = 0$ for every $v \in \mathbb{R}^k$ and $t \in [0, T]$. □

Lemma 5.2 implies the following necessary condition on M for an optimum to exist, which we assume throughout the rest of this section.[1]

Condition 4.1.

$$v' M = 0 \text{ for some } v \geq \beta. \tag{4.11}$$

[1]By Farkas' lemma, Condition 4.1 rules out the existence of any (Pareto improving) fixed consumption increment $x \in \mathbb{R}^k$ satisfying $Mx \in \mathbb{R}_+^k$ and $\beta' M x > 0$; that is, by quasilinearity (4.8), an increment that reduces no agent's utiilty but strictly improves at least one's.

Remark 4.1. By Condition 4.1 we can choose some $\hat{v} \geq \beta$ satisfying $\hat{v}'M = 0$; let $\hat{\alpha} = \min\{\alpha \geq 0 : \alpha\hat{v} \geq \beta\}$, and define $v = \hat{\alpha}\hat{v}$ (i.e., we choose the "smallest" v satisfying (4.11)). Supposing a solution to problem (4.1) exists, we get that $v^i > \beta^i$ implies that i^{th} constraint is binding. In Example 4.2, with M given by (4.9) and $\beta = \boldsymbol{e}_N$, then $v = \mathbf{1}$ and the first $N-1$ constraints must therefore (be nonvoid and) bind at any solution.

It follows from the next lemma that the existence of a unique solution to problem (4.1) implies that in the TI class at most $N-k$ constraints in (4.1) are nonbinding at the optimum. Note that the Lemma applies even with void constraints, implying that there cannot be a unique solution with more than $N-k$ void constraints.

Lemma 4.2. *Under the TI aggregators class, if a solution to* (4.1) *exists, and if more than* $N-k$ *constraints are nonbinding, the solution is not unique.*

Proof. Let $\hat{c}$ be a solution to (4.1) and suppose exactly $n \leq N$ constraints in (4.1) are nonbinding; without loss of generality assume that these correspond to $i = 1, \ldots, n$. That is, $Z_t^i(\hat{c}) > K^i$, $i = 1, \ldots, n$; and $Z_t^i(\hat{c}) = K^i$, $i = n+1, \ldots, N$. Decompose $M = \left[M^{a\prime}, M^{b\prime}\right]'$ where $M^a \in \mathbb{R}^{n \times k}$ and $M^b \in \mathbb{R}^{(N-n)\times k}$; similarly, decompose $Z = \left[Z^{a\prime}, Z^{b\prime}\right]'$, $\beta = \left[\beta^{a\prime}, \beta^{b\prime}\right]'$, $K = \left[K^{a\prime}, K^{b\prime}\right]'$ and $\varepsilon_0 = \left[\varepsilon_0^{a\prime}, \varepsilon_0^{b\prime}\right]'$. Defining $\tilde{M} \in \mathbb{R}^{(N-n+1)\times k}$ by $\tilde{M} = \begin{pmatrix} \beta^{a\prime}M^a \\ M^b \end{pmatrix}$, then nonuniqueness is implied if $\text{rank}\left(\tilde{M}\right) < k$. This follows because then there is a $x \in \mathbb{R}^k$ satisfying $\tilde{M}x = 0$, and therefore (using (4.8)) $Z_t^b(\hat{c} + \alpha x) - Z_t^b(\hat{c}) = \alpha M^b x = 0$ and $\beta^{a\prime} Z_t^a(\hat{c} + \alpha x) - \beta^{a\prime} Z_t^a(\hat{c}) = \alpha\beta^{a\prime}M^a x = 0$. Choose $\alpha \in \mathbb{R}$ sufficiently small that $Z_t^a(\hat{c} + \alpha x) = Z_t^a(\hat{c}) + \alpha M^a x \geq K^a$. From $\varepsilon_0^a = \beta^a$ (from (4.2) and the supposition that the first n constraints are nonbinding) we get that $\varepsilon_0' M = 0$ (which is implied by optimality of $\hat{c}$ and (4.10)) is equivalent to $\left(1, \varepsilon_0^{b\prime}\right)\tilde{M} = 0$, which implies $\text{rank}\left(\tilde{M}\right) \leq N - n$. Therefore $n > N - k$ implies $\text{rank}\left(\tilde{M}\right) < k$ and therefore nonuniqueness. $\square$

The main result of this section is Theorem 4.2 below, which provides a sufficiency proof together with a method for constructing a solution to the problem (4.1) under TI aggregators. We first provide a brief sketch of the solution method based on the results of Theorem 4.1 and Lemma 5.2. But we will see that the conditions required for the proof of Theorem 4.2 are considerably weaker than the sufficiency conditions for Theorem 4.1.

We assume throughout the rest of this section that $k = N - 1$, in which case a unique solution implies at most one nonbinding constraint; the key simplification in this case is that the null space of M', in which ε_t lies, is one dimensional (that is, λ is a constant vector). In light of Lemma 4.2 we rearrange the equations so that $K^i \in \mathbb{R}$, $i = 1, \ldots, N-1$, and $K^N = -\infty$. Therefore if a unique optimum exists, the first $N-1$ constraints bind.

Choose v as in Remark 4.1. Defining

$$Y_t = v' Z_t, \quad x_t = Mc_t - Z_t, \quad t \in [0, T], \tag{4.12}$$

we have the FOC condition, from (4.2) and (4.10).

$$v' \psi_x \left(t, x_t, \Sigma_t\right) M = 0, \quad t \in [0, T]. \tag{4.13}$$

Assuming invertibility, the $N-1$ conditions in (4.13), and the identity $Y_t = -v' x_t$, which is implied by the identities in (4.12), together imply

$$x_t = \phi \left(t, Y_t, \Sigma_t\right), \quad t \in [0, T],$$

for some $\phi : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{N \times d} \to \mathbb{R}^N$.

A constant λ also implies a zero diffusion term in (4.7):

$$\sum_{j=1}^{N} v^j \left\{ \psi^j_{\Sigma^i}\left(t, \phi\left(t, Y_t, \Sigma_t\right), \Sigma_t\right) - \left(\frac{v^i}{v^1}\right) \psi^j_{\Sigma^1}\left(t, \phi\left(t, Y_t, \Sigma_t\right), \Sigma_t\right) \right\} = 0, \qquad (4.14)$$

$$i = 2, \ldots, N.$$

The restrictions (4.14) together with the identity $v'\Sigma_t = \Sigma_t^{Y\prime}$ can, assuming invertibility, allow us to obtain $\Sigma_t = \theta\left(t, Y_t, \Sigma_t^Y\right)$ for some $\theta : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{N \times d}$. We then solve the BSDE for $\left(Y, \Sigma^Y\right)$:

$$dY_t = -v'\psi\left(t, \phi\left(t, Y_t, \theta\left(t, Y_t, \Sigma_t^Y\right)\right), \theta\left(t, Y_t, \Sigma_t^Y\right)\right) dt + \Sigma_t^Y dB_t, \quad Y_T = v'\zeta. \qquad (4.15)$$

The solution for Σ^Y gives us the diffusion coefficients of Z. We solve the forward equations corresponding to the binding constraints:

$$dZ_t^i = -\psi^i\left(t, \phi\left(t, Y_t, \theta\left(t, \Sigma_t^Y\right)\right), \theta\left(t, \Sigma_t^Y\right)\right) dt + \Sigma_t^{i\prime} dB_t, \quad Z_0^i = K^i,$$
$$i = 1, \ldots, N-1.$$

For a vector Z (matrix M), we denote $Z^{(-i)}(M^{(-i)})$ to be the vector (matrix) with ith element (row) removed. We solve for optimal consumption $\hat{c}$ from $Z^{(-N)}$ in (4.18) below.

The solution method is made more transparent and simple in the following theorem.

Theorem 4.2. *Suppose, for all* $t \in [0,T]$*,*

$$\left(\hat{x}_t, \hat{\Sigma}_t\right) = \arg \max_{(x,\Sigma)\in\mathbb{R}^N \times \mathbb{R}^{N\times d}} v'\psi\left(t, x, \Sigma\right) \qquad (4.16)$$

subject to

$$v'x = -Y_t, \quad v'\Sigma = \Sigma_t^{Y\prime},$$

and $\left(Y, \Sigma^Y\right)$ *uniquely solves the BSDE*

$$dY_t = -v'\psi\left(t, \hat{x}_t, \hat{\Sigma}_t\right) dt + \Sigma_t^{Y\prime} dB_t, \quad Y_T = v'\zeta. \tag{4.17}$$

Then the optimal policy is

$$\begin{aligned}\hat{c}_t &= \left(M^{(-N)}\right)^{-1}\left(Z_t^{(-N)} + \hat{x}_t^{(-N)}\right), \quad t \in [0, T), \\ \hat{c}_T &= \left(M^{(-N)}\right)^{-1}\left(Z_T^{(-N)} - \zeta^{(-N)}\right),\end{aligned} \tag{4.18}$$

where $Z^{(-N)}$ *solves the forward SDE system*

$$dZ_t^i = -\psi^i\left(t, \hat{x}_t, \hat{\Sigma}_t\right) dt + \hat{\Sigma}_t^i dB_t, \quad Z_0^i = K^i, \quad i = 1, \ldots, N-1. \tag{4.19}$$

Furthermore, the optimal objective function is $\beta' Z_0(\hat{c}) = Y_0 - (v - \beta)' K$.

We first prove an envelope-theorem type result that implies Lipschitz continuity of the drift of Y under uniform Lipschitz continuity of $\psi^i(\omega, t, x, \Sigma)$ in (x, Σ) for all i. Note that the uniform Lipschitz condition is weaker than the assumption of uniformly bounded derivatives of ψ^i assumed in Condition 3.1.

Lemma 4.3. *If* $\psi^i(t, \cdot)$ *is uniformly Lipschitz for* $i = 1, \ldots, N$*, then* $\mu^Y(t, \cdot)$ *defined by*

$$\mu^Y\left(\omega, t, Y, \Sigma^Y\right) = \max_{(x,\Sigma) \in \mathbb{R}^N \times \mathbb{R}^{N\times d}} v'\psi(\omega, t, x, \Sigma) \tag{4.20}$$
$$\text{subject to } v'x = -Y \text{ and } v'\Sigma = \Sigma^{Y\prime},$$

is uniformly Lipschitz.

Proof. For simplicity of notation we will omit the (ω, t) arguments. By the uniform Lipschitz property for ψ^i there exists a $C_1 \in \mathbb{R}_+$ such that, for each $i = 1, \ldots, N$,

$$\left|\psi^i\left(\tilde{x}, \tilde{\Sigma}\right) - \psi^i(x, \Sigma)\right| \leq C_1 \left\{\|\tilde{x} - x\| + \left\|\tilde{\Sigma} - \Sigma\right\|\right\}$$
$$\text{for all } (\omega, t) \in \Omega \times [0, T], \tilde{x}, x \in \mathbb{R}^N \text{ and } \tilde{\Sigma}, \Sigma \in \mathbb{R}^{N \times d}.$$

Fix (ω, t), and choose any $\left(Y, \Sigma^Y\right)$ and $\left(\tilde{Y}, \tilde{\Sigma}^Y\right)$ (both in $\mathbb{R} \times \mathbb{R}^d$) and suppose the maximizing arguments in (4.20) are (x, Σ) and $\left(\tilde{x}, \tilde{\Sigma}\right)$, respectively (both are in $\mathbb{R}^N \times \mathbb{R}^{N \times d}$). Denote

$$\mu^Y\left(Y, \Sigma^Y\right) = v'\psi(x, \Sigma), \quad \mu^Y\left(\tilde{Y}, \tilde{\Sigma}^Y\right) = v'\psi\left(\tilde{x}, \tilde{\Sigma}\right)$$

(and note that $v'x = -Y_t$, $v'\tilde{x} = -\tilde{Y}_t$, $v'\Sigma = \Sigma_t^{Y\prime}$, $v'\tilde{\Sigma} = \tilde{\Sigma}_t^{Y\prime}$). Choose i corresponding to some $v_i > 0$. Because (x, Σ) maximizes (4.20) for $\left(Y, \Sigma^Y\right)$ and $\left(\tilde{x}, \tilde{\Sigma}\right)$ for $\left(\tilde{Y}, \tilde{\Sigma}^Y\right)$ we have

$$\mu^Y\left(\tilde{Y}, \tilde{\Sigma}^Y\right) = v'\psi\left(\tilde{x}, \tilde{\Sigma}\right) \geq v'\psi\left(x + \frac{1}{v_i}\mathbf{e}_i\left\{Y - \tilde{Y}\right\}, \Sigma + \frac{1}{v_i}\mathbf{e}_i\left\{\Sigma^{Y\prime} - \tilde{\Sigma}^{Y\prime}\right\}\right),$$
$$\mu^Y\left(Y, \Sigma^Y\right) = v'\psi(x, \Sigma) \geq v'\psi\left(\tilde{x} + \frac{1}{v_i}\mathbf{e}_i\left\{\tilde{Y} - Y\right\}, \tilde{\Sigma} + \frac{1}{v_i}\mathbf{e}_i\left\{\tilde{\Sigma}^{Y\prime} - \Sigma^{Y\prime}\right\}\right).$$

By Lipschitz continuity there exists a constant $C_2 \in \mathbb{R}_+$, independent (ω, t), such that

$$v'\psi\begin{pmatrix} x + \frac{1}{v_i}\mathbf{e}_i\left\{Y - \tilde{Y}\right\}, \\ \Sigma + \frac{1}{v_i}\mathbf{e}_i\left\{\Sigma^{Y\prime} - \tilde{\Sigma}^{Y\prime}\right\} \end{pmatrix} \geq v'\psi(x, \Sigma) - C_2\left\{\left|\tilde{Y} - Y\right| + \left\|\tilde{\Sigma}^{Y\prime} - \Sigma^{Y\prime}\right\|\right\},$$
$$v'\psi\begin{pmatrix} \tilde{x} + \frac{1}{v_i}\mathbf{e}_i\left\{\tilde{Y} - Y\right\}, \\ \tilde{\Sigma} + \frac{1}{v_i}\mathbf{e}_i\left\{\tilde{\Sigma}^{Y\prime} - \Sigma^{Y\prime}\right\} \end{pmatrix} \geq v'\psi\left(\tilde{x}, \tilde{\Sigma}\right) - C_2\left\{\left|\tilde{Y} - Y\right| + \left\|\tilde{\Sigma}^{Y\prime} - \Sigma^{Y\prime}\right\|\right\}.$$

Combining the results yields

$$\left|\mu^Y\left(Y, \Sigma^Y\right) - \mu^Y\left(\tilde{Y}, \tilde{\Sigma}^Y\right)\right| \leq C_2\left\{\left\|\tilde{Y} - Y\right\| + \left\|\tilde{\Sigma}^{Y\prime} - \Sigma^{Y\prime}\right\|\right\}.$$

□

We now prove Theorem 4.2.

Proof of Theorem 4.2. Suppose $\left(\hat{x}, \hat{\Sigma}\right)$ solve (4.16) and let $\left(Y, \Sigma^Y\right)$ solve the BSDE (5.36). The existence and uniqueness of the solution is implied by Lemma 4.2. Let $Z^{(-N)}$ and $\hat{c}$ be computed as in (4.19) and (4.18), and define $Z^N = \left(Y - v^{(-N)\prime} Z^{(-N)}\right)/v^N$. It is straightforward to confirm that $\left(Z, \hat{\Sigma}\right)$ so constructed solves the BSDE system

$$dZ_t = -\psi\left(t, M\hat{c}_t - Z_t, \hat{\Sigma}_t\right) dt + \hat{\Sigma}_t dB_t, \quad Z_T = M\hat{c}_T + \zeta. \tag{4.21}$$

Now consider any $\tilde{c} \in \mathcal{C}$ and let $\left(\tilde{Z}, \tilde{\Sigma}\right)$ denote the solution to the BSDE (4.21), with $\left(\tilde{Z}, \tilde{\Sigma}, \tilde{c}\right)$ replacing $\left(Z, \hat{\Sigma}, \hat{c}\right)$, and define $\tilde{x}_t = M\tilde{c}_t - \tilde{Z}_t$. Defining $\tilde{Y}_t = v'\tilde{Z}_t$ and $\tilde{\Sigma}_t^{Y\prime} = v'\tilde{\Sigma}_t$, and letting $\tilde{\Sigma}_t^{(-N)}$ denote the first $N-1$ rows of $\tilde{\Sigma}_t$, then $\left(\tilde{Y}, \tilde{\Sigma}^Y\right)$ solves the BSDE

$$d\tilde{Y}_t = -\left\{v'\psi\left(t, \tilde{x}_t, \tilde{\Sigma}_t^{(-N)}, \left\{\tilde{\Sigma}_t^{Y\prime} - \sum_{i=1}^{N-1} v^i \tilde{\Sigma}^i\right\}/v^N\right)\right\} dt + \tilde{\Sigma}_t^{Y\prime} dB_t, \quad \tilde{Y}_T = v'\zeta. \tag{4.22}$$

By (4.16) we have

$$v'\psi\left(t, \hat{x}_t, \hat{\Sigma}_t\right) = p_t + v'\psi\left(t, \tilde{x}_t, \tilde{\Sigma}_t^{(-N)}, \left\{\Sigma_t^{Y\prime} - \sum_{i=1}^{N-1} v^i \tilde{\Sigma}^i\right\}/v^N\right), \quad t \in [0, T],$$

for some nonnegative process p, and therefore

$$dY_t = -\left\{p_t + v'\psi\left(t, \tilde{x}_t, \tilde{\Sigma}_t^{(-N)}, \left\{\Sigma_t^{Y\prime} - \sum_{i=1}^{N-1} v^i \tilde{\Sigma}^i\right\}/v^N\right)\right\} dt + \Sigma_t^{Y\prime} dB_t, \tag{4.23}$$

$$Y_T = v'\zeta.$$

The comparison lemma of [17] applied to (4.23) and (4.22) implies $Y_0 \geq \tilde{Y}_0$ and therefore (because constraints $1, \ldots, N-1$ are binding) $Z_0^N(\hat{c}) \geq Z_0^N(\tilde{c})$. Because this holds for all $\tilde{c} \in \mathcal{C}$, $\hat{c}$ must be optimal. $\square$

Constructing a solution essentially amounts to maximizing a linear combination of drift terms. The solution $\left(\hat{x}_t, \hat{\Sigma}_t\right)$, if it exists, is shown in the Lemma above to be a Lipschitz-continuous function of $\left(Y_t, \Sigma_t^Y\right)$; the existence and uniqueness of the BSDE solution $\left(Y, \Sigma^Y\right)$ follows from standard results.

The following example illustrates the construction of a solution.

Example 4.3. Suppose agent i's utility process satisfies the TI specification[2]

$$dZ_t^i = -\left\{h^i(t, x_t) - \frac{1}{2}\sum_{j=1}^N \Sigma_t^j Q_t^{ij} \Sigma_t^{j\prime}\right\} dt + \Sigma_t^{i\prime} dB_t, \quad Z_T^i = M^i c_T^i + \zeta^i, \quad (4.24)$$

$$i = 1, \ldots, N,$$

where $x_t = Mc_t - Z_t$ and where $Q^{ij} \in \mathcal{L}\left(\mathbb{R}^{d\times d}\right)$ is assumed bounded, symmetric and positive definite for all (w, t) and every i, j. Defining $h(t,x) = \left[h^1(t,x), \ldots, h^N(t,x)\right]'$ then (assuming the argmax is well defined)

$$\hat{x}_t = \arg\max_{x\in\mathbb{R}^N} \sum_{i=1}^N v^i h^i(t,x), \quad \text{subject to} \quad v'x = -Y_t,$$

has a solution of the form $\hat{x}_t = \phi(t, Y_t)$ for some $\phi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}^N$. Defining

$$\bar{Q}_t^j = \sum_{i=1}^N v^i Q_t^{ij}, \quad Q_t^Y = \left(\sum_{j=1}^N \left(v^j\right)^2 \left(\bar{Q}_t^j\right)^{-1}\right)^{-1}, \quad t \in [0,T],$$

[2] This example violates our Lipschitz continuity condition, but existence and uniqueness results of [4] can be used here.

then $\hat{\Sigma}$ in (4.16) is

$$\hat{\Sigma}_t^{i\prime} = v^i \left(\bar{Q}_t^i\right)^{-1} Q_t^Y \Sigma_t^Y, \quad i = 1, \ldots, N,$$

and the BSDE (4.15) for $\left(Y, \Sigma^Y\right)$ becomes

$$dY_t = -\left\{v' h\left(t, \phi\left(t, Y_t\right)\right) - \frac{1}{2} \Sigma_t^{Y\prime} Q_t^Y \Sigma_t^Y\right\} dt + \Sigma_t^{Y\prime} dB_t, \quad Y_T = v'\zeta. \tag{4.25}$$

4.4 Pareto optimality under linked recursive utility

In this section we use Theorem 4.1 to characterize Pareto optimal allocations with linked recursive preferences. We fix the consumption space $\mathcal{C} = \mathcal{L}_2\left(\mathbb{R}_{++}^N\right)$, which Example 3.2 shows is extended convex, as well as the aggregate consumption process $C \in \mathcal{L}_2\left(\mathbb{R}_{++}\right)$.

Definition 4.2. *A consumption plan* $c \in \mathcal{C}$ *is called feasible if* $\sum_{i=1}^{N} c_t^i = C_t$. *A feasible allocation* c *is Pareto optimal if there is no feasible allocation* $\tilde{c}$ *such that* $Z_0^i(c) \leq Z_0^i(\tilde{c}), 1 \leq i \leq N$, *with strict inequality for at least one* i.

For any nonzero set of weights $\beta \in \mathbb{R}_+^N$ we say (as in [14]) that the consumption plan c is β-efficient if it solves the following optimization problem:

$$\max_{c \in \mathcal{C}} \beta' Z_0(c) \text{ subject to } \sum_{i=1}^{N} c_t^i = C_t \quad t \in [0, T]. \tag{4.26}$$

It is well known that under monotonicity and concavity of the utility functions $Z_0^i(\cdot)$, Pareto optimality is equivalent β-efficiency. In the unlinked case, with each agent i's aggregator a function of only i's consumption, utility and diffusion, concavity of the aggregator and monotonicity of the aggregator in consumption imply concavity and monotonicity of $Z_0^i(\cdot)$ (see [13] for the SDU case). In the linked case, however, current comparison theorems impose additional restrictions on the aggregators to obtain these properties:

Condition 4.2. *a)* $E\left\{sup_{t\leq T}\|\Phi(t,0,0,0))\|^2\right\}<\infty$. *b) (Quasi-monotonicity*[3]*) Fix any* $(\omega,t,c)\in\Omega\times[0,T]\times\mathbb{R}^k$*;* $z_1,z_2\in\mathbb{R}^N$*, and* $\Sigma_1,\Sigma_2\in\mathbb{R}^{N\times d}$*. Then, for each* $k=1,\ldots,N$,

$$\Phi^k(\omega,t,c,z_1,\Sigma_1)\leq\Phi^k(\omega,t,c,z_2,\Sigma_2)$$

if[4] $\Sigma_1^k=\Sigma_2^k, z_1^k=z_2^k$ *and* $z_1\leq z_2$.

Lemma 4.4. *Suppose Condition 4.2 holds.*

a) If $\Phi^i(\omega,t,\cdot)$ *is a concave function for all* (ω,t) *and* i *then* $Z_0^i(\cdot)$ *is concave for all* i.

b) If $\Phi^i(\omega,t,c,Z,\Sigma)$ *is nondecreasing in* c *for all* (ω,t,Z,Σ) *and* i*, then* $Z_0^i(c)$ *is nondecreasing in* c *for all* i.

Proof. a) Let $c_a,c_b\in\mathcal{C}$ and let (Z_a,Σ_a) and $\left(Z_b,\Sigma_b\right)$ denote the solutions to the BSDEs

$$dZ_j(t)=-\Phi\left(t,c_j(t),Z_j(t),\Sigma_j(t)\right)dt+\Sigma_j(t)\,dB_{t,}\quad Z_j(T)=\Phi\left(T,c_j(T)\right),$$
$$j\in\{a,b\}.$$

Let $\alpha\in(0,1)$ and define $\tilde{c}(t)=\alpha c_a(t)+(1-\alpha)c_b(t)$, $\tilde{Z}(t)=\alpha Z_a(t)+(1-\alpha)Z_b(t)$, $\tilde{\Sigma}(t)=\alpha\Sigma_a(t)+(1-\alpha)\Sigma_b(t)$. Then concavity of Φ^i for all i implies

$$d\tilde{Z}(t)=-\left\{\Phi\left(t,\tilde{c}(t),\tilde{Z}(t),\tilde{\Sigma}(t)\right)-p_t\right\}dt+\tilde{\Sigma}(t)\,dB_{t,}\quad \tilde{Z}(T)=\Phi(T,\tilde{c}(T))-p_T,$$

for some nonnegative process p. Also,

$$dZ_\alpha(t)=-\left\{\Phi\left(t,\tilde{c}(t),Z_\alpha(t),\Sigma_\alpha(t)\right)\right\}dt+\Sigma_\alpha(t)dB_{t,}\quad Z_\alpha(T)=\Phi\left(T,c\left(T\right)\right),$$

[3]The quasi-monotonicity assumption implies that $\Phi^k(\omega,t,c,z,.)$ can only depend on the k^{th} row of Σ.

[4]Recall that Σ^k denotes the kth row of Σ.

Using Condition 4.2 and Condition 3.1(a), the Comparison Theorem A.1 of [50] implies $Z_\alpha(0) \geq \tilde{Z}(0)$, which proves concavity of $Z_0^i(\cdot)$ for each i.

b) We again use the comparison theorem of [50]. □

With sufficient conditions for concavity of $Z^i(\cdot)$, the usual separating hyperplane argument shows that Pareto optimality implies β-efficiency.

Proposition 4.1. *Suppose Condition 4.2 and that $\Phi^i(t, \omega, \cdot)$ is concave for all i. If c is Pareto optimal then c is also β-efficient for some $\beta \in \mathbb{R}_+^N$.*

The converse of the proposition is trivial if $\beta \in \mathbb{R}_{++}^N$. However, if $\beta \in \mathbb{R}_+^N$ but not strictly positive, we must rely on the strong monotonicity conditions assumed for Lemma 4.4(b).

Necessary and sufficient conditions for β-efficiency are obtained from Theorem 4.1.

Proposition 4.2. *a) If $c \in \mathcal{C}$ is β-efficient then*

$$\varepsilon_t' \Phi_{c^1}(t, c_t, Z_t, \Sigma_t) = \varepsilon_t' \Phi_{c^2}(t, c_t, Z_t, \Sigma_t) = \cdots = \varepsilon_t' \Phi_{c^N}(t, c_t, Z_t, \Sigma_t), \quad t \in [0, T], \tag{4.27}$$

where ε is the solution of SDE (3.3) with $\varepsilon_0 = \beta$.

b) Suppose $\beta \in \mathbb{R}_+^N \setminus 0$, $\Phi^i(t, \omega, \cdot)$ is concave and $\Phi^j_{Z^i}(t) \geq 0$ for all $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$. If (4.27) holds and $c \in \mathcal{C}$, then c is β-efficient.

Proof. We apply Theorem 4.1(a) after substituting $c^N = C - \sum_{i=1}^{N-1} c^i$. Then $\varepsilon_t' \Phi_C(t) = 0$ in (4.2) is equivalent to

$$\varepsilon_t' \frac{\partial \Phi(t, c_t, Z_t, \Sigma_t)}{\partial c^i} = \varepsilon_t' \left\{ \Phi_{c^i}(t, c_t, Z_t, \Sigma_t) - \Phi_{c^N}(t, c_t, Z_t, \Sigma_t) \right\} = 0,$$
$$i \in \{1, 2, \ldots, N-1\}.$$

□

Corollary 4.1. *Suppose for each i, Φ^i depends on c^i (the agent's own consumption) but not on other agents' consumption. Then Proposition 4.2 holds with condition (4.27) replaced by*

$$\varepsilon_t^1 \Phi_{c^1}^1(t, c_t^1, Z_t, \Sigma_t) = \varepsilon_t^2 \Phi_{c^2}^2(t, c_t^2, Z_t, \Sigma_t) = \cdots = \varepsilon_t^N \Phi_{c^N}^N(t, c_t^N, Z_t, \Sigma_t), \quad t \in [0, T]\,. \tag{4.28}$$

The following example obtains a β-efficient allocation for a simple quadratic aggregator.[5]

Example 4.4 (quadratic aggregator)**.** Suppose

$$\Phi^i\,(t, c, Z, \Sigma) = -\frac{1}{2}\,(c-p)'\,Q^i\,(c-p) - q^{i\prime} Z, \quad t < T, \quad i = 1, \ldots, N,$$

where $Q^i \in \mathbb{R}^{N\times N}$ is symmetric and positive definite, $q^i = \left(q_1^i, \ldots, q_N^i\right)' \in \mathbb{R}^N$ and $p \in \mathbb{R}^N$. The adjoint processes satisfy the linear SDE

$$d\varepsilon_t^i = \sum_{j=1}^N \varepsilon_t^j q_i^j\, dt, \quad \varepsilon_0 = \beta, \quad i = 1, \ldots, N. \tag{4.29}$$

Letting Q_j^i denote jth row of Q^i, Proposition 4.2(a) gives the FOCs

$$\sum_{j=1}^N \varepsilon_t^j Q_1^j\,(c_t - p) = \cdots = \sum_{j=1}^N \varepsilon_t^j Q_N^j\,(c_t - p)\,.$$

These $N-1$ inequalities together with the constraint $\mathbf{1}' c_t = C_t$ can be used to solve for c.

---

[5] In this example and Example 4.6 below we assume a linear dependence of the aggregator on Z to obtain a simple expression for ε. If we generalize to the additively-separable form

$$\Phi^i\,(t, c, Z, \Sigma) = -\frac{1}{2}\,(c-p)'\,Q^i\,(c-p) + g^i\,(t, Z, \Sigma)$$

(and analogously for Example 4.6), we obtain the same expression for c, but an adjoint ε, from equation (3.3), with coefficients that depends on the solution of the BSDE (3.2).

For example if Q^i =diag$\left(\bar{q}_1^i, \ldots, \bar{q}_N^i\right)$ for each i, and defining $\alpha_i(t) = \sum_{j=1}^N \varepsilon_t^j \bar{q}_i^j$, then β-efficiency of c implies

$$c^i(t) - p^i = \left(\alpha_i(t) \sum_{j=1}^N \frac{1}{\alpha_j(t)}\right)^{-1} \left(C_t - \mathbf{1}'p\right) \quad i = 1, \ldots, N.$$

By Proposition 4.2(b), if $q_j^i \geq 0$ for all $i \neq j$ then we have sufficiency of the solution.

We show in the next example that the problem of β-efficiency under TI preferences (and no positivity constraint on consumption) results in either no (finite) solution, or an infinite number of solutions.

Example 4.5 (TI preferences)**.** We relax the requirement of strictly positive consumption and let $\mathcal{C} = \mathcal{L}_2\left(\mathbb{R}^N\right)$ and $C \in \mathcal{L}_2(\mathbb{R})$. We now apply results in Section 5.5 to the control to the $k = N - 1$ dimensional control $c^{(-N)}$ (the first $N - 1$ elements of c). If each agent's aggregator depends only on his/her own consumption, then M is given by (4.9), but we make no such assumption here. There are two cases:

a) If $\beta' M \neq 0$ then no solution to (4.26) exists. This is seen by applying the quasilinearity property (4.8) to $\alpha = M'\beta$ to get $\beta' Z_t\left(c^{(-N)} + M'\beta\right) = \beta' Z_t\left(c^{(-N)}\right) + \left\|M'\beta\right\|^2$ for any $c \in \mathcal{C}$. Thus no allocation can be β-efficient.

b) If $\beta' M = 0$ then $\beta' Z_t\left(c^{(-N)} + \alpha\right) = \beta' Z_t\left(c^{(-N)}\right)$ for all $\alpha \in \mathbb{R}^{N-1}$, and so if a solution exists it cannot be unique. A solution exists if there is a solution to (4.16). This is shown by imposing $(N-1)$ arbitrary finite constraints on all but agent i, letting $K^i = -\infty$, and applying Theorem 4.2 to construct a solution. The optimum, $Y_0 = \beta' Z_0(\hat{c})$, is independent of these constraints (note that $\beta' M = 0$ implies that $v = \beta$). We therefore get an infinite number of solutions to the β-efficiency problem, one for each arbitrary set of constraints.

4.5 Optimal consumption with altruism

In this section we examine the optimal consumption and portfolio problem of an agent when the agent's aggregator depends on the other agents' consumption, utility and utility diffusion. That is, each agent is altruistic in the sense that he/she cares about the consumption, utility and risk levels of the other agents. Although the consumption processes of the other agents are given, the consumption process chosen by agent i can impact the utility processes of the other agents, which feeds back into agent i's aggregator, making the problem nonstandard.

Throughout we assume that the dimension of the consumption plan (of all agents) is N, but the dimension of the control of agent i is one. Each agent i trades in a complete securities market, which contains a money-market security with short-term interest rate process $r \in \mathcal{L}(\mathbb{R})$, and a set of d risky assets. We denote $\phi^i \in \mathcal{L}\left(\mathbb{R}^d\right)$ the trading plan of agent i with ϕ_t^i representing the vector of time-t market values of the risky asset investments. Let $\mu^R \in \mathcal{L}_1\left(\mathbb{R}^d\right)$ represent the excess (above the riskless rate) instantaneous expected returns process of the risky assets, and $\sigma^R \in \mathcal{L}_2\left(\mathbb{R}^{d\times d}\right)$ the returns diffusion process, which is assumed to be invertible for all (ω, t). The planned consumption, trading and wealth for agent i is feasible if $c \in \mathcal{C}$ and the usual *budget equation* is satisfied:

$$dW_t^i = \left(W_t^i r_t + \phi_t^{i\prime}\mu_t^R - c_t^i\right)dt + \phi_t^{i\prime}\sigma_t^{R\prime}dB_t, \quad c_T^i = W_T^i, \tag{4.30}$$

as well as the integrability conditions (the latter is to rule out doubling-type strategies)

$$\int_0^t \left(\left|\phi_s^{i\prime}\mu_s^R\right| + \phi_s^{i\prime}\sigma^{R\prime}\sigma^R\phi_s^i\right)ds < \infty, \quad t \in [0,T),$$
$$E\left[\sup_{t\in[0,T]}\left\{\max\left(0, -W_t^i\right)\right\}^2\right] < \infty.$$

We can view the wealth process (4.30) as a forward equation, starting at an initial wealth level w_0^i with the terminal lump-sum balance W_T consumed at T; or we can define agent i's wealth process $W^i = W^i\left(c^i\right)$ as part of the pair $\left(W^i, \sigma^i\right)$ solving the BSDE

$$dW_t^i = \left(W_t^i r_t + \sigma_t^{i\prime}\eta_t - c_t^i\right)dt + \sigma_t^{i\prime}dB_t, \quad c_T^i = W_T^i; \tag{4.31}$$

where $\eta_t = \left(\sigma_t^{R\prime}\right)^{-1}\mu_t^R$ is the market price of risk, and the trading strategy financing c^i is $\phi_t^i = \left(\sigma_t^R\right)^{-1}\sigma_t^i$. Thus $W_t^i\left(c^i\right)$ represents the time-t cost of financing $\left\{c_s^i; s\in[t,T]\right\}$. We assume that η is bounded, and then by Novikov's condition there is a unique *state-price density* $\pi \in \mathcal{L}_2(\mathbb{R})$ satisfying

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t' dB_t, \quad \pi_0 = 1, \tag{4.32}$$

such that $W_t^i\left(c^i\right) = \frac{1}{\pi_t}E_t\left(\int_t^T c_s^i\pi_s ds + \pi_T c_T^i\right)$ for every $c^i \in \mathcal{L}_2(\mathbb{R})$ (see [17]). By linearity it follows that π is the gradient density of $W_0^i\left(c^i\right) = \left(\pi|c^i\right)$.

Agent i's problem is to choose a consumption process c^i to maximize utility subject to the wealth constraint, taking as given c^{-i}, the consumption processes of the other agents:

$$\max_{c^i: c\in\mathcal{C}} Z_0^i\left(c^i, c^{-i}\right) \text{ subject to } \left(\pi|c^i\right) \le w_0^i, \tag{4.33}$$

where $Z_0(c)$ is the initial utility specified by (3.2). The problem is nonstandard because of the possible dependence of agent i's aggregator, $\Phi^i(t)$, on $\left\{Z_t^j, \Sigma_t^j;\, j \ne i\right\}$. A perturbation in i's consumption plan can affect the other agent's utility and utility-diffusion processes, which in turn indirectly impacts agent i's utility process.

We can adapt Theorem 4.1 to the problem as follows.

Corollary 4.2. *a) (Necessity) If* $c \in \mathcal{C}$ *solves the problem* (4.33) *then there is some* $\kappa \in \mathbb{R}_+$

such that[6]

$$\varepsilon_0 = \boldsymbol{e}_i, \quad \Phi_{c^i}(t, c_t, Z_t, \Sigma_t)\varepsilon_t = \kappa\pi_t, \quad t \in [0, T],$$
$$\kappa \left\{ w_0^i - \left(\pi | c^i\right) \right\} = 0,$$

where (Z, Σ) *solve the BSDE* (3.2) *and* $\varepsilon \in \mathcal{L}\left(\mathbb{R}^N\right)$ *solves the SDE* (3.3).

b) (Sufficiency) Assume $\Phi^i(t, \cdot)$ *is a concave function,* $\Phi^j_{Z^i}(t) \geq 0$ *and* $\Phi^j_{\Sigma^i}(t) = 0$ *for all* $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$ *. If* (4.2) *holds then* c *is optimal.*

If the ith aggregator Φ^i depends only on Z_t^i and Σ_t^i (but can depend on the vector c_t), then $\varepsilon^j = 0$ for $j \neq i$, and the FOC reduces to the standard result for generalized recursive utility (see, for example, [18] and, in the SDU case, [15]):

$$\Phi^i_{c^i}(t, c_t, Z_t^i, \Sigma_t^i)\varepsilon_t^i = \kappa\pi_t$$
$$\text{where } \frac{d\varepsilon_t^i}{\varepsilon_t^i} = \Phi^i_{Z^i}\left(t, c_t, Z_t^i, \Sigma_t^i\right) dt + \Phi^i_{\Sigma^i}\left(t, c_t, Z_t^i, \Sigma_t^i\right)' dB_t, \quad \varepsilon_0^i = 1. \tag{4.34}$$

The following example applies Corollary 4.2 to a continuous-time version of [1].

Example 4.6. (Catching Up with Joneses) Letting $c^{-i} = \sum_{j \neq i} c^j$, and suppose

$$\Phi^i(t, c, Z, \Sigma) = u^i(t, c^i, c^{-i}) + q^{i\prime} Z, \quad t < T, \quad i = 1, \ldots, N, \tag{4.35}$$

where $q^i = \left(q_1^i, \ldots, q_N^i\right)' \in \mathbb{R}^N$ and $u^i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For example, the form

$$u^i(c^i, c^{-i}) = \frac{1}{1 - \gamma_i}(c^i)^{1-\gamma^i}(c^{-i})^{\mu^i}, \quad \mu^i \in \mathbb{R}, \gamma^i \in \mathbb{R}_{++} \tag{4.36}$$

(though with $q^i = 0$) is used in many papers including [1] and [22]. When $\mu_i > 0$ the marginal utility of i's consumption is increasing in the consumption of others, resulting in

[6] Recall that $\boldsymbol{e}_i$ is a length-N vector with one in the i^{th} position and zeros elsewhere.

higher consumption by i (the reverse holds for $\mu_i < 0$). To jointly solve for the agents' optimal consumption, we define the matrix $Q \in \mathbb{R}^{N \times N}$ with typical element $Q^{kj} = q_k^j$, and let $X \in \mathcal{L}\left(\mathbb{R}^{N \times N}\right)$ satisfy the SDE

$$dX_t = QX_t dt, \quad X_0 = I$$

(so that ε_t in Corollary 4.2 for agent i's problem is the ith column of X_t). Applying Corollary 4.2 we get the FOCs for optimality of c^i for agent i, $i = 1, \ldots, N$:

$$(c_t^i)^{-\gamma_i} \left(c_t^{-i}\right)^{\gamma_i \mu_i} X_t^{ii} + \sum_{j \neq i} \frac{\mu^j}{1 - \gamma^j} (c_t^j)^{1-\gamma^j} \left(c_t^{-j}\right)^{\mu^j - 1} X_t^{ji} = \kappa^i \pi_t,$$

$$\kappa^i \left\{w_0^i - \left(\pi | c^i\right)\right\} = 0, \quad w_0^i \geq \left(\pi | c^i\right), \quad \kappa^i \geq 0, \quad i = 1, \ldots, N.$$

4.6 Optimal portfolio with direct utility for wealth

We consider the portfolio maximization problem of a single agent in complete markets with an aggregator that depends on current wealth. Special cases include [49], which examines time-additive HARA utility, but with a linear combination of consumption and wealth replacing consumption in the aggregator; and [28], which examines (in discrete time) an aggregator that is a function of wealth and consumption. We first present the FOCs in the general recursive case, and consider some specializations. Example 4.7 considers the case where consumption and wealth enter the aggregator as a linear combination, as in [49], but with a general aggregator also dependent on utility and utility diffusion, and shows that the solution can be obtained by solving the problem without wealth dependence after modifying the short-rate process. Example 4.9 solves the optimal consumption/portfolio problem for a general homothetic class of recursive utility with wealth dependence and constrained trading.

We let $N = 2$ (two BSDEs) and $k = 1$, but for notational clarity let Z be one-dimensional and introduce the additional BSDE (4.31) representing the agent's wealth. The (scalar-valued) utility satisfies the BSDE

$$dZ_t(c) = -\Phi(t, c_t, Z_t, W_t, \Sigma_t)\,dt + \Sigma_t dB_t, \quad Z_T = \Phi\left(T, c_T, W_T\right).$$

where $\Phi : \Omega \times [0,T] \times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. The agent's problem is to maximize utility subject to the budget constraint:

$$\max_{c\in\mathcal{C}} Z_0(c) \text{ subject to } (\pi|c) \leq w_0. \tag{4.37}$$

We solve the problem with the following corollary to Theorem 4.1:

Corollary 4.3. *a) (Necessity) If $c \in \mathcal{C}$ solves the problem (4.37) then there is some $\kappa \in \mathbb{R}_+$ such that*

$$\begin{aligned} \Phi_c(t, c_t, Z_t, W_t, \Sigma_t) &= \varepsilon_t^2/\varepsilon_t^1, \quad t \in [0,T], \\ \kappa\left\{(\pi|c) - w_0\right\} &= 0, \quad (\pi|c) \leq w_0, \end{aligned} \tag{4.38}$$

where (Z,Σ) solves the BSDE (3.2), and $\varepsilon = \left(\varepsilon^1, \varepsilon^2\right)$ solves the SDE system[7]

$$\begin{aligned} \frac{d\varepsilon_t^1}{\varepsilon_t^1} &= \Phi_Z(t, c_t, Z_t, W_t, \Sigma_t)\,dt + \Phi_\Sigma(t, c_t, Z_t, W_t, \Sigma_t)\,dB_t, \quad \varepsilon_0^1 = 1, \\ d\varepsilon_t^2 &= -\left\{\varepsilon_t^2 r_t + \varepsilon_t^1 \Phi_W(t, c_t, Z_t, W_t, \Sigma_t)\right\}dt - \varepsilon_t^2 \eta_t' dB_t, \quad \varepsilon_0^2 = \kappa. \end{aligned}$$

b) (Sufficiency) Assume $\Phi(t,\cdot)$ is a concave function, and $\Phi_W(t) \geq 0$. If (4.38) holds then c is optimal.

The first adjoint process, ε^1, is the standard one for unlinked recursive utility as given

[7] Note the reversal in the sign of Φ_W, which follows because we apply Theorem 4.1 to $-W$ to get the correct inequality constraint.

in (4.34) (with $i = 1$). The dynamics of the second adjoint process, ε^2, are the same as the state-price density π in (4.32) after adjusting the short rate for the incremental impact of wealth on the aggregator, which is accomplished by replacing the short rate r with $r + \Phi_W \varepsilon^1/\varepsilon^2$ (assuming $\varepsilon^2 > 0$). Just as with a higher interest rate, $\Phi_W > 0$ has the effect of deferring more consumption to the future, reducing current consumption and increasing wealth.

Example 4.7. Suppose the aggregator depends only on a linear combination of consumption and wealth:[8]

$$\Phi(t, c_t, Z_t, W_t, \Sigma_t) = \psi\left(t, x_t, Z_t, \Sigma_t\right), \quad t \in [0, T), \quad \Phi(T, c_T, W_T) = \psi\left(T, x_T\right),$$

where $x_t = c_t + \delta W_t$, $t \in [0, T)$, for some $\delta > -1$, $x_T = c_T$, and $\psi(t, \cdot)$ is a concave function. Then the FOC (necessary and sufficient) is

$$\psi_x\left(t, x_t, Z_t, \Sigma_t\right) = \varepsilon_t^2/\varepsilon_t^1, \quad t \in [0, T].$$

Substituting $\Phi_W(t) = \delta\psi_x(t) = \delta\varepsilon_t^2/\varepsilon_t^1$, the dynamics of ε_t^2 simplify to

$$\frac{d\varepsilon_t^2}{\varepsilon_t^2} = -\left(r_t + \delta\right) dt - \eta_t' dB_t, \quad \varepsilon_0^2 = \kappa,$$

which are the same as the dynamics of the state-price density π defined in (4.32), but with an interest rate of $r + \delta$ instead of r.[9] The optimal consumption problem is therefore the same as in the case of recursive utility without wealth dependence, but with the interest rate changed from r to $r + \delta$, and the budget constraint changed from $(\pi|c) \le w_0\pi_0$ to

[8] Note that this specification does not fall within the TI class, which requires the time-t aggregator to depend only on a linear combination of (c_t, W_t, Z_t), and requires the terminal utility be a linear in c_T.

[9] That is, $\varepsilon_t^2 = e^{-\delta t}\kappa\pi_t$, $t \in [0, T]$. An alternative approach to the problem is to use an isomorphism as in [42].

$\left(\varepsilon^2|x\right) \leq w_0 \varepsilon_0^2$.

Example 4.8. Suppose the aggregator takes the form

$$\Phi(t, c_t, Z_t, \Sigma_t) = f(t, Z_t, \Sigma_t) + \frac{(c_t)^{1-\gamma} + \delta(W_t)^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$ controls the curvature of utility over both consumption and wealth, $\delta > 0$ is a scaling parameter, which allows us to control the "intensity" of the agents' direct wealth preference. Corollary 4.3 implies that optimal consumption satisfies

$$c_t = \left(\varepsilon_t^1 / \varepsilon_t^2\right)^{1/\gamma}.$$

The final example uses Theorem 4.2 in Section 5.5 to solve the optimal consumption and portfolio problem with homothetic preferences.

Example 4.9 (Homothetic wealth-dependent utility)**.** Suppose the homothetic specification

$$\frac{dU_t}{U_t} = -\left\{ g\left(t, \frac{c_t}{U_t}, \frac{W_t}{U_t}\right) + q\left(t, \sigma_t^U\right) \right\} dt + \sigma_t^{U\prime} dB_t, \quad U_T = C_T, \tag{4.39}$$

where $g : \Omega \times [0, T] \times \mathbb{R}_{++}^2 \to \mathbb{R}$, $q : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $g(t, \cdot)$ and $q(t, \cdot)$ are concave, and g satisfies the Inada conditions in the c_t/U_t argument. The special case with no dependence on W_t/U_t is examined in [43]. Epstein-Zin utility corresponds to the power form of g in (4.42) below with $\alpha = 0$ (no dependence on W_t/U_t) and $q\left(t, \sigma_t^U\right) = \delta \sigma_t^{U\prime} \sigma_t^U$ for some $\delta > 0$. We also relax the assumption of complete markets.

Defining the investment proportion process ψ by $\psi_t^i = \phi_t^i / W_t$, and the consumption-to-wealth ratio $\rho_t = c_t / W_t$, the budget equation (4.30) can be written

$$\frac{dW_t}{W_t} = \left(r_t - \rho_t + \psi_t' \mu_t^R\right) dt + \psi_t' \sigma_t^{R\prime} dB_t, \quad W_T = C_T. \tag{4.40}$$

Because the utility and wealth aggregators fall within the TI class (after transforming

W, U, and c to logs),[10] the problem can be solved using the dynamic programming approach of Section 5.5. We impose possible trading restrictions by assuming that $\psi_t \in K$, $t \in [0, T]$, for some convex set K. The homothetic form implies $U_t = \lambda_t W_t$ for some λ satisfying

$$\frac{d\lambda_t}{\lambda_t} = \mu_t^\lambda dt + \sigma_t^{\lambda\prime} dB_t, \quad \lambda_T = 1.$$

The optimality condition (from Theorem 4.2) is

$$-\mu_t^\lambda = \max_{x>0,\, y\in K} \left\{ r_t - x + y' \left(\mu_t^R + \sigma_t^{R\prime} \sigma_t^\lambda \right) + g\left(t, \frac{x}{\lambda_t}, \frac{1}{\lambda_t} \right) + q\left(t, \sigma_t^\lambda + \sigma_t^R y \right) \right\}, \tag{4.41}$$

with the optimal (ρ_t, ψ_t) representing the maximizing arguments. The additive separability of the aggregator implies that we can separately solve for ρ_t and ψ_t:

$$\begin{aligned}
\hat{\rho}_t &= \arg\max_{x>0} \left\{ g\left(t, \frac{x}{\lambda_t}, \frac{1}{\lambda_t} \right) - x \right\}, \\
\hat{\psi}_t &= \arg\max_{y\in K} \left\{ y' \left(\mu_t^R + \sigma_t^{R\prime} \sigma_t^\lambda \right) + q\left(t, \sigma_t^\lambda + \sigma_t^R y \right) \right\}.
\end{aligned}$$

Given these solutions, we obtain μ_t^λ as a function of σ_t^λ, and then solve for the BSDE for $\left(\lambda, \sigma^\lambda\right)$ to complete the solution.

If g has the power form

$$g(t, z_1, z_2) = \begin{cases} \beta_t \frac{1}{1-\gamma} \left\{ \left(z_1 z_2^\alpha \right)^{\frac{1-\gamma}{1+\alpha}} - 1 \right\} & \text{if } 1 \neq \gamma > 0, \\ \beta_t \left\{ \frac{1}{1+\alpha} \ln\left(\frac{c_t}{U_t} \right) + \frac{\alpha}{1+\alpha} \ln\left(\frac{W_t}{U_t} \right) \right\} & \text{if } \gamma = 1, \end{cases} \tag{4.42}$$

[10] Let $Z_t = [\ln(U_t), -\ln(W_t)]'$ and apply a logarithmic transformation to consumption. Then $M = (1, -1)'$, $v = (1, 1)'$ and $Y_t = v' Z_t = \ln(\lambda_t)$. As stated in Section 4.18, the dynamic programming approach in Theorem 4.2 extends easily to additional controls, such as the constrainted portfolio choice we introduce here.

with $\alpha + \gamma > 0$, we get

$$\hat{\rho}_t = \left(\frac{\beta_t}{1 + \alpha} \lambda_t^{\gamma - 1} \right)^{\left(\frac{1+\alpha}{\alpha+\gamma} \right)}.$$

In the log case ($\gamma = 1$) this simplifies to $\hat{\rho}_t = \beta_t / (1 + \alpha)$, which is invariant to λ (and therefore invariant to the dynamics of μ^R, r and σ^R), and decreasing in α, reflecting a desire to postpone consumption and increase wealth when more weight is placed on wealth in the aggregator.

5

Continuous Time Principal-Agent Problem

5.1 Introduction

We study the principal-agent problem with moral hazard in a continuous-time Brownian filtration with recursive preferences on the part of both principal and agent, and pay over the lifetime of the contract. Previous work has considered only additive utility, which, as is well known, arbitrarily links intertemporal substitution and risk aversion (see, for example, [19]). Yet time-additivity offers essentially no advantage in tractability because agent optimality induces recursivity to the principal's preferences even in the additive case. We allow both principal and agent preferences to be within the generalized recursive utility class, which was introduced in [34] to unify the stochastic differential utility (SDU) formulation of [13], and the multiple-prior formulation of [5]. Unlike the additive class, the recursive class allows distinct treatment of aversion to variability in consumption across states and across time. Furthermore, the class can accommodate source-dependent (domestic versus foreign, for example) risk aversion, differences in agent and principal beliefs, as well as first-order risk aversion (which imposes a higher penalty for small levels of risk) in addition to the standard second-order risk aversion.[1] Also, [47] shows that SDU, a special case of the recursive class, includes the robust control formulations of [2]; [24]; and [37].

In the principal-agent problem with moral hazard, a utility maximizing principal pays a compensation process to an agent in order to induce effort (which typically increases expected future cash flows). However the principal faces two constraints. First, because effort is assumed noncontractible, the contract must satisfy an incentive compatibility condition that the agent, faced with a particular compensation process, will choose effort that maximizes his/her own utility. Second, because the agent has employment opportunities elsewhere, the agent's initial utility must exceed some fixed amount. In the continuous-time Brownian version, first examined in [27], the impact of effort choice is typically modeled as an equivalent change of measure (that is, the agent's efforts change the probabilities of the states), which changes the drift of the driving Brownian processes. This is a convenient way to model, for example, the impact of effort on the growth rate of a cash flow process.

We derive necessary and sufficient conditions for both agent and principal optimality, and show that the first-order conditions (FOCs) for the principal's problem take the form of a forward-backward stochastic differential equation (FBSDE). The utility processes are backward systems, and these are coupled together with a forward equation which incor-

[1] See [44] and [48]. We consider only second-order risk aversion in this paper, but extensions to the first-order case (modeled by kinks in the aggregator) can be handled along the lines of [44].

porates both the impact of agent effort on principal utility and the agent's participation constraint. We also provide a dynamic programming proof of sufficiency.

When agent and principal preferences are translation invariant (TI), a class of preferences introduced in [44] as an extension of time-additive exponential utility, the system uncouples and dramatically simplifies to the solution of a single backward stochastic differential equation (BSDE). This BSDE can be interpreted as a subjective cash-flow present-value process, and incorporates a mixture of agent and principal preferences. Construction of a solution in the TI case is straightforward and the required technical conditions are less stringent. We illustrate with a number of examples with quadratic risk-aversion and effort penalties, and obtain closed-form solutions for some parametric examples, including Ornstein-Uhlenbeck and square-root cash-flow dynamics. In the quadratic class we obtain a simple sharing rule for the volatility the subjective present-value process. These sharing rules depend only on risk aversion and effort penalty processes, not on preferences for intertemporal substitution. In only very special cases do these volatility sharing rules imply linear sharing rules for the cash flows themselves.

As in [41], we consider a general (non-Markovian) Brownian setting, and derive first-order conditions for the agent and principal's problems. [41] use the martingale approach for stochastic control theory to solve for the first-order condition of optimality under exponential utility and terminal consumption only. Our paper considers generalized recursive preferences and lifetime and terminal consumption. Methodologically, we rely on a combination of the utility gradient approach and dynamic programming.

Many other papers have considered variations of the continuous-time principal-agent problem, but have been focused on particular applications in a Markovian setting. See, for example, [40] (the agent controls the drift of an output process with constant volatility), [11] (extends [10] to continuous setting), and [25] and [26] (the agent controls the drift of the firm's cash flow process with a binary effort choice).

[9] consider a general Brownian setting, but with agent and principal maximizing ex-

pected utility of lump-sum consumption. They provide a necessary condition for optimality in the general case as a solution of system of coupled FBSDEs and a maximal principle; these are also sufficient under regularity conditions. They obtain an essentially closed-form solution in the case of quadratic effort penalty (see Example 5.4 below).

5.2 Setup and Statement of the Problem

We will use the same setting as Chapter 3. The definition of utility from Chapter 3 will be assumed in this chapter. The agents and the principal's preferences will be assumed to follow generalized recursive utility.

For any subset S of Euclidean space, let $\mathcal{L}(S)$ denote the set of S-valued processes, and, for any $p \geq 1$,

$$\mathcal{L}_p^-(S) = \left\{ x \in \mathcal{L}(S) : \; E\left[\int_0^T \|x_t\|^p \, dt \right] < \infty \right\},$$
$$\mathcal{L}_p(S) = \left\{ x \in \mathcal{L}_p^-(S) : \; E\left[\|x_T\|^p \right] < \infty \right\},$$

where $\|x_t\|$ denotes Euclidean norm. Note that $\mathcal{L}_2(\mathbb{R})$ is a Hilbert space with the inner product

$$(x\,|y) = E\left[\int_0^T x_t y_t dt + x_T y_T \right], \quad x, y \in \mathcal{L}_2(\mathbb{R}).$$

Finally, define

$$\mathcal{E}^* = \left\{ x \in \mathcal{L}(\mathbb{R}) : \; E\left[\exp\left(\kappa \sup_{t\in[0,T]} |x_t| \right) \right] < \infty \text{ for all } \kappa > 0 \right\}.$$

We re-define the set of *consumption plans* as the set $\widetilde{\mathcal{C}} \subset \mathcal{L}_2(\boldsymbol{C})$ where $\boldsymbol{C} \subset \mathbb{R}$ (in typical applications, either $\boldsymbol{C} = \mathbb{R}$ or $\boldsymbol{C} = \mathbb{R}_{++}$). For any $c \in \widetilde{\mathcal{C}}$, we interpret c_t as a consumption rate for $t < T$, and c_T as lump-sum terminal consumption. Let $\widetilde{\mathcal{C}}^-$ denote the set of intermediate consumption plans (i.e. c_t, $0 \leq t < T$). We define the set of *effort plans* as $\widetilde{\mathcal{E}} = \{e \in \mathcal{L}_2^-(\boldsymbol{E})\}$ for some convex set $\boldsymbol{E} \subset \mathbb{R}^d$ (there is no lump-sum terminal effort).

The impact of agent effort is modeled as a change of probability measure. Let

$$Z_t^e = \exp\left(\int_0^t e_s' dB_s - \frac{1}{2}\int_0^t \|e_s\|^2 \, ds\right).$$

We assume throughout that Z^e is a martingale (equivalently, $EZ_T^e = 1$) for all $e \in \widetilde{\mathcal{E}}$. One of the well known condition for Z^e to be a martingale is that e satisfies Novikov's condition. Define the probability measure P^e (with expectation operator E^e) corresponding to effort e by

$$\frac{dP^e}{dP} = Z_T^e.$$

Girsanov's Theorem implies $B_t^e = B_t - \int\limits_0^t e_s ds$ is standard Brownian motion under P^e with respect to the filtration $\{\mathcal{F}_t : t \in [0, T]\}$.

Preferences are assumed to be in the generalized recursive utility class. Given the consumption stream $c \in \widetilde{\mathcal{C}}$ paid by the principal, and effort level $e \in \widetilde{\mathcal{E}}$ chosen by the agent, the agent's utility $U(c, e)$ is the first element of the pair $\left(U, \Sigma^U\right)$ assumed to uniquely satisfy the BSDE

$$dU_t = -F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t^e, \quad U_T = F\left(T, c_T\right). \tag{5.1}$$

The function $F : \Omega \times [0, T] \times \boldsymbol{C} \times \boldsymbol{E} \times \mathbb{R}^{1+d} \to \mathbb{R}$ is called the *aggregator* and is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{2+2d})$ measurable, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0, T]$. Utility of lump-sum terminal consumption depends on only Ω and c_T (there is no lump-sum terminal effort).

Agent effort is not contractible, but can be influenced by the principal through the consumption process paid to the agent. For any agent effort choice $e \in \widetilde{\mathcal{E}}$ and consumption stream $c \in \widetilde{\mathcal{C}}$ paid to the agent, the principal's utility $V(c, e)$ is the first element of the pair $\left(V, \Sigma^V\right)$ assumed to uniquely satisfy the BSDE

$$dV_t = -G\left(t, c_t, V_t, \Sigma_t^V\right) dt + \Sigma_t^{V\prime} dB_t^e, \quad V_T = G(T, c_T), \tag{5.2}$$

where $G : \Omega \times [0, T] \times \boldsymbol{C} \times \mathbb{R}^{1+d} \to \mathbb{R}$ is the principal's aggregator and is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{2+d})$ measurable. Terminal utility again depends on only Ω and c_T.

We will assume throughout this paper that F and G are concave and differentiable in c, e, U, V, Σ. Also $F_c > 0$ and $G_c < 0$ (the agent's consumption is paid by the principal).

Example 5.1 (Time-Additive Utility)**.** The case of time-additive preferences with quadratic penalty for agent effort corresponds to

$$F(t, c, e, U, \Sigma) = u_t(c) - \beta_t^U U - \frac{1}{2} q e^{\prime} e, \quad G(t, c, V, \Sigma) = v_t(X_t - c) - \beta_t^V V, \quad t < T$$
$$F(T, c_T) = u_T(c_T), \quad G(T, c_T) = v_T(X_T - c_T),$$

for some increasing and concave functions $v_t(\cdot)$ and $u_t(\cdot)$, some $q \in \mathbb{R}_{++}$, discount processes β^U and β^V, and cash-flow-rate process X satisfying

$$dX_t = \mu_t^X dt + \Sigma_t^{X\prime} dB_t,$$

with $\mu^X \in \mathcal{L}_1(\mathbb{R})$ and $\Sigma^X \in \mathcal{L}_2^-\left(\mathbb{R}^d\right)$. With sufficient integrability, agent utility satisfies

$$U_t = E_t^e \left\{ \int_t^T e^{-\int_t^s \beta_u^U du} \left(u_s(c_s) - \frac{1}{2} q e_s^{\prime} e_s \right) ds + e^{-\int_t^T \beta_u^U du} u_s(c_T) \right\},$$

with an analogous expression (with $q = 0$) holding for principal utility. The principal consumes at the rate $X_t - c_t$, for $t < T$, and the lump-sum amount $X_T - c_T$, which is what remains of X after paying the agent. Agent effort e changes the probability measure to P^e, resulting in a change in the drift of the cash-flow process of $e_t' \Sigma_t^X$.

We next define the set of feasible consumption and effort plans. We will use the theory on BSDEs with aggregators which have quadratic growth in volatility. This theory was developed in [32] for the case of uniformly bounded terminal values and later extended in [4] to the case of terminal values with exponential moments. Their results show that our definition of feasibility given below implies existence and uniqueness of solutions to the BSDEs (5.1) and (5.2) (as well as allow the application later of a comparison theorem for BSDEs). We note that stricter conditions on the aggregators (such as Lipschitz continuity) are associated with weaker integrability conditions on U_T and V_T for existence and uniqueness to hold (see [17]). Furthermore, as less restrictive conditions for existence and uniqueness of BSDE solutions are developed (hopefully), the restrictions in the following definition of a feasible plan can be relaxed, and our results will hold with similar proofs.

Definition 5.1. $(c, e) \in \widetilde{\mathcal{C}} \times \widetilde{\mathcal{E}}$ *will be called a* feasible *consumption and effort plan with respect to the aggregators* F *and* G *if there exist a process* $\alpha(t) \geq 0$ *and two constants* $\beta, \gamma > 0$ *such that following hold:*

a) for all $(t, u, v, u', v', \Sigma) \in [0, T] \times \mathbb{R}^{d+4}$,

$$\left|F(t, c_t, e_t, u, \Sigma) - F(t, c_t, e_t, u', \Sigma)\right| + \left|G(t, c_t, v, \Sigma) - G(t, c_t, v', \Sigma)\right| \\ \leq \beta(\left|u - u'\right| + \left|v - v'\right|);$$

b) for all $(t, u, v, \Sigma) \in [0, T] \times \mathbb{R}^{d+2}$,

$$|F(t, c_t, e_t, u, \Sigma)| + |G(t, c_t, v, \Sigma)| \leq \alpha(t) + \beta(|u| + |v|) + \frac{\gamma}{2} |\Sigma|^2 ;$$

c) $\int_0^T \alpha(s) ds$ *,* $\left|U_T\right|$*, and* $\left|V_T\right|$ *have exponential moments of all order.*

Define $\mathcal{C} = \left\{ c \in \widetilde{\mathcal{C}} \mid \exists e \in \widetilde{\mathcal{E}} \text{ s.t. } (c,e) \text{ is feasible} \right\}$ and

$\mathcal{E} = \left\{ e \in \widetilde{\mathcal{E}} \mid \exists c \in \widetilde{\mathcal{C}} \text{ s.t. } (c,e) \text{ is feasible} \right\}$. For simplicity of presentation we will assume the product structure, i.e.

$\mathcal{C} \times \mathcal{E} = \left\{ (c,e) \in \widetilde{\mathcal{C}} \times \widetilde{\mathcal{E}} \mid \ (c,e) \text{ are feasible} \right\}$, but the proofs all go through without this assumption.

Remark 5.1. Let A be progressively measurable subset of $\Omega \times [0,T]$. Given two plans $(c_1, e_1), (c_2, e_2) \in \mathcal{C} \times \mathcal{E}$, we can merge them into one plan defined as: $(c,e) = (c_1, e_1) \cdot 1_A + (c_2, e_2) \cdot 1_{A^c}$. It is easy to see that $(c,e) \in \mathcal{C} \times \mathcal{E}$ as well.

Remark 5.2. As was mentioned before it follows from [4] that for any $(c,e) \in \mathcal{C} \times \mathcal{E}$ there is a unique solution $\left(U, \Sigma^U\right) \in \mathcal{E}^* \times \mathcal{L}_2^- \left(\mathbb{R}^d\right)$ to the BSDE (5.1) and $\left(V, \Sigma^V\right) \in \mathcal{E}^* \times \mathcal{L}_2^- \left(\mathbb{R}^d\right)$ to the BSDE (5.2).

Principal optimality, in Theorem 5.2 below, is obtained using an extension of the Kuhn-Tucker Theorem (see Chapter 3), which relies on the assumption that $\mathcal{C}$ is an "extended convex" set, which we define as follows.

Definition 5.2. *X, a collection of stochastic processes, is extended convex if for all $x_1, x_2 \in X$ there is a process $\delta = \delta(\omega, t; x_1, x_2) > 0$ such that*

$$\alpha x_1 + (1-\alpha) x_2 \in X$$

for each process $\alpha = \alpha(\omega, t)$ that satisfies $-\delta \leq \alpha \leq 1 + \delta$.

Given any $c \in \mathcal{C}$, the agent chooses effort to maximize utility:

$$U_0(c) = \sup_{e \in \mathcal{E}} U_0(c,e).$$

Let

$$e(c) = \{ e \in \mathcal{E} \mid U_0(c) = U_0(c,e) \} \tag{5.3}$$

denote the set of optimal agent effort processes induced by the consumption process c. The principal's problem is to choose the optimal consumption of the agent subject to the participation constraint that the initial agent utility must be at least K:

$$\sup_{c \in \mathcal{C}, e \in e(c)} V_0(c, e) \text{ subject to } U_0(c) \geq K.$$

5.3 Agent Optimality

This section derives a necessary and sufficient condition for agent optimality given any consumption plan offered by the principal. We show that optimality is essentially equivalent to choosing effort to minimize the instantaneous drift of U (that is, maximizing $F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} e_t$ at each (ω, t)). A necessary and sufficient characterization of agent optimality is given in the following theorem.

Theorem 5.1 (Agent Optimality)**.** *Fix some* $c \in \mathcal{C}$ *. Then* $e \in \mathcal{E}$ *is optimal if and only if*

$$F\left(c_t, e_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} e_t \geq F\left(c_t, \tilde{e}_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} \tilde{e}_t, \quad t \in [0, T), \quad \textit{for all } \tilde{e} \in \mathcal{E}, \tag{5.4}$$

where $U_t = U_t(c, e)$ *and* $\Sigma_t^U = \Sigma_t^U(c, e)$ *solve the BSDE* (5.1).

Proof. See the appendix. □

If the optimal effort is interior (that is $e_t \in$int($\mathbf{E}$), $t \in [0, T)$), then (5.4) is equivalent to

$$-F_e\left(t, c_t, e_t, U_t, \Sigma_t^U\right) = \Sigma_t^U. \tag{5.5}$$

At each (ω, t), the agent chooses the effort level that, at the margin, equates the instantaneous cost, $-F_e$, and the instantaneous measure-change benefit, Σ_t^U, per additional unit effort. That is, optimal effort equates incremental cost to the sensitivity of the agent's utility to unit Brownian shocks. The policy is particularly simple because it depends only on the current time, state and values of $\left(c_t, U_t, \Sigma_t^U\right)$.

Example 5.2. Suppose $\boldsymbol{E} = \mathbb{R}^d$ and the aggregator is separable in effort with a quadratic penalty:

$$F\left(\omega, t, c, e, U, \Sigma^U\right) = h\left(\omega, t, c, U, \Sigma^U\right) - \frac{1}{2} e' Q^e(\omega, t)\, e,$$

where $Q^e \in \mathcal{L}\left(\mathbb{R}^{d \times d}\right)$ is positive definite. Optimal agent effort is then linear in Σ^U:

$$e_t = \left(Q_t^e\right)^{-1} \Sigma_t^U.$$

If F is a deterministic function (does not depend on ω), then all uncertainty and therefore all effort is driven by the consumption process. For example, if F is deterministic and c satisfies the Markovian SDE $dc_t = \mu^c dt + \Sigma^{c\prime} dB_t$ for some $\mu^c \in \mathbb{R}$ and $\Sigma^c \in \mathbb{R}^d$, then agent utility under optimal effort will take the form $U_t = g(t, c_t)$ for some deterministic function g increasing in consumption, and optimal effort will satisfy $e_t = g_c(t, c_t)\left(Q_t^e\right)^{-1} \Sigma_t^c$.

Our framework is sufficiently flexible such that factors other than consumption can induce effort. For example, reputation concerns or some stochastic cash flow process not provided by the principal can be incorporated.

The following example considers a quadratic risk-aversion penalty combined with a quadratic effort penalty.

Example 5.3. Suppose we further specialize Example 5.2 by assuming a quadratic utility penalty:

$$F\left(\omega, t, c, e, U, \Sigma^U\right) = h(\omega, t, c, U) - \frac{1}{2} e' Q^e(\omega, t)\, e + p(\omega, t)' \Sigma^U - \frac{1}{2} \Sigma^{U\prime} Q^U(\omega, t)\, \Sigma^U,$$

where $Q^e, Q^U \in \mathcal{L}\left(\mathbb{R}^{d \times d}\right)$ are positive definite and $p \in \mathcal{L}\left(\mathbb{R}^d\right)$. The process Q^U represents aversion towards the d-dimensions of risk, while p represents beliefs different from the true probability measure P (the agent views P^p as the true probability measure). Then under the optimal effort process $e_t = \left(Q_t^e\right)^{-1} \Sigma_t^U$, agent utility satisfies

$$dU_t = -\left\{ h\left(\omega, t, c, U\right) + p_t \Sigma_t^U - \frac{1}{2} \Sigma_t^{U\prime} \left(Q_t^U - \left(Q_t^e\right)^{-1} \right) \Sigma_t^U \right\} dt + \Sigma_t^{U\prime} dB_t,$$
$$U_T = F\left(T, c_T\right),$$

which is the same as the zero-effort utility function except with a modified risk-aversion coefficient.

On one hand, the quadratic effort penalty combines with the risk aversion penalty to increase the quadratic penalty on Σ_t^U. On the other hand, an increase in Σ_t^U also increases the private benefit to agent effort (in typical applications, this benefit is through the share of some cash flow process promised to the agent, and the impact of effort on the cash flow drift), which, together with the higher level of effort induced, results in a positive contribution to utility, also quadratic in Σ_t^U. If Q^e is small enough (that is, if $Q_t^U - \left(Q_t^e\right)^{-1}$ is negative definite), then effort choice can induce a preference for risk.

The dependence of agent effort on the volatility term Σ^U, combined with the dependence of principal utility on effort (via the change of measure), causes a dependence of the principal's utility on the agent's volatility, and induces a (bivariate) recursiveness to utility under optimal agent effort even if none existed for a given effort process. We will see more about this later in the paper.

5.4 Principal Optimality

The principal's problem, of choosing the consumption/compensation process c to maximize the principal's utility given the agent's optimal effort, is far more complicated than the agent's problem of choosing effort taking the consumption process as given. The principal must balance the reduction in his own utility from paying an extra dollar against the benefits of increased agent effort as well as the slackened participation constraint resulting from the increased utility of the agent. We express the solution as an FBSDE, which links the BSDEs of both principal and agent utilities together with a forward equation representing the sensitivity of principal's utility to a unit change in the agent's utility. Optimal consumption at each (ω, t) depends not only on the utility and utility diffusions of both principal and agent, but also on the value of this Lagrange multiplier process, which incorporates the shadow price of the participation constraint as well as the dependence of the principal's utility on agent effort.

We first obtain necessary and sufficient conditions for optimality using the utility gradient approach which originated in the portfolio optimization problem in [8] and [30] for the case of additive-utility, and extended for non-additive utilities in [46]; [15]; [18] (which allows non-linear wealth dynamics); [43]; [44] (the latter two allow constrained markets); and [45] (which considers constraints and a jump-diffusion setting). The gradient approach applied in the portfolio setting essentially amounts to choosing the consumption stream to match marginal utility (the utility gradient density) to marginal price (the state-price density). In the principal-agent setting, the agent utility process plays a role analogous to the that of the wealth process, and the optimality conditions take a similar form. In addition to utility gradient approach to be developed in Section 5.4.1, we will present a dynamic programming approach in Section 5.4.2, from which we obtain sufficient conditions for principal optimality under different regularity conditions.

5.4.1 Utility gradient density approach

We begin by substituting the agent's optimal effort into the utility BSDEs. Define $I : \Omega \times [0,T] \times \boldsymbol{C} \times \mathbb{R}^{1+d} \to \boldsymbol{E}$ by

$$I\left(\omega, t, c, U, \Sigma^U\right) = \arg\max_{e \in \boldsymbol{E}} \left\{ F\left(\omega, t, c, e, U, \Sigma^U\right) + \Sigma^{U\prime} e \right\}, \quad t \in [0,T). \tag{5.6}$$

By Theorem 5.1, if I is well-defined (the $\arg\max$ above exists and is unique) then $e_t = I\left(\omega, t, c, U, \Sigma^U\right)$ is the optimal choice of the agent effort. That is why we assume throughout the following condition.[2]

Condition 5.1. *For each* $(\omega, t, c, u, \sigma) \in \Omega \times [0, T] \times \boldsymbol{C} \times \mathbb{R}^{d+1}$, $I(\omega, t, c, u, \sigma)$ *is well defined and* I $\in$*interior*$(\boldsymbol{E})$*. Also for each* $c \in \mathcal{C}$ *,* $\left\{I\left(t, c_t, U_t, \Sigma_t^U\right); t \in [0, T]\right\} \in \mathcal{E}$, *where* $\left(U, \Sigma^U\right)$ *is solution of* (5.1) *with* e_t *replaced by* $I\left(t, c_t, U_t, \Sigma_t^U\right)$*(the solution to equation* (5.8) *below).*

It follows from Condition 5.1 that $e(c)$, the set of optimal effort processes induced by c (defined in (5.3)), contains exactly one process.

Sufficient conditions for a unique interior (with respect to **E**) solution are that F is strictly concave in e and that $F_e\left(\omega, t, c, \cdot, U, \Sigma^U\right)$ maps $\boldsymbol{E}$ onto $\mathbb{R}^d$ for each $(\omega, t, c, U, \Sigma)$.

Substituting optimal effort into the BSDEs (5.1) and (5.2), the principal's problem is

$$\sup_{c \in \mathcal{C}} V_0(c) \text{ subject to } U_0(c) \geq K \tag{5.7}$$

where $\left(U, \Sigma^U, V, \Sigma^V\right)$ satisfy the BSDE system

$$\begin{aligned} dU_t &= -\bar{F}\left(t, c_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t, \quad U_T = F\left(T, c_T\right), \\ dV_t &= -\bar{G}\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right) dt + \Sigma_t^{V\prime} dB_t, \quad V_T = G\left(T, c_T\right), \end{aligned} \tag{5.8}$$

and where we have used the abbreviations $V_t = V_t(c)$, $U_t = U_t(c)$, $\Sigma_t^V = \Sigma_t^V(c)$ and

[2] Extensions to corner solutions in (5.6) as well as nondifferentiable aggregators are straightforward. The utility gradient approach can be handled along the lines of [43], replacing derivatives with appropriately defined superdifferentials. The dynamic programming approach below doesn't use the differentiability assumption, and constraints are simple to impose.

$\Sigma_t^U = \Sigma_t^U(c)$, and the modified aggregators

$$\bar{F}\left(\omega,t,c,u,\Sigma^1\right) = F\left(\omega,t,c,I\left(\omega,t,c,u,\Sigma^1\right),u,\Sigma^1\right) + \Sigma^{1\prime} I\left(\omega,t,c,u,\Sigma^1\right),$$
$$\bar{G}\left(\omega,t,c,v,\Sigma^2,u,\Sigma^1\right) = G\left(\omega,t,c,v,\Sigma^2\right) + \Sigma^{2\prime} I\left(\omega,t,c,u,\Sigma^1\right).$$

Condition 5.1 implies a unique solution of (5.8) for each $c \in \mathcal{C}$.

The agency problem induces a dependence of the principal's aggregator on the utility and utility-diffusion term of the agent. This follows because agent effort, which is a function of both the agent's utility and utility-diffusion term, affects the Brownian motion drift, which impacts the principal's utility.

The principal's optimality conditions below are expressed in terms of the gradient density (or Gateaux derivative) or supergradient density of a linear combination of agent and principal utilities. The general definitions of these densities follow.

Definition 5.3. *Let $\nu : \mathcal{C} \to \mathbb{R}$ be a functional. For any $c \in \mathcal{C}$, the process $\pi \in \mathcal{L}_2(\mathbb{R})$ is a* supergradient density *of ν at c if*

$$\nu(c+h) - \nu(c) \leq (\pi \,|h) \quad \textit{for all } h \textit{ such that } c+h \in \mathcal{C},$$

and $\pi \in \mathcal{L}_2(\mathbb{R})$ is a gradient density *at c if*

$$(\pi \,|h) = \lim_{\alpha \downarrow 0} \frac{\nu(c+\alpha h) - \nu(c)}{\alpha} \quad \textit{for all } h \textit{ such that } c+\alpha h \in \mathcal{C} \textit{ for some } \alpha > 0.$$

The computation of the gradient and supergradient densities requires the following $\mathbb{R}^2$-valued adjoint process $\varepsilon_t = \left(\varepsilon_t^V, \varepsilon_t^U\right)'$, with some initial value $\varepsilon_0 \in \mathbb{R}^2$ and dynamics

$$d\varepsilon_t = \begin{pmatrix} \bar{G}_V(t) & 0 \\ \bar{G}_U(t) & \bar{F}_U(t) \end{pmatrix} \varepsilon_t dt + \begin{pmatrix} \bar{G}_{\Sigma V}(t)' \, dB_t & 0 \\ \bar{G}_{\Sigma U}(t)' \, dB_t & \bar{F}_{\Sigma U}(t)' \, dB_t \end{pmatrix} \varepsilon_t, \qquad (5.9)$$

where $\bar{G}_V(t) = \frac{\partial \overline{G}}{\partial V}(t, c_t, U, \Sigma^U, V, \Sigma^V)$ and similar definitions for rest of the terms in (5.9).

The following condition imposes sufficient smoothness and integrability on the aggregators for existence of gradient densities and supergradient densities.

Condition 5.2. *Define* $\Phi(t) = (\bar{F}(t), \bar{G}(t))$*, and let* ε *satisfy the SDE* (5.9) *with initial value* ε_0*. The conditions below hold* $\forall c \in \mathcal{C}$*.*

a) *Gradient density conditions:*

- *For every* $t \in [0, T]$*,* $\Phi(t, \cdot)$ *has uniformly bounded continuous partial derivatives with respect to* $(c, U, V, \Sigma^U, \Sigma^V)$*. (The bounds do not depend on* ω*.)*
- $\{\Phi(t, c_t, 0, 0, 0, 0),\ t \in [0, T)\} \in \mathcal{L}_2^-(\mathbb{R}^2)$.
- $E[\ \|\Phi(T, c_T)\|^2] < \infty$.

  (The last two conditions follow from Definition 5.1 and are brought here for convenience of presentation.)

b) *Supergradient density conditions:*

- $\Phi^i(t, \cdot)$ *is a concave function in* $(c, U, V, \Sigma^U, \Sigma^V)$*, for* $i \in \{1, 2\}$*.*
- $\left\{\Phi_Z^i(t);\ t \in [0, T]\right\}$ *is locally bounded*[3] *for* $i \in \{1, 2\},\ Z \in \left\{U, V, \Sigma^U, \Sigma^V\right\}$*, where* $\Phi_Z^i(t)$ *is calculated in* $\left(U(c), \Sigma^U(c), V(c), \Sigma^V(c)\right)$*, a solution of* (5.8).
- $\left\{h_t \Phi_c^i(t);\ t \in [0, T]\right\} \in \mathcal{L}_2(\mathbb{R}),\ E\left[\sup_{t \in [0,T]} \left(\varepsilon_t^i\right)^2\right] < \infty$ *for all* h *such that* $c + h \in \mathcal{C}$*,* $i \in \{1, 2\}$*.*

It is convenient to express the FOCs in terms of the gradient of a linear combination of utilities (though, as remarked below, it could instead be stated in terms of the individual gradients of the agent and principal utilities).

Lemma 5.1 (Gradient and supergradient densities)**.** *Suppose* $c \in \mathcal{C}$*,* $\left(U, \Sigma^U, V, \Sigma^V\right)$ *satisfies the BSDE system* (5.8)*, and* ε *satisfies* (5.9) *with initial value* $\varepsilon_0 \in \mathbb{R}_+^2$*.*

---

[3] A process $X(t, \omega)$ $0 \le t \le T$, is called locally bounded if $\exists\ \tau_k \uparrow T,\ \tau_k$ are stopping times such that $X(t \wedge \tau_k, \omega)$ is bounded $\forall k$.

a) If Condition 5.2 (a) holds, then $\left\{\left[\bar{G}_c(t), \bar{F}_c(t)\right]\varepsilon_t;\ \ t \in [0,T]\right\}$ *is a gradient density of* $[V_0(c), U_0(c)]\,\varepsilon_0$ *at* c *for any* $\varepsilon_0 \in \mathbb{R}_+^2$.

b) If Condition 5.2 (b) holds, then $\left\{\left[\bar{G}_c(t), \bar{F}_c(t)\right]\varepsilon_t;\ \ t \in [0,T]\right\}$ *is a supergradient density of* $[V_0(c), U_0(c)]\,\varepsilon_0$ *at* c *for any* $\varepsilon_0 \in \mathbb{R}_+^2$.

Proof. See Lemma 3.1 and 3.3 in chapter 3. □

The main result of the section, providing the first-order conditions (FOCs) for principal optimality, follows.

Theorem 5.2. *Let* $c \in \mathcal{C}$*,* $\left(U, \Sigma^U, V, \Sigma^V\right)$ *solve the BSDE system* (5.8)*, and suppose* ε *solves the SDE* (5.9)*.*

a) (Necessity) Suppose Condition 5.2 (a). If c *solves the principal's problem* (5.7) *then there is some* $\kappa \in \mathbb{R}_+$ *such that*

$$\begin{gathered}\varepsilon_0 = (1,\kappa)', \quad \left[\bar{G}_c(t), \bar{F}_c(t)\right]\varepsilon_t = 0, \quad t \in [0,T], \\ \kappa\left\{U_0(c) - K\right\} = 0, \quad U_0(c) \geq K.\end{gathered} \tag{5.10}$$

b) (Sufficiency) Suppose Condition 5.2 (b). If (5.10) *holds then* c *is optimal.*

Proof. Follows from theorem 4.1 in chapter 4. □

Remark 5.3. The FOCs (5.10) can be restated as follows. If $\pi^U, \pi^V \in \mathcal{L}_2(\mathbb{R})$ are gradient densities or supergradient densities of U and V, respectively, at c, then (5.10) is equivalent to

$$\pi^V = -\kappa\pi^U, \quad \kappa\left\{U_0(c) - K\right\} = 0. \tag{5.11}$$

The equivalence follows because for $i \in \{1,2\}$, $\pi_t^V = \left[\bar{G}_c(t), \bar{F}_c(t)\right]\varepsilon_t^1$, and $\pi_t^U = \left[\bar{G}_c(t), \bar{F}_c(t)\right]\varepsilon_t^2$ and ε^i satisfies (5.9) with initial values $\varepsilon_0^1 = (1,0)'$ and $\varepsilon_0^2 = (0,1)'$. The linearity of the adjoint processes (5.9) implies that $\varepsilon_t = \varepsilon_t^1 + \kappa\varepsilon_t^2$ where $\varepsilon_0 = (1,\kappa)'$.

The FOCs are the natural extension of the static problem, with κ representing the shadow price of the participation constraint.

Concavity of $\Phi(t) = (\bar{F}(t), \bar{G}(t))$, which is assumed in Condition 5.2 (b) for the sufficiency part of the Theorem, is violated in many applications. A dynamic programming derivation of sufficiency based on weaker conditions (see Remark 5.5) is introduced in Section 5.4.2 below.

The dimensionality of the solution can be reduced by expressing the FOCs (5.10) in terms of a ratio. Define the process[4]

$$\lambda_t = -\frac{\bar{G}_c\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right)}{\bar{F}_c\left(t, c_t, U_t, \Sigma_t^U\right)}, \quad t < T, \quad \lambda_T = -\frac{\bar{G}_c\left(T, c_T\right)}{\bar{F}_c\left(T, c_T\right)} = -\frac{G_c\left(T, c_T\right)}{F_c\left(T, c_T\right)}, \tag{5.12}$$

which under the FOCs (5.10) satisfies

$$\lambda_t = \frac{\varepsilon_t^U}{\varepsilon_t^V}, \tag{5.13}$$

where $\varepsilon_0 = (1, \kappa)'$ for some $\kappa \geq 0$. The dynamic programming argument in Section 5.4.2 justifies the interpretation of λ_t as the Lagrangian of the time-t principal optimization problem, representing the rate of change in V_t per unit reduction in U_t. At the margin it is given by the ratio of the sensitivities of the corresponding utilities to unit changes in time-t consumption, which, in turn, equals the ratio of the marginal aggregators. Ito's lemma implies the following dynamics of λ:

$$d\lambda_t = \left\{\lambda_t \bar{F}_U(t) - \lambda_t \bar{G}_V(t) + \bar{G}_U(t) - \bar{G}'_{\Sigma V}(t)\,\Sigma_t^\lambda\right\} dt + \Sigma_t^{\lambda\prime} dB_t, \tag{5.14}$$
$$\text{where } \Sigma_t^\lambda = \lambda_t\left\{\bar{F}_{\Sigma U}(t) - \bar{G}_{\Sigma V}(t)\right\} + \bar{G}_{\Sigma U}(t).$$

We will assume that (5.12) can be inverted to solve consumption as a function of

[4]Note that $\bar{F}_c = F_c$ and recall the assumption $F_c > 0$.

$\left(\lambda_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right)$:

Condition 5.3. *There exists* $\phi : \Omega \times [0,T] \times \mathbb{R}^{2d+3} \to \mathbb{R}$ *so that* $(t, c_t, \lambda_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U)$ *satisfies* (5.12) *with* $c_t = \phi\left(\omega, t, \lambda, V, \Sigma^V, U, \Sigma^U\right)$.

Given Condition 5.3 we can express the first-order conditions (5.10) as an FBSDE system for $\left(U, \Sigma^U, V, \Sigma^V, \lambda\right)$:

$$
\begin{aligned}
dU_t &= -\bar{F}\left(t, c_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t, \quad U_T = F\left(T, c_T\right), \qquad (5.15)\\
dV_t &= -\bar{G}\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right) dt + \Sigma_t^{V\prime} dB_t, \quad V_T = G\left(T, c_T\right),\\
d\lambda_t &= \left\{ \begin{array}{c} \lambda_t \bar{F}_U(t) - \lambda_t \bar{G}_V(t) + \bar{G}_U(t) \\ -\bar{G}'_{\Sigma^V}(t)\left(\lambda_t \bar{F}_{\Sigma^U}(t) - \lambda_t \bar{G}_{\Sigma^V}(t) + \bar{G}_{\Sigma^U}(t)\right) \end{array} \right\} dt\\
&\quad + \left(\lambda_t \bar{F}_{\Sigma^U}(t) - \lambda_t \bar{G}_{\Sigma^V}(t) + \bar{G}_{\Sigma^U}(t)\right)' dB_t, \quad \lambda_0 = \kappa \geq 0,\\
U_0 &\geq K, \quad \kappa\left(U_0 - K\right) = 0,\\
c_t &= \phi\left(t, \lambda_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right), \quad c_T = \phi\left(t, \lambda_T\right).
\end{aligned}
$$

The solution is complicated because the backward utility equations and the forward equation for λ are coupled, requiring the system to be simultaneously solved.

Remark 5.4. (a) The optimality conditions can be expressed in terms of the original aggregators F and G as follows. The equation $-\bar{F}_c(t)\lambda_t = \bar{G}_c(t)$ is equivalent to (substituting $F_e(t) = -\Sigma^U$)

$$0 = F_c(t)\lambda_t + G_c(t) + \Sigma_t^{V\prime} I_c(t). \qquad (5.16)$$

Furthermore, $\bar{G}_{\Sigma^U}(t) = I_\Sigma(t)' \Sigma^V$ implies

$$d\lambda_t = \left\{\lambda_t F_U(t) - \lambda_t G_V(t) + \Sigma_t^{V\prime} I_U(t) - \left(G_\Sigma(t) + I(t)\right)' \Sigma_t^\lambda\right\} dt + \Sigma_t^{\lambda\prime} dB_t, \qquad (5.17)$$

where $\Sigma_t^\lambda = \lambda_t\left\{F_\Sigma(t) - G_\Sigma(t)\right\} + I_\Sigma(t)' \Sigma_t^V$.

(b) A sufficient condition for the participation constraint to bind is easily obtained from (5.16). If $\Sigma_t^{V\prime} I_c(t) \leq 0$ for all t (for example, if $I_c = 0$), then (5.16) implies $\lambda > 0$ (recall $F_c > 0$ and $G_c < 0$). In particular, $\lambda_0 > 0$ which implies a binding constraint.

The first-order condition for optimal consumption (5.16) equates the instantaneous net benefit per unit change of time-t pay to zero. The term $G_c(t) + \Sigma_t^{V\prime} I_c(t)$ is the direct impact on principal utility per unit extra consumption for the agent, the second term representing the direct effect of the incremental pay on agent effort and its impact on the principal's utility (via the change of measure). In typical applications, however, $I_c = 0$ and the incentive effect of compensation is through the agent's volatility, Σ^U (see Example 5.3 above in which $I\left(\omega, t, \Sigma_t^U\right) = \{Q^e(\omega, t)\}^{-1} \Sigma_t^U$). Increasing agent pay by promising, say, a larger share of the cash flows increases the (absolute) sensitivity of agent volatility; this induces additional effort, which increases the drift of the cash flow process and increases both agent and principal utility. At the optimum, this effect is captured by the term $F_c(t)\lambda_t$, with the Lagrangian λ_t representing the benefit to the principal from increasing agent utility, which incorporates incentive effects of extra pay as well as the shadow price of the participation constraint.

5.4.2 Dynamic Programming Approach

Our first goal is to reformulate the agent's utility process as a forward equation. Recall that for any consumption plan $c \in C$ paid by the principal there is a unique agent utility and volatility pair $\left(U, \Sigma^U\right) \in \mathcal{E}^* \times \mathcal{L}_2^-\left(\mathbb{R}^d\right)$ solving the BSDE (5.1) with, by Theorem 5.1 and Condition 5.1, optimal agent effort $e_t = I\left(t, c_t, U_t, \Sigma_t^U\right)$. In Remark 5.4(b) we presented a sufficient condition for $\lambda > 0$, which implies a binding participation constraint. The principal's problem is made amenable to the dynamic programming approach by reformulating the agent's utility equation as a forward SDE (starting at K), and changing the principal's controls from lifetime consumption to intermediate consumption and agent utility volatility. That is, we proceed *as if* the principal can control the agent's utility-volatility process Σ^U in addition to intermediate consumption. The terminal value of the agent's utility gives the unique c_T that corresponds to these controls, which, together with intermediate consumption, forms the pay package offered to the agent. This approach is in the spirit of [41] and [27]. The problem is analogous to the optimal portfolio/consumption problem, with the forward equation U playing the role of a non-linear wealth equation, K representing initial wealth, and Σ^U the portfolio vector.

We start with the primitive intermediate consumption space $\widetilde{\mathcal{C}}^-$ (recall that terminal consumption is not included in $c \in \widetilde{\mathcal{C}}^-$) and the space of agent utility-volatility processes as $\mathcal{L}_2^-\left(\mathbb{R}^d\right)$. A principal's plan will be some pair $\left(c, \Sigma^U\right) \in \widetilde{\mathcal{C}}^- \times \mathcal{L}_2^-\left(\mathbb{R}^d\right)$, and for any such plan agent utility solves the forward SDE

$$dU_t = -F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_{t,}^e, \quad U_0 = K, \tag{5.18}$$

where optimal agent effort is $e_t = I\left(t, c_t, U_t, \Sigma_t^U\right)$. The lump-sum terminal consumption corresponding to the plan is $c_T = F^{-1}\left(T, U_T\right)$ (in the language of the analogous portfolio problem, c_T is "financed" by the diffusion process $\left(c, \Sigma^U\right)$).

We would like the utility gradient density approach which we developed in Section 5.4.1 to be consistent with the dynamic programming approach. However using (5.18) with any $\Sigma^U \in \mathcal{L}_2^-\left(\mathbb{R}^d\right)$ will give $c_T = F^{-1}\left(T, U_T\right)$, which combined with $c \in \widetilde{\mathcal{C}}^-$ doesn't necessarily belong to $\mathcal{C}$ (i.e. the solution is not feasible). The following definition guarantees that the solution we get from current approach is feasible.

Definition 5.4. $\left(c, \Sigma^U\right) \in \widetilde{\mathcal{C}}^- \times \mathcal{L}_2^-\left(\mathbb{R}^d\right)$ *will be called a* viable *principal's plan if* $(c, e) \in \mathcal{C} \times \mathcal{E}$*, where* $c_T = F^{-1}\left(T, U_T\right)$ *and* $e_t = I\left(t, c_t, U_t, \Sigma_t^U\right)$*, with* U_t *satisfying equation* (5.18)*. We will denote the class of viable principal's plans by* $\mathcal{A}$*.*

The principal's problem in the current setting is

$$\sup_{\left(c,\Sigma^U\right)\in\mathcal{A}} V_0\left(c,\Sigma^U\right). \tag{5.19}$$

The restriction in (5.19) to $\left(c,\Sigma^U\right)\in\mathcal{A}$ ensures that in the case of a binding participation constraint, the solution we get from the two approaches (gradient density and dynamic programming) will be identical.

The solution of (5.19) is implemented as follows: The principal specifies the intermediate consumption plan and the agent's volatility. Then the agent maximizes his/her utility by choosing the optimal effort $e_t = I\left(t,c_t,U_t,\Sigma_t^U\right)$. The principal computes the terminal consumption consistent with $\left(c,\Sigma^U\right)$ and a binding participation constraint using equation (5.18), yielding $c_T = F^{-1}\left(T,U_T\right)$. At this stage the principal's utility, $V(t)$, solves equation (5.2) with $e_t = I\left(t,c_t,U_t,\Sigma_t^U\right)$.

Having reformulated the problem, we will use dynamic programming to solve the principal's problem. We start by defining

$$Y_t = V_t + \lambda_t\left(U_t - K\right), \quad t\in[0,T],$$

where λ satisfies the BSDE

$$d\lambda_t = \mu_t^\lambda dt + \Sigma_t^{\lambda\prime} dB_t, \quad \lambda_T = -\frac{G_c\left(T,c_T\right)}{F_c\left(T,c_T\right)},$$

where μ_t^λ will be specified below as part of the solution (note that $\left(\lambda,\Sigma_t^\lambda\right)$ is now a solution of a backward equation). We can think of Y as a Lagrangian with λ as the shadow price of agent utility (reflecting both the participation constraint and incentive effect of consumption on effort). Denote the optimal policy and corresponding agent utility (solving (5.18)) by

$\left(\hat{c}, \hat{\Sigma}^{U}, \hat{U}\right)$, and write the dynamics of Y at the optimum as

$$dY_t = \mu_t^Y dt + \Sigma_t^{Y\prime} dB_t, \quad Y_T = G\left(T, \hat{c}_T\right) + \lambda_T \left\{F\left(T, \hat{c}_T\right) - K\right\}. \tag{5.20}$$

Apply Ito's lemma and substitute $\Sigma_t^Y = \Sigma_t^V + \lambda_t \hat{\Sigma}_t^U + \left(\hat{U}_t - K\right)\Sigma_t^\lambda$ to get, for $t \in [0, T)$,

$$\begin{aligned} -\mu_t^Y = {} & G\left(t, \hat{c}_t, Y_t - \lambda_t\left(\hat{U}_t - K\right), \Sigma_t^Y - \lambda_t \hat{\Sigma}_t^U - \left(\hat{U}_t - K\right)\Sigma_t^\lambda\right) - \left(\hat{U}_t - K\right)\mu_t^\lambda \\ & - \Sigma_t^{\lambda\prime}\hat{\Sigma}_t^U + \lambda_t F\left(t, \hat{c}_t, I\left(t, \hat{c}_t, \hat{U}_t, \hat{\Sigma}_t^U\right), \hat{U}_t, \hat{\Sigma}_t^U\right) \\ & + \left\{\Sigma_t^Y - \left(\hat{U}_t - K\right)\Sigma_t^\lambda\right\}' I\left(t, \hat{c}_t, \hat{U}_t, \hat{\Sigma}_t^U\right). \end{aligned} \tag{5.21}$$

Theorem 5.3 below shows that optimality follows if for any other viable plan $\left(c, \Sigma^U\right)$ and corresponding U calculated from (5.18), we have for $t \in [0, T)$:

$$\begin{aligned} -\mu_t^Y \geq {} & G\left(t, c_t, Y_t - \lambda_t\left(U_t - K\right), \Sigma_t^Y - \lambda_t \Sigma_t^U - \left(U_t - K\right)\Sigma_t^\lambda\right) - \left(U_t - K\right)\mu_t^\lambda \\ & - \Sigma_t^{\lambda\prime}\Sigma_t^U + \lambda_t F\left(t, c_t, I\left(t, c_t, U_t, \Sigma_t^U\right), U_t, \Sigma_t^U\right) \\ & + \left\{\Sigma_t^Y - \left(U_t - K\right)\Sigma_t^\lambda\right\}' I\left(t, c_t, U_t, \Sigma_t^U\right). \end{aligned} \tag{5.22}$$

Note that the optimality conditions jointly specify the optimal principal policy $\left(\hat{c}, \hat{\Sigma}^U\right)$, the corresponding Lagrangian drift process μ^λ, as well as μ^Y.

The following Theorem provides the sufficient condition for principal's problem under less restrictive condition than Theorem 5.2 (see Remark 5.5 below).

Theorem 5.3. *Assume the participation constraint binds. Let* $\left(\hat{c}, \hat{\Sigma}^U\right)$ *be a viable principal's plan such that* $\left(Y, \Sigma^Y, \lambda, \Sigma^\lambda, \hat{U}\right)$ *solves the following FBSDE system for some*

μ^{λ}*:*

$$d\hat{U}_t = -F\left(t, \hat{c}_t, \hat{e}_t, \hat{U}_t, \hat{\Sigma}_t^U\right) dt + \hat{\Sigma}_t^{U\prime}\left(dB_t - \hat{e}_t dt\right), \tag{5.23}$$
$$\hat{c}_T = F^{-1}\left(T, \hat{U}_T\right),$$
$$d\lambda_t = \mu_t^{\lambda} dt + \Sigma_t^{\lambda\prime} dB_t, \quad \lambda_T = -\frac{G_c\left(T, \hat{c}_T\right)}{F_c\left(T, \hat{c}_T\right)},$$
$$dY_t = \mu_t^{Y} dt + \Sigma_t^{Y\prime} dB_t, \quad Y_T = G\left(T, \hat{c}_T\right) + \lambda_T\left\{F\left(T, \hat{c}_T\right) - K\right\},$$
$$\hat{e}_t = I\left(t, \hat{c}_t, \hat{U}_t, \hat{\Sigma}_t^U\right),$$

where μ^Y *satisfies* (5.21)*. If* (5.22) *holds for any other viable* $\left(c, \Sigma^U\right)$ *then* $\left(\hat{c}, \hat{\Sigma}^U\right)$ *is optimal.*

Proof. See the appendix. □

Remark 5.5. Observe that the proof of Theorem 5.3 works if we assume concavity of $G\left(t, c, \cdot\right)$, $t \in [0, T)$, $G\left(T, \cdot\right)$ and $F\left(T, \cdot\right)$. These concavity assumptions are less restrictive than the concavity assumption on $\bar{G}$ and $\bar{F}$ imposed to obtain sufficiency in Theorem 5.2.

The rest of this section restates the optimality condition of Theorem 5.3 and then reconciles it with the FOC in Theorem 5.2. Finally we illustrate with two examples.

Define $\Theta_t = \left(Y_t, \Sigma_t^Y, \lambda_t, \Sigma_t^{\lambda}\right)$ and, for any $\left(\tilde{c}, \tilde{\Sigma}, \tilde{U}\right) \in \mathbb{R}^{d+2}$,

$$H\left(t, \tilde{c}, \tilde{\Sigma}, \tilde{U}, \Theta_t\right) = G\left(t, \tilde{c}, Y_t - \lambda_t\left(\tilde{U} - K\right), \Sigma_t^Y - \lambda_t\tilde{\Sigma} - \left(\tilde{U} - K\right)\Sigma_t^{\lambda}\right)$$
$$+ \lambda_t F\left(t, \tilde{c}, I\left(t, \tilde{c}, \tilde{U}, \tilde{\Sigma}\right), \tilde{U}, \tilde{\Sigma}\right)$$
$$+ \left\{\Sigma_t^Y - \left(\tilde{U} - K\right)\Sigma_t^{\lambda}\right\}' I\left(t, \tilde{c}, \tilde{U}, \tilde{\Sigma}\right).$$

Then Theorem 5.3 shows that optimality of $\left(\hat{c}, \hat{\Sigma}^U\right)$ is implied if, for all $t \in [0, T]$,

$$H\left(t, \tilde{c}, \tilde{\Sigma}, \tilde{U}, \Theta_t\right) - H\left(\hat{c}_t, \hat{\Sigma}_t^U, \hat{U}_t, \Theta_t\right) \leq \mu_t^\lambda \left(\tilde{U} - \hat{U}_t\right) + \Sigma_t^{\lambda\prime}\left(\tilde{\Sigma} - \hat{\Sigma}_t^U\right), \quad (5.24)$$
$$\text{all } \left(\tilde{c}, \tilde{\Sigma}, \tilde{U}\right) \in \mathbb{R}^{d+2}.$$

Given any Θ_t and U_t, (5.24) can be used to solve for $\left(\hat{c}_t, \hat{\Sigma}_t^U, \mu_t^\lambda\right)$ as functions of (Θ_t, U_t) (the resulting $\hat{U}$ then solves (5.18)). Given $\left(\hat{c}_t, \hat{\Sigma}_t^U, \mu_t^\lambda\right)$ and $\hat{U}_t$ we obtain
$\mu_t^Y = H\left(\hat{c}_t, \hat{\Sigma}_t^U, \hat{U}_t, \Theta_t\right) - \left(\hat{U}_t - K\right)\mu_t^\lambda - \Sigma_t^{\lambda\prime}\hat{\Sigma}_t^U$.

The resulting linked FBSDE system characterizes the solution to the principal's problem.

If H is differentiable and concave in $\left(\tilde{c}, \tilde{\Sigma}, \tilde{U}\right)$ then optimality is implied if $\left(\hat{c}_t, \hat{\Sigma}_t^U, \mu_t^\lambda\right)$ satisfy

$$H_c\left(t, \hat{c}_t, \hat{\Sigma}_t^U, \hat{U}_t, \Theta_t\right) = 0, \quad H_\Sigma\left(t, \hat{c}_t, \hat{\Sigma}_t^U, \hat{U}_t, \Theta_t\right) = \Sigma_t^\lambda, \quad (5.25)$$
$$H_U\left(t, \hat{c}_t, \hat{\Sigma}_t^U, \hat{U}_t, \Theta_t\right) = \mu_t^\lambda.$$

Remark 5.6. The FOCs for an interior solution which was achieved by using the utility gradient approach are (recall that $F_e = -\Sigma_t^U$)

$$\begin{aligned} 0 &= G_c(t) + \lambda_t F_c(t) + \Sigma_t^{V\prime} I_c(t), \quad (5.26)\\ \Sigma_t^\lambda &= \lambda_t \left\{F_\Sigma(t) - G_\Sigma(t)\right\} + \Sigma_t^{V\prime} I_\Sigma(t),\\ \mu_t^\lambda &= \lambda_t \left\{F_U(t) - G_V(t)\right\} - \Sigma_t^{\lambda\prime}\left\{G_\Sigma(t) + I(t)\right\} + \Sigma_t^{V\prime} I_U(t), \end{aligned}$$

which are identical to (5.25). Also note that concavity of H in $\left(\tilde{c}, \tilde{\Sigma}, \tilde{U}\right)$ is implied by the concavity of $\bar{F}$ and $\bar{G}$ assumed in Condition 5.2 (b).

The following example shows how the result in [9], with expected utility for terminal consumption only, is obtained in our setting. Example 5.5 which follows, shows a simple extension to a recursive principal utility.

Example 5.4 (Terminal consumption only)**.** Suppose there is no intermediate consumption and the penalty for agent effort is quadratic:

$$F(t,c,e,\Sigma) = -\frac{1}{2}qe'e, \quad G(t,c,V,\Sigma) = 0, \quad t < T,$$
$$F(T,c_T) = f(c_T), \quad G(T,c_T) = g(X_T - c_T),$$

for some $q > 0$. From (5.5) optimal agent effort is $e_t = \Sigma_t^U/q$, and (5.26) implies the following dynamics of λ:

$$d\lambda_t = \Sigma_t^{\lambda\prime} dB_t^e, \quad \lambda_T = \frac{g'(X_T - c_T)}{f'(c_T)}.$$

Optimal agent volatility is $\Sigma_t^V = q\Sigma_t^\lambda$ which implies the key simplification $dV_t = qd\lambda_t$, and therefore $V_t - q\lambda_t = \beta$ for some constant β. Substituting the terminal conditions for V and λ gives

$$\beta = g(X_T - c_T) - q\left(\frac{g'(X_T - c_T)}{f'(c_T)}\right), \tag{5.27}$$

which can be used to solve implicitly for optimal c_T as a function of β and X_T. To solve for β, apply Ito's lemma to $u_t = \exp(U_t/q)$ to get $du_t = u_t e_t' dB_t$ and therefore (using $u_0 = e^K$)

$$\exp(K) = E\exp\left(\frac{f(c_T)}{q}\right).$$

The martingale representation theorem gives the optimal effort e.

Example 5.5. Suppose the agent's utility is the same as in Example 5.4, but the principal's is of the recursive form

$$G(\omega,t,c,\Sigma) = -\frac{1}{2}q^V\Sigma'\Sigma, \quad G(T,c_T) = g(X_T - c_T),$$

where $q^V > 0$. Defining the ordinally equivalent transformation $v_t = -\exp\left(-q^V V_t\right)$, Ito's lemma implies

$$v_0 = -E_0 \exp\left\{-q^V g\left(X_T - c_T\right)\right\}.$$

The solution of Example 5.4 therefore applies after replacing $g(\cdot)$ with $\tilde{g}(\cdot) = -\exp\left\{-q^V g(\cdot)\right\}$, and the optimality condition (5.27) becomes

$$\beta = -\exp\left\{-q^V g\left(X_T - c_T\right)\right\}\left[1 + aq^V\left(\frac{g'\left(X_T - c_T\right)}{f'\left(c_T\right)}\right)\right].$$

5.5 Translation-Invariant Preferences

Section 5.4 characterized the solution to the principal problem as an FBSDE system. The solution is complicated by the links among the utility and Lagrange multiplier processes. Under translation-invariant (TI) preferences, the Lagrange multiplier process is a constant and the solution dramatically simplifies to a two-step problem: first solve a single unlinked BSDE (which yields the optimal principal plan), and then plug the optimal plan into a single forward SDE. The solution can be constructed in a straightforward manner, and optimality easily confirmed.

[44] introduce TI recursive utility as a generalization of time-additive exponential utility. TI utility has the tractability of the latter, but allows more flexible modeling of risk aversion and intertemporal substitution. For the optimal portfolio choice problem, [44] show that the solution under TI preferences reduces to the solution of a single backward equation, even in the presence of constrained markets and non-traded income. We show below that the same simplification is achieved in the principal-agent problem.

We assume throughout this section that the set of intermediate consumption plans is $\widetilde{\mathcal{C}} = \mathcal{L}_2^-(\mathbb{R})$. Next we define TI preferences:

Definition 5.5. *We will say that the agent's and principal's preferences are TI if*

$$F(\omega,t,c,e,U,\Sigma)=f\left(\omega,t,\frac{c}{\gamma^U}-U,e,\Sigma\right),\quad F(T,c)=\frac{c}{\gamma^U},$$
$$G(\omega,t,c,V,\Sigma)=g\left(\omega,t,\frac{X(\omega,t)-c}{\gamma^V}-V,\Sigma\right),\quad G(\omega,T,c)=\frac{X(\omega,T)-c}{\gamma^V},$$

for some constants $\gamma^U,\gamma^V>0$ *and some functions* $f:\Omega\times[0,T]\times\mathbb{R}^{1+2d}\to\mathbb{R}$ *and* $g:\Omega\times[0,T]\times\mathbb{R}^{1+d}\to\mathbb{R}$*. The process* $X\in\mathcal{L}(\mathbb{R})$ *is interpreted as the principal's cash flow process.*

The dependence of the principal's aggregator on X is convenient for our applications, but neither extends nor restricts the generality because the aggregators are still allowed to depend on ω. For any given effort process $e\in\mathcal{E}$, it is easy to confirm the quasilinear relationships

$$U_t\left(c+k\gamma^U,e\right)=U_t(c,e)+k,\quad V_t\left(c+k\gamma^V,e\right)=V_t(c,e)-k,\quad \text{for all } c\in\mathcal{C},\ k\in\mathbb{R}, \tag{5.28}$$

where U and V solve the BSDE's (5.1) and (5.2), respectively. The tractability of the TI class all derives from (5.28).

Let

$$x_t^U=\frac{c_t}{\gamma^U}-U_t,\quad x_t^V=\frac{X_t-c_t}{\gamma^V}-V_t. \tag{5.29}$$

Redefining I from (5.6), the agent's optimal effort is given by $e_t=I\left(t,x_t^U,\Sigma_t^U\right)$ where $I:\Omega\times[0,T]\times\mathbb{R}^{d+1}\to\mathbb{R}^d$ solves

$$I(\omega,t,x,\Sigma)=\arg\max_{e\in\boldsymbol{E}}\left\{f(\omega,t,x,e,\Sigma)+\Sigma'e\right\},\quad t\in[0,T),$$

and the utility processes under agent optimality are therefore

$$dU_t = -\left\{ f\left(t, x_t^U, I\left(t, x_t^U, \Sigma_t^U\right), \Sigma_t^U\right) + \Sigma_t^{U\prime} I\left(t, x_t^U, \Sigma_t^U\right)\right\} dt + \Sigma_t^{U\prime} dB_t, \tag{5.30}$$

$$U_T = \frac{c_T}{\gamma^U},$$

$$dV_t = -\left\{ g\left(t, x_t^V, \Sigma_t^V\right) + \Sigma_t^{V\prime} I\left(t, x_t^U, \Sigma_t^U\right)\right\} dt + \Sigma_t^{V\prime} dB_t, \quad V_T = \frac{X_T - c_T}{\gamma^V}.$$

Because optimal effort is invariant to constant shifts in consumption (which follows because the agent's utility diffusion is invariant to such shifts), the quasilinear property is preserved.

The following examples give some special cases of agent TI preferences, each with an aggregator separable in consumption, effort and volatility. The effort-penalty function is given by $\psi : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ (typically assumed convex in e), and the volatility penalty is assumed either zero or quadratic. (The case of the principal is analogous, but with no effort penalty.)

Example 5.6 (risk-neutral agent)**.** Suppose the agent's aggregator is

$$f(\omega, t, x, e, \Sigma) = \beta x - \psi(\omega, t, e), \quad \beta \in \mathbb{R}.$$

Then time-t agent utility is expected (under P^e) discounted future consumption minus effort penalty:

$$U_t = E_t^e \left\{ \int_t^T e^{-\left(\beta/\gamma^U\right)(s-t)} \left(\frac{\beta}{\gamma^V} c_s^V - \psi(s, e_s)\right) ds + e^{-\left(\beta/\gamma^U\right)(T-t)} \frac{c_T^V}{\gamma^V} \right\}.$$

The following example obtains, as a special case, time-additive exponential utility with an effort penalty in the discounting term. This is the class examined, for example, in [27], [41] and [33]. The example also shows that recursive utility in the special case of no intermediate consumption and quadratic volatility penalty is equivalent to time-additive utility.

Example 5.7 (additive exponential with no intermediate consumption)**.** Suppose

$$f(\omega, t, x, e, \Sigma) = -\frac{q}{2}\Sigma'\Sigma - \psi(\omega, t, e), \quad q > 0.$$

Then the ordinally equivalent utility processes $u_t = -\exp(-qU_t)$ satisfies (under sufficient integrability)

$$u_t = -E_t^e \left\{ \exp\left(-q \left[\frac{c_T}{\gamma^U} - \int_t^T \psi(s, e_s)\, ds \right] \right) \right\}.$$

That is, u is standard additive exponential utility with coefficient of absolute risk aversion q/γ^U, and with effort affecting utility through the change of measure and the discounting of terminal consumption by the effort penalty.

The next example is the special case of additive exponential utility with intermediate consumption:

Example 5.8 (additive exponential)**.** Suppose

$$f(\omega, t, x, e, \Sigma) = -\exp(-x) - \frac{1}{2}\Sigma'\Sigma - \psi(\omega, t, e).$$

Then the ordinally equivalent process $u_t = -\exp(-U_t)$ has the solution (under sufficient integrability)

$$u_t = -E_t^e \left\{ \begin{array}{c} \int_t^T \exp\left(-\left[\frac{c_s}{\gamma^U} - \int_t^s \psi(r, e_r)\, dr \right] \right) ds \\ + \exp\left(-\left[\frac{c_T}{\gamma^U} - \int_t^T \psi(r, e_r)\, dr \right] \right) \end{array} \right\}.$$

5.5.1 Optimality

The following lemma shows the key simplifying property of the TI class: that the Lagrange multiplier process λ is a constant, equal to the ratio of the γ parameters (the same result can be deduced from the dynamic programming result in Theorem 5.3).[5]

Lemma 5.2. *Assume Condition 5.2 (a). Under the TI aggregators of Definition* 5.5*, at the optimum* (c, e)

$$\lambda_t = \frac{\gamma^U}{\gamma^V}, \quad t \in [0, T].$$

In particular, the participation constraint always binds.

Proof. On the one hand, letting $\varepsilon_0 = (1, \kappa)'$, Lemma 5.1 and Theorem 5.2 imply[6]

$$\lim_{\alpha \downarrow 0} \frac{\varepsilon_t^V \{V_t(c+\alpha) - V_t(c)\} + \varepsilon_t^U \{U_t(c+\alpha) - U_t(c)\}}{\alpha} = 0.$$

On the other hand, by quasilinearity (5.28), the left-hand side above equals $\varepsilon_t^U/\gamma^U - \varepsilon_t^V/\gamma^V$. So we get $\lambda_t = \varepsilon_t^U/\varepsilon_t^V = \gamma^U/\gamma^V$. The participation constraint binds because $\kappa = \lambda_0 = \gamma^U/\gamma^V > 0$ (see Theorem 5.2). □

The fact that the participation constraint binds is an intuitive result because constant shifts in consumption do not affect optimal agent effort. Any consumption plan resulting in a slack constraint could be improved upon by reducing the agent pay by a small constant process. In view of this we will pursue the dynamic programming approach of Section 5.4.2.

As explained in Section 5.4.2, the principal solves the pay process by choosing the intermediate consumption stream $\{c_t;\ t < T\}$, as well as the agent utility diffusion process Σ^U. This implies that lump-sum terminal consumption is $c_T = \gamma^U U_T$, where U_T is the

[5]Note that the proof of Theorem 5.4 doesn't rely on the lemma, and therefore doesn't assume Condition 5.2.

[6]It is trivial to extend Lemma 5.1 to obtain the time-t gradient needed here.

terminal value of the SDE

$$dU_t = -\left\{ f\left(t, x_t^U, I\left(t, x_t^U, \Sigma_t^U\right), \Sigma_t^U\right) + \Sigma_t^{U\prime} I\left(t, x_t^U, \Sigma_t^U\right)\right\} dt + \Sigma_t^{U\prime} dB_t, \tag{5.31}$$

$$U_0 = K.$$

To motivate the optimality result in Theorem 5.4 below, we suppose $\left(\hat{c}, \hat{\Sigma}^U\right)$ is the optimal plan, and (similar to Section 5.4.2) define for this plan the linear combination

$$Y_t = V_t + \lambda U_t, \quad \text{where } \lambda = \frac{\gamma^U}{\gamma^V},$$

which satisfies

$$dY_t = \mu_t^Y dt + \Sigma_t^{Y\prime} dB_t, \quad Y_T = X_T/\gamma^V. \tag{5.32}$$

Using the identity $x_t^V = -\lambda x_t^U + X_t/\gamma^V - (V_t + \lambda U_t)$ we get

$$\begin{aligned} -\mu_t^Y &= g\left(t, -\lambda \hat{x}_t^U + X_t/\gamma^V - Y_t, \Sigma_t^Y - \lambda \hat{\Sigma}_t^U\right) + \Sigma_t^{Y\prime} I\left(t, \hat{x}_t^U, \hat{\Sigma}_t^U\right) \\ &+ \lambda f\left(\hat{x}_t^U, I\left(t, \hat{x}_t^U, \hat{\Sigma}_t^U\right), \hat{\Sigma}_t^U\right), \qquad t \in [0, T), \\ &\text{where } \hat{x}_t^U = \frac{\hat{c}_t}{\gamma^U} - \hat{U}_t. \end{aligned} \tag{5.33}$$

A dynamic programming argument implies that for any other viable plan $\left(c, \Sigma^U\right)$ the drift term is smaller:

$$\begin{aligned} -\mu_t^Y &\geq g\left(t, -\lambda x_t^U + X_t/\gamma^V - Y_t, \Sigma_t^Y - \lambda \Sigma_t^U\right) + \Sigma_t^{Y\prime} I\left(t, x_t^U, \Sigma_t^U\right) \\ &+ \lambda f\left(x_t^U, I\left(t, x_t^U, \Sigma_t^U\right), \Sigma_t^U\right), \qquad t \in [0, T), \\ &\text{where } x_t^U = \frac{c_t}{\gamma^U} - U_t. \end{aligned} \tag{5.34}$$

Theorem 5.4. *Assume the TI aggregators of Definition* 5.5*. Suppose* $\left(Y, \Sigma^{Y}\right)$ *solves the BSDE* (5.32) *where* μ^{Y} *solves* (5.33) *for some viable* $\left(\hat{c}, \hat{\Sigma}^{U}\right)$*. Then* $\left(\hat{c}, \hat{\Sigma}^{U}\right)$ *is optimal if and only if* (5.34) *holds for any other viable plan* $\left(c, \Sigma^{U}\right)$*.*

Proof. See the appendix. □

It follows from the theorem that optimality is essentially equivalent to solving the Bellman equation

$$-\mu_t^Y = \max_{\left(x_t^U, \Sigma_t^U\right) \in \mathbb{R} \times \mathbb{R}^d} \{ g\left(t, -\lambda x_t^U + X_t/\gamma^V - Y_t, \Sigma_t^Y - \lambda \Sigma_t^U\right) + \Sigma_t^{Y\prime} I\left(t, x_t^U, \Sigma_t^U\right) + \lambda f\left(x_t^U, I\left(t, x_t^U, \Sigma_t^U\right), \Sigma_t^U\right) \}, \qquad \text{all } t \in [0, T). \tag{5.35}$$

That is, optimality reduces to maximizing, for each (ω, t), the negative of the drift of the linear combination of the aggregators. Writing the maximizing arguments of (5.35) as $x_t^U = \phi\left(t, X_t/\gamma^V - Y_t, \Sigma_t^Y\right)$ and $\Sigma_t^U = \psi\left(t, X_t/\gamma^V - Y_t, \Sigma_t^Y\right)$ for some functions $\phi : \Omega \times [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R}$ and $\psi : \Omega \times [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R}^d$, we obtain the following BSDE for $\left(Y, \Sigma^Y\right)$:

$$\begin{aligned} dY_t &= -\left\{ g\left(t, -\lambda x_t^U + X_t/\gamma^V - Y_t, \Sigma_t^Y - \lambda \Sigma_t^U\right) + \lambda f\left(t, x_t^U, e_t, \Sigma_t^U\right) \right\} dt \\ &\qquad + \Sigma_t^{Y\prime}\left(dB_t - I\left(t, x_t^U, \Sigma_t^U\right) dt\right), \\ Y_T &= X_T/\gamma^V, \\ x_t^U &= \phi\left(t, X_t/\gamma^V - Y_t, \Sigma_t^Y\right), \quad \Sigma_t^U = \psi\left(t, X_t/\gamma^V - Y_t, \Sigma_t^Y\right). \end{aligned} \tag{5.36}$$

If the maximization problem in (5.35) is well defined, and the BSDE has a unique solution, then optimality of $\left(x^U, \Sigma^U\right)$ follows.

The BSDE does not depend on either the agent or principal's utility process or utility diffusion. If f is a deterministic function (that is, f depends on ω only through its other arguments), then so is I, and if g is also deterministic, then X is the only source of uncertainty driving Y.

Let $Y_t(X)$ denote the solution to (5.36) corresponding to the cash-flow process X. This solution can be interpreted as a subjective time-t present value of the cash-flow process; a present value that depends on the preferences of both principal and agent. It is easily seen that Y inherits the quasilinearity property of the principal and agent, but with respect to the cash-flow process instead of consumption:

$$Y_t\left(X + k\gamma^V\right) = Y_t(X) + k, \quad \text{for all } k \in \mathbb{R} \text{ and } t \in [0, T].$$

It follows that the optimal $\left(x^U, e\right)$ is invariant to constant shifts in X, and, from the SDE (5.31), so are the utility process U and the optimal consumption plan c. Therefore any constant unit shift in the cash flow process all accrues to the principal, whose utility process increases by $1/\gamma^V$.

Solving the optimal principal utility under TI preferences in our moral hazard problem is therefore equivalent to solving a simpler TI utility problem with given consumption process X and modified TI aggregator. The solution immediately yields the optimal $\left(x^U, \Sigma^U\right)$, which can then be substituted into the forward equation (5.31) to solve for the agent's utility process, U, and the optimal consumption plan

$$c_t = \gamma^U \left\{x_t^U + U_t\right\}, \quad c_T = \gamma^U U_T. \tag{5.37}$$

The following example shows that additively separable agent and principal aggregators implies that Y also has a separable form.

Example 5.9 (separable absolute aggregators)**.** Suppose the aggregators are separable in x

and (e, Σ):

$$f\left(\omega, t, x^U, e, \Sigma\right) = h^U\left(\omega, t, x^U\right) + k^U\left(\omega, t, e, \Sigma\right), \tag{5.38}$$
$$g\left(\omega, t, x^V, \Sigma\right) = h^V\left(\omega, t, x^V\right) + k^V\left(\omega, t, \Sigma\right).$$

Optimal agent effort takes the form $e_t = I\left(t, \Sigma_t^U\right)$ where $I : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. Defining, for all $t \in [0,T)$,

$$H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) = \max_{x_t^U \in \mathbb{R}} \left\{ h^V\left(t, -\lambda x_t^U + \frac{X_t}{\gamma^V} - Y_t\right) + \lambda h^U\left(t, x_t^U\right)\right\}, \tag{5.39}$$
$$Q\left(t, \Sigma_t^Y\right) = \max_{\Sigma_t^U \in \mathbb{R}^d} \left\{ \begin{array}{c} k^V\left(t, \Sigma_t^Y - \lambda \Sigma_t^U\right) + \lambda k^U\left(t, I\left(t, \Sigma_t^U\right), \Sigma_t^U\right) \\ +\Sigma_t^{Y\prime} I\left(t, \Sigma_t^U\right) \end{array} \right\},$$

then the BSDE for Y is

$$dY_t = -\left\{ H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) + Q\left(t, \Sigma_t^Y\right)\right\} dt + \Sigma_t^Y dB_t, \quad Y_T = X_T/\gamma^V. \tag{5.40}$$

Example 5.10. If we assume (5.38) with the added restriction that $h^U = h^V$ (common rankings of deterministic consumption plans), then, denoting by h the common aggregator function, the optimum is $x_t^U = (1+\lambda)^{-1}\left(X_t/\gamma^V - Y_t\right)$ and therefore

$$H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) = (1+\lambda)\, h\left(t, (1+\lambda)^{-1}\left(\frac{X_t}{\gamma^V} - Y_t\right)\right).$$

The next example shows that if either principal or agent exhibits infinite elasticity of intertemporal substitution (h is affine) then H in (5.40) is affine (the case of no intermediate consumption, which corresponds to h^U and h^V depending only on (ω, t), is a special case).

Example 5.11 (infinite elasticity)**.** Let the aggregators take the separable form (5.38). We show that if either principal or agent's aggregator is affine in intermediate consumption (i.e., either exhibits infinite elasticity of intertemporal substitution):

$$h^{i}\left(\omega,t,x^{i}\right)=\alpha\left(\omega,t\right)+\beta\left(\omega,t\right)x^{i},\quad \text{for } i=U \text{ or } i=V,\quad \beta\in\mathcal{L}\left(\mathbb{R}^{+}\right), \tag{5.41}$$

then

$$H\left(\omega,t,x\right)=\kappa\left(\omega,t\right)+\beta\left(\omega,t\right)x \tag{5.42}$$

for some process $\kappa\in\mathcal{L}\left(\mathbb{R}\right)$, which we now specify. Optimality of the first equation in (5.39) is equivalent to (if $i=U$ then $-i$ denotes V, and vice versa) $\beta_{t}=h_{x}^{-i}\left(t,x_{t}^{-i}\right)$ and inverting we get $x_{t}^{-i}=\phi_{t}$ for some $\phi\in\mathcal{L}\left(\mathbb{R}\right)$. If $i=U$ then $\kappa_{t}=\lambda\alpha_{t}-\beta_{t}\phi_{t}+h^{V}\left(t,\phi_{t}\right)$. If instead $i=V$ then $\kappa_{t}=\alpha_{t}-\lambda\beta_{t}\phi_{t}+\lambda h^{U}\left(t,\phi_{t}\right)$.

5.5.2 Quadratic Penalties

We specialize the TI preferences to the case of quadratic volatility and effort penalties:

$$\begin{aligned} f\left(\omega,t,x^{U},e,\Sigma\right) &= h^{U}\left(\omega,t,x^{U}\right)+p^{U}\left(\omega,t\right)'\Sigma-\frac{1}{2}\Sigma' Q^{U}\left(\omega,t\right)\Sigma-\frac{1}{2}e' Q^{e}\left(\omega,t\right)e, \\ g\left(\omega,t,x^{V},\Sigma\right) &= h^{V}\left(\omega,t,x^{V}\right)+p^{V}\left(\omega,t\right)'\Sigma-\frac{1}{2}\Sigma' Q^{V}\left(\omega,t\right)\Sigma, \end{aligned} \tag{5.43}$$

where $Q^{e},Q^{U},Q^{V}\in\mathcal{L}\left(\mathbb{R}^{d\times d}\right)$ are assumed symmetric positive definite, and represent the effort and risk-aversion penalties; and $p^{U},p^{V}\in\mathcal{L}\left(\mathbb{R}^{d}\right)$ can be interpreted as differences in beliefs of the agent and principal from the true probability measure. We can interpret p^{U}, for example, as a measure of agent optimism in the sense that under the agent's subjective probability measure $P^{p^{U}}$ the drift of the Brownian increment dB_{t} is $p^{U}dt$ (that is, $dB_{t}^{p^{U}}=dB_{t}-p_{t}^{U}dt$ is standard Brownian motion under $P^{p^{U}}$).

Recall from Example 5.8 that the case of additive exponential utility is the special case with $p^{U}=p^{V}=0$ and $Q^{U}=Q^{V}=\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix, and h^{U} and h^{V}

are exponential functions. Let us define weight matrix $W \in \mathcal{L}\left(\mathbb{R}^{d\times d}\right)$ as

$$W_t = \left(\{Q_t^e\}^{-1} + \lambda Q_t^V + Q_t^U\right)^{-1}\left(\{Q_t^e\}^{-1} + \lambda Q_t^V\right), \quad t \in [0, T],$$

Lemma 5.3. *The BSDE for* $\left(Y, \Sigma^Y\right)$ *given the aggregators* (5.43) *is*

$$dY_t = -\left\{H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) + \mu_t^Y + p_t^{Y\prime}\Sigma_t^Y - \frac{1}{2}\Sigma_t^{Y\prime}Q_t^Y\Sigma_t^Y\right\}dt + \Sigma_t^{Y\prime}dB_t, \tag{5.44}$$

$$Y_T = X_T/\gamma^V,$$

where H *is defined in* (5.39) *and*

$$\begin{aligned} Q_t^Y &= \lambda^{-1}\left(Q_t^U W_t - \{Q_t^e\}^{-1}\right), \\ p_t^Y &= (\boldsymbol{I} - W_t)' p_t^V + W_t' p_t^U, \\ \mu_t^Y &= \frac{\lambda}{2}\left(p_t^U - p_t^V\right)'(\boldsymbol{I} - W_t)\left(Q_t^U\right)^{-1}\left(p_t^U - p_t^V\right). \end{aligned}$$

Optimal agent effort satisfies $e_t = \left(Q_t^e\right)^{-1}\Sigma_t^U$*. The optimal* x^U *satisfies the first equation in* (5.39)*, and the optimal agent utility diffusion is*

$$\Sigma_t^U = \lambda^{-1}W_t\Sigma_t^Y + (\boldsymbol{I} - W_t)\left(Q_t^U\right)^{-1}\left(p_t^U - p_t^V\right). \tag{5.45}$$

Lump-sum terminal consumption is given by $c_T = \gamma^U U_T$*, where* U_T *is the terminal value of the SDE*

$$dU_t = -\left\{h^U\left(t, x^U\right) + p_t^U\Sigma_t^U - \frac{1}{2}\Sigma_t^{U\prime}\left(Q_t^U - \left(Q_t^e\right)^{-1}\right)\Sigma_t^U\right\}dt + \Sigma_t^{U\prime}dB_t, \tag{5.46}$$

$$U_0 = K.$$

Proof. See the appendix. □

Recalling that the subjective cash-flow volatility satisfies $\Sigma_t^Y = \Sigma_t^V + \lambda\Sigma_t^U$, then (5.46) in the case $p^U = p^V$ gives the optimal volatility-sharing rule[7] $\Sigma_t^V = (\boldsymbol{I} - W_t)\,\Sigma_t^Y$ and $\lambda\Sigma_t^U = \Sigma_t^Y$, $t \in [0, T]$. If the agent is risk neutral, $Q_t^U = 0$, then $W_t = \boldsymbol{I}$ and all the time-t risk is optimally borne by the agent. If the principal is risk-neutral and agent effort is infinitely expensive, $Q_t^V = \{Q_t^e\}^{-1} = 0$, then $W_t = 0$ and the principal bears all the time-t risk.

In the case of scalar penalties we obtain below a convenient expression for the lump-sum terminal pay, as well as simplified comparative statics.[8]

Example 5.12. Suppose $Q_t^U = q^U\boldsymbol{I}$, $Q_t^V = q^V\boldsymbol{I}$, $Q_t^e = q^e\boldsymbol{I}$, where $\boldsymbol{I}$ is the $d \times d$ identity matrix and $q^e, q^U, q^V \in \mathbb{R}_+$. Then $W_t = w\boldsymbol{I}$ and $Q_t^Y = q^Y\boldsymbol{I}$ where

$$w = \frac{1 + \lambda q^e q^V}{1 + \lambda q^e q^V + q^e q^U}, \quad q^Y = \frac{1}{\lambda}\left(q^U w - \frac{1}{q^e}\right), \tag{5.47}$$

and optimal effort satisfies

$$e_t = \frac{w}{\lambda q^e}\Sigma_t^Y + \frac{1 - w}{q^e q^U}\left(p_t^U - p_t^V\right). \tag{5.48}$$

The coefficient of Σ^Y in (5.48) is decreasing in q^e and q^U, but increasing in q^V. We can rearrange (5.44) to obtain an expression for $\Sigma_t^{Y\prime} dB_t$, and substitute into (5.46) together

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[7] Note that both W_t and $\boldsymbol{I} - W_t$ are positive definite for all (ω, t).

[8] See the appendix for a derivation of the example.

with $\Sigma_t^U = q^e e_t$, with e_t given by (5.48), to get[9]

$$c_T = w \left(X_T - E\left(X_T\right) + \gamma^V \int_0^T \left\{ H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) - E\, H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) \right\} dt \right) \tag{5.49}$$

$$+ \gamma^U \left(K - \int_0^T h^U\left(t, x^U\right) dt \right)$$

$$+ \frac{\gamma^U}{2}\left(q^U - \frac{1}{q^e}\right) \int_0^T \left\| \frac{w}{\lambda}\Sigma_t^Y + \left(\frac{1-w}{q^U}\right)\left(p_t^U - p_t^V\right) \right\|^2 dt$$

$$- w\gamma^V \int_0^T \Sigma_t^{Y\prime} p_t^U \, dt + \gamma^U \left(\frac{1-w}{q^U}\right) \int_0^T \left(p_t^U - p_t^V\right)\left(dB_t - p_t^U \, dt\right)$$

$$+ \gamma^V w \left\{ \int_0^T \left[\begin{array}{c} \left\{ \mu_t^Y + p_t^{Y\prime}\Sigma_t^Y - \frac{q^Y}{2}\Sigma_t^{Y\prime}\Sigma_t^Y \right\} \\ -E\left\{ \mu_t^Y + p_t^{Y\prime}\Sigma_t^Y - \frac{q^Y}{2}\Sigma_t^{Y\prime}\Sigma_t^Y \right\} \end{array} \right] dt \right\}.$$

The first component of the agent's terminal pay is a fixed share of the unexpected component of terminal cash flow, as well as a share of the unexpected cumulative transformed intermediate consumption. The second compensation term is the participation constraint adjusted for utility of intermediate agent consumption. When principal and agent agree ($p^U = p^V$), the third term increases terminal agent pay proportional to the quadratic variation of Y, which in typical models is driven by uncertainty about the cash flow process. The next two terms represent adjustments for agent optimism as well as an additional disagreement term. The final term compensates for the unexpected cumulative drift in the process Y. The proportion w (from (5.47)) is decreasing in q^e and q^U but increasing in λq^V. The more risk averse the agent and the more costly the effort, the greater the share is needed to incentivize effort. A more risk averse principal will optimally keep a smaller share of the risky cash flow.

[9] Recall $U_T = c_T/\gamma^U$, $Y_0 = V_0 + \lambda K$, and $x_t^V = X_t/\gamma^V - Y_t - \lambda x_t^U$.

Before presenting the main example of this section, we give the solution to the BSDE (5.44) in the special case when H is affine and $Q^Y = 0$.

Example 5.13 ($Q^Y = 0$)**.** Suppose the conditions of Example 5.11 (infinite elasticity), which implies the affine form of H in (5.42), and suppose $Q^Y_t = 0$ for $t \in [0, T]$. Then Y is obtained via risk neutral discounting:

$$Y_t = E^{p^Y}_t \left\{ \int_t^T e^{-\int_t^s \beta_u du} \left(\beta_s \frac{X_s}{\gamma^V} + \kappa_s + \mu^Y_s \right) ds + e^{-\int_t^T \beta_u du} \frac{X_T}{\gamma^V} \right\}. \quad (5.50)$$

When H is affine (see Example 5.11), the BSDE (5.44) (after substituting (5.42)) is essentially the same as the BSDE (21) in [44] (which applies to the optimal portfolio problem). They provide sufficient conditions on the BSDE parameters and Markovian state-variable processes such that the Y will be an affine function of the state variables, the coefficients of which satisfy a set of Riccati ODEs. The following simple example considers a one-dimensional state variable representing the cash-flow process X.

Example 5.14. Assume that the cash-flow process satisfies the forward SDE

$$dX_t = \left(\mu^X - \beta^X X_t \right) dt + \sum_{i=1}^{d} \sigma^{Xi} \sqrt{a^i + b^i X_t} dB^i_t,$$

for some $\mu^X, \beta^X, b^i \in \mathbb{R}_+$, and a^i, $\sigma^{Xi} \in \mathbb{R}$, $i = 1, \ldots, d$. Let $\sigma^X = \left(\sigma^{X1}, \ldots, \sigma^{Xd} \right)'$. Suppose h^U and h^V are deterministic functions, and assume the conditions in Example 5.11 which imply the affine form for H in (5.42); for simplicity we assume β is constant. Also, suppose all the preference parameters (Q^U, Q^V, Q^e, p^U, p^V) are deterministic. The solution for Y in the two cases below is affine in the state variable:

$$Y_t = \Phi_0 (t) + \Phi_1 (t) X_t, \quad (5.51)$$

where the deterministic processes Φ_0 and Φ_1 are given in closed form below.

a) (Ornstein-Uhlenbeck dynamics) Let $a^i = 1$ and $b^i = 0$, $i = 1, \ldots, d$. Then

$$\Phi_1(t) = \frac{1}{\gamma^V} e^{-\left(\beta^X+\beta\right)(T-t)} + \frac{\beta}{\gamma^V\left(\beta^X+\beta\right)}\left\{1 - e^{-\left(\beta^X+\beta\right)(T-t)}\right\},$$

$$\Phi_0(t) = \int_t^T e^{-\beta(s-t)}\begin{pmatrix} \Phi_1(s)\,\mu^X + \kappa_s + \mu_s^Y + \Phi_1(s)\,p_s^{Y\prime}\sigma^X \\ -\frac{1}{2}\Phi_1(s)^2\,\sigma^{X\prime}Q_s^Y\sigma^X \end{pmatrix} ds.$$

Substituting $\Sigma_t^Y = \Phi_1(t)\,\sigma^X$, then optimal agent utility diffusion is deterministic:

$$\Sigma_t^U = \lambda^{-1}W_t\Phi_1(t)\,\sigma^X + (\boldsymbol{I} - W_t)\left(Q_t^U\right)^{-1}\left(p_t^U - p_t^V\right).$$

As cash-flow mean reversion blows up, $\beta^X \to \infty$, then $\Phi_1 \to 0$ and, if $p^U = p^V$, we get $\Sigma^U \to 0$; that is, the impact of effort on cash-flow drift becomes more transient, and the optimal contract transfers no cash-flow risk to the agent.. If there is neither mean reversion nor intermediate consumption, $\beta = \beta^X = 0$, then $\Phi_1 = 1/\gamma^V$ and therefore $\Sigma^Y = \sigma^X/\gamma^V$.

b) (Square-root dynamics) We further assume unbiased beliefs ($p^V = p^U = 0$), and constant diagonal preferences ($Q_t^i = q^i\boldsymbol{I}$, $q^i > 0$, $i \in \{U, V, e\}$). Using the notation (5.47) we have

$$\Phi_1(t) = \frac{r^+\left(1/\gamma^V - r^-\right) - e^{-\left(r^+-r^-\right)(T-t)}\left(1/\gamma^V - r^+\right)r^-}{1/\gamma^V - r^- - e^{-\left(r^+-r^-\right)(T-t)}\left(1/\gamma^V - r^+\right)},$$

$$\Phi_0(t) = \int_t^T e^{-\beta(s-t)}\left(\Phi_1(s)\,\mu^X + \kappa_s + \frac{1}{2}q^Y\Phi_1(s)^2\sum_{i=1}^d\left(\sigma^{Xi}\right)^2 a^i\right) ds,$$

where (assuming the expression in the radical is positive)

$$r^{\pm} = \frac{\left(\beta^X + \beta\right) \pm \sqrt{\left(\beta^X + \beta\right)^2 + 2\beta\bar{q}/\gamma^V}}{\bar{q}}, \quad \bar{q} = q^Y \sum_{i=1}^{d} \left(\sigma^{Xi}\right)^2 b^i.$$

Substituting

$$\Sigma_t^{Yi} = \Phi_1(t)\,\sigma^{Xi}\sqrt{a^i + b^i X_t},$$

then optimal agent utility diffusion is

$$\Sigma_t^{Ui} = \lambda^{-1} w \Phi_1(t)\,\sigma^{Xi}\sqrt{a^i + b^i X_t}, \quad i = 1, \ldots, d.$$

and therefore effort, $e_t = (q^e)^{-1}\,\Sigma_t^U$, is increasing in the cash-flow process. Terminal consumption is

$$\begin{aligned} c_T = {} & w\left(X_T - E\left(X_T\right) + \beta\int_0^T \left\{X_t - E\left(X_t\right)\right\} dt\right) + \gamma^U\left(K - \int_0^T h^U\left(t, x^U\right) dt\right) \\ & - \frac{q^Y}{2}\gamma^V w \sum_{i=1}^{d}\left(\sigma^{Xi}\right)^2 \int_0^T \Phi_1(t)^2\, b^i \left\{X_t - E\left(X_t\right)\right\} dt \\ & + \frac{\gamma^U}{2}\left(q^U - \frac{1}{q^e}\right)\left(\frac{w}{\lambda}\right)^2 \int_0^T \sum_{i=1}^{d}\left\{\sigma^{Xi}\Phi_1(t)\right\}^2 \left(a^i + b^i X_t\right) dt. \end{aligned}$$

Terminal consumption is a fixed fraction w of terminal cash flow plus a cumulative weighted average of the cash flow over the agent's lifetime. The last term on the right represents compensation for risk bearing, which could be negative if q^e is small enough (inducing a desire for risk by the agent), and includes a stochastic component compensating the agent for the volatility of the cash-flow process over the life of the contract when b is nonzero.

For the special case of no intermediate consumption ($\beta = \kappa = 0$) we have

$$\Phi_1(t) = \begin{cases} \left\{ e^{\beta^X (T-t)} \gamma^V + \frac{\bar{q}}{2\beta^X} \left(e^{\beta^X (T-t)} - 1 \right) \right\}^{-1} & \text{if} \quad \beta^X \neq 0 \\ \left\{ \gamma^V + \frac{\bar{q}}{2} (T-t) \right\}^{-1} & \text{if} \quad \beta^X = 0 \end{cases}$$

5.6 Appendix

5.6.1 Proofs

We will start by presenting a Comparison Lemma for BSDE. It is based on [4]. The following lemma is a variation of their result with a similar proof.

Lemma 5.4 (Comparison)**.** *Suppose* $\left(U^i, \Sigma^i\right)$*,* $i \in \{a, b\}$*, solve the BSDE*

$$dU_t^i = -f^i\left(t, U_t^i, \Sigma_t^i\right) dt + \Sigma_t^{i\prime} dB_t, \quad U_T^i = f^i(T), \quad i \in \{a, b\},$$

where $f^i : \Omega \times [0, T) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *and* $f^i(T) : \Omega \to \mathbb{R}$*. Assume that there exist a process* $\alpha(t) \geq 0$ *and two constants* $\beta \geq 0, \gamma > 0$ *such that* $f^i,\ i \in \{a, b\}$ *satisfies the following conditions,*

a) for all $(t, \Sigma, u, u') \in [0, T] \times \mathbb{R}^{d+2}$,

$$\left| f^i(t, u, \Sigma) - f^i(t, u', \Sigma) \right| \leq \beta \left| u - u' \right|;$$

b) for all $(t, u, \Sigma) \in [0, T] \times \mathbb{R}^{d+1}$,

$$\left| f^i(t, u, \Sigma) \right| \leq \alpha(t) + \beta |u| + \frac{\gamma}{2} \|\Sigma\|^2;$$

c) $\int_0^T \alpha(s) ds$ *,* $\left| f^i(T) \right|$ *has exponential moments of all order;*

d) $f^i(\omega, t, u, \cdot)$ *is concave for all* $\omega, t, u \in \Omega \times [0, T) \times \mathbb{R}$.

If

$$f^a(t,u,\Sigma) \geq f^b(t,u,\Sigma), \quad (t,u,\Sigma) \in [0,T) \times \mathbb{R}^{d+1}, \quad f^a(T) \geq f^b(T), \tag{5.52}$$

then $U_t^a \geq U_t^b$*,* $t \in [0,T]$*. If the inequalities* (5.52) *are reversed then* $U_t^a \leq U_t^b$*,* $t \in [0,T]$*.*

Lemma 5.5. *Suppose* $\left(U, \Sigma^U\right)$ *and* $\left(\tilde{U}, \tilde{\Sigma}^U\right)$ *solve the BSDE* (5.1) *for some feasible plans* (c,e) *and* $(\tilde{c},\tilde{e})$*, respectively. If*

$$F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} e_t \geq F\left(t, \tilde{c}_t, \tilde{e}_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} \tilde{e}_t, \quad t \in [0,T), \tag{5.53}$$
$$F\left(T, c_T\right) \geq F\left(T, \tilde{c}_T\right),$$

then $U_t \geq \tilde{U}_t$*,* $t \in [0,T]$*. If the inequalities in* (5.53) *are reversed then* $U_t \leq \tilde{U}_t$*.*

Proof. Define the nonnegative process η as the difference between the left and right-hand sides of (5.53). From (5.1) we have

$$dU_t = -\left\{\eta_t + F\left(t, \tilde{c}_t, \tilde{e}_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} \tilde{e}_t\right\} dt + \Sigma_t^{U\prime} dB_t, \quad U_T = F\left(T, c_T\right).$$

Compare to

$$d\tilde{U}_t = -\left\{\eta_t + F\left(t, \tilde{c}_t, \tilde{e}_t, \tilde{U}_t, \Sigma_t^U\right) + \tilde{\Sigma}_t^{U\prime} \tilde{e}_t\right\} dt + \tilde{\Sigma}_t^{U\prime} dB_t, \quad U_T = F\left(T, c_T\right),$$

and apply Lemma 5.4 (the required conditions of the lemma are satisfied by Definition 5.1; note that $F(t,c,e,U,\Sigma)$ is concave in (U,Σ)). □

Proof of Theorem 5.1

1) (Sufficiency) Lemma 5.5 implies that $U_0(c,e) \geq U_0(c,\tilde{e})$ for any $\tilde{e} \in \mathcal{E}$, and therefore e is optimal.

2) (Necessity) Suppose e is optimal and (5.4) is violated for some $\bar{e} \in \mathcal{E}$. Let

$$\tilde{e}_t = \begin{cases} e_t & \text{if } F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} e_t \geq F\left(t, c_t, \bar{e}_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} \bar{e}_t, \\ \bar{e}_t & \text{otherwise.} \end{cases}$$

By Remark 5.1 it follows that $\tilde{e} \in \mathcal{E}$. Then (5.53) holds with the inequalities reversed (the inequalities are strict). Lemma 5.5 implies $U_0(c, e) < U_0(c, \tilde{e})$. But this contradicts the supposed optimality of e.

Proof of Theorem 5.3

Consider any viable plan $\left(c, \Sigma^U\right)$, and let $V = V(c) \in \mathcal{E}^*$ and $J_t = Y_t - \lambda_t (U_t - K) \in \mathcal{E}^*$. We will show that $V_0 \leq J_0$, with equality holding if $\left(c, \Sigma^U\right) = \left(\hat{c}, \hat{\Sigma}^U\right)$. The terminal value is

$$J_T = G(T, \hat{c}_T) + \lambda_T \left\{F(T, \hat{c}_T) - F(T, c_T)\right\} = G(T, c_T) + \eta_T$$

where (using the concavity of G and F in c, and $\lambda_T > 0$)

$$\begin{aligned} \eta_T &= G(T, \hat{c}_T) - G(T, c_T) + \lambda_T \left\{F(T, \hat{c}_T) - F(T, c_T)\right\} \\ &\geq G_c(T, \hat{c}_T)(\hat{c}_T - c_T) + \lambda_T F_c(T, \hat{c}_T)(\hat{c}_T - c_T) \\ &= 0. \end{aligned}$$

Now define η_t, $t \in [0, T)$, as the nonnegative process which is the difference between left and right-hand sides of (5.22). Applying Ito's lemma we get

$$dJ_t = dY_t - \lambda_t dU_t - (U_t - K)\, d\lambda_t - d\lambda_t dU_t.$$

Substituting $\mu_t^J = \mu_t^Y - \mu_t^\lambda (U_t - K) + \lambda_t F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) - \Sigma_t^{\lambda\prime} \Sigma_t^U$, using definition of η_t and (5.22), we get

$$dJ_t = -\left\{\eta_t + G\left(t, c_t, J_t, \Sigma_t^J\right)\right\} dt + \Sigma_t^{J\prime}\left(dB_t - I\left(t, c_t, U_t, \Sigma_t^U\right) dt\right),$$
$$J_T = G\left(T, c_T\right) + \eta_T.$$

Comparing to the BSDE

$$dV_t = -G\left(t, c_t, V_t, \Sigma_t^V\right) dt + \Sigma_t^{V\prime}\left(dB_t - I\left(t, c_t, U_t, \Sigma_t^U\right) dt\right), \quad V_T = G\left(T, c_T\right)$$

and applying Lemma 5.4 implies that $J_0 \geq V_0\left(c, \Sigma^U\right)$, with strict equality holding for the optimal plan. (Note that a binding participation constraint implies $J_0 = Y_0$.)

Proof of Theorem 5.4

Sufficiency: This is a special case of Theorem 5.3; however we give a self contained proof for completeness. Consider any viable plan $\left(c, \Sigma^U\right)$, and let $V = V\left(c, \Sigma^U\right)$ denote the principal's utility (from the solution $\left(V, \Sigma^V\right)$ to (5.30)). Define $J_t = Y_t - \lambda U_t$. We will show that $V_0 \leq J_0$, with equality holding if $\left(c, \Sigma^U\right) = \left(\hat{c}, \hat{\Sigma}^U\right)$. Define nonnegative process η as the difference between left and right-hand sides of (5.34). Substituting the dynamics of Y and U into $dJ_t = dY_t - \lambda dU_t$, together with $\Sigma^J = \Sigma^Y - \lambda\Sigma^U$, yields

$$dJ_t = -\left\{\eta_t + g\left(t, \frac{X_t - c_t}{\gamma^V} - J_t, \Sigma_t^J\right) + \Sigma_t^{J\prime} I\left(t, x_t^U, \Sigma_t^U\right)\right\} dt + \Sigma_t^{J\prime} dB_t,$$
$$J_T = \frac{X_T - c_T}{\gamma^V}.$$

Comparing to the BSDE

$$dV_t = -\left\{g\left(t, \frac{X_t - c_t}{\gamma^V} - V_t, \Sigma_t^V\right) + \Sigma_t^{V\prime} I\left(t, x_t^U, \Sigma_t^U\right)\right\} dt + \Sigma_t^{V\prime} dB_t,$$
$$V_T = \frac{X_T - c_T}{\gamma^V},$$

and applying Lemma 5.4 implies that $J_0 \geq V_0\left(c^V, e\right)$, where $e_t = I\left(t, x_t^U, \Sigma_t^U\right)$, with strict equality holding for the optimal plan.

Necessity: Let $\left(\hat{c}, \hat{\Sigma}^U\right)$ be the optimal viable plan. Consider some viable plan $\left(c, \Sigma^U\right)$. Define η as the difference between left and right-hand sides of (5.34). Suppose that $\eta_t < 0$ on some subset of $\Omega \times [0, T]$ that belongs to $\mathcal{F} \times B_{[0,T]}$ with a strictly positive $P \otimes l$ (where l is the Lebesgue measure on $[0, T]$) measure. Define, for $t \in [0, T)$,

$$\left(\tilde{x}_t^U, \tilde{\Sigma}_t^U\right) = \begin{cases} \left(x_t^U, \Sigma_t^U\right) & \text{if } \eta_t < 0, \\ \left(\hat{x}_t^U, \hat{\Sigma}_t^U\right) & \text{otherwise;} \end{cases}$$

and $\tilde{e}_t = I\left(t, \tilde{x}_t^U, \tilde{\Sigma}_t^U\right)$. By Remark 5.1 it follows that $(\tilde{c}, \tilde{e}) \in \mathcal{C} \times \mathcal{E}$. Let $\tilde{U}$ solve the SDE

$$d\tilde{U}_t = -\left\{f\left(t, \tilde{x}_t^U, \tilde{e}_t, \tilde{\Sigma}_t^U\right) + \tilde{\Sigma}_t^{U\prime}\tilde{e}_t\right\}dt + \tilde{\Sigma}_t^{U\prime}dB_t, \quad U_0 = K;$$

and define $\tilde{c}_t = \gamma^U\left(\tilde{x}_t^U + \tilde{U}_t\right)$. Then $\tilde{J}_t = Y_t - \lambda\tilde{U}_t$ satisfies the BSDE

$$\begin{aligned} d\tilde{J}_t &= -\left\{\min\left(0, \eta_t\right) + g\left(t, \frac{X_t - \tilde{c}_t}{\gamma^V} - \tilde{J}_t, \tilde{\Sigma}_t^J\right) + \tilde{\Sigma}_t^{J\prime}\tilde{e}_t\right\}dt + \tilde{\Sigma}_t^{J\prime}dB_t, \\ \tilde{J}_T &= \frac{X_T - \tilde{c}_T}{\gamma^V}, \end{aligned}$$

where $\tilde{c}_T = \gamma^U\tilde{U}_T$. Comparing to the BSDE

$$d\tilde{V}_t = -\left\{g\left(t, \frac{X_t - \tilde{c}_t}{\gamma^V} - \tilde{V}_t, \tilde{\Sigma}_t^V\right) + \tilde{\Sigma}_t^{V\prime}\tilde{e}_t\right\}dt + \tilde{\Sigma}_t^{V\prime}dB_t, \quad \tilde{V}_T = \frac{X_T - \tilde{c}_T}{\gamma^V},$$

Lemma 5.4 implies $\tilde{J}_0 < \tilde{V}_0(\tilde{c}, \tilde{e})$. That is, $Y_0 - \lambda K = V_0\left(\hat{c}, \hat{e}\right) < \tilde{V}_0\left(\tilde{c}, \tilde{e}\right)$, where $\hat{e}_t = I\left(t, \hat{x}_t^U, \hat{\Sigma}_t^U\right)$, which contradicts optimality.

Proof of Lemma 5.3

Substitute $f_\Sigma = -Q^U\Sigma^U + p^U$, $g_\Sigma = -Q^V\Sigma^V + p^V$, and $I_e = Q^e$ into the second

FOC in (5.26) (recall $\Sigma^U = I = -f_e$) to get (omitting time subscripts throughout)

$$(Q^e)^{-1}\Sigma^V = \lambda\left(Q^U\Sigma^U - Q^V\Sigma^V - p^U + p^V\right)$$

and therefore

$$\Sigma^V = \lambda\left(\left[Q^e\right]^{-1} + \lambda Q^V\right)^{-1}\left(Q^U\Sigma^U - p^U + p^V\right). \tag{5.54}$$

Substitute (5.54) into $\Sigma^Y = \Sigma^V + \lambda\Sigma^U$ to get

$$\Sigma^Y = -\lambda\left(\left[Q^e\right]^{-1} + \lambda Q^V\right)^{-1}\left(p^U - p^V\right) + \lambda\left\{\boldsymbol{I} + \left(\left[Q^e\right]^{-1} + \lambda Q^V\right)^{-1} Q^U\right\}\Sigma^U.$$

Substituting $\left\{\boldsymbol{I} + \left(\left[Q^e\right]^{-1} + \lambda Q^V\right)^{-1} Q^U\right\} = W^{-1}$ and solving for Σ^U we get

$$\Sigma^U = \lambda^{-1} W \Sigma^Y + \left(\boldsymbol{I} - W\right)\left(Q^U\right)^{-1}\left(p^U - p^V\right) \tag{5.55}$$

where we have used the identity $W\left(\left[Q^e\right]^{-1} + \lambda Q^V\right)^{-1} = \left(\boldsymbol{I} - W\right)\left(Q^U\right)^{-1}$. Substituting the same identity into (5.54) we get

$$\begin{aligned}\Sigma^V &= \left(\boldsymbol{I} - W\right)\Sigma^Y + \\ &\quad \lambda W^{-1}\left(\boldsymbol{I} - W\right)\left(Q^U\right)^{-1}\left\{Q^U\left(\boldsymbol{I} - W\right)\left(Q^U\right)^{-1} - \boldsymbol{I}\right\}\left(p^U - p^V\right)\end{aligned}$$

and therefore

$$\Sigma^V = \left(\boldsymbol{I} - W\right)\Sigma^Y - \lambda\left(\boldsymbol{I} - W\right)\left(Q^U\right)^{-1}\left(p^U - p^V\right). \tag{5.56}$$

Substituting $e = (Q^e)^{-1} \Sigma^U$, the dynamics of Y are (omitting the arguments to H)

$$dY = -\left\{ \begin{array}{c} H + p^{V\prime}\Sigma^V + \lambda p^{U\prime}\Sigma^U - \frac{1}{2}\Sigma^{V\prime} Q^V \Sigma^V \\ -\frac{\lambda}{2}\Sigma^{U\prime}\left\{(Q^e)^{-1} + Q^U\right\}\Sigma^U + \Sigma^{Y\prime}(Q^e)^{-1}\Sigma^U \end{array} \right\} dt + \Sigma^Y dB.$$

Substituting (5.55) and (5.56) we obtain

$$p^{V\prime}\Sigma^V + \lambda p^{U\prime}\Sigma^U = p^{Y\prime}\Sigma^Y + \lambda\left(p^U - p^V\right)(\boldsymbol{I} - W)\left(Q^U\right)^{-1}\left(p^U - p^V\right) \quad (5.57)$$

and[10]

$$\begin{aligned} & -\frac{1}{2}\left\{\Sigma^{V\prime}Q^V\Sigma^V + \lambda\Sigma^{U\prime}\left\{(Q^e)^{-1} + Q^U\right\}\Sigma^U - 2\Sigma^{Y\prime}(Q^e)^{-1}\Sigma^U\right\} \quad (5.58) \\ & = -\frac{1}{2}\Sigma^{Y\prime}Q^Y\Sigma^Y \\ & -\frac{\lambda}{2}\left(p^U - p^V\right)'\left(Q^U\right)^{-1}(\boldsymbol{I} - W)'\left\{ \begin{array}{c} \lambda Q^V \\ + (Q^e)^{-1} + Q^U \end{array} \right\}(\boldsymbol{I} - W)\left(Q^U\right)^{-1}\left(p^U - p^V\right), \end{aligned}$$

where

$$\begin{aligned} Q^Y &= (\boldsymbol{I} - W)' Q^V (\boldsymbol{I} - W) \\ &+ \lambda^{-1} W'\left\{(Q^e)^{-1} + Q^U\right\} W - \lambda^{-1}(Q^e)^{-1} W - \lambda^{-1} W'(Q^e)^{-1} \\ &= Q^V(\boldsymbol{I} - W) - \lambda^{-1}(Q^e)^{-1} W \\ &= \lambda^{-1}\left(Q^U W - (Q^e)^{-1}\right), \end{aligned}$$

where the last equality is easily confirmed (and the symmetry of Q^Y is also easily con-

[10]It is easy to confirm that the cross term of the form $\Sigma^Y \{\} \left(p^U - p^V\right)$ (and its transpose) is zero using

$$\lambda (I - W)' Q^V - W'\left\{(Q^e)^{-1} + Q^U\right\} + (Q^e)^{-1} = 0.$$

firmed).

Finally, substitute[11]

$$\left(Q^U\right)^{-1}(\boldsymbol{I}-W)'\left\{(Q^e)^{-1}+\lambda Q^V+Q^U\right\}(\boldsymbol{I}-W)\left(Q^U\right)^{-1}=\left(Q^U\right)^{-1}(\boldsymbol{I}-W)'$$

(obtained using the identity $\left\{(Q^e)^{-1}+\lambda Q^V+Q^U\right\}(\boldsymbol{I}-W)\left(Q^U\right)^{-1}=\boldsymbol{I}$) into the last term of (5.58) and add to (5.57) to get

$$\mu^Y=\frac{\lambda}{2}\left(p^U-p^V\right)(\boldsymbol{I}-W)\left(Q^U\right)^{-1}\left(p^U-p^V\right).$$

5.6.2 Derivation of Examples

Derivation of Example 5.12

Substitute optimal agent utility diffusion

$$\Sigma_t^U=\lambda^{-1}w\Sigma_t^Y+\left(\frac{1-w}{q^U}\right)\left(p_t^U-p_t^V\right)$$

into the agent's SDE (5.46) and integrate to get

$$\begin{aligned}U_T-K=&-\int_0^T\left\{h^U\left(t,x^U\right)-\frac{1}{2}\left(q_t^U-(q_t^e)^{-1}\right)\left\|\begin{matrix}\lambda^{-1}w\Sigma_t^Y\\+\left(\frac{1-w}{q^U}\right)\left(p_t^U-p_t^V\right)\end{matrix}\right\|^2\right\}dt\\&+\int_0^T\left(\frac{1-w}{q^U}\right)\left(p_t^U-p_t^V\right)\left(dB_t-p_t^U dt\right)+\lambda^{-1}w\int_0^T\Sigma_t^Y\left(dB_t-p_t^U dt\right).\end{aligned}$$

[11] Note that the following implies symmetry of $(I-W)\left(Q^U\right)^{-1}$.

Now integrate the BSDE for Y to obtain

$$\int_0^T \Sigma_t^Y \left(dB_t - p_t^U dt\right) = \int_0^T \left\{ \begin{array}{c} H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) + \mu_t^Y \\ +(1-w)\left(p_t^V - p_t^U\right)' \Sigma_t^Y - \frac{1}{2} q^Y \Sigma_t^{Y\prime} \Sigma_t^Y \end{array} \right\} dt$$
$$+ X_T/\gamma^V - Y_0,$$

and substitute, along with $c_T = \gamma^U U_T$, and (from (5.44))

$$Y_0 = E \int_0^T \left\{ H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) + \mu_t^Y + p_t^{Y\prime} \Sigma_t^Y - \frac{q^Y}{2} \Sigma_t^{Y\prime} \Sigma_t^Y \right\} dt + E\left(X_T/\gamma^V\right)$$

to get the result.

Derivation of Example 5.14

Define $\sigma^i = \sigma^{Xi} \sqrt{a^i + b^i X_t}$, $i = 1, \ldots, d$, and apply Ito's lemma to $Y_t = \Phi_0(t) + \Phi_1(t) X_t$ to get

$$0 = \dot{\Phi}_0(t) + \dot{\Phi}_1(t) X_t + \Phi_1(t)\left(\mu^X - \beta^X X_t\right) + \kappa_t + \beta_t \left(\frac{X_t}{\gamma^V} - \Phi_0(t) - \Phi_1(t) X_t\right)$$
$$+ \mu_t^Y + \Phi_1(t) p_t^{Y\prime} \sigma - \frac{1}{2} \Phi_1(t)^2 \sigma' Q_t^Y \sigma.$$

For the case $Q_t^Y = q^Y I$ and $p^Y = 0$ (the extension to nonzero p^Y and nondiagonal Q_t^Y when $b = 0$ is obvious), the resulting Riccati system is

$$0 = \dot{\Phi}_0(t) + \Phi_1(t) \mu^X + \kappa_t + \mu_t^Y - \beta \Phi_0(t) - \frac{1}{2} q^Y \Phi_1(t)^2 \sum_{i=1}^d \left(\sigma^{Xi}\right)^2 a^i,$$

$$\Phi_0(T) = 0,$$

$$0 = \dot{\Phi}_1(t) + \frac{\beta}{\gamma^V} - \Phi_1(t)\left(\beta^X + \beta\right) - \frac{1}{2} q^Y \Phi_1(t)^2 \sum_{i=1}^d \left(\sigma^{Xi}\right)^2 b^i,$$

$$\Phi_1(T) = 1/\gamma^V.$$

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