EFFECTS UPON THE FACTORIAL SOLUTION OF ROTATING VARYING NUMBERS OF FACTORS WITH DIFFERING INITIAL COMMUNALITY ESTIMATES

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ABSTRACT

EFFECTS UPON THE FACTORIAL SOLUTION OF ROTATING VARYING NUMBERS OF FACTORS WITH DIFFERING INITIAL COMMUNALITY ESTIMATES

by Donald F. Kiel

The most crucial problem in the entire field of factor analysis has been its subjectivity. While a number of issues must be resolved in order to arrive at a truly objective method of factor analysis this study concerned itself with two key issues -- (1) the number of factors which should be considered for a particular set of data and, (2) the effect of different initial communality estimates on the final solution.

Four correlation matrices, well known in factor analytic literature, were selected for study -- the eight physical variables, eight political variables and twenty-four psychological variable matrices from Harman (1) and the eleven Air-Force classification tests from Fruchter (2).

Three principal axes factor analyses were carried out on each matrix, one using unities (1.0) in the diagonals of the correlation matrix, another using "Guttman communalities" and a third using squared multiple correlations. The unrotated factors from each of the twelve analyses were ordered from largest to smallest on their corresponding latent roots and an extensive series of rotations were calculated using the normalized Varimax method. The two largest factors were rotated, then the three largest, and so on until all real factors were included in the

set of solutions.

The findings indicated that there is very little difference in the numeric values of the largest factors regardless of the initial communality estimates used. A point is reached, however, at which unique factors begin to appear when unities are used which are not present when squared multiple correlations are used as initial communality estimates.

The factors have a tendency to split as additional factors are rotated, emerging in a definite hierarchical pattern. For example, a large verbal-deductive factor present in one solution may split into verbal and deductive factors when an additional unrotated factor is included in the solution.

As a result of this study, a criterion has been proposed (frequently called the Kiel-Wrigley Criterion) for the number of factors which should be included in a factor analytic report, i.e. "when to stop factoring," It suggests that unities are satisfactory as diagonal entries in the correlation matrix to be factored. A series of rotations should be carried out, starting with the two largest factors, then the three largest, etc. until a solution is reached which includes a rotated factor on which fewer than three of the variables have their highest loadings. This is based on the theory that three or more points are necessarily to define a hyperplane in n-dimensional space.

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CHAPTER I

INTRODUCTION

Factor analysis is a mathematical method by which a large number of variables can be resolved, i.e. described, in terms of a small number of underlying categories or "factors". The technique is often mistakenly labeled as psychological theory primarily because it had its foundations among psychologists who were attempting to develop mathematical models for the description of human intellectual ability and its development for many years continued to be a psychological problem. Only in fairly recent years has there been a concerted effort to bring factor analysis into the realm of a recognized statistical theory.

The father of factor analysis is generally conceded to be Charles Spearman, who in 1904 published a paper (41) on the nature of intelligence in which he described a factor analytical investigation of various cognitive tests. He spent the rest of his life developing his Two-Factor Theory which stated that intellect consists of a general ability factor and a number of specific factors -- one for each test used. In other words, his theory was that each test contains two factors, a factor common to all tests and a factor specific to the individual test.

By the 1930's it became apparent that a single factor was not adequate to describe many batteries of psychological tests and group factor theory developed. While many investigators contributed to the development of multiple-factor theory, the name of

L. L. Thurstone is most widely recognized for popularization of the theory. His pioneering work, "Vectors of the Mind", published in 1935 (42) set forth a set of principles which have become popular among factor analysts and known as Simple Structure Theory.

Implicit in this brief history is the controversy which has been generated over the years as to the proper system of factor analysis. The controversy has been primarily between British and American schools of factor analytic thought and -- of particular salience to this thesis -- the key issue has been the number of factors worth extracting from a given battery of variables. The British have traditionally tended to stop with fewer factors than have the Americans.

Excellent presentations of details of the various factor theories are available elsewhere (17, 7, 11, 18), and will be discussed where applicable in this thesis. The overriding problem, however, with the entire field of factor analysis has been its subjectivity. As Wrigley wryly states it (52):

> ... In most statistical work two persons who start with the same data and calculate correctly will reach the same answer. This is not necessarily the case for factor analysis. This remains a method which depends upon arbitrary judgments by the investigator, so that skill is acquired only after long experience in estimating communalities, deciding upon the number of factors to be extracted, selecting pairs of factors for rotation, and so on. ... In general, mathematical statisticians have ignored factor analysis, treating it as a jungle of the psychologists' making, into which any self-respecting statistician would be most unwise to stray.

While there are a number of issues which must be resolved in order to arrive at an objective method of factor analysis -- that is, a method which does not depend primarily on human judgment -- this study will concern itself with only two of these issues, namely,

the determination of the number of factors which should be considered for a particular set of data and the effect of different initial communality estimates on the solution.

Various criteria have been proposed in the past for either determining the number of factors which should be extracted in factoring a correlation matrix or determining the number of extracted factors which should be rotated. Philip Vernon, in an unpublished manuscript (44) in 1949, listed twenty-four indices of significance for centroid factors. Other indices have since been advanced. The question of how many factors to extract was more important with non-computerized, short-cut methods of factor analysis when factors were determined serially, one at a time. Since desk calculator computing methods were extremely laborious for even small selections of variables, investigators usually preferred to stop with as few factors as possible. Present day analytic methods using digital computers usually extract as many factors as variables, making more important the question of determining which of these factors should be considered meaningful for further analysis, i.e. how many should be rotated?

Two general classes of criteria have been suggested: (1) Judgmental methods which are based on arbitrary judgments of such things as proportion of variance accounted for by each factor and accepting only those which account for more than, say, five or ten or twenty percent of the total variance; clarity of the factor solution which, unfortunately, depends upon some vague definition of the concept of clarity; or size of the sums of squares of the loadings. Kaiser, for example, proposes that only those principal-axes factors, obtained from a correlation matrix with unities in the diagonals, which have latent roots greater than 1.0 should be accepted on the theory that it is a doubtful gain to accept into the system any factors which contribute less information than a single test. (25)

(2) Statistical criteria which involve tests with an associated probability level to determine if factor loadings or residuals are significantly different from zero. Such statistical criteria have the disadvantage of being dependent upon the size of the sample from which the correlation matrix was calculated and usually result in a number of statistically significant factors which have no practical value (28, 19). A number of factor analysts have found that empirical tests of significance frequently lead to about the same results as the more proper statistical tests (17, p. 363).

Factor analysts, following the Thurstonian concept of simple structure, have tended to regard each factor as having equal status with every other factor, and that there is therefore only one "correct" solution. It has also been the tradition for published factor analyses to provide only one of a series of interpretable solutions -- the particular one depending upon the investigator's criterion for, or judgment about, completeness of factor extraction. However, little seems to be known about the effects upon the final solution of varying the number of factors (52). There have been a few suggestions in the literature which suggest that a hierarchical organization would be more effective (40, 15, 3, 48) and a few prior studies have noticed that when the number of factors is increased the larger factors may split into smaller

ones (45, 51). No systematic study of this phenomenon has been made.

Some newer methods of factor analysis introduced in the past twenty-five years, such as Lawley's Maximum Likelihood method (30) and Rao's Canonical Analysis (38), attempt to bypass the separate extraction of factors followed by rotation to a meaningful and interpretable solution by going directly to a "final" solution. Such methods have the disadvantage that they also depend upon a statistical criterion making use of sample size and in actual practice tend to produce far more factors than are considered meaningful by most practicing psychologists or others using factor analytic methods.

The problem of the number of factors which should be accepted in a factor analysis is difficult to isolate from the problem of communality, that is, the initial entries in the diagonal cells of the correlation matrix. In the Thurstonian simple structure model the two problems are interrelated. This problem is discussed in greater detail in Chapter IV.

The controversy between psychologists about the factors of mental ability and the foundations of factor analysis among psychologists should not be construed as limiting factor analysis to usefulness only by psychologists. Indeed, quite the contrary is the case. Among the examples used in this study is an early application in the field of physiology and one from political science. Some stimulating applications have been and are being made in the fields of medicine for classification of symptoms and clinical tests as an aid to diagnosis, (4, 8, 35) in urban and

regional planning for comparison of American cities (37, 2), in advertising and marketing research (6), political science (53) and in many other disciplines.

The use of factor analysis in communications research is not new, probably because many of the theoreticians in communications are also psychologists and have introduced the techniques into other communications oriented areas. Extensive use of factor analysis has been made by C. E. Osgood, et. al. (34) in the study of the measurement of meaning and the development of the widely used semantic differential technique, by Berlo, Lemert and Mertz in the study of source credibility (1) and Kumata (29) in the cross-cultural analysis of meaning. In the field of journalism some of the studies using factor analysis are those of Nafziger, MacLean and Engstrom on tools for newspaper readership data (32) and Deutschmann and Kiel on attitudes toward the mass media (9).

The present study was undertaken for two reasons: (1) "traditional" methods of factor analysis involving the rotation of factors are most common and have a vast background in empirical studies -- they will not soon be completely obsolete; and (2) no systematic study of the effect of rotating increasing numbers of factors has been done. The results of such a study may shed more light on efforts to reach the ephemeral goal of a "definitive" direct and completely objective method of factor analysis or at least encourage researchers to report factor analytic results in a way which would make possible more and better comparisons between similar studies.

CHAPTER II

DATA AND ANALYTICAL METHODS

Introduction

Since the purpose of this study was to investigate various factorial solutions in the hope that an objective criterion for "when to stop factoring" could be developed, particularly one which was not dependent upon strictly statistical criteria, methods of analysis were selected which did not require subjective decisions at any point in the analysis; and, since it was also intended that the findings of this investigation should have general applicability to the wide range of disciplines now using factor analysis, the data for this study were chosen not because of the relevance of the variables to any particular discipline but, rather, because the matrices had been intensively studied and reported on by other investigators and were therefore well-known, even classics, in factor analytic literature.

Correlation Matrices Used in the Study

Initially ten matrices were considered for analysis and some work was done on all of them. However, it became evident early in the study that many of them would merely duplicate the results of other matrices and unnecessarily confuse the presentation of results, so the list was narrowed to four matrices which clearly illustrated different types of factor solutions, and the major emphasis was placed upon the intensive study of these four.

In two of the four examples the matrices were made up of subsets of variables chosen from larger sets of variables for computational convenience by earlier investigators. This fact does not reduce their value for the purposes of the present study, but it would be of interest to apply the methods of this study to the larger matrices at some future time to validate the results of this study and to further insure that the criterion of "factorial invariance" is met.

Matrix I: Eight Physical Variables. -- This is a matrix of intercorrelations of eight physical measurements made on 305 girls between the ages of seven and seventeen. They were chosen by Holzinger and Harman (21, p. 80) from a larger set of seventeen variables reported by Mullen (31) to be representative of two distinct factors which were bi-polar. It has been intensively analyzed by Holzinger and Harman and by Harman (17, p. 82).as an example of a rank two matrix. The numbering and description of the variables and the complete correlation matrix is shown in Table 2.1. Most of the measurements are self-explanatory except for "bitrochanteric diameter" which is, in laymen's terms, hip measurement.

The first four variables were chosen to be measures of longitudinal growth or "lankiness" and the second four as measures of horizontal growth or "stockiness". Observe that the first four are more highly intercorrelated with each other than are the second four and that, in comparison with most empirical matrices, part-

TABLE 2.1 Intercorrelations Among Eight Physical Variables for 305 Girls

Variable	1	2	3	4	5	6	7	8
1. Height	1.000							
2. Arm span	.846	1.000						
3. Length of forearm	.805	.881	1.000					
4. length of lower leg	.859	.826	.801	1.000				
5. Weight	.473	.376	.301	.436	1.000			
6. Bitrochanteric diam.	.398	.326	.319	.329	.762	1.000		
7. Chest girth	.301	.277	.237	.327	.730	.583	1.000	
8. Chest width	.382	.415	.345	.365	.629	.577	.539	1.000

From : Mullen, Frances. "Factors in the Growth of Girls Seven to Seventeen Years of Age." Ph.D. Dissertation, Department of Education, U. of Chicago, 1939.

icularly in the fields of psychology and communications, the correlations are very high.

Matrix II: Eight Political Variables. -- This set of variables has also been used by Holzinger and Harman as an example of the applicability of factor analysis to a non-psychological field. It is, again, a subset of seventeen political variables collected by Gosnell and Schmidt (13) in 147 Chicago election areas following the 1932 presidential election. It is well to remember, in interpreting the results, that this was at the height of the Depression, with high unemployment and a great deal of "it's time for a change" sentiment among the population.

The complete matrix of intercorrelations is given in Table 2.2 and a brief description of each of the variables follows:

1. Lewis -- percentage of the total Democratic and Republican vote cast for Lewis, the Democratic candidate for mayor of Chicago in the 1932 election.

2. Roosevelt -- the corresponding percentage of the total vote cast for Franklin Delano Roosevelt, the Democratic candidate for President of the U. S. (and land-slide victor) in 1932.

3. Party Voting -- percentage that the straight-party votes were of the total vote in the election area.

4. Median Rental -- median rental price in the election area (in dollars).

5. Home Ownership -- percentage of the total families in the election area owning their own homes.

6. Unemployment -- percentage unemployed in the election area in 1931 of the gainful workers ten years of age and over.

7. Mobility -- percentage of total families in the area at the time of the election who had lived less than one year at the present address.

TABLE 2.2 Intercorrelations of Eight Political Variables for 147 Election Areas (1932 Presidential Election)

Variable	1	2	3	4	5	6	7	8
1. Lewis	1.00							
2. Roosevelt	.84	1.00						
3. Party Voting	.62	.84	1.00					
4. Median Rental	-.53	-.68	-.76	1.00				
5. Homownership	.03	-.05	.08	-.25	1.00			
6. Unemployment	.57	.76	.81	-.80	.25	1.00		
7. Mobility	-.33	-.35	-.51	.62	-.72	-.58	1.00	
8. Education	-.63	-.73	-.81	.88	-.36	-.84	.68	1.00

8. Education -- percentage of population, eighteen years of age and older, which had completed more than ten grades of school.

Matrix III: Eleven Air Force Classification Tests. -- In Table 2.3 are the intercorrelations of eleven tests which were part of a battery of tests used by the United States Army Air Force during World War II to classify aviation cadets into training assignments for air-crew positions such as pilot, bombardier, and naviagtor. The product moment correlations are based on a sample of 8,158 unclassified aviation students. The complete description of these tests may be found in Guilford (14) although the matrix has been intensively studied by Fruchter (11, pp. 69-72) and it is Fruchter's analysis which has been used for comparison. A brief description of the eleven tests follows:

1. Dial and Table Reading -- the test consists of two parts, the first of which measures how quickly and accurately the examinee can read the dials on an instrument panel; the second involves locating specific values within the body of tables.

2. Spatial Orientation I -- a perceptual-speed test in which the subject is required to locate small sections of an aerial photograph within a larger picture.

3. Reading Comprehension -- a test designed to measure understanding of paragraph material and the ability to make inferences based on the material read. An attempt was made to minimize mechanical and numerical content in the material presented.

4. Instrument Comprehension -- each of the 60 items consisted of pictures of two instruments, and artificial horizon and compass, followed by pictures of a plane in five different attitudes. The problem was to determine which of the five planes had a position and direction consistent with the instrument readings.

TABLE 2.3 Intercorrelations of Eleven Air Force Aptitude Tests

Test	1	2	3	4	5	6	7	8	9	10	11
1. Dial and table reading	1.00										
2. Spatial orientation I	.40	1.00									
3. Reading comprehension	.41	.18	1.00								
4. Instrument comprehension	.37	.28	.33	1.00							
5. Menchanical principles	.20	.15	,32	.38	1.00						
6. Speed of identification	.32	.46	.18	.28	.16	1.000					
7. Numerical operations I	.52	.23	.29	.17	-.02	.15	1.00				
8. Numerical operations II	.55	.23	.35	.22	.08	.18	.67	1.00			
9. Mechanical information	.02	.01	.18	.22	.47	.07	-.13	-.06	1.00		
10. Practical judgment	.27	.14	.41	.26	.33	.14	.13	.20	.25	1.000	
11. Complex coordination	.36	.30	.20	.36	.31	.28	.16	.12	.18	.20	1.00

5. Mechanical Principles -- each item pictured a mechanical stiuation, with questions designed to test the subject's ability to understand mechanical forces and movements.

6. Speed of Identification -- the subject was required to match airplane silhouettes.

7. Numerical Operations I -- 100 simple numerical-computation items involving addition and multiplication.

8. Numerical Operations II -- same as number 7 except the problems involved subtraction and division.

9. Mechanical Information -- a verbally stated mechanical knowledge test, relating particularly to operation of parts of automobiles. The items were quite brief, calling for only a limited amount of reading and requiring quite specific mechanical knowledge.

10. Practical Judgment -- a test requiring the subject to determine the most practical course of action to a verbally presented problem situation.

11. Complex Coordination -- an apparatus test of the speed and accuracy of hand and foot adjustments to a complex perceptual stimulus. The subject was faced with a panel containing three rows of red lights and three rows of corresponding green lights. When a particular stimulus pattern of red lights was presented the subject was required to move controls similar to those used in an airplane in flight so as to turn on the green lights corresponding to each of the red lights. As soon as the match had been completed, a new set of red lights was automatically presented.

Matrix IV: Twenty-four Psychological Tests. -- This example consists of twenty-four psychological tests administered to 145 seventh and eighth grade pupils of a suburban Chicago school in the late 1930's. The data was gathered by Holzinger and Swineford (20) and subsequent analyses by Holzinger and Harman (21), Harman (17), Kaiser (27), Neuhaus and Wrigley (33) and others have made it a classic in factor analytic literature. The complete correlation

matrix is presented in Table 2.4 and a brief description of the tests follows:

1. Visual Perception Test -- a non-language multiple-choice test composed of items from Spearman's Visual Perception Test.

2. Cubes -- A simplification of Brigham's test of spatial relations.

3. Paper Form Board -- A revised multiple-choice test of spatial imagery, with dissected squares, triangles, hexagons, and trapezoids.

4. Flags -- Adapted from a test by Thurstone. Requires visual imagery in two or three dimensions.

5. General Information -- A multiple-choice test of a wide variety of simple scientific and social facts.

6. Paragraph Comprehension -- Comprehension of written material measured by completion and multiple-choice questions.

7. Sentence Completion -- A multiple-choice test in which "correct" answers reflect good judgment on the part of the subject.

8. Word Classification -- Sets of five words one of which is to be indicated as not belonging with the other four.

9. Word Meaning -- A multiple-choice vocabulary test.

10. Add -- Speed of adding pairs of one-digit numbers.

11. Code -- A simple code of three characters is presented and exercise therein given to measure perceptual speed.

12. Counting Groups of Dots -- Four to seven dots, arranged in random patterns, to be counted by the subject. A test of perceptual speed.

13. Straight and Curved Capitals -- A series of capital letters. The subject is required to distinguish between those composed of straight lines only and those containing curved lines. A test of perceptual speed.

TABLE 2.4 INTERCORRELATIONS OF TWENTY-FOUR PSYCHOLOGICAL TESTS FOR 145 CHILDREN

Test	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1.000																							
2	.318	1.000																						
3	.403	.317	1.000																					
4	.468	.230	.305	1.000																				
5	.321	.285	.247	.227	1.000																			
6	.335	.234	.268	.327	.611	1.000																		
7	.304	.157	.223	.335	.656	.722	1.000																	
8	.332	.157	.382	.391	.578	.527	.619	1.000																
9	.326	.195	.184	.325	.723	.714	.685	.532	1.000															
10	.116	.057	-.075	.099	.311	.203	.246	.285	.170	1.000														
11	.308	.150	.091	.110	.344	.353	.232	.300	.280	.484	1.000													
12	.314	.145	.140	.160	.215	.095	.181	.271	.113	.585	.428	1.000												
13	.489	.239	.321	.327	.344	.309	.345	.395	.280	.408	.535	.512	1.000											
14	.125	.103	.177	.066	.280	.292	.236	.252	.260	.172	.350	.131	.195	1.000										
15	.238	.131	.065	.127	.229	.251	.172	.175	.248	.154	.240	.173	.139	.370	1.000									
16	.414	.272	.263	.322	.187	.291	.180	.296	.242	.124	.314	.119	.281	.412	.325	1.000								
17	.176	.005	.177	.187	.208	.273	.228	.255	.274	.289	.362	.278	.194	.341	.345	.324	1.000							
18	.368	.255	.211	.251	.263	.167	.159	.250	.208	.317	.350	.349	.323	.201	.334	.344	.448	1.000						
19	.270	.112	.312	.137	.190	.251	.226	.274	.274	.190	.290	.110	.263	.206	.192	.258	.324	.358	1.000					
20	.365	.292	.297	.339	.398	.435	.451	.427	.446	.173	.202	.246	.241	.302	.272	.388	.262	.301	.167	1.000				
21	.369	.306	.165	.349	.318	.263	.314	.362	.266	.405	.399	.355	.425	.183	.232	.348	.173	.357	.331	.413	1.000			
22	.413	.232	.250	.380	.441	.386	.396	.357	.483	.160	.304	.193	.279	.243	.246	.283	.273	.317	.342	.463	.374	1.000		
23	.474	.348	.383	.335	.435	.431	.405	.501	.504	.262	.251	.350	.382	.242	.256	.360	.287	.272	.303	.509	.451	.503	1.000	
24	.282	.211	.203	.248	.420	.433	.437	.388	.424	.531	.412	.414	.358	.304	.165	.262	.326	.405	.374	.366	.448	.375	.434	1.000

Harman, Harry H. Modern Factor Analysis. Chicago, Ill.: U. of Chicago Press. 1960. p. 134.

14. Word Recognition -- Twenty-five four-letter words are studied for three minutes. These words are then to be checked from memory on a hundred-word list.

15. Number Recognition -- Similar to test 14. Fifteen three-digit numbers.

16. Figure Recognition -- Similar to test 14. Fifteen geometric designs.

17. Object-Number -- Twenty pairs of names of familiar objects and two-digit numbers are studied for three minutes. The words only are then presented to the subject, who is required to supply the proper numbers.

18. Number-Figure -- Similar to test 17. Ten pairs of numbers and geometric figures.

19. Figure-Word -- Similar to test 17. Ten pairs of geometric figures and words studied for one minute.

20. Deduction -- Logical deduction test using the symbols) and (and the letters A, B, C, and D.

21. Numerical Puzzles -- A numerical deduction test, the object being to supply four numbers which will produce four given answers employing the operations of addition, multiplication, or division.

22. Problem Reasoning -- A reasoning test in completion form. Each problem lists the steps in obtaining a required amount of water using two or three vessels of given capacity.

23. Series Completion -- From a series of five numbers the subject is supposed to deduce the rule of procedure from one number to the next, and thus supply the sixth number in the series.

24. Woody-McCall Mixed Fundamentals: Form I -- A series of thirty-five arithmetic problems, graduated for difficulty.

Computing Procedures Used In This Investigation

The decision as to the procedures to be used in this investigation was based on four criteria: (1) the solution produced should be as mathematically precise as possible, (2) no subjective

decisions should be required in arriving at a solution, such as estimate of rank, significance of residual matrix, etc., (3) the solution should be psychologically acceptable, and (4) the methods should be programmable for computation on an electronic computer.

All of the original calculation for this study were made on MISTIC, Michigan State University's original electronic digital computer. Several new computer programs were written for MISTIC in conjunction with this study and, since MISTIC is now obsolete, most of them have been reprogrammed for or incorporated into existing programs for the Control Data Corporation 3600 Computer currently in use at Michigan State University. A 3600 FORTRAN program (called FACTORA), incorporating all of the methods used in this study as well as provision for calculating from the raw data matrix, is available from the Michigan State University Computer Laboratory. All of the data for this study have been recalculated on the CDC 3600 and the complete set of results and a listing of the program are on file in the Michigan State University Library.

It is interesting to compare calculation time as an example of the growth of the computer "art". On MISTIC, which required laborious punching of paper tape for input and output and some desk calculator work between runs, the complete eigenvalue-factor loading solutions for each of three different communality estimates (discussed in Chapter 5) and all sets of rotations for all three sets of factors required approximately <u>fifteen hours</u> of computer time over a period of several months. All of the same results were obtained on the 3600 <u>in one computer run</u> in

less than ten minutes.

For each of the four examples used in this study, factoring started from a previously calculated matrix of product-moment correlations. For the methods used in this study no assumptions are made about the statistical distribution of the variables, hence the matrices could have been any other of the frequently used non-parametric indices of relationship, i.e. the tetrachoric correlation coefficient, phi coefficient, biserial or point biserial correlation, etc. Whenever the assumptions of the product-moment correlations can be met, however, they are the preferred starting point for factor analysis, since they allow the calculation of additional statistical tests for the significance of factor loadings, etc. Most modern factor analysis programs for digital computers provide for starting with "raw" data; that is, the individual measurements of each of the variables for each observation point (subject).

For the initial phase of this investigation unities were used in the leading diagonals of the correlation matrices. The controversy over the appropriate initial diagonal entries, the so-called "communality question", is a burning issue in factor analytic literature and will be discussed further in Chapter 5. Unities, which are the self-correlations of each variable and represent the total variance of each variable in the linear factor model, were chosen because (1) they were computationally most convenient since they were already in the diagonals in the output of the correlation program on MISTIC, (2) they

preserve the Gramian property of the matrix, producing n real, positive roots, and (3) they are one of the few unique communality estimates.

From both a mathematical and statistical point of view the preferred method of extracting the initial factors is the method of principal axes, first proposed by Pearson in 1901 (36) and developed by Hotelling in the 1930's (22). While this method was recognized as superior for many years, the computational method was so laborious that it was not until high-speed electronic computers came into general use in the 1960's that the method became practical for use with other than very small matrices.

The principal axes method has a number of properties which make it ideal from a factor analytic standpoint: (1) it produces a unique solution for any given correlation matrix; (2) the first factor extracted in the sequential method (the factor corresponding to the largest eigenvalue in the Jacobi method) accounts for the maximum possible proportion of variance in the matrix, the second factor for the maximum proportion of the remaining variance, the third factor for the maximum remaining after the first two were extracted, and so on (usually a small number of the total roots will account for almost all of the total communality); (3) the resulting columns of factor coefficients are "orthogonal"; that is, the correlation between any two pairs of factors is zero meaning that all factors are independent. This property, expressed mathematically, is:

$$a'_p a_q = \delta_{pq} \lambda_p$$

where a_p is a vector of factor loadings, λ_p is an eigenvalue of the correlation matrix and the Kronecker $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$.

The original method proposed by Hotelling (22), involving the sequential extraction of factors, is extremely arduous and inefficient even on high-speed calculators (although it has been programmed and used on MISTIC for factoring very large matrices when only a relatively small number of the total possible factors was desired and when capacity would otherwise have been exceeded). For computer applications the relationship between factor analytic theory and the mathematical problem of determining the eigenvalues and eigenvectors of a square, symmetric matrix has been taken into consideration. The most frequently used method, and the method programmed originally for MISTIC and more recently for the CDC 3600, makes use of work done by Jacobi (24) in the 1840's -- commonly known as the Jacobi method. It is an iterative method which produces the complete matrix of eigenvalues (latent roots) and eigenvectors without producing a residual matrix after each eigenvalue. The eigenvectors are transformed into factors by scaling each eigenvector by the square root of the corresponding eigenvalue.

Even the Jacobi method has disadvantages. For large matrices the number of iterations necessary frequently leads to very great rounding errors. Recent work by Householder (33), Givens (12), Wilkinson (46), and others is leading to more efficient methods of solving the eigenvalue-eigenvector problem on

electronic computers. These methods are being rapidly added to the repertory of computer methods of factor analysis.

<u>Factor Rotation</u>

While an occasional factor analyst will stop once the principal axes factors have been extracted, most psychologists as well as users of factor analysis in other disciplines do not find the end product of the principal axes method useful or acceptable from the standpoint of interpretability. Most prefer to rotate some subset of the principal axes factors to some reference structure which provides more psychological "meaningfulness".

For years the methods of rotation were crude, subjective methods in which the final solution depended on the judgment of the investigator. Any two investigators analyzing the same data were not likely to reach the same solution (52). Just over ten years ago the first significant developments were reported of analytical, objective methods of rotation. Carroll (5), Neuhaus and Wrigley (33), Saunders (39), and Ferguson (10) almost simultaneously, although independently, developed criteria which were very similar. Neuhaus and Wrigley, who were the first to program and use the method on electronic computers, coined the now well-known name, Quartimax, for this method. It attempts to minimize the complexity of the individual variables, that is, to approach a unifactorial structure in which each variable has a high loading on only one factor. This is extremely difficult to achieve with empirical data, especially for orthogonal structures, and usually leads to a fairly large general factor.

Since the quartimax method did not completely meet one of the important requirements for simple structure in the Thurstonian sense, namely that the number of zero loadings on a factor should be maximized, Kaiser (25) developed a modification of the quartimax method which he called the varimax method. Where the quartimax method placed the emphasis on the simplification of each <u>row</u> (variable) of the factor matrix, the varimax method places more emphasis on simplifying the <u>columns</u> (26).

It was also Kaiser who noted a disadvantage in both his original varimax method as well as in the quartimax method, namely, that even after rotation there was more disparity in the variance contributions (sums of squares of the loadings on a factor) of the different factors than is desirable. In other words, one objective of rotation should be to give equal weight to each factor. Kaiser attributed this disparity to the fact that in either method of rotation, each variable contributes to the function being maximized as the square of its communality. In other words, a variable with a communality twice as great as another variable will influence the rotation four times as much. For this reason, Kaiser modified his original method (referred to as the "raw" varimax method) by weighting each variable so that it contributes equally to the rotation.

This procedure, known as normalizing, involves dividing each loading before rotation by the square root of the sum of squares of all of the loadings for that variable (the observed communality of the variable for that solution) which extends the

vector representation of that variable in common factor space to unit length, performing the rotation on the "normalized" loadings, and then reducing the vectors back to their original length by multiplying all of the rotated loadings for a variable by the original scaling factor. Notice that under orthogonal rotation the communalities remain constant even though the individual loadings change. The method using this weighting process is known as "normal" varimax and requires that the final loadings be such as to maximize the following function:

$$V = n \sum_{p=1}^{m} \sum_{j=1}^{n} (b_{jp}/h_j)^4 - \sum_{p=1}^{m} \left(\sum_{j=1}^{n} b_{jp}^2/h_j^2 \right)^2$$

The mathematical details for achieving this maximization is available elsewhere (25).

The same weighting or normalizing process is applicable to the quartimax method as well, and results in a considerable improvement in evenness of the factors over the "raw" quartimax. The normalized quartimax and varimax are the methods presently programmed for the CDC 3600.

The question of which rotational method is "correct" is unresolvable. The normal varimax method was selected for this study because it results in a solution which is probably closest to the concept of simple structure preferred by most American factor analysts. After comparing varimax, quartimax and subjective graphical solutions for four factors of the twenty-four psychological tests, Harman concludes (17, p. 306) that "the varimax solution seems to be the 'best' parsimonious analytical solution in the

sense that it correlates best with the intuitive concept of that term as exemplified by the graphical solution." It has an additional attribute which is considered by Kaiser as of primary importance; namely, that of "factorial invariance." As Thurstone describes this principle, it is that "the factorial description of a test must remain invariant when the test is moved from one battery to another which involves the same common factors." (43, p. 361)

For the purposes of this study, the unrotated principal axes factors were ranked in order by decreasing size of their corresponding eigenvalues and then the first two, the first three, four, five, and so on, factors were rotated. In the case of the factors obtained using squared multiple correlations as initial communalities (Chapter 5), only those factors corresponding to positive eigenvalues were rotated.

CHAPTER III

THE EFFECT ON THE FACTORIAL STRUCTURE OF INCREASING THE NUMBER OF FACTORS IN THE ROTATED FACTOR SOLUTION

Introduction

The initial phase of this investigation was concerned with observing the effects of rotating the first two, then the first three, etc. factors where first refers to the unrotated factor corresponding to the largest eigenvalue, second to the next largest eigenvalue, and so on as described in the preceding chapter. In this phase only those factors obtained using unities in the principal diagonals of the various test correlation matrices were studied.

Eight Physical Variables

The eight variables in this matrix were specifically chosen by Holzinger and Harman to represent four "longitudinal" and four "horizontal" variables. In previously published analyses of this matrix only two principal axes factors have ever been given, and these using communalities calculated by estimating the rank. Table 3.1 shows the unrotated principal axes factors for this matrix. The maximum discrepancy between the unrotated factors calculated by Harman (17, p. 173) and those obtained in this study is only .09. As might be expected, since the variables were chosen to represent only two factors, slightly more than 80% of the total variance is accounted for by the first two factors. The first factor is a large general factor with the first four variables (hypothesized

TABLE 3.1 UNROTATED PRINCIPAL AXES FACTORS FOR EIGHT PHYSICAL VARIABLES -- UNITIES USED IN LEADING DIAGONALS OF CORRELATION MATRIX

Variable	I	II	III	IV	V	VI	VII	VIII	Comm. (h^2)
1. Height	.859	-.372	-.070	-.070	-.197	.066	-.235	-.120	1.000
2. Arm Span	.842	-.441	.079	.044	.163	.031	-.103	.225	1.000
3. Forearm length	.813	-.459	.010	-.031	.262	-.170	.090	-.149	1.000
4. Lower leg length	.840	-.395	-.101	.081	-.222	.108	.245	.034	1.000
5. Weight	.758	.525	-.148	-.075	-.143	-.311	-.010	.073	1.000
6. Bitrochanteric diameter	.674	.533	-.051	-.462	.106	.178	.041	-.002	1.000
7. Chest girth	.617	.580	-.292	.409	.124	.111	-.030	-.039	1.000
8. Chest width	.670	.419	.592	.143	-.053	.018	.013	-.037	1.000
Contribution of Factor (Eigenvalue)	4.673	1.771	.481	.421	.233	.187	.137	.097	8.000
Percent of total variance	58.41	22.14	6.01	5.27	2.92	2.33	1.72	1.21	100.01*
Cumulative percentage	58.41	80.55	86.56	91.83	94.74	97.08	98.79	100.00	

*Individual percentages add to 100.01 due to rounding.

TABLE 3.2 TWO, THREE, FOUR, FIVE, AND SIX FACTOR VARIMAX ROTATION SOLUTIONS FOR EIGHT PHYSICAL VARIABLES
(Leading decimal points omitted from values; i.e. read 893 as 0.893.)

Variable	Two Factors I	II	h^2	Three Factors I	II	III	h^2	Four Factors I	II	III	IV	h^2
1	900	259	877	903	246	079	882	901	-239	108	079	887
2	930	195	903	925	127	192	909	924	-092	071	209	911
3	919	164	871	918	126	116	872	916	-136	035	118	873
4	899	229	861	904	231	039	871	906	-112	199	073	878
5	251	887	850	260	867	229	872	259	-666	554	245	878
6	181	940	739	185	787	296	741	175	-905	237	220	955
7	107	840	718	123	885	072	803	136	-297	908	198	970
8	251	749	625	223	447	851	975	217	-312	271	882	995
Factor Variance	3.50	2.95	6.44	3.50	2.50	0.93	6.92	3.49	1.55	1.32	.99	7.35

Variable	Five Factors I	II	III	IV	V	h^2	Six Factors I	II	III	IV	V	VI	h^2
1	892	-231	080	110	-241	926	900	076	-202	112	-153	-192	930
2	928	-102	089	196	139	938	924	098	-105	202	032	152	939
3	923	-152	066	094	229	941	909	053	-109	101	-102	329	970
4	896	-103	168	106	-271	927	907	165	-083	106	-107	-243	939
5	249	-656	528	277	-224	898	243	471	-472	292	-637	001	995
6	173	-909	243	222	048	966	176	285	-897	244	-143	011	998
7	136	-301	914	201	013	986	137	923	-251	220	-129	-006	998
8	212	-304	260	891	-007	998	212	250	-256	900	-124	002	998
Factor Variance	3.47	1.54	1.29	1.03	.26	7.58	3.47	1.26	1.23	1.08	.50	.23	7.77

as "lankiness" variables) most highly loaded. The second unrotated factor is of the bipolar type; that is, the two groups of variables have opposite signs.

Table 3.2 contains the rotational solutions for two, three, four, five and six factors.

When rotated the two factors originally hypothesized are clearly evident, the longitudinal variables (1, 2, 3, and 4) appearing highly loaded on the first factor and the four horizontal variables (5, 6, 7, and 8) highly loaded on the second factor. Notice that the observed communalities (columns headed h^2, the usual abbreviation) indicate that 85% or more of the total original communality of the first five variables has been extracted in the first two factors. (Since the initial communality was 1.0 for each variable, the observed communality is a proportion.)

Notice that only 62% of the total variance of variable 8 appears in the first two factors and that the loading of this variable on the third unrotated factor is .60, an additional variance contribution of approximately 36%. The effect of including this third factor in a rotation is to split the eighth variable away from the group of "stockiness" variables to form a new factor on which only the eighth variable has a high loading although there is still an appreciable loading of variable 8 on the second factor.

On the unrotated factors, variables 6 and 7 have appreciable loadings on the fourth factor, although of opposite signs, and each of the other variables also have relatively higher loadings on at least one of the remaining factors. These additional non-zero

loadings are probably attributable to "unique and specific" variance, since communality estimates were not used in the diagonals of the correlation matrix.

In the four factor solution, variables 6 and 7 do not form a new factor but, instead, because of the opposite signs, cause the formation of two new factors, doublets of variables 5 and 6 and another of variables 5 and 7 (variable 5 has its highest loading on the factor with variable 6). Additional examples of bipolar factors splitting into either specific factors or doublets will be seen in other test matrices.

The rotation of the five largest factors produces no new factors. The fifth factor is of the type we shall call a "null" factor, signifying that none of the variables have their highest loading on that factor and, in most cases, none of the variables have any appreciable loading on such a factor.

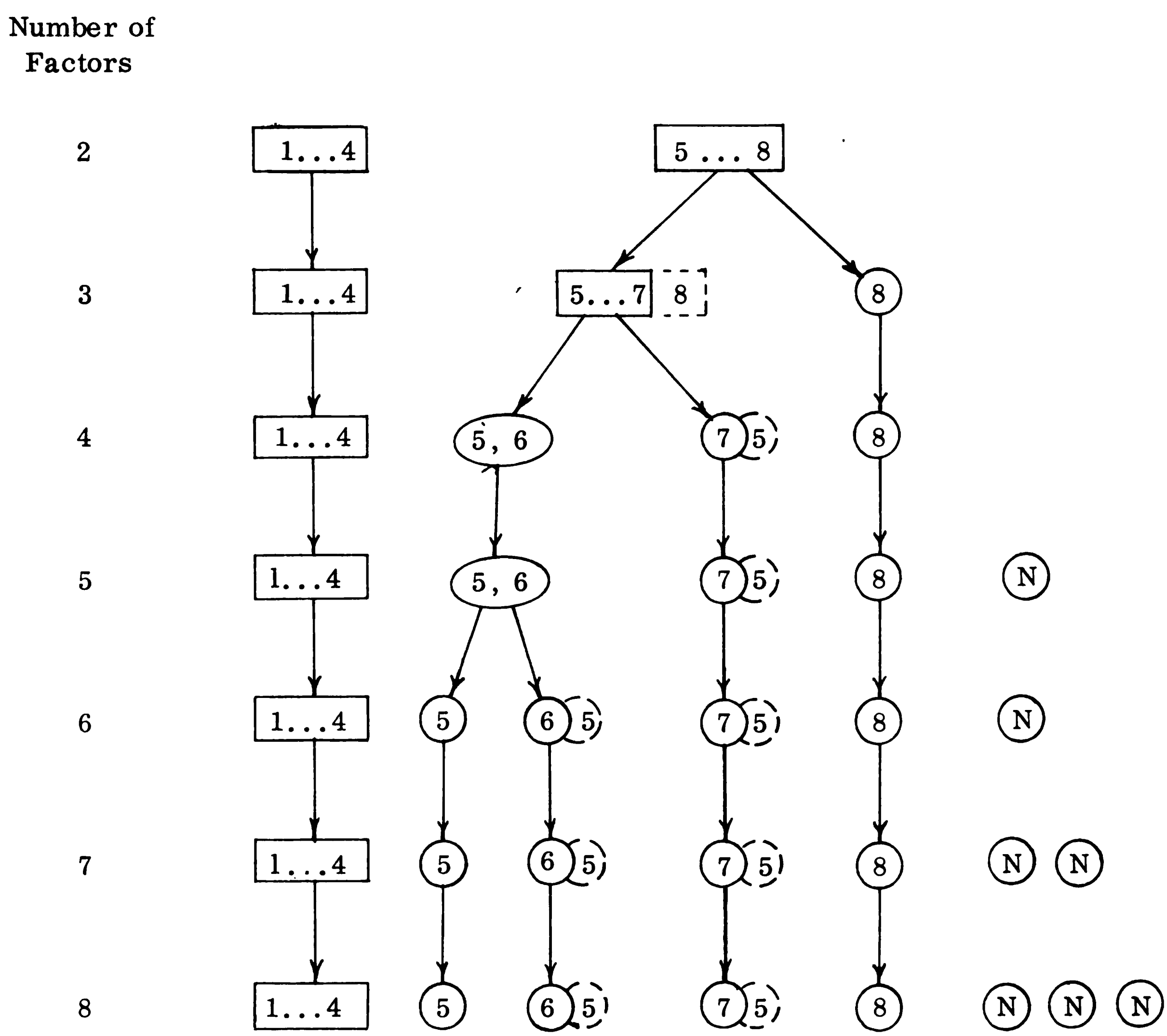
The addition of the sixth factor to the rotational solution results in a specific factor on which variable five has its highest loading although variable 5 still retains appreciable loadings on the two doublet factors produced in the four factor solution. The seven and eight factor solutions produce no new factors on which any other variable has its highest loading, although two factors appear which have relatively high loadings on variables 1 and 4. Variables 1 through 4, the "lankiness" or longitudinal variables, which had the highest correlation coefficients and the largest loadings on the first unrotated factor, remain grouped into a single factor.

Figure 3.1 gives a hierarchical diagram of the results of the seven sets of rotations. The variable numbers in solid boxes indicate that those variables had their highest loadings on that particular factor. When a variable had a loading greater than .40 (16% of the variance of that variable) on a particular factor it is indicated in dotted lines for that factor. The symbol N indicates a "null" factor, one on which no variable had any appreciable loading. Factors on which at least three variables were most highly loaded are noted by rectangular boxes, those with less than three highest loadings are enclosed within circles.

From a physiological point of view the hierarchy seems to be eminently sensible. The "lankiness" variables are related to bone structure which normally is proportional in an individual and is independent of leanness or obesity. The arm span and length of forearm (variables 2 and 3) would naturally be most closely related. The "stockiness" variables, on the other hand, are less closely related. It would not be unusual for girls seven to seventeen to have varying chest and hip girths, and the chest width and chest girth are measures of somewhat different types of bodily development. Additional speculation is not germane to this discussion.

FIGURE 3.1

HIERARCHICAL STRUCTURE OF EIGHT PHYSICAL VARIABLES -- STARTING WITH UNITIES IN CORRELATION MATRIX



Eight Political Variables

The unrotated principal axes factors for this matrix are shown in Table 3.3. In this example over 85% of the total variance is contributed by the first two factors, although almost 30% of the variance of the first variable appears in the third factor.

The results when the two largest, the three largest, etc. factors are rotated is presented in Table 3.4. The order of the variables has been rearranged in the table to group the variables in the factors in which they appear. Figure 3.2 presents the same information in the form of a hierarchical structure. Notice that it is much easier to interpret in this form.

When only two factors are rotated, six of the eight variables have their highest loadings on the first factor, and the other two have their highest loadings on the second factor. Variables 4 and 8 also have appreciable loadings on the second factor. Observation of the signs of the loadings is necessary for interpretation of these factors, since both are of the bipolar type. These two factors have previously been identified by Holzinger and Harman (17, p. 178) as a large "Traditional Democratic Voting" factor (relating high Democratic party as well as straight ticket vote to high unemployment, high residential mobility, and low median rental and low education) and a smaller factor which has been called a "Home Permanency" factor (high home ownership negatively related to home mobility, education, and median rental).

TABLE 3.3 EIGHT POLITICAL VARIABLES -- UNROTATED PRINCIPAL AXES FACTORS
UNITIES USED IN LEADING DIAGONALS OF CORRELATION MATRIX

Variable	I	II	III	IV	V	VI	VII	VIII	h^2
1. Percent Lewis Vote	74	-36	54	-12	00	-04	-04	-10	1.00
2. Percent Roosevelt vote	86	-43	16	10	01	01	14	17	1.00
3. Percent straight party vote	89	-21	-17	26	-16	17	-00	-10	1.00
4. Median rental	-89	-04	28	32	03	-00	-15	05	1.00
5. Percent home ownership	32	89	23	06	16	15	08	-01	1.00
6. Percent unemployment	91	-02	-19	14	29	-19	-03	-04	1.00
7. Residential mobility	-71	-62	-07	-06	27	20	03	-03	1.00
8. Education	-95	-09	09	15	-04	-13	21	-09	1.00
Contribution of factor (Eigenvalue)	5.19	1.54	.53	.24	.21	.14	.10	.06	8.00
Percent of total variance	64.8	19.2	6.6	3.0	2.6	1.8	1.2	0.8	100.0
Cumulative percentage	64.8	84.0	90.6	93.6	96.2	98.0	99.2	100.0	

TABLE 3.4 EIGHT POLITICAL VARIABLES -- VARIMAX ROTATION SOLUTIONS

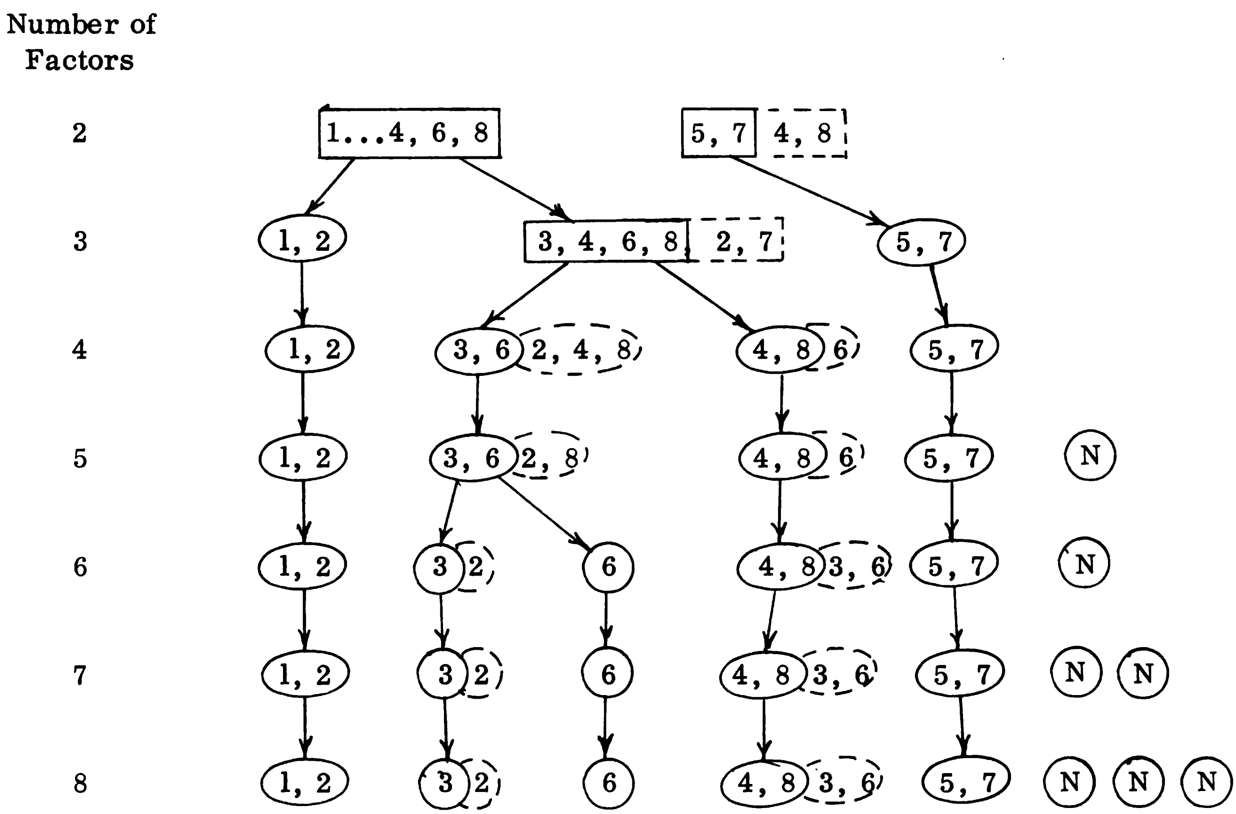
FOR TWO, THREE, FOUR, FIVE AND SIX FACTORS

Variable	Two Factors I	II	h^2	Three Factors I	II	III	h^2	Four Factors I	II	III	IV	h^2
1	823	-031	.679	319	067	931	.973	269	060	932	-205	.987
2	957	-046	.918	643	-068	725	.943	642	-040	682	-272	.954
3	902	170	.842	838	049	405	.869	836	098	340	-334	.936
4	-797	-399	.795	-876	-242	-213	.871	-443	-217	-248	818	.974
5	-064	940	.888	038	969	-036	.942	-035	969	-028	-066	.945
6	838	345	.821	847	212	311	.859	740	245	269	-446	.880
7	-394	-853	.882	-470	-804	-138	.886	-326	-814	-132	322	.890
8	-828	-463	.900	-808	-355	-361	.909	-515	-350	-370	637	.930
Factor Variance	4.59	2.14	6.72	3.58	1.83	1.85	7.25	2.30	1.85	1.74	1.61	7.49

Variable	Five Factors I	II	III	IV	V	h^2	Six Factors I	II	III	IV	V	VI	h^2
1	225	053	941	-217	-016	.987	-230	952	046	131	-082	-057	.988
2	599	-054	705	-303	-058	.954	-357	734	-070	449	-282	-050	.954
3	776	073	371	-386	-259	.961	-472	396	053	735	-232	-118	.990
4	-386	-198	-265	824	063	.975	873	-284	-162	-220	184	152	.975
5	-004	924	-029	-072	125	.970	-097	-039	990	-017	-044	012	.993
6	744	239	294	-477	195	.962	-532	347	193	331	-663	-085	.997
7	-272	-793	-146	361	325	.960	363	-156	-705	-204	127	537	.998
8	-474	-335	-388	666	015	.931	721	-406	-325	-339	203	043	.947
Factor Variance	2.01	1.80	1.85	1.81	.23	7.70	2.11	1.99	1.66	1.07	.67	.34	7.84

FIGURE 3.2

HIERARCHICAL STRUCTURE OF EIGHT POLITICAL VARIABLES -- STARTING WITH UNITIES IN CORRELATION MATRIX



When three factors are rotated the six variables which previously clustered on the first factor subdivide into two groups. The first variable (percent voting for Lewis) shifts completely to the third factor and "percent Roosevelt vote" also has its highest loading on this third factor, although still retaining a high loading on the first factor. While two high loadings do not uniquely determine a factor, we can see that there is a tendency for the "Traditional Democratic Vote" to subdivide into what might be interpreted as "Local" and "National" components. It is possible that had there been additional measures of local voting behavior in the selection of variables, this third factor might have been better established. In Chapter 5 it will be shown that when lower communality estimates are used, rotational stability is established with the three factor solution.

When the four largest factors obtained with unities in the diagonals of the original correlation matrix are rotated the first factor, on which variables 3, 4, 6, and 8 had their highest loadings in the three factor solution, splits with variables 4 and 8 forming a new factor on which variable 6 also has an appreciable negative loading. In this solution four factors are produced, on each of which two variables have their highest loadings. It should be noted that each of the four factors so obtained seem to "make sense" from the standpoint of interpretability.

The addition of a fifth factor produces only a "null" factor. The six factor solution, however, results in the splitting of variables 3 and 6 into two unique factors. Rotational stability is

achieved at this point, with the seven and eight factor solutions adding only additional null factors.

Eleven Air Force Classification Tests

The analysis of this matrix is particularly interesting because it illustrates quite a different effect than previously observed, namely, the complete disintegration of the factorial structure. In each of the two preceding examples, at least one group of variables remained clustered in a single factor; that is, at least one factor remained on which more than one variable had its highest loading even after all factors had been included in the rotational solution. In this example, when unities are used in the diagonals of the original correlation matrix, complete fissioning takes place, eventually resulting in eleven unique factors.

The difference is apparent even with the unrotated factors (Table 3.5). Unlike the two preceding examples, in which the first two factors accounted for from eighty to ninety percent of the total variance, with this matrix the first two factors account for less than 50% of the total variance; in fact, the first five factors (this matrix has been used elsewhere in the literature (11, p. 149-151) as an example of a five factor test battery) account for only about 75% of the total variance. The total variance is apportioned much more evenly among the eleven unrotated factors, with the later factors accounting for a higher percentage of the variance than had been the case in the two preceding examples.

Figure 3.3 shows the hierarchical structure for the Eleven Air Force Classification Tests matrix. The effect of rotating the first two factors is that those variables (1, 2, 6, 7, 8) which had negative or zero loadings on the second factor grouped on one factor and those with positibe loadings grouped on the other factor; this in spite of the fact that variables 5 and 9 had higher loadings on the second unrotated factor than on the first.

When three factors are rotated, the three variables which had the largest negative loadings on the third unrotated factor (2, 6, 11) have split away from the two preceding factors and formed a new factor. Variable 4, which was also negatively loaded on the third unrotated factor, and variable 1 which had a zero loading on the third factor also have appreciable loadings on this new third factor.

The four factor solution results in splitting of both the second and third factors of the three factor solution. It is at this point that the solution begins to stabilize. In this example less than fifty percent of the total variance of variables 2, 3, 4, 6, 10 and 11 were accounted for in the first three factors. (The communalities shown for each solution may be interpreted as proportions, since the total original communality was 1.0 for each variable.) Notice that there was a great deal of factorial instability between the two, three and four factor solutions -- that, in general, the variables which had the greatest increase in proportion of variance with the addition of a new factor tended to have the greatest influence in the formation of a new factor.

The five factor solution is that presented in the literature by Fruchter. It would be of little value to present a detailed

TABLE 3.5 ELEVEN AIR FORCE CLASSIFICATION TESTS -- UNROTATED PRINCIPAL AXES FACTORS
UNITIES USED IN LEADING DIAGONALS OF CORRELATION MATRIX

Variable	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	h^2
1. Dial and Table Reading	77	-30	01	-10	04	05	-02	-07	-04	-55	-08	1.00
2. Spatial Orientation I	56	-12	-56	23	-10	07	20	-46	-14	13	03	1.00
3. Reading Comprehension	64	08	38	26	13	-21	-50	-18	-14	10	01	1.00
4. Instrument Comprehension	63	25	-10	-23	12	-60	26	15	-13	05	-01	1.00
5. Mechanical Principles	50	62	12	-13	-18	-02	00	-21	51	02	-07	1.00
6. Speed of Identification	52	00	-58	35	-21	-02	-21	41	14	-00	-03	1.00
7. Numerical Operations I	56	-61	23	-18	-14	12	05	08	-01	24	-36	1.00
8. Numerical Operations II	62	-53	30	-09	-23	04	08	08	10	08	40	1.00
9. Mechanical Information	26	70	18	-10	-46	22	01	09	-37	-03	00	1.00
10. Practical Judgment	52	29	36	46	34	24	33	16	03	01	-02	1.00
11. Complex Coordination	55	22	-32	-45	41	34	-17	07	-03	12	09	1.00
Contribution of Factor (Eigenvalue)	3.55	1.82	1.23	.79	.69	.64	.55	.54	.48	.40	.31	11.00
Percent of Total Variance	32.3	16.5	11.2	7.2	6.3	5.8	5.0	4.9	4.4	3.6	2.8	100.0
Cumulative Percentage	32.3	48.8	60.0	67.2	73.5	79.3	84.3	89.2	93.6	97.6	100.0	

FIGURE 3.3

HIERARCHICAL STRUCTURE OF ELEVEN AIR FORCE CLASSIFICATION TESTS -- WITH UNITIES AS INITIAL COMMUNALITY ESTIMATES

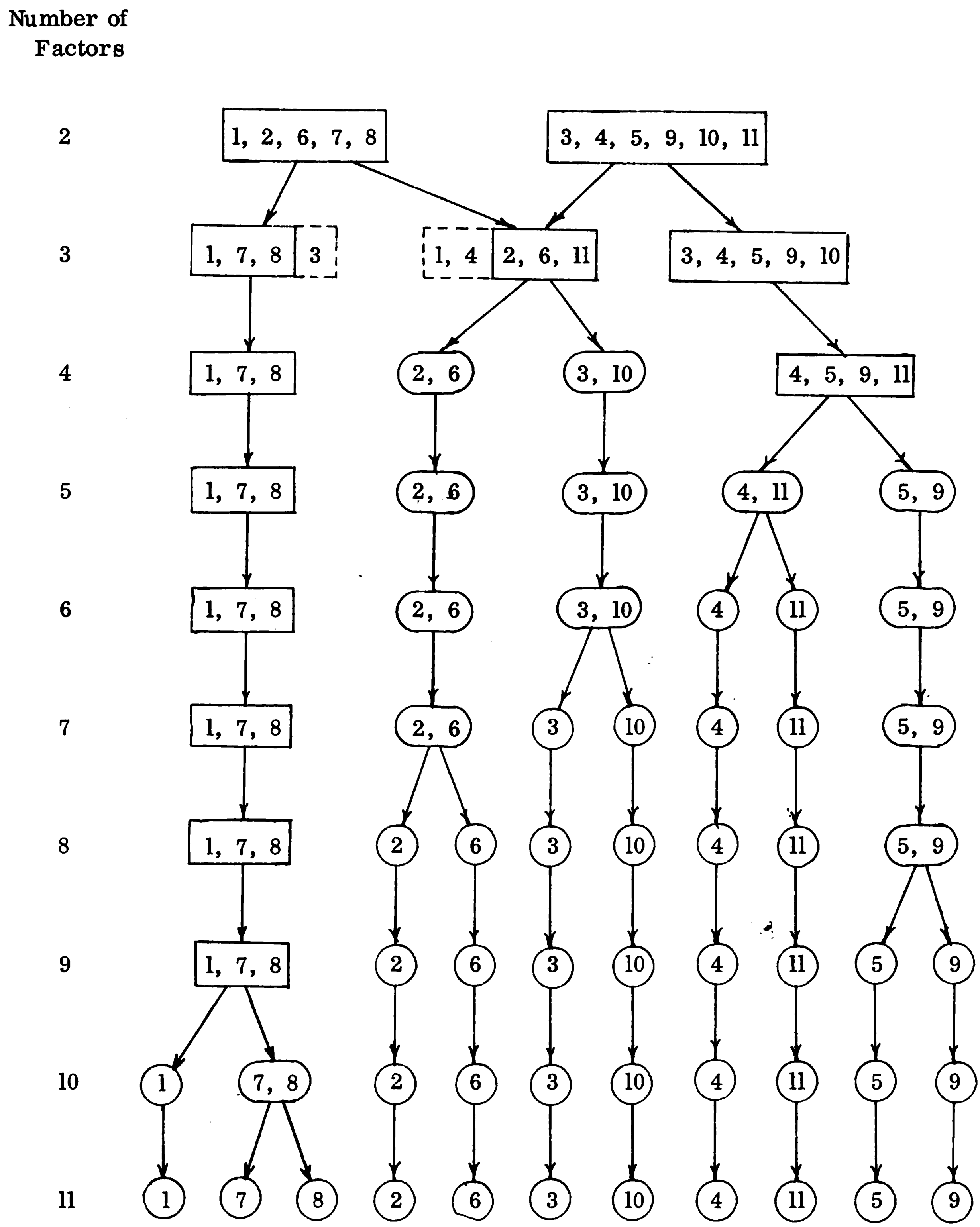


TABLE 3.6 ELEVEN AIR FORCE CLASSIFICATION TESTS --VARIMAX ROTATIONAL SOLUTIONS

FOR TWO, THREE, FOUR, FIVE and SIX FACTORS

Variable	Two Factors I	II	h^2	Three Factors I	II	III	h^2	Four Factors I	II	III	IV	h^2
1	777	268	.676	678	174	431	.675	695	247	341	160	.686
2	502	270	.324	186	-020	776	.637	181	110	803	049	.692
3	439	467	.411	517	536	060	.558	375	187	090	666	.627
4	323	593	.458	192	474	455	.469	247	614	258	134	.522
5	-015	790	.625	-033	772	205	.640	-044	712	022	384	.657
6	397	328	.266	077	034	773	.605	041	099	834	130	.725
7	821	-110	.687	847	-089	122	.740	874	-034	069	035	.772
8	810	-005	.657	858	035	101	.747	846	-005	069	185	.755
9	-251	704	.558	-214	736	026	.589	-237	629	-128	362	.599
10	211	553	.350	293	626	017	.478	094	153	112	804	.691
11	286	520	.352	071	322	590	.457	218	695	313	-171	.658
Factor Variance	2.83	2.53	5.364	2.39	2.19	2.01	6.595	2.31	1.91	1.67	1.49	7.383

Variable	Five Factors I	II	III	IV	V	h^2	Six Factors I	II	III	IV	V	VI	h^2
1	650	290	019	231	356	.687	654	294	016	213	-228	278	.689
2	176	789	-011	060	209	.701	184	793	-009	059	-078	183	.706
3	352	073	189	685	103	.645	332	068	134	638	-379	-081	.689
4	192	193	310	199	572	.537	124	181	172	104	-879	175	.893
5	019	068	734	247	294	.691	009	074	710	221	-326	164	.691
6	078	860	098	071	091	.769	075	860	081	059	-112	031	.769
7	876	064	-105	045	076	.791	886	066	-088	040	-005	080	.805
8	883	097	027	134	-006	.809	883	095	024	114	-073	-048	.810
9	-083	002	893	081	019	.812	-070	014	919	088	009	033	.858
10	031	067	136	880	103	.810	065	077	178	897	017	134	.865
11	063	154	088	058	888	.828	101	177	148	089	-164	919	.944
Factor Variance	2.18	1.54	1.52	1.44	1.41	8.077	2.18	1.55	1.46	1.35	1.13	1.05	8.720

interpretation in this study, but for reference they have been identified by Fruchter as: (numbered in decreasing order of factor variance)

I -- Numerical. The three highest loadings are for numerical operations (1, 7, 8)

II -- Perceptual Speed, consisting of the spatial orientation test (a misnomer?) and speed of identification test (variables 2 and 6), both of which involve perception of small detail working against a time limit.

III -- Mechanical Experience, with highest loadings on the mechanical principles and mechanical information tests (variables 5 and 9)

IV -- Verbal Comprehension, with highest loadings on the reading comprehension (3) and practical judgment (10) tests

V -- Spatial Relations, consisting of the tests of instrument comprehension (4) and complex coordination (11).

Fruchter points out that "in a new area of investigation it would ordinarily not be feasible to identify five factors derived from only eleven tests." (11, p. 149) This is certainly true. This was a matrix based on a very large sample and consisting of test items which had been validated in many preceding analyses, therefore it was useful to demonstrate the effects of increasing the number of factors.

The solutions for six through all eleven factors each result in the splitting of one of the preceding factors. The first factor, that composed of variables 1, 7 and 8, persists longest, through the nine factor solution.

Table 3.6 gives the rotational solutions for two, three, four, five and six factors. The remaining solutions are on file in the Michigan State University library.

Twenty-four Psychological Tests

Each of the preceding examples has been a very well structured matrix, with variables chosen from larger batteries of tests to illustrate particular types of solutions. The twenty-four psychological test matrix was for a number of years considered a prime example of a large matrix for principal axes factor analysis, and has been extensively studied and factored in different ways by Holzinger and Harman, Wrigley and Neuhaus, Kaiser and many others. With the comparatively recent advent of large-capacity electronic computers, it can no longer be considered exceedingly large, since it is now possible to accurately factor matrices of many more variables which psychologists and others have long desired. It does, however, serve as an excellent example of a large matrix containing variables which are not necessarily clearly representative of any single factor and for which more than one solution has been proposed in the past.

Table 3.7 gives the unrotated principal axes solution with unities as initial communality estimates. Five factors have sums of squares (eigenvalues) greater than 1.0 and account for approximately 60% of the total variance.

In prior analyses of this matrix it has always been presented as either a four or five factor set of variables. It is interesting to notice that three of the variables have their highest unrotated loadings on factors other than the first four. Variable 19 (figure-word) is most highly loaded on the fifth factor, variable 2 (cubes) on the seventh factor, and variable 15 (number recognition) on the eleventh factor.

The two, three and four factor solutions are shown in Table 3.8.

The complete set of 23 different rotational solutions is on file in the Michigan State University Library. With the rotation of only the two largest factors, tests which are essentially verbal and spatial are divided from those which are perceptual and numeric. Only a small proportion of the total variance of many of the variables is included in the two-factor solution, however. When three factors are rotated, verbal and spatial factors appear separately with deductive tests common to both factors. With the addition of a fourth factor, a group of memory and recognition tests is isolated. The addition of a fifth factor results in a new factor on which only variable 19 (figure-word) has its highest loading, although variables 3 and 17 (Paper Form Board and Object-Number) also have quite high loadings on this factor. The six factor solution illustrates an effect which is frequently observed with large matrices, namely, that segments of two or more former factors will sometimes combine to form an additional factor. In the six factor solution, the factor containing variables 14 through 17 of the five factor solution splits into two new factors, isolating the recognition tests (14 through 16) from the memory tests (17 through 19). Notice that variable 19 has recombined with the other memory tests and that variable 3 has split away from the other spatial tests into an additional factor on which variables 1 and 13 also have high loadings. The seven, eight and nine factor solutions result in the splitting of variable 2 away from the spatial

TABLE 3.7 TWENTY-FOUR PSYCHOLOGICAL TESTS -- UNROTATED PRINCIPLE AXES FACTORS
UNITIES IN LEADING DIAGONALS OF CORRELATION MATRIX

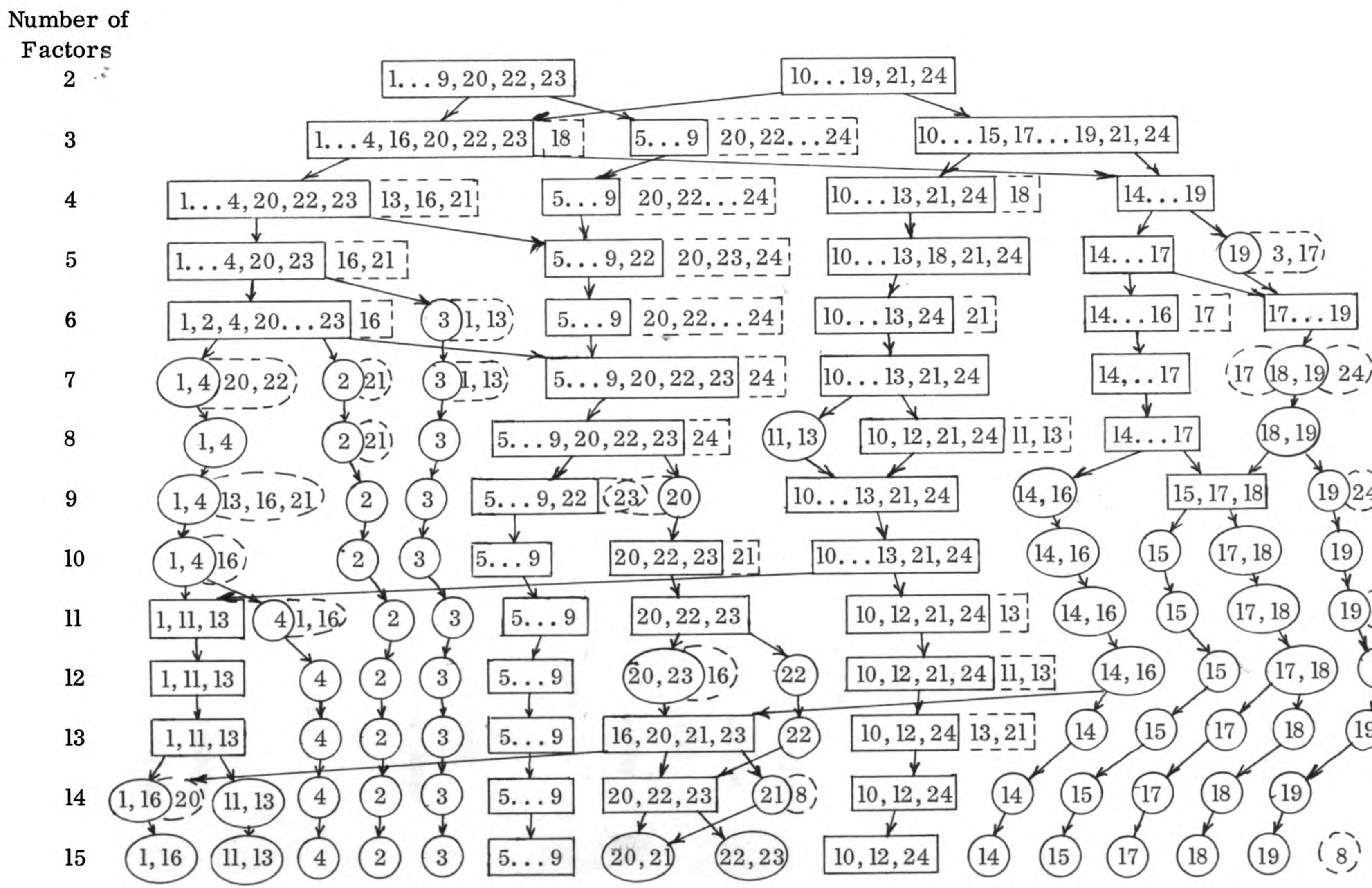
Variable	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	IXX	XX	XXI	XXII	XXIII	XXIV
1. Visual Perception	62	-01	43	-20	-01	07	20	22	-15	16	05	13	-04	32	-04	02	12	-27	12	-01	-13	-08	05	04
2. Cubes	40	-08	40	-20	35	09	-51	-02	-26	-26	07	02	-12	-22	04	-09	-12	-07	07	-07	-02	-07	02	04
3. Paper Form Board	45	-19	48	-11	-38	33	-08	-36	-06	02	-05	=13	-03	-01	10	24	07	00	-16	09	02	08	-04	09
4. Flags	51	-18	34	-22	-01	-19	46	14	14	-27	04	-21	-28	-16	-00	-20	-04	03	10	14	06	03	-07	-06
5. General Information	70	-32	-34	-05	08	08	-12	03	-23	-01	00	-12	13	-01	-20	03	08	-15	-13	19	08	-08	-05	-22
6. Paragraph Comprehension	69	-42	-27	08	-01	12	00	13	-05	-13	02	16	-11	04	21	10	-02	05	02	-04	-08	25	20	-11
7. Sentence Completion	68	-43	-36	-07	-04	01	08	01	00	-09	-10	-03	01	08	14	-14	-05	-10	-09	-24	-08	05	-25	06
8. Word Classification	69	-24	-14	-12	-14	12	16	-17	12	-07	-26	-08	22	-15	-23	10	-11	01	22	-16	-04	-12	12	03
9. Word Meaning	69	-45	-29	08	-01	-07	-01	12	-12	03	07	11	-02	02	-07	-11	06	08	01	15	23	-02	06	26
10. Addition	47	54	-45	-20	08	-09	-01	-08	08	-10	-06	04	-10	-03	-10	08	-17	-11	-10	22	-24	06	01	12
11. Code	58	43	-21	03	00	30	-04	32	-00	06	21	02	14	-15	13	24	03	06	21	02	05	00	-15	02
12. Counting Dots	48	55	-13	-34	10	04	16	-30	-13	16	02	02	-06	06	03	-07	-12	-12	04	-11	30	14	03	-04
13. Straight-Curved Capitals	62	28	04	-37	-08	36	13	18	-04	10	-02	-05	03	-01	09	-26	-04	26	-18	01	-08	-13	07	-03
14. Word Recognition	45	09	-06	56	16	38	-08	-13	26	06	11	-31	-14	06	-09	-21	09	-08	11	02	-05	07	03	02
15. Number Recognition	42	14	08	53	31	-06	13	07	-30	21	-44	-07	-18	-01	07	14	-06	08	-04	-02	01	-07	-02	01
16. Figure Recognition	53	09	39	33	17	17	08	13	30	-21	02	29	11	08	-21	06	-16	-00	-20	-05	11	03	-03	-01
17. Object-Number	49	28	-05	47	-26	-11	25	-21	-15	-14	18	19	-00	-29	12	-08	13	-13	-07	-04	-04	-13	04	-01
18. Number-Figure	54	39	20	15	-10	-25	-02	-00	-35	-29	01	-17	27	18	-13	-07	07	16	05	-02	-06	17	-02	01
19. Figure-Word	48	14	12	19	-60	-14	-34	19	10	11	-21	05	-04	02	06	-14	-21	-09	09	09	06	00	-02	-06
20. Deduction	64	-19	13	07	29	-19	03	-29	18	06	08	00	28	15	34	-02	-14	03	08	17	-02	-10	-00	-03
21. Numerical Puzzles	62	23	10	-20	17	-23	-16	18	32	-00	-27	-07	13	-15	12	-01	32	-10	-10	-05	06	06	07	03
22. Problem Reasoning	64	-15	11	06	-02	-33	-05	13	02	37	36	-23	01	-11	-09	10	-15	-00	-15	-15	-06	03	06	02
23. Series Completion	71	-11	15	-10	06	-11	-08	-25	07	30	-02	31	-10	-12	-20	-08	14	18	10	03	-11	10	-12	-07
24. Arithmetic Problems	67	20	-23	-06	-10	-17	-23	-12	15	-19	10	-02	-29	29	-01	16	10	14	-00	-11	06	-21	01	-04
Contribution of Factor (Eignevalue)	8.14	2.10	1.69	1.50	1.03	.94	.90	.82	.79	.71	0.64	0.54	0.53	0.51	0.48	0.39	0.38	0.34	0.33	0.32	0.30	0.27	0.19	0.17
Percent of Total Variance	33.9	8.7	7.1	6.3	4.3	3.9	3.8	3.4	3.3	3.0	2.7	2.3	2.2	2.1	2.0	1.6	1.6	1.4	1.4	1.3	1.2	1.1	.8	.7
Cumulative Percentage	33.9	42.6	49.7	55.9	60.2	64.1	67.9	71.3	74.6	77.5	80.2	82.5	84.7	86.8	88.8	90.4	92.0	93.4	94.8	96.1	97.4	98.5	99.3	100.

TABLE 3.8 TWO, THREE AND FOUR FACTOR VARIMAX ROTATION SOLUTIONS FOR TWENTY-FOUR PSYCHOLOGICAL TESTS
(Leading decimal points omitted from values; i.e., read 476 as 0.476.)

Variable	Two Factors I	Two Factors II	Two Factors h^2	Three Factors I	Three Factors II	Three Factors III	Three Factors h^2	Four Factors I	Four Factors II	Four Factors III	Four Factors IV	Four Factors h^2
1	476	390	379	162	244	690	562	151	713	231	141	604
2	358	195	166	090	069	560	326	082	593	085	028	367
3	464	138	234	147	-011	663	461	137	664	-038	108	472
4	506	190	292	251	072	580	404	243	619	092	027	451
5	740	198	586	789	235	147	699	785	157	217	117	701
6	798	121	651	810	140	215	721	803	183	069	212	728
7	792	107	639	851	149	134	765	849	156	149	062	770
8	689	258	541	643	248	294	561	638	315	239	109	575
9	822	099	685	845	123	201	770	839	171	056	201	776
10	017	720	518	181	806	-186	718	181	-135	833	118	758
11	165	702	520	190	722	080	564	182	044	631	363	566
12	019	730	534	020	737	082	550	019	163	798	046	666
13	296	610	460	187	565	328	461	182	411	626	030	595
14	285	358	209	243	344	186	212	227	009	063	682	521
15	228	375	193	126	331	272	199	109	080	054	674	476
16	352	412	293	071	281	602	447	052	458	055	582	554
17	198	524	314	154	509	184	316	138	010	248	675	537
18	170	645	445	-006	564	407	484	-021	319	405	491	507
19	277	410	245	143	351	341	260	130	252	202	420	297
20	613	269	448	439	193	436	466	427	437	093	299	471
21	329	576	440	185	514	389	450	177	418	506	167	491
22	595	298	430	423	228	460	443	411	416	134	293	446
23	614	375	517	422	292	526	540	411	527	249	204	550
24	391	581	491	406	599	147	545	400	144	551	255	549
Factor Variance	5.66	4.57	10.23	4.23	4.08	3.62	11.92	4.11	3.33	3.25	2.74	13.43

FIGURE 3.4

HIERARCHICAL STRUCTURE OF TWENTY-FOUR PSYCHOLOGICAL TESTS -- STARTING WITH UNITIES IN CORRELATION MATRIX



tests, variables 11 and 13 away from the numerical tests and re-isolation of variables 19 and 20 into unique factors. The solutions for ten through thirteen factors are interesting because they illustrate the stabilizing, for several solutions, of two new combinations of variables not previously grouped. These are the combinations of variables 1 (visual perception test), 11 (code) and 13 (straight and curved capitals) into a much clearer Perceptual-Speed factor, and the stabilizing of tests 10 (addition), 12 (counting dots), 21 (numerical puzzles), and 24 (Woody-McCall "arithmetic") into a possibly clearer numerical ability factor. Test 21 splits away on the thirteenth factor, since it is a deductive test requiring numerical operation. Most important, however, is the strong relationship of test 12, which was intended as a perceptual speed test, with test 10 (addition). These two tests remain on the same factor until the twenty-one factor solution -- one of the strongest linkages among the twenty-four tests.

The strongest factor, in the sense of being most resistent to splitting, is the verbal factor (variables 5 through 9). Variable 8 starts to split away on the nineteen factor solution, but it is not until the last two solutions that this factor breaks up into unique factors.

This matrix is an additional example of the complete disintegration of the factorial structure, with the complete splitting into twenty-four unique factors. In the process, however, there is a great deal of structural instability, with variables splitting away from one factor and becoming part of another factor, etc.

Figure 3.4 shows the hierarchical structure of the first sixteen rotational solutions obtained with unities in the diagonals of the original correlation matrix. The diagram shows in solid boxes those variables which have their highest loading on a particular factor and, in dotted lines attached to the solid boxes, those other variables which have loadings greater than 0.40 on the factor.

CHAPTER IV

THE EFFECT OF DIFFERENT COMMUNALITY ESTIMATES ON VARYING ROTATIONAL SOLUTIONS

One of the most perplexing problems in the field of factor analysis, and one of the greatest obstacles in reaching agreement on an objective procedure for factor analyzing a given correlation matrix, has been the question of what starting values should be inserted in the principal diagonal elements of the matrix prior to extracting any factors -- the so-called "communality problem". The problem stems from basic factor theory which assumes that any variable can be described in terms of two basic types of factors: (1) _common_ factors which are present in more than one variable of a set of variables, and (2) _unique_ factors which are present only in that particular variable. This is further complicated by the fact that the uniqueness of a variable can be subdivided into two components: (1) _error variance_, due to imperfections in measurement, and (2) _specific variance_, which is reliable but due to the specific selection of tests in the battery. The specific variance is sometimes added to the common variance, the sum being termed the _reliability_ of the variable.

Factor analysts in the behavioral sciences have usually preferred to start factoring with values in the diagonals of the correlation matrix which represented the proportion of the variance of each particular variable which was common to all of the other variables in the test battery -- the "communalities". Unfortunately there is no *a priori* knowledge of the exact values for the communalities, hence the proliferation of many suggestions as to the approximations which should be used.

Distinction should be made here between different uses of the word "communality". As Wrigley points out (51):

> The term "communality" has been defined by various writers in different and sometimes conflicting ways. Through all their definitions, however, there runs the notion that communalities are reduced values to be inserted in the leading diagonal, and that, after extraction of *k* common factors ($k < p$, the number of tests), they become either zero or very small.
>
> ....
>
> Various sets of communalities, based on varying numbers of common factors, will satisfy the theory, i.e. provide residual correlations of exactly zero. The solution with the fewest common factors is generally assumed to be the best.

Another use of the word communality is in referring to the sum of squares of loadings across factors for each variable when presenting a particular factor solution. These should perhaps be called the "observed communalities" for a particular solution. In actual practice, the exact meaning is usually clear from the context.

Because of the controversy over the appropriate initial values to place in the principal diagonals of the correlation matrix prior to factoring, and because it had been suggested by some authors (17) that when the number of variables was large (say 20 or more) the values in the diagonal made little difference, an additional phase was added to this study to compare different rotational solutions, obtained in the same manner as in the preceding chapter, when communality estimates other than unities were used in the diagonals of the test matrices.

Since the intention was still to use only methods which were objective, two methods of directly calculating communality estimates were chosen:

1. <u>Squared multiple correlations</u> -- Wrigley, in a number of excellent papers (47, 48, 51), has urged the use of squared multiple correlation of a variable with each of the other n-1 variables as the best value to be inserted in the diagonal for that variable prior to factoring.

> Until the communality problem is stated in such a way that exact and unique values can be found, we shall do better to insert the squared multiple correlations in the diagonal. The SMCs might be called the observed communalities, in distinction from the theoretical communalities . . ., since they and not the minimal rank communalities measure the predictable common variance in the observed correlations. They are objective, unique and obtainable rapidly and without iteration with modern computational equipment. Then all factors with positive roots should be rotated. That is to say, to avoid the downward bias of dimensionality when sample size is small, decisions upon the number of factors and the diagonal values will be made algebraically rather than statistically. Since the SMCs can be proved to be lower bounds for the minimal rank communalities, and the number of SMC factors with positive roots is a lower bound for the minimal rank number of factors, this proposal follows the conservative policy of operating exclusively with observed common variance instead of trying to determine what would happen to common variance in a domain of tests. (52, p. 472)

The calculation of squared multiple correlations, SMCs for short, as Wrigley said is very simple on a modern electronic computer, and the SMCs for all of the variables in a matrix can be calculated simultaneously. The procedure is simple to calculate the inverse, R^{-1}, of the original correlation matrix with unities in the diagonals. The SMC for variable z_i is then:

$$SMC_i = \frac{r^{ii} - 1}{r^{ii}}$$ where r^{ii} is the diagonal element of the inverse corresponding to variable z_i.

2. <u>Guttman communalities</u> -- As implied in the preceding quotation, the insertion of SMC's in the diagonals of the correlation matrix produces a matrix which is not Gramian, i.e. has a number of eigenvalues which will be negative (imaginary). Guttman proposed a method (16), very similar to the SMC calculations, which results in communality estimates which have two very desirable properties: (a) the matrix in which they are inserted remains Gramian, and (b) the rank of the matrix is reduced by exactly one, i.e. the resulting principal axes solution results in $(n-1)$ positive roots and one zero root.

The calculation is somewhat more complicated than that for SMCs but very similar, it is direct, and the solution is unique for any matrix. The starting point is, again, the inverse, R^{-1}, of the original correlation matrix, R, with unities in the diagonals. A diagonal matrix, D, is then formed as follows:

$$d_{ij} = \sqrt{r^{ii}} \qquad \text{for} \quad i = j$$

$$d_{ij} = 0 \qquad \text{for} \quad i \neq j$$

Let λ^* be the smallest eigenvalue (latent root) of the triple-product matrix DRD. The Guttman communality for variable z_j is then

$$GC_j = \frac{r^{jj} - \lambda^*}{r^{jj}}$$ which is similar to the formula for SMCs.

As in the preceding chapter, the procedure for this phase of the investigation was to insert the new communality estimates into the diagonals of the test matrices, obtain the principal axes factors, order the factors in descending order by size of the corresponding latent root, and then rotate the two largest factors, then the three largest factors, and so on until all factors with positive roots had been rotated.

Eight Physical Variables

A comparison of the two communality estimates is given in Table 4.1:

TABLE 4.1

COMMUNALITY ESTIMATES FOR EIGHT PHYSICAL VARIABLES

Variable	Guttman Communality	SMC
1	.9088	.8162
2	.9252	.8493
3	.9011	.8006
4	.8950	.7884
5	.8754	.7488
6	.8036	.6042
7	.7828	.5622
8	.7409	.4778
Total	6.8328	5.6475
% Total Variance	85.5	70.7

Since the SMCs are a measure of the predictable common variance in the observed correlations for a particular variable, they may also be considered a proportion of the total variance factored. Notice that in the above example each of the first five variables has at least 75% or more of its variance in common with the other tests. The proportion falls considerably for the last three variables; in fact, less than 50% of the total variance of variable eight is in common with the rest of the variables.

One might expect to find quite different factor loadings in all of the solutions using the three different estimates. This is

not the case, however, Table 4.2 shows a comparison of the first three unrotated principal axes factors from each of the three solutions. It is evident that the additional unique variance present when unities were used in the diagonals makes little difference in the first two factors, and becomes evident primarily in the later factors. This is particularly true of variable eight. With unities in the diagonal, the total variance is almost equally divided between the first and third factors, which this third "unique" factor practically disappears with the use of SMCs. The maximum difference between highest and lowest loading for any variable on either of the first two factors is only 0.06 or .36% of the total variance. Notice that the sum of the two largest eigenvalues, 5.87, is greater than the sum of the original communalities in the SMC solution. Table 4.3 gives the remaining four factors with Guttman communalities, only four positive eignevalues resulted using SMCs.

TABLE 4.2

EFFECTS OF DIFFERING COMMUNALITY ESTIMATES
ON THE FIRST FOUR UNROTATED FACTORS
(EIGHT PHYSICAL VARIABLES)

Variable	I				II				III			
	U	GC	SMC	Diff	U	GC	SMC	Diff	U	GC	SMC	Diff
1	86	86	86	00	-37	-34	-30	05	-07	-08	-06	04
2	84	84	84	01	-44	-42	-38	04	08	11	14	04
3	81	81	83	00	-46	-43	-42	06	01	05	03	08
4	84	84	86	01	-40	-37	-35	06	-10	-14	-14	04
5	76	74	72	03	52	53	55	02	-15	-12	-06	10
6	67	65	62	04	53	52	50	04	-05	03	04	08
7	62	59	56	05	58	55	53	06	-29	-21	-06	25
8	67	64	60	06	42	39	37	06	60	39	13	48
Eigenvalue	4.67	4.54	4.45	.26	1.77	1.62	1.49	.31	.48	.25	.07	.41
% of Total Original Communality	58.4	66.4	77.1	3.2	22.1	23.7	25.8	3.9	6.0	3.7	1.2	5.1

TABLE 4.3

REMAINING UNROTATED PRINCIPAL AXES FACTORS
WITH GUTTMAN COMMUNALITIES -- (EIGHT PHYSICAL VARIABLES)

Variable	IV	V	VI	VII
1	-09	16	05	-10
2	06	-10	10	-03
3	-03	-21	-09	01
4	07	15	-03	12
5	-07	01	-12	-06
6	-32	-04	08	07
7	27	-08	07	00
8	14	10	-03	00
Eigenvalue	.22	.12	.05	.04
% of Total Variance	3.2	1.8	0.7	0.6

Figures 4.1 and 4.2 show the hierarchical structures which appear when the factors obtained with Guttman communalities and SMCs are rotated. These may be compared with Figure 3.1, the hierarchical structure of the solution starting with unities. In all three cases variables one through four remain grouped into a single factor through the entire series of rotations. The two factor solution is the same with all three communality estimates. The basic differences between the use of different communality estimates is that with unities variables five through eight eventually split into four unique factors. With Guttman communalities, variable eight splits away from variables five through seven and with SMCs only two factors ever appear, the rotation of the two additional factors with positive eigenvalues producing only null factors.

Eight Political Variables

Tables 4.4 and 4.5 show similar analyses for the eight political variable matrix. In this example there is even less discrepancy between

FIGURE 4.1

HIERARCHICAL STRUCTURE OF EIGHT PHYSICAL VARIABLES -- STARTING WITH GUTTMAN COMMUNALITIES

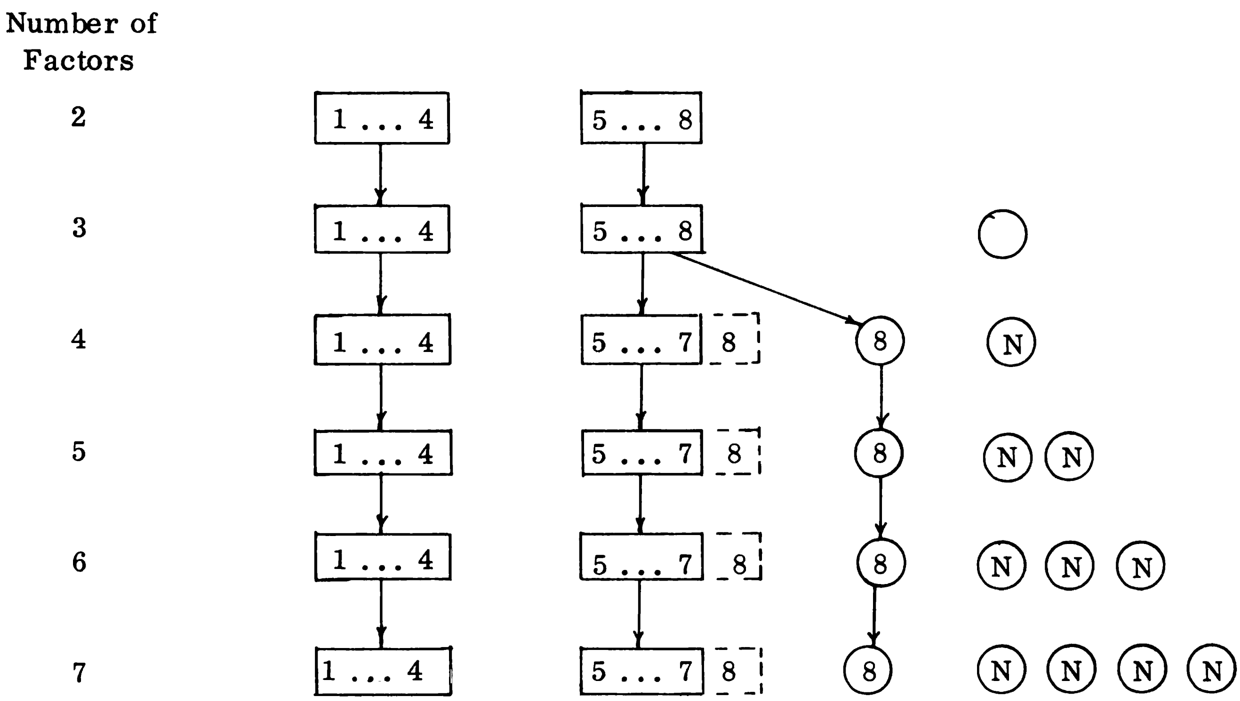
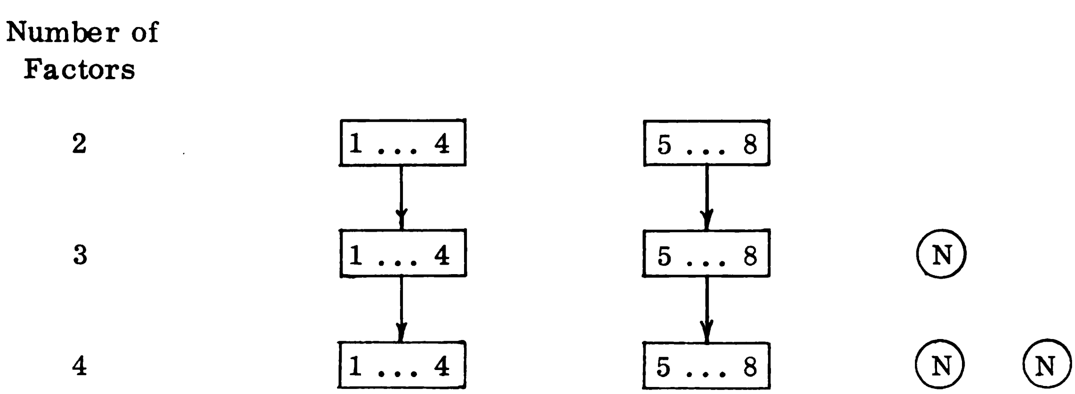


FIGURE 4.2

HIERARCHICAL STRUCTURE OF EIGHT PHYSICAL VARIABLES -- STARTING WITH SQUARED MULTIPLE CORRELATIONS



unities and SMCs over all eight of the variables, with the maximum uniqueness of 33% for variable 5. Table 4.5 shows the comparison of the first four factors with different initial communalities. In this example the first factor is almost identical with all three initial estimates. There is, however, slightly greater discrepancy in at least one variable on each of the other three factors.

TABLE 4.4

COMMUNALITY ESTIMATES FOR
EIGHT POLITICAL VARIABLES

Variable	Guttman Communality	Squared Multiple Correlation
1	.9074	.7709
2	.9591	.8989
3	.9330	.8343
4	.9212	.8051
5	.8684	.6745
6	.9150	.7896
7	.9016	.7565
8	.9516	.8801
Total	7.3574	6.4099
% of Total Variance	91.97	80.13

TABLE 4.5

EFFECTS OF DIFFERING COMMUNALITY ESTIMATES
ON THE FIRST THREE UNROTATED FACTORS
FOR EIGHT POLITICAL VARIABLES

Variable	I				II				III			
	U	GC	SMC	Diff	U	GC	SMC	Diff	U	GC	SMC	Diff
1	74	73	72	02	-36	-35	-33	03	54	49	39	15
2	86	86	86	00	-43	-43	-43	00	16	15	14	02
3	89	89	88	01	-21	-20	-19	02	-16	-16	-14	02
4	-89	-88	-87	02	-04	-05	-06	02	28	26	22	06
5	32	31	30	02	89	84	77	12	23	21	18	05
6	91	90	88	03	-02	-02	-01	01	-19	-18	-14	05
7	-70	-70	-68	02	-62	-61	-58	04	-06	-08	-08	02
8	-94	-94	-94	00	-09	-10	-11	02	09	09	08	01
Eigenvalue	5.19	5.11	5.01	.18	1.54	1.43	1.28	.26	.53	.44	.31	.22
% of Total orig. comm.	64.8	69.5	78.2		19.2	19.5	19.9		6.6	5.9	4.8	

Figures 4.3 and 4.4 diagram the hierarchical structures obtained with the two communality estimates (they may be compared with Figure 3.2, the structure with unities). Three factors stabilize through the entire series of rotational solutions for both Guttman communalities and SMCs, the addition of factors resulting only in null factors.

FIGURE 4.3

HIERARCHICAL STRUCTURE OF EIGHT POLITICAL VARIABLES STARTING WITH GUTTMAN COMMUNALITIES

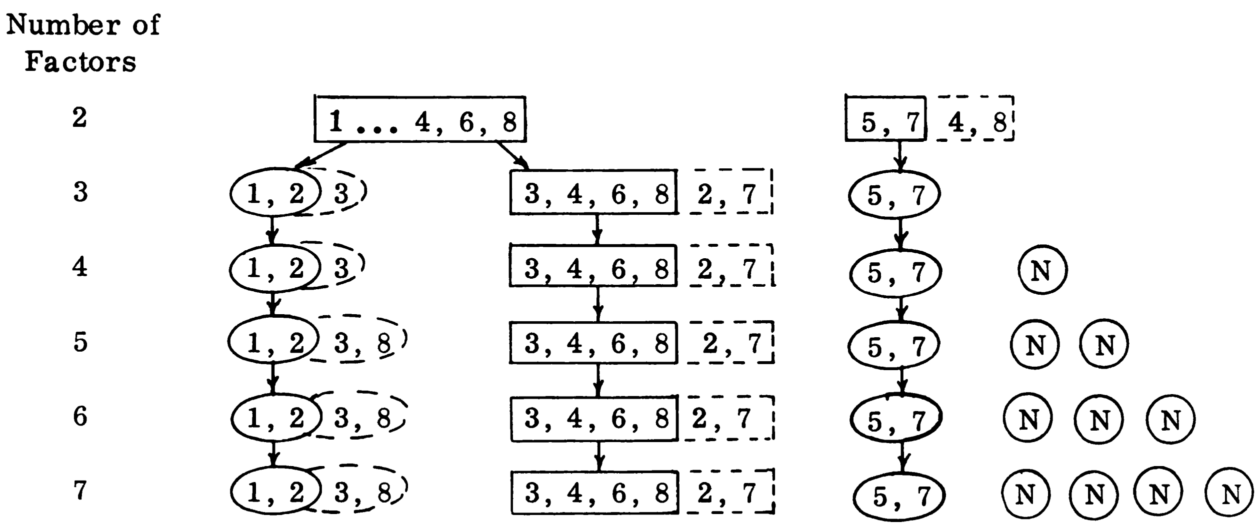
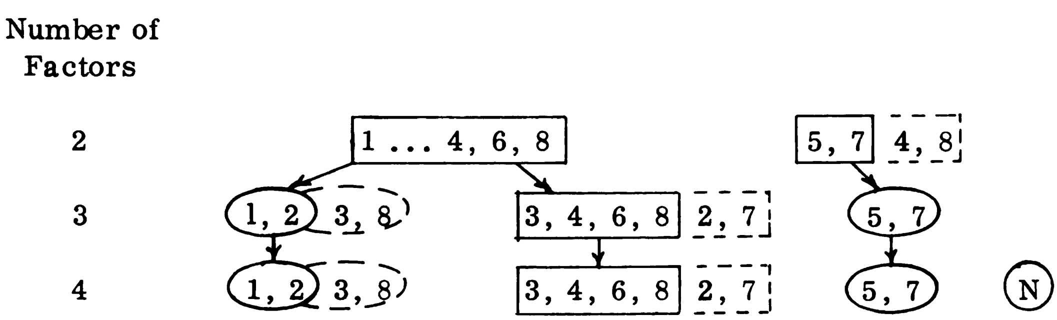


FIGURE 4.4

HIERARCHICAL STRUCTURE OF EIGHT POLITICAL VARIABLES -- STARTING WITH SQUARED MULTIPLE CORRELATIONS



Eleven Air Force Classification Tests

In the two preceding examples, both the Guttman communalities and the SMCs were unusually large -- for most variables the initial values in the diagonals of the correlation matrix were greater than 0.75. With the eleven variable classification test matrix there is a much greater discrepancy between unities and the other two communality estimates, as illustrated in Table 4.6.

TABLE 4.6

COMMUNALITY ESTIMATES FOR ELEVEN AIR FORCE CLASSIFICATION TESTS

Variable	Guttman Communality	Squared Multiple Correlation
1	.693	.500
2	.570	.301
3	.584	.323
4	.568	.296
5	.609	.364
6	.543	.257
7	.695	.503
8	.706	.522
9	.547	.262
10	.531	.237
11	.544	.258
Total	6.590	3.824
% of Total Variance	59.91	34.75

In spite of this discrepancy in initial diagonal entries, there is less difference in unrotated loadings on the first three factors than might be expected. The difference is much more marked in factors four and five. Table 4.7 give the comparison figures for the first five factors for all three starting communalities. Notice that the percentage of total original communality accounted for by the first three factors actually increases with lower communalities.

TABLE 4.7 ELEVEN AIR FORCE CLASSIFICATION TESTS EFFECTS OF DIFFERENT COMMUNALITY ESTIMATES ON THE FIRST FIVE UNROTATED P.A. FACTORS

Variable	I				II				III				IV				V			
	U	CC	SMC	DIFF.	U	CC	SMC	DIFF.	U	CC	SMC	DIFF.	U	CC	SMC	DIFF.	U	CC	SMC	DIFF.
1	77	75	73	.04	-29	-25	-21	.08	01	-03	-04	.05	-10	-05	-02	.12	03	08	02	.06
2	56	52	49	.07	-12	-06	-04	.08	-56	-46	-39	.07	23	15	06	.29	-09	-10	-02	.09
3	64	60	57	.07	07	08	09	.02	38	30	24	.14	26	24	13	.13	13	12	02	.12
4	63	58	55	.08	25	24	23	.02	-10	-08	-06	.04	-23	-13	-06	.29	12	14	02	.10
5	50	46	44	.06	62	56	51	.09	12	13	12	.01	-13	-12	06	.19	-18	-12	-02	.16
6	52	47	45	.07	00	03	05	.05	-58	-46	-37	.21	35	20	08	.43	-21	-17	-02	.19
7	56	55	55	.01	-61	-56	-52	.09	23	17	14	.09	-18	-14	-07	.25	-14	-07	-01	.13
8	62	61	60	.02	-53	-48	-45	.08	30	25	21	.09	-09	-07	-03	.12	-23	-17	-02	.21
9	26	23	21	.05	70	59	52	.18	18	17	15	.03	-10	-11	06	.16	-46	-26	-03	.44
10	52	47	44	.08	29	25	23	.06	36	27	21	.15	46	30	14	.60	34	12	01	.33
11	55	51	48	.07	21	21	20	.01	-32	-26	-20	.12	-45	-26	-12	.57	41	23	03	.38
Contribution of Factor	3.55	3.16	2.92	.63	1.82	1.44	1.20	.62	1.23	.80	.54	.69	.79	.35	.08	.71	.69	.26	.00	.69
% Total Var.	32.3	28.7	26.5		16.5	13.1	10.9		11.2	7.3	4.9		7.2	3.2	0.7		6.3	2.4	0.0	
% Total Orig Comm.	32.3	48.0	76.4		16.5	21.8	31.5		11.2	12.2	14.1		7.2	5.3	2.0		6.3	4.0	0.1	

FIGURE 4.5

HIERARCHICAL STRUCTURE OF ELEVEN AIR FORCE CLASSIFICATION TESTS -- STARTING WITH GUTTMAN COMMUNALITIES

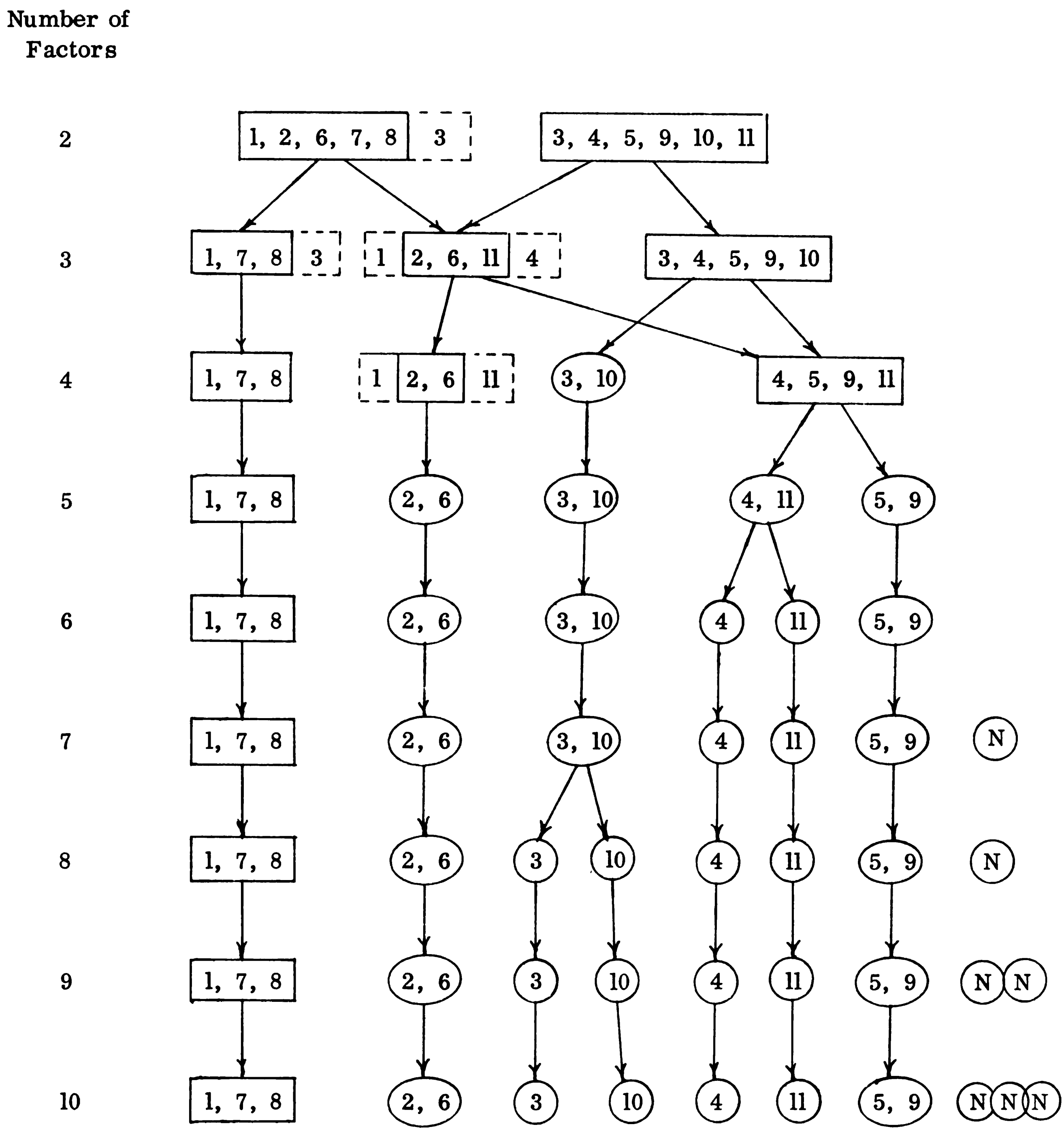
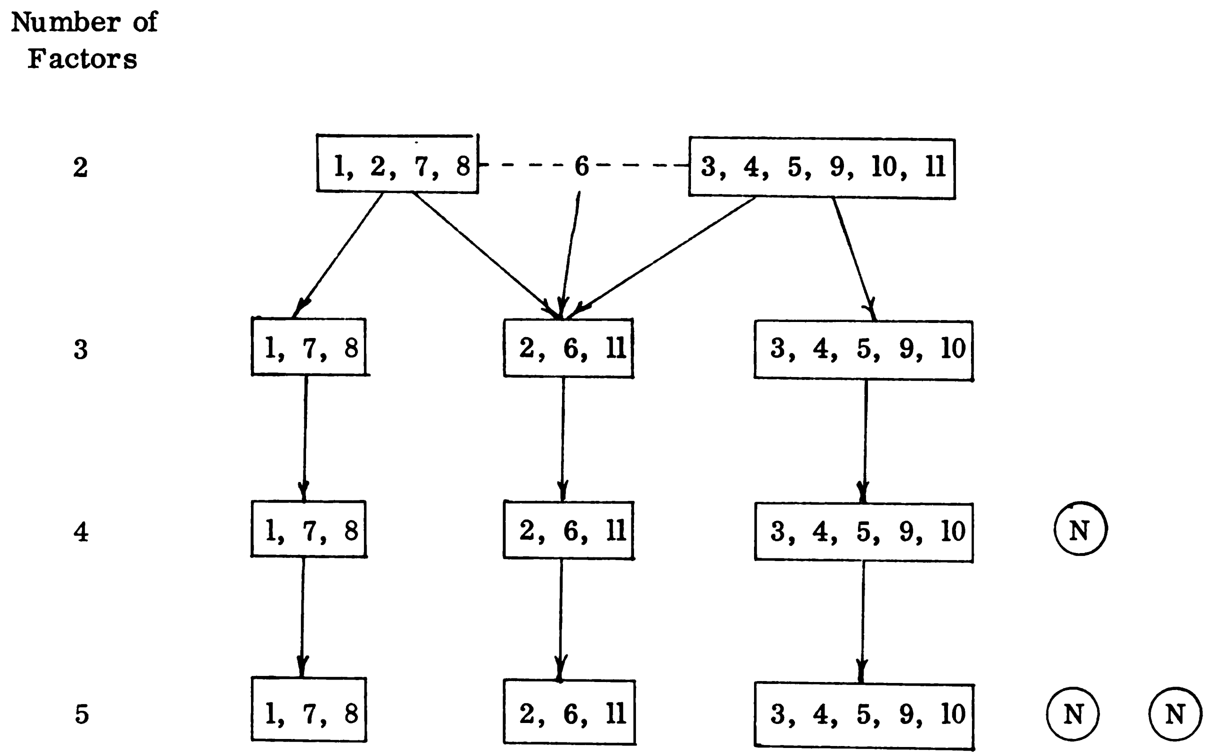


FIGURE 4.6

HIERARCHICAL STRUCTURE OF ELEVEN AIR FORCE CLASSIFICATION TESTS -- STARTING WITH SQUARED MULTIPLE CORRELATIONS



The hierarchical structure for different rotational solutions for both initial communality estimates is shown in Figures 4.5 and 4.6. The structure starting with Guttman communalities is the same as that with unities (Figure 3.3) through the six factor solution. The addition of a seventh factor produces only a null factor. Variables 3 and 10 split on the eight factor solution rather than on the seven factor solution as they did with unities in the diagonals. The addition of other factors results in no further splitting, merely the addition of null factors.

With SMCs as the starting communality estimates only five factors have positive latent roots. Rotation of these factors results in only three factors, corresponding to the three factor solutions with both unities and Guttman communalities. Of particular importance is the fact that the five factors which have been presented in the past for this matrix never appear.

Twenty-four Psychological Variables

Guttman communalities and the squared multiple correlations for the twenty-four variables are given in Table 4.8. As in the previous examples, the different initial communality estimates make relatively little difference in the first few unrotated factors; the maximum difference between loadings on the fourth factor being only 0.16 in this example.

Of greater significance is the proportion of the total original communality represented by the eigenvalues (variance contributions) of the unrotated factors with different communality estimates. Table 4.9 shows this comparison. Notice that for the first four factors,

TABLE 4.7 TWENTY-FOUR PSYCHOLOGICAL VARIABLES

INITIAL COMMUNALITY ESTIMATES

Variable	Guttman Communalities	Squared Multiple Correlations
1	.7835	.5107
2	.6900	.2996
3	.7524	.4405
4	.7385	.4092
5	.8551	.6726
6	.8569	.6766
7	.8602	.6840
8	.8069	.5638
9	.8731	.7133
10	.8137	.5790
11	.7969	.5410
12	.7950	.5368
13	.7961	.5393
14	.7161	.3584
15	.6871	.2929
16	.7471	.4286
17	.7399	.4124
18	.7533	.4425
19	.7199	.3671
20	.7626	.4637
21	.7669	.4733
22	.7563	.4494
23	.8058	.5611
24	.7905	.5265
Total	18.6636	11.9424
Percent of Total Variance	77.8	49.8

the percent of variance contribution increases as the size of communality decreases. From this point on the percentage decreases.

TABLE 4.9

VARIANCE CONTRIBUTION (EIGENVALUES) OF FIRST EIGHT UNROTATED FACTORS AS A PERCENT OF TOTAL ORIGINAL COMMUNALITY FOR TWENTY-FOUR PSYCHOLOGICAL VARIABLES

Factor	Unities		Guttman Communalities		Squared Multiple Correlations	
	λ	% Total Comm.	λ	% Total Comm.	λ	% Total Comm.
1	8.14	33.9	7.93	42.5	7.66	64.2
2	2.10	8.7	1.91	10.2	1.67	14.0
3	1.69	7.1	1.48	7.9	1.21	10.1
4	1.50	6.3	1.24	6.7	.92	7.7
5	1.03	4.3	.76	4.1	.45	3.7
6	.94	3.9	.71	3.8	.41	3.4
7	.90	3.8	.64	3.4	.32	2.7
8	.82	3.4	.59	3.2	.31	2.6
Total Orig. Communality	24.00	100.0	18.66	77.8*	12.94	53.9*

*Percent total original communality is of the total variance (24.0).

Figures 4.7 and 4.8 give the hierarchical structure of the solutions with Guttman communalities and SMCs through the fifteen and fourteen factor solutions, respectively. Through the four factor solutions they are almost identical to the hierarchy obtained with unities in the diagonals. (The appearance of variable 15 in the same factor with variables 1 through 4 in the three factor solution with SMCs is somewhat misleading -- actually it was almost equally loaded on both the first and third factors.) The addition of a fifth factor with all three communality estimates results in either a null factor (with SMCs) or a factor with only one or two highest loadings (with Guttman communalities or unities). This is primarily due to the great discrepancy in the coefficients for variable 19 on the fifth unrotated factor with the three different initial diagonals.

A comparison of the complete hierarchical structure for all three sets of solutions reveals that with unities the factors continue fissioning until the final twenty-four factor rotation in which each variable is loaded most highly on a unique factor. With Guttman communalities, there is still considerable factorial instability, with some variables splitting into unique factors on one solution then combining again with other variables on a later solution, etc. Eventually, however, when all twenty-three factors with positive eigenvalues are rotated, variables 5 through 9 (the verbal tests) remain in a single factor, tests 10 and 12 (addition and counting dots) cluster on a single factor, the remaining seventeen variables each have their highest loading on a single unique factor, and four null factors make up the total of twenty-three. The complete set of rotations starting with SMCs in the diagonals (only fourteen factors had positive eigenvalues and were included in the solutions) shows much more stability. Some splitting continues through the ten factor solution. From this point on (factors 10 through 14) no further splitting takes place, merely the addition of null factors. The verbal-deductive, numerical and memory factors remain undivided.

FIGURE 4.7

HIERARCHICAL STRUCTURE OF TWENTY-FOUR PSYCHOLOGICAL TESTS -- STARTING WITH GUTTMAN COMMUNALITIES

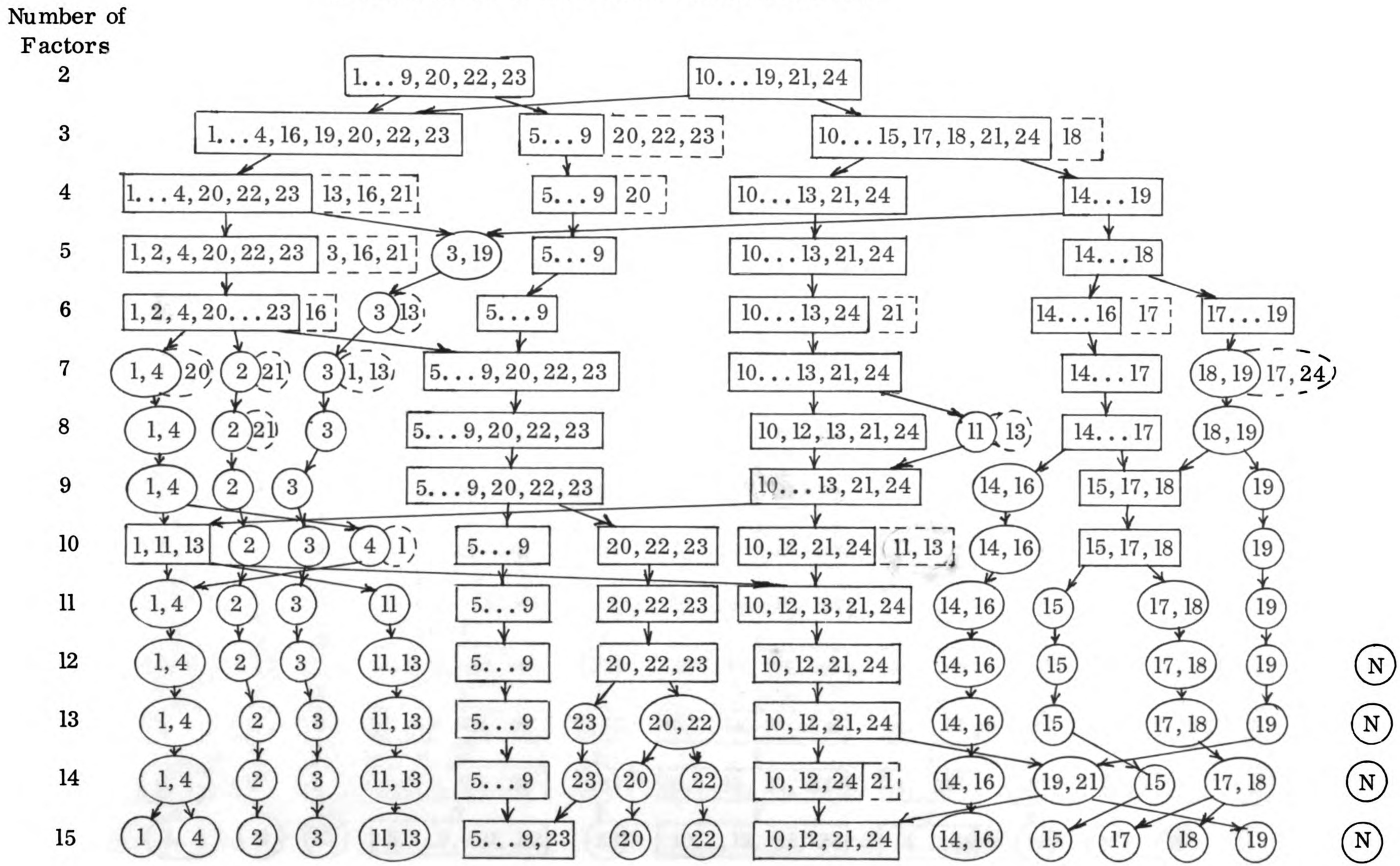
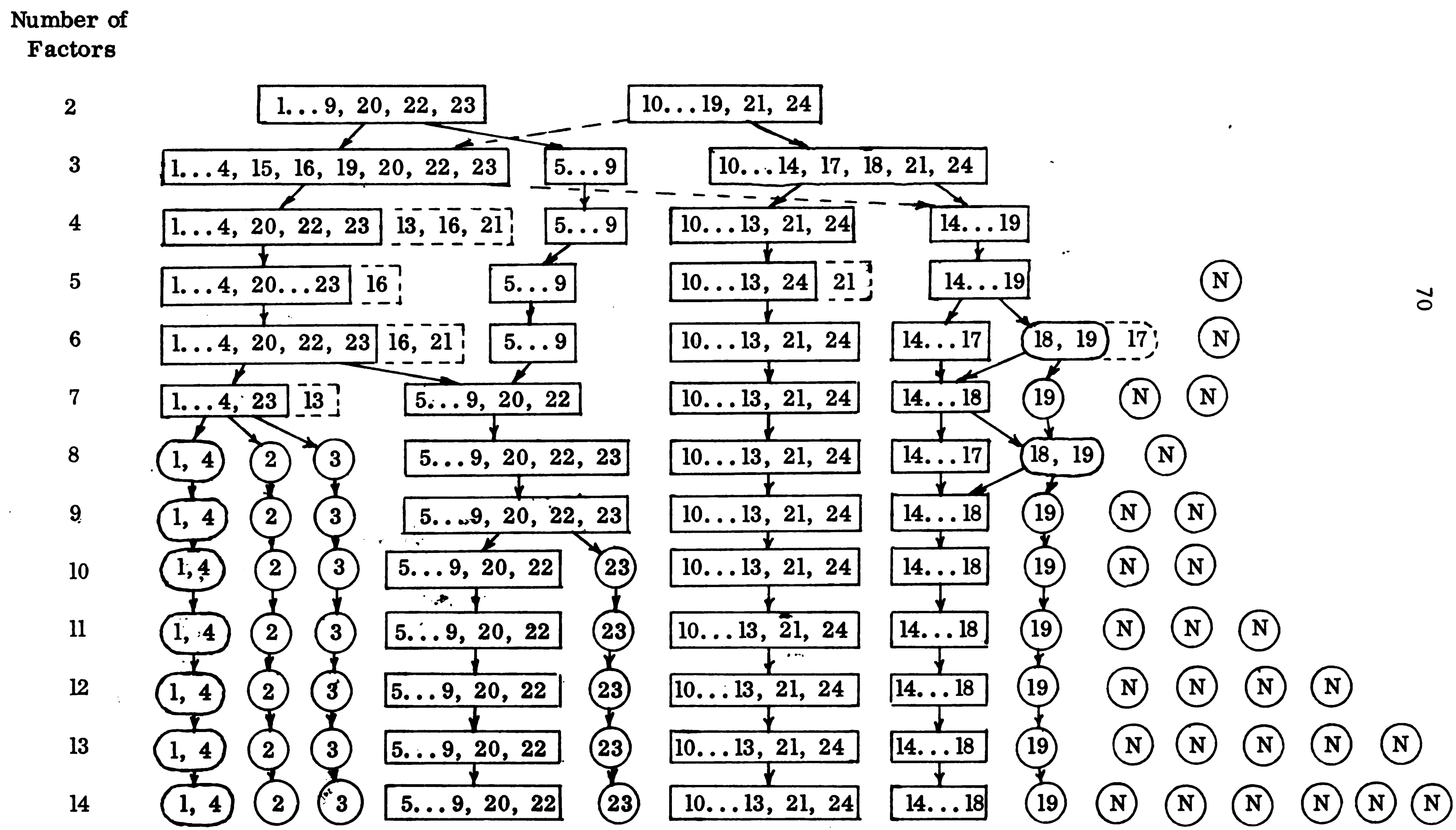


FIGURE 4.8

HIERARCHICAL STRUCTURE OF TWENTY-FOUR PSYCHOLOGICAL TESTS -- STARTING WITH SQUARED MULTIPLE CORRELATIONS



CHAPTER V

CONCLUSIONS

One of the major results of this study is the confirmation of previously published reports of factor splitting when different numbers of factors are rotated and the hierarchical structure which is therefore obtained.

In this study we have defined four types of factors: (1) group factors -- factors on which three or more of the variables in the analysis have their highest loadings; (2) binary factors -- factors on which only two of the variables have their highest loadings and which are therefore insufficiently determined in three or higher dimensional hyperspace; (3) unique or specific factors -- on which only one variable has its highest loading; and (4) null factors -- those factors on which none of the variables have their highest loadings.

The tendency for factors to emerge hierarchically as additional factors are rotated follows several different patterns. With few variables of high reliability such as those in the physical and political variable matrices, the splitting is "clean", with little or no tendency for variables to split away from one factor and recombine with others. The point at which no further splitting takes place is usually reached before all factors are rotated and usually the solution includes some null factors. In the two examples mentioned, however, the matrices consisted of variables especially selected to illustrate two-factor sets and from 80 to 84 percent of the total variance was accounted for by the first two factors.

With larger matrices consisting of sets of variables with lower reliabilities (which is usually the case in exploratory factor analysis) there are usually more than two factors inherent in the data. In these cases variables which have small loadings on all of the first few factors will be quite unstable until unrotated factors on which the variables have relatively high loadings are included in the rotational solution. Another phenomenon observed is that segments of two or more earlier factors will sometimes combine to form new factors. This effect is much more noticeable when unities, which include a large amount of unique variance, are used in the diagonals of the matrix than when smaller initial communality estimates such as squared multiple correlations are used.

The Effect of Different Communality Estimates on Factor Fissioning

With large diagonal entries such as unities in the correlation matrix, the factors will usually continue to split into many unique factors. This effect seems to vary inversely with the reliability of the variables, a rough measure of which is the discrepancy between unities and squared multiple correlations $(1 - R_j^2(n-1))$. When there is relatively little difference between unities and squared multiple correlations, as illustrated by most of the variables in the physical and political variable matrices, most of the variance for a particular variable tends to be concentrated in a single, early factor and the addition of later factors into the rotational solution adds little additional variance (in the form of high loadings) which seems to be the primary cause of factor fissioning.

When there is a great discrepancy between unities and squared multiple correlations for any of the variables, there will usually be at least two unrotated factors on which that variable has a high loading when unities are used in the leading diagonals, an early group factor and a later unique factor. The introduction of the later factors containing unique loadings into the rotational solutions results in unique rotated factors.

When squared multiple correlations are used in the leading diagonals as initial communality estimates, the rotation of additional factors results in few unique factors. Instead most of the later unrotated factors contain very small loadings and result only in the formation of null factors.

With Guttman communalities, which are numerically less than unities but greater than squared multiple correlations, the effect of increasing the number of factors in the rotational solution is a combination of the two above types of results. There are more unique factors formed than with squared multiple correlations but there are also a number of group factors which remain in the final solution (with all factors rotated) as well as some null factors.

The findings of this study indicate that <u>there is relatively little difference in the numerical values of the first few factors extracted by the principal axes method regardless of the initial communality estimates</u>. For example, a comparison of the first four unrotated factors obtained with unities and squared multiple correlations for the twenty-four psychological variables indicated that of the ninety-six pairs of loadings only three had a difference greater than 0.1, one of these was on the third factor and the other two on the fourth factor. The maximum discrepancy was 0.161 and the average difference over the ninety-six pairs was 0.035.

One of the best indications of the point at which unique variance begins to appear when unities are used, hence the point at which the factorial structure will begin to show differences with different initial communality estimates, is to compare the percentage of the total original communality accounted for by the eigenvalues of each of the factors with different communalities. The smaller the initial communality estimates the larger will be the percentage of the total accounted for by the earlier factors. This is illustrated graphically in Figures 5.1, 5.2, 5.3 and 5.4, the data for which is compiled from Tables 4.2, 4.5, 4.7, and 4.9. As long as the percentages for squared multiple correlations are higher than those for unities the factorial solutions are almost identical. The point at which the percentage for unities is greater than that for the lower communality estimates is the point at which the factors begin to include unique loadings with unities which are not present in the factors derived from the matrix with squared multiple correlations. This cross-over point is also the point at which the factorial solution begins to differ.

Probably the most significant result of this study is the confirmation that the factorial structure -- that is, the grouping of variables on the different factors -- is identical regardless of the initial communality estimates up to the cross-over point where binary or unique factors begin splitting off with unities or where null factors are formed when starting with squared multiple correlations. The controversy over the proper communality estimates which should be used and the insistence on the use of values other than unities now seems somewhat unnecessary. The "common" factors are those corresponding to the largest latent roots with the principal axes method. Any specific variance which is included due to the use of unities in the correlations matrix appears only in the "later" factors corresponding to the smaller latent roots and may be excluded by simply not including such factors in the rotational solution.

The comparison of percentages such as those in Figures 5.1 through 5.4 might well be used as a criterion for "when to stop rotating". This would involve doing two principal axes analyses, one with unities as initial communality estimates and another using squared multiple correlations. Those factors on which the eigenvalues obtained with squared multiple correlations accounted for a greater percentage of the total initial communalities than those with unities would be rotated. While this method would be feasible for very large, fast electronic computers on which it would be possible to save two different sets of factors, it would be inordinately expensive and time consuming for the average researcher. However, the results of this study do suggest an alternative criterion which is both useful and

FIGURE 5.1

EIGHT PHYSICAL VARIABLES --

Percent of Total Initial Communality Accounted for by Latent Roots of Four Largest Factors.

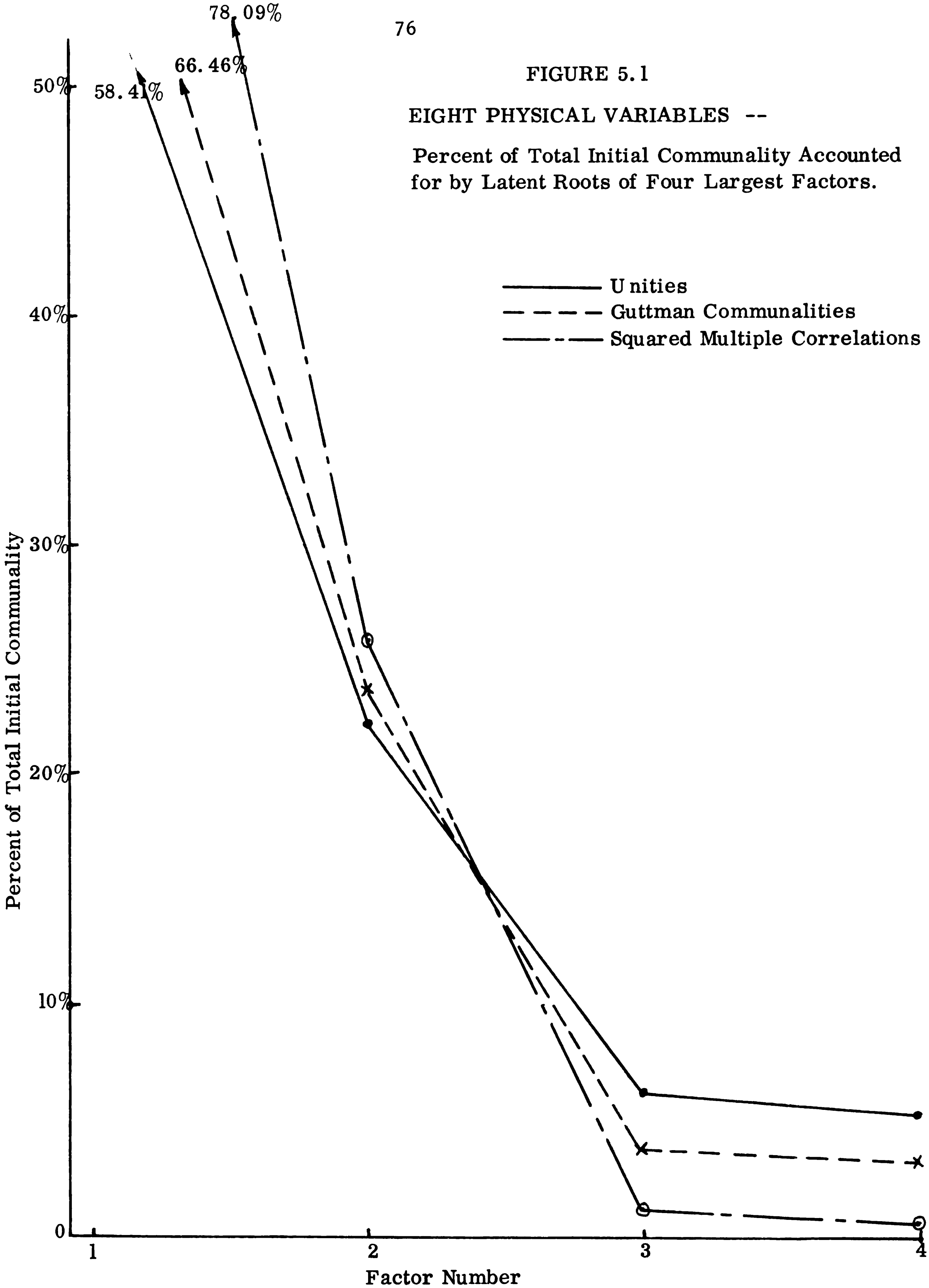


FIGURE 5.2

EIGHT POLITICAL VARIABLES —
Percent of Total Initial Communality
Accounted for by Latent Roots of Four
Largest Factors.

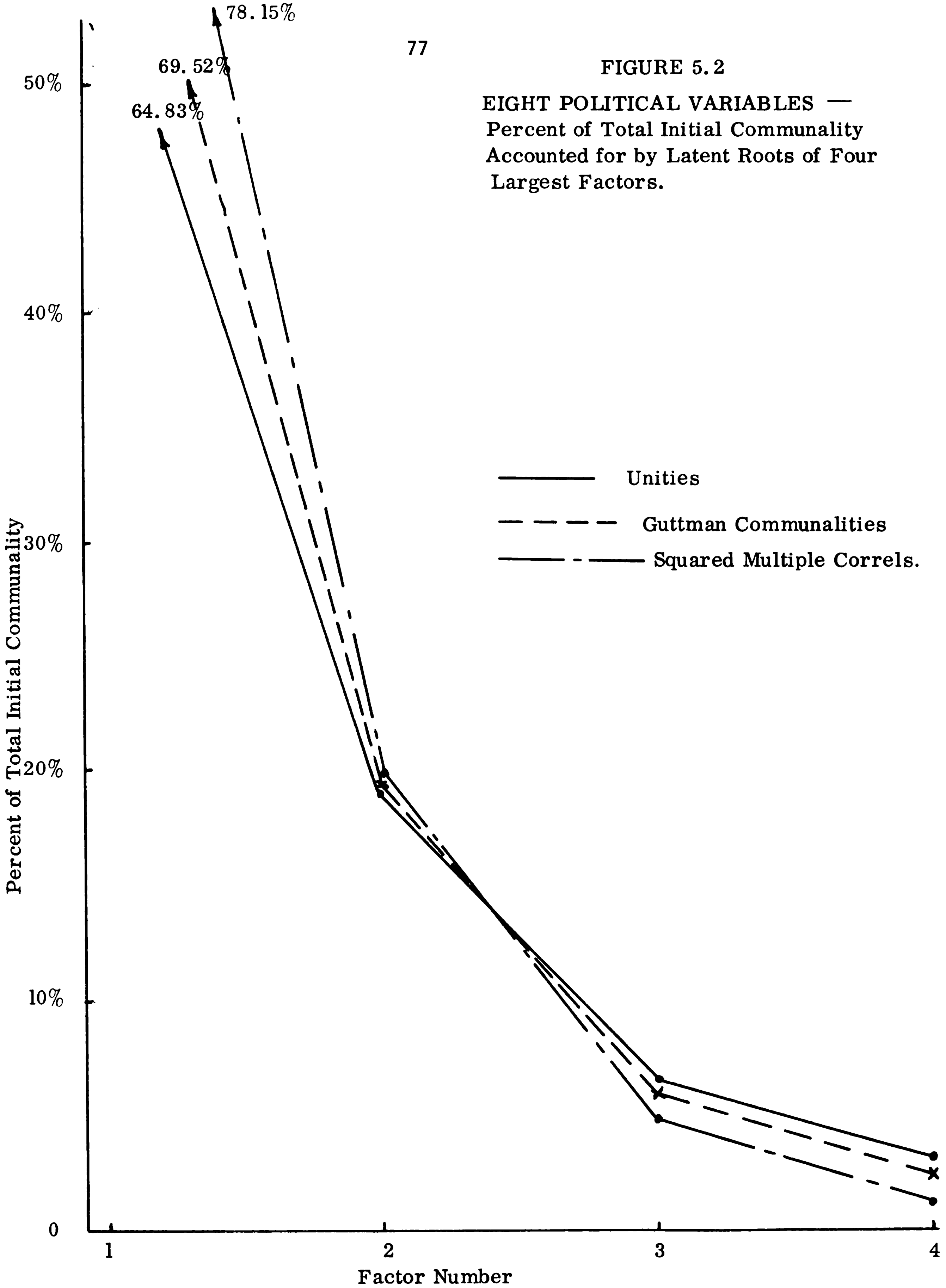


FIGURE 5.3

ELEVEN AIR-FORCE CLASSIFICATION TESTS --

Percent of Total Initial Communality Accounted for by Latent Roots of the Five Largest Factors.

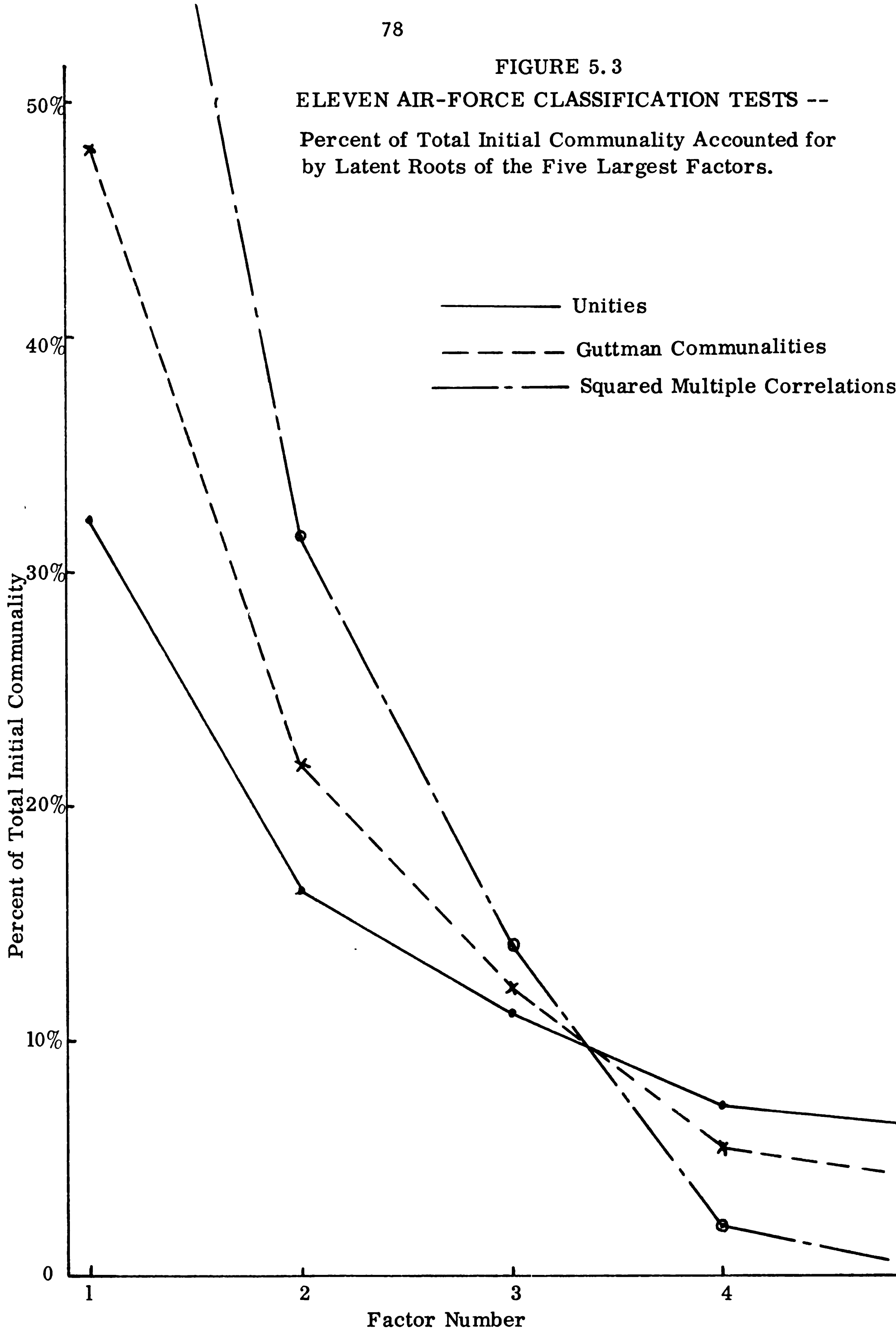
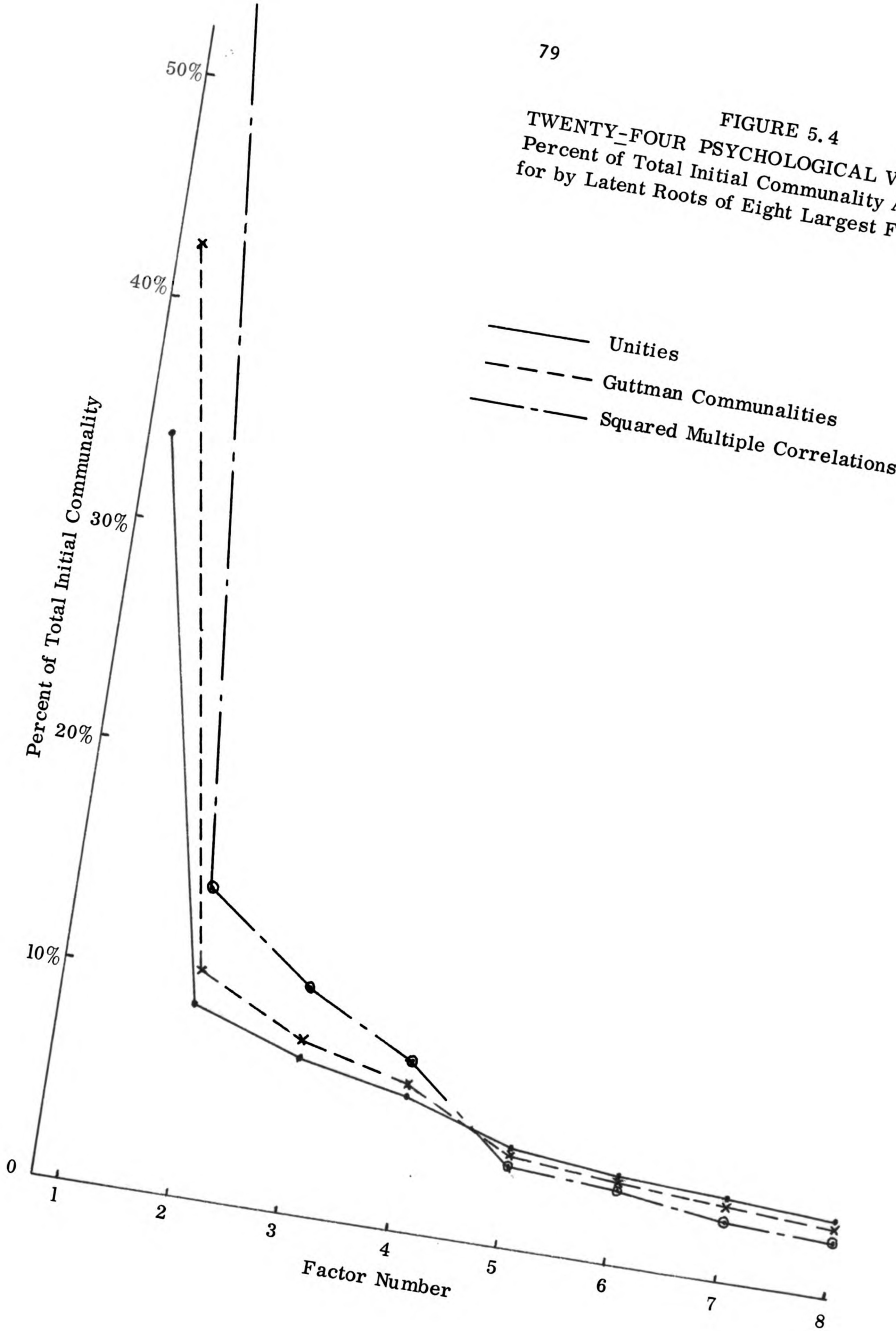


FIGURE 5.4
TWENTY-FOUR PSYCHOLOGICAL VARIABLES:
Percent of Total Initial Communality Accounted
for by Latent Roots of Eight Largest Factors.



practical. The results also suggest a guide for reporting the results of factor analyses.

In every case so far studied, the solution following the above cut-off point had at least one factor on which less than three of the variables had their largest loadings. The following method of factoring any set of variables is therefore recommended. It is completely objective, with no need for subjective decisions at any point during the calculations, and is practical and fast on a modern electronic digital computer.

1. Since there is little difference between unities and other communality estimates as far as the factorial structure is concerned, unities are recommended for the entries in the leading diagonals of the correlation matrix to be factored.
2. Carry out a complete principal axes analysis by the best eigenvalue-eigenvector method available.
3. Rank the principal axes factors from largest to smallest on the basis of their corresponding eigenvalues.
4. Rotate the factors by an acceptable analytical procedure (the varimax method is recommended) starting with the two largest factors, then the three largest factors, and so on, continuing to rotate with additional factors as long as each factor in the solution includes the highest loadings of at least three variables. In other words, terminate rotations when a solution contains a factor on which less than three variables have their largest loadings.

5. In reporting the results of an analysis, the complete hierarchical structure should be given as well as the final factorial solution.

For descriptive or exploratory factor analyses, particularly when little is known from other work about the factorial structure of the variables, it might be useful to rotate several additional factors beyond the cut-off point at which binary or unique factors begin to split off. The additional information obtained could suggest additional factors which might be significant if additional variables were to be included in the test battery.

The requirement that each interpretable factor should include the highest loadings for at least three variables is not simply an arbitrary choice. Other investigators have also suggested that each factor should be highly loaded on at least three variables (18, 17). It is based on the mathematical axiom that at least three points are rquired to define a plane in three dimensional space. (In N-dimensional space we would theoretically require at least N non-coplanar points to determine a hyperplane. An investigation is currently underway to determine the feasibility of using a variable test for cut-off rather than the constant three. In each of the four examples used in this study the cut-off point would be the same.)

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