

GEOMETRIC EVOLUTION OF SINGLE-LAYER INTERFACES IN THE  
FUNCTIONALIZED CAHN-HILLIARD EQUATION

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## ABSTRACT

# GEOMETRIC EVOLUTION OF SINGLE-LAYER INTERFACES IN THE FUNCTIONALIZED CAHN-HILLIARD EQUATION

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We study the Functionalized Cahn-Hilliard Energy (FCH), which is a higher-order reformulation of the Cahn-Hilliard energy, as a model for network formation in polymer-solvent mixtures. The model affords a finite interfacial width, accommodates merging and other topological reorganization, and couples naturally to momentum balance and other macroscopic mass transport equations.

The corresponding constrained  $L^2$  gradient flow has a rich family of approximately steady-state solutions that include not only the single-layer heteroclinic front profile seen in gradient flows of the Cahn-Hilliard energy, but also a novel one parameter family of homoclinic bi-layer solutions. In this thesis we rigorously derive the geometric evolution of the single-layer polymer-solvent interface.

We form a manifold of quasi-equilibria by “dressing” a large family of co-dimension one interfaces immersed in  $R^d$  with heteroclinic solutions of a one-dimensional equilibrium equation derived from the first variation of the FCH energy. We show that solutions of the gradient flow that start sufficiently close to the manifold remain close, and moreover the flow can be decomposed, at leading order, as a normal velocity for the underlying co-dimension one interface. Assuming the smoothness of the interface under this flow, we develop rigorous estimates on the proximity of the true solution to the manifold, in an appropriate norm, for long time.

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# Chapter 1

## Introduction

A central goal of polymer chemistry is to design materials with novel macroscopic properties by controlling the spontaneous generation of nanoscaled, phase separated networks [4]. A primary mechanism to generate such networks is through the *functionalization* of hydrophobic polymer chains and nanoparticles by the addition of acid or alkaline terminated side-chains. The end-groups interact exothermically with solvents, driving the generation of polymer-solvent or nanoparticle-solvent interface generating as much as  $1000\text{m}^2/\text{gram}$  of solvated interface, the largest density known. Modeling the generation of these polymer networks is a critical need in chemistry.

A key step is establishing an expression for the ‘free energy’ functional of the system which decreases along solutions of corresponding gradient flows. The Canham-Helfrich free energy is a broadly applicable template for sharp-interface energies of an interface  $\Gamma \subset \mathbb{R}^3$  written in

terms of its mean,  $H$ , and Gaussian,  $K$ , curvatures,

$$\mathcal{E}_{CH}(\Gamma) = \int_{\Gamma} \sigma + \kappa_b(H - H_{\text{int}})^2 + \kappa_s K dS. \quad (1.0.1)$$

The parameter,  $\sigma$ , denotes the energy density per unit surface area,  $\kappa_b$  and  $\kappa_s$ , denote the energy density attributed to extension and splay deformations, and  $H_{\text{int}}$  denotes the intrinsic, or zero-energy, value of the mean curvature. In the context of functionalized polymers, these parameters incorporate the role of solvation, electrostatic energy, and polymer backbone stiffness into the energy associated with an interface between the solvent and polymer backbone regions. However, the Canham-Helfrich energy is a sharp interface energy and as such is not useful in describing the merging and pinch-off events that play a crucial role in polymer network formation. We avoid these difficulties by regularizing the Canham-Helfrich energy into a finite-width phase field model. To this end the ‘functionalized’ Cahn-Hilliard Energy (FCH) was introduced in [18]

$$\mathcal{E}_F[u] = \int_{\Omega} \frac{1}{2} \left( \frac{\delta \mathcal{E}}{\delta u} \right)^2 - \eta \left( \frac{\varepsilon}{2} |\nabla u|^2 + \varepsilon^{-1} W(u) \right). \quad (1.0.2)$$

Here

$$\mathcal{E}[u] = \int_{\Omega} \frac{\varepsilon}{2} |\nabla u|^2 + \varepsilon^{-1} W(u) \quad (1.0.3)$$

is the well studied Cahn-Hilliard energy,  $\varepsilon \ll 1$  is a small positive parameter,  $\eta > 0$  represents the strength of functionalization, and  $u$  is a phase function that differentiates between the solvent and polymer backbone regions (two minima of a symmetric double-well potential  $W$ ).

Without loss of generality, we assume that the minima occur at  $u = \pm 1$  and

$$\mu_0 := W''(\pm 1) > 0. \quad (1.0.4)$$

We have shown that there exist families of functions,  $\{u_\varepsilon(x)\}_{\varepsilon>0}$ , for which in the  $\varepsilon \rightarrow 0$  limit the energy  $\mathcal{E}_F$  recovers a Canham-Helfrich energy of the form (1.0.1) [18]. However, unlike for the Cahn-Hilliard energy, several distinct forms of the limiting energy are possible. A conjecture of De Giorgi, concerning the  $\Gamma$ -convergence of (1.0.2) in the case  $\eta = -\varepsilon^2$ , when both terms are positive, has recently been established, [31]; however the nature of the energy and the dynamics of the associated gradient flows are fundamentally different in the case  $\eta > 0$ .

The Functionalized Cahn-Hilliard energy, (1.0.2), represents not a perturbation, but a substantial highly-nonlinear reformulation of the Cahn-Hilliard energy. The Cahn-Hilliard equation, the  $H^{-1}$  gradient flow on the Cahn-Hilliard Energy, drives the system to minimize interface area while preserving the volume fraction. As such it is a model for coarsening processes of binary alloys, see [7], [9], [10]. Network formation in functionalized polymers is a fundamentally different process, and the FCH energy (1.0.2) takes this into consideration by balancing the solvation energy released by formation of water-acid interface *against* the elastic energy required to bend the interface, and the associated polymer backbones. This ‘functionalization’ of the Cahn-Hilliard functional (1.0.3) through sign inversion and regularization via the square of the variational derivative fundamentally changes the nature of the energy landscape.

The simplest mass-preserving gradient flow associated with the FCH energy is the non-local

$L^2$  flow on  $\Omega \subset \mathbb{R}^d$ ,

$$\partial_t u = -\Pi_0 F(u), \quad (1.0.5)$$

where

$$F(u) = (\varepsilon^2 \Delta - W''(u) + \varepsilon^2 \eta)(\varepsilon^2 \Delta u - W'(u)) \quad (1.0.6)$$

and the zero-mass projection  $\Pi_0$  subtracts the average value

$$\Pi_0 f = f - \frac{1}{|\Omega|} \int_{\Omega} f dx. \quad (1.0.7)$$

The evolution presented in Figure 1.1 shows two distinct stages. The initial stage corresponds to the single-layer front solutions that separate the two distinct phases. These class of quasi-steady solutions have analogues in the Cahn-Hilliard equation. In the second stage the single-layer fronts come close to merging, but are arrested by the formation of a thin bi-layer, which subsequently grows and dominates the flow, until it arrives at the quasi-steady equilibrium depicted in the final frame. Understanding of the generation and evolution of these interfaces is integral to predicting the spontaneous generation of polymer solvent networks.

Equation (1.0.5) has a rich family of approximately steady state solutions that include not only the single-layer heteroclinic front profile seen in gradient flows of the Cahn-Hilliard energy, but also a novel one parameter family of homoclinic bi-layer solutions [18]. The single-layer quasi-steady states  $\Phi(x; \Gamma, b)$  are formed by ‘dressing’ an admissible interface  $\Gamma$  (see Section 2.4) and parameterized by a background state  $b \in \mathbb{R}$  (for bi-layers there is an additional parameter  $\alpha$  which represents the width of the homoclinic profile). The class of ‘admissible

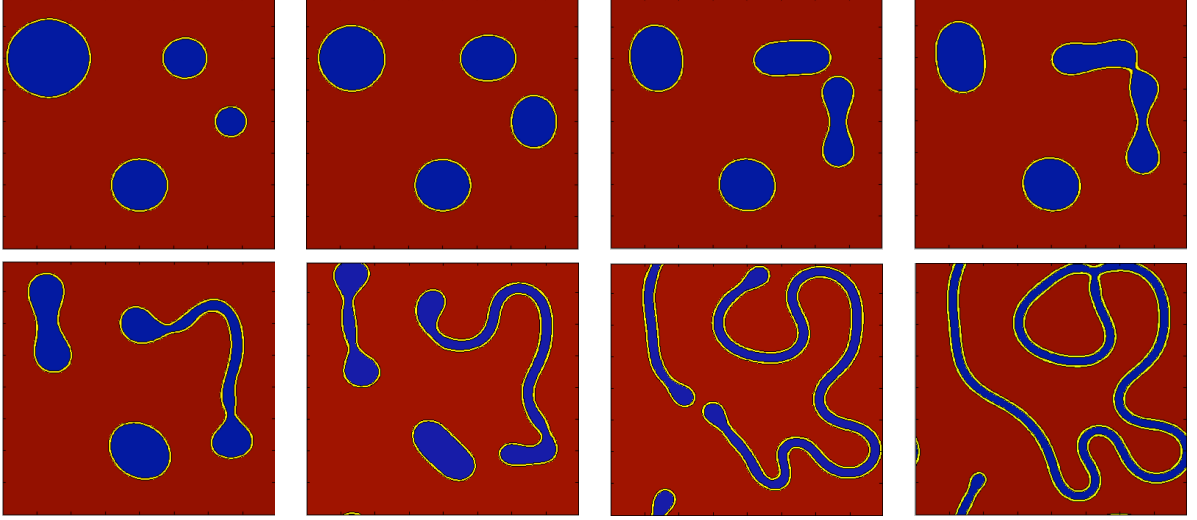


Figure 1.1: The spontaneous network formation in  $\Omega \subset \mathbb{R}^2$  which is typical of (1.0.5). Reading left to right, the initial data is four circles of “water” ( $u = 1$ ) represented by the blue color within a background of “polymer” ( $u = -1$ ) in red. The boundary between the two domains is given by a front or single-layer solution which it shares with the Cahn-Hilliard equation. In the initial stage the higher curvature circles grow at the expense of the lower curvature ones, however, as can be seen in the second frame, the circular domains are unstable to an antipodal elongation. The elongated circle assumes a dumb-bell shape which stretches and merges with the adjacent circles in frames 3 and 4, forming a narrow bi-layer capped by two semi-circular endcaps. In frames 4-8 the bi-layer interface narrows and meanders as it is driven to its optimal width, which is stationary on the  $O(\varepsilon^{-3})$  time scale. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this thesis.

interface'  $\mathcal{G}_K$  corresponds to hypersurfaces in  $\mathbb{R}^d$  which are far from self-intersection and have curvatures bounded in  $L^\infty(\Gamma)$  by a prescribed constant  $K$ . In particular, we assume that the interfaces are free of junctions and provide conditions which insure that no merging or self-intersections take place.

In the present work we are concerned with co-dimension one interfaces embedded in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  dressed with a single-layer ansatz. The construction of the single-layer ansatz begins with the choice of an equilibrium of an associated 2nd-order, one-dimensional differential equation. Via a formal expansion we determine higher-order correction terms that adapt the solution to the curvature of the interface and the desired background (far-field) value. The resulting function  $\Phi = \Phi(x; \Gamma, b)$  has a small residual, indeed  $F(\Phi) = O(\varepsilon^3)$  in  $H^4(\Omega)$ . However, the residual cannot be further reduced since solvability conditions fail at the next order in the formal expansion. The precise construction of  $\Phi$  exploits the local normal-tangential coordinate system about  $\Gamma$ , which we call the 'whiskered coordinates' as presented in Chapter 2. The resulting manifold

$$\mathcal{M}_K := \{\Phi(x; \Gamma, b) : \Gamma \in \mathcal{G}_K, b \in (-\bar{b}\varepsilon^3, \bar{b}\varepsilon^3)\}, \quad (1.0.8)$$

is not invariant, however by utilizing the methods from [30] we show that a solution that starts in a sufficiently small neighborhood of  $\mathcal{M}_K$  may be decomposed as  $u(x, t) = \Phi(x; \Gamma_t; b(t)) + w(x, t)$  where the time-dependent interface  $\Gamma_t$  and the background state  $b(t)$  shadow the slow evolution of  $u$  up to a small remainder term  $w$ . In particular, we will show in our main result, Theorem 3.7, that assuming the continued smoothness of the interface, there exists a time  $T_b > 0$ , which depends only upon  $K$ , such that if the evolution of the local parametrization

$\gamma(s, t)$  of the interface  $\Gamma_t$  is described by

$$\partial_t \gamma(s, t) = \mathbf{V}_n(s, t) \nu(s, t), \quad (1.0.9)$$

$$\gamma(s, 0) = \gamma_0(s), \quad (1.0.10)$$

for an appropriate choice of normal velocity  $\mathbf{V}_n$  then the remainder  $w$  will remain small, in an appropriate norm, for all  $t \in [0, T_b]$ . Indeed, we show that the appropriate choice of the normal velocity is given, to leading order, by the expression

$$\mathbf{V}_n = -\Pi_{0,\Gamma} \left( (\Delta_s + \eta)H - \frac{H^3}{2} + H \operatorname{tr}(A^2) \right), \quad (1.0.11)$$

where  $H$  is the mean curvature,  $A$  is the Weingarten map whose eigenvalues are the curvatures of  $\Gamma$ , and  $\Pi_{0,\Gamma}$  is the zero-mass projection associated to the surface integral over  $\Gamma$ . We also express the normal velocity as a higher-order Ricci Flow on the fundamental forms associated to  $\Gamma$ . The reduction of the FCH to a curvature driven flow requires a detailed analysis of the linearization  $\mathcal{L}_\phi$  of  $F$  about each  $\Phi \in \mathcal{M}_K$ . We show that  $\mathcal{L}_\phi$  has an  $M$ -dimensional slow eigenspace, corresponding to eigenvalues that are asymptotically small, and a remainder that is uniformly bounded in the left-half complex plane. Moreover the slow-eigenspace, spanned by  $\Psi_1, \dots, \Psi_M$ , can be well approximated by a convenient basis  $\mathcal{Z}_M = \{Z_i\}_{i=1}^M$ . The decomposition  $\mathcal{Z}_M \oplus \mathcal{Z}_M^\perp$  breaks  $L^2(\Omega)$  into two approximately- $\mathcal{L}_\phi$  invariant subspaces. Moreover, the dimension  $M$  of  $\mathcal{Z}_M$  may be chosen so that the bilinear form which we associated with  $\mathcal{L}_\phi$  is uniformly coercive on  $\mathcal{Z}_M^\perp$ .

The full program outlined above has not been achieved in this thesis. A key gap in the

proof is our inability to produce closed estimates on the smoothness of the interface as it evolves under the normal flow (1.0.9). There are several obstacles to this effort, not the least of which is the complexity of the associated Ricci-curvature equations. We outline the difficulties in the paragraphs below.

The first difficulty is a loss of one derivative of spatial regularity when decomposing the initial data. More specifically, any initial data  $u_0$  which is close to the manifold  $\mathcal{M}_K$  can be (trivially) written as

$$u_0 = \Phi(x; \Gamma_*, b_*) + w_*(x), \quad (1.0.12)$$

for some  $\Gamma_* \in \mathcal{G}_K$  with  $w_*$  small in an appropriate norm. To obtain decay of  $w$ , we look for an orthogonal decomposition of the form

$$u_0 = \Phi(x; \Gamma_0, b) + w_0(x), \quad (1.0.13)$$

with the additional condition that  $w_0$  lie in the fast space, that is  $w_0 \in \mathcal{Z}_M^\perp(\Gamma_0)$ . We search for  $\Gamma_0$  among interfaces  $\Gamma_p$  near  $\Gamma_*$  in the form

$$\gamma p(s) = \gamma_*(s) + \nu_*(s)R(s), \quad (1.0.14)$$

where  $\nu_*(s)$  is the normal to  $\Gamma_*$  at  $\gamma_*(s)$ . We take candidates for  $R$  as among the Galerkin sums on the first  $M$  Laplace-Beltrami eigenmodes,  $\{\Theta_i\}_{i=1}^M$  of  $\Gamma_*$ ,

$$R = \sum_{i=1}^M p_i \Theta_i(s), \quad (1.0.15)$$

and determine the parameters  $p = (p_1, \dots, p_M)$  to impose the orthogonality condition

$$w_0 := u_0 - \Phi_0 \in \mathcal{Z}_M^\perp(\Gamma_0). \quad (1.0.16)$$

However it is implicit in the construction (1.0.14) that  $\gamma_p$  loses one derivative compared to  $\gamma_*$ , since the normal  $\nu_*$  is one derivative less smooth than the  $\gamma_*$ .

A second, principle difficulty is to understand the smoothness of the interfaces under the flow (1.0.9). The regularity of the interface is best understood by recasting it as a Ricci-curvature driven flow on the first and second fundamental forms of  $\Gamma$ . The analysis of the resulting PDEs is indeed nontrivial, even for  $d = 2$ , when the fundamental forms are scalar, the Ricci flow reduces, at leading order, to a generalized Kuramoto-Sivashinsky equation, see (3.3.17), whose regularity is (just) outside the known results. However the situation is complicated by the fact that we are only able to obtain tight bounds on the higher-order corrections to the normal velocity, which enter into the Ricci-flow, in relatively weak norms, such as  $L^2$  and  $L^\infty$ . Moreover, while the leading order terms in the normal velocity are independent of the remainder,  $w$ , the correction terms do depend upon  $w$ , thus the smoothness of the interface couples to the norm bounds obtainable on  $w$ . Indeed, we would like the correction terms to be asymptotically small in  $H^6(\mathbf{S})$ , where  $\mathbf{S} \subset \mathbf{R}^{d-1}$  is the reference set for the interface parameterization  $\gamma$ . In that case the perturbations to the evolution of the second fundamental form would be small in  $H^4$ , and hence the corresponding parameterization  $\gamma$  would reside in  $H^6(\mathbf{S})$ .

There are two other obstacles to showing regularity in our orthogonal decomposition of the solution  $u$ . The first is that the interface  $\Gamma_t$ , which we characterize via its second fundamental

form, could generate a self-intersection in finite time, at which point our decomposition would break down. However, with a strong bound on the  $L^\infty$  norm of the normal velocity it is straightforward to obtain a sharp estimate on a possible time to self-intersection. The second mechanism is more technical, our estimates require that the gap between the  $M$ 'th and  $M+1$ 'st Laplace-Beltrami eigenvalues for the interface  $\Gamma_t$  be  $O(1)$ , where  $M \approx \varepsilon^{-2}$  is the dimension of the slow eigenspace. This generically holds, according to Weyl's pioneering estimates, however for certain degenerate interfaces the gaps between particular eigenvalues may close. We can prevent this by estimating the rate of motion of the eigenvalues in terms of the normal velocity. We find that lower bounds on the time to a "spectral-collapse" scale like  $t = O(\varepsilon^{-1})$ . However we may avoid the spectral collapse at a time  $t = t_s$  by decomposing the solution  $u(t_s)$  as in (1.0.13) and adjusting the slow dimension  $M$  to avoid proximate eigenvalues. However this too requires smoothness of the interface  $\Gamma_t$ .

Thus the central theorem of this thesis, Theorem 3.7, is stated under the assumption of smoothness of the underlying interface  $\Gamma_t$  for a range of  $t \in [0, T_f \varepsilon^{-4}]$  for some  $T_f > 0$ .

# Chapter 2

## Geometry of Interfaces

This chapter lays out the necessary framework on the differential geometry of co-dimension one interfaces in  $\mathbb{R}^d$  that is required to state the main results. The cognizant reader may skip directly to chapter 3 to find the main result, Theorem 3.7.

The main goal of our work is to describe the geometric evolution of the functionalized polymer-solvent interface. Mathematically, we take such an interface to be represented by a smooth, closed (compact and without boundary), oriented  $d - 1$  dimensional manifold  $\Gamma$  embedded in  $\mathbb{R}^d$ . We denote the family of such manifolds by  $\mathcal{G}$ . In addition, we consider only the interfaces that are far from self-intersection, and whose curvatures are  $O(1)$  with respect to  $\varepsilon$ . We defer the precise definition of admissible interfaces to Chapter 3. We will show that a large class of solutions  $u$  of the FCH gradient flow

$$\partial_t u = -\Pi_0 F(u) = -\Pi_0(\varepsilon^2 \Delta - W''(u) + \varepsilon^2 \eta_2)(\varepsilon^2 \Delta u - W'(u)). \quad (2.0.1)$$

have an asymptotic limit captured by a higher-order Ricci curvature flow on the fundamental

forms associated to  $\Gamma$ . More precisely, equation (2.0.1) has a rich family of quasi-steady states represented by  $\Gamma \in \mathcal{G}$  *dressed* in the normal direction with either single or bi-layer profiles. In the present work we study the evolution of initial datum in a small enough neighborhood of a *single-layer* manifold  $\mathcal{M}_K$ . In particular, for the single-layer solutions the interface is represented by the  $u = 0$  level sets. In order to make this construction precise we need several concepts from differential geometry which we review in the next section.

## 2.1 Weingarten map and fundamental forms

In this section we summarize some key definitions from Differential Geometry (for more details see [25]).

Let  $\Gamma$  be a  $d - 1$  dimensional smooth manifold embedded in  $\mathbb{R}^d$  with a chosen orientation. Let  $\mathbf{S}$  be an open set in  $\mathbb{R}^{d-1}$  and  $\gamma : \mathbf{S} \rightarrow \Gamma$  be a local parametrization. Then the tangent space  $T_{\gamma(s)}\Gamma$  is defined as the image of  $T_s\mathbf{S} \cong \mathbb{R}^{d-1}$  under the map  $D\gamma|_s$ . Denote by  $\mathbf{S}^{d-1}$  a sphere  $\|x\| = 1$  in  $\mathbb{R}^d$ .

**Definition 2.1.** *The Gauss map,  $\nu : \mathbf{S} \rightarrow \mathbf{S}^{d-1}$  maps points of  $\mathbf{S}$  into normal unit vectors  $\nu(s)$  orthogonal to  $T_{\gamma(s)}\Gamma$ .*

**Definition 2.2.** *The Weingarten map,  $A : T_{\gamma(s)}\Gamma \rightarrow T_{\gamma(s)}\Gamma$  is defined by*

$$A = -D\nu \circ (D\gamma)^{-1}. \quad (2.1.1)$$

**Remark 2.3.** *Letting  $e_i$  be the standard basis element in  $\mathbb{R}^{d-1}$ , we note that  $D\nu : T_s\mathbf{S} \cong \mathbb{R}^{d-1} \rightarrow T_{\gamma(s)}\Gamma$  follows from the fact that  $\frac{\partial \nu}{\partial s_i} = (D\nu)e_i$  belongs to the tangent space  $T_{\gamma(s)}\Gamma$ .*

This can be seen from differentiating  $\nu(s) \cdot \nu(s) = 1$  with respect to  $s_i$ .

**Definition 2.4.** For  $X, Y \in T_{\gamma(s)}\Gamma$  the  $k$ -th fundamental form of  $\Gamma$  is defined by

$$\langle A^{k-1}X, Y \rangle, \quad (2.1.2)$$

where the brackets represent the euclidean inner product in  $\mathbb{R}^d$ .

**Definition 2.5.** The principal curvatures  $k_1, \dots, k_{d-1}$  of  $\Gamma$  are defined as the eigenvalues of the Weingarten map  $A$ .

**Weingarten Map and Fundamental Forms in Local Coordinates.** From definitions of  $D\gamma$  and  $D\nu$  we have

$$(D\gamma)e_i = \frac{\partial \gamma}{\partial s_i}, \quad (2.1.3)$$

and

$$(D\nu)e_i = \frac{\partial \nu}{\partial s_i}, \quad (2.1.4)$$

where the vector  $e_i$  contains 1 in the  $i$ -th position. It follows from the definition of  $\Gamma$  that  $D\gamma$  is full-rank and so the vectors  $\left\{ \frac{\partial \gamma}{\partial s_1}, \dots, \frac{\partial \gamma}{\partial s_{d-1}} \right\}$  form a basis for the tangent space  $T_{\gamma(s)}\Gamma$ . Assume that  $X, Y \in T_{\gamma(s)}\Gamma$  are given by

$$X = \sum_{i=1}^{d-1} \xi^i \frac{\partial \gamma}{\partial s_i}, \quad (2.1.5)$$

$$Y = \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j}. \quad (2.1.6)$$

Then,

$$\langle X, Y \rangle = \left\langle \sum_{i=1}^{d-1} \xi^i \frac{\partial \gamma}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j g_{ij}, \quad (2.1.7)$$

where

$$g_{ij} = \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle, \quad (2.1.8)$$

is the representation of the first fundamental form in local coordinates. In addition, by definition, the Weingarten map  $A$  maps

$$\frac{\partial \gamma}{\partial s_i} \rightarrow -\frac{\partial \nu}{\partial s_i}, \quad (2.1.9)$$

and we have

$$\langle AX, Y \rangle = \left\langle \sum_{i=1}^{d-1} \xi^i A \frac{\partial \gamma}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = - \left\langle \sum_{i=1}^{d-1} \xi^i \frac{\partial \nu}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j h_{ij}, \quad (2.1.10)$$

where

$$h_{ij} = - \left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle = \left\langle \nu, \frac{\partial^2 \gamma}{\partial s_i \partial s_j} \right\rangle, \quad (2.1.11)$$

is the representation of the second fundamental form in local coordinates. The last equality results from differentiating

$$\left\langle \nu, \frac{\partial \gamma}{\partial s_i} \right\rangle = 0 \quad (2.1.12)$$

with respect to  $s_j$ . Similarly,

$$\langle A^2 X, Y \rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j e_{ij}, \quad (2.1.13)$$

where

$$e_{ij} = \left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \nu}{\partial s_j} \right\rangle = - \left\langle \nu, \frac{\partial^2 \nu}{\partial s_i \partial s_j} \right\rangle, \quad (2.1.14)$$

is the representation of the third fundamental form.

In addition if we write  $\frac{\partial \nu}{\partial s_i}$  in the tangent basis as

$$\frac{\partial \nu}{\partial s_i} = - \sum_{j=1}^{d-1} \frac{\partial \gamma}{\partial s_j} h_i^j, \quad (2.1.15)$$

then referring to (2.1.11) we see that

$$h_{ij} = \sum_{j=1}^{d-1} \left\langle \frac{\partial \gamma}{\partial s_j} h_i^j, \frac{\partial \gamma}{\partial s_j} \right\rangle = \sum_{j=1}^{d-1} h_i^j g_{ij}, \quad (2.1.16)$$

and

$$h_i^j = \sum_k h_{ik} g^{kj}, \quad (2.1.17)$$

where  $g^{kj}$  represents the elements of the inverse matrix of  $g_{ij}$ . The key relation (2.1.15) shows that  $h_i^j$  are the coefficients that express the linear variation of the normal vector in terms of

the tangent basis. Moreover, since

$$AX = \sum_{i=1}^{d-1} \xi^i A \frac{\partial \gamma}{\partial s_i} = - \sum_{i=1}^{d-1} \xi^i \frac{\partial \nu}{\partial s_i} = \sum_{i=1}^{d-1} \xi^i h_i^j \frac{\partial \gamma}{\partial s_i}, \quad (2.1.18)$$

$\{h_i^j\}$  is the matrix representation of the linear operator  $A$  in the basis  $\left\{\frac{\partial \gamma}{\partial s_1}, \dots, \frac{\partial \gamma}{\partial s_{d-1}}\right\}$  and from definition (2.1.2),  $k_i$  is the  $i$ -th eigenvalue of the matrix  $\{h_i^j\}$ . The matrix operator norm of  $h_i^j$  is then

$$||\{h_i^j\}|| = \max_{1 \leq i \leq d-1} k_i. \quad (2.1.19)$$

Finally, defining

$$e_i^j := \sum_k e_{ik} g^{kj}, \quad (2.1.20)$$

we use (2.1.15) to express  $e_i^j$  in terms of the Weingarten matrix  $h_i^j$

$$\begin{aligned} e_i^j &= \sum_k e_{ik} g^{kj} = \sum_k \left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \nu}{\partial s_k} \right\rangle g^{kj} = \sum_{k,l,m} h_i^l h_k^m \left\langle \frac{\partial \gamma}{\partial s_l}, \frac{\partial \gamma}{\partial s_m} \right\rangle g^{kj} \\ &= \sum_{k,l,m} h_i^l h_k^m g_{lm} g^{kj} = \sum_{k,l} h_i^l h_{kl} g^{kj} = \sum_l h_i^l h_l^j. \end{aligned} \quad (2.1.21)$$

Since the principal curvatures  $k_i$  are the eigenvalues of  $h_i^j$  it implies the following estimate for the operator norm of the matrix  $\{e_i^j\}$

$$||\{e_i^j\}|| = \max_{1 \leq i \leq d-1} k_i^2. \quad (2.1.22)$$

## 2.2 Whiskered Coordinates

For  $\Gamma \in \mathcal{G}$  we introduce the whiskered coordinates in a neighborhood of  $\Gamma$ . Although these coordinates have been introduced before in the context of geometric evolution, see [12] for example, their application to the fourth-order problems hinges upon higher order terms which we make precise in this section.

For  $x \in \mathbb{R}^d$  belonging to a neighborhood of some point  $\gamma(s) \in \Gamma$ , we introduce the following change of variables  $\varphi : (s, z) \rightarrow x$ , defined by

$$\varphi(s, z) = \gamma(s) + \varepsilon z \nu(s), \quad (2.2.1)$$

where the *Gauss map*,  $\nu : \Gamma \subset \mathbb{R}^d \rightarrow S^{d-1}$ , gives the outward pointing unit normal vector perpendicular to the tangent plane to  $\Gamma$  at  $\gamma(s)$ . This change of variables is a diffeomorphism in the neighborhood

$$\Gamma(\ell) := \{\varphi(s, z) | s \in \mathbf{S}, -\ell/\varepsilon \leq z \leq \ell/\varepsilon\}, \quad (2.2.2)$$

of  $\Gamma$ , so long as  $\ell$  is sufficiently small, as measured against the curvatures of  $\Gamma$ , but independent of  $\varepsilon$ . We will call the line segments  $\{\gamma(s) \times [-\ell/\varepsilon, \ell/\varepsilon] | \gamma(s) \in \Gamma\}$  the *whiskers* of  $\Gamma$ , and refer to  $(s, z)$  as the whiskered coordinate system.

If we introduce the variables  $y = (s_1, \dots, s_{d-1}, z)$  then  $x = \varphi(y)$  and  $\varphi^{-1}$  defines a chart for  $\Gamma(\ell)$ . The gradient and the Laplace operator in  $\mathbb{R}^d$  can be written in the  $y$ -coordinates as

$$\nabla_x^i = \sum_{j=1}^d G^{ij} \frac{\partial}{\partial y_j}, \quad i = 1, \dots, d \quad (2.2.3)$$

and

$$\Delta x = \frac{1}{\sqrt{\det(\mathbf{G})}} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial y_i} G^{ij} \sqrt{\det(\mathbf{G})} \frac{\partial}{\partial y_j}, \quad (2.2.4)$$

where  $\mathbf{G}$  is the metric tensor

$$G_{ij} = \left\langle \frac{\partial x}{\partial y_i}, \frac{\partial x}{\partial y_j} \right\rangle_{\mathbb{R}^d}, \quad (2.2.5)$$

and  $G^{ij}$  is the  $ij$  component of the inverse of the  $\mathbf{G}$ . Letting  $\mathbf{J}$  denote the Jacobian matrix for  $\varphi = \varphi(s, z)$ , we have,

$$\mathbf{G} = \mathbf{J}^T \mathbf{J}, \quad (2.2.6)$$

and consequently,  $\det(\mathbf{G}) = J^2$ , where  $J = J(s, z)$  is the associated Jacobian. The relation between  $\mathbf{G}$  and  $\mathbf{g}$  is clarified by the following Lemma.

**Lemma 2.6.**

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_0 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \quad (2.2.7)$$

and

$$\mathbf{G}_0(s, 0) = \mathbf{g}(s), \quad (2.2.8)$$

where  $\mathbf{g} = \{g_{ij}\}$  is the first fundamental form (metric tensor) on  $\Gamma$ .

**Proof:** The expression (2.2.7) comes from differentiating (2.2.1) with respect to  $z$  and  $s_i$

$$\frac{\partial \varphi}{\partial z} = \varepsilon \nu, \quad (2.2.9)$$

$$\frac{\partial \varphi}{\partial s_i} = \frac{\partial \gamma}{\partial s_i} + \varepsilon z \frac{\partial \nu}{\partial s_i} \in T_{\gamma(s)} \Gamma, \quad (2.2.10)$$

(see Remark 2.3). To obtain (2.2.8) we observe that for  $i, j = 1, \dots, d-1$ ,

$$(G_0)_{ij}|_{z=0} = G_{ij}|_{z=0} = \left\langle \frac{\partial \varphi}{\partial s_i}, \frac{\partial \varphi}{\partial s_j} \right\rangle \Big|_{z=0} = \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle = g_{ij}. \quad (2.2.11)$$

□

In particular  $\det \mathbf{g} = \varepsilon^2 \det \mathbf{G}_0$ . With this  $\mathbf{G}$ , in the whiskered variables the gradient and the Laplacian take the form,

$$\nabla_x^i = \sum_{j=1}^{d-1} G^{ij} \frac{\partial}{\partial s_j} \text{ for } i = 1, \dots, d-1, \quad \nabla_x^d = \varepsilon^{-2} \frac{\partial}{\partial z}, \quad (2.2.12)$$

and

$$\Delta_x = \varepsilon^{-2} J^{-1} \frac{\partial}{\partial z} J \frac{\partial}{\partial z} + \Delta_G, \quad (2.2.13)$$

where  $\Delta_G$  is

$$\Delta_G = \frac{1}{\sqrt{\det(\mathbf{G})}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} G^{ij} \sqrt{\det(\mathbf{G})} \frac{\partial}{\partial s_j} = J^{-1} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} G^{ij} J \frac{\partial}{\partial s_j}. \quad (2.2.14)$$

When  $z = 0$ ,  $\Delta_G$  is simply the Laplace-Beltrami operator on  $\Gamma$ , which we will denote by  $\Delta_s$ . The precise relation between  $\Delta_G$  and  $\Delta_s$  will be given in Chapter 3 we define the class of admissible interfaces.

To simplify the  $z$  derivatives in (2.2.13) we derive an expression for  $J$  in terms of the

principal curvatures  $\{k_i\}_{i=1}^{d-1}$  of  $\Gamma$ . We first observe that identity (2.1.15) implies,

$$\frac{\partial \varphi}{\partial s_i} = \frac{\partial \gamma}{\partial s_i} + \varepsilon z \frac{\partial \nu}{\partial s_i} = \frac{\partial \gamma}{\partial s_i} - \varepsilon z \sum_{j=1}^{d-1} \frac{\partial \gamma}{\partial s_j} h_i^j. \quad (2.2.15)$$

From (2.2.15) we see that the Jacobian matrix takes the form,

$$\mathbf{J} = \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} & \nu \end{pmatrix} \begin{pmatrix} I_{d-1} - \varepsilon z h_i^j & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad (2.2.16)$$

where  $I_{d-1}$  is the  $(d-1) \times (d-1)$  identity matrix. Taking the determinant of the Jacobian matrix yields

$$\begin{aligned} J(s, z) &= \det \mathbf{J} = \varepsilon J_0(s) \det(I_{d-1} - \varepsilon z h_i^j) \\ &= \varepsilon J_0(s) \prod_{i=1}^{d-1} (1 - \varepsilon z k_i) = J_0(s) \sum_{j=0}^d \varepsilon^{j+1} K_j z^j, \end{aligned} \quad (2.2.17)$$

where

$$K_0 = 1, \quad K_i := (-1)^i \sum_{j_1 < \cdots < j_i} k_{j_1} \cdots k_{j_i}, \quad (2.2.18)$$

and, since  $\frac{\partial \gamma}{\partial s_i} \in T_{\gamma(s)}$  and  $\nu$  is a unit normal,

$$\begin{aligned}
J_0 &= \det \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} & \nu \end{pmatrix} \\
&= \left( \det \left[ \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} & \nu \end{pmatrix} \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} & \nu \end{pmatrix}^T \right] \right)^{1/2} \\
&= \left( \det \left[ \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} \end{pmatrix} \begin{pmatrix} \frac{\partial \gamma}{\partial s_1} & \frac{\partial \gamma}{\partial s_2} & \cdots & \frac{\partial \gamma}{\partial s_{d-1}} \end{pmatrix}^T \right] \right)^{1/2}. \quad (2.2.19)
\end{aligned}$$

Therefore, from (2.2.19), we have

$$J_0 = \sqrt{\det(\mathbf{g})}, \quad (2.2.20)$$

where  $\mathbf{g}$  is the metric tensor for  $\Gamma$ . Taking the  $z$  derivative of the product form of the Jacobian expression in (2.2.17) we obtain the identity,

$$\partial_z J = -\varepsilon^2 J_0(s) \sum_{i=1}^{d-1} k_i \prod_{j \neq i} (1 - \varepsilon z k_j) = \varepsilon \kappa J, \quad (2.2.21)$$

where the extended curvature

$$\kappa(s, z) := \partial_z J / (\varepsilon J) = - \sum_{i=1}^{d-1} \frac{k_i}{1 - \varepsilon z k_i} = \sum_{j=0}^{\infty} \kappa_j \varepsilon^j z^j, \quad (2.2.22)$$

is expressed in terms of the coefficients

$$\kappa_j(s) = - \sum_{i=1}^{d-1} k_i^{j+1}(s) = -\text{tr}(A^{j+1}), \quad (2.2.23)$$

where  $A$  is the Weingarten map (see Section 2.1). In particular,

$$H = \sum_{i=1}^{d-1} k_i = -\kappa_0, \quad (2.2.24)$$

is the mean curvature.

We remark that while the Jacobian remains smooth, in fact it is a polynomial in  $z$ , the extended curvature becomes singular when the whiskers intersect. Distributing the  $z$  derivative in (2.2.13) and using the identity (2.2.22) yields the central result of this subsection, the local decomposition of the Laplacian in the whiskered coordinates,

$$\Delta_x = \varepsilon^{-2} \partial_z^2 + \varepsilon^{-1} \kappa(s, z) \partial_z + \Delta_G. \quad (2.2.25)$$

We call  $\kappa$  the extended mean curvature since  $\kappa = \nabla_x \cdot \nu$  is the Cartesian divergence of the normal to  $\Gamma$  when the normal is extended off of  $\Gamma$  as a constant along whiskers. We conclude the chapter with the proof of this fact.

**Proposition 2.7.** *Let  $\nu(x) = \nu(s)$  be the unit normal to  $\Gamma$  extended to the whiskered region  $\Gamma(\ell)$  as a constant along whiskers. Then, the following holds for the extended mean curvature  $\kappa$  defined in (2.2.22).*

$$\kappa = \nabla_x \cdot \nu. \quad (2.2.26)$$

**Proof:**

$$\nabla_x \cdot \nu = \sum_{i=1}^d \frac{\partial \nu_i}{\partial x_i} = \sum_{i,j=1}^d \frac{\partial \nu_i}{\partial s_j} \frac{\partial s_j}{\partial x_i}, \quad (2.2.27)$$

and

$$x = \gamma(s) + \varepsilon z \nu(s), \quad (2.2.28)$$

implies

$$\frac{\partial \nu}{\partial s_j} = - \sum_{k=1}^{d-1} h_k^j \frac{\partial \gamma}{\partial s_k} = - \sum_{k=1}^{d-1} h_k^j \left( \frac{\partial x}{\partial s_k} - \varepsilon z \frac{\partial \nu}{\partial s_k} \right). \quad (2.2.29)$$

Introducing matrix notation:

$$S = (S)_{ij} = \frac{\partial s_i}{\partial x_j}, \quad (2.2.30)$$

$$X = (X)_{ij} = \frac{\partial x_i}{\partial s_j}, \quad (2.2.31)$$

$$N = (N)_{ij} = \frac{\partial \nu_i}{\partial s_j}, \quad (2.2.32)$$

we rewrite (2.2.29) in the form,

$$N = -(X - \varepsilon z N)B, \quad (2.2.33)$$

where  $B = \{h_i^j\}$  is the matrix representation of the Weingarten map in the tangent basis (see (2.1.18)). However,

$$(SX)_{ij} = \sum_k^d \frac{\partial s_i}{\partial x_k} \frac{\partial x_k}{\partial s_j} = \delta_{ij}, \quad (2.2.34)$$

and (2.2.33) yields after pre-multiplying both sides by  $S$

$$SN = -(I - \varepsilon z SN)B. \quad (2.2.35)$$

Isolating  $SN$  on the left-hand side we obtain

$$SN(I - \varepsilon z B) = -B, \quad (2.2.36)$$

and

$$SN = -B(I - \varepsilon z B)^{-1}. \quad (2.2.37)$$

Returning this expression for  $SN$  to (2.2.27) and using a Neumann Expansion we obtain

$$\nabla_x \cdot \nu = \sum_{i,j=1}^d S_{ji} N_{ij} = \sum_{j=1}^d (SN)_{jj} = \text{tr}(SN) = - \sum_{j=0}^{\infty} \varepsilon^j z^j \text{tr}(B^{j+1}). \quad (2.2.38)$$

Finally noting that  $k_i$  is the  $i$ -th eigenvalue of  $B = \{h_i^j\}$  and so  $\text{tr}(B^{j+1}) = \sum_{i=1}^{d-1} k_i^{j+1}$  yields

$$\nabla_x \cdot \nu = - \sum_{j=1}^{\infty} \sum_{i=1}^{d-1} \varepsilon^j z^j k_i^{j+1} = \sum_{j=0}^{\infty} \varepsilon^j z^j \kappa_j = \kappa, \quad (2.2.39)$$

where we used definitions of  $\kappa$  and  $\kappa_j$  in (2.2.22) and (2.2.23) respectively.  $\square$

## 2.3 Example: Spherical Coordinates

Let  $S_R^2$  be a sphere of radius  $R$  centered at the origin. Parametrizing  $S_R^2$  using spherical coordinates gives

$$\gamma(\phi, \Theta) = \begin{pmatrix} \gamma_1(\phi, \Theta) \\ \gamma_2(\phi, \Theta) \\ \gamma_3(\phi, \Theta) \end{pmatrix} = \begin{pmatrix} R \sin \phi \cos \Theta \\ R \sin \phi \sin \Theta \\ R \cos \phi \end{pmatrix}. \quad (2.3.1)$$

In addition, the normal vector  $\nu$  takes the form

$$\nu(\phi, \Theta) = \begin{pmatrix} \sin \phi \cos \Theta \\ \sin \phi \sin \Theta \\ \cos \phi \end{pmatrix}, \quad (2.3.2)$$

and following (2.2.1) we may introduce the whiskered coordinates in a neighborhood of  $S_R^2$  by

$$x = \gamma(s) + \epsilon z \nu(s) = (R + \epsilon z) \begin{pmatrix} \sin \phi \cos \Theta \\ \sin \phi \sin \Theta \\ \cos \phi \end{pmatrix}. \quad (2.3.3)$$

The metric tensor  $\mathbf{G}$  takes the form

$$\mathbf{G} = \begin{pmatrix} (R + \epsilon z)^2 & 0 & 0 \\ 0 & (R + \epsilon z)^2 \sin^2 \phi & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}, \quad (2.3.4)$$

and consequently

$$J = \sqrt{\det \mathbf{g}} = \epsilon (R + \epsilon z)^2 \sin \phi. \quad (2.3.5)$$

Setting  $y = [\phi, \Theta, z]$ ,

$$\begin{aligned}
\Delta &= \frac{1}{\sqrt{\det(\mathbf{G})}} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial y_i} \mathbf{G}^{ij} \sqrt{\det(\mathbf{G})} \frac{\partial}{\partial y_j} \\
&= \frac{1}{(R + \epsilon z)^2 \sin \phi} \sum_{i=1}^d \frac{\partial}{\partial y_i} \mathbf{G}^{ii} ((R + \epsilon z)^2 \sin \phi) \frac{\partial}{\partial y_j} \\
&= \underbrace{\frac{1}{(R + \epsilon z)^2} \frac{\partial^2}{\partial \phi^2} + \frac{\cos \phi}{(R + \epsilon z)^2 \sin \phi} \frac{\partial}{\partial \phi} + \frac{1}{(R + \epsilon z)^2 \sin^2 \phi} \frac{\partial^2}{\partial \theta^2}}_{\Delta_G} + \frac{1}{\epsilon^2} \frac{\partial^2}{\partial z^2} + \frac{2}{R + \epsilon z} \frac{\partial}{\partial z}.
\end{aligned} \tag{2.3.6}$$

Restriction of  $\Delta_G$  to  $\Gamma$  is

$$\Delta_s = \frac{1}{R^2} \frac{\partial^2}{\partial \phi^2} + \frac{\cos \phi}{R^2 \sin \phi} \frac{\partial}{\partial \phi} + \frac{1}{R^2 \sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \tag{2.3.7}$$

and the eigenfunctions  $\Theta_i$  of  $\Delta_s$  are orthonormal with respect to the inner product

$$(\Theta_i, \Theta_j)_\Gamma = \int_\Gamma \Theta_i \Theta_j J_0(s) ds = \int_\Gamma \Theta_i \Theta_j R^2 \sin \phi ds. \tag{2.3.8}$$

## 2.4 Admissible Interfaces

**Definition 2.8.** Denote by  $\mathcal{G}$  the family of smooth compact oriented  $(d-1)$ -dimensional manifolds without boundary embedded in  $\mathbb{R}^d$ .

**Definition 2.9.** For  $K > 0$  denote by  $\mathcal{G}_K$  the set of manifolds  $\Gamma \in \mathcal{G}$  satisfying the following assumptions:

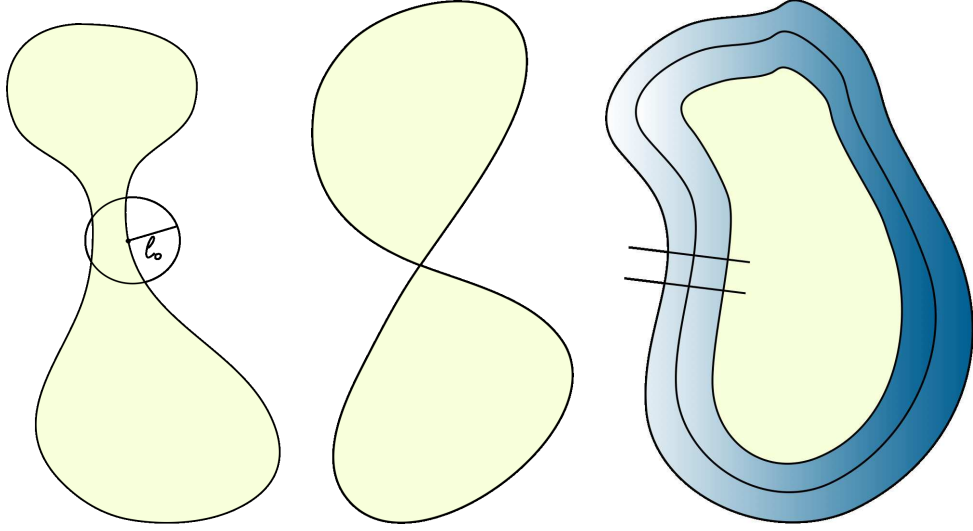


Figure 2.1: The figure on the left shows an interface  $\Gamma \subset \mathbb{R}^2$  that does not belong to  $\mathcal{G}_K$  due to its closeness to self-intersection. The center figure shows an interface that is not even in  $\mathcal{G}$  since it is not an embedding. The figure on the right shows an admissible interface together with its 'whiskered neighborhood'  $\Gamma(\ell)$ . For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

- (i) the principal curvatures and their derivatives up to the fourth order are bounded in  $L^\infty(\Gamma)$  norm by  $K$ ,
- (ii) the whiskers of length  $1/K$  (in the unscaled distance) do not intersect each-other,
- (iii) the volume  $\text{vol}(\Gamma)$  of  $\Gamma$  is bounded by  $K$ .

In the following we fix  $K > 0$  and choose  $\ell \leq \min\{K, \frac{1}{4K}\}$ .

**Lemma 2.10.** *Let  $G$  be the metric tensor for the whiskered coordinates defined in (2.2.5). Let  $\Delta_G$  be the laplacian in whiskered coordinates given in (2.2.14) and let  $\Delta_s$  be the Laplace-Beltrami operator on the interface  $\Gamma$ . Then the following relationship holds between the operators  $\Delta_G$  and  $\Delta_s$*

$$\Delta_G = \Delta_s + \varepsilon z D_{s,2}, \quad (2.4.1)$$

where

$$D_{s,2} := \sum_{i,j=1}^{d-1} d_{ij}(s,z) \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} d_j(s,z) \frac{\partial}{\partial s_j}, \quad (2.4.2)$$

and

$$\max_{ij} \left( \|\partial_z^m d_{ij}\|_{L^\infty(\Gamma(2\ell))}, \|\partial_z^m d_j\|_{L^\infty(\Gamma(2\ell))} \right) \leq C(K) \varepsilon^m, \quad (2.4.3)$$

where the constant  $C(K)$  is independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** We recall the definition of  $\Delta_G$  from (2.2.14)

$$\Delta_G = J^{-1} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} G^{ij} J \frac{\partial}{\partial s_j} \quad (2.4.4)$$

and observe that (2.2.1) and (2.2.5) imply

$$G_{ij} = \left\langle \frac{\partial \gamma(s)}{\partial s_i} + \varepsilon z \frac{\partial \nu(s)}{\partial s_i}, \frac{\partial \gamma(s)}{\partial s_j} + \varepsilon z \frac{\partial \nu(s)}{\partial s_j} \right\rangle = g_{ij} - 2\varepsilon z h_{ij} + \varepsilon^2 z^2 e_{ij}, \quad (2.4.5)$$

where we used the definitions of the fundamental forms in (2.1.8), (2.1.11), and (2.1.14).

In addition, the Jacobian is given in (2.2.17) as

$$J(s,z) = J_0(s) \sum_{j=0}^d \varepsilon^{j+1} K_j z^j. \quad (2.4.6)$$

Writing

$$J^{-1} = \frac{1}{J_0(s) \sum_{j=0}^d \varepsilon^{j+1} K_j z^j} = J_0^{-1} \sum_{m=0}^{\infty} a_m z^m \varepsilon^m, \quad (2.4.7)$$

we deduce

$$\sum_{j=0}^d \sum_{m=0}^{\infty} K_j a_m (\varepsilon z)^{m+j} = 1. \quad (2.4.8)$$

Reindexing so that  $m + j = i$  yields

$$\sum_{i=0}^{\infty} \sum_{m=0}^i a_m K_{i-m} (\varepsilon z)^i = 1, \quad (2.4.9)$$

and the following formulas for  $a_i$

$$a_0 = 1 \quad (2.4.10)$$

and

$$\sum_{m=0}^i a_m K_{i-m} = 0. \quad (2.4.11)$$

In particular,

$$a_i = - \sum_{m=0}^{i-1} a_m K_{i-m}, \quad (2.4.12)$$

and a simple inductive argument shows

$$|a_i| \leq C(2K)^m, \quad (2.4.13)$$

where  $C$  depends on  $d$  only.

Furthermore, written in matrix notation

$$G = g(I - 2\varepsilon z g^{-1} h + \varepsilon^2 z^2 g^{-1} e) = g(I - 2\varepsilon \tilde{h} + \varepsilon^2 z^2 \tilde{e}), \quad (2.4.14)$$

where  $\tilde{h} = g^{-1}h = \{h_i^j\}$  and  $\tilde{e} = g^{-1}e = \{e_i^j\}$  and

$$(I - 2\varepsilon\tilde{h} + \varepsilon^2 z^2 \tilde{e})^{-1} = \sum_{m=0}^{\infty} \varepsilon^m z^m A_m, \quad (2.4.15)$$

where

$$A_0 = I, \quad (2.4.16)$$

$$A_1 = 2\tilde{h}, \quad (2.4.17)$$

and

$$A_m = 2\tilde{h}A_{m-1} - \tilde{e}A_{m+2}. \quad (2.4.18)$$

Since  $\|\tilde{h}\| \leq K$  and  $\|\tilde{e}\| \leq K^2$  (see 2.1.19 and 2.1.22) we again use a recursive argument to establish

$$\|A_m\| \leq (3K)^m. \quad (2.4.19)$$

Since, we assumed that  $\varepsilon z \leq \ell \leq \frac{1}{4K}$  the series (2.4.7) and (2.4.15) are convergent and the result of the proposition follows by substituting the expressions for  $J^{-1}$  and  $G^{ij}$  into (2.4.4).

□

**Lemma 2.11.** *Fix  $K > 0$ . Let  $g = \{g_{ij}\}$  and  $h = \{h_{ij}\}$  be the first and second fundamental forms of the interface  $\Gamma \in \mathcal{G}_K$ . Then the following estimate holds*

$$\|(g - \varepsilon zh)^{-1}\| \leq C(K). \quad (2.4.20)$$

where  $C(K)$  is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** Letting  $\tilde{h} := g^{-1}h$ , we have from the Neumann expansion of  $(I - \varepsilon z \tilde{h})^{-1}$

$$[g(I - \varepsilon z \tilde{h})]^{-1} = \left( \sum_{m=0}^{\infty} \varepsilon^m z^m \tilde{h}^m \right) g^{-1}, \quad (2.4.21)$$

and since  $\ell \leq \frac{1}{4K}$ , (2.4.20) follows from the bound on  $\tilde{h}$  in (2.1.19)

$$\|\varepsilon z \tilde{h}\| \leq \ell K \leq \frac{1}{4}. \quad (2.4.22)$$

□

**Proposition 2.12.** *Let  $\Gamma \in \mathcal{G}_K$ . Then, for  $\varepsilon$  sufficiently small and  $\ell \leq \frac{1}{4K}$ , the following estimates hold for the metric tensor  $\mathbf{G}$ , the Jacobian  $J$ , and the extended mean curvature  $\kappa$*

$$\sup_{i,j \leq d-1} \|\partial_z^m G^{ij}\|_{L^\infty(\Gamma(2\ell))} \leq C(K) \varepsilon^m, \quad (2.4.23)$$

$$\sup_{i,j \leq d-1} \|\partial_z^m G_{ij}\|_{L^\infty(\Gamma(2\ell))} \leq C(K) \varepsilon^m, \quad (2.4.24)$$

$$\varepsilon \left(\frac{1}{2}\right)^{d-1} \leq J \leq \varepsilon \left(\frac{3}{2}\right)^{d-1}, \quad (2.4.25)$$

$$\sup_{|\alpha| \leq 2} \|D_s^\alpha \partial_z^m J\|_{L^\infty(\Gamma(2\ell))} \leq C(K) \varepsilon^{m+1}, \quad (2.4.26)$$

$$\sup_{|\alpha| \leq 2} \|D_s^\alpha \partial_z^m \kappa\|_{L^\infty(\Gamma(2\ell))} \leq C(K) \varepsilon^m. \quad (2.4.27)$$

Moreover, the constants in these estimates depend on  $K$  only.

**Proof:** The result follows from formulas for  $G^{ij}$ ,  $G_{ij}$  and  $J$  in the proof of the previous proposition. □

A function  $f(x)$  which decays exponentially to zero away from  $\Gamma$  at an  $O(1)$  rate in  $z$  will be said to be ‘localized on the surface’ or merely ‘localized’. Up to exponentially small terms localized functions can be written in the whiskered variables, so that  $f(x) = f(s, z)$ . Functions  $h(z)$  of one variable which decay exponentially to constant values at  $\pm\infty$  can be extended to  $\Omega$  by setting them to their limiting value for  $z > 2\ell/\varepsilon$  and smooth for  $|z| \in (\ell/\varepsilon, 2\ell/\varepsilon)$ . The expression *the interface  $\Gamma$  dressed by  $h$*  refers to the extended function  $h(x)$ , understood to mean  $h(z(x))$ .

# Chapter 3

## Overview of Results

This chapter provides an overview of the derivation of the curvature driven flow on a single-layer dressed interface, which we express in our main result, Theorem 3.7.

### 3.1 Single-layer Dressed Ansatz

Our goal in this section is to construct a family of approximate solutions to the steady-state equation corresponding to (1.0.5). Recall that the set of admissible interfaces  $\mathcal{G}_K$  consists of smooth compact oriented  $(d-1)$ -dimensional manifolds without boundary embedded in  $\mathbb{R}^d$  satisfying the following assumptions:

- (i) the principal curvatures and their derivatives up to the fourth order are bounded in  $L^\infty(\Gamma)$  norm by  $K$ ,
- (ii) the whiskers of length  $1/K$  (in the unscaled distance) do not intersect each-other,
- (iii) the volume  $vol(\Gamma)$  of  $\Gamma$  is bounded by  $K$ .

Fix  $K > 0$  and let  $\Gamma \in \mathcal{G}_K$  be an admissible interface. We use the whiskered coordinates

$(z, s)$  defined in Section 2.2 in the whiskered neighborhood of  $\Gamma$ ,  $\Gamma(2\ell)$ , of width  $2\ell$  and the  $x$  coordinates outside of this neighborhood. Recall that the orientation of  $\Gamma$  determines the direction of the normal vector in the two connected components of  $\Omega$ . We will denote these components as  $\Omega_{\pm}$  respectively. Therefore  $z > 0$  in  $\Omega_+ \cap \Gamma(2\ell)$  and  $z < 0$  in  $\Omega_- \cap \Gamma(2\ell)$  (see (2.2.2)).

In order to transition between Cartesian and whiskered coordinates we introduce a smooth cutoff function  $\eta_1$  which is equal to one near the interface and is zero away from it. Similarly we introduce another smooth cutoff function  $\bar{\eta}_1(x)$  which takes the values  $\pm 1$  away from the interface and is zero near the interface. More precisely, we define

$$\eta_0(z) = \begin{cases} 1, & |z| \leq \ell, \\ 0, & |z| \geq 2\ell, \\ \in (0, 1), & \ell < |z| < 2\ell; \end{cases} \quad (3.1.1)$$

$$\eta_1(x) = \begin{cases} 1, & x \in \Gamma(\ell), \\ 0, & x \in \Omega \setminus \Gamma(2\ell), \\ \eta_0(z), & x \in \Gamma(2\ell) \setminus \Gamma(\ell); \end{cases} \quad (3.1.2)$$

$$\bar{\eta}_1(x) = \begin{cases} -1 + \eta_1(x), & x \in \Omega_-, \\ 1 - \eta_1(x), & x \in \Omega_+. \end{cases} \quad (3.1.3)$$

We remark that  $\bar{\eta}_1$  is smooth since  $\bar{\eta}_1 \equiv 0$  in  $\Gamma(\ell)$ .

We associate to each  $\Gamma \in \mathcal{G}_K$ ,  $b \in \mathbb{R}$  the corresponding single-layer dressed ansatz  $\Phi \in$

$H^5(\Omega)$ ,

$$\Phi(x; \Gamma, b) := \eta_1(z(x))(\phi(z(x)) + \varepsilon^2 \phi_2(s(x), z(x))) + b + \bar{\eta}_1(z(x)). \quad (3.1.4)$$

Here  $b = \varepsilon^3 b_3$  is an  $O(\varepsilon^3)$  parameter that incorporates the small, spatially-constant variation of the background state of  $\Phi$  away from the limiting values of  $\pm 1$ .

The function  $\phi$  is the heteroclinic solution to the one-dimensional Allen-Cahn equation,

$$\phi_{zz} - W'(\phi) = 0, \quad \lim_{z \rightarrow \pm\infty} \phi = \pm 1, \quad (3.1.5)$$

and  $\phi_2$  incorporates curvature induced shape corrections into the ansatz,

$$\phi_2(s, z) := -\left(\frac{1}{2}H^2 + \text{tr}(A^2)\right)(\Pi_1 L_\phi)^{-1}(z\phi'), \quad (3.1.6)$$

where  $\Pi_1$  is the orthogonal projection off of the kernel of  $L_\phi$ ,

$$L_\phi := \partial_{zz} - W''(\phi), \quad (3.1.7)$$

is the linearization of (3.1.5) about  $\phi$ , and

$$H = \text{tr}(A) = -\kappa_0 = \sum k_i, \quad (3.1.8)$$

$$\text{tr}(A^2) = -\kappa_1 = \sum k_i^2, \quad (3.1.9)$$

are the traces of the first two powers of the Weingarten map  $A$  (see Chapter 2). As we will show in Proposition 3.1, the form of  $\phi_2$  guarantees that the residual  $F(\Phi)$  defined in (1.0.6)

is  $O(\varepsilon^3)$ .  $(\Pi_1 L_\phi)^{-1}(z\phi')$  is well defined, since from Lemma 9.1 the operator  $L_\phi$ , acting on  $L^2(\mathbb{R})$  has a simple kernel, spanned by  $\phi'$ , since  $z\phi'$  is orthogonal to  $\phi'$ , there is a unique function  $f$ , also orthogonal to  $\phi'$  such that  $L_\phi f = z\phi'$ .

**Proposition 3.1.** *Fix  $K > 0$ , then there exists  $C(K) > 0$  such that for all  $\Gamma \in \mathcal{G}_K$ , the single-layer dressed ansatz  $\Phi = \Phi(\cdot; \Gamma, b)$  defined in (3.1.4) satisfies*

$$\|F(\Phi)\|_{L^\infty(\Omega)} \leq C(K)\varepsilon^3. \quad (3.1.10)$$

**Proof:** We recall the definition of  $F$  from (1.0.6)

$$F(u) = (\varepsilon^2 \Delta - W''(u) + \varepsilon^2 \eta)(\varepsilon^2 \Delta u - W'(u)). \quad (3.1.11)$$

Since  $W'(\pm 1) = 0$  and as  $z \rightarrow \pm\infty$ ,  $\phi(z) + \varepsilon^2 \phi_2(s, z)$  approaches  $\pm 1$  exponentially fast in  $|z|$ , we need only consider  $\|F(\Phi)\|_{L^\infty(\Gamma(\ell))}$ . For  $x \in \Gamma(\ell)$ ,

$$\Phi(x) = \phi(z) + \varepsilon^2 \phi_2(s, z) + \varepsilon^3 b_3, \quad (3.1.12)$$

and we may change to whiskered variables to obtain

$$F(\Phi) = \underbrace{(\partial_z^2 + \varepsilon \kappa \partial_z + \varepsilon^2 \Delta_G - W''(\Phi) + \varepsilon^2 \eta)}_A \underbrace{(\partial_z^2 \Phi + \varepsilon \kappa \partial_z \Phi + \varepsilon^2 \Delta_G \Phi - W'(\Phi))}_B. \quad (3.1.13)$$

In addition Taylor expanding  $W'$  and  $W''$  about  $\phi$  we obtain,

$$W'(\Phi) = W'(\phi) + (\varepsilon^2\phi_2 + \varepsilon^3b_3)W''(\phi) + O(\varepsilon^4), \quad (3.1.14)$$

and

$$W''(\Phi) = W''(\phi) + (\varepsilon^2\phi_2 + \varepsilon^3b_3)W'''(\phi) + O(\varepsilon^4). \quad (3.1.15)$$

Here and in the remainder of the proof  $O(\varepsilon^4)$  is in  $L^\infty$  norm. Substituting in the expressions for  $W'(\Phi)$  and  $W''(\Phi)$  yields

$$\begin{aligned} A &= \partial_z^2 + \varepsilon\kappa\partial_z + \varepsilon^2\Delta_G - W''(\Phi) + \varepsilon^2\eta \\ &= \partial_z^2 + \varepsilon\kappa\partial_z + \varepsilon^2\Delta_G - W''(\phi) - (\varepsilon^2\phi_2 + \varepsilon^3b_3)W'''(\phi) + \varepsilon^2\eta + O(\varepsilon^4) \\ &= L_\phi + \varepsilon\kappa\partial_z + \varepsilon^2\Delta_G - \varepsilon^2\phi_2W'''(\phi) + \varepsilon^2\eta + O(\varepsilon^3), \end{aligned}$$

Since  $\phi$  is a function of only  $z$ , we have

$$\|\varepsilon^2\Delta_G\Phi\|_{C^2(\Gamma(\ell))} = \|\varepsilon^4\Delta_G\phi_2\|_{C^2(\Gamma(\ell))} = O(\varepsilon^4) \quad (3.1.16)$$

and

$$\begin{aligned} B &= \partial_z^2\Phi + \varepsilon\kappa\partial_z\Phi + \varepsilon^2\Delta_G\Phi - W'(\Phi) \\ &= \phi'' + \varepsilon^2\partial_z^2\phi_2 + \varepsilon\kappa\phi' + \varepsilon^3\kappa\partial_z\phi_2 - W'(\phi) - (\varepsilon^2\phi_2 + \varepsilon^3b_3)W''(\phi) + O(\varepsilon^4) \\ &= \varepsilon\kappa\phi' + \varepsilon^2L_\phi\phi_2 + \varepsilon^3(\kappa\partial_z\phi_2 - b_3W''(\phi)) + O(\varepsilon^4), \end{aligned} \quad (3.1.17)$$

where we used  $\phi'' - W'(\phi) = 0$ . Using the expansions for  $A$  and  $B$  in (3.1.13) and in addition using (2.2.22) to expand the extended curvature  $\kappa$  in powers of  $\varepsilon$  we obtain

$$F(\Phi) = \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R_3 + O(\varepsilon^4) \quad (3.1.18)$$

where by Lemma 9.1

$$R_1 = \kappa_0 L_\phi \phi' = 0, \quad (3.1.19)$$

$$R_2 = \kappa_1 L_\phi(z\phi') + L_\phi^2 \phi_2 + \kappa_0^2 \phi z z = L_\phi^2 \phi_2 + \left(\frac{1}{2}\kappa_0^2 + \kappa_1\right) L_\phi(z\phi'), \quad (3.1.20)$$

and

$$\begin{aligned} R_3 = & L_\phi(\kappa_2 z^2 \phi' + \kappa_0 \partial_z \phi_2) + L_\phi^2 b_3 + \kappa_0 \kappa_1 z \phi'' + \kappa_0 \partial_z(\kappa_1 z \phi' + L_\phi \phi_2) \\ & + \phi'(\Delta_G + \eta)\kappa_0 - \kappa_0 W'''(\phi)\phi_2 \phi'. \end{aligned} \quad (3.1.21)$$

We note that  $\phi_2$  was selected in (3.1.6) precisely to make

$$R_2 = L_\phi^2 \phi_2 + \left(\frac{1}{2}\kappa_0^2 + \kappa_1\right) L_\phi(z\phi') = 0. \quad (3.1.22)$$

One can easily verify that the constants in all asymptotic estimates depend on  $K$  only.  $\square$

The following corollary follows from Proposition 3.1 and the fact that  $F(\Phi)$  is localized on  $\Gamma$ .

**Corollary 3.2.** *Fix  $K > 0$ , then there exists  $C(K) > 0$  such that for all  $\Gamma \in \mathcal{G}_K$ , the*

single-layer dressed ansatz  $\Phi = \Phi(\cdot; \Gamma, b)$  defined in (3.1.4) satisfies

$$\|\Pi_0 F(\Phi)\|_{L^2(\Omega)} \leq C(K)\varepsilon^{7/2}. \quad (3.1.23)$$

## 3.2 Linearization and the Functionalized Operator

For each  $K, \bar{b} > 0$ , we define the single-layer dressed manifold

$$\mathcal{M}_{K, \bar{b}} := \{\Phi(x; \Gamma, b) : \Gamma \in \mathcal{G}_K, b \in (-\bar{b}\varepsilon^3, \bar{b}\varepsilon^3)\}, \quad (3.2.1)$$

where  $K$  is the admissible interface parameter (section 2.4),  $\bar{b}$  is a positive constant and  $\Phi$  is defined in (3.1.4). By Proposition 3.1 the manifold  $\mathcal{M}_{K, \bar{b}}$  is comprised of small residual functions  $\Phi$  obtained by dressing an interface  $\Gamma \in \mathcal{G}_K$  with the heteroclinic solution  $\phi$  to the one-dimensional Allen-Cahn equation, and then correcting for higher-order effects, such as interfacial curvature. We also introduce the linearization

$$\mathcal{L}_\phi := (-\varepsilon^2 \Delta + W''(\Phi) - \varepsilon^2 \eta)(-\varepsilon^2 \Delta + W''(\Phi)) - W'''(\Phi)(\varepsilon^2 \Delta \Phi - W'(\Phi)), \quad (3.2.2)$$

of the vector field

$$F(u) = (\varepsilon^2 \Delta - W''(u) + \varepsilon^2 \eta)(\varepsilon^2 \Delta u - W'(u)), \quad (3.2.3)$$

about  $\Phi$ , see (1.0.6). It will be convenient to write  $\mathcal{L}_\phi$  in the following factored form

$$\begin{aligned}\mathcal{L}_\phi &= (\varepsilon^2 \Delta - W''(\Phi) + \varepsilon^2 \eta)^2 + \varepsilon^2 \eta (W''(\Phi) - \varepsilon^2 \eta) - W'''(\Phi) (\varepsilon^2 \Delta \Phi - W'(\Phi)) \\ &= \mathcal{A}_\phi^2 + \varepsilon \tilde{q},\end{aligned}\tag{3.2.4}$$

where

$$\mathcal{A}_\phi := \varepsilon^2 \Delta - W''(\Phi) + \varepsilon^2 \eta,\tag{3.2.5}$$

is the Allen-Cahn operator and

$$\tilde{q} := \varepsilon \eta (W''(\Phi) - \varepsilon^2 \eta) - \frac{1}{\varepsilon} W'''(\Phi) (\varepsilon^2 \Delta \Phi - W'(\Phi)).\tag{3.2.6}$$

We remark that the last term on the right-hand side of (3.2.6) is  $O(1)$  (see calculations in Proposition 3.1). In particular,  $\|\tilde{q}\|_{L^\infty(\Gamma)} \leq C(K)$  where  $C(K)$  is a positive constant that depends only on the admissible interface parameter  $K$ .

In addition, we define the  $\mathcal{A}$ -norm of  $u \in H^2(\Omega)$  as

$$\|u\|_{\mathcal{A}} := \|\Delta u\|_{L^2(\Omega)} + \varepsilon^{-2} \|u\|_{L^2(\Omega)}.\tag{3.2.7}$$

The objective of this section is to characterize the asymptotically small eigenspace of the operator  $\mathcal{L}_\phi : H^2(\Omega) \rightarrow L^2(\Omega)$ . To this end, we first construct a convenient ‘separated variables’ basis for this eigenspace. We start with the Laplace-Beltrami operator  $\Delta_\mathcal{S} : H^2(\Gamma) \rightarrow L^2(\Gamma)$

given locally on  $\Gamma$  by

$$\Delta_s = \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} g^{ij} \sqrt{\det \mathbf{g}} \frac{\partial}{\partial s_j}, \quad (3.2.8)$$

and denote the (nonnegative) eigenvalues of  $-\Delta_s$  by  $\{\beta_{j,\Gamma}\}$  and the corresponding eigenfunctions by  $\{\Theta_{j,\Gamma}\}$ ,

$$-\Delta_s \Theta_{j,\Gamma} = \beta_{j,\Gamma} \Theta_{j,\Gamma}. \quad (3.2.9)$$

The Laplace-Beltrami eigenvalues  $\{\beta_{j,\Gamma}\}$  play an important role in our analysis. We discuss them in more detail in Remark 3.4 as well as Chapter 5. We introduce the following subspaces of  $L^2(\Omega)$ .

**Definition 3.3.** *For each  $M > 0$ , the  $M + 1$  dimensional slow space associated to  $\mathcal{L}_\phi$  is defined by*

$$\mathcal{Z}_M := \text{span} \left( \{Z_i\}_{i=1\dots M} \cup \{1\} \right), \quad (3.2.10)$$

where

$$Z_i(x) := \eta_1(x) \Theta_{i,\Gamma}(s) \phi'(z); \quad (3.2.11)$$

$\mathcal{Z}_M^\perp$  is the orthogonal complement of  $\mathcal{Z}_M$  in  $L^2(\Omega)$ .

**Remark:** We will show that  $\mathcal{Z}_M^\perp$  corresponds to the fast eigenspace of  $\mathcal{L}_\phi$ , in the sense that the bilinear form we associated with  $\mathcal{L}_\phi$  is coercive on  $\mathcal{Z}_M^\perp$ .

Recall that  $\eta_1$  is a cutoff function that is identically zero outside the ‘whiskered’ region  $\Gamma(2\ell)$  and  $\phi'$  is the ground-stable eigenfunction of the one-dimensional Allen-Cahn operator and corresponds to the zero eigenvalue (see Lemma 9.1). Let  $[a]$  represent the greatest integer

less than or equal to  $a$ . We will show in Chapter 8 that selecting  $M = [M_1 \varepsilon^{-(d-1)}]$ , where  $M_1$  introduced in Theorem 3.5 is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ , guarantees the coercivity of the bilinear form associated with  $\mathcal{L}_\phi$  on the orthogonal complement to  $\mathcal{Z}_M$ . The proof uses the following key relation on the asymptotics of the large eigenvalues of the Laplace-Beltrami operator, first proved by Weyl (see [11]),

$$(\beta_m)^{(d-1)/2} \sim \frac{(2\pi)^{d-1}/\omega_d}{\text{vol}\Gamma} m, \quad \text{as } m \rightarrow \infty \quad (3.2.12)$$

to obtain a uniform bound for  $\beta_M$ . The Laplace-Beltrami operator and the formula above are discussed in more detail in Chapter 5.

**Remark 3.4.** *We note that the Weyl asymptotic formula implies that for  $M = [M_1 \varepsilon^{-(d-1)}]$*

$$\beta_M \sim \frac{(2\pi)^2}{(\text{vol}\Gamma \omega_d)^{2/(d-1)}} \varepsilon^{-2}. \quad (3.2.13)$$

*Moreover, for each admissible interface parameter  $K > 0$  there exists  $c_g > 0$  and an interval  $[M_{1-}, M_{1+}]$ , independent of  $\varepsilon < \varepsilon_0$ , such that for all  $\Gamma \in \mathcal{G}_K$  there exists  $M_1 \in [M_{1-}, M_{1+}]$  for which the corresponding  $M$  satisfies*

$$\beta_{M+1} - \beta_M > c_g. \quad (3.2.14)$$

*In addition, the sum of the squares of the Laplace-Beltrami eigenmodes satisfy the following bound (see [19], [21])*

$$\sup_{s \in \Gamma} \sum_{i=1}^M |\Theta_i(s)|^2 \leq C(K, M_1) \varepsilon^{-(d-1)}. \quad (3.2.15)$$

The following theorem is proved in Chapter 8.

**Theorem 3.5. Coercivity Estimates for the Linear Operator.** *Fix  $K, \bar{b} > 0$  and  $\varepsilon > 0$  sufficiently small. Recall the definition of  $\mathcal{A}_\phi$  from (3.2.5). There exists  $M_{1-} > 0$  independent of  $\varepsilon$ , such that for all  $M_1 \geq M_{1-}$  the following bounds hold for all  $w$  from the associated fast space,  $\mathcal{Z}_M^\perp$ ,*

$$\|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)} \geq \frac{1}{32} \varepsilon^4 \beta_{M+1}^2 \|w\|_{L^2(\Omega)}, \quad (3.2.16)$$

$$\|\mathcal{A}_\phi w\|_{L^2(\Omega)} \geq \frac{1}{8} \varepsilon^2 \beta_{M+1} \|w\|_{L^2(\Omega)}, \quad (3.2.17)$$

$$\|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)} \geq \frac{1}{8} \varepsilon^2 \beta_{M+1} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \quad (3.2.18)$$

$$\|\mathcal{A}_\phi w\|_{L^2(\Omega)} \geq C \varepsilon^4 \beta_{M+1} \|w\|_{H^2(\Omega)}, \quad (3.2.19)$$

$$C_a \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)} \leq \|w\|_{\mathcal{A}} \leq C_b \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \quad (3.2.20)$$

where  $C_a$  and  $C_b$  are constants that depend on  $M_1$ , but are independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

Moreover,  $\{\Theta_{i,\Gamma}\}_{i=1}^M$  form an approximate basis of  $L^2(\Gamma)$ , in the sense that any function  $f$  which is orthogonal to this set in  $L^2(\Gamma)$  satisfies the inequality

$$\|f\|_{L^2(\Gamma)} \leq C \varepsilon \|\nabla_s f\|_{L^2(\Gamma)}, \quad (3.2.21)$$

where the constant  $C$  depends on  $M_1$ , but is independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Remark:** As will be shown in the proof of Theorem 3.5, the positivity estimates (3.2.16)-(3.2.18) are consequences of the coercivity property of the bilinear form associated with  $\mathcal{L}_\phi$ , when restricted to  $\mathcal{Z}_M^\perp$ .

### 3.3 Main Results

#### 3.3.1 Statement of the Main Theorem

In this section we state our main result. We wish to decompose solutions  $u(x, t)$  to (1.0.5) which are in a vicinity of the single-layer manifold  $\mathcal{M}_{K, \bar{b}}$  (see 3.2.1) as

$$u(x, t) = \Phi(x; \Gamma_t, b) + w(x, t), \quad (3.3.1)$$

where  $w \in \mathcal{Z}_M^\perp$ ,  $\Gamma_t$  denotes the interface at time  $t$  and the dressed ansatz  $\Phi$  is determined from  $\Gamma_t \in \mathcal{G}_K$  and  $b \in (-\bar{b}\varepsilon^3, \bar{b}\varepsilon^3)$  via (3.1.4). Moreover we will show that if the evolution of the local parametrization  $\gamma(s, t)$  of  $\Gamma_t$  is given by

$$\begin{aligned} \partial_t \gamma(s, t) &= V_n(s, t) \cdot \nu(s, t) \\ \gamma(s, 0) &= \gamma_0(s) \end{aligned} \quad (3.3.2)$$

for an appropriate choice of normal velocity  $V_n$ , then  $w$  will remain small in the  $\|\cdot\|_{\mathcal{A}}$  norm. Moreover we show that the choice of normal velocity depends at (formal) leading order only upon the second fundamental form of the interface, and not upon the remainder  $w$ . In particular,  $V_n = \varepsilon^4 V_n^0(h) + \varepsilon^5 V_n^c(h, w)$ , with  $V_n^0$  given in (3.3.11).

In the following theorem we derive an expression for the leading term  $V_n^0$  of  $V_n$  for which the remainder  $w$  stays small under the associated flow. An equivalent description of the evolution of the interface will be given as a Ricci-curvature flow involving the first two fundamental forms  $g = g_{ij}$  and  $h = h_{ij}$  of the interface  $\Gamma_t$ . In what follows,  $g^{ij}$  is the inverse of  $g_{ij}$  and

$\nabla_j$  is the operator of covariant differentiation on the Riemannian manifold  $\Gamma$ :

$$\nabla_j X^i = \frac{\partial}{\partial s_j} X^i + \sum_{k=1}^{d-1} \Gamma_{jk}^i X^k, \quad (3.3.3)$$

and  $\Gamma_{jk}^i$  are the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{d-1} \left( \frac{\partial}{\partial s_i} g_{jl} + \frac{\partial}{\partial s_j} g_{il} - \frac{\partial}{\partial s_l} g_{ij} \right). \quad (3.3.4)$$

**Assumption 3.6.** *We assume that the Ricci-curvature driven flow (time has been rescaled to  $\tau = \varepsilon^4 t$ ), given in terms of the rescaled normal velocity  $\bar{V}_n := \varepsilon^{-4} V_n = V_n^0(h) + \varepsilon V_n^c(h, w)$  (see (3.3.26) for the defining equation of  $V_n$ ),*

$$\partial_\tau g_{ij} = -2\bar{V}_n h_{ij}, \quad (3.3.5)$$

$$\partial_\tau h_{ij} = -\nabla_i \nabla_j V_n + \sum_{l,m} \bar{V}_n h_{il} g^{lm} h_{mj}, \quad (3.3.6)$$

is locally well posed in  $H^4(\Gamma)$ . That is we assume that the second fundamental form  $h(t)$  of the interface  $\Gamma_t$  remains uniformly bounded in  $H^4(\mathbf{S})$  for all scaled time  $\tau \in [0, T_f]$ .

**Theorem 3.7.** *Let the space dimension  $d = 2$  or  $3$  and let Assumption 3.6 hold. Fix the admissible interface parameters  $\bar{b}, K > 0$  and  $\bar{b}_0 < \bar{b}$ ,  $K_0 < K$ . Fix the slow-space dimension parameter  $M_1 > 0$ . Then there exist  $\frac{1}{2}\bar{B}_0 > \bar{B} > 0$  and  $U, \varepsilon_0, T_f > 0$  such that for all*

$\varepsilon \in (0, \varepsilon_0)$  and for all  $u_0$  satisfying

$$\inf_{\substack{\Gamma \in \mathcal{G}_{K_0} \\ b \in (-\bar{b}_0 \varepsilon^3, \bar{b}_0 \varepsilon^3)}} \|u_0 - \Phi(\cdot, \Gamma, b)\|_{\mathcal{A}} < \bar{B}_0 \varepsilon^{d-3/2}, \quad (3.3.7)$$

then for  $t < T_f \varepsilon^{-4}$  (see Assumption 3.6) we have a decomposition of the solution  $u(x, t)$

$$u(x, t) = \Phi(x, \Gamma_t, b(t)) + w(x, t) \quad (3.3.8)$$

where  $b(t) = \varepsilon^3 b_3(t)$ ,  $w$  lies in the  $L^2$  orthogonal complement of the slow space,  $w \in \mathcal{Z}_M^\perp$ , with the slow dimension  $M = [M_1 \varepsilon^{-(d-1)}]$  and

$$\|w\|_{\mathcal{A}} \leq \bar{B}_0 \varepsilon^{d-3/2} e^{-\frac{U}{2}t} + \bar{B} \varepsilon^{3/2}. \quad (3.3.9)$$

Moreover, the remainder term in the normal velocity  $V_n = \varepsilon^4 V_n^0 + \varepsilon^5 V_n^c$  of the interface  $\Gamma_t$  satisfies

$$\|V_n^c\|_{L^2(\Gamma)} = \varepsilon^{-5} \|V_n - \varepsilon^4 V_n^0\|_{L^2(\Gamma)} \leq C(K) \sqrt{M_1} \left( \varepsilon^{(1-d)/2} + \|w\|_{\mathcal{A}} \varepsilon^{(-2-d)/2} \right) \quad (3.3.10)$$

where  $V_n^0$  takes the following form

$$V_n^0 = - \left[ \left( \Delta_s + \eta \right) H - \frac{H^3}{2} + H \text{tr}(A^2) \right] - \frac{2}{\sqrt{\det(\mathbf{g})} \|\phi'\|_{L^2(\mathbb{R})}^2} \left( \mu_0^2 b_3 - \partial_t b_3 \right), \quad (3.3.11)$$

and the background state evolves according to

$$\begin{aligned} \partial_t b_3 + \varepsilon \frac{4\mu_0^2 |\Gamma|}{\sqrt{\det(\mathbf{g})} \|\phi'\|_{L^2(\mathbb{R})}^2 |\Omega|} b_3 &= -\frac{2\varepsilon}{|\Omega|} \int_{\Gamma} \left( \Delta_s + \eta - \frac{H^2}{2} + \text{tr}(A^2) \right) H ds \\ &+ O\left(\sqrt{M_1} \varepsilon^{(8-d)/2} \|w\|_{\mathcal{A}}, \sqrt{M_1} \varepsilon^{(11-d)/2}\right). \end{aligned} \quad (3.3.12)$$

Moreover, the evolution equations (3.3.11) and (3.3.12) result in a two time-scale system, which, after an  $O(\varepsilon^{-1})$  time drives the background to its quasi-equilibrium, and  $V_n^0$  relaxes to

$$V_n^0 = -\Pi_{0,\Gamma} \left( \Delta_s + \eta - \frac{H^2}{2} + \text{tr}(A^2) \right) H, \quad (3.3.13)$$

where  $\Pi_{0,\Gamma}$  denotes the zero-mass projection over the interface  $\Gamma$ .

**Remark 3.8.** Under the assumptions of Theorem 3.7, we can formally reformulate the evolution in terms of an equivalent (up to rigid body transformations) higher-order Ricci flow given by the following leading-order closed system of equations for the fundamental forms of the interface  $\Gamma_t$

$$\partial_\tau g_{ij} = -2V_n^0 h_{ij}, \quad (3.3.14)$$

$$\partial_\tau h_{ij} = -\nabla_i \nabla_j V_n^0 + \sum_{l,m} V_n^0 h_{il} g^{lm} h_{mj}. \quad (3.3.15)$$

In particular,

$$\partial_\tau H = (\Delta_s + \text{tr}(A^2)) V_n^0, \quad (3.3.16)$$

where time has been rescaled to  $\tau = \varepsilon^4 t$  to remove all leading-order  $\varepsilon$  dependence.

In  $\mathbb{R}^2$ , choosing an arc-length parametrization of the one-dimensional manifold  $\Gamma_t$  we

can write the leading-order evolution equation for the interface as a generalized Kuramoto-Sivashinsky equation

$$\partial_\tau H - (\partial_s H) \int_0^s V_n^0 H ds = - \left( \partial_s^2 + H^2 \right) \Pi_{0,\Gamma} \left( \partial_s^2 + \eta + \frac{H^2}{2} \right) H. \quad (3.3.17)$$

**Remark 3.9.** Theorem 3.7 gives the normal velocity  $V_n$  of the interface  $\Gamma_t$  as an asymptotic expression (3.3.10) with an  $O(\varepsilon^4)$  leading term and asymptotic constants in the remainder proportional to  $\sqrt{M_1}$ . We note that the slow-space dimension parameter  $M_1$  (see Remark 3.4) can be chosen arbitrarily small, independent of  $\varepsilon$ .

### 3.3.2 The time to exit the Manifold

Before we develop the technical estimates, we first address the salient issue of developing a uniform bound on the time  $T_f$  to exit the Manifold. This happens when the interface approaches self-intersection, the spectral gap,  $\beta_{M+1} - \beta_M$ , collapses, or curvature singularity develops.

The uniform bound on  $V_n$  which we obtain in Proposition 3.11 implies a lower bound on the time to self-intersection which depends only upon the admissible interface parameter  $K$ . Since the distance to self-intersection is  $O(1)$  in the unscaled variables, the time to self-intersection scales like  $\varepsilon^{-4}$ .

From the large eigenvalue asymptotics of the Laplace-Beltrami operator (see Remark 3.4), for  $d = 3$ , we know that the gaps in the spectrum are generically  $O(1)$ , and moreover one generically expects the eigenvalues to maintain this separation. However in degenerate cases, the spectral gap may collapse (in squares for example). However we may address this collapse.

From the estimate (3.3.9) we see that the norm of the remainder,  $\|w\|_{\mathcal{A}}$ , becomes less than  $\bar{B}_0$  on an  $O(1)$  time scale, while from Proposition 5.6 we see that  $\beta_{M+1}$  can become close to  $\beta_M$  only on an  $O(\varepsilon^{-1})$  time-scale. However, if the gap between  $\beta_M$  and  $\beta_{M+1}$  closes at a time  $t_1$ , then since  $u(t_1)$  will be sufficiently close to the manifold  $\mathcal{M}_K$  we may re-decompose  $u(t_1)$  according to (3.3.1) with the slow manifold dimension  $M = [M_1 \varepsilon^{-(d-1)}]$  taking a slightly different value, corresponding to an adjustment of  $M_1$ , for which there is a gap of at least size  $c_g$  between  $\beta_M$  and  $\beta_{M+1}$ . In this manner we renormalize through any possible gap-collapse in the Laplace-Beltrami spectrum, by dynamically adjusting the dimension of the slow-manifold.

Regarding the development of curvature singularities in finite-time, we first observe that the dominant,  $-H^5$ , term in the Kuramoto-Sivashinsky type equation (3.3.17) is a favorable term, and one does not anticipate finite-time singularities. Indeed the Kuramoto-Sivashinsky equation is globally well-posed [34]. However a full analysis of this problem, in either dimension 2 or 3, is beyond the scope of this thesis. We thus make Assumption 3.6, that the higher-order Ricci curvature flow leads to uniform bounds on the  $H^4(\mathbf{S})$  norm of the fundamental forms.

### 3.3.3 Interface Evolution

In the remainder of Section 3.3 we give an outline of the proof of Theorem 3.7. We recall that the dynamics are governed by the FCH gradient flow

$$\begin{aligned}\partial_t u(x, t) &= -\Pi_0 F(u), \\ u(x, 0) &= u_0(x),\end{aligned}\tag{3.3.18}$$

where  $F$  and the zero-mass projection  $\Pi_0$  are defined in (1.0.6) and (1.0.7) respectively.

By the assumption of Theorem 3.7 there exists  $\Gamma_0 \in \mathcal{G}_{K_0}$  and  $b_0 \in (-\bar{b}_0\varepsilon^3, \bar{b}_0\varepsilon^3)$  such that

$$\|u_0 - \Phi(\cdot, \Gamma_0, b_0)\|_{\mathcal{A}} < \bar{B}_0\varepsilon^{d-3/2}. \quad (3.3.19)$$

We show in Chapter 6 that for  $\varepsilon$  sufficiently small we may represent  $u_0$  as a sum of an element  $\Phi \in \mathcal{M}_{K_0}$  and a perturbation belonging to  $Z_{M, \Gamma_0}^\perp$ , defined in Definition 3.3, which approximates the ‘fast’ eigenspaces of the linearization  $\mathcal{L}_\phi$  about  $\Phi(x; \Gamma_0, b_0)$ . This decomposition allows us to separate the evolution into its projections onto the ‘slow’ and ‘fast’ eigenspaces.

We represent a point in the ‘whiskered neighborhood’  $\Gamma(2\ell)$  of  $\Gamma_t$  as  $(z(x; t), s(x; t))$ . Differentiating the ansatz (3.1.4) with respect to  $t$ , the chain rule yields

$$\partial_t \Phi(x; \Gamma_t; b) = \partial_z \Phi(z(x; t), s(x; t)) \frac{\partial z(x; t)}{\partial t} + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} + \partial_t b. \quad (3.3.20)$$

We simplify the time derivatives of the whiskered coordinates with the aid of the following proposition.

**Proposition 3.10.** *Let  $(z, s)$  be the whiskered coordinates of a point  $x \in \Gamma_t(2\ell)$  with respect to interface  $\Gamma_t$ . Then, the following relation holds for the derivative of  $z$  with respect to time,*

$$\frac{\partial z(x; t)}{\partial t} = -\frac{1}{\varepsilon} V_n(s(x; t)). \quad (3.3.21)$$

**Proof:** Recall the evolution equation

$$\frac{\partial \gamma(s, t)}{\partial t} = V_n(s, t) \cdot \nu(s, t), \quad (3.3.22)$$

given in (3.3.2). Differentiating  $\nu(s, t) \cdot \nu(s, t) = 1$  with respect to  $t$  and  $s_i$  ( $i = 1, \dots, d-1$ ) gives

$$\frac{\partial \nu(s, t)}{\partial t} \cdot \nu(s, t) = 0, \quad \frac{\partial \nu(s, t)}{\partial s_i} \cdot \nu(s, t) = 0. \quad (3.3.23)$$

Differentiating

$$z(x; t) = \frac{1}{\varepsilon} \left( x - \gamma(s(x; t), t) \right) \cdot \nu(s(x; t), t). \quad (3.3.24)$$

with respect to  $t$  and using  $\frac{\partial \gamma}{\partial s_i} \cdot \nu = 0$ , we obtain

$$\begin{aligned} \frac{\partial z(x; t)}{\partial t} &= \frac{1}{\varepsilon} \left( -D_s \gamma \cdot \frac{\partial s(x; t)}{\partial t} - V_n \nu \right) \cdot \nu + \frac{1}{\varepsilon} \left( x - \gamma(s(x; t), t) \right) \cdot \frac{d}{dt} \nu(s(x; t), t) \\ &= -\frac{1}{\varepsilon} V_n + \frac{1}{\varepsilon} \left( x - \gamma(s(x; t), t) \right) \cdot \frac{d}{dt} \nu(s(x; t), t). \end{aligned} \quad (3.3.25)$$

Since  $\frac{1}{\varepsilon} (x - \gamma(s(x), t)) = z\nu$ , (3.3.23) implies that the last term in (3.3.25) is zero.  $\square$

Substituting the decomposition (3.3.1) into (2.0.1), suppressing the dependence of coordinates on  $x$ , and using (3.3.20) and Proposition 3.10, we obtain,

$$\begin{aligned} -\frac{1}{\varepsilon} V_n(s, t) \partial_z \Phi + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} + \partial_t b + \partial_t w &= -\Pi_0 F(\Phi + w), \\ &= -\Pi_0 F(\Phi) - \Pi_0 \mathcal{L}_\phi w - \Pi_0 \mathcal{N}(w), \end{aligned} \quad (3.3.26)$$

where

$$\mathcal{N}(w) = F(\Phi + w) - F(\Phi) - \mathcal{L}_\phi w, \quad (3.3.27)$$

represents the nonlinear terms in  $w$ .

To obtain an evolution equation for  $\Gamma_t$ , we project (3.3.26) onto the ‘slow space’  $Z_{M, \Gamma_t}$

(see section 3.2). In particular taking the  $L^2(\Omega)$  inner product of (3.3.26) with  $Z_i = \eta_1 \phi' \Theta_{i,\Gamma_t}$  and changing to whiskered variables we obtain

$$\begin{aligned} & \underbrace{-\frac{1}{\varepsilon} \int_{\Gamma} V_n \Theta_{i,\Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 (\partial_z \Phi) \phi' J dz ds}_I = - \underbrace{(\Pi_0 F(\Phi), Z_i)_{L^2(\Omega)} - \partial_t b \int_{\Omega} Z_i dx}_{II} \\ & - (\Pi_0 \mathcal{L}_{\phi} w, Z_i)_{L^2(\Omega)} - (\Pi_0 \mathcal{N}(w), Z_i)_{L^2(\Omega)} - \int_{\Omega} \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} Z_i dx + (\partial_t w, Z_i)_{L^2(\Omega)}. \end{aligned} \quad (3.3.28)$$

We look for  $V_n$  of the form

$$V_n = \sum_{i=1}^M v_i \Theta_{i,\Gamma_t} \quad (3.3.29)$$

and substitute (3.3.29) into (3.3.28). We remark, that we can define the Euclidean norm,

$$|v|^2 = \sum_{i=1}^M v_i^2. \quad (3.3.30)$$

for  $v = (v_1, \dots, v_M)$  and it follows from orthonormality of eigenfunctions  $\Theta_{i,\Gamma_t}$  that

$$\|V_n\|_{L^2(\Gamma)} = |v|. \quad (3.3.31)$$

We show in Chapter 9 that to the leading-order  $I$  corresponds to the projection

$v_i = \left( V_n, \Theta_{i,\Gamma_t} \right)_{L^2(\Gamma)}$  of the normal velocity  $V_n$  onto the  $i$ th Laplace-Beltrami mode  $\Theta_{i,\Gamma_t}$ . Similarly  $II$  corresponds to the projection  $\varepsilon^4 v_i^0 := \varepsilon^4 \left( V_n^0, \Theta_{i,\Gamma_t} \right)_{L^2(\Gamma)}$  of  $\varepsilon^4 V_n^0$  onto  $\Theta_{i,\Gamma_t}$ .

Obtaining bounds on the remaining terms in (3.3.28) and thus showing that  $I$  and  $II$  are indeed leading-order is somewhat technical and relies on the estimates derived in the following

chapters. For this reason we state the result below and defer the proof until Chapter 9 when all of the required estimates have been established.

**Proposition 3.11.** *Assuming that (3.3.28) holds for each  $i = 1, \dots, M$ , and that  $w$  satisfies*

$$\|w\|_{\mathcal{A}} \leq M_b \varepsilon^{d-3/2}, \quad (3.3.32)$$

*for some constant  $M_b$  which depends only on the admissible interface parameter  $K$ , then*

$$\left| v_i - \varepsilon^4 v_i^0 \right| \leq C(K) \left( \varepsilon^5 + \varepsilon^{7/2} \|w\|_{\mathcal{A}} \right), \quad (3.3.33)$$

$$\|V_n\|_{L^\infty(\Gamma)} \leq M_1 C(K) \left( \varepsilon^{5-(d-1)} + \varepsilon^{7/2-(d-1)} \|w\|_{\mathcal{A}} \right), \quad (3.3.34)$$

$$\left\| V_n - \varepsilon^4 V_n^0 \right\|_{L^2(\Gamma)} \leq C(K) \sqrt{M_1} \left( \varepsilon^{(11-d)/2} + \varepsilon^{(8-d)/2} \|w\|_{\mathcal{A}} \right), \quad (3.3.35)$$

where

$$V_n^0 = - \left[ \left( \Delta_s + \eta \right) H - \frac{H^3}{2} + H \operatorname{tr}(A^2) \right] - \frac{2}{\|\phi'\|_{L^2(\mathbb{R})}^2} \left( \mu_0^2 b_3 - \partial_t b_3 \right). \quad (3.3.36)$$

and  $M_1$  is the slow-space dimension parameter. In particular,

$$\|V_n\|_{L^2(\Gamma)} \leq C(K) \sqrt{M_1} \left( \varepsilon^{(11-d)/2} + \varepsilon^{(8-d)/2} \|w\|_{\mathcal{A}} \right). \quad (3.3.37)$$

Our objective in this section is to obtain an energy inequality for  $w$ , which we will use

to show that  $\|w\|_{\mathcal{A}}$  decays to  $O(\varepsilon^{3/2})$ . This, in turn, implies that the right-hand side of (3.3.35) is  $C(K)\sqrt{M_1}\varepsilon^{(11-d)/2}$  and  $V_n$  is given at leading-order by  $\varepsilon^4 V_n^0$ . In the following proposition we use  $C(K)$  to denote a constant that depends only on the admissible interface parameter  $K$ .

**Proposition 3.12.** *Under the assumptions of Proposition 3.11, there exist constants  $\frac{1}{2}\bar{B}_0 > \bar{B} > 0$ , such that the perturbation  $w$  satisfied the following estimate in the  $\|\cdot\|_{\mathcal{A}}$  norm,*

$$\|w\|_{\mathcal{A}} \leq \bar{B}_0 \varepsilon^{d-3/2} e^{-\frac{U}{2}t} + \bar{B} \varepsilon^{3/2}. \quad (3.3.38)$$

**Proof:** We first rewrite equation (3.3.26) in the following form

$$\partial_t w = -\Pi_0 \mathcal{L}_\phi w - \Pi_0 \mathcal{N}w + R, \quad (3.3.39)$$

where

$$R := -\partial_t \Phi - \Pi_0 F(\Phi). \quad (3.3.40)$$

Recall the definition of  $\mathcal{A}_\phi$  in (3.2.5). The idea is to use the coercivity of the operator  $\mathcal{L}_\phi$  to show that  $\|w\|_{\mathcal{A}}$  (and due to Theorem 3.5 also  $\varepsilon^{-2}\|\mathcal{A}_\phi w\|_{L^2(\Omega)}$ ) decay to  $O(\varepsilon^{3/2})$ . Similarly to the previous section we defer the more technical estimates to Chapter 9.

Multiplying both sides of (3.3.39) by  $\mathcal{A}_\phi^2 w$  and writing out  $\mathcal{L}_\phi = \mathcal{A}_\phi^2 + \varepsilon \tilde{q}$  from (3.2.4), we

obtain

$$\begin{aligned}
(\partial_t w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} &= -(\mathcal{A}_\phi^2 w + \varepsilon \tilde{q} w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + \left( \langle \mathcal{L}_\phi w \rangle, \mathcal{A}_\phi^2 w \right)_{L^2(\Omega)} \\
&\quad - (\Pi_0 \mathcal{N}(w), \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + (R, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} \\
&= -(\mathcal{A}_\phi^2 w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} - \varepsilon (\tilde{q} w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + |\Omega| \langle \mathcal{L}_\phi w \rangle \langle \mathcal{A}_\phi^2 w \rangle \\
&\quad - (\Pi_0 \mathcal{N}(w), \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + (R, \mathcal{A}_\phi^2 w)_{L^2(\Omega)}. \tag{3.3.41}
\end{aligned}$$

Integrating by parts on the left-hand side we obtain

$$\begin{aligned}
\frac{1}{2} \partial_t \left\| \mathcal{A}_\phi w \right\|_{L^2(\Omega)}^2 &= -(\mathcal{A}_\phi^2 w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} - \varepsilon (\tilde{q} w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + |\Omega| \langle \mathcal{L}_\phi w \rangle \langle \mathcal{A}_\phi^2 w \rangle \\
&\quad - (\Pi_0 \mathcal{N}(w), \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + (R, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} + \left( \left[ \partial_t, \mathcal{A}_\phi \right] w, \mathcal{A}_\phi w \right)_{L^2(\Omega)}. \tag{3.3.42}
\end{aligned}$$

We show in Lemma 9.13 that

$$\left| \left( \left[ \partial_t, \mathcal{A}_\phi \right] w, \mathcal{A}_\phi w \right)_{L^2(\Omega)} \right| \leq C(K) \varepsilon^{1/2} \left\| \mathcal{A}_\phi w \right\|_{L^2(\Omega)}^2. \tag{3.3.43}$$

Using coercivity estimates from Theorem 3.5 and  $\beta_{M+1} \sim \varepsilon^{-2}$ ,

$$\left| \left( \left[ \partial_t, \mathcal{A}_\phi \right] w, \mathcal{A}_\phi w \right)_{L^2(\Omega)} \right| \leq C(K) \varepsilon^{1/2} \left\| \mathcal{A}_\phi^2 w \right\|_{L^2(\Omega)}^2 \leq \frac{1}{16} \left\| \mathcal{A}_\phi^2 w \right\|_{L^2(\Omega)}^2 \tag{3.3.44}$$

for  $\varepsilon$  sufficiently small. Similarly, since  $\|\tilde{q}\|_{L^\infty(\Omega)} = O(1)$  and  $\|w\|_{L^2(\Omega)} \leq U_1 \left\| \mathcal{A}_\phi^2 w \right\|_{L^2(\Omega)}$

we have

$$\begin{aligned}
\varepsilon \left| (\tilde{q}w, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} \right| &\leq \varepsilon \|\tilde{q}\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)} \\
&\leq C(K)\varepsilon \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2 \leq \frac{1}{16} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.3.45}$$

By Young's Inequality

$$\begin{aligned}
\left| (\Pi_0 \mathcal{N}(w), \mathcal{A}_\phi^2 w)_{L^2(\Omega)} \right| &\leq 2 \|\Pi_0 \mathcal{N}(w)\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2 \\
&\leq 2 \|\mathcal{N}(w)\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2
\end{aligned} \tag{3.3.46}$$

and

$$\left| (R, \mathcal{A}_\phi^2 w)_{L^2(\Omega)} \right| \leq 2 \|R\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2. \tag{3.3.47}$$

In addition, for  $\varepsilon$  sufficiently small, the following estimate holds and is shown in Proposition 9.4

$$\left| \langle \mathcal{L}_\phi w \rangle \langle \mathcal{A}_\phi^2 w \rangle \right| \leq C(K)\varepsilon \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2 \leq \frac{1}{8} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2. \tag{3.3.48}$$

Substituting (3.3.44)-(3.3.48) into (3.3.42), we obtain

$$\frac{1}{2} \partial_t \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 \leq -\frac{1}{2} \|\mathcal{A}_\phi^2 w\|_{L^2(\Omega)}^2 + 2 \|\mathcal{N}w\|_{L^2(\Omega)}^2 + 2 \|R\|_{L^2(\Omega)}^2. \tag{3.3.49}$$

Coercivity estimate on  $\mathcal{A}_\phi^2 w$  from Theorem 3.5 and  $\beta_{M+1} \sim \varepsilon^{-2}$  imply

$$\partial_t \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 \leq -U_1 \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + 4 \|\mathcal{N}w\|_{L^2(\Omega)}^2 + 4 \|R\|_{L^2(\Omega)}^2, \tag{3.3.50}$$

where  $U_1$  is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ . In addition, we can bound  $R = -\partial_t \Phi - \Pi_0 F(\Phi)$  using Lemma 9.12 and Corollary 3.2,

$$\|R\|_{L^2(\Omega)} = \|-\partial_t \Phi - \Pi_0 F(\Phi)\|_{L^2(\Omega)} \leq C(K) \left( \varepsilon^{-1/2} \|V_n\|_{L^2(\Gamma)} + \varepsilon^{7/2} \right) \quad (3.3.51)$$

and the nonlinear terms using Lemma 9.2

$$\|\mathcal{N}(w)\|_{L^2(\Omega)} \leq C(K) \left( \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + \varepsilon^{-4} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^3 \right), \quad (3.3.52)$$

to obtain

$$\begin{aligned} \partial_t \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 &\leq -U_1 \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + C_n(K) \left( \varepsilon^{-4} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^4 + \varepsilon^{-8} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^6 \right. \\ &\quad \left. + \varepsilon^{-1} \|V_n\|_{L^2(\Gamma)}^2 + \varepsilon^7 \right), \end{aligned} \quad (3.3.53)$$

where  $C_n(K)$  is a positive constant that depends only on  $K$ . From (3.2.20) and (3.3.32)

$$\|\mathcal{A}_\phi w\|_{L^2(\Omega)} \leq C(K) \varepsilon^2 \|w\|_{\mathcal{A}} \leq M_b C(K) \varepsilon^{d+1/2}, \quad (3.3.54)$$

and it follows that for  $\varepsilon$  sufficiently small

$$C_n(K) \left( \varepsilon^{-4} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^4 + \varepsilon^{-8} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^6 \right) \leq \frac{U_1}{2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2, \quad (3.3.55)$$

and from (3.3.53) we obtain

$$\partial_t \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 \leq -\frac{U_1}{2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + C_n(K) \left( \varepsilon^{-1} \|V_n\|_{L^2(\Gamma)}^2 + \varepsilon^7 \right). \quad (3.3.56)$$

From Proposition 3.11

$$\|V_n\|_{L^2(\Gamma)} \leq C(K) \sqrt{M_1} \left( \varepsilon^{(11-d)/2} + \varepsilon^{(4-d)/2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)} \right), \quad (3.3.57)$$

where we used the equivalence of  $\varepsilon^2 \|w\|_{\mathcal{A}}$  and  $\|\mathcal{A}_\phi w\|_{L^2(\Omega)}$  from Theorem 3.5, and returning this estimate to (3.3.56) gives for  $\varepsilon$  sufficiently small

$$\partial_t \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 \leq -\frac{U_1}{4} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + C(K) \varepsilon^7, \quad (3.3.58)$$

Gronwall's Lemma now implies

$$\|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 \leq \|\mathcal{A}_\phi w_0\|_{L^2(\Omega)}^2 e^{-Ut} + C(K) \varepsilon^7, \quad (3.3.59)$$

where  $U := \frac{U_1}{4}$ .

Using equivalence of  $\varepsilon^2 \|w\|_{\mathcal{A}}$  and  $\|\mathcal{A}_\phi w\|_{L^2(\Omega)}$  from Theorem 3.5 as well as the initial condition (3.3.19) we obtain from (3.3.59) the energy inequality for  $\|w\|_{\mathcal{A}}$

$$\|w\|_{\mathcal{A}} \leq \bar{B}_0 \varepsilon^{d-3/2} e^{-\frac{U}{2}t} + C(K) \varepsilon^{3/2}. \quad (3.3.60)$$

□

By the assumption of Theorem 3.7 there exists  $\Gamma_0 \in \mathcal{G}_{K_0}$  and  $b_0 \in (-\bar{b}_0\varepsilon^3, \bar{b}_0\varepsilon^3)$  such that

$$\|w_0\|_{\mathcal{A}} \leq \bar{B}_0\varepsilon^{d-3/2}. \quad (3.3.61)$$

It follows from the decay inequality on  $\|w\|_{\mathcal{A}}$  that (3.3.32) is always satisfied and (3.3.9) follows.

### 3.3.4 Background Evolution

**Proposition 3.13.** *The leading order evolution of the background state  $b = \varepsilon^3 b_3$  is given by*

$$\partial_t b_3 + \frac{4\varepsilon\mu_0^2|\Gamma|}{\|\phi'\|_{L^2(\mathbb{R})}^2|\Omega|} b_3 = -\frac{2\varepsilon}{|\Omega|} \int_{\Gamma} \left[ (\Delta_s + \eta)H - \frac{H^3}{2} + H \text{tr}(A^2) \right] ds. \quad (3.3.62)$$

**Proof:** The evolution of the background state is determined from the total mass conservation.

Projecting (3.3.26) onto  $Z_0 = 1$ , the right-hand side is identically zero and we obtain

$$\int_{\Omega} -\frac{1}{\varepsilon} V_n(s, t) \Phi_z + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} + \partial_t b + \partial_t w dx = 0. \quad (3.3.63)$$

It also follows from  $w \in \mathcal{Z}_M^\perp$  that  $\int_{\Omega} \partial_t w dx = \partial_t \int_{\Omega} w dx = 0$ . Replacing  $V_n$  with its leading-order term  $\varepsilon^4 V_n^0$  and using Corollary 9.6 to bound  $\frac{\partial s}{\partial t}$ , as well as using  $[\|\phi\|] = 2$  we obtain

$$\begin{aligned} \partial_t b &= \frac{\varepsilon^3}{|\Omega|} \int_{\Omega} V_n^0 \Phi_z dx + O(\varepsilon \|V_n\|_{L^2(\Gamma)}) = -\frac{\varepsilon^3}{|\Omega|} \int_{\Gamma} V_n^0 \int_{-\infty}^{\infty} \phi' J dz ds + O(\varepsilon \|V_n\|_{L^2(\Gamma)}) \\ &= \frac{\varepsilon^4}{|\Omega|} \int_{\Gamma} V_n^0 \left( \int_{-\infty}^{\infty} \phi' dz \right) J^0 ds + O(\varepsilon \|V_n\|_{L^2(\Gamma)}) = \frac{2\varepsilon^4}{|\Omega|} \int_{\Gamma} V_n^0 ds + O(\varepsilon \|V_n\|_{L^2(\Gamma)}). \end{aligned} \quad (3.3.64)$$

Substituting the normal velocity, (3.3.36), into this expression and dropping the lower order terms yields an ordinary differential equation for  $b_3$ ,

$$\partial_t b_3 + \frac{4\varepsilon\mu_0^2|\Gamma|}{\|\phi'\|_{L^2(\mathbb{R})}^2|\Omega|} b_3 = -\frac{2\varepsilon}{|\Omega|} \int_{\Gamma} \left[ (\Delta_s + \eta)H - \frac{H^3}{2} + H\text{tr}(A^2) \right] ds. \quad (3.3.65)$$

□

We note that the equilibrium value of  $b_3$  is given by

$$b_3 = -\frac{\|\phi'\|_{L^2(\mathbb{R})}^2}{2\mu_0^2|\Gamma|} \int_{\Gamma} \left[ (\Delta_s + \eta)H - \frac{H^3}{2} + H\text{tr}(A^2) \right] ds. \quad (3.3.66)$$

Substituting the expression for  $b_3$  into the equation for the normal velocity (3.3.36) we obtain (3.3.13).

### 3.4 Evolution in $\mathbb{R}^2$

**Proof of Corollary 3.8:** In  $\mathbb{R}^2$ , the fundamental forms  $g$  and  $h$  become scalar functions on a one-dimensional manifold  $\Gamma$ . We have

$$g = |\partial_s \gamma|^2, \quad (3.4.1)$$

$$h = \langle \nu, \partial_s^2 \gamma \rangle, \quad (3.4.2)$$

$$H = k = h/g = \frac{h}{|\partial_s \gamma|^2}, \quad (3.4.3)$$

and the leading-order evolution equations take the following form

$$\partial_\tau |\partial_s \gamma|^2 = -2V_n^0 H |\partial_s \gamma|^2, \quad (3.4.4)$$

$$\partial_\tau H = \left( \frac{1}{|\partial_s \gamma|^2} \partial_s^2 - \frac{1}{|\partial_s \gamma|^2} \langle \partial_s \gamma, \partial_s^2 \gamma \rangle \partial_s + H^2 \right) V_n^0. \quad (3.4.5)$$

To obtain (3.3.17) we choose an arc-length parametrization

$$s_1 = \int_0^s \sqrt{g(\xi, t)} d\xi. \quad (3.4.6)$$

Then  $g(s_1) \equiv 1$ . Let  $H_1(s_1, t)$  be the mean curvature in new variable  $s_1$ ,

$$H(s, t) = H_1(s_1, t). \quad (3.4.7)$$

Then

$$\frac{\partial H}{\partial t} = \frac{\partial H_1}{\partial t} + \frac{\partial H_1}{\partial s_1} \frac{\partial s_1}{\partial t}. \quad (3.4.8)$$

and

$$\frac{\partial s_1}{\partial t} = \int_0^s \frac{\partial}{\partial t} (\sqrt{g(\xi, t)}) d\xi = - \int_0^s V_n(\xi, t) H(\xi, t) \sqrt{g(\xi, t)} d\xi, \quad (3.4.9)$$

where we used  $\frac{\partial}{\partial t} \sqrt{\det(g)} = -V_n H \sqrt{\det(g)}$  (see Proposition 4.6).

Introducing a new variable

$$\rho = \int_0^\xi \sqrt{g(\xi_1, t)} d\xi_1, \quad (3.4.10)$$

we have

$$\frac{\partial s_1}{\partial t} = - \int_0^{s_1} V_n(\rho, t) H(\rho, t) d\rho. \quad (3.4.11)$$

Therefore, rewriting (3.4.5) in terms of the new variable  $s_1$  and employing  $g(s_1) = 1$ , we obtain from (3.4.8) and (3.4.11), that

$$\frac{\partial H_1}{\partial t} - \frac{\partial H_1}{\partial s_1} \int_0^{s_1} V_n(\rho, t) H(\rho, t) d\eta = (\partial_{s_1}^2 + H^2) V_n^0. \quad (3.4.12)$$

□

# Chapter 4

## Evolution by Normal Velocity

In this chapter we expand on key differential geometry concepts that relate to interface evolution. In particular we derive equations (3.3.14)-(3.3.16) as well as evolution equations for the principal curvatures of the interface. These relations are key to the analysis of the spectrum of Laplace-Beltrami operator in the following chapter.

### 4.1 Evolution Equations for the Fundamental Forms

We consider a family of interfaces  $\{\Gamma_t : t \in [0, T]\}$  evolving by normal velocity  $V_n(s, t)$  according to the evolution equation given by

$$\begin{aligned}\partial_t \gamma(s, t) &= V_n(s, t) \cdot \nu(s, t) \\ \gamma(s, 0) &= \gamma_0(s),\end{aligned}\tag{4.1.1}$$

where  $\gamma(s, t)$  gives a local parametrization of  $\Gamma_t$ .

**Proposition 4.1.** (See [23], [25]) *The following equations hold:*

i) *Weingarten Equation*

$$\frac{\partial \nu}{\partial s_i} = - \sum_{k=1}^{d-1} h_i^k \frac{\partial \gamma}{\partial s_k}, \quad (4.1.2)$$

ii) *Gauss Formula*

$$\frac{\partial^2 \gamma}{\partial s_i \partial s_j} = \sum_{k=1}^{d-1} \Gamma_{ij}^k \frac{\partial \gamma}{\partial s_k} + h_{ij} \cdot \nu, \quad (4.1.3)$$

where  $\Gamma_{ij}^k$  is the Christoffel symbol

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{d-1} \left( \frac{\partial}{\partial s_i} g_{jl} + \frac{\partial}{\partial s_j} g_{il} - \frac{\partial}{\partial s_l} g_{ij} \right). \quad (4.1.4)$$

**Proposition 4.2.** *Under the evolution equation given in (4.1.1) the first fundamental form satisfies*

$$\frac{\partial}{\partial t} g_{ij} = -2V_n h_{ij}. \quad (4.1.5)$$

**Proof:** Differentiating expression (2.1.8) for the first fundamental form, we obtain

$$\frac{\partial}{\partial t} g_{ij} = \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle = \left\langle \frac{\partial^2 \gamma}{\partial t \partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle + \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial^2 \gamma}{\partial t \partial s_j} \right\rangle. \quad (4.1.6)$$

Using evolution equation (4.1.1) to replace  $\frac{\partial \gamma}{\partial t}$  yields

$$\frac{\partial}{\partial t} g_{ij} = \left\langle \frac{\partial(V_n \cdot \nu)}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle + \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial(V_n \cdot \nu)}{\partial s_j} \right\rangle. \quad (4.1.7)$$

When the derivatives in the right-hand side of (4.1.7) fall on  $V_n$  the resulting inner product

is zero since  $\nu$  is orthogonal to the tangent space and we conclude

$$\frac{\partial}{\partial t}g_{ij} = V_n \left( \left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle + \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \nu}{\partial s_j} \right\rangle \right) = -2V_n h_{ij}, \quad (4.1.8)$$

where we used the definition of the second fundamental form in (2.1.11).  $\square$

**Proposition 4.3.** *Under the evolution equation given in (4.1.1) the inverse of the first fundamental form satisfies*

$$\frac{\partial}{\partial t}g^{ij} = 2V_n \sum_{k=1}^{d-1} h_k^i g^{kj}. \quad (4.1.9)$$

**Proof:** Differentiating

$$\sum_{k=1}^{d-1} g^{ik} g_{kj} = \delta_{ij}, \quad (4.1.10)$$

we obtain

$$\sum_{k=1}^{d-1} \left( (\partial_t g^{ik}) g_{kj} + g^{ik} \partial_t (g_{kj}) \right) = 0. \quad (4.1.11)$$

Using Proposition 4.2 we substitute the expression for  $\partial_t(g_{kj})$  to obtain

$$\sum_{k=1}^{d-1} \left( (\partial_t g^{ik}) g_{kj} - 2V_n g^{ik} h_{kj} \right) = 0, \quad (4.1.12)$$

and

$$\sum_{k=1}^{d-1} (\partial_t g^{ik}) g_{kj} = 2V_n h_{kj}^i, \quad (4.1.13)$$

where we used (2.1.16) to raise the index on  $h_{kj}$ .  $\square$

**Proposition 4.4.** *Under the evolution equation given in (4.1.1) the unit normal vector  $\nu$*

satisfies

$$\frac{\partial}{\partial t}\nu = - \sum_{i,j=1}^{d-1} g^{ij} \frac{\partial V_n}{\partial s_i} \frac{\partial \gamma}{\partial s_j}. \quad (4.1.14)$$

**Proof:** Differentiating the relation  $\langle \nu, \nu \rangle = 1$  with respect to  $t$  implies that  $\frac{\partial \nu}{\partial t} \in T_{\gamma(s)}\Gamma$  and we may express  $\frac{\partial \nu}{\partial t}$  in terms of the tangent basis as

$$\frac{\partial \nu}{\partial t} = \sum_{j=1}^{d-1} a_j \frac{\partial \gamma}{\partial s_j}. \quad (4.1.15)$$

To find the coefficients  $a_j$  we take the inner product of (4.1.15) with  $\frac{\partial \gamma}{\partial s_i}$  to obtain

$$\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial \gamma}{\partial s_i} \right\rangle = \sum_{j=1}^{d-1} a_j g_{ji}. \quad (4.1.16)$$

and

$$a_j = \sum_{i=1}^{d-1} \left\langle \frac{\partial \nu}{\partial t}, \frac{\partial \gamma}{\partial s_i} \right\rangle g^{ij}. \quad (4.1.17)$$

Since  $\left\langle \nu, \frac{\partial \gamma}{\partial s_i} \right\rangle = 0$

$$a_j = - \sum_{i=1}^{d-1} \left\langle \nu, \frac{\partial^2 \gamma}{\partial t \partial s_i} \right\rangle g^{ij} = - \sum_{i=1}^{d-1} \left\langle \nu, \frac{\partial (V_n \cdot \nu)}{\partial s_i} \right\rangle g^{ij} = - \sum_{i=1}^{d-1} \frac{\partial V_n}{\partial s_i} g^{ij}. \quad (4.1.18)$$

□

**Proposition 4.5.** *Under the evolution equation given in (4.1.1) the second fundamental form*

satisfies

$$\frac{\partial}{\partial t} h_{ij} = \frac{\partial^2 V_n}{\partial s_i \partial s_j} - \sum_{k=1}^{d-1} \Gamma_{ij}^k \frac{\partial V_n}{\partial s_k} - e_{ij} V_n. \quad (4.1.19)$$

**Proof:** Differentiating expression (2.1.11) for the second fundamental form and using the evolution equation (4.1.1) to replace  $\frac{\partial \gamma}{\partial t}$  and Proposition 4.4 to replace  $\frac{\partial \nu}{\partial t}$  we obtain

$$\frac{\partial}{\partial t} h_{ij} = \frac{\partial}{\partial t} \left\langle \frac{\partial^2 \gamma}{\partial s_i \partial s_j} \cdot \nu \right\rangle = \frac{\partial^2}{\partial s_i \partial s_j} (V_n \cdot \nu) \cdot \nu - \sum_{k,l=1}^{d-1} \frac{\partial^2 \gamma}{\partial s_i \partial s_j} g^{kl} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_l}. \quad (4.1.20)$$

Using  $\left\langle \nu, \frac{\partial \gamma}{\partial s_i} \right\rangle = 0$  we distribute the derivatives in the first term on the right-hand side of (4.1.20) to conclude

$$\frac{\partial}{\partial t} h_{ij} = \frac{\partial^2 V_n}{\partial s_i \partial s_j} + V_n \left\langle \frac{\partial^2 \nu}{\partial s_i \partial s_j}, \nu \right\rangle - \sum_{k,l=1}^{d-1} \frac{\partial^2 \gamma}{\partial s_i \partial s_j} g^{kl} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_l}, \quad (4.1.21)$$

and

$$\frac{\partial}{\partial t} h_{ij} = \frac{\partial^2 V_n}{\partial s_i \partial s_j} - e_{ij} V_n - \sum_{k,l=1}^{d-1} \frac{\partial^2 \gamma}{\partial s_i \partial s_j} g^{kl} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_l}, \quad (4.1.22)$$

where we used the expression for the third fundamental form  $e_{ij}$  in (2.1.14). To simplify the second term on the right-hand side of (4.1.22) we use the Gauss Formula of Proposition 4.1 to obtain

$$\begin{aligned} \sum_{k,l=1}^{d-1} \frac{\partial^2 \gamma}{\partial s_i \partial s_j} g^{kl} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_l} &= \sum_{k,l,m=1}^{d-1} \Gamma_{ij}^m \frac{\partial \gamma}{\partial s_m} g^{kl} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_l} = \sum_{k,l,m=1}^{d-1} \Gamma_{ij}^m \frac{\partial V_n}{\partial s_k} g^{kl} g_{ml} \\ &= \sum_{k,l=1}^{d-1} \Gamma_{ij}^k \frac{\partial V_n}{\partial s_k}. \end{aligned} \quad (4.1.23)$$

□

**Proposition 4.6.** *Under the evolution equation given in (4.1.1) the metric  $\sqrt{\det(g)} = \sqrt{\det(\{g_{ij}\})}$  satisfies*

$$\frac{\partial}{\partial t} \sqrt{\det(g)} = -V_n H \sqrt{\det(g)}. \quad (4.1.24)$$

**Proof:** To evaluate the time derivative of  $\det(g)$  we utilize the Leibniz formula for determinant of  $g$

$$\det(g) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{d-1} g_{i, \sigma_i}, \quad (4.1.25)$$

where the sum is computed over all permutations  $\sigma$  of the set  $\{1, 2, \dots, d-1\}$ . Differentiating (4.1.25) with respect to  $t$  and using the evolution equation for  $g_{ij}$  given in (4.2) we obtain

$$\frac{\partial}{\partial t} \det(g) = -2V_n \sum_{\sigma \in S_n} \sum_{i=1}^{d-1} \text{sgn}(\sigma) h_{i, \sigma_i} \prod_{j=1, j \neq i}^{d-1} g_{j, \sigma_j}. \quad (4.1.26)$$

Using (2.1.16) we write the above expression in terms of the Weingarten matrix  $\{h_i^k\}$

$$\frac{\partial}{\partial t} \det(g) = -2V_n \sum_{\sigma \in S_n} \sum_{i, k=1}^{d-1} \text{sgn}(\sigma) h_i^k g_{k, \sigma_i} \prod_{j=1, j \neq i}^{d-1} g_{j, \sigma_j} \quad (4.1.27)$$

For  $k \neq i$ , the expression  $\sum_{\sigma \in S_n} g_{k, \sigma_i} \prod_{j=1, j \neq i}^{d-1} g_{j, \sigma_j}$  is equal to the determinant of a

matrix with two identical rows and is identically 0. The remaining terms give

$$\begin{aligned}
\frac{\partial}{\partial t} \det(g) &= -2V_n \sum_{\sigma \in S_n} \sum_{i=1}^{d-1} \operatorname{sgn}(\sigma) h_i^i g_{i, \sigma_i} \prod_{j=1, j \neq i}^{d-1} g_{j, \sigma_j} \\
&= -2V_n \sum_{i=1}^{d-1} h_i^i \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{d-1} g_{j, \sigma_j} = -2V_n H \det(g), \quad (4.1.28)
\end{aligned}$$

where we used the definition of  $H$  as the trace of the Weingarten matrix  $\{h_i^j\}$ . Relation (4.1.24) follows easily.  $\square$

## 4.2 Evolution of Principal Curvatures

Recall that the principal curvatures  $k$  of  $\Gamma$  are defined to be the eigenvalues of the Weingarten map  $\{h_i^j\}$ . Setting  $g = \{g_{ij}\}$  and  $h = \{h_{ij}\}$  for the matrices of the first two fundamental forms of  $\Gamma$  and using  $\{h_i^j\} = g^{-1}h$  (see (2.1.16)), the corresponding eigenvalue equation is

$$hv = kgv, \quad (4.2.1)$$

where  $v$  is the normalized eigenvector corresponding to the eigenvalue  $k$ .

**Theorem 4.7.** *Under the evolution equation given in (4.1.1) the principal curvatures  $k_i$  satisfy*

$$\frac{\partial}{\partial t} k_i = 2V_n k_i^2 + \frac{((\partial_t h)v_i, v_i)}{(gv_i, v_i)}, \quad (4.2.2)$$

where  $v_i \in \mathbb{R}^{d-1}$  is the corresponding eigenvector.

**Proof:** To obtain an expression for  $\frac{\partial}{\partial t}k$  we differentiate the eigenvalue equation (4.2.1) with respect to  $t$

$$(\partial_t h)v + h(\partial_t v) = (\partial_t k)gv + k(\partial_t g)v + kg(\partial_t v), \quad (4.2.3)$$

and take the inner product with  $v$  to conclude

$$((\partial_t h)v, v) + (h\partial_t v, v) = \partial_t k(gv, v) + k((\partial_t g)v, v) + k(g\partial_t v, v). \quad (4.2.4)$$

Since  $g$  and  $h$  are self-adjoint, the eigenvalue equation (4.2.1) implies

$$(h\partial_t v, v) = (\partial_t v, hv) = k(\partial_t v, gv) = k(g\partial_t v, v), \quad (4.2.5)$$

and we may cancel the corresponding terms in (4.2.4) yielding

$$((\partial_t h)v, v) = \partial_t k(gv, v) + k((\partial_t g)v, v). \quad (4.2.6)$$

Solving (4.2.6) for  $\partial_t k$  and using Proposition 4.2 we obtain

$$\begin{aligned} \partial_t k &= \frac{((\partial_t h)v, v) - k((\partial_t g)v, v)}{(gv, v)} = \frac{((\partial_t h)v, v) + 2V_n k(hv, v)}{(gv, v)} \\ &= \frac{((\partial_t h)v, v) + 2V_n k^2(gv, v)}{(gv, v)} = \frac{((\partial_t h)v, v)}{(gv, v)} + 2V_n k^2. \end{aligned} \quad (4.2.7)$$

□

**Corollary 4.8.** *We have the following bound*

$$\left\| \frac{\partial}{\partial t} k_i(t) \right\|_{L^2(\Gamma)} \leq C \|V_n\|_{H^2(\Gamma)}, \quad (4.2.8)$$

where  $t \in [0, T]$  and  $C$  is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** The estimate follows from Theorem 4.7 and the evolution equation in Proposition 4.5.

## Chapter 5

# Eigenvalues of the Laplace-Beltrami Operator

### 5.1 Weyl's Asymptotic Formula for the Eigenvalues

Recall that we consider a family of interfaces  $\{\Gamma_t : t \in [0, T]\}$  evolving by normal velocity  $V_n(s, t)$  according to the evolution equation given by

$$\begin{aligned}\partial_t \gamma(s, t) &= V_n(s, t) \cdot \nu(s, t) \\ \gamma(s, 0) &= \gamma_0(s),\end{aligned}\tag{5.1.1}$$

where  $\gamma(s, t)$  gives a local parametrization of  $\Gamma_t$ .

In Chapter 3 we introduced the Laplace-Beltrami operator

$$-\Delta_s = -\frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} g^{ij} \sqrt{\det \mathbf{g}} \frac{\partial}{\partial s_j}. \quad (5.1.2)$$

on  $\Gamma_t$  and the corresponding eigenvalue/eigenfunction pairs  $\{\Theta_{i,\Gamma_t}, \beta_{i,\Gamma_t}\}_{i=1}^M$ . In Chapter 8 we show that selecting

$$M := [M_1 \varepsilon^{-(d-1)}], \quad (5.1.3)$$

where  $M_1 > 0$  is a constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$  and  $[a]$  represents the greatest integer less than or equal to  $a$ , guarantees that the bilinear form associated with the linearization  $\mathcal{L}_\phi$  (see (3.2.2)) is coercive on the orthogonal complement to the ‘center space’  $\mathcal{Z}_M = \text{span}\{1, \eta_1 \phi' \Theta_{i,\Gamma_t}\}_{i=1}^M$ . The proof relies on the following asymptotic bound for the Laplace-Beltrami eigenvalue  $\beta_{M,\Gamma_t}$  (see [11])

$$(\beta_M)^{(d-1)/2} \sim \frac{(2\pi)^{d-1}/\omega_d}{\text{vol} \Gamma_t} M, \quad (5.1.4)$$

as  $M \rightarrow +\infty$ , where  $\omega_d$  is the volume of the unit disk in  $\mathbb{R}^{d-1}$ .

## 5.2 Rayleigh Characterization of Eigenvalues

We start this section with the Rayleigh characterization of eigenvalues of  $-\Delta_s$ . The following can be found in [11] and [14] for example.

**Proposition 5.1.** *The  $i$ -th eigenvalue of the weighted Laplace-Beltrami operator  $-\Delta_s$  is char-*

actarized by the following Rayleigh principle

$$\beta_i = \min_{\substack{\Theta \in V_i^\perp \\ \|\Theta\|_\Gamma^2 = 1}} (\nabla_s \Theta, \nabla_s \Theta)_\Gamma, \quad (5.2.1)$$

where  $V_i := \text{span}\{\Theta_1, \dots, \Theta_{i-1}\}$ ,

$$\nabla_s^j = \sum_{i=1}^{d-1} g^{ij} \frac{\partial}{\partial s_i} \quad (5.2.2)$$

$$(\nabla_s f, \nabla_s g)_\Gamma := \int_\Gamma \langle \nabla_s f, \nabla_s g \rangle \xi(s) ds. \quad (5.2.3)$$

**Proposition 5.2.** *The following identities hold for the tangential derivatives of the eigenfunctions  $\Theta_i$ .*

$$\|\nabla_s \Theta_{i,\Gamma_t}\|_\Gamma = \beta_{i,\Gamma_t}^{1/2}, \quad (5.2.4)$$

$$\|\Delta_s \Theta_{i,\Gamma_t}\|_\Gamma = \beta_{i,\Gamma_t}, \quad (5.2.5)$$

$$\|\nabla_s (\Delta_s \Theta_{i,\Gamma_t})\|_\Gamma = \beta_{i,\Gamma_t}^{3/2}. \quad (5.2.6)$$

**Proof:** From the definition of  $\Theta_{i,\Gamma_t}$  as the normalized eigenfunction of  $-\Delta_s$  corresponding to eigenvalue  $\beta_{i,\Gamma_t}$  we have

$$\Delta_s^2 \Theta_{i,\Gamma_t} = \beta_{i,\Gamma_t}^2 \Theta_{i,\Gamma_t}, \quad (5.2.7)$$

Since  $\|\Theta_{i,\Gamma_t}\|_\Gamma = 1$  taking the  $L^2(\Gamma)$  inner product of (5.2.7) with  $\Theta_{i,\Gamma_t}$  yields

$$(\Delta_s^2 \Theta_{i,\Gamma_t}, \Theta_{i,\Gamma_t})_\Gamma = \beta_{i,\Gamma_t}^2. \quad (5.2.8)$$

Since  $\Gamma$  has no boundary, integrating by parts we obtain

$$(\Delta_s \Theta_{i,\Gamma_t}, \Delta_s \Theta_{i,\Gamma_t})_\Gamma = \beta_{i,\Gamma_t}^2, \quad (5.2.9)$$

and

$$\|\Delta_s \Theta_{i,\Gamma_t}\|_\Gamma = |\beta_{i,\Gamma_t}| = \beta_{i,\Gamma_t}. \quad (5.2.10)$$

Similarly we integrate by parts and use (5.2.10) to obtain

$$\begin{aligned} \|\nabla_s \Theta_{i,\Gamma_t}\|_\Gamma^2 &= (\nabla_s \Theta_{i,\Gamma_t}, \nabla_s \Theta_{i,\Gamma_t})_\Gamma = (-\Delta_s \Theta_{i,\Gamma_t}, \Theta_{i,\Gamma_t})_\Gamma = \beta_{i,\Gamma_t} \|\Theta_{i,\Gamma_t}\|_\Gamma = \beta_{i,\Gamma_t}, \end{aligned} \quad (5.2.11)$$

and

$$\begin{aligned} \|\nabla_s(\Delta_s \Theta_{i,\Gamma_t})\|_\Gamma^2 &= (\nabla_s(\Delta_s \Theta_{i,\Gamma_t}), \nabla_s(\Delta_s \Theta_{i,\Gamma_t}))_\Gamma = -(\Delta_s^2 \Theta_{i,\Gamma_t}, \Delta_s \Theta_{i,\Gamma_t})_\Gamma \\ &= \beta_{i,\Gamma_t}^3 \|\Theta_{i,\Gamma_t}\|_\Gamma = \beta_{i,\Gamma_t}^3. \end{aligned} \quad (5.2.12)$$

□

### 5.3 Smooth Dependence of Eigenfunctions on Time

In order to show that the term involving  $\partial_t w$  in formula (3.3.28) is lower order we need the following technical estimate.

**Theorem 5.3.** *For any differentiable function  $w$  which satisfies  $w(x, t) \in \mathcal{Z}_M^\perp$  we have the*

estimate

$$\left| (\partial_t w, Z_i)_{L^2(\Omega)} \right| \leq C(K) \left( \frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\beta_i} \sqrt{\varepsilon}}{\sqrt{\beta_{M+1}} - \sqrt{\beta_i}} \right) \|V_n\|_{L^\infty(\Gamma)} \|w\|_{L^2(\Omega)}, \quad (5.3.1)$$

where  $Z_i = \eta_1 \phi' \Theta_{i, \Gamma_t}$  span the ‘slow’ space  $Z_M$  and  $C(K)$  is a positive constant that depends only on the admissible interface parameter  $K$ .

**Proof:** First, we note that differentiating the identity  $(w, Z_i)_{L^2(\Omega)} = 0$  with respect to  $t$  yields

$$(\partial_t w, Z_i)_{L^2(\Omega)} + (w, \partial_t Z_i)_{L^2(\Omega)} = 0, \quad (5.3.2)$$

Using  $Z_i = \eta_1 \phi' \Theta_{i, \Gamma_t}$  it follows from (5.3.2) that

$$\begin{aligned} (\partial_t w, Z_i)_{L^2(\Omega)} &= -(w, \partial_t Z_i)_{L^2(\Omega)} = -(w, \partial_t (\eta_1 \phi' \Theta_{i, \Gamma_t}))_{L^2(\Omega)} \\ &= - \left( w, \partial_z (\eta_1 \phi') \frac{\partial z}{\partial t} \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)} - \left( w, \eta_1 \phi' \partial_t \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)} \\ &= \frac{1}{\varepsilon} \left( w, \partial_z (\eta_1 \phi') V_n \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)} - \left( w, \eta_1 \phi' \partial_t \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)}, \end{aligned} \quad (5.3.3)$$

where we used Proposition 3.10 in the last step.

We next decompose  $\partial_t \Theta_{i, \Gamma_t}$  into its projections onto the first  $M$  eigenfunctions of  $-\Delta_s$

$$\partial_t \Theta_{i, \Gamma_t} = \sum_{j=1}^M a_{ij} \Theta_{j, \Gamma_t} + \Theta_i^\perp, \quad (5.3.4)$$

where  $(\Theta_{j, \Gamma_t}, \Theta_i^\perp)_\Gamma = 0$ . Since  $w \in \mathcal{Z}_M^\perp$  where  $\mathcal{Z}_M = \text{span}\{\eta_1 \phi' \Theta_{j, \Gamma_t}\}$ , the only contribution to  $\left( w, \eta_1 \phi' \partial_t \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)}$  comes from the  $\Theta_i^\perp$  term in (5.3.4) and after replacing

$\partial_t \Theta_{i, \Gamma_t}$  with  $\Theta_i^\perp$  in (5.3.3) we obtain

$$(\partial_t w, Z_i)_{L^2(\Omega)} = \frac{1}{\varepsilon} \left( w, \partial_z(\eta_1 \phi') V_n \Theta_{i, \Gamma_t} \right)_{L^2(\Omega)} - \left( w, \eta_1 \phi' \Theta_i^\perp \right)_{L^2(\Omega)}. \quad (5.3.5)$$

It follows that

$$\begin{aligned} \left| (\partial_t w, Z_i)_{L^2(\Omega)} \right| &\leq \frac{1}{\varepsilon} \|V_n\|_{L^\infty(\Gamma)} \|w\|_{L^2(\Omega)} \|\partial_z(\eta_1 \phi') \Theta_{i, \Gamma_t}\|_{L^2(\Omega)} \\ &\quad + \|w\|_{L^2(\Omega)} \|\eta_1 \phi' \Theta_i^\perp\|_{L^2(\Omega)}. \end{aligned} \quad (5.3.6)$$

We estimate the two terms on the right-hand side of (5.3.6) with the aid of the following lemmas.

**Lemma 5.4.**

$$\left\| \partial_z(\eta_1 \phi') \Theta_{i, \Gamma_t} \right\|_{L^2(\Omega)} \leq C(K) \sqrt{\varepsilon}. \quad (5.3.7)$$

**Proof of Lemma 5.4:** We observe that since  $\|\Theta_{i, \Gamma_t}\|_{L^2(\Gamma)} = 1$  and the Jacobian  $J$  is  $O(\varepsilon)$  in  $L^\infty(\Gamma(2\ell))$ ,

$$\begin{aligned} \left\| \partial_z(\eta_1 \phi') \Theta_{i, \Gamma_t} \right\|_{L^2(\Omega)}^2 &= \int_{\Gamma} \Theta_{i, \Gamma_t}^2 \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\partial_z(\eta_1 \phi'))^2 J dz ds \\ &\leq C\varepsilon \|\Theta_{i, \Gamma_t}\|_{L^2(\Gamma)}^2 = C\varepsilon. \end{aligned} \quad (5.3.8)$$

□

**Lemma 5.5.** *Consider the decomposition of  $\partial_t \Theta_{i, \Gamma_t}$  given in (5.3.4). Under the assumptions*

of Theorem 5.3 we have the estimate

$$\left\| \eta_1 \phi' \Theta_i^\perp \right\|_{L^2(\Omega)} \leq \frac{\sqrt{\beta_i \varepsilon}}{\sqrt{\beta_{M+1}} - \sqrt{\beta_i}} C(K) \|V_n\|_{L^\infty(\Gamma)}. \quad (5.3.9)$$

where  $C(K)$  is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

### Proof of Lemma 5.5:

To obtain an expression for the derivative of  $\Theta_{i,\Gamma_t}$  with respect to time we differentiate the eigenvalue equation

$$-\Delta_s \Theta_i = \beta_i \Theta_i \quad (5.3.10)$$

with respect to  $t$  to obtain

$$-\Delta_s \partial_t \Theta_i - [\partial_t, \Delta_s] \Theta_i = (\partial_t \beta_i) \Theta_i + \beta_i \partial_t \Theta_i, \quad (5.3.11)$$

where (5.1.2) implies

$$\begin{aligned} [\partial_t, \Delta_s] \Theta_i &:= \partial_t \Delta_s \Theta_i - \Delta_s (\partial_t \Theta_i) = -\frac{1}{2} \frac{\partial_t \sqrt{\det \mathbf{g}}}{\sqrt{\det \mathbf{g}}} \Delta_s \Theta_i \\ &+ \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{k=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_k} \left( \frac{\partial}{\partial t} \sqrt{\det \mathbf{g}} g^{kj} \right) \frac{\partial \Theta_i}{\partial s_j}. \end{aligned} \quad (5.3.12)$$

Using decomposition (5.3.4) in (5.3.11) we obtain

$$\sum_{j=1}^M a_{ij} \beta_j \Theta_j - \Delta_s \Theta_i^\perp - [\partial_t, \Delta_s] \Theta_i = \beta_i \left( \sum_{j=1}^M a_{ij} \Theta_j + \Theta_i^\perp \right) + (\partial_t \beta_i) \Theta_i \quad (5.3.13)$$

Taking the  $(\cdot, \cdot)_\Gamma$  inner product with  $\Theta_i^\perp$  and recalling that  $(\Theta_j, \Theta_i^\perp)_\Gamma = 0$  gives

$$-(\Delta_s \Theta_i^\perp, \Theta_i^\perp)_\Gamma - ([\partial_t, \Delta_s] \Theta_i, \Theta_i^\perp)_\Gamma = \beta_i \|\Theta_i^\perp\|_{L^2(\Gamma)}^2. \quad (5.3.14)$$

Integrating by parts in  $s$  we obtain

$$\beta_i \|\Theta_i^\perp\|_{L^2(\Gamma)}^2 - \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}^2 = - ([\partial_t, \Delta_s] \Theta_i, \Theta_i^\perp)_\Gamma \quad (5.3.15)$$

and

$$\frac{\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}^2}{\|\Theta_i^\perp\|_{L^2(\Gamma)}^2} = \beta_i + \frac{([\partial_t, \Delta_s] \Theta_i, \Theta_i^\perp)_\Gamma}{\|\Theta_i^\perp\|_{L^2(\Gamma)}^2}. \quad (5.3.16)$$

We will use the above inequality to obtain an estimate on  $\|\Theta_i^\perp\|_{L^2(\Gamma)}$ . First, we observe that

$$\begin{aligned} ([\partial_t, \Delta_s] \Theta_i, \Theta_i^\perp)_\Gamma &= -\frac{1}{2} \left( \frac{\partial_t \sqrt{\det \mathbf{g}}}{\sqrt{\det \mathbf{g}}} \Delta_s \Theta_i, \Theta_i^\perp \right)_\Gamma \\ &+ \left( \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{k=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_k} \left( \frac{\partial}{\partial t} \sqrt{\det \mathbf{g}} g^{kj} \right) \frac{\partial \Theta_i}{\partial s_j}, \Theta_i^\perp \right)_\Gamma \\ &= -\frac{1}{2} \beta_i (V_n H \Theta_i, \Theta_i^\perp)_\Gamma \\ &- \left( \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{k=1}^{d-1} \sum_{j=1}^{d-1} \left( \frac{\partial}{\partial t} \sqrt{\det \mathbf{g}} g^{kj} \right) \frac{\partial \Theta_i}{\partial s_j}, \frac{\partial \Theta_i^\perp}{\partial s_k} \right)_\Gamma, \end{aligned} \quad (5.3.17)$$

where we used Proposition 4.6 to evaluate  $\partial_t \sqrt{\det \mathbf{g}}$ . In addition, applying Proposition 4.3 to

estimate the derivative with respect to  $t$  of  $g^{kj}$  in (5.3.17) we obtain,

$$\begin{aligned}
\left| ([\partial_t, \Delta_s] \Theta_i, \Theta_i^\perp)_\Gamma \right| &\leq C(K) \left( \beta_i \|V_n\|_{L^\infty(\Gamma)} \|\Theta_i^\perp\|_{L^2(\Gamma)} \right. \\
&\quad \left. + \|V_n\|_{L^\infty(\Gamma)} \|\nabla_s \Theta_i\|_{L^2(\Gamma)} \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} \right) \\
&\leq C(K) \left( \beta_i \|V_n\|_{L^\infty(\Gamma)} \|\Theta_i^\perp\|_{L^2(\Gamma)} \right. \\
&\quad \left. + \sqrt{\beta_i} \|V_n\|_{L^\infty(\Gamma)} \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} \right), \tag{5.3.18}
\end{aligned}$$

where we used Proposition 5.2 to bound  $\|\nabla_s \Theta_i\|_{L^2(\Gamma)}$ . Substituting the estimate in (5.3.18) into (5.3.16) yields

$$\frac{\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}^2}{\|\Theta_i^\perp\|_{L^2(\Gamma)}^2} \leq \beta_i + \frac{\beta_i C(K) \|V_n\|_{L^\infty(\Gamma)}}{\|\Theta_i^\perp\|_{L^2(\Gamma)}} + C(K) \|V_n\|_{L^\infty(\Gamma)} \sqrt{\beta_i} \frac{\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}}{\|\Theta_i^\perp\|_{L^2(\Gamma)}^2} \tag{5.3.19}$$

and

$$\begin{aligned}
\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}^2 &\leq \beta_i \|\Theta_i^\perp\|_{L^2(\Gamma)}^2 + \beta_i C(K) \|V_n\|_{L^\infty(\Gamma)} \|\Theta_i^\perp\|_{L^2(\Gamma)} \\
&\quad + C(K) \|V_n\|_{L^\infty(\Gamma)} \sqrt{\beta_i} \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}. \tag{5.3.20}
\end{aligned}$$

We rewrite (5.3.20) as

$$\begin{aligned}
&\left( \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} - \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} \right) \left( \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} + \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} \right) \\
&\leq \sqrt{\beta_i} C(K) \|V_n\|_{L^\infty(\Gamma)} \left( \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} + \|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} \right). \tag{5.3.21}
\end{aligned}$$

It follows that

$$\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} - \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} \leq \sqrt{\beta_i} C(K) \|V_n\|_{L^\infty(\Gamma)} \quad (5.3.22)$$

and

$$\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)} \leq \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} + \sqrt{\beta_i} C(K) \|V_n\|_{L^\infty(\Gamma)}. \quad (5.3.23)$$

Letting  $V_i := \text{span}\{\Theta_1, \dots, \Theta_M\}$ , it follows from (5.3.15) and the Rayleigh characterization in Proposition 5.1 that

$$\beta_{M+1} \leq \frac{\|\nabla_s \Theta_i^\perp\|_{L^2(\Gamma)}^2}{\|\Theta_i^\perp\|_{L^2(\Gamma)}^2}. \quad (5.3.24)$$

Using (5.3.24) in (5.3.23) yields

$$\sqrt{\beta_{M+1}} \|\Theta_i^\perp\|_{L^2(\Gamma)} \leq \sqrt{\beta_i} \|\Theta_i^\perp\|_{L^2(\Gamma)} + \sqrt{\beta_i} C(K) \|V_n\|_{L^\infty(\Gamma)} \quad (5.3.25)$$

and

$$\|\Theta_i^\perp\|_{L^2(\Gamma)} \leq \frac{\sqrt{\beta_i}}{\sqrt{\beta_{M+1}} - \sqrt{\beta_i}} C(K) \|V_n\|_{L^\infty(\Gamma)}. \quad (5.3.26)$$

Finally, since the Jacobian  $J$  is  $O(\varepsilon)$  in  $L^\infty(\Gamma(2\ell))$ ,

$$\begin{aligned} \left\| \eta_1 \phi' \Theta_i^\perp \right\|_{L^2(\Omega)}^2 &= \int_{\Gamma} (\Theta_i^\perp)^2 \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\eta_1 \phi')^2 J dz ds \\ &\leq C\varepsilon \|\Theta_i^\perp\|_{L^2(\Gamma)}^2 \leq \frac{\beta_i \varepsilon}{(\sqrt{\beta_{M+1}} - \sqrt{\beta_i})^2} C(K) \|V_n\|_{L^\infty(\Gamma)}^2 \end{aligned} \quad (5.3.27)$$

and (5.3.9) follows. □

The result of the proposition follows by using the estimates in Lemmas 5.4 and 5.5 in (5.3.6). □

## 5.4 Smooth Dependence of Eigenvalues on Time

We conclude the chapter with an estimate on the derivative of the Laplace-Beltrami eigenvalues  $\beta_{i,\Gamma_t}$  with respect to time.

**Proposition 5.6.** *Under the evolution equation given in (4.1.1) we have the following bound on the derivative of the Laplace-Beltrami eigenvalues with respect to time*

$$|\partial_t \beta_i| \leq C(K) \beta_i \|V_n\|_{L^\infty(\Gamma)}, \quad (5.4.1)$$

where  $t \in [0, T]$  and  $C$  is a positive constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** Taking take the  $(\cdot, \cdot)_\Gamma$  inner product of (5.3.13) with  $\Theta_i$  yields

$$a_{ii}\beta_i - (\Delta_s \Theta_i^\perp, \Theta_i)_\Gamma - ([\partial_t, \Delta_s] \Theta_i, \Theta_i)_\Gamma = \beta_i a_{ii} + \partial_t \beta_i \quad (5.4.2)$$

where recall that

$$(u, v)_\Gamma = \int_\Gamma uv \sqrt{\det \mathbf{g}} ds. \quad (5.4.3)$$

It follows after integrating by parts in the second term on the left-hand side of (5.4.2) that

$$\partial_t \beta_i = -([\partial_t, \Delta_s] \Theta_i, \Theta_i)_\Gamma. \quad (5.4.4)$$

As in the proof of Lemma 5.5 we have

$$\begin{aligned}
([\partial_t, \Delta_s] \Theta_i, \Theta_i)_\Gamma &= -\frac{1}{2} \left( \frac{\partial_t \sqrt{\det \mathbf{g}}}{\sqrt{\det \mathbf{g}}} \Delta_s \Theta_i, \Theta_i \right)_\Gamma \\
&+ \left( \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} \left( \frac{\partial}{\partial t} \sqrt{\det \mathbf{g}} g^{ij} \right) \frac{\partial \Theta_i}{\partial s_j}, \Theta_i \right)_\Gamma \\
&= -\frac{1}{2} \beta_i (V_n H \Theta_i, \Theta_i)_\Gamma \\
&- \left( \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \left( \frac{\partial}{\partial t} \sqrt{\det \mathbf{g}} g^{ij} \right) \frac{\partial \Theta_i}{\partial s_j}, \frac{\partial \Theta_i}{\partial s_i} \right)_\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
|([\partial_t, \Delta_s] \Theta_i, \Theta_i)_\Gamma| &\leq C(K) \left( \beta_i \|V_n\|_{L^\infty(\Gamma)} \|\Theta_i\|_{L^2(\Gamma)} + \|V_n\|_{L^\infty(\Gamma)} \|\nabla_s \Theta_i\|_{L^2(\Gamma)}^2 \right) \\
&\leq C(K) \beta_i \|V_n\|_{L^\infty(\Gamma)},
\end{aligned} \tag{5.4.5}$$

where we used Proposition 5.2 to bound  $\|\nabla_s \Theta_i\|_{L^2(\Gamma)}$ . □

# Chapter 6

## Initial Data Decomposition

### 6.1 Normal Variation

We wish to decompose the initial data as

$$u_0 = \Phi(x; \Gamma_0, b_0) + w_0(x), \quad (6.1.1)$$

where  $w_0 \in Z_{M, \Gamma_0}$ . However, first we study normal variations of an interface  $\Gamma$ . More precisely, we make the following definition

**Definition 6.1.** Fix  $K > 0$ , let  $\Gamma \in \mathcal{G}_K$  be given locally by a parametrization  $\gamma : S \rightarrow \Gamma \subset \mathbb{R}^d$ .

For each  $R \in C^2(\Gamma)$  we define the  $R$ -variation of  $\Gamma$  as

$$\Gamma_R := \{\gamma(s) + \nu(s)R(s) : s \in S\}, \quad (6.1.2)$$

where  $\nu(s)$  is the unit normal vector at  $s$ .

If  $\|R\|_{C^2(\Gamma)}$  is sufficiently small, then  $\Gamma_R$  is locally represented by the diffeomorphism

$$\gamma_R(s) = \gamma(s) + \nu(s)R(s). \quad (6.1.3)$$

Indeed, let  $x = \gamma(s) + \nu(s)R(s) \in \Gamma_R$  and consider the mapping in (6.1.3). We will show that  $\gamma_R^{-1}$  is a chart for  $\Gamma_R$ . Let

$$g_{R,ij} = \left\langle \frac{\partial \gamma_R}{\partial s_i}, \frac{\partial \gamma_R}{\partial s_j} \right\rangle \quad (6.1.4)$$

be the first fundamental form associated with  $\gamma_R$ . Then substituting (6.1.3) into (6.1.4) and taking into account that  $\frac{\partial \nu}{\partial s_j} \in T_{\gamma(s)}\Gamma$  one obtains

$$\begin{aligned} g_{R,ij} &= g_{ij} - 2Rh_{ij} + R^2e_{ij} + \frac{\partial R}{\partial s_i} \frac{\partial R}{\partial s_j} \\ &= g_{ij} - 2R \sum_k g_{ik} h_j^k + R^2 \sum_k g_{ik} e_j^k + \sum_{k,l} g_{ik} g^{kl} \frac{\partial R}{\partial s_l} \frac{\partial R}{\partial s_j} \\ &= \sum_k g_{ik} \left( \delta_j^k - 2Rh_j^k + R^2 e_j^k + \sum_l g^{kl} \frac{\partial R}{\partial s_l} \frac{\partial R}{\partial s_j} \right), \end{aligned}$$

where  $g^{ij}$  is the inverse of  $g_{ij}$ . Bounds (2.1.19) and (2.1.22) for the norms of the second and third fundamental forms imply that  $g_{R,ij}$  is invertible for  $R$  with  $\|R\|_{C^2(\Gamma)}$  less than some constant depending on  $K$  only. Therefore the Jacobian of  $\gamma_R$  doesn't vanish and  $\gamma_R^{-1}$  defines a diffeomorphism of an open neighborhood  $M_x$  of  $x$  into  $\mathbb{R}^{d-1}$  and so is a chart for  $\Gamma_R$ . The manifold  $\Gamma_R$  is compact by compactness of  $\Gamma$  and continuity of the map (6.1.3).

Since parametrizations  $\gamma$  and  $\gamma_R$  are diffeomorphisms they induce a diffeomorphism

$\gamma_R \circ \gamma^{-1} : \gamma(s) \rightarrow \gamma_R(s)$ . Since the first, second, third fundamental forms are smooth on  $\Gamma$ , they will be smooth on  $\Gamma_R$  as well. Since  $\Gamma$  is a compact manifold and  $\Gamma \in \mathcal{G}_K$ , there exists a constant  $\bar{K} > 0$  such that  $\Gamma_R \in \mathcal{G}_{\bar{K}}$  for all  $R \in C^\infty(\Gamma)$ ,  $\|R\|_{C^2(\Gamma)}$  less than some constant depending on  $\bar{K}$ .

## 6.2 Main Theorem

In Chapter 3 we introduced the Laplace-Beltrami operator  $\Delta_S$  and the corresponding eigenvalue/eigenfunction pairs  $\{\Theta_i, \beta_i\}_{i=1}^M$ . We constructed the ‘center space’

$\mathcal{Z}_M = \text{span}\{1, \eta_1 \phi' \Theta_i\}$  and will show in Chapter 8 that selecting  $M = [M_1 \varepsilon^{-(d-1)}]$  with a positive constant  $M_1$  independent on  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ , guarantees the coercivity of the bilinear form associated with  $\mathcal{L}$  on the orthogonal complement to  $\mathcal{Z}_M$ .

In order to track the evolution of an element of  $\mathcal{M}_K$  (see (3.2.1)), for each  $p \in \mathbb{R}^M$ , we define the normal p-variant  $\Gamma_p$  of  $\Gamma_0$  given locally by the parametrization

$$\gamma_p(s) = \gamma_0(s) + \nu_0(s) \sum_{i=1}^M p_i \Theta_{i, \Gamma_0}(s), \quad (6.2.1)$$

where  $\nu_0(s)$  is a unit normal vector at  $\gamma_0(s)$  to  $\Gamma_0$ . This corresponds to selecting  $R(s) = \sum_{i=1}^M p_i \Theta_{i, \Gamma_0}(s)$  in Definition 6.1.

**Theorem 6.2.** *Let  $u_0 \in H^2(\Omega)$ . Fix  $M_1 > 0$  sufficiently small and let  $M = [M_1 \varepsilon^{-(d-1)}]$ . There exists  $\bar{B}_0 > 0$  such that if  $u_0$  is  $\bar{B}_0 \varepsilon^{d-3/2}$  away from the set of single-layer dressed*

interfaces,  $\mathcal{M}_K$  in the sense that for some  $\Gamma_0 \in \mathcal{G}_K$  and  $b_0 \in \mathbb{R}$

$$u_0(x) = \Phi(x; \Gamma_0, b_0) + w_*, \quad (6.2.2)$$

with  $\|w_*\|_{L^2(\Omega)} \leq \bar{B}_0 \varepsilon^{d-3/2}$ , then there exist  $p = p(w_*) \in \mathbb{R}^M$  and  $b = b(w_*) \in \mathbb{R}$  such that

$$u_0(x) = \Phi(x; \Gamma_p, b) + w_0, \quad (6.2.3)$$

with

$$w_0 \in \mathcal{Z}_{\Gamma_p, M}^\perp. \quad (6.2.4)$$

**Proof:** Fix  $\Gamma_0 \in \mathcal{G}_K$ . For  $i = 0, \dots, M$  let  $F : U \subset L^2(\Omega) \times \mathbb{R}^{M+1} \rightarrow \mathbb{R}^{M+1}$  be defined by

$$F_i(w_*, [b, p]) := \int_{\Omega} \underbrace{(\Phi(x; \Gamma_0, b_0) + w_*(x) - \Phi(x; \Gamma_p, b))}_{:=w_0} Z_i dx, \quad (6.2.5)$$

where  $Z_i = \eta_1 \phi' \Theta_{i, \Gamma_p}$  span the ‘center space’  $Z_{\Gamma_p, M}$ . Then, the condition that  $w_0 := \Phi(x; \Gamma_0, b_0) + w_* - \Phi(x; \Gamma_p, b)$  belongs to  $\mathcal{Z}_{\Gamma_p, M}^\perp$  is equivalent to  $F_i(w_*, [b, p]) = 0$  for  $i = 0, \dots, M$ . Clearly,  $F_i(0, [b_0, 0]) = 0$ , or in other words if  $w_* \equiv 0$ , decomposition (6.2.3) is achieved with  $p = 0, b = b_0$  and  $w_0 \equiv 0$ .

We will show that if  $\|w_*\|_{L^2(\Omega)}$  is sufficiently small, we may select  $[p(w_*), b(w_*)] \in \mathbb{R}^{M+1}$  satisfying

$$F_i(w_*, [b, p]) = 0 \text{ for } i = 0, \dots, M. \quad (6.2.6)$$

In order to use the Implicit Function Theorem we need to calculate the variation of  $F$  with

respect to  $p$  and  $b$ . The dependence on  $p$  comes from the fact that both the ansatz  $\Phi$  and  $Z_i = \eta_1 \phi' \Theta_{i, \Gamma_p}$  depend on  $p$  through the interface  $\Gamma$ . The first fundamental form of  $\Gamma_p$  depends analytically on  $p$  while classical results [24] show that the Laplace-Beltrami modes  $\Theta_{i, \Gamma_p}$  smoothly depend on  $p$ .

The ansatz  $\Phi(x; \Gamma_p, b)$  is defined in terms of the whiskered coordinates  $s$  and  $z$  (see (3.1.4)) and so to understand the dependence of  $\Phi(x)$  on  $p$  one needs to first analyze the variations of the coordinates  $s$  and  $z$  with respect to  $p$ .

Recall that in a neighborhood of  $\Gamma_p$  one can introduce whiskered coordinates  $(s_p, z_p)$ , where  $s_p \in S$ . In this coordinate system the domain  $\Gamma(\ell)$  is parametrized by

$$x = \gamma_p(s_p) + \varepsilon z_p \nu_p(s_p), \quad (6.2.7)$$

where  $\nu_p(s_p)$  is a unit normal vector at  $\gamma_p(s_p)$ .

**Lemma 6.3.** *We have the following formulas*

$$\frac{\partial \gamma_p(s_p)}{\partial p_i} \cdot \nu_p(s_p) = \nu_0(s_p) \cdot \nu_p(s_p) \Theta_{i, \Gamma_0}(s_p), \quad (6.2.8)$$

$$\frac{\partial \nu_p(s_p)}{\partial p_i} \cdot \nu_p(s_p) = 0, \quad (6.2.9)$$

$$\frac{\partial z_p}{\partial p_i} = -\varepsilon^{-1} \nu_0(s_p) \cdot \nu_p(s_p) \Theta_{i, \Gamma_0}(s_p), \quad i = 1, \dots, M. \quad (6.2.10)$$

**Proof:** Since

$$\gamma_p(s_p) = \gamma_0(s_p) + \nu_0(s_p) \sum_{i=1}^M p_i \Theta_{i, \Gamma_0}(s_p), \quad (6.2.11)$$

(see (6.2.1)), differentiating (6.2.11) with respect to  $p_i$  gives

$$\frac{\partial \gamma_p(s_p)}{\partial p_i} = D_s \gamma_p(s_p) \cdot \frac{\partial s_p}{\partial p_i} + \nu_0(s_p) \Theta_{i, \Gamma_0}(s_p). \quad (6.2.12)$$

Taking the inner product with  $\nu_p(s_p)$  gives

$$\frac{\partial \gamma_p(s_p)}{\partial p_i} \cdot \nu_p(s_p) = \nu_0(s_p) \cdot \nu_p(s_p) \Theta_{i, \Gamma_0}(s_p), \quad (6.2.13)$$

where we used  $\frac{\partial \gamma_p(s_p)}{\partial s_j} \cdot \nu_p(s_p) = 0$  since  $\frac{\partial \gamma_p(s_p)}{\partial s_j}$  is in the tangent plane of  $\Gamma_p$ . Identity (6.2.9) comes from differentiating

$$\nu_p(s_p) \cdot \nu_p(s_p) = 1, \quad (6.2.14)$$

with respect to  $p_i$ . The final formula of the Lemma is obtained by fixing  $x$ , taking the derivative of

$$z_p = \frac{1}{\varepsilon} (x - \gamma_p(s_p)) \cdot \nu_p(s_p) \quad (6.2.15)$$

with respect to  $p_i$ , and using (6.2.8) and (6.2.9).

□

There is one more identity we need before proceeding to the Implicit Function Theorem. Lemma 6.4 gives the derivatives with respect to  $p_i$  of the local coordinates  $s_{p,j}$  of the touch-down point  $\gamma_p(s_p)$ .

**Lemma 6.4.** *Let  $x \in \Gamma_p(\ell)$  be represented locally by the whiskered coordinates  $z_p(x)$  and*

$sp(x)$ . Then

$$\frac{\partial s_{p,j}}{\partial p_i} = O(\varepsilon^{-1}). \quad (6.2.16)$$

**Proof:** We differentiate

$$x = \gamma_p(sp) + \varepsilon z_p \nu_p(sp) \quad (6.2.17)$$

with respect to  $p_i$  to obtain

$$D_s \gamma_p(sp) \cdot \frac{\partial sp}{\partial p_i} + \nu_0(sp) \Theta_{i, \Gamma_0}(sp) + \varepsilon \frac{\partial z_p}{\partial p_i} \nu_p + \varepsilon z_p D_s \nu_p(sp) \cdot \frac{\partial sp}{\partial p_i} + \varepsilon z_p \frac{\partial \nu_p}{\partial p_i} = 0. \quad (6.2.18)$$

Expanding the products,

$$D_s \gamma_p(sp) \cdot \frac{\partial sp}{\partial p_i} = \sum_{k=1}^{d-1} \frac{\partial \gamma_p}{\partial s_k} \frac{\partial s_{p,k}}{\partial p_i} \quad (6.2.19)$$

and

$$D_s \nu_p(sp) \cdot \frac{\partial sp}{\partial p_i} = \sum_{k=1}^{d-1} \frac{\partial \nu_p}{\partial s_k} \frac{\partial s_{p,k}}{\partial p_i}, \quad (6.2.20)$$

multiplying (6.2.18) by  $\frac{\partial \gamma_p}{\partial s_j}$  and using the definitions of the fundamental forms in (2.1.8) and (2.1.11) yields

$$\sum_{k=1}^{d-1} \left( g_{jk} - \varepsilon z_p h_{jk} \right) \frac{\partial s_{p,k}}{\partial p_i} + \nu_0(sp) \Theta_{i, \Gamma_0}(sp) \frac{\partial \gamma_p}{\partial s_j} + \varepsilon z_p \frac{\partial \nu_p}{\partial p_i} \frac{\partial \gamma_p}{\partial s_j} = 0. \quad (6.2.21)$$

To obtain an expression for  $\frac{\partial \nu_p}{\partial p_i}$ , we first express it in terms of the tangent basis

$$\frac{\partial \nu_p}{\partial p_i} = \sum_{k=1}^{d-1} a_k \frac{\partial \gamma_p}{\partial s_k}. \quad (6.2.22)$$

Taking the inner product of both sides of the above relation with  $\frac{\partial \gamma_p}{\partial s_m}$  and using (2.1.8) yields

$$\left(\frac{\partial \nu_p}{\partial p_i}, \frac{\partial \gamma_p}{\partial s_m}\right) = \sum_{k=1}^{d-1} a_k g_{km}. \quad (6.2.23)$$

Inverting  $g$  we solve for  $a$

$$a_k = \sum_{m=1}^{d-1} \left(\frac{\partial \nu_p}{\partial p_i}, \frac{\partial \gamma_p}{\partial s_m}\right) g^{km} \quad (6.2.24)$$

which when substituted into (6.2.22) yields

$$\begin{aligned} \frac{\partial \nu_p}{\partial p_i} &= \sum_{k,m=1}^{d-1} \left(\frac{\partial \nu_p}{\partial p_i}, \frac{\partial \gamma_p}{\partial s_m}\right) \frac{\partial \gamma_p}{\partial s_k} g^{km} = - \sum_{k,m=1}^{d-1} \left(\nu_p, \frac{\partial^2 \gamma_p}{\partial p_i \partial s_m}\right) \frac{\partial \gamma_p}{\partial s_k} g^{km} \\ &= - \sum_{k,m=1}^{d-1} \left(\nu_p, \frac{\partial(\Theta_{i,\Gamma_0} \nu_0)}{\partial s_m}\right) \frac{\partial \gamma_p}{\partial s_k} g^{km} \end{aligned} \quad (6.2.25)$$

Substituting this expression for  $\frac{\partial \nu_p}{\partial p_i}$  into (6.2.21) we obtain

$$\sum_{k=1}^{d-1} \left(g_{jk} - \varepsilon z_p h_{jk}\right) \frac{\partial s_{p,k}}{\partial p_i} + \nu_0(s_p) \Theta_{i,\Gamma_0}(s_p) \frac{\partial \gamma_p}{\partial s_j} - \varepsilon z_p \sum_{k,m=1}^{d-1} \left(\nu_p, \frac{\partial(\Theta_{i,\Gamma_0} \nu_0)}{\partial s_m}\right) g_{kj} g^{km} = 0, \quad (6.2.26)$$

and canceling the  $g, g^{-1}$  in the last term,

$$\sum_{k=1}^{d-1} \left(g_{jk} - \varepsilon z_p h_{jk}\right) \frac{\partial s_{p,k}}{\partial p_i} + \nu_0(s_p) \Theta_{i,\Gamma_0}(s_p) \frac{\partial \gamma_p}{\partial s_j} - \varepsilon z_p \left(\nu_p, \frac{\partial(\Theta_{i,\Gamma_0} \nu_0)}{\partial s_j}\right) = 0. \quad (6.2.27)$$

Since  $|z| \leq 2\ell/\varepsilon$ , estimate (6.2.16) follows from uniform invertibility of  $\mathbf{g} - \varepsilon z \mathbf{h}$  (see Lemma 2.11) and Proposition 5.2 since  $\beta_M = O(\varepsilon^{-2})$ .  $\square$

We return to the proof of Theorem 6.2. For  $u_0 = \Phi(x; \Gamma_0, b_0) + w_*(x)$ , the condition  $u_0(x) - \Phi(x; \Gamma_p, b) \in Z_{M, \Gamma_p}^\perp$  is equivalent to the equations

$$F_i(w_*, [b, p]) := \int_{\Omega} (\Phi(x; \Gamma_0, b_0) + w_*(x) - \Phi(x; \Gamma_p, b)) Z_{i, \Gamma_p} dx = 0, \quad (6.2.28)$$

for  $i = 0, \dots, M$ . Here  $Z_0 := 1$ ,  $Z_{i, \Gamma_p} = \eta_1 \phi' \Theta_{i, \Gamma_p}$  for  $i = 1, \dots, M$ . For  $w_* \equiv 0$  one solution is  $[b, p] = [b_0, 0]$ . To show invertibility, for  $i, j = 1 \dots M$ , we calculate

$$\partial_{p_j} F_i = \int_{\Omega} -\partial_{p_j} \Phi(x; \Gamma_p, b) Z_{i, \Gamma_p} + (\Phi(x; \Gamma_0, b_0) + w_*(x) - \Phi(x; \Gamma_p, b)) \partial_{p_j} Z_{i, \Gamma_p} dx, \quad (6.2.29)$$

and in particular

$$\partial_{p_j} F_i|_{0, [b_0, 0]} = \int_{\Omega} -\frac{\partial \Phi}{\partial p_j}(x; \Gamma_0, b_0) Z_{i, \Gamma_0} dx. \quad (6.2.30)$$

To simplify, we observe that for  $i, j = 1, \dots, M$  expressions (6.2.10) and (6.2.16) for the derivatives of the whiskered coordinates with respect to  $p_j$  imply

$$\begin{aligned} & - \int_{\Omega} \frac{\partial \Phi}{\partial p_j}(x; \Gamma_0, b_0) Z_{i, \Gamma_0} dx = - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial \Phi(z, s)}{\partial p_j} \eta_1 \phi' \Theta_{i, \Gamma_0} J dz ds \\ & = - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial \Phi(z, s)}{\partial z} \frac{\partial z(x)}{\partial p_j} + D_s \Phi(z, s) \cdot \frac{\partial s(x)}{\partial p_j} \right) \eta_1 \phi' \Theta_{i, \Gamma_0} J dz ds \\ & = \int_{\Gamma} \Theta_{i, \Gamma_0}(s) \Theta_{j, \Gamma_0}(s) J_0(s) \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \cdot (\phi' + \varepsilon^2 \partial_z \phi_2) \phi' (1 + \varepsilon K_1 z) dz ds + O(\varepsilon^2) \\ & = \|\phi'\|_{L^2(\mathbb{R})}^2 (\Theta_{i, \Gamma_0}, \Theta_{j, \Gamma_0})_{\Gamma_0} + O(\varepsilon^2) = \|\phi'\|_{L^2(\mathbb{R})}^2 \delta_{ij} + O(\varepsilon^2). \end{aligned} \quad (6.2.31)$$

In addition we obtain with the aid of (3.1.4)

$$\partial_b F_i \Big|_{0, [b_0, 0]} = \int_{\Omega} -\frac{\partial \Phi}{\partial b} Z_{i, \Gamma_0} dx = - \int_{\Omega} Z_{i, \Gamma_0} dx = - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \phi' \Theta_{i, \Gamma_0} J dz ds = O(\varepsilon), \quad (6.2.32)$$

due to the  $\varepsilon$  in the Jacobian term. Similarly

$$\begin{aligned} \partial_{p_j} F_0 \Big|_{0, [b_0, 0]} &= - \int_{\Omega} \frac{\partial \Phi}{\partial p_j} (x; \Gamma_0, b_0) dx, \quad (6.2.33) \\ &= - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial \Phi(z, s)}{\partial z} \frac{\partial z(x)}{\partial p_j} + D_s \Phi(z, s) \cdot \frac{\partial s(x)}{\partial p_j} \right) J dz ds \\ &= \int_{\Gamma} \Theta_{i, \Gamma_0}(s) J_0(s) \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \cdot \left( \phi' + \varepsilon^2 \phi_2 + \varepsilon^3 b \right) (1 + \varepsilon K_1 z) dz ds + O(\varepsilon^2) \\ &= [|\phi|] \int_{\Gamma} \Theta_{i, \Gamma_0}(s) J_0(s) ds + O(\varepsilon^2) = 2(\Theta_{i, \Gamma_0}, 1)_{\Gamma_0} + O(\varepsilon^2) = O(\varepsilon^2). \quad (6.2.34) \end{aligned}$$

and

$$\partial_b F_0 \Big|_{0, [b_0, 0]} = - \int_{\Omega} \frac{\partial \Phi}{\partial b} dx = - \int_{\Omega} dx = -|\Omega|. \quad (6.2.35)$$

Combining (6.2.29) - (6.2.35) we obtain

$$\nabla_{b,p} F(0, [0, b_0]) = \begin{pmatrix} -|\Omega| & O(\varepsilon^2) \\ O(\varepsilon) & \|\phi'\|_{L^2(\mathbb{R})}^2 I_{M \times M} + O(\varepsilon^2) \end{pmatrix}. \quad (6.2.36)$$

Since  $M = [M_1 \varepsilon^{-(d-1)}]$ , it follows that for  $d = 2, 3$  and each  $i$

$$\sum_{j=1, j \neq i}^M \{\nabla_{b,p} F(0, [0, b_0])\}_{ij} \leq C \varepsilon^2 M_1 \varepsilon^{-(d-1)} \leq C M_1, \quad (6.2.37)$$

and for  $M_1$  sufficiently small  $\nabla_{b,p} F(0, [0, b_0])$  is diagonally dominant and so invertible. In addition,

$$\begin{aligned} \partial_{p_j p_k} F_i &= \int_{\Omega} -\partial_{p_j p_k} \Phi(x; \Gamma_p, b) Z_{i, \Gamma_p} - \partial_{p_j} \Phi(x; \Gamma_p, b) \partial_{p_k} Z_{i, \Gamma_p} - \partial_{p_k} \Phi(x; \Gamma_p, b) \partial_{p_j} Z_{i, \Gamma_p} \\ &+ \left( \Phi(x; \Gamma_0, b_0) + w_*(x) - \Phi(x; \Gamma_p, b) \right) \partial_{p_j p_k} Z_{i, \Gamma_p} dx. \end{aligned} \quad (6.2.38)$$

It follows from (6.2.38) that the second derivatives of  $F_i$  with respect to  $[b, p]$  are uniformly bounded. For  $\|w\|_{L^2(\Omega)}$  sufficiently small, the Implicit Function Theorem guarantees the existence of smooth functions  $[b, p]$  such that  $u_0 - \Phi(x; \Gamma_p, b) \in Z_{\Gamma_p, M}^\perp$ .

In particular,  $y(w_*) := [b(w_*), p(w_*)]$  is given as a fixed point of the map

$$G_{w_*}(y) := y - \nabla_y^{-1} F(0, [0, b_0]) F(w_*, y). \quad (6.2.39)$$

Using the definition above, we obtain for  $i = 0$ , (and  $Z_0 = 1$ )

$$\begin{aligned} \|G_{w_*}^i(y)\| &\leq \left| G_{w_*}^i([b_0, 0]) \right| + \left| G_{w_*}^i(y) - G_{w_*}^i([b_0, 0]) \right| \\ &= \left| \int_{\Omega} w_* dx \right| + \left| b - b_0 - \left( \nabla_y^{-1} F(0, [0, b_0]) [F(w_*, y) - F(w_*, [b_0, 0])] \right)_i \right|. \end{aligned} \quad (6.2.40)$$

Similarly, for  $i = 1, \dots, M$

$$\left\| G_{w_*}^i(y) \right\| \leq \left| \int_{\Omega} w_* Z_{i, \Gamma_0} dx \right| + \left| y_i - \left( \nabla_y^{-1} F(0, [0, b_0]) [F(w_*, y) - F(w_*, [b_0, 0])] \right)_i \right|. \quad (6.2.41)$$

Taylor expanding  $F(w_*, [b, p])$ , the linear terms cancel and we obtain

$$\begin{aligned} \left\| G_{w_*}^i(y) \right\| &\leq \|w_*\|_{L^2(\Omega)} \|Z_i\|_{L^2(\Omega)} + C_1 \left[ \sum_{k,m} (\partial_{y_k} y_m F_i(\tilde{y}) y_k y_m) \right] \\ &\leq C(K) \left( \|w_*\|_{L^2(\Omega)} \varepsilon^{1/2} + \left[ \sum_{k,m} (\partial_{y_k} y_m F_i(\tilde{y}) y_k y_m) \right] \right), \end{aligned} \quad (6.2.42)$$

where the extra  $\varepsilon^{1/2}$  in the first term came from the fact that  $Z_i$  is localized on  $\Gamma$ . Similarly

$$\left\| G_{w_*}^0(y) \right\| \leq C(K) \left( \|w_*\|_{L^2(\Omega)} + \left[ \sum_{k,m} (\partial_{y_k} y_m F_i(\tilde{y}) y_k y_m) \right] \right) \quad (6.2.43)$$

It follows that there exists  $\bar{M}$  such that for each  $w_*$  satisfying

$$\|w_*\|_{L^2} \leq \bar{M}_0 \varepsilon^{d+1/2} \quad (6.2.44)$$

the function  $G_{w_*}$  maps  $\{[b, p] : |b| \leq \bar{B}_1 \varepsilon^{d+1/2}, |p_i| \leq \bar{B}_2 \varepsilon^{d+1}\}$  into itself and in particular the fixed point satisfies

$$|p_i| \leq \bar{B} \varepsilon^{d+1}, \text{ for } i = 0, \dots, M. \quad (6.2.45)$$

We note that the bound above implies

$$\|R\|_{H^2(\Gamma)} = \sum_{i=1}^M |p_i| \|\Theta_i\|_{H^2(\Gamma)} \leq C \sum_{i=1}^M \varepsilon^{d+1} |\beta_i| \leq CM_1 \varepsilon^{-(d-1)} \varepsilon^{d-1} \leq CM_1, \quad (6.2.46)$$

where we used  $\beta_i \sim \varepsilon^{-2}$  and  $M \sim M_1 \varepsilon^{-(d-1)}$ . In addition,

$$w_0 = \Phi(x; \Gamma_0, b_0) + w_* - \Phi(x, \Gamma_p, b) \quad (6.2.47)$$

and

$$\begin{aligned} \|w_0\|_{L^2(\Omega)} &\leq \|w_*\|_{L^2(\Omega)} + \|\Phi(x; \Gamma_0, b_0) - \Phi(x, \Gamma_p, b)\|_{L^2(\Omega)} \\ &\leq \bar{B}_0 \varepsilon^{d+1/2} + C(K) \varepsilon^{1/2} |[b, p]| \leq C(K) \varepsilon^{d+1/2}, \end{aligned} \quad (6.2.48)$$

where the extra  $\varepsilon^{1/2}$  comes from the Jacobian in the  $L^2$  norm of the localized term. Similarly,

$$\begin{aligned} \|\varepsilon^2 \Delta w_0\|_{L^2(\Omega)} &\leq \|\varepsilon^2 \Delta w_*\|_{L^2(\Omega)} + \left\| \varepsilon^2 \Delta \Phi(x; \Gamma_0, b_0) - \varepsilon^2 \Delta \Phi(x, \Gamma_p, b) \right\|_{L^2(\Omega)} \\ &\leq \|\varepsilon^2 \Delta w_*\|_{L^2(\Omega)} + \left\| \partial_z^2 \Phi(x; \Gamma_0, b_0) - \partial_z^2 \Phi(x, \Gamma_p, b) \right\|_{L^2(\Omega)} \\ &\quad + \varepsilon \left\| \kappa \partial_z \Phi(x; \Gamma_0, b_0) - \kappa \partial_z \Phi(x, \Gamma_p, b) \right\|_{L^2(\Omega)} \\ &\quad + \varepsilon^2 \left\| \Delta_s \Phi(x; \Gamma_0, b_0) - \Delta_s \Phi(x, \Gamma_p, b) \right\|_{L^2(\Omega)} \\ &\leq \bar{B}_0 \varepsilon^{d+1/2} + C(K) \varepsilon^{1/2} |[b, p]|. \end{aligned} \quad (6.2.49)$$

It follows that

$$\|w\|_{\mathcal{A}} = \|\Delta w\|_{L^2(\Omega)} + \varepsilon^{-2} \|w\|_{L^2(\Omega)} \leq C \varepsilon^{d-3/2}. \quad (6.2.50)$$



# Chapter 7

## Analysis of the Full Linearized Operator

The goal of the next two chapters is the analysis of the linear operator  $\mathcal{L}_\phi$  defined in (3.2.2), and in particular the proof of Theorem 3.5. With a slight manipulation we rewrite the operator as

$$\mathcal{L}_\phi = (\varepsilon^2 \Delta - W''(\Phi) + \varepsilon^2 \eta)^2 + (W''(\Phi) - \varepsilon^2 \eta) \varepsilon^2 \eta - W'''(\Phi) (\varepsilon^2 \Delta \Phi - W''(\Phi)). \quad (7.0.1)$$

In this chapter we characterize the small energy eigenspace of a generalization of  $\mathcal{L}_\phi$ , which we define in the next section. The organization of the chapter is as follows. Section 7.1 is used to define the needed notation. Next, in Section 7.2 we prove several technical propositions which follow from basic definitions in Chapter 3, but were moved here for clarity. The last two sections contain the statement and the proof of the main result of this chapter.

## 7.1 Functionalized Operator: Overview of Results

Fix  $K > 0$  and let  $\Gamma \in \mathcal{G}_K$  be a smooth closed hypersurface as described in Chapter 3. We consider a linear operator  $\mathcal{L} : L^2(\Omega) \rightarrow L^2(\Omega)$  with domain  $D(\mathcal{L}) = H^2(\Omega) \subset L^2(\Omega)$  and natural boundary conditions having the form

$$\mathcal{L} := \left( \varepsilon^2 \Delta - q(x; \varepsilon) \right)^2 + \varepsilon \tilde{q}(x; \varepsilon), \quad (7.1.1)$$

where  $q(x)$ ,  $\tilde{q}(x)$  are smooth functions. In addition, in  $\Gamma(\ell)$

$$q(x) = q(s, z) = q_0(s, z; \varepsilon) + \varepsilon q_1(s, z; \varepsilon), \quad (7.1.2)$$

with

$$q(x) > q^+ > 0, \text{ in } \Omega \setminus \Gamma(\ell), \quad (7.1.3)$$

$q - q_+$  and  $\tilde{q}$  are localized on  $\Gamma$ , and

$$\sup_{|\alpha| \leq 2} \|D_s^\alpha \partial_z^m q_i\|_{L^\infty(\Gamma(2\ell))} \leq C(K), \quad (7.1.4)$$

$$\sup_{|\alpha| \leq 2} \|D^\alpha q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \leq C(K), \quad (7.1.5)$$

$$\|q_i\|_{L^\infty(\Omega)} \leq C(K), \quad (7.1.6)$$

$$\|\tilde{q}\|_{L^\infty(\Omega)} \leq C(K), \quad (7.1.7)$$

where  $C(K)$  is a constant independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Definition 7.1.** Let  $f \in C(\Omega)$ . If there exist a constant  $f_+$  and  $m > 0$  such that

$$f = f_+ \text{ on } \Omega \setminus \Gamma(2\ell), \quad (7.1.8)$$

and in  $\Gamma(2\ell)$

$$\sup_s |f(s, z) - f_+| \leq C e^{-m|z|} \quad (7.1.9)$$

then we say  $f - f_+$  is localized on  $\Gamma$ .

**Remark 7.2.** Operator  $\mathcal{L}_\phi$  is a special case of  $\mathcal{L}$  with

$$q(x) = W''(\Phi) - \varepsilon^2 \eta = W''(\phi) + O(\varepsilon^2) \quad (7.1.10)$$

and

$$\tilde{q} = \varepsilon^{-1}(W''(\Phi) - \varepsilon^2 \eta) \varepsilon^2 \eta - W'''(\Phi)(\varepsilon^2 \Delta \Phi - W''(\Phi)) = W'''(\phi) \kappa \phi' + O(\varepsilon). \quad (7.1.11)$$

In particular,  $q^+ = W''(\pm 1)$  and (7.1.4)-(7.1.7) hold for the Functionalized Operator  $\mathcal{L}_\phi$  as long as  $\Gamma \in \mathcal{G}_K$ .

One can write  $\mathcal{L}$  in terms of the Allen-Cahn operator  $\mathcal{A}$  as

$$\mathcal{L} = \mathcal{A}^2 + \varepsilon \tilde{q}(x), \quad (7.1.12)$$

where

$$\mathcal{A} = -\varepsilon^2 \Delta + q(x). \quad (7.1.13)$$

In  $\Gamma(\ell)$  we may use (2.2.25) and (7.1.2) to write  $\mathcal{A}$  in the local coordinates as

$$\mathcal{A} = -\partial_z^2 - \varepsilon \kappa(s, z) \partial_z + q_0(s, z) + \varepsilon q_1(s, z) - \varepsilon^2 \Delta_G. \quad (7.1.14)$$

The leading-order approximations are

$$L_0[s] := -\partial_z^2 + q_0(s, z) \quad (7.1.15)$$

and

$$L_a[s] := -J^{-1} \frac{\partial}{\partial z} J \frac{\partial}{\partial z} + q(s, z) = -\partial_z^2 - \varepsilon \kappa(s, z) \partial_z + q(s, z). \quad (7.1.16)$$

Similarly

$$L := L_a^2 + \varepsilon \tilde{q}(s, z) = (L_0 - \varepsilon \kappa \partial_z + \varepsilon q_1)^2 + \varepsilon \tilde{q}, \quad (7.1.17)$$

is the leading-order part of  $\mathcal{L}$ , and in local coordinates

$$\mathcal{L} = \mathcal{A}^2 + \varepsilon \tilde{q} = L - \varepsilon^2 \Delta_G L_a - \varepsilon^2 L_a \Delta_G + \varepsilon^4 \Delta_G^2. \quad (7.1.18)$$

We have a proliferation of operators on different domains, which we summarize in the table below. Here  $I$  denotes the interval  $(-2\ell/\varepsilon, 2\ell/\varepsilon)$ , and the boundary conditions are natural for the appropriate bilinear form. The precise definitions of the corresponding bilinear forms will be given as needed.

Note that the differential operators  $L_0$ ,  $L_a$ , and  $L$  depend upon  $s$  as a parameter, and for a fixed  $s$  may be considered as operators acting on  $H^2(-2\ell/\varepsilon, 2\ell/\varepsilon)$ . We will sometimes use  $'$

| Operator                                                              | Region                        | Description                  | Eigenpairs                |
|-----------------------------------------------------------------------|-------------------------------|------------------------------|---------------------------|
| $L_0 = -\partial_z^2 + q_0(s, z)$                                     | $I \subset \mathbb{R}$        | 1D Allen-Cahn Operator       | $(\psi_i^0, \lambda_i^0)$ |
| $L_a = -\partial_z^2 - \varepsilon \kappa(s, z) \partial_z + q(s, z)$ | $I \subset \mathbb{R}$        | 1D Allen-Cahn Lead Order     | $(\psi_i^a, \lambda_i^a)$ |
| $L = L_a^2 + \varepsilon \tilde{q}(s, z)$                             | $I \subset \mathbb{R}$        | 1D Functionalized Lead Order |                           |
| $\mathcal{A} = -\varepsilon^2 \Delta + q(x)$                          | $\Omega \subset \mathbb{R}^d$ | Allen-Cahn Operator          |                           |
| $\mathcal{L} = \mathcal{A}^2 + \varepsilon \tilde{q}(x)$              | $\Omega \subset \mathbb{R}^d$ | Functionalized Operator      | $(\Psi_i, \mu_i)$         |

Table 7.1: Definitions of the operators that are used.

| Operator      | Inner Product                                           | Bilinear Form                                                                                              |
|---------------|---------------------------------------------------------|------------------------------------------------------------------------------------------------------------|
| $L_0$         | $\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uv dz$   | $b_0[u, v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (u'v' + q_0 uv) dz$                             |
| $L_a$         | $\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uv J dz$ | $b_a[u, v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (u'v' + quv) J dz$                              |
| $L$           | $\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uv J dz$ | $b[u, v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} ((L_a u)(L_a v) + \varepsilon \tilde{q} uv) J dz$ |
| $\mathcal{A}$ | $\int_{\Omega} uv dx$                                   | $B_a[u, v] = \int_{\Omega} (\varepsilon^2 \nabla u \cdot \nabla v + quv) dx$                               |
| $\mathcal{L}$ | $\int_{\Omega} uv dx$                                   | $B[u, v] := \int_{\Omega} ((\mathcal{A}u)(\mathcal{A}v) + \varepsilon \tilde{q} uv) dx$                    |

Table 7.2: Definitions of the bilinear forms that are used.

to represent derivative with respect to the dominant variable  $z$ . We make several assumptions about the structure of the spectrum of  $L_0$ .

**Assumption (H):** There exist  $C > 0$  independent of  $\varepsilon$  and  $s$  such that  $L_0$  has  $r$  eigenvalues satisfying  $\sup_s |\lambda_i^0(s)| \leq C\varepsilon$  for  $1 \leq i \leq r$ , and the rest of the spectrum is bounded below by some value  $\nu_0 > 0$  which is independent of  $\varepsilon$  and  $s$ .

**Remark:** The leading-order operator  $L_0$  corresponds to the negative of the 1D Allen-Cahn operator defined in (3.1.7) and by Lemma 9.1, Assumption (H) holds for  $L_0$  with  $r = 1$ .

The following lemma is a classical result.

**Lemma 7.3.** *There exists  $m > 0$  such that the eigenfunctions  $\{\psi_i^0\}_{i=1}^r$  of  $L_0$  satisfy*

$$\sup_{n \leq 4} \sup_s \partial_z^n \psi_i^0(s, \pm \ell/\varepsilon) = O(e^{-m/\varepsilon}), \quad (7.1.19)$$

and

$$\sup_{|\alpha| \leq 2, n \leq 4} \sup_s \|D_s^\alpha \partial_z^n \psi_i^0\|_0 = O(1). \quad (7.1.20)$$

## 7.2 Additional Estimates

We introduce additional notation used in the following chapters. The  $L^2$  inner product over  $\Omega$  or  $\Gamma$  will be denoted  $(f, g)_\Omega$  or  $(f, g)_\Gamma$  respectively. The induced norms will be denoted by  $\|f\|_{L^2(\Omega)}$  and  $\|f\|_{L^2(\Gamma)}$ . The  $L^2$  inner product of a localized function over a particular whisker will be unadorned,

$$(f, g)_0(s) = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s, z)g(s, z) dz. \quad (7.2.1)$$

The corresponding whiskered  $L^2$  norm is denoted  $\|f\|_0$ . Volume integrals of  $f$  in the whiskered variables take the form

$$\int_{\Gamma(2\ell)} f(x) dx = \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s, z) J(s, z) dz ds. \quad (7.2.2)$$

Consequently, we will also use a  $J$  inner product on each whisker denoted by

$$(f, g)_J = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s, z) g(s, z) J(s, z) dz. \quad (7.2.3)$$

The norm induced by this inner product will be denoted:  $\|\cdot\|_J$ . The Jacobian  $J$  is  $O(\varepsilon)$  in  $L^\infty(\Gamma(2\ell))$  and introduces a factor of  $\varepsilon^{1/2}$  into the  $J$ -norm of localized functions.

**Proposition 7.4.** *The inverse of the first fundamental form  $G$  may be written as*

$$G^{ij} = \sum_{m=1}^{d-1} \vartheta_{mi} \vartheta_{mj}, \quad (7.2.4)$$

where

$$\vartheta = (\mathbf{J}^T)^{-1}. \quad (7.2.5)$$

**Proof:** From (2.2.6) and (2.2.7)

$$G_{ij} = (\mathbf{J}^T \mathbf{J})_{ij} \quad (7.2.6)$$

and

$$G^{ij} = (\mathbf{J}^{-1} (\mathbf{J}^T)^{-1})_{ij} = \sum_{m=1}^{d-1} \vartheta_{mi} \vartheta_{mj}, \quad (7.2.7)$$

where

$$\vartheta_{ij} = ((\mathbf{J}^T)^{-1})_{ij} \text{ for } i, j \leq d-1. \quad (7.2.8)$$

□

Relations (7.2.5) and (2.2.16) also allow us to obtain bounds on the  $z$  derivatives of  $\vartheta_{ij}$ .

**Lemma 7.5.** *Fix  $K > 0$ , then for all  $m \in \mathbb{N}$  there exists  $C(K) > 0$  such that for all  $\Gamma \in G_K$*

$$\sup_{i,j \leq d-1} \|\partial_z^m \vartheta_{ij}\|_{L^\infty(\Gamma(2\ell))} \leq C(K)\varepsilon^m. \quad (7.2.9)$$

**Proposition 7.6.** *Assume  $f \in L^2(\Omega)$  and  $f - f_+$  is localized on  $\Gamma$ . Then*

$$\langle f \rangle := \frac{1}{|\Omega|} \int_{\Omega} f dx = f_+ + O(\varepsilon). \quad (7.2.10)$$

**Proof:** Since  $f_+$  is constant,  $\int_{\Omega} f_+ dx = f_+ |\Omega|$  and

$$\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega} f dx = \frac{1}{|\Omega|} \int_{\Omega} (f - f_+) dx + \frac{1}{|\Omega|} \int_{\Omega} f_+ dx = I + f_+, \quad (7.2.11)$$

where

$$I := \frac{1}{|\Omega|} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (f - f_+) J dz ds. \quad (7.2.12)$$

However  $f - f_+$  is localized on  $\Gamma$  and

$$|I| \leq \frac{1}{|\Omega|} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} |f - f_+| J dz ds \leq \frac{C}{|\Omega|} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} e^{-m|z|} J dz ds = O(\varepsilon), \quad (7.2.13)$$

where we used (2.4.25) bound  $J$ . □

**Proposition 7.7.** *Assume  $f \in L^2(\Omega)$  and  $f - f_+$  is localized on  $\Gamma$ . Then*

$$\|\Pi_0 f\|_{L^2(\Omega)}^2 = O(\varepsilon). \quad (7.2.14)$$

**Proof:** Using the definition of  $\Pi_0$

$$\begin{aligned} \|\Pi_0 f\|_{L^2(\Omega)}^2 &= \int_{\Omega} (f - \langle f \rangle)^2 dx = \int_{\Omega} (f - f_+ + f_+ - \langle f \rangle)^2 dx \\ &= \int_{\Omega} (f - f_+)^2 dx + 2(f_+ - \langle f \rangle) \int_{\Omega} (f - f_+) dx + \int_{\Omega} (f_+ - \langle f \rangle)^2 dx \\ &\leq C^2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} e^{-2m|z|} J dz ds + 2C(f_+ - \langle f \rangle) \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} e^{-m|z|} J dz ds \\ &\quad + |\Omega| (f_+ - \langle f \rangle)^2 dx = O(\varepsilon), \end{aligned} \quad (7.2.15)$$

where we used (2.4.25) to bound terms involving  $J$  and Proposition 7.6 to bound  $f_+ - \langle f \rangle$ . □

As a consequence of (2.2.1) we also have the following inequalities.

**Proposition 7.8.** *For any function  $f(x) = f(s, z)$*

$$\|\partial_z f\|_{L^2(\Gamma(2\ell))} \leq \varepsilon \|f\|_{H^1(\Gamma(2\ell))}, \quad (7.2.16)$$

$$\|\partial_z^2 f\|_{L^2(\Gamma(2\ell))} \leq C(K) \varepsilon^2 \|f\|_{H^2(\Gamma(2\ell))}, \quad (7.2.17)$$

$$\|\partial_{s_i} f\|_{L^2(\Gamma(2\ell))} \leq C(K) \|f\|_{H^1(\Gamma(2\ell))}. \quad (7.2.18)$$

**Proof:** Since

$$x = \gamma(s) + \varepsilon z \nu(s), \quad (7.2.19)$$

we have

$$\partial_z f = \varepsilon \nu \cdot \nabla_x f, \quad (7.2.20)$$

and

$$\begin{aligned} \|\partial_z f\|_{L^2(\Gamma(2\ell))}^2 &= \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\partial_z f)^2 J dz ds \leq \varepsilon^2 |\nu|^2 \sum_{i,j=1}^d \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} |\nabla_x f|^2 J dz ds \\ &\leq \varepsilon^2 \|f\|_{H^1(\Gamma(2\ell))}^2. \end{aligned} \quad (7.2.21)$$

The remaining estimates follow similarly. □

### 7.3 Main Results

We now give definitions analogous to those in Chapter 3, but for the more general operator  $\mathcal{L}$ . Consider the bilinear form associated with  $\mathcal{L}$  of (7.1.12)

$$B[u, v] := (\mathcal{A}u, \mathcal{A}v) + \varepsilon(\tilde{q}u, v), \quad (7.3.1)$$

in  $H^2(\Omega)$ , where recall that

$$\mathcal{A} = -\varepsilon^2 \Delta + q(x), \quad (7.3.2)$$

and  $\mathcal{A}$ 's leading-order '1D' form on each 'whisker' is

$$L_a[s] = -J^{-1} \frac{\partial}{\partial z} J \frac{\partial}{\partial z} + q(s, z) = -\partial_z^2 - \varepsilon \kappa(s, z) \partial_z + q(s, z). \quad (7.3.3)$$

Let  $\Psi \in H^2(\Omega)$ . Let  $\eta_1(x)$  be a smooth cutoff function equal to one in  $\Gamma(\ell)$  and zero outside  $\Gamma(2\ell)$ . Set  $\eta_2(x) = 1 - \eta_1(x)$ . Then we have the decomposition

$$\Psi = \Psi_1 + \Psi_2, \quad (7.3.4)$$

where  $\Psi_1 := \eta_1 \Psi$  is supported inside  $\Gamma(2\ell)$  and  $\Psi_2 := \eta_2 \Psi$  is supported inside  $\Omega \setminus \Gamma(\ell)$ . For the remainder of this section we assume that  $\Psi$  and consequently  $\Psi_2$  satisfies Neumann boundary conditions on  $\partial\Omega$ .

### 7.3.1 Whiskered Operator

For each fixed  $s \in S$ , we would first like to understand the spectra of the 1D operators

$$L_a[s] = L_0[s] - \varepsilon \kappa(s, z) \partial_z + \varepsilon q_1(s, z), \quad (7.3.5)$$

and

$$L[s] = L_a^2[s] + \varepsilon \tilde{q}(s, z), \quad (7.3.6)$$

as perturbations of  $L_0 = -\partial_z^2 + q_0(s, z)$  acting on  $H^2(-\ell/\varepsilon, \ell/\varepsilon)$ . We will see that the curvature perturbation  $\varepsilon \kappa(s, z)$  causes an  $O(\varepsilon^2)$  shifting of the asymptotically small eigenvalues. In addition,  $\varepsilon q_1$  and  $\varepsilon \tilde{q}$  perturbation will result in  $O(\varepsilon)$  perturbations of the point spectrum

unless  $q_1$  and  $\tilde{q}$  satisfy certain orthogonality properties.

Recall that  $b_0$  and  $b_a$  stand for bilinear forms corresponding to  $L_0$  and  $L_a$  respectively (see Table 7.1). Under Neumann boundary conditions  $\partial_z \psi(\pm 2\ell/\varepsilon) = 0$ ,  $L_a$  and  $L$  are self-adjoint in the weighted  $(\cdot, \cdot)_J$  inner product defined in (7.2.3). Indeed, considering the  $J$ -inner product of  $L_a u$  with  $v$ , we integrate by parts and use the relation  $J_z = \varepsilon \kappa J$  to find,

$$b_a[u, v] = (L_a u, v)_J = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (-u_{zz} - \varepsilon \kappa u_z + qu)v J dz = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (u_z v_z + quv) J dz = (u, L_a v), \quad (7.3.7)$$

where  $b_a[u, v] = (L_a u, v)_J$  is the associated bilinear form.

**Lemma 7.9.** *Let  $(\cdot, \cdot)_J$  be the weighted inner product defined in (7.2.3). Then*

$$b_a[u, v] = b_0[J^{1/2}u, J^{1/2}v] + \left( \left[ \frac{(\sqrt{J})''}{\sqrt{J}} + \varepsilon q_1 \right] u, v \right)_J, \quad (7.3.8)$$

where the bilinear forms,  $b_a$  and  $b_0$  are given in Table 7.1.

**Proof:** Simple differentiation establishes the following relation between the bilinear forms

$$\begin{aligned} b_0[J^{1/2}u, J^{1/2}v] &= \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} ((J^{1/2}u)'(J^{1/2}v)' + q_0 uv J) dz \\ &= \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \left( \frac{1}{2} J^{-1/2} J' u + J^{1/2} u' \right) \left( \frac{1}{2} J^{-1/2} J' v + J^{1/2} v' \right) + q_0 uv J \right) dz \\ &= \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{1}{4} J^{-1} (J')^2 uv + \frac{1}{2} J' uv' + \frac{1}{2} J' u' v + u' v' J + q_0 uv J \right) dz. \end{aligned} \quad (7.3.9)$$

Integrating by parts in the second term on the right-hand side of (7.3.9) we obtain

$$b_0[u, v] = b_a[u, v] + \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left[ \frac{1}{4} J^{-1} (J')^2 - \frac{1}{2} J'' - \varepsilon q_1 \right] u v dz. \quad (7.3.10)$$

Relation (7.3.8) then follows from the identity

$$-(1/4) J^{-2} (J')^2 + (1/2) J^{-1} J'' = \frac{(\sqrt{J})''}{\sqrt{J}}. \quad (7.3.11)$$

□

Let  $\lambda_1^0(s)$  be the ground eigenvalue of  $L_0$ . The next proposition gives the well-known lower bound estimates on the Allen-Cahn bilinear form in terms of  $\inf_{s \in \Gamma} \lambda_1^0$ , (see [12]).

**Proposition 7.10.** *(The deMottoni-Schatzman Estimates on the Allen-Cahn Form)*

Fix  $K > 0$  and let  $\Gamma \in \mathcal{G}_K$ . There exists a positive constant  $C(K)$  independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$  such that the following lower bounds hold for the bilinear forms associated with  $L_a$  and  $\mathcal{A}$

$$\inf_{\substack{\psi \in H^1(-2\ell/\varepsilon, \ell/\varepsilon) \\ \partial_z \psi(\pm 2\ell/\varepsilon) = 0}} \frac{(L_a \psi, \psi)_J}{\|\psi\|_J^2} \geq \inf_s \lambda_1^0(s) - C(K)\varepsilon, \quad (7.3.12)$$

$$\inf_{\substack{\Psi \in H^1(\Omega) \\ \partial_\nu \Psi = 0 \text{ on } \partial\Omega}} \frac{(\mathcal{A}\Psi, \Psi)_{L^2(\Omega)}}{\|\Psi\|_{L^2(\Omega)}^2} \geq \inf_s \lambda_1^0(s) - C(K)\varepsilon. \quad (7.3.13)$$

**Proof:** Fix  $s \in \Gamma$  and let  $\psi(s, z) \in H^1(-2\ell/\varepsilon, 2\ell/\varepsilon)$  and define

$$\psi^0 := J^{1/2} \psi. \quad (7.3.14)$$

Then by Lemma 7.3.9

$$(L_a\psi, \psi)_J = (L_0\psi^0, \psi^0)_0 + \left( \left[ \frac{(\sqrt{J})''}{\sqrt{J}} + \varepsilon q_1 \right] \psi, \psi \right)_J. \quad (7.3.15)$$

Proposition 2.12 implies that there exists a constant  $C(K) > 0$  that depends only on  $K$  such that

$$\left\| \frac{(\sqrt{J})''}{\sqrt{J}} \right\|_{L^\infty(\Gamma(2\ell))} = \left\| -(1/4)J^{-2}(J')^2 + (1/2)J^{-1}J'' \right\|_{L^\infty(\Gamma(2\ell))} \leq C(K)\varepsilon^2. \quad (7.3.16)$$

Using (7.3.16) and  $\|q_1\|_{L^\infty(\Gamma(2\ell))} \leq C(K)$  (see (7.1.4)) allows us to bound the second term on the right-hand side of (7.3.15)

$$\left| \left( \left[ \frac{(\sqrt{J})''}{\sqrt{J}} + \varepsilon q_1 \right] \psi, \psi \right)_J \right| \leq C(K)\varepsilon \|\psi\|_J^2. \quad (7.3.17)$$

Returning this estimate to (7.3.15) and using the minimax characterization for eigenvalues of  $L_0$  gives

$$(L_a\psi, \psi)_J \geq (L_0\psi_0, \psi_0)_0 - C\varepsilon \|\psi\|_J^2 \geq \lambda_1^0(s) \|\psi_0\|_0^2 - C(K)\varepsilon \|\psi\|_J^2 \geq (\inf_s \lambda_1^0(s) - C\varepsilon) \|\psi\|_J^2, \quad (7.3.18)$$

where we used  $\|\psi_0\|_0 = \|J^{1/2}\psi\|_0 = \|\psi\|_J$ .

To obtain (7.3.13), we take  $\Psi \in H^1(\Omega)$  with  $\partial_\nu \Psi|_{\partial\Omega} = 0$ , form the bilinear form  $b_a[\Psi, \Psi]$

and integrate by parts using the relation  $\partial_z J = \varepsilon \kappa J$  to find,

$$(L_a \Psi, \Psi)_J = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (-\partial_z^2 \Psi - \varepsilon \kappa \partial_z \Psi + q \Psi) \Psi J dz = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} ((\partial_z \Psi)^2 + q \Psi^2) J dz. \quad (7.3.19)$$

Finally, integrating by parts and using  $q(x) = q^+ > 0$  in  $\Omega \setminus \Gamma(2\ell)$  gives

$$\begin{aligned} (\mathcal{A}\Psi, \Psi)_{L^2(\Omega)} &= \int_{\Omega} (\varepsilon^2 |\nabla \Psi|^2 + q \Psi^2) dx \geq \int_{\Gamma(2\ell)} (\varepsilon^2 |\nabla \Psi|^2 + q \Psi^2) dx \\ &= \int_{\Gamma(2\ell)} (\Psi_z^2 + \varepsilon^2 |\nabla_G \Psi|^2 + q \Psi^2) dx \\ &\geq \int_{\Gamma} (L_a \Psi, \Psi)_J ds \geq \int_{\Gamma} (\inf_s \lambda_1^0(s) - C(K)\varepsilon) \|\Psi\|_J^2 ds \\ &\geq (\inf_s \lambda_1^0(s) - C(K)\varepsilon) \|\Psi\|_{L^2(\Omega)}^2. \end{aligned} \quad (7.3.20)$$

□

### 7.3.2 Low Energy Function Decomposition

We introduce the following near-field, far-field decomposition

$$\Psi = \Psi_1 + \Psi_2, \quad (7.3.21)$$

where

$$\Psi_1 = \eta_1 \Psi, \Psi_2 = (1 - \eta_1) \Psi \quad (7.3.22)$$

and  $\eta_1$  is given by (3.1.2). Let  $\{\psi_k^0\}_{k=1}^r$  be the normalized ( $\|\psi_k^0\|_0 = 1$ ) small eigenvalue eigenfunctions of  $L_0 = -\partial_z^2 + q(s, z)$  on  $(-2\ell/\varepsilon, 2\ell/\varepsilon)$ . Recall that the rest of the eigenvalues

are bounded below by  $\nu_0 > 0$  which is independent of  $\varepsilon$  and  $s$ . Denote  $\psi_k := J^{-1/2}\psi_k^0$  and

$$c_k(s) := (\Psi_1, \psi_k)_J. \quad (7.3.23)$$

The following theorem gives a ‘separated variables’ decomposition for low energy functions  $\Psi$  in terms of  $O(\varepsilon)$  eigenvalue eigenfunctions of  $L_0$ .

**Theorem 7.11.** *Let  $U_0 = \frac{1}{32} \min\{q_+^2, \nu_0^2\}$ . Assume  $\Psi \in H^2(\Omega)$  (with  $\|\Psi\|_{L^2(\Omega)} = 1$ ,  $\partial_\nu \Psi = 0$  on  $\partial\Omega$ ) satisfies the bound*

$$B[\Psi] \leq U < U_0, \quad (7.3.24)$$

*for the bilinear form  $B$  given in (7.3.1). Then  $c_k(s) \in H^2(\Gamma)$  and*

$$\Psi_1(x) = \Psi_1(s, z) = \sum_{k=1}^r c_k(s) \psi_k(s, z) + \Psi^\perp(x), \quad (7.3.25)$$

*where*

$$(\psi_k, \Psi^\perp)_J = 0, \quad (7.3.26)$$

$$\|\Psi^\perp\|_{L^2(\Gamma(2\ell))}^2 \leq \frac{U}{\nu_0^2} + O(\varepsilon), \quad (7.3.27)$$

$$\|\Delta_s c_k\|_{L^2(\Gamma)}^2 \leq U\varepsilon^{-4} + O(\varepsilon^{-3}), \quad (7.3.28)$$

$$\|\Psi_2\|_{L^2(\Omega)}^2 \leq \frac{U}{q_+^2} + O(\varepsilon), \quad (7.3.29)$$

$$\int_{\Gamma} \sum_{k=1}^r c_k^2 ds \geq \frac{1}{4}. \quad (7.3.30)$$

## 7.4 Proof of Theorem 7.11

We present the proof in a series of lemmas.

**Lemma 7.12.** *Under the conditions of Theorem 7.11, there exists  $\varepsilon_0 > 0$  and  $C_1 > 0$  such that for  $\varepsilon < \varepsilon_0$*

$$B[\Psi_2] \geq C_1 \varepsilon^4 \|\Psi_2\|_{H^2(\Omega)}^2 + q_+ \varepsilon^2 \|\Psi_2\|_{H^1(\Omega)}^2 + \frac{q_+^2}{2} \|\Psi_2\|_{L^2(\Omega)}^2. \quad (7.4.1)$$

**Proof:**

$$\begin{aligned} B[\Psi_2] &= \int_{\Omega} [(-\varepsilon^2 \Delta \Psi_2 + q \Psi_2)^2 + \varepsilon \tilde{q} \Psi_2^2] dx \\ &= \varepsilon^4 \int_{\Omega} |\Delta \Psi_2|^2 dx - 2\varepsilon^2 \int_{\Omega} \Delta \Psi_2 q \Psi_2 dx + \int_{\Omega} q^2 \Psi_2^2 dx + \varepsilon \int_{\Omega} \tilde{q} \Psi_2^2 dx. \end{aligned} \quad (7.4.2)$$

We use Neumann boundary conditions on  $\Psi$  (and thus on  $\Psi_2$ ) to integrate by parts in the second term to yield

$$\begin{aligned} B[\Psi_2] &= \varepsilon^4 \int_{\Omega} |\Delta \Psi_2|^2 dx + 2\varepsilon^2 \int_{\Omega} q |\nabla \Psi_2|^2 dx + 2\varepsilon^2 \int_{\Omega} (\nabla q \cdot \nabla \Psi_2) \Psi_2 dx \\ &\quad + \int_{\Omega} q^2 \Psi_2^2 dx + \varepsilon \int_{\Omega} \tilde{q} \Psi_2^2 dx. \end{aligned} \quad (7.4.3)$$

Since, by (7.1.3) ,  $q > q_+$  in  $\Omega \setminus \Gamma(\ell)$  and  $\Psi_2$  is supported inside  $\Omega \setminus \Gamma(\ell)$ ,

$$\begin{aligned}
B[\Psi_2] &\geq \varepsilon^4 \|\Delta \Psi_2\|_{L^2(\Omega)}^2 + 2\varepsilon^2 q_+ \|\nabla \Psi_2\|_{L^2(\Omega)}^2 + q_+^2 \|\Psi_2\|_{L^2(\Omega)}^2 \\
&\quad - 2\varepsilon^2 \|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \|\nabla \Psi_2\|_{L^2(\Omega)} \|\Psi_2\|_{L^2(\Omega)} \\
&\quad - \varepsilon \|\tilde{q}\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \|\Psi_2\|_{L^2(\Omega)}^2 \\
&\geq \varepsilon^4 \|\Delta \Psi_2\|_{L^2(\Omega)}^2 + 2\varepsilon^2 q_+ \|\nabla \Psi_2\|_{L^2(\Omega)}^2 + q_+^2 \|\Psi_2\|_{L^2(\Omega)}^2 \\
&\quad - \varepsilon^2 \|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \left( \varepsilon \|\nabla \Psi_2\|_{L^2(\Omega)}^2 \right. \\
&\quad \left. + \varepsilon^{-1} \|\Psi_2\|_{L^2(\Omega)}^2 \right) - \varepsilon \|\tilde{q}\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \|\Psi_2\|_{L^2(\Omega)}^2 \\
&\geq \varepsilon^4 \|\Delta \Psi_2\|_{L^2(\Omega)}^2 + \varepsilon^2 \left( 2q_+ - \varepsilon \|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \right) \|\nabla \Psi_2\|_{L^2(\Omega)}^2 \\
&\quad + \left( q_+^2 - \varepsilon \|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} - \varepsilon \|\tilde{q}\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \right) \|\Psi_2\|_{L^2(\Omega)}^2.
\end{aligned} \tag{7.4.4}$$

We use elliptic regularity theory (Evans, Theorem 4, p.317) to estimate

$$\|\Psi_2\|_{H^2(\Omega)}^2 \leq C(\|\Delta \Psi_2\|_{L^2(\Omega)}^2 + \|\Psi_2\|_{L^2(\Omega)}^2) \tag{7.4.5}$$

Using (7.4.5) we obtain the bound

$$\varepsilon^4 \|\Delta \Psi_2\|_{L^2(\Omega)}^2 \geq \frac{\varepsilon^4}{C} \|\Psi_2\|_{H^2(\Omega)}^2 - \varepsilon^4 \|\Psi_2\|_{L^2(\Omega)}^2 \tag{7.4.6}$$

Returning this estimate to (7.4.4) gives

$$\begin{aligned}
B[\Psi_2] &\geq \frac{\varepsilon^4}{C} \|\Psi_2\|_{H^2(\Omega)}^2 + \varepsilon^2 \left( 2q_+ - \varepsilon \|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} \right) \|\nabla \Psi_2\|_{L^2(\Omega)}^2 \\
&\quad + \left( q_+^2 - \varepsilon \|\tilde{q}\|_{L^\infty(\Omega \setminus \Gamma(\ell))} - \varepsilon^4 \right) \|\Psi_2\|_{L^2(\Omega)}^2
\end{aligned} \tag{7.4.7}$$

and (7.4.1) follows for  $\varepsilon$  sufficiently small.  $\square$

**Lemma 7.13.** *Under the conditions of Theorem 7.11, there exist positive constants  $\varepsilon_0, C_1, C_2, C_b$  such that for  $\varepsilon < \varepsilon_0$*

$$\|\Psi\|_{H^1(\Omega)} \leq C_1 \varepsilon^{-1}, \tag{7.4.8}$$

$$\|\Psi\|_{H^2(\Omega)} \leq C_2 \varepsilon^{-2}, \tag{7.4.9}$$

$$B[\Psi_1, \Psi_2] \geq -C_b \varepsilon^2, \tag{7.4.10}$$

for some  $C > 0$  independent of  $\varepsilon$ .

**Proof:** Neumann boundary conditions on  $\Psi$  allow us to integrate by parts yielding

$$(\mathcal{A}\Psi, \Psi)_{L^2(\Omega)} = \int_{\Omega} [\varepsilon^2 |\nabla \Psi|^2 + q \Psi^2] dx. \tag{7.4.11}$$

Using (7.3.24) and the definition of  $B$  in (7.3.1)

$$\|\mathcal{A}\Psi\|_{L^2(\Omega)}^2 = B[\Psi] - \varepsilon(\tilde{q}\Psi, \Psi) \leq U_0, \tag{7.4.12}$$

for  $\varepsilon$  sufficiently small, since  $\|\tilde{q}\|_{L^\infty(\Omega)} = O(1)$  and  $\|\Psi\|_{L^2(\Omega)} = 1$ . Using (7.4.11) and (7.4.12) gives

$$\begin{aligned}\varepsilon^2 \|\nabla \Psi\|_{L^2(\Omega)}^2 &= (\mathcal{A}\Psi, \Psi)_{L^2(\Omega)} - (q\Psi, \Psi)_{L^2(\Omega)} \\ &\leq \left( \|\mathcal{A}\Psi\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)} \right) \|\Psi\|_{L^2(\Omega)} \leq \sqrt{U_0} + O(1) = O(1),\end{aligned}\tag{7.4.13}$$

since  $\|q\|_{L^\infty(\Omega)} = O(1)$  and  $\|\Psi\|_{L^2(\Omega)} = 1$ . Estimate (7.4.13) implies (7.4.8). To obtain (7.4.9) note that

$$\Delta \Psi = -\frac{1}{\varepsilon^2} \mathcal{A}\Psi + \frac{1}{\varepsilon^2} q\Psi^2\tag{7.4.14}$$

Using elliptic regularity (Evans, Theorem 4, p.317)

$$\|\Psi\|_{H^2(\Omega)} \leq C \left( \|\Psi\|_{L^2(\Omega)} + \|-\varepsilon^{-2} \mathcal{A}\Psi + \varepsilon^{-2} q\Psi^2\|_{L^2(\Omega)} \right) \leq C(1 + O(\varepsilon^{-2})) = O(\varepsilon^{-2}),\tag{7.4.15}$$

since  $\|\mathcal{A}\Psi\|_{L^2(\Omega)} \leq \sqrt{U_0}$  by (7.4.12) and  $\|q\|_{L^\infty(\Omega)} = O(1)$ .

To establish (7.4.10) we write

$$\begin{aligned}B[\Psi_1, \Psi_2] &= \int_{\Omega} (-\varepsilon^2 \Delta \Psi_1 + q\Psi_1)(-\varepsilon^2 \Delta \Psi_2 + q\Psi_2) dx + \varepsilon \int_{\Omega} \tilde{q} \Psi_1 \Psi_2 dx \\ &= \int_{\Omega} (-\varepsilon^2 \Delta(\eta_1 \Psi) + q\eta_1 \Psi)(-\varepsilon^2 \Delta(\eta_2 \Psi) + q\eta_2 \Psi) dx + \varepsilon \int_{\Omega} \tilde{q} \eta_1 \eta_2 \Psi^2 dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7\end{aligned}\tag{7.4.16}$$

where

$$I_1 = \int_{\Omega} (-\varepsilon^2 \Delta \eta_1 + q\eta_1)(-\varepsilon^2 \Delta \eta_2 + q\eta_2) \Psi^2 dx,\tag{7.4.17}$$

$$I_2 = -2\varepsilon^2 \int_{\Omega} \Psi((q\eta_2 + \Delta\eta_2)\nabla\eta_1 + (q\eta_1 + \Delta\eta_1)\nabla\eta_2) \cdot \nabla\Psi dx, \quad (7.4.18)$$

$$I_3 = 2\varepsilon^4 \int_{\Omega} (\nabla\eta_1 \cdot \nabla\Psi)(\nabla\eta_2 \nabla\Psi) dx, \quad (7.4.19)$$

$$I_4 = \varepsilon^4 \int_{\Omega} (\eta_1 \Delta\eta_2 + \eta_2 \Delta\eta_1) \Psi \Delta\Psi dx - \varepsilon^2 \int_{\Omega} 2q(\eta_1 \eta_2) \Psi \Delta\Psi dx, \quad (7.4.20)$$

$$I_5 = \varepsilon^4 \int_{\Omega} (\eta_1 \nabla\eta_2 + \eta_2 \nabla\eta_1) \cdot \Psi \Delta\Psi dx, \quad (7.4.21)$$

$$I_6 = \varepsilon^4 \int_{\Omega} \eta_1 \eta_2 |\Delta\Psi|^2 dx, \quad (7.4.22)$$

$$I_7 = \varepsilon \int_{\Omega} \tilde{q} \eta_1 \eta_2 \Psi^2 dx. \quad (7.4.23)$$

We use  $\eta_i \geq 0$ ,  $\|\eta_i\|_{L^\infty(\Omega)}$ ,  $\|\nabla\eta_i\|_{L^\infty(\Omega)}$ ,  $\|q\|_{L^\infty(\Omega)}$ ,  $\|\nabla q\|_{L^\infty(\Omega \setminus \Gamma(\ell))} = O(1)$  and  $\|\Psi\|_{L^2(\Omega)} = 1$  to estimate  $I_1$  through  $I_7$ .

$$\begin{aligned} I_1 &= \varepsilon^4 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \Delta\eta_1 \Delta\eta_2 \Psi^2 dx - \varepsilon^2 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} q(\eta_1 \Delta\eta_2 + \eta_2 \Delta\eta_1) \Psi^2 dx \\ &\quad + \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} q^2 \eta_1 \eta_2 \Psi^2 dx \geq -C\varepsilon^2, \end{aligned} \quad (7.4.24)$$

for some  $C$ . Similarly, since

$$\Psi \left( (q\eta_2 + \Delta\eta_2)\nabla\eta_1 + (q\eta_1 + \Delta\eta_1)\nabla\eta_2 \right) \cdot \nabla\Psi = \frac{1}{2} \left( (q\eta_2 + \Delta\eta_2)\nabla\eta_1 + (q\eta_1 + \Delta\eta_1)\nabla\eta_2 \right) \cdot \nabla\Psi^2, \quad (7.4.25)$$

after integration by parts

$$I_2 = \varepsilon^2 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \nabla \cdot \left( (q\eta_2 + \Delta\eta_2)\nabla\eta_1 + (q\eta_1 + \Delta\eta_1)\nabla\eta_2 \right) \Psi^2 dx = O(\varepsilon^2). \quad (7.4.26)$$

Using  $\|\Psi\|_{H^1(\Omega)} = O(\varepsilon^{-1})$  from (7.4.8) gives

$$I_3 = 2\varepsilon^4 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} (\nabla \eta_1 \cdot \nabla \Psi)(\nabla \eta_2 \cdot \nabla \Psi) dx = O(\varepsilon^2). \quad (7.4.27)$$

Using  $q, \eta_1, \eta_2 \geq 0$  in  $\Gamma(2\ell) \setminus \Gamma(\ell)$  and  $\|\Psi\|_{H^2(\Omega)} = O(\varepsilon^{-2})$  from (7.4.9) gives

$$\begin{aligned} I_4 &= O(\varepsilon^2) + 2\varepsilon^2 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} q\eta_1\eta_2 |\nabla \Psi|^2 dx + 2\varepsilon^2 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} (\nabla(q\eta_1\eta_2) \cdot \nabla \Psi) \Psi dx \\ &\geq O(\varepsilon^2) - \varepsilon^2 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \Delta(q\eta_1\eta_2) \Psi^2 dx \geq -C\varepsilon^2, \end{aligned} \quad (7.4.28)$$

for some  $C$ .

$$I_5 = \varepsilon^4 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \sum_{i,j=1}^d \tilde{\eta}_i \frac{\partial \Psi}{\partial x_i} \frac{\partial^2 \Psi}{\partial x_j^2} dx, \quad (7.4.29)$$

where  $\tilde{\eta} = \eta_1 \nabla \eta_2 + \eta_2 \nabla \eta_1$ . Therefore after integration by parts

$$\begin{aligned} I_5 &= -\varepsilon^4 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \tilde{\eta}_i \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \frac{\partial \Psi}{\partial x_j} dx - \varepsilon^4 \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \sum_{i,j=1}^d \frac{\partial \tilde{\eta}_i}{\partial x_j} \frac{\partial \Psi}{\partial x_i} \frac{\partial \Psi}{\partial x_j} dx \\ &= -\frac{\varepsilon^4}{2} \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} \sum_{i,j=1}^d \tilde{\eta}_i \frac{\partial}{\partial x_i} \left( \frac{\partial \Psi}{\partial x_j} \right)^2 dx + O(\varepsilon^2) \\ &= \frac{\varepsilon^4}{2} \int_{\Gamma(2\ell) \setminus \Gamma(\ell)} (\nabla \cdot \tilde{\eta}) |\nabla \Psi|^2 dx + O(\varepsilon^2) = O(\varepsilon^2). \end{aligned} \quad (7.4.30)$$

Finally,

$$I_6 \geq 0, \quad (7.4.31)$$

and

$$I_7 = O(\varepsilon^2), \quad (7.4.32)$$

since  $\tilde{q}$  is localized. □

The following technical integration by parts lemma is key in the proof of Proposition 7.15.

**Lemma 7.14.** *Under the conditions of Theorem 7.11, there exist positive constants  $\varepsilon_0, C$  such that for  $\varepsilon < \varepsilon_0$*

$$\left| \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( (-L_a \Psi_1) \Delta_G \Psi_1 - \sum_{i,j}^{d-1} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} \right) J dz ds \right| < C(K), \quad (7.4.33)$$

where  $L_a$  is given in (7.3.3).

**Proof:** Letting

$$S := \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (-L_a \Psi_1) \Delta_G \Psi_1 J dz ds, \quad (7.4.34)$$

and substituting in the expressions for  $L_a$  from (7.3.3) and  $\Delta_G$  from (2.2.14) we have

$$S = \sum_{i,j=1}^d \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \left( \frac{\partial}{\partial z} J \frac{\partial \Psi_1}{\partial z} \right) J^{-1} - q \Psi_1 \right) \frac{\partial}{\partial s_i} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds. \quad (7.4.35)$$

Since  $\Psi_1$  is compactly supported inside  $\Gamma(2\ell)$  we may integrate by parts in  $z$

$$\begin{aligned} S &= \sum_{i,j=1}^d \left( - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial \Psi_1}{\partial z} \frac{\partial^2}{\partial z \partial s_i} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds \right. \\ &\quad + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} J \frac{\partial \Psi_1}{\partial z} \frac{\partial z J}{J^2} \frac{\partial}{\partial s_i} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds \\ &\quad \left. - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} q \Psi_1 \frac{\partial}{\partial s_i} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds \right). \end{aligned} \quad (7.4.36)$$

Next, integrating by parts with respect to  $s_i$  gives

$$\begin{aligned}
S &= \sum_{i,j=1}^d \left( \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial}{\partial z} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds \right. \\
&\quad + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial_z J}{J} \frac{\partial \Psi_1}{\partial z} \frac{\partial}{\partial s_i} \left( G^{ij} J \frac{\partial \Psi_1}{\partial s_j} \right) dz ds \Big) \\
&\quad + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial q}{\partial s_i} \Psi_1 \frac{\partial \Psi_1}{\partial s_j} G^{ij} J dz ds + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} q G^{ij} J \frac{\partial \Psi_1}{\partial s_i} \frac{\partial \Psi_1}{\partial s_j} dz ds \quad (7.4.37)
\end{aligned}$$

Distributing the derivatives and integrating by parts with respect to  $s_i$  in the second term we obtain

$$\begin{aligned}
S &= \sum_{i,j=1}^d \left( \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} G^{ij} J dz ds + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} G^{ij} \partial_z J \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial \Psi_1}{\partial s_j} dz ds \right. \\
&\quad + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\partial_z G^{ij}) \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial \Psi_1}{\partial s_j} J dz ds - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} G^{ij} \partial_z J \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial \Psi_1}{\partial s_j} dz ds \\
&\quad - \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \partial_{s_i} \left( \frac{\partial_z J}{J} \right) G^{ij} \frac{\partial \Psi_1}{\partial z} \frac{\partial \Psi_1}{\partial s_j} J dz ds \Big) \\
&\quad + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial q}{\partial s_i} \Psi_1 \frac{\partial \Psi_1}{\partial s_j} G^{ij} J dz ds + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} q G^{ij} J \frac{\partial \Psi_1}{\partial s_i} \frac{\partial \Psi_1}{\partial s_j} dz ds. \quad (7.4.38)
\end{aligned}$$

Simplifying

$$\begin{aligned}
S &= \sum_{i,j}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds \\
&+ \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\partial_z G^{ij}) \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial \Psi_1}{\partial s_j} J dz ds \\
&- \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \partial_{s_i} \left( \frac{Jz}{J} \right) G^{ij} \frac{\partial \Psi_1}{\partial z} \frac{\partial \Psi_1}{\partial s_j} J dz ds + \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \frac{\partial q}{\partial s_i} \Psi_1 \frac{\partial \Psi_1}{\partial s_j} G^{ij} J dz ds \\
&= \sum_{i,j}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds + O(\varepsilon^{-1}),
\end{aligned}$$

where we used (2.4.23) and (2.4.25) as well as estimates (7.4.8) and (7.4.9).  $\square$

**Proposition 7.15.** *Under the conditions of Theorem 7.11, there exist positive constants  $\varepsilon_0, C$  such that for  $\varepsilon < \varepsilon_0$*

$$\begin{aligned}
B[\Psi_1] &= \int_{\Gamma} b[\Psi_1] ds + 2\varepsilon^2 \sum_{i,j} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds \\
&+ \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \varepsilon^4 (\Delta_G \Psi_1)^2 J dz ds + O(\varepsilon),
\end{aligned} \tag{7.4.39}$$

where

$$b[u, v] = (L_a u, L_a v)_J + \varepsilon(\tilde{q}u, v)_J, \tag{7.4.40}$$

is the bilinear form associated with  $L = L_a^2 + \varepsilon \tilde{q}$  on  $H^2(-\ell/\varepsilon, \ell/\varepsilon)$ .

**Proof:** Recalling definition (7.3.1) we calculate

$$\begin{aligned}
B[\Psi_1] &:= B[\Psi_1, \Psi_1] = \int_{\Gamma(2\ell)} (-\varepsilon^2 \Delta \Psi_1 + q \Psi_1)^2 dx + \int_{\Omega} \varepsilon \tilde{q} \Psi_1^2 dx \\
&= \varepsilon^4 \int_{\Gamma(2\ell)} (\Delta \Psi_1)^2 dx - 2\varepsilon^2 \int_{\Omega} q(\Delta \Psi_1) \Psi_1 dx + \int_{\Omega} q^2 \Psi_1^2 dx + \varepsilon \int_{\Gamma(2\ell)} \tilde{q} \Psi_1^2 dx \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{7.4.41}$$

We first use (2.2.25) to rewrite  $A_1$  and  $A_2$  in local coordinates

$$\begin{aligned}
A_1 &= \int_{\Gamma(2\ell)} \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} + \varepsilon^2 \Delta_G \Psi_1 \right)^2 dx \\
&= \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left[ \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} \right)^2 + 2\varepsilon^2 \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} \right) \Delta_G \Psi_1 \right. \\
&\quad \left. + \varepsilon^4 (\Delta_G \Psi_1)^2 \right] J dz ds,
\end{aligned}$$

$$\begin{aligned}
A_2 &= -2 \int_{\Gamma(2\ell)} q \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} + \varepsilon^2 \Delta_G \Psi_1 \right) \Psi_1 J dz ds \\
&= -2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} q \Psi_1 \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} \right) J dz ds \\
&\quad - 2\varepsilon^2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} q \Psi_1 (\Delta_G \Psi_1) dz ds.
\end{aligned} \tag{7.4.42}$$

Combining the expressions for  $A_1$  and  $A_2$  we obtain

$$\begin{aligned}
B[\Psi_1] &= \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} - q \Psi_1 \right)^2 J dz ds + \varepsilon \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \tilde{q} \Psi_1^2 J dz ds \\
&+ 2\varepsilon^2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z^2} + \varepsilon \kappa \frac{\partial \Psi_1}{\partial z} - q \Psi_1 \right) \Delta_G \Psi_1 J dz ds \\
&+ \varepsilon^4 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G \Psi)^2 J dz ds \\
&= \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (L_a \Psi_1)^2 + \varepsilon \tilde{q} \Psi_1^2 J dz ds + \varepsilon^4 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G \Psi)^2 J dz ds \\
&+ 2\varepsilon^2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (-L_a \Psi_1) \Delta_G \Psi_1 J dz ds \\
&= \int_{\Gamma} b[\Psi_1] ds + \varepsilon^4 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G \Psi)^2 J dz ds \\
&+ 2\varepsilon^2 \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (-L_a \Psi_1) \Delta_G \Psi_1 J dz ds,
\end{aligned}$$

and (7.4.39) follows from (7.4.33). □

**Lemma 7.16.** *Under the conditions of Theorem 7.11, there exist positive constants  $\varepsilon_0, C(K)$  such that for  $\varepsilon < \varepsilon_0$*

$$S := \sum_{i,j=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds \geq -C(K) \varepsilon^{-1}. \quad (7.4.43)$$

**Proof:** The idea is to rewrite  $S$  as  $(L_a f, f)$  for an appropriate function  $f$  and use the deMottoni-Schatzman estimates from Proposition 7.10. Indeed, expanding  $G^{ij}$  with the help

of Proposition 7.4 yields

$$\begin{aligned}
S &= \sum_{i,j=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds \\
&= \sum_{m,i,j=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \vartheta_{mi} \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \vartheta_{mj} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_j} \vartheta_{mj} \frac{\partial \Psi_1}{\partial s_i} \right) J dz ds \\
&= \sum_m^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \right)^2 + q \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right)^2 \right) J dz ds \\
&= \sum_m^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \left( \sum_{i=1}^{d-1} \frac{\partial}{\partial z} \left( \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right) \right)^2 + q \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right)^2 \right) J dz ds \\
&\quad - \sum_{m=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi_1}{\partial s_i} \right)^2 J dz ds \\
&\quad - 2 \sum_{m=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi_1}{\partial s_i} \right) \left( \sum_{j=1}^{d-1} \vartheta_{mj} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} \right) J dz ds \\
&= \sum_m^{d-1} \int_{\Gamma} \left( La \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right), \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right) \right)_J \\
&\quad - \sum_{m=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi_1}{\partial s_i} \right)^2 J dz ds \\
&\quad - 2 \sum_{m=1}^{d-1} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi_1}{\partial s_i} \right) \left( \sum_{j=1}^{d-1} \vartheta_{mj} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} \right) J dz ds. \tag{7.4.44}
\end{aligned}$$

From Lemma 7.13 we have

$$\left\| \frac{\partial \Psi_1}{\partial s_i} \right\|_{L^2(\Omega)} \leq C(K) \varepsilon^{-1}. \tag{7.4.45}$$

Lemmas 7.5 and Proposition 7.8 then imply that there exists a constant  $C(K) > 0$  such that

$$\left\| \sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi_1}{\partial s_i} \right\|_{L^2(\Omega)} \leq \sum_{i=1}^{d-1} \left\| \frac{\partial \vartheta_{mi}}{\partial z} \right\|_{L^\infty(\Gamma(2\ell))} \left\| \frac{\partial \Psi_1}{\partial s_i} \right\|_{L^2(\Omega)} \leq C(K), \quad (7.4.46)$$

$$\left\| \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial^2 \Psi_1}{\partial s_i \partial z} \right\|_{L^2(\Omega)} \leq \sum_{i=1}^{d-1} \|\vartheta_{mi}\|_{L^\infty(\Gamma(2\ell))} \left\| \frac{\partial^2 \Psi_1}{\partial s_i \partial z} \right\|_{L^2(\Omega)} \leq C(K) \varepsilon^{-1}, \quad (7.4.47)$$

and

$$\left\| \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right\|_{L^2(\Omega)} \leq \sum_{i=1}^{d-1} \|\vartheta_{mi}\|_{L^\infty(\Gamma(2\ell))} \left\| \frac{\partial \Psi_1}{\partial s_i} \right\|_{L^2(\Omega)} \leq C(K) \varepsilon^{-1}, \quad (7.4.48)$$

In addition, it follows from Proposition 7.10 that

$$\begin{aligned} \sum_m \int_\Gamma \left( L_a \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right), \left( \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right) \right)_J &\geq \frac{\nu_0}{2} \sum_m \int_\Gamma \left\| \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i} \right\|_J^2 \\ &\geq -C(K) \varepsilon^{-1}. \end{aligned}$$

Combining (7.4.45)-(7.4.49) into (7.4.44) yields (7.4.43). □

**Lemma 7.17.** *Under the conditions of Theorem 7.11, there exist positive constants  $\varepsilon_0, C$  such that for  $\varepsilon < \varepsilon_0$ ,  $\Psi_1 = \eta_1 \Psi$  satisfies*

$$-C\varepsilon \leq \int_\Gamma b[\Psi_1] ds \leq U + O(\varepsilon), \quad (7.4.49)$$

and

$$\|\Delta_G \Psi_1\|_{L^2(\Gamma(2\ell))}^2 \leq U\varepsilon^{-4} + O(\varepsilon^{-3}), \quad (7.4.50)$$

where

$$b[u, v] = (L_a u, L_a v)_J + \varepsilon(\tilde{q}u, v)_J, \quad (7.4.51)$$

is the bilinear form associated with  $L = L_a^2 + \varepsilon\tilde{q}$  on  $H^2(-\ell/\varepsilon, \ell/\varepsilon)$ .

**Proof:** From assumption (7.3.24)

$$B[\Psi] = B[\Psi_1] + 2B[\Psi_1, \Psi_2] + B[\Psi_2] \leq U. \quad (7.4.52)$$

Lower bounds on  $B[\Psi_2]$  and  $B[\Psi_1, \Psi_2]$  in Lemmas 7.12 and 7.13 imply

$$B[\Psi_1] \leq U + O(\varepsilon^2). \quad (7.4.53)$$

Substituting expression (7.4.39) for  $B[\Psi_1]$  gives

$$\begin{aligned} \int_{\Gamma} b[\Psi_1] ds &+ 2\varepsilon^2 \sum_{i,j} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial^2 \Psi_1}{\partial z \partial s_i} \frac{\partial^2 \Psi_1}{\partial z \partial s_j} + q \frac{\partial \Psi_1}{\partial s_j} \frac{\partial \Psi_1}{\partial s_i} \right) G^{ij} J dz ds \\ &+ \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \varepsilon^4 (\Delta_G \Psi_1)^2 J dz ds + O(\varepsilon^2) \leq U + O(\varepsilon^2). \end{aligned} \quad (7.4.54)$$

Using (7.4.43) in (7.4.54) and dropping the positive  $(\Delta_G \Psi_1)^2$  integral yields the upper bound in (7.4.49). To obtain the lower bound in (7.4.49) we observe that the definition of  $b$  in (7.4.51)

and  $\|\tilde{q}\|_{L^\infty(\Gamma(2\ell))} = O(1)$  imply

$$\int_{\Gamma} b[\Psi_1] ds \geq \varepsilon \int_{\Gamma} (\tilde{q}\Psi_1, \Psi_1)_J ds \geq -\varepsilon \|\tilde{q}\|_{L^\infty(\Gamma(2\ell))} \int_{\Gamma} \|\Psi_1\|_J^2 ds \geq -C\varepsilon, \quad (7.4.55)$$

since

$$\int_{\Gamma} \|\Psi_1\|_J^2 ds = \|\Psi_1\|_{L^2(\Gamma(2\ell))}^2 \leq \|\Psi_1\|_{L^2(\Omega)}^2 = 1. \quad (7.4.56)$$

The last estimate (7.4.50) follows from (7.4.54) and Lemma 7.16.  $\square$

### Proof of Theorem 7.11:

By Lemma 7.17,  $\Psi_1 = \eta_1 \Psi$  satisfies estimate (7.4.49). Substituting the expression for the bilinear form  $b$  (see Table 7.1) into the estimate (7.4.49) yields

$$\int_{\Gamma} (L_a \Psi_1, L_a \Psi_1)_J ds + \varepsilon \int_{\Gamma} (\tilde{q}\Psi_1, \Psi_1)_J ds \leq U + O(\varepsilon). \quad (7.4.57)$$

We use  $\|\tilde{q}\|_{L^\infty(\Omega)} = O(1)$  and  $\|\Psi\|_{L^2(\Omega)} = 1$  to estimate the second term on the left-hand side of (7.4.57)

$$\varepsilon \left| \int_{\Gamma} (\tilde{q}\Psi_1, \Psi_1)_J ds \right| = \varepsilon \left| \int_{\Gamma(2\ell)} \tilde{q}\Psi_1^2 dx \right| \leq \varepsilon \|\tilde{q}\|_{L^\infty(\Omega)} \|\Psi_1\|_{L^2(\Omega)}^2 = O(\varepsilon). \quad (7.4.58)$$

Returning this estimate to (7.4.57) yields the upper bound

$$\int_{\Gamma} (L_a \Psi_1, L_a \Psi_1)_J ds \leq U + O(\varepsilon). \quad (7.4.59)$$

Substituting the definition of  $L_a$ , from (7.1.16), into (7.4.59) yields

$$\int_{\Gamma} \left( (L_0 + \varepsilon(q_1 - \kappa \partial_z)) \Psi_1, (L_0 + \varepsilon(q_1 - \kappa \partial_z)) \Psi_1 \right)_J ds \leq U + O(\varepsilon), \quad (7.4.60)$$

and expanding the  $J$ -inner product we obtain

$$\begin{aligned} \int_{\Gamma} (L_0 \Psi_1, L_0 \Psi_1)_J + 2\varepsilon \left( (q_1 - \kappa \partial_z) \Psi_1, L_0 \Psi_1 \right)_J + \varepsilon^2 \left( (q_1 - \kappa \partial_z) \Psi_1, (q_1 - \kappa \partial_z) \Psi_1 \right)_J ds \\ \leq U + O(\varepsilon). \end{aligned} \quad (7.4.61)$$

Using estimates (7.2.16) and (7.2.17) of Proposition 7.8 in (7.4.8) and (7.4.9) yields control of  $z$  derivatives of  $\Psi_1$

$$\|\partial_z \Psi_1\|_{L^2(\Gamma(2\ell))}^2 = O(1), \quad (7.4.62)$$

and

$$\|\partial_z^2 \Psi_1\|_{L^2(\Gamma(2\ell))}^2 = O(1). \quad (7.4.63)$$

Using these derivative bounds and  $\|q_0\|_{L^\infty(\Gamma(2\ell))}, \|q_1\|_{L^\infty(\Gamma(2\ell))} = O(1)$ , we estimate

$$\begin{aligned} \|L_0 \Psi_1\|_{L^2(\Gamma(2\ell))} &= \| -\partial_z^2 \Psi_1 + q_0 \Psi_1 \|_{L^2(\Gamma(2\ell))} \\ &\leq \|\partial_z^2 \Psi_1\|_{L^2(\Gamma(2\ell))} + \|q_0\|_{L^\infty(\Gamma(2\ell))} \|\Psi_1\|_{L^2(\Gamma(2\ell))} = O(1), \end{aligned} \quad (7.4.64)$$

and

$$\begin{aligned}
\left| \int_{\Gamma} \left( (q_1 - \kappa \partial_z) \Psi_1, L_0 \Psi_1 \right)_J ds \right| &= \left| \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} ((q_1 - \kappa \partial_z) \Psi_1) (L_0 \Psi_1) J dz ds \right| \\
&\leq \|q_1\|_{L^\infty(\Gamma(2\ell))} \|\Psi_1\|_{L^2(\Gamma(2\ell))} \|L_0 \Psi_1\|_{L^2(\Gamma(2\ell))} \\
&\quad + \|\kappa\|_{L^\infty(\Gamma(2\ell))} \|\partial_z \Psi_1\|_{L^2(\Gamma(2\ell))} \|L_0 \Psi_1\|_{L^2(\Gamma(2\ell))} \\
&= O(1).
\end{aligned} \tag{7.4.65}$$

Similar estimates imply

$$\int_{\Gamma} \left( (q_1 - \kappa \partial_z) \Psi_1, (q_1 - \kappa \partial_z) \Psi_1 \right)_J ds = O(1). \tag{7.4.66}$$

Returning (7.4.65) and (7.4.66) to (7.4.61) yields

$$\int_{\Gamma} (L_0 \Psi_1, L_0 \Psi_1)_J ds \leq U + O(\varepsilon). \tag{7.4.67}$$

However, the definition  $L_0$  allows us to expand

$$\begin{aligned}
L_0(J^{1/2} \Psi_1) &= (-\partial_z^2 + q_0)(J^{1/2} \Psi_1) \\
&= -\partial_z \left( (1/2) J^{-1/2} (\partial_z J) \Psi_1 + J^{1/2} \partial_z (\Psi_1) \right) + q_0 J^{1/2} \Psi_1 \\
&= \frac{1}{4} J^{-3/2} (\partial_z J)^2 \Psi_1 - \frac{1}{2} J^{-1/2} (\partial_z^2 J) \Psi_1 \\
&\quad - J^{-1/2} (\partial_z J) (\partial_z \Psi_1) - J^{1/2} (\partial_z^2 \Psi_1) + q_0 J^{1/2} \Psi_1 \\
&= J^{1/2} L_0 \Psi_1 + \frac{1}{4} J^{-3/2} (\partial_z J)^2 \Psi_1 - \frac{1}{2} J^{-1/2} (\partial_z^2 J) \Psi_1 - J^{-1/2} \partial_z J \partial_z \Psi_1
\end{aligned}$$

to obtain

$$\left( L_0(J^{1/2}\Psi_1), L_0(J^{1/2}\Psi_1) \right)_0 = (L_0\Psi_1, L_0\Psi_1)_J + O(\varepsilon), \quad (7.4.68)$$

where we used (2.4.26) to estimate the terms involving the derivatives of  $J$ . Decompose  $J^{1/2}\Psi_1$  into tangential and normal modes as in (7.3.25)

$$J^{1/2}\Psi_1 = \sum_{k=1}^r c_k(s) \psi_k^0(s) + \psi^\perp(s), \quad (7.4.69)$$

where

$$(\psi_k^0(s), \psi^\perp(s))_0 = 0, \quad (7.4.70)$$

for each  $s \in S$  and since  $\int_\Gamma \|\Psi_1\|_J^2 ds = \|\Psi_1\|_{L^2(\Gamma(2\ell))}^2 \leq 1$ ,

$$\int_\Gamma \sum_{k=1}^r c_k^2 ds + \int_\Gamma \|\psi^\perp\|_0^2 ds = 1. \quad (7.4.71)$$

Substituting the decomposition (7.4.69) into (7.4.68) yields,

$$\begin{aligned} (L_0\Psi_1, L_0\Psi_1)_J &= \left( L_0 \left( \sum_{k=1}^r c_k \psi_k^0 + \psi^\perp \right), L_0 \left( \sum_{l=1}^r c_l \psi_l^0 + \psi^\perp \right) \right) + O(\varepsilon) \\ &= \left( \sum_{k=1}^r c_k \lambda_k^0 \psi_k^0 + L_0 \psi^\perp, \sum_{l=1}^r c_l \lambda_l^0 \psi_l^0 + L_0 \psi^\perp \right)_0 + O(\varepsilon) \\ &= \sum_{k=1}^r c_k^2 (\lambda_k^0)^2 + (L_0 \psi^\perp, L_0 \psi^\perp)_0 + O(\varepsilon), \end{aligned} \quad (7.4.72)$$

and the bound,

$$\int_{\Gamma} \sum_{k=1}^r c_k^2(s) (\lambda_k^0(s))^2 ds + \int_{\Gamma} (L_0 \psi^{\perp}, L_0 \psi^{\perp}) ds \leq U + O(\varepsilon), \quad (7.4.73)$$

Since the first integral in (7.4.73) is nonnegative,

$$\int_{\Gamma} (L_0 \psi^{\perp}, L_0 \psi^{\perp}) \leq U + O(\varepsilon). \quad (7.4.74)$$

In addition, choosing  $V = \text{span}\{\psi_1^0, \dots, \psi_r^0\}$  in the mini-max characterization for  $L_0$  and recalling the lower bound on the rest of the spectrum in Hypothesis (H) yields

$$\nu_0 \leq \lambda_{r+1}^0 \leq \frac{(L_0 \psi^{\perp}, \psi^{\perp})_0}{\|\psi^{\perp}\|_0^2}. \quad (7.4.75)$$

from which we deduce the bound

$$\nu_0 \|\psi^{\perp}\|_0^2 \leq (L_0 \psi^{\perp}, \psi^{\perp})_0 \leq \|L_0 \psi^{\perp}\|_0 \|\psi^{\perp}\|_0. \quad (7.4.76)$$

Dividing by  $\|\psi^{\perp}\|_0$ , squaring both sides and integrating with respect to  $\Gamma$  yields the estimate

$$\int_{\Gamma} \|\psi^{\perp}\|_0^2 ds \leq \frac{1}{\nu_0^2} \int_{\Gamma} \|L_0 \psi^{\perp}\|_0^2 ds \leq \frac{U}{\nu_0^2} + O(\varepsilon). \quad (7.4.77)$$

Equivalently we may write this as

$$\|J^{-1/2} \psi^{\perp}\|_{L^2(\Gamma(2\ell))}^2 = \int_{\Gamma} \|\psi^{\perp}\|_0^2 ds \leq \frac{U}{\nu_0^2} + O(\varepsilon). \quad (7.4.78)$$

Setting  $\Psi^\perp = J^{-1/2}\psi^\perp$  yields (7.3.27).

To address the  $H^2$  bounds of (7.3.28) we take  $\Delta_s$  of (7.3.23), yielding

$$\begin{aligned}
\Delta_s c_k &= (\Delta_s \Psi_1, \psi_k)_J + (\Psi_1, \Delta_s \psi_k)_J + (\Psi_1, \psi_k J^{-1}(\Delta_s J))_J \\
&+ 2(\nabla_s \Psi_1, \nabla_s \psi_k)_J + 2(\nabla_s \Psi_1 \cdot J^{-1} \nabla_s J, \psi_k)_J \\
&+ 2(\Psi_1, \nabla_s \psi_k \cdot J^{-1} \nabla_s J)_J.
\end{aligned} \tag{7.4.79}$$

Utilizing the relation between  $\Delta_G$  and  $\Delta_s$  shown in Lemma 2.10 we further obtain

$$\begin{aligned}
|(\Delta_s \Psi_1, \psi_k)_J| &= \left| \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_s \Psi_1) \psi_k J dz \right| = \left| \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G \Psi_1 + \varepsilon z D \Psi_1) \psi_k J dz \right| \\
&\leq \|\Delta_G \Psi_1\|_J \|\psi_k\|_J + \varepsilon \|D \Psi_1\|_J \|z \psi_k\|_J,
\end{aligned} \tag{7.4.80}$$

where  $D$  represents a second order differential operator

$$D := \sum_{i,j=1}^{d-1} d_{ij} \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} d_j \frac{\partial}{\partial s_j}, \tag{7.4.81}$$

with coefficients bounded by some constant  $C(K)$ .

Substituting the bound on the inner product in (7.4.80) into (7.4.79) we deduce

$$\begin{aligned}
|\Delta_s c_k| &\leq \|\Delta_G \Psi_1\|_J + \varepsilon \|D\Psi_1\|_J \|z\psi_k\|_J + \|\Psi_1\|_J \|\Delta_s \psi_k\|_J \\
&+ \|J^{-1} \Delta_s J\|_{L^\infty(\Gamma(2\ell))} \|\Psi_1\|_J \\
&+ 2\|\nabla_s \Psi_1\|_J \|\nabla_s \psi_k\|_J + 2\|J^{-1} \Delta_s J\|_{L^\infty(\Gamma(2\ell))} \|\nabla_s \Psi_1\|_J \\
&+ 2\|J^{-1} \Delta_s J\|_{L^\infty(\Gamma(2\ell))} \|\Psi_1\|_J \|\nabla_s \psi_k\|_J,
\end{aligned} \tag{7.4.82}$$

where we used  $\|\psi_k\|_J = 1$ . Squaring both sides of (7.4.82) and integrating over  $\Gamma$ , we note that due to inequalities (7.1.19), (7.1.20), (2.4.26), and (7.4.8) the leading-order contribution comes from the  $\Delta_G \Psi_1$  term, which yields the desired result after applying (7.4.50),

$$\|\Delta_s c_k\|_{L^2(\Gamma)}^2 \leq \|\Delta_G \Psi_1\|_{L^2(\Gamma(2\ell))}^2 + O(\varepsilon^{-3}) \leq U\varepsilon^{-4} + O(\varepsilon^{-3}). \tag{7.4.83}$$

Addressing (7.3.29) we first observe that, taking collectively, the expression (7.4.39) and estimates (7.4.43) and (7.4.49) afford the bound

$$B[\Psi_1] \geq -C\varepsilon. \tag{7.4.84}$$

Decomposing  $\Psi$  into near and far-field parts  $\Psi = \Psi_1 + \Psi_2$  and using (7.4.10) and (7.4.84) we find that

$$B[\Psi_2] = B[\Psi] - 2B[\Psi_1, \Psi_2] - B[\Psi_1] \leq B[\Psi] + C\varepsilon. \tag{7.4.85}$$

However, inequality (7.4.1) and assumption (7.3.24) imply that

$$\frac{q_+^2}{2} \|\Psi_2\|_{L^2(\Omega)}^2 \leq B[\Psi_2] \leq B[\Psi] + C\varepsilon \leq U + O(\varepsilon). \quad (7.4.86)$$

Dividing by  $\frac{q_+^2}{2}$  yields (7.3.29).

To address (7.3.30) we again decompose  $\Psi$  as  $\Psi = \Psi_1 + \Psi_2$ , and recall that  $\|\Psi_1\|_{L^2(\Omega)} = \|\eta_1 \Psi\|_{L^2(\Omega)} \leq 1$ . In addition, (7.4.86) implies that

$$\|\Psi_2\|_{L^2(\Omega)} \leq \frac{2}{q_+^2} B[\Psi_2] \leq \frac{4}{q_+^2} U_0 \leq \frac{1}{4}, \quad (7.4.87)$$

where we used the assumption that  $U_0 \leq \frac{1}{32} q_+^2$ . This observation leads to the inequalities

$$\begin{aligned} \|\Psi_1\|_{L^2(\Omega)}^2 &= \|\Psi\|_{L^2(\Omega)}^2 - \|\Psi_2\|_{L^2(\Omega)}^2 - 2 \int_{\Omega} \Psi_1 \Psi_2 dx \\ &\geq 1 - \frac{1}{16} - 2 \|\Psi_1\|_{L^2(\Omega)} \|\Psi_2\|_{L^2(\Omega)} > \frac{1}{4}. \end{aligned} \quad (7.4.88)$$

Rewriting (7.4.71), we isolate the  $c_k$  terms, and use the estimate above and (7.4.78) to obtain the lower bound

$$\int_{\Gamma} \sum_{k=1}^r c_k^2 ds = \|\Psi_1\|_{L^2(\Omega)}^2 - \|J^{-1/2} \psi^{\perp}\|_{L^2(\Omega)}^2 \geq \frac{1}{4} - \frac{U}{\nu_0^2} - O(\varepsilon). \quad (7.4.89)$$

However, by assumption  $U \leq \frac{1}{32} \nu_0^2$ , which yields (7.3.30). □

## Chapter 8

# Analysis of the Full Linearized Operator - Coercivity of The Bilinear Form

The main result of Chapter 7, presented in Theorem 7.11, was the bound on the remainder  $\Psi^\perp$  in the decomposition

$$\Psi(x) = \Psi(s, z) = \sum_{k=1}^r c_k(s) \psi_k(s, z) + \Psi^\perp(x), \quad (8.0.1)$$

of ‘low energy’ functions  $\Psi$ . It follows that a norm-1  $w$  that is orthogonal to  $\sum_{k=1}^r c_k(s) \psi_k(s, z)$ , is approximately orthogonal to  $\Psi$ . In fact, the size of the projection  $(w, \Psi)_{L^2(\Omega)}$  is then less than  $\|\Psi^\perp\|_{L^2(\Omega)}$ .

In this chapter we use the minimax principle for eigenfunctions  $\Theta_i$  of  $\Delta_S$  to control the

size of projections  $(w, \Psi)_{L^2(\Omega)}$  for  $w \in \mathcal{Z}_M^\perp$ , where

$$Z_{M,r} := \text{span} \left( \{ \eta_1 \Theta_j(s) H_k(s, z) \}_{j=1 \dots M, k=1 \dots r} \cup \{1\} \right) \quad (8.0.2)$$

is a finite dimensional subspace of  $L^2(\Gamma)$  and  $H_k(s, z)$  are any functions supported in  $\Gamma(2\ell)$  that satisfy

$$\sup_s \|H_k - \psi_k\|_J = O(\varepsilon). \quad (8.0.3)$$

**Remark:** Definition of  $\mathcal{Z}_M$  in (3.3) is a special case and corresponds to  $r = 1$  and  $H_1 = \varepsilon^{-1/2} \phi'$ .

In Proposition 8.1 we establish ‘approximate orthogonality’ of  $w \in Z_{M,r}^\perp$  to the low energy eigenspace of  $\mathcal{L}$ . Next, in Proposition 8.2 we show that this implies that the bilinear form associated with  $\mathcal{L}$  is coercive on  $Z_{M,r}^\perp$ . We conclude the chapter with the proof of Theorem 3.5.

## 8.1 Orthogonality to Low Energy Eigenspace

The next proposition shows that the functions orthogonal to the ‘slow space’  $Z_{M,r}$  have small projections onto low energy functions.

**Proposition 8.1.** *Let  $U_0 = \frac{1}{32} \min\{q_+^2, \nu\}$ . Assume  $\Psi \in H^2(\Omega)$  with  $\|\Psi\|_{L^2(\Omega)} = 1$ ,  $\partial_\nu \Psi = 0$  on  $\partial\Omega$  satisfies the bound*

$$B[\Psi] \leq U < U_0, \quad (8.1.1)$$

for the bilinear form  $B$  given in (7.3.1). Recall the definition of  $Z_{M,r}$  in (8.0.2). Then for any  $w \in Z_{M,r}^\perp$ , satisfying  $\|w\|_{L^2(\Omega)} = 1$ , the following holds true

$$(w, \Psi)_{L^2(\Omega)}^2 \leq \left( \frac{4r\varepsilon^{-4}}{\beta_{M+1}^2} + \frac{4}{\nu} + \frac{8}{q_+^2} \right) U + O\left(\frac{\varepsilon^{-3}}{\beta_{M+1}^2}, \varepsilon\right), \quad (8.1.2)$$

where  $\beta_i$  is the  $i$ th eigenvalue of  $-\Delta_S : H^2(\Gamma) \rightarrow L^2(\Gamma)$ .

**Proof:**

We use Theorem 7.11 to decompose  $\Psi$  into near and far-field parts

$$\begin{aligned} \Psi(x) &= \Psi_1(x) + \Psi_2(x) = \eta_1 \Psi(x) + \Psi_2(x) = \eta_1 \Psi_1(x) + \Psi_2(x) \\ &= \sum_{k=1}^r \eta_1 c_k(s) \psi_k(s, z) + \eta_1 \Psi^\perp(s, z) + \Psi_2(x) \end{aligned} \quad (8.1.3)$$

where

$$\|\Psi^\perp\|_{L^2(\Gamma(2\ell))}^2 \leq \frac{U}{\nu} + O(\varepsilon), \quad (8.1.4)$$

$$\|\Psi_2\|_{L^2(\Omega)}^2 \leq \frac{U}{q_+^2} + O(\varepsilon), \quad (8.1.5)$$

$$\|\Delta_S c_k(s)\|_{L^2(\Gamma)}^2 \leq U\varepsilon^{-4} + O(\varepsilon^{-3}). \quad (8.1.6)$$

Since  $\eta_1$  is supported inside  $\Gamma(2\ell)$  we may write the inner product of (8.1.3) with  $w$  as,

$$(w, \Psi)_{L^2(\Omega)} = \left( w, \sum_{k=1}^r \eta_1 c_k \psi_k \right)_{L^2(\Gamma(2\ell))} + \left( w, \eta_1 \Psi^\perp \right)_{L^2(\Gamma(2\ell))} + \left( w, \Psi_2 \right)_{L^2(\Omega)}. \quad (8.1.7)$$

With the aid of (8.0.3) we may replace  $\psi_k$  in (8.1.7) with  $H_k$  to obtain

$$(w, \Psi)_{L^2(\Omega)} = \left( w, \sum_{k=1}^r \eta_1 c_k H_k \right)_{L^2(\Gamma(2\ell))} + \left( w, \eta_1 \Psi^\perp \right)_{L^2(\Gamma(2\ell))} + \left( w, \Psi_2 \right)_{L^2(\Omega)} + O(\varepsilon). \quad (8.1.8)$$

In addition, we decompose  $c_k$  into its projections onto the Laplace-Beltrami eigenfunctions  $\Theta_i$  to obtain

$$c_k(s) = \sum_{j=1}^M (c_k, \Theta_j)_\Gamma \Theta_j + c_k^\perp. \quad (8.1.9)$$

Substituting the expression for  $c_k$  into (8.1.8) gives

$$(w, \Psi)_{L^2(\Omega)} = A_1 + A_2 + A_3 + A_4 + O(\varepsilon), \quad (8.1.10)$$

where

$$A_1 := \sum_{j=1}^M \sum_{k=1}^r (c_k, \Theta_j)_\Gamma (w, \eta_1 \Theta_j H_k)_{L^2(\Gamma(2\ell))}, \quad (8.1.11)$$

$$A_2 := \left( w, \sum_{k=1}^r \eta_1 c_k^\perp(s) \psi_k \right)_{L^2(\Gamma(2\ell))}, \quad (8.1.12)$$

$$A_3 := (w, \eta_1 \Psi^\perp)_{L^2(\Gamma(2\ell))}, \quad (8.1.13)$$

$$A_4 := (w, \Psi_2)_{L^2(\Omega)}. \quad (8.1.14)$$

It follows from (8.1.10) that

$$(w, \Psi)_{L^2(\Omega)}^2 = (A_1 + A_2 + A_3 + A_4)^2 + O(\varepsilon^2) \leq 4A_1^2 + 4A_2^2 + 4A_3^2 + 4A_4^2 + O(\varepsilon^2). \quad (8.1.15)$$

We will estimate each term separately. Due to the fact that  $w \in Z_{M,r}^\perp$ ,

$$A_1 = 0. \quad (8.1.16)$$

To bound  $A_3$  and  $A_4$ , we use  $\|w\|_{L^2(\Omega)} = 1$  and (8.1.4) to obtain

$$A_3^2 = (w, \eta_1 \Psi^\perp)_{L^2(\Omega)}^2 \leq \|w\|_{L^2(\Omega)}^2 \|\Psi^\perp\|_{L^2(\Omega)}^2 \leq \frac{U}{\nu} + O(\varepsilon), \quad (8.1.17)$$

and (8.1.5) to bound

$$A_4^2 = (w, \Psi_2)_{L^2(\Omega)}^2 \leq 2\|w\|_{L^2(\Omega)}^2 \|\Psi_2\|_{L^2(\Omega)}^2 \leq \frac{2U}{q_+^2} + O(\varepsilon). \quad (8.1.18)$$

It remains to bound  $A_2$ . Since  $(c_k^\perp, \Theta_j)_\Gamma = 0$  for  $j = 1, \dots, M$  and  $\Theta_j$  is the  $j$ th eigenfunction of  $\Delta_s^2$  corresponding to eigenvalue  $\beta_j^2$  we have from the minimax characterization of eigenvalues of  $\Delta_s^2$

$$\beta_{M+1}^2 \leq \frac{(\Delta_s c_k^\perp, \Delta_s c_k^\perp)_\Gamma}{\|c_k^\perp\|_\Gamma^2}. \quad (8.1.19)$$

In addition, decomposition (8.1.9) yields

$$\Delta_s^2 c_k = \sum_{j=1}^M (c_k, \Theta_j)_\Gamma \Delta_s^2 \Theta_j + \Delta_s^2 c_k^\perp = \sum_{j=1}^M \beta_j^2 (c_k, \Theta_j)_\Gamma \Theta_j + \Delta_s^2 c_k^\perp \quad (8.1.20)$$

so that we may expand the inner product as

$$(\Delta_s c_k, \Delta_s c_k)_\Gamma = (\Delta_s^2 c_k, c_k)_\Gamma = \sum_{j=1}^M \beta_j^2 (c_k, \Theta_j)_\Gamma^2 + (\Delta_s^2 c_k^\perp, c_k)_\Gamma \quad (8.1.21)$$

Using (8.1.20) to expand the last term yields

$$\begin{aligned}
(\Delta_s c_k, \Delta_s c_k)_\Gamma &= \sum_{j=1}^M \beta_j^2 (c_k, \Theta_j)_\Gamma^2 + \sum_{j=1}^M \beta_j^2 (c_k, \Theta_j)_\Gamma (c_k^\perp, \Theta_j)_\Gamma + (\Delta_s c_k^\perp, \Delta_s c_k^\perp)_\Gamma \\
&= \sum_{j=1}^M \beta_j^2 (c_k, \Theta_j)_\Gamma^2 + (\Delta_s c_k^\perp, \Delta_s c_k^\perp)_\Gamma,
\end{aligned} \tag{8.1.22}$$

since  $(c_k^\perp, \Theta_j)_\Gamma = 0$ . However, since  $\beta_j^2 \geq 0$  for all  $j$  we conclude from (8.1.22) that

$$\|\Delta_s c_k^\perp\|_\Gamma^2 \leq \|\Delta_s c_k\|_\Gamma^2. \tag{8.1.23}$$

Using (8.1.6) in (8.1.23) implies

$$\|\Delta_s c_k^\perp\|_\Gamma^2 \leq U\varepsilon^{-4} + O(\varepsilon^{-3}). \tag{8.1.24}$$

It then follows from (8.1.19) that

$$\|c_k^\perp\|_\Gamma^2 \leq \frac{U\varepsilon^{-4}}{\beta_{M+1}^2} + O\left(\frac{\varepsilon^{-3}}{\beta_{M+1}^2}\right), \tag{8.1.25}$$

and

$$A_2^2 = \left(w, \sum_{k=1}^r \eta_1 c_k^\perp(s) \psi_k\right)_{L^2(\Gamma(2\ell))}^2 \leq \frac{rU\varepsilon^{-4}}{\beta_{M+1}^2} + O\left(\frac{\varepsilon^{-3}}{\beta_{M+1}^2}\right). \tag{8.1.26}$$

Returning the estimates for  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  to (8.1.15) gives (8.1.2). □

## 8.2 Coercivity of the Bilinear Form

The previous proposition gives an upper bound on the projection of  $Z_{M,r}^\perp$  onto the small eigenspace of  $\mathcal{L}$ . In this section we show that if this upper bound is sufficiently small, then the operator  $\mathcal{L}$  satisfies a coercivity estimate on  $Z_{M,r}^\perp$ . Let  $\Psi_i$  be the  $i$ th normalized eigenfunction of  $\mathcal{L}$ . Fix  $U_0 > 0$  independent of  $\varepsilon$ . For  $0 < U < U_0$  let

$$\sigma_U(\mathcal{L}) := \sigma(\mathcal{L}) \cap \{\lambda < U\} = \{\mu_1, \dots, \mu_{N_U}\}. \quad (8.2.1)$$

**Proposition 8.2.** *Let  $U_0 = \frac{1}{32} \min\{q_+^2, \nu\}$ . Recall definition of  $N_U$  from (8.2.1) and  $Z_{M,r}$  from (8.0.2). If  $U < U_0$  and  $M > 0$  satisfy*

$$\left( \frac{4r\varepsilon^{-4}}{\beta_{M+1}^2} + \frac{4}{\nu} + \frac{8}{q_+^2} \right) U + O\left( \frac{\varepsilon^{-3}}{\beta_{M+1}^2}, \varepsilon \right) < \frac{1}{2}, \quad (8.2.2)$$

then for any  $w \in Z_{M,r}^\perp$  satisfying

$$\partial_\nu w|_{\partial\Omega} = 0, \quad (8.2.3)$$

the following coercivity property holds

$$B[w] \geq \left( \frac{U}{2} + O(\varepsilon) \right) \|w\|_{L^2(\Omega)}^2 \quad (8.2.4)$$

where recall that  $B$  is the bilinear form for  $\mathcal{L}$  defined in (7.3.1). In addition, we have the bounds

$$\|\mathcal{A}w\|_{L^2(\Omega)} \geq \frac{\sqrt{U}}{2} \|w\|_{L^2(\Omega)}, \quad (8.2.5)$$

$$\|\mathcal{A}^2 w\|_{L^2(\Omega)} \geq \frac{U}{4} \|w\|_{L^2(\Omega)}, \quad (8.2.6)$$

$$\|\mathcal{A}^2 w\|_{L^2(\Omega)} \geq \frac{\sqrt{U}}{2} \|\mathcal{A} w\|_{L^2(\Omega)}, \quad (8.2.7)$$

and

$$\|\mathcal{A} w\|_{L^2(\Omega)} \geq C\sqrt{U}\varepsilon^2 \|w\|_{H^2(\Omega)}. \quad (8.2.8)$$

**Remark:** In particular, if  $M = [M_1 \varepsilon^{-(d-1)}]$  then  $\beta_M = O(\varepsilon^{-2})$  and the coercivity estimates above are valid with  $U$  sufficiently small, but independent of  $\varepsilon$ .

**Proof:** We denote by  $\Psi_i$  the eigenfunction of  $\mathcal{L}$  corresponding to the eigenvalue  $\mu_i \in \sigma_U(\mathcal{L})$ , defined in (8.2.1). If  $Z_{M,r}$  was exactly the eigenspace spanned by eigenfunctions corresponding to  $\sigma_U(\mathcal{L})$ , the coercivity estimates above would follow immediately from the minimax principle. However,  $Z_{M,r}$  only ‘approximates’ the small eigenspace of  $\mathcal{L}$ . Define the projection of  $w \in Z_{M,r}^\perp$  onto the eigenspace  $\{\Psi_i\}_{i=1}^{N_U}$

$$P_U w := \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)} \Psi_i. \quad (8.2.9)$$

For any  $w \in H^2(\Omega)$ , the minimax characterization of eigenvalues of  $\mathcal{L}$  gives

$$B[w - P_U w] \geq U \|w - P_U w\|_{L^2(\Omega)}^2. \quad (8.2.10)$$

Our goal is to show that  $P_U w$  is small in an appropriate norm. Expanding the bilinear term

and the norm, and using orthogonality we obtain

$$B[w] - 2B[w, P_U w] + B[P_U w] \geq U \left( \|w\|_{L^2(\Omega)}^2 - \sum_{i=1}^{N_U} (w, \Psi_i)^2 \right). \quad (8.2.11)$$

We may simplify the second term of (8.2.11)

$$\begin{aligned} B[w, P_U w] &= \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)} B[w, \Psi_i] = \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)} \mu_i (w, \Psi_i)_{L^2(\Omega)} \\ &= \sum_{i=1}^{N_U} \mu_i (w, \Psi_i)_{L^2(\Omega)}^2. \end{aligned} \quad (8.2.12)$$

Using orthonormality of eigenfunctions  $\Psi_i$  we estimate the third term of (8.2.11)

$$B[P_U w] = B \left[ \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)} \Psi_i, \sum_{j=1}^{N_U} (w, \Psi_j)_{L^2(\Omega)} \Psi_j \right] = \sum_{i=1}^{N_U} \mu_i (w, \Psi_i)_{L^2(\Omega)}^2. \quad (8.2.13)$$

Using (8.2.12) and (8.2.13) in (8.2.11) gives

$$\begin{aligned} B[w] &\geq \sum_{i=1}^{N_U} \mu_i (w, \Psi_i)_{L^2(\Omega)}^2 + U \left( \|w\|_{L^2(\Omega)}^2 - \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)}^2 \right) \\ &\geq U \|w\|_{L^2(\Omega)}^2 + \sum_{i=1}^{N_U} (\mu_i - U) (w, \Psi_i)_{L^2(\Omega)}^2 \\ &\geq U \|w\|_{L^2(\Omega)}^2 + (\mu_1 - U) \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)}^2, \end{aligned} \quad (8.2.14)$$

since  $\mu_1 \leq \mu_i$ . In addition,

$$\|P_U w\|_{L^2(\Omega)}^2 = \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)}^2. \quad (8.2.15)$$

and since  $B[\Psi_i, \Psi_j] = \mu_i \delta_{ij}$ , (8.2.9) gives

$$B\left[\frac{P_U w}{\|P_U w\|_{L^2(\Omega)}}\right] = \frac{1}{\|P_U w\|_{L^2(\Omega)}^2} B[P_U w] = \frac{1}{\|P_U w\|_{L^2(\Omega)}^2} \sum_{i=1}^{N_U} \mu_i (w, \Psi_i)_{L^2(\Omega)}^2 \leq U, \quad (8.2.16)$$

since  $\mu_i \leq U$  for  $i = 1, \dots, N_U$ . Applying Proposition 8.1 with  $\Psi = \frac{P_U w}{\|P_U w\|_{L^2(\Omega)}}$

$$\begin{aligned} \sum_{i=1}^{N_U} (w, \Psi_i)_{L^2(\Omega)}^2 &= \left(w, \frac{P_U w}{\|P_U w\|_{L^2(\Omega)}}\right)_{L^2(\Omega)}^2 \\ &\leq \left(\frac{rU\varepsilon^{-4}}{\beta_{M+1}^2} + \frac{U}{\nu} + \frac{U}{q_+^2} + O(\varepsilon) + O\left(\frac{\varepsilon^{-3}}{\beta_{M+1}^2}\right)\right) \|w\|_{L^2(\Omega)}^2 \\ &< \frac{1}{2} \|w\|_{L^2(\Omega)}^2, \end{aligned} \quad (8.2.17)$$

by assumption (8.2.2). Returning this estimate to (8.2.14) and using  $-C\varepsilon \leq \mu_1 \leq U$  gives

$$B[w] \geq U \|w\|_{L^2(\Omega)}^2 - \frac{(U - \mu_1)}{2} \|w\|_{L^2(\Omega)}^2 \geq \left(\frac{U}{2} + O(\varepsilon)\right) \|w\|_{L^2(\Omega)}^2. \quad (8.2.18)$$

This gives (8.2.4). To obtain the remaining coercivity estimates note that using the definition of  $B$  in (7.3.1)

$$\|\mathcal{A}w\|_{L^2(\Omega)}^2 = B[w] - \varepsilon(\tilde{q}w, w), \quad (8.2.19)$$

and if  $U$  is independent of  $\varepsilon$

$$\|\mathcal{A}w\|_{L^2(\Omega)}^2 \geq \frac{U}{4} \|w\|_{L^2(\Omega)}^2 \quad (8.2.20)$$

follows from (8.2.4) and  $\|\tilde{q}\|_{L^\infty(\Omega)} = O(1)$ . Similarly

$$\|\mathcal{A}^2 w\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \geq (\mathcal{A}^2 w, w) = \|\mathcal{A}w\|_{L^2(\Omega)}^2 \geq \frac{U}{4} \|w\|_{L^2(\Omega)}^2, \quad (8.2.21)$$

and (8.2.5) follows by dividing by  $\|w\|_{L^2(\Omega)}$ . Similarly, (8.2.7) follows from

$$\|\mathcal{A}w\|^2 = (\mathcal{A}w, \mathcal{A}w) = (\mathcal{A}^2 w, w) \leq \|\mathcal{A}^2 w\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \leq \frac{4}{U} \|\mathcal{A}^2 w\|_{L^2(\Omega)}^2, \quad (8.2.22)$$

To obtain the  $H^2$  coercivity in (8.2.6) we write  $\Delta w$  in terms of  $\mathcal{A}w$

$$\Delta w = -\frac{1}{\varepsilon^2} \mathcal{A}w + \frac{1}{\varepsilon^2} qw, \quad (8.2.23)$$

and use elliptic regularity (Evans, Theorem 4, p.317) to obtain

$$\|w\|_{H^2(\Omega)} \leq C \left( \|w\|_{L^2(\Omega)} + \varepsilon^{-2} \|\mathcal{A}w\| + \varepsilon^{-2} \|q\|_\infty \|w\|_{L^2(\Omega)} \right). \quad (8.2.24)$$

The  $H^2$  coercivity in (8.2.8) follows from (8.2.5). □

### 8.3 Proof of Theorem 3.5

In the case of linearization  $\mathcal{L}_\phi$  about a single-layer interface,  $r = 1$ . We choose  $U = \frac{1}{16}\varepsilon^4\beta_{M+1}^2$  and  $M = \lfloor M_1\varepsilon^{-(d-1)} \rfloor$ . Weyl asymptotic formula implies that  $\beta_{M+1} = O(\varepsilon^{-2})$  and the assumption of Proposition 8.2 is satisfied. Coercivity estimates (3.2.17)-(3.2.19) follow from Proposition 8.2 by taking  $H_1 := \varepsilon^{-1/2}\phi'$ .

In addition, addressing (3.2.20) we observe that since  $\|W''(\Phi)\|_{L^\infty(\Omega)} = O(1)$ ,

$$\begin{aligned} \|\mathcal{A}_\phi w\|_{L^2(\Omega)} &= \|\varepsilon^2 \Delta w - W''(\Phi)w + \varepsilon^2 \eta w\|_{L^2(\Omega)} \\ &\leq \varepsilon^2 \left( \|\Delta w\|_{L^2(\Omega)} + \varepsilon^{-2} \|W''(\Phi) + \varepsilon^2 \eta\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \right) \\ &\leq C\varepsilon^2 \|w\|_{\mathcal{A}}. \end{aligned} \tag{8.3.1}$$

We also deduce from (3.2.17) and (3.2.19) that

$$\begin{aligned} \|w\|_{\mathcal{A}} &= \|\Delta w\|_{L^2(\Omega)} + \varepsilon^{-2} \|w\|_{L^2(\Omega)} \leq \|w\|_{H^2(\Omega)} + \varepsilon^{-2} \|w\|_{L^2(\Omega)} \\ &\leq C\varepsilon^{-4} \frac{1}{\beta_{M+1}} \|\mathcal{A}_\phi w\|_{L^2(\Omega)} \leq C_b \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \end{aligned} \tag{8.3.2}$$

where in the last step we used  $\beta_{M+1} = O(\varepsilon^{-2})$ . □

# Chapter 9

## Interface Evolution - Bounds on Lower Order Terms

### 9.1 Linear Operator

The following lemma is established by differentiation of equation (3.1.5).

**Lemma 9.1.** *The spectrum of  $L_\phi$  near the origin consists of a single eigenvalue at zero corresponding to the translational eigenfunction  $\phi'$ . The remainder of the spectrum of  $L_\phi$  is real and uniformly negative. Moreover the following identities hold,*

$$L_\phi \phi' = 0, \tag{9.1.1}$$

$$L_\phi \phi_{zz} = W'''(\phi) \phi'^2, \tag{9.1.2}$$

$$L_\phi(z\phi') = 2\phi_{zz}. \tag{9.1.3}$$

## 9.2 Nonlinearity Control

Our main goal in this section is to establish energy inequality (3.3.53). We first need the following bound on the nonlinearity  $\mathcal{N}(w)$  defined in (3.3.27).

**Lemma 9.2.** *The nonlinearity  $\mathcal{N}(w)$  defined in (3.3.27) satisfies*

$$\|\mathcal{N}(w)\|_{L^2(\Omega)} \leq C(\varepsilon^{-2}\|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2 + \varepsilon^{-4}\|\mathcal{A}_\phi w\|_{L^2(\Omega)}^3). \quad (9.2.1)$$

**Proof:** Utilizing  $W^{(m)}(u) = 0$  for  $m > 4$ , it is straightforward to write  $\mathcal{N}(w)$  in the following form

$$\mathcal{N}(w) = a_1 w^2 + a_2 w^3 + \varepsilon^2 a_3 |\nabla w|^2 + \varepsilon^2 a_4 w \Delta w + \varepsilon^2 a_5 w^2 \Delta w + \varepsilon^2 a_6 w |\nabla w|^2, \quad (9.2.2)$$

where  $a_i(x)$  are smooth  $O(1)$  functions. We use the coercivity estimates in Proposition 8.2 and the following Sobolev Inequalities ( $d \leq 3$ )

$$\|w\|_{L^\infty(\Omega)} \leq \|w\|_{H^2(\Omega)} \leq \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \quad (9.2.3)$$

$$\|w\|_{W^{1,4}(\Omega)} \leq \|w\|_{H^2(\Omega)} \leq \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \quad (9.2.4)$$

to estimate

$$\|a_1 w^2\|_{L^2(\Omega)} \leq \|a_1\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \leq C \varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2, \quad (9.2.5)$$

$$\|a_2 w^3\|_{L^2(\Omega)} \leq \|a_2\|_{L^\infty(\Omega)} \|w^2\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)} \leq C \varepsilon^{-4} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^3, \quad (9.2.6)$$

$$\|\varepsilon^2 a_3 |\nabla w|^2\|_{L^2(\Omega)} \leq \varepsilon^2 \|a_3\|_{L^\infty(\Omega)} \|w\|_{W^{1,4}(\Omega)}^2 \leq C\varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2, \quad (9.2.7)$$

$$\|\varepsilon^2 a_4 w \Delta w\|_{L^2(\Omega)} \leq \varepsilon^2 \|a_4\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} \|w\|_{H^2(\Omega)} \leq C\varepsilon^{-2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2. \quad (9.2.8)$$

The rest of the estimates follow similarly and we obtain (9.2.1).  $\square$

**Corollary 9.3.** *The following estimate holds:*

$$\left| (\Pi_0 \mathcal{N}(w), Z_i)_{L^2(\Omega)} \right| \leq C(K) \varepsilon^{5/2} \left( \|w\|_{\mathcal{A}}^2 + \|w\|_{\mathcal{A}}^3 \right), \quad (9.2.9)$$

where  $Z_i = \eta_1 \Theta_{i, \Gamma_t} \phi'$  was defined in Definition 3.3 and  $C(K) > 0$  is independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** The result follows from (9.2.1) and

$$\begin{aligned} \|\mathcal{A}_\phi w\|_{L^2(\Omega)} &= \|\varepsilon^2 \Delta w - W''(\Phi)w + \varepsilon^2 \eta w\|_{L^2(\Omega)} \\ &\leq C \left( \varepsilon^2 \|\Delta w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \right) \\ &\leq C\varepsilon^2 \|w\|_{\mathcal{A}} \end{aligned} \quad (9.2.10)$$

and

$$\|Z_i\|_{L^2(\Omega)}^2 = \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\eta_1 \phi' \Theta_i)^2 J dz ds = O(\varepsilon), \quad (9.2.11)$$

due to the  $\varepsilon$  in the Jacobian.  $\square$

**Proposition 9.4.** *There exists  $C > 0$  such that for all  $w \in \mathcal{Z}_M^\perp$ ,*

$$\left| \langle \mathcal{A}_\phi^2 w \rangle \right| \leq C \|\mathcal{A}_\phi w\|_{L^2(\Omega)} \varepsilon^{1/2}. \quad (9.2.12)$$

$$\left| \langle \mathcal{L}w \rangle \right| \leq C \|\mathcal{A}_\phi w\|_{L^2(\Omega)} \varepsilon^{1/2}, \quad (9.2.13)$$

where

$$\langle f \rangle := \frac{1}{|\Omega|} \int_{\Omega} f(x) dx. \quad (9.2.14)$$

**Proof:** Using  $\partial_\nu \mathcal{A}_\phi w = 0$  on  $\partial\Omega$  we obtain

$$\left| \int_{\Omega} \mathcal{A}_\phi^2 w dx \right| = \left| \int_{\Omega} (-\varepsilon^2 \Delta + q) \mathcal{A}_\phi w dx \right| = \left| \int_{\Omega} q \mathcal{A}_\phi w dx \right| \leq \|q\|_{L^2(\Omega)} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}, \quad (9.2.15)$$

where  $q$  is given for the functionalized operator in Remark 7.2.

Since  $q$  is localized on  $\Gamma$ , the extra factor of  $\varepsilon$  that the  $\|q\|_{L^2(\Omega)}^2$  acquires from the Jacobian yields (9.2.12). Similarly, since  $(w, 1)_{L^2(\Omega)} = 0$  and consequently  $\Pi_0 w = w$ , we obtain

$$\int_{\Omega} \tilde{q} w dx = \int_{\Omega} (\Pi_0 \tilde{q}) w + \int_{\Omega} \langle \tilde{q} \rangle w dx = \int_{\Omega} (\Pi_0 \tilde{q}) w, \quad (9.2.16)$$

and since  $\tilde{q}$  is localized on  $\Gamma$ , (9.2.13) follows from  $\mathcal{L}_\phi = \mathcal{A}_\phi^2 + \varepsilon \tilde{q}$  and coercivity estimate  $\|w\|_{L^2(\Omega)} \leq U \|\mathcal{A}_\phi w\|_{L^2(\Omega)}$ .  $\square$

### 9.3 Normal Velocity

**Proposition 9.5.** *Fix  $K > 0$ . Let  $z(x; t) \in \mathbb{R}$  and  $s(x; t) \in \mathbb{R}^{d-1}$  be the whiskered coordinates of a point  $x \in \Gamma_t(2\ell)$  with respect to interface  $\Gamma_t$ . Then, there exists  $C(K) > 0$  such that for*

all  $\Gamma \in \mathcal{G}_K$  the following relations hold for the derivatives of  $s_j(x, t)$  with respect to time,

$$\left\| \frac{\partial s_j(x; t)}{\partial t} \right\|_{L^2(\Omega)} \leq C(K) \left\| \nabla_s V_n(s) \right\|_{L^2(\Gamma)}. \quad (9.3.1)$$

**Proof:** We differentiate the change of coordinates

$$x = \gamma(s(t); t) + \varepsilon z(t) \nu(s(t), t) \quad (9.3.2)$$

with respect to  $t$  to obtain

$$\sum_{j=1}^{d-1} \frac{\partial \gamma}{\partial s_j} \frac{ds_j}{dt} + \frac{\partial \gamma}{\partial t} + \varepsilon \frac{dz}{dt} \nu + \varepsilon z(t) \sum_{j=1}^{d-1} \frac{\partial \nu}{\partial s_j} \frac{ds_j}{dt} + \varepsilon z \frac{\partial \nu}{\partial t} = 0. \quad (9.3.3)$$

Multiplying by  $\frac{\partial \gamma}{\partial s_i}$  and recalling  $\frac{\partial \gamma}{\partial t} = V_n \nu$  yields

$$\sum_{j=1}^{d-1} \left( g_{ij} - \varepsilon z h_{ij} \right) \frac{ds_j}{dt} + \varepsilon z \frac{\partial \nu}{\partial t} \frac{\partial \gamma}{\partial s_i} = 0. \quad (9.3.4)$$

From Proposition 4.4 we have

$$\frac{\partial \nu}{\partial t} = \frac{\partial}{\partial t} \nu = - \sum_{k,m=1}^{d-1} g^{km} \frac{\partial V_n}{\partial s_k} \frac{\partial \gamma}{\partial s_m}. \quad (9.3.5)$$

Returning this estimate to (9.3.4)

$$\sum_{j=1}^{d-1} \left( g_{ij} - \varepsilon z h_{ij} \right) \frac{ds_j}{dt} = \varepsilon z \sum_{k,m=1}^{d-1} \frac{\partial V_n}{\partial s_i} g^{km} g_{mi}. \quad (9.3.6)$$

Canceling the  $g, g^{-1}$  in the last term yields

$$\sum_{j=1}^{d-1} \left( g_{ij} - \varepsilon z h_{ij} \right) \frac{ds_j}{dt} = \varepsilon z \frac{\partial V_n}{\partial s_i}, \quad (9.3.7)$$

and (9.3.1) follows from uniform invertibility of  $g - \varepsilon z h$  (see Lemma 2.11).  $\square$

**Corollary 9.6.** *Let*

$$V_n := \sum_{k=1}^M v_i \Theta_{i, \Gamma_t}. \quad (9.3.8)$$

*Then*

$$\left\| \frac{\partial s_j(x; t)}{\partial t} \right\|_{L^2(\Omega)} \leq C(K) \varepsilon^{-1} \|V_n\|_{L^2(\Gamma)} \quad (9.3.9)$$

*and*

$$\left| \int_{\Omega} \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} Z_i dx \right| \leq C(K) \varepsilon^2 \|V_n\|_{L^2(\Gamma)}, \quad (9.3.10)$$

**Proof:** We note that Proposition 9.5, as well as

$$-\Delta_s \Theta_i = \beta_i \Theta_i, \quad (9.3.11)$$

and orthonormality of eigenfunctions  $\Theta_i$  with respect to the  $(\cdot, \cdot)_{\Gamma}$  inner product implies

$$\begin{aligned} \left\| \frac{\partial s_j(x; t)}{\partial t} \right\|_{L^2(\Omega)}^2 &\leq C(K) \|\nabla_s V_n\|_{L^2(\Gamma)}^2 = C(K) \int_{\Gamma} \left( \sum_{i=1}^M v_i \nabla_s \Theta_{i, \Gamma_t} \right)^2 ds \\ &= C(K) \sum_{i=1}^M v_i^2 \beta_i \leq C(K) \varepsilon^{-2} |v|^2 = C(K) \varepsilon^{-2} \|V_n\|_{L^2(\Gamma)}^2 \end{aligned} \quad (9.3.12)$$

where we used  $|\beta_i| \leq C(K) \varepsilon^{-2}$  (see Remark 3.4) and  $|v| = \|V_n\|_{L^2(\Gamma)}$ . Similarly (9.3.10)

follows from (9.3.12) and  $J = O(\varepsilon)$ . □

### 9.3.1 Leading Order Dynamics

We will denote by  $C(K)$  the different constants depending on  $K$  only. We start with the following lemma, which addresses  $I$ , the first term on the left-hand side of (3.3.28).

**Lemma 9.7.** *Fix  $K > 0$ . Let the normal velocity  $V_n$  of interface  $\Gamma_t \in \mathcal{G}_K$  be given by (3.3.29). Then the following bound holds*

$$\left| \frac{1}{\varepsilon} \int_{\Gamma} V_n \Theta_{i, \Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(\partial_z \Phi) \phi' J dz ds - \|\phi'\|_{L^2(\mathbb{R})}^2 v_i \right| \leq C(K) \varepsilon^2 \|V_n\|_{L^2(\Omega)}. \quad (9.3.13)$$

**Proof of Lemma 9.7:** Due to the expansion of the Jacobian  $J(s, z) = J_0(s) \sum_{j=0}^d \varepsilon^{j+1} K_j z^j$  (2.2.17) we obtain

$$\begin{aligned} & -\frac{1}{\varepsilon} \int_{\Gamma} V_n \Theta_{i, \Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) (\partial_z \Phi) \phi' J dz ds \\ &= -\frac{1}{\varepsilon} \int_{\Gamma} V_n \Theta_{i, \Gamma_t} J_0(s) \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) (\phi' + \varepsilon^2 \partial_z \phi_2) \phi' \sum_{j=0}^d \varepsilon^{j+1} K_j z^j dz ds. \end{aligned} \quad (9.3.14)$$

The exponential decay of  $\phi'$  when  $z \rightarrow \pm\infty$  implies the following asymptotic estimate,

$$\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(\phi')^2 dz = -\|\phi'\|_{L^2(\mathbb{R})}^2 + O(\varepsilon^N), \quad (9.3.15)$$

where  $N$  is any positive integer.

Therefore, the leading term becomes

$$\begin{aligned}
& - \int_{\Gamma} V_n \Theta_{i, \Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) (\phi')^2 J^0 dz ds = \\
& (-\|\phi'\|_{L^2(\mathbb{R})}^2 + O(\varepsilon^N)) \int_{\Gamma} \left( \sum_{k=1}^M v_k \Theta_{k, \Gamma_t} \right) \Theta_{i, \Gamma_t} J^0 ds \\
& = (-\|\phi'\|_{L^2(\mathbb{R})}^2 + O(\varepsilon^N)) v_i.
\end{aligned} \tag{9.3.16}$$

The next term of the asymptotic expansion is zero since  $\eta_1(z)(\phi'(z))^2 z$  is an odd function on the symmetric interval  $(-2\ell/\varepsilon, 2\ell/\varepsilon)$

$$-\varepsilon \int_{\Gamma} K_1 V_n \Theta_{i, \Gamma_t} J^0 \left( \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(\phi')^2 z dz \right) ds = 0. \tag{9.3.17}$$

Employing again the exponential decay of  $\phi'(z)$  and boundness of  $\partial_z \phi_2$  and  $K_j(s)$  we obtain asymptotic estimates uniform with respect to  $s \in \Gamma_t$  with constants depending on  $K$  only

$$\left| \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) (\phi')^2 \sum_{j=0}^{d-2} \varepsilon^j K_{j+2} z^{j+2} dz \right| \leq C(K), \tag{9.3.18}$$

and

$$\left| \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) \phi' \partial_z \phi_2 \sum_{j=1}^d \varepsilon^j dz \right| \leq C(K). \tag{9.3.19}$$

Now we can estimate the sum of the remaining terms in (9.3.14)

$$\left| \varepsilon^2 \left( - \int_{\Gamma} V_n \Theta_{i, \Gamma_t} J_0(s) \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) (\phi')^2 \sum_{j=0}^{d-2} \varepsilon^j K_{j+2} z^{j+2} dz ds \right. \right. \tag{9.3.20}$$

$$\left| - \int_{\Gamma} V_n \Theta_{i,\Gamma_t} J_0(s) \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1(z) \phi' \partial_z \phi_2 \sum_{j=0}^d \varepsilon^j K_j z^j dz ds \right| \leq C(K) \varepsilon^2 \|V_n\|_{L^2(\Gamma)}. \quad (9.3.21)$$

□

Referring to (3.3.28), we next estimate  $II = (\Pi_0 F(\Phi), Z_i)_{L^2(\Omega)} + \partial_t b \int_{\Omega} Z_i dx$ .

**Lemma 9.8.** *Fix  $K > 0$ . The following estimate holds*

$$\left| (\Pi_0 F(\Phi), Z_i)_{L^2(\Omega)} + \partial_t b \int_{\Omega} Z_i dx - \varepsilon^4 (V_n^0, \Theta_{i,\Gamma_t})_{L^2(\Gamma)} \right| \leq C(K) \varepsilon^5, \quad (9.3.22)$$

where

$$V_n^0 := - \left[ (\Delta_s + \eta) H - \frac{H^3}{2} + H \text{tr}(A^2) \right] - \frac{2}{\|\phi'\|_{L^2(\mathbb{R})}^2} (\mu_0^2 b_3 - \partial_t b_3). \quad (9.3.23)$$

**Proof of Lemma 9.8:** Since  $\phi'$  is exponentially decaying and  $\phi(\infty) - \phi(-\infty) = 2$  we first observe that

$$\begin{aligned} \int_{\Omega} Z_i dx &= \int_{\Gamma} \Theta_{i,\Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \phi' J dz ds = \varepsilon \int_{\Gamma} \Theta_{i,\Gamma_t} \left( \int_{-\infty}^{\infty} \phi' dz \right) J^0 ds + O(\varepsilon^2) \\ &= 2\varepsilon \int_{\Gamma} \Theta_{i,\Gamma_t} J^0 ds + O(\varepsilon^2), \end{aligned} \quad (9.3.24)$$

and

$$\partial_t b \int_{\Omega} Z_i dx = 2\varepsilon \partial_t b \int_{\Gamma} \Theta_{i,\Gamma_t} ds + O(\varepsilon^2 |\partial_t b|) = 2\varepsilon^4 \partial_t b_3 \int_{\Gamma} \Theta_{i,\Gamma_t} ds + O(\varepsilon^5 |\partial_t b_3|), \quad (9.3.25)$$

where we used  $b = \varepsilon^3 b_3$  and (2.2.17) to expand the Jacobian. In addition, letting

$$\langle F(\Phi) \rangle := \frac{1}{|\Omega|} \int_{\Omega} \Phi dx, \quad (9.3.26)$$

we obtain

$$\begin{aligned} (\Pi_0 F(\Phi), Z_i)_{L^2(\Omega)} &= (F(\Phi), Z_i)_{L^2(\Omega)} - \langle F(\Phi) \rangle \int_{\Omega} Z_i dx \\ &= (F(\Phi), Z_i)_{L^2(\Omega)} - 2\varepsilon \int_{\Gamma} \Theta_{i,\Gamma_t} ds \langle F(\Phi) \rangle + O(\varepsilon^5), \end{aligned} \quad (9.3.27)$$

where we used (9.3.24) and  $\|F(\Phi)\|_{L^\infty(\Omega)} = O(\varepsilon^3)$ .

Again, using the definition of  $Z_i$  and exponential decay of  $\phi'$  away from the interface, similarly to the previous lemma we have

$$\begin{aligned} (F(\Phi), Z_i)_{L^2(\Omega)} &= \int_{\Gamma} \Theta_{i,\Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \phi' F(\Phi) J dz ds \\ &= \varepsilon \int_{\Gamma} \Theta_{i,\Gamma_t} \left( \int_{-\infty}^{\infty} \phi' F(\Phi) dz \right) J^0 ds + O(\varepsilon^2) \|F(\Phi)\|_{L^\infty(\Omega)} \end{aligned} \quad (9.3.28)$$

From the proof of Proposition 3.1 we have

$$\begin{aligned} F(\Phi) &= \varepsilon^3 R_3 + O(\varepsilon^4) = \varepsilon^3 L_\phi \left( \kappa_2 z^2 \phi' + \kappa_0 \partial_z \phi_2 + L_\phi b_3 + \frac{1}{2} \kappa_0 \kappa_1 z \phi' \right) \\ &+ \varepsilon^3 \left( \kappa_0 \kappa_1 z \phi_{zz} + \kappa_0 \partial_z (\kappa_1 z \phi' + L_\phi \phi_2) + \phi' (\Delta_s + \eta) \kappa_0 - \kappa_0 W'''(\phi) \phi_2 \phi' \right) + O(\varepsilon^4), \end{aligned} \quad (9.3.29)$$

where  $O(\varepsilon^4)$  is in  $L^\infty(\Omega)$  norm and the asymptotic constant is independent of  $\Gamma \in \mathcal{G}_K$ . We

take the  $L^2(\mathbb{R})$  inner product of  $F(\Phi)$  with  $\phi'$  and integrate by parts to cancel the  $\kappa_0\kappa_1$  terms

$$\begin{aligned} \int_{-\infty}^{\infty} \phi' F(\Phi) dz &= \varepsilon^3 \left[ -\kappa_0 (L_\phi \phi_2, \phi'')_{L^2(\mathbb{R})} + (\Delta_s + \eta)(\kappa_0) \|\phi'\|_{L^2(\mathbb{R})}^2 \right. \\ &\quad \left. - \kappa_0 (W'''(\phi)(\phi')^2, \phi_2)_{L^2(\mathbb{R})} \right] + O(\varepsilon^4). \end{aligned} \quad (9.3.30)$$

Using the definition (3.1.6) of  $\phi_2$  and properties of  $L_\phi$  in Lemma 9.1 we evaluate

$$(L_\phi \phi_2, \phi'')_{L^2(\mathbb{R})} = -\left(\frac{\kappa_0^2}{2} + \kappa_1\right) (z\phi', \phi_{zz})_{L^2(\mathbb{R})} = \frac{1}{2} \left(\frac{\kappa_0^2}{2} + \kappa_1\right) \|\phi'\|_{L^2(\mathbb{R})}^2, \quad (9.3.31)$$

and

$$(W'''(\phi)\phi'^2, \phi_2) = -\left(\frac{\kappa_0^2}{2} + \kappa_1\right) (L_\phi \phi'', L^{-1} z\phi')_{L^2(\mathbb{R})} = \frac{1}{2} \left(\frac{\kappa_0^2}{2} + \kappa_1\right) \|\phi'\|_{L^2(\mathbb{R})}^2. \quad (9.3.32)$$

Combining these results gives the leading-order expression

$$\begin{aligned} \int_{-\infty}^{\infty} \phi' F(\Phi) dz &= \varepsilon^3 \|\phi'\|_{L^2(\mathbb{R})}^2 \left[ (\Delta_s + \eta)\kappa_0 - \left(\frac{\kappa_0^3}{2} + \kappa_0\kappa_1\right) \right] + O(\varepsilon^4) \\ &= -\varepsilon^3 \|\phi'\|_{L^2(\mathbb{R})}^2 \left[ (\Delta_s + \eta)H - \frac{H^3}{2} + Htr(A^2) \right] + O(\varepsilon^4). \end{aligned} \quad (9.3.33)$$

and

$$\begin{aligned}
(F(\Phi), Z_i)_{L^2(\Omega)} &= \int_{\Gamma} \Theta_{i,\Gamma_t} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \phi' F(\Phi) J dz ds \\
&= \varepsilon \int_{\Gamma} \Theta_{i,\Gamma_t} \left( \int_{-\infty}^{\infty} \phi' F(\Phi) dz \right) J^0 ds + O(\varepsilon^5) \\
&= -\varepsilon^4 \|\phi'\|_{L^2(\mathbb{R})}^2 \left( (\Delta_s + \eta) H - \frac{H^3}{2} + H \text{tr}(A^2), \Theta_{i,\Gamma_t} \right)_{L^2(\Gamma)} \\
&\quad + O(\varepsilon^5). \tag{9.3.34}
\end{aligned}$$

In addition, to evaluate  $\langle F(\Phi) \rangle_{\Omega}$  we return to (9.3.29) and observe that all terms except those involving the background perturbation,  $b$ , are localized on  $\Gamma$ . We integrate over  $\Omega$  and change to the whiskered variables for the localized terms. So long as  $\Gamma \in \mathcal{G}_K$ , we have  $|\Gamma| < K$ , and the extra factor of  $\varepsilon$  that the localized terms acquire from the Jacobian renders them lower order than the far-field terms. At the leading-order we have

$$\langle F(\Phi) \rangle_{\Omega} = |\Omega| \int_{\Omega} L_{\phi}^2 b dx + O(\varepsilon^4) = \frac{b}{|\Omega|} \int_{\Omega} (W''(\phi))^2 dx + O(\varepsilon^4) = b \mu_0^2 + O(\varepsilon^4), \tag{9.3.35}$$

where recall from (1.0.4) that  $\mu_0 = W''(\pm 1) > 0$ . Combining (9.3.35) with (9.3.34) and (9.3.27) yields (9.3.22). □

## 9.4 Development of a priori estimates

**Proposition 9.9.** *Assuming that (3.3.28) holds for each  $i = 1, \dots, M$ , then the coefficients of  $V_n$  must satisfy*

$$\begin{aligned} \left| v_i - \varepsilon^4 (V_n^0, \Theta_{i, \Gamma_t})_{L^2(\Gamma)} \right| &\leq C(K) \varepsilon^2 \left[ \left( \frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\beta_{i, \Gamma_t}} \sqrt{\varepsilon}}{\sqrt{\beta_{M+1, \Gamma_t}} - \sqrt{\beta_{i, \Gamma_t}}} \right) \|V_n\|_{L^\infty(\Gamma)} \|w\|_{\mathcal{A}} \right. \\ &\quad \left. + \left( \varepsilon^3 + |v| + \varepsilon^{3/2} \|w\|_{\mathcal{A}} + \varepsilon^{1/2} \|w\|_{\mathcal{A}}^2 + \varepsilon^{1/2} \|w\|_{\mathcal{A}}^3 \right) \right], \quad (9.4.1) \end{aligned}$$

where

$$V_n^0 = - \left[ (\Delta_s + \eta) H - \frac{H^3}{2} + H \text{tr}(A^2) \right] - \frac{2}{\|\phi'\|_{L^2(\mathbb{R})}^2} (\mu_0^2 b_3 - \partial_t b_3), \quad (9.4.2)$$

and  $C > 0$  is independent of  $\varepsilon$  and  $\Gamma \in \mathcal{G}_K$ .

**Proof:** The idea is to simplify equation (3.3.28) and isolate the leading-order expression for  $v_i$ .

Referring to (3.3.28), we first estimate  $(\partial_t w, Z_i)_{L^2(\Omega)}$  term in the following lemma.

**Lemma 9.10.** *For any differentiable function  $w$  which satisfies  $w(x, t) \in Z_M^\perp$  we have the estimate*

$$\left| (\partial_t w, Z_i)_{L^2(\Omega)} \right| \leq C(K) \varepsilon^2 \left( \frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\beta_{i, \Gamma_t}} \sqrt{\varepsilon}}{\sqrt{\beta_{M+1, \Gamma_t}} - \sqrt{\beta_{i, \Gamma_t}}} \right) \|V_n\|_{L^\infty(\Gamma)} \|w\|_{\mathcal{A}}, \quad (9.4.3)$$

where  $\beta_{i, \Gamma_t}$  is the  $i$ th eigenvalue of  $-\Delta_s$ .

**Proof of Lemma 9.10:** The estimate follows from Theorem 5.3 and

$$\|w\|_{L^2(\Omega)} \leq \varepsilon^2 \|w\|_{\mathcal{A}}. \quad (9.4.4)$$

□

In addition, we summarize bounds on lower order terms in (3.3.28) with the aid of the following table. We also give references to the locations of the corresponding proofs.

|                                                                                                                                                          |                  |
|----------------------------------------------------------------------------------------------------------------------------------------------------------|------------------|
| $\left  \int_{\Omega} \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} Z_i dx \right  \leq C(K) \varepsilon^2 \ V_n\ _{L^2(\Omega)}$    | Corollary 9.6    |
| $\left  (\Pi_0 \mathcal{L}_{\phi} w, Z_i)_{L^2(\Omega)} \right  \leq C(K) \varepsilon^{7/2} \ w\ _{\mathcal{A}}$                                         | Proposition 9.15 |
| $\left  (\Pi_0 \mathcal{N}(w), Z_i)_{L^2(\Omega)} \right  \leq C(K) (\varepsilon^{5/2} \ w\ _{\mathcal{A}}^2 + \varepsilon^{5/2} \ w\ _{\mathcal{A}}^3)$ | Corollary 9.3    |

Combining the lemmas with the above estimates as well as leading-order calculations in Lemmas 9.7 and 9.8 we obtain (9.4.1). □

**Proposition 9.11.** *Assume that  $w$  satisfies*

$$\|w\|_{\mathcal{A}} \leq 2\bar{B}_0 \varepsilon^{d-3/2}. \quad (9.4.5)$$

Assuming that (3.3.28) holds for each  $i = 1, \dots, M$ , then the coefficients of  $V_n$  must satisfy

$$\|V_n\|_{L^\infty(\Gamma)} \leq C(K)\varepsilon^3, \quad (9.4.6)$$

and

$$\left| v_i - \varepsilon^4 (V_n^0, \Theta_{i,\Gamma_t})_{L^2(\Gamma)} \right| \leq C(K) \left( \varepsilon^5 + \varepsilon^{7/2} \|w\|_{\mathcal{A}} \right). \quad (9.4.7)$$

**Proof:** First, utilizing Proposition 9.9 we obtain

$$\begin{aligned} \left| V_n - \varepsilon^4 \sum_{i=1}^M (V_n^0, \Theta_i)_{\Gamma} \Theta_i \right| &\leq \left| \sum_{i=1}^M v_i \Theta_i - \sum_{i=1}^M (\varepsilon^4 V_n^0, \Theta_i)_{\Gamma} \Theta_i \right| \\ &\leq C(K) \varepsilon^2 \sum_{i=1}^M \left[ \left( \frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\beta_i} \sqrt{\varepsilon}}{\sqrt{\beta_{M+1}} - \sqrt{\beta_i}} \right) \|V_n\|_{L^\infty(\Gamma)} \|w\|_{\mathcal{A}} \right. \\ &\quad + \left( \|V_n\|_{L^\infty(\Gamma)} + \varepsilon^3 + \varepsilon^{3/2} \|w\|_{\mathcal{A}} + \varepsilon^{1/2} \|w\|_{\mathcal{A}}^2 \right. \\ &\quad \left. \left. + \varepsilon^{1/2} \|w\|_{\mathcal{A}}^3 \right) \right] |\Theta_i| \end{aligned} \quad (9.4.8)$$

In addition,

$$\begin{aligned} &\varepsilon^2 \sum_{i=1}^M \left( \frac{\sqrt{\beta_i} \sqrt{\varepsilon}}{\sqrt{\beta_{M+1}} - \sqrt{\beta_i}} \right) |\Theta_i| \\ &= \varepsilon^{5/2} \left( \sum_{i=1}^M \frac{\beta_i}{(\sqrt{\beta_{M+1}} - \sqrt{\beta_i})^2} \right)^{1/2} \left( \sum_{i=1}^M |\Theta_i|^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq C(K)\varepsilon^{5/2} \left( \sum_{i=1}^M \beta_i (\sqrt{\beta_{M+1}} + \sqrt{\beta_i})^2 \right)^{1/2} \left( \sum_{i=1}^M |\Theta_i|^2 \right)^{1/2} \\
&\leq C(K)\varepsilon^{5/2} (M\beta_M^2)^{1/2} \left( \sum_{i=1}^M |\Theta_i|^2 \right)^{1/2} \\
&\leq C(K)\sqrt{M_1}\varepsilon^{3/2-d}, \tag{9.4.9}
\end{aligned}$$

where we used  $M = [M_1\varepsilon^{-(d-1)}]$ ,  $\beta_i \leq \beta_M \leq C(K)\varepsilon^{-2}$ ,  $\left(\sum_{i=1}^M |\Theta_i|^2\right)^{1/2} \leq C(K)\varepsilon^{-(d-1)/2}$  and the fact that the eigenvalue gap  $\beta_{M+1} - \beta_M > cg$  stays  $O(1)$ . (see Remark 3.4) Similarly, setting

$$Rw = \|V_n\|_{L^\infty(\Gamma)} + \varepsilon^3 + \varepsilon^{3/2}\|w\|_{\mathcal{A}} + \varepsilon^{1/2}\|w\|_{\mathcal{A}}^2 + \varepsilon^{1/2}\|w\|_{\mathcal{A}}^3 \tag{9.4.10}$$

we deduce

$$\varepsilon^2 \sum_{i=1}^M R_w |\Theta_i| \leq \varepsilon^2 \left( \sum_{i=1}^M R_w^2 \right)^{1/2} \left( \sum_{i=1}^M |\Theta_i|^2 \right)^{1/2} \leq C(K)M_1 R_w \varepsilon^{2-(d-1)}. \tag{9.4.11}$$

Using (9.4.9) and (9.4.11) in (9.4.8) gives

$$\begin{aligned}
\left| V_n - \varepsilon^4 \sum_{i=1}^M (V_n^0, \Theta_{i,\Gamma_t})_{\Gamma} \Theta_{i,\Gamma_t} \right| &\leq M_1 C(K) \left( \varepsilon^{3/2-d} \|V_n\|_{L^\infty(\Gamma)} \|w\|_{\mathcal{A}} \right. \\
&\quad + \varepsilon^{2-(d-1)} (\|V_n\|_{L^\infty(\Gamma)} + \varepsilon^3 + \varepsilon^{3/2}\|w\|_{\mathcal{A}} \\
&\quad \left. + \varepsilon^{1/2}\|w\|_{\mathcal{A}}^2 + \varepsilon^{1/2}\|w\|_{\mathcal{A}}^3) \right). \tag{9.4.12}
\end{aligned}$$

As long as  $\|w\|_{\mathcal{A}} \leq 2\bar{B}_0\varepsilon^{d-3/2}$  and  $M_1$  is sufficiently small, we have

$$\left| V_n - \varepsilon^4 \sum_{i=1}^M (V_n^0, \Theta_{i,\Gamma_t})_{\Gamma} \Theta_{i,\Gamma_t} \right| \leq C(K)\varepsilon^3, \quad (9.4.13)$$

and (9.4.6) follows. Using (9.4.6) in the bound in Proposition 9.9 yields (9.4.7).  $\square$

**Proof of Proposition 3.11:** The  $L^\infty$  bound on  $V_n$  was shown in the previous proposition.

In addition, we showed that

$$\left| v_i - \varepsilon^4 v_i^0 \right| \leq C(K) \left( \varepsilon^5 + \varepsilon^{7/2} \|w\|_{\mathcal{A}} \right) \quad (9.4.14)$$

and from (9.4.14)

$$\begin{aligned} \|V_n\|_{L^2(\Omega)}^2 &= \|v\|^2 = \sum_{i=1}^M v_i^2 \leq \varepsilon^8 \sum_{i=1}^M (v_i^0)^2 + MC(K) \left( \varepsilon^{10} + \varepsilon^7 \|w\|_{\mathcal{A}}^2 \right) \\ &\leq \varepsilon^8 \sum_{i=1}^M (v_i^0)^2 + M_1 C(K) \left( \varepsilon^{11-d} + \varepsilon^{8-d} \|w\|_{\mathcal{A}}^2 \right), \end{aligned} \quad (9.4.15)$$

where we used  $M = M_1[\varepsilon^{-(d-1)}]$ . In addition, the approximate basis property (3.2.21)

implies

$$\sum_{i=1}^M (v_i^0)^2 = \sum_{i=1}^M (V_n^0, \Theta_{i,\Gamma_t})^2 = \|V_n^0\|_{L^2(\Gamma)}^2 + O(\varepsilon^2), \quad (9.4.16)$$

which combined with (9.4.15) yields (3.3.37).

## 9.5 Other Estimates

**Lemma 9.12.** *Fix  $K > 0$ . Then for all  $\Gamma \in \mathcal{G}_K$ , the following bound holds for the derivative of the ansatz  $\Phi$  with respect to  $t$*

$$\|\partial_t \Phi\|_{L^2(\Omega)} \leq C(K) \varepsilon^{-1/2} \|V_n\|_{L^2(\Omega)}. \quad (9.5.1)$$

**Proof of Lemma 9.12:** To bound the time derivative of the ansatz  $\Phi$  we refer to (3.1.4) and note that the dependence on  $t$  comes from the whiskered coordinates  $(s(x; t), z(x; t))$  and the background state  $b$ . It follows from Proposition 3.10 that

$$\frac{\partial z(x; t)}{\partial t} = \varepsilon^{-1} V_n. \quad (9.5.2)$$

Similarly, we show in Proposition 9.5 that

$$\left\| \frac{\partial s_j(x; t)}{\partial t} \right\|_{L^2(\Omega)} \leq C(K) \left\| \nabla_s V_n(s) \right\|_{L^2(\Gamma)} \leq C(K) \varepsilon^{-1} \|V_n\|_{L^2(\Gamma)}. \quad (9.5.3)$$

The evolution of the background state is determined from the total mass conservation. Projecting (3.3.26) onto  $Z_0 = 1$ , the right-hand side is identically zero and we obtain

$$\int_{\Omega} \frac{\partial z(x; t)}{\partial t} \Phi_z + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x; t)}{\partial t} + \partial_t b dx = 0, \quad (9.5.4)$$

where we used  $\int_{\Omega} \partial_t w dx = \partial_t \int_{\Omega} w dx = 0$ , since  $w \in \mathcal{Z}_M^{\perp}$ . In addition, it follows from (9.5.4)

and estimates (9.5.2) and (9.5.3) and  $J = O(\varepsilon)$  that

$$|\partial_t b| = \frac{1}{|\Omega|} \left| \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial z(x;t)}{\partial t} \Phi_z + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} \right) J dz ds \right| \leq C(K) \varepsilon^{-1/2} \|V_n\|_{L^2(\Gamma)}. \quad (9.5.5)$$

Recall from (3.3.20) that  $\partial_t \Phi$  is given by

$$\partial_t \Phi(x; \Gamma_t; b) = \partial_z \Phi \frac{\partial z}{\partial t} + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s}{\partial t} + \partial_t b. \quad (9.5.6)$$

Combining (9.5.2), (9.5.3), and (9.5.5), we obtain

$$\begin{aligned} \|\partial_t \Phi\|_{L^2(\Omega)} &= \left( \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \frac{\partial z(x;t)}{\partial t} \partial_z \Phi + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} + \partial_t b \right)^2 J dz ds \right)^{1/2} \\ &\leq \left( \int_{\Gamma} (\varepsilon^{-1} V_n)^2 \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\partial_z \Phi)^2 J dz ds \right)^{1/2} \\ &\quad + \left( \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left( \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} + \partial_t b \right)^2 J dz ds \right)^{1/2} \\ &\leq C(K) \varepsilon^{-1/2} \|V_n\|_{L^2(\Gamma)}, \end{aligned} \quad (9.5.7)$$

where we again used  $J = O(\varepsilon)$ . □

**Lemma 9.13.** *Fix  $K > 0$ . Then for all  $\Gamma \in \mathcal{G}_K$ , the following bound holds*

$$\left( \left[ \partial_t, \mathcal{A}_\phi \right] w, \mathcal{A}_\phi w \right)_{L^2(\Omega)} \leq C(K) \varepsilon^{1/2} \|\mathcal{A}_\phi w\|_{L^2(\Omega)}^2. \quad (9.5.8)$$

**Proof:** We calculate

$$\partial_t \mathcal{A}_\phi w = \partial_t (\varepsilon^2 \Delta w - W''(\Phi)w + \varepsilon^2 \eta w) = \mathcal{A}_\phi \partial_t w + (\partial_t W''(\Phi))w, \quad (9.5.9)$$

and

$$\begin{aligned} \left\| [\partial_t, \mathcal{A}_\phi] w \right\|_{L^2(\Omega)} &= \left\| (\partial_t W''(\Phi))w \right\|_{L^2(\Omega)} \leq \left\| W'''(\Phi) \right\|_{L^\infty} \left\| \partial_t \Phi \right\|_{L^2(\Omega)} \|w\|_{L^\infty(\Omega)} \\ &\leq C \varepsilon^{-2} \left\| \partial_t \Phi \right\|_{L^2(\Omega)} \left\| \mathcal{A}_\phi w \right\|_{L^2(\Omega)}, \end{aligned} \quad (9.5.10)$$

where the estimate of the  $L^\infty$  norm of  $w$  in terms of the  $L^2$  norm of  $\mathcal{A}_\phi w$  is a consequence of the Sobolev Inequality (see Lemma 9.2).

Using the bound on  $\partial_t \Phi$  from Lemma 9.12 in (9.5.10) we obtain

$$\left( \left[ \partial_t, \mathcal{A}_\phi \right] w, \mathcal{A}_\phi w \right)_{L^2(\Omega)} \leq C(K) \varepsilon^{-5/2} \|V_n\|_{L^2(\Gamma)} \left\| \mathcal{A}_\phi w \right\|_{L^2(\Omega)}^2. \quad (9.5.11)$$

In particular, since by Proposition 9.11

$$\|V_n\|_{L^2(\Gamma)} \leq C \|V_n\|_{L^\infty(\Gamma)} \leq C(K) \varepsilon^3, \quad (9.5.12)$$

we have (9.5.8). □

**Proposition 9.14.** *Fix  $K > 0$ , let  $L_0 = -\partial_z^2 + q_0(s, z)$  and  $L = L_a^2 + \varepsilon \tilde{q}$ , where  $L_a = -\partial_z^2 - \varepsilon \kappa(s, z) \partial_z + q(s, z) = L_0 - \varepsilon (\kappa(s, z) \partial_z + q_1(s, z))$ . Then there exists  $C(K) > 0$*

such that for all  $\Gamma \in \mathcal{G}_K$  the following estimates hold

$$\sup_{|\alpha| \leq 4, i=0,1} \sup_s \|z^i D_s^\alpha L_a \phi'\|_J \leq C(K) \varepsilon^{3/2}, \quad (9.5.13)$$

$$\sup_{|\alpha| \leq 4, i=0,1} \sup_s \|z^i D_s^\alpha L \psi_z\|_J \leq C(K) \varepsilon^{3/2}. \quad (9.5.14)$$

**Proof:** From the definition of  $L_a$  in (7.1.16) we have

$$L_a \phi' = L_0 \phi' - \varepsilon \kappa \phi_{zz} + \varepsilon q_1 \phi' = \varepsilon (-\kappa \phi_{zz} + q_1 \phi'), \quad (9.5.15)$$

$$L \phi' = L_a^2 \phi' + \varepsilon \tilde{q} \phi' = (-\partial_z^2 - \varepsilon \kappa \partial_z + q) L_a \phi' + \varepsilon \tilde{q} \phi'. \quad (9.5.16)$$

Both terms are localized on  $\Gamma$  and acquire an extra factor of  $\varepsilon^{1/2}$  from the Jacobian. The result follows from bounds on  $\kappa$  in Proposition 2.12 as well as bounds on  $q$  and  $\tilde{q}$  in (7.1.4)-(7.1.7).

□

We conclude the chapter with the following proposition which is used in bounding the lower order terms in Theorem 3.7.

**Proposition 9.15.** *Let  $w \in \mathcal{Z}_M^\perp$  and  $Z_i = \eta_1 \Theta_i \phi' \in \mathcal{Z}_M$ . Then,*

$$\left| (\Pi_0 \mathcal{L}_\phi w, Z_i)_\Omega \right| \leq C(K) \varepsilon^{3/2} \|w\|_{L^2(\Omega)}. \quad (9.5.17)$$

**Proof:** Using the fact that  $\Pi_0$  and  $\mathcal{L}_\phi$  are self-adjoint

$$(\Pi_0 \mathcal{L}_\phi w, \eta_1 \Theta_i \phi')_{L^2(\Omega)} = (w, \mathcal{L}_\phi \Pi_0 (\eta_1 \Theta_i \phi'))_{L^2(\Omega)}. \quad (9.5.18)$$

From the definition of  $\Pi_0$

$$\Pi_0(\eta_1 \Theta_i \phi') = \eta_1 \Theta_i \phi' - \langle \eta \Theta_i \phi' \rangle \quad (9.5.19)$$

and

$$\mathcal{L}_\phi \Pi_0(\eta_1 \Theta_i \phi') = \mathcal{L}_\phi(\eta_1 \Theta_i \phi') - \mathcal{L}_\phi \langle \eta \Theta_i \phi' \rangle. \quad (9.5.20)$$

However  $w \in \mathcal{Z}_M^\perp$  and so  $(w, 1)_\Omega = 0$  implies

$$w = \Pi_0 w. \quad (9.5.21)$$

Using (9.5.20) and (9.5.21) in (9.5.18) gives

$$(\Pi_0 \mathcal{L}_\phi w, \eta_1 \Theta_i \phi')_\Omega = (w, \mathcal{L}_\phi(\eta_1 \Theta_i \phi'))_\Omega - (w, \mathcal{L}_\phi \langle \eta_1 \Theta_i \phi' \rangle)_\Omega. \quad (9.5.22)$$

We will estimate the two terms on the right-hand side of (9.5.22) separately. Using  $J = O(\varepsilon)$ ,

$$\langle \eta_1 \Theta_i \phi' \rangle = \frac{1}{|\Omega|} \int_{\Gamma(2\ell)} \eta_1 \Theta_i \phi' dx = \frac{1}{|\Omega|} \int_\Gamma \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \eta_1 \Theta_i \phi' J dz ds = O(\varepsilon), \quad (9.5.23)$$

In addition, recalling the definition of  $\mathcal{L}_\phi$  in (3.2.2) we obtain

$$\mathcal{L}_\phi 1 = (-\varepsilon^2 \Delta + W''(\Phi) - \varepsilon^2 \eta) (W''(\Phi)) - W'''(\Phi) (\varepsilon^2 \Delta \Phi - W''(\Phi)), \quad (9.5.24)$$

Since  $W''(\Phi) \rightarrow q_+$  as  $z \rightarrow \pm\infty$ ,  $\mathcal{L}_\phi 1 - q_+^2$  is localized on  $\Gamma$  we apply Proposition 7.7 to get

$$\left| (w, \Pi_0 \mathcal{L}_\phi 1)_\Omega \right| \leq \|w\|_{L^2(\Omega)} \|\Pi_0 \mathcal{L}_\phi 1\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \|w\|_{L^2(\Omega)}. \quad (9.5.25)$$

Combining estimates on  $\langle \eta_1 \Theta_i \phi' \rangle$  and  $|(w, \Pi_0 \mathcal{L}_\phi 1)_\Omega|$  into (9.5.22) yields

$$\left| (\Pi_0 \mathcal{L}_\phi w, \eta_1 \Theta_i \phi')_\Omega \right| \leq \left| (w, \mathcal{L}_\phi (\eta_1 \Theta_i \phi'))_\Omega \right| + C\varepsilon^{3/2} \|w\|_{L^2(\Omega)}. \quad (9.5.26)$$

Since  $\eta_1$  is compactly supported, we use the exponential decay of eigenfunctions  $\phi'$  in (7.1.19) and expansion of  $\mathcal{L}_\phi$  in (7.1.18)

$$\mathcal{L}_\phi (\eta_1 \Theta_i \phi') = \eta_1 \mathcal{L}_\phi (\Theta_i \phi') + O(e^{-m/\varepsilon}) = \eta_1 (I_1 - I_2 - I_3 + I_4) + O(e^{-m/\varepsilon}), \quad (9.5.27)$$

where

$$I_1 = L(\Theta_i \phi'), \quad (9.5.28)$$

$$I_2 = \varepsilon^2 \Delta_G L_a (\Theta_i \phi'), \quad (9.5.29)$$

$$I_3 = \varepsilon^2 L_a \Delta_G (\Theta_i \phi'), \quad (9.5.30)$$

$$I_4 = \varepsilon^4 \Delta_G^2 (\Theta_i \phi'). \quad (9.5.31)$$

We estimate each term separately. Bound on  $L\phi'$  in (9.5.14) give

$$\|I_1\|_{L^2(\Omega)} = \|L(\Theta_i \phi')\|_{L^2(\Omega)} = \|\Theta_i L\phi'\|_{L^2(\Omega)} \leq \|\Theta_i\|_{L^2(\Gamma)} \sup_s \|L\phi'\|_J \leq C(K) \varepsilon^{3/2}. \quad (9.5.32)$$

Addressing the next term  $I_2$ , we use Lemma 2.10 to write

$$\Delta_G L_a(\Theta_i \phi') = \Delta_s L_a(\Theta_i \phi') + \varepsilon z D_{s,2}(\Theta_i L_a \phi'), \quad (9.5.33)$$

where  $D_{s,2}$  represents a second order differential operator

$$D_{s,2} := \sum_{i,j=1}^{d-1} d_{ij} \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} d_j \frac{\partial}{\partial s_j}, \quad (9.5.34)$$

with coefficients bounded by some constant  $C(K)$ . In particular,

$$\begin{aligned} I_2 &= \varepsilon^2 \Delta_G L_a(\Theta_i \phi') = \varepsilon^2 \Delta_s L_a(\Theta_i \phi') + \varepsilon^3 D_{s,2}(\Theta_i z L_a \phi') \\ &= \varepsilon^2 (\Delta_s \Theta_i) L_a \phi' + 2\varepsilon^2 (\nabla_s \Theta_i \cdot \nabla_s L_a \phi') + \varepsilon^2 \Theta_i \Delta_s (L_a \phi') + \varepsilon^3 D_{s,2}(\Theta_i z L_a \phi') \\ &\quad . \end{aligned} \quad (9.5.35)$$

We note that Proposition 5.2 implies

$$\begin{aligned} \int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} ((\Delta_s \Theta_i) L_a \phi')^2 J dz ds &= \int_{\Gamma} (\Delta_s \Theta_i)^2 \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (L_a \phi')^2 J dz ds \\ &\leq \|\Delta_s \Theta_i\|_{L^2(\Gamma)}^2 \sup_s \|L_a \phi'\|_J^2 = \beta_i^2 \sup_s \|L_a \phi'\|_J^2 \\ &\leq C(K) \varepsilon^3 \beta_i^2. \end{aligned} \quad (9.5.36)$$

Similarly, one can show that

$$\begin{aligned}
\int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\nabla_s \Theta_i \cdot \nabla_s L_a \phi')^2 J dz ds &\leq \|\nabla_s \Theta_i\|_{L^2(\Gamma)}^2 \sup_s \|L_a \phi'\|_J^2 \\
&\leq \beta_i \sup_s \|L_a \phi'\|_J^2 \leq C(K) \varepsilon^3 \beta_i, \quad (9.5.37)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Gamma} \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Theta_i \Delta_s (L_a \phi'))^2 J dz ds &\leq \|\Theta_i\|_{L^2(\Gamma)}^2 \sup_s \|\Delta_s L_a \phi'\|_J^2 \\
&\leq C(K) \varepsilon^3, \quad (9.5.38)
\end{aligned}$$

where we used  $\|\Theta_i\|_{L^2(\Gamma)} = 1$  as well as Proposition 9.14. It follows from estimates (9.5.36)-(9.5.38) that

$$\left\| \varepsilon^2 (\Delta_s \Theta_i) L_a \phi' + 2\varepsilon^2 (\nabla_s \Theta_i \cdot \nabla_s L_a \phi') + \varepsilon^2 \Theta_i \Delta_s (L_a \phi') \right\|_{\Gamma(2\ell)} \leq C(K) \varepsilon^{7/2} (\beta_i + \beta_i^{1/2} + 1). \quad (9.5.39)$$

Similar calculation show that

$$\|\varepsilon^2 D_{s,2}(\Theta_i z L_a \phi')\|_{\Gamma(2\ell)} \leq C(K) \varepsilon^{7/2} (\beta_i + \beta_i^{1/2} + 1), \quad (9.5.40)$$

from which we deduce the bound

$$\|I_2\|_{\Gamma(2\ell)} \leq C(K) \varepsilon^{7/2} (\beta_i + \beta_i^{1/2} + 1) \leq C(K) \varepsilon^{3/2}, \quad (9.5.41)$$

where we used  $\beta_i \leq \beta_M \leq C(K)\varepsilon^{-2}$  (see Remark 3.4). Next, addressing  $I_3$  we observe that

$$\begin{aligned} I_3 &= \varepsilon^2 L_a \Delta_G(\Theta_i \phi') = \varepsilon^2 L_a(\phi' \Delta_G \Theta_i) = \varepsilon^2 (\Delta_s \Theta_i)(L_a \phi') + \varepsilon^3 (L_a(z \phi' D \Theta_i)) \\ &= \varepsilon^2 \beta_i \cdot \Theta_i(L_a \phi') + \varepsilon^3 (L_a(z \phi' D \Theta_i)) \end{aligned} \quad (9.5.42)$$

and

$$\|I_3\|_{L^2(\Omega)} \leq C(K)\varepsilon^{7/2}\beta_i \leq C(K)\varepsilon^{3/2}. \quad (9.5.43)$$

Finally,

$$I_4 = \varepsilon^4 \Delta_G^2(\Theta_i \phi') = \varepsilon^4 \phi'(\Delta_G^2 \Theta_i) = \varepsilon^4 \phi'(\Delta_s^2 \Theta_i) + \varepsilon z \phi' D_{s,2} \Theta_i \quad (9.5.44)$$

and estimates (5.2.11), (5.2.10), and (5.2.12) yield

$$I_4 = \varepsilon^4 \phi' \beta_i^2 \Theta_i + O(\varepsilon^5 \beta_i^{3/2}) = \varepsilon^4 \phi' \beta_i^2 \Theta_i + O(\varepsilon^2). \quad (9.5.45)$$

However,  $w \in \mathcal{Z}_M^\perp$  implies

$$(w, \eta_1 \Theta_i \phi')_{L^2(\Omega)} = 0, \quad (9.5.46)$$

It follows that

$$(w, I_4)_{L^2(\Omega)} \leq C(K)\varepsilon^2 \quad (9.5.47)$$

and returning the estimates on  $I_1$  through  $I_4$  to (9.5.26) gives (9.5.17).

## BIBLIOGRAPHY

## BIBLIOGRAPHY

- [1] Shmuel Agmon. *Lectures on Elliptic Boundary Value Problems*. AMS Chelsea Publishing, 1965.
- [2] N. Alikakos, P. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Archive for Rational Mechanics and Analysis*, 128:165–205, 1994.
- [3] N. Alikakos and G. Fusco. The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions.
- [4] B. Ameduri. From Vinylidene Fluoride (VDF) to the applications of VDF-containing polymers and co-polymers: Recent developments and future trends. *Chem. Rev.*, 109:6632–6686, 2009.
- [5] P. W. Atkins. *Physical Chemistry*. W. H. Freeman and Company, New York, 1994.
- [6] P. Bates and P. Fife. Spectral comparison principles for the Cahn-Hilliard and phase-field equations, and time scales for coarsening. *Physica D: Nonlinear Phenomena*, 43:334–348, 1990.
- [7] P. Bates and P. Fife. The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM J. Appl. Math.*, 53:990–1008, 1993.
- [8] L. Bronsard and R. Kohn. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math.*, 43:983–997, 1990.
- [9] J. Cahn. On the spinodal decomposition. *Acta Metall.*, 9:795–801, 1961.
- [10] J. Cahn and J. Hilliard. Free energy of a nonuniform system. I. Interfacial energy. *J. Chem. Phys.*, 28:258–, 1958.
- [11] Isaac Chavel. *Eigenvalues in Riemannian Geometry*. Academic Press, 1984.
- [12] X. Chen. Spectrum for the Allen-Chan, Chan-Hilliard, and phase-field equations for generic interfaces. *Communications in Partial Differential Equations*, 19:1371 – 1395, 1994.
- [13] Ward Cheney. *Analysis for Applied Mathematics*, volume 208 of *Graduate texts in mathematics*. Springer, 2001.
- [14] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Springer, 1953.

- [15] Q. Du, C. Liu, and X. Wang. Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions. *J. Comp. Physics*, 212:757–777, 2006.
- [16] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45:1097–1123, 1992.
- [17] P. Fife. Models for phase separation and their mathematics. *Electronic Journal of Differential Equations*, 48:1–26, 2000.
- [18] N. Gavish, G. Hayrapetyan, K. Promislow, and L. Yang. Curvature driven flow of bi-layer interfaces. *Physica D: Nonlinear Phenomena.*, 2011.
- [19] D. Grieser. Uniform bounds for eigenfunctions of the laplacian on manifolds with boundary. *Comm. P.D.E.*, 27:1283–1299, 2002.
- [20] W. Helfrich. Elastic properties of lipid bilayers - theory and possible experiments. *Zeitschrift für naturforschung C*, 28:693–703, 1973.
- [21] Lars Hörmander. *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag, 1985.
- [22] W. Hsu and T. Gierke. Ion transport and clustering in nafion perfluorinated membranes. *J. Membrane Science*, 13:307–326, 1983.
- [23] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geometry*, 20:237–266, 1984.
- [24] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer, 1980.
- [25] Wolfgang Kuhnel. *Differential Geometry. Curves - Surfaces - Manifolds*. AMS, 2006.
- [26] K. Lee, C. Hsu, W. Chiu, M. Tsui, Y. Tung, S. Tsai, and K. Ho. Dye-sensitized solar cells with micro-porous TiO<sub>2</sub> electrode and gel polymer electrolytes prepared by in situ cross-link reaction. *Solar Energy Materials & Solar cells*, 93:2003–2007, 2009.
- [27] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational. Mech. Anal.*, 98:123–142.
- [28] R. Moore and K. Promislow. Renormalization group reduction of pulse dynamics in thermally loaded optical parametric oscillators. *Physica D*, 206:62–81, 2005.
- [29] R. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. R. Soc. Lond.*, 422, 1989.

- [30] K. Promislow. A renormalization method for modulational stability of quasi-steady patterns in dispersive systems. *Siam J. Math. Anal.*, 33:1455–1482, 2002.
- [31] M. Roger and R. Schatzle. On a modified conjecture of De Giorgi. *Math. Z.*, 254:675–714, 2006.
- [32] J. Rubinstein and P. Sternberg. Nonlocal reaction-diffusion equations and nucleation. *IMA Journal of Applied Mathematics*, 48:249–264, 1992.
- [33] B. Sandstede and A. Scheel. Glueing unstable fronts and backs together can produce stable pulses. *Nonlinearity*, 13:233–277, 2000.
- [34] E. Tadmor. The well-posedness of the Kuramoto-Sivashinsky equation. *SIAM J. MATH. ANAL.*, 17:884–893, 1986.