THE PROOF IS IN THE PRACTICE? GRADUATE TEACHING ASSISTANTS AND FUTURE TEACHERS

By

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ABSTRACT

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In mathematics, engaging in reasoning-and-proving (RP) includes investigating mathematical relationships, formulating conjectures, evaluating others' conjectures or arguments, generating arguments, and communicating mathematical knowledge. These mathematical processes are key habits of mind that help learners think critically within and outside mathematics classrooms. Researchers and policymakers have reflected the centrality of RP by emphasizing the importance of helping students in all mathematical content areas and in all grade levels to engage in rich proving activities and become competent at generating and evaluating mathematical arguments (NCTM, 2000, 2009; Yackel & Hanna, 2003). In contrast to these recommendations, however, K-12 students' initial experiences with formal proof traditionally occur in high school geometry classes, where students often experience only one slice of the range of RP processes with a focus on proving given statements using a formal two-column proof format (Harel & Sowder, 1998; Herbst, 2002). Prospective teachers of elementary grades (PTEs) also typically have limited experiences developing and using proof, which could hinder their future elementary students' engagement in these RP processes (Balacheff, 1988; Martin & Harel, 1989). How are PTEs currently being prepared to provide rich opportunities for their future students experience and understand RP?

Mathematics for elementary teachers courses are the primary site for supporting this development of PTEs' mathematical knowledge for teaching. Geometry and Measurement is a

common content slice for these courses, with explicit attention to the work of proof (Cannata & McCrory, 2007; McCrory, Siedel, & Stylianides, 2008). This attention on RP is important for developing PTEs' ability to explain why mathematical relationships are true, as well as for preparing them to help their students develop RP-abilities. By examining how graduate teaching assistants (TAs) implemented RP mathematical tasks when teaching a Geometry and Measurement course for PTEs, this study investigated the opportunities for PTEs to learn about RP in such courses.

Specifically, six TAs, each assigned to teach their own sections of the course, participated in the study. Through classroom observations and interviews, the way these TAs engaged PTEs in RP and how TAs' conceptions of RP illuminate their instructional decisions around RP tasks was examined. RP tasks were defined as tasks with potential to engage PTEs in RP processes. Findings indicate that TAs engaged PTEs in a range of RP processes. For a plurality of observed tasks, however, opportunities for PTEs to engage in RP were decreased. TAs often decreased RP opportunities by stating a conjecture or proof before PTEs had a chance to generate them. There are multiple factors (e.g., TAs' conceptions of the purposes for teaching about RP and how to facilitate class discussions) that influenced their instructional decisions. This research has implications for professional development to support college mathematics instructors' teaching.

I dedicate this work to
my gracious Lord and Savior, who inspires me;
my supportive husband, who encourages me;
and my loving family, who believes in me.
Thank You!

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CHAPTER 1

INTRODUCTION

The process of reasoning-and-proving is fundamental to mathematics, both for professional mathematicians and in mathematics classrooms. Descriptions of mathematicians' practice and K-12 standards documents note that the proving process involves exploration of patterns, which can lead to the generation of conjectures, and can then be tested and revised or proven informally or formally (Lakatos, 1976; NCTM, 2009). The hyphenated term *reasoning-and-proving* (Stylianides, 2008) denotes the range of activities including investigating patterns, formulating conjectures, generating arguments, evaluating others' arguments, and communicating mathematical knowledge, which are "frequently involved in the process of making sense of and establishing mathematical knowledge" (Stylianides, 2009, p. 259). These reasoning-and-proving (RP) processes are key habits of mind that equip students to think critically within and outside mathematics classrooms. Researchers and policymakers also emphasize the centrality of RP through recommendations for students in all mathematical content areas and in all grade levels, including elementary-grades, to engage in proving activities and become competent at generating and evaluating mathematical arguments (Ball & Bass, 2000; CCSSI, 2010; NCTM, 1989, 2000, 2009; Stylianides, 2007a; Yackel & Hanna, 2003).

The treatment of proof in K-12 classrooms, however, is often restricted to verifying given statements using a two-column format (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Harel & Sowder, 1998; Herbst, 2002). This emphasis represents only one aspect of the range of RP processes, and contributes to the pervasive difficulties and limited views of proof held both by K-12 students and their teachers (Alibert & Thomas, 1991; Balacheff, 1988; de Villiers, 1998;

Herbst, 2002; Martin & Harel, 1989). For example, teachers often favor empirical arguments over deductive proofs, finding them more convincing or easier to follow, and are more likely to favor form over explanatory power (Knuth, 2002a; Steele, 2006; Stylianides & Stylianides, 2009). These conceptions extend to pedagogical choices, as teachers are more likely to use an empirical argument as compared to a proof if they find it more convincing (Knuth, 2002a; Steele, 2006). The full range of RP processes of noticing patterns, making conjectures, and forming arguments, however, can be accessible at the elementary (grades K-8) level, and as such, elementary teachers should be equipped to teach RP in meaningful ways. It is crucial, therefore, to help prospective teachers of elementary grades (PTEs) move from conceptions of proof as empirical arguments or as having one particular form, towards an understanding of the role of proof as a process in which one engages to make meaning in mathematics.

To address this need, the Conference Board of the Mathematical Sciences (2012) argues for undergraduate education across the nation to focus on the development of PTEs' abilities to reason and think mathematically. The authors specifically contend that "preparation . . . for teachers must provide opportunities to *do* mathematics and to develop mathematical habits of mind. . . . Teachers should have time and opportunity to reason abstractly and quantitatively, to construct viable arguments, to listen carefully to other people's reasoning, and to discuss and critique it" (p. 16). This need for supporting the development of PTEs' understanding of mathematical content and processes related to RP is particularly important when considering the fact that they will ultimately help students in grades K-8 makes sense of mathematics. Engaging elementary students in a range of RP processes would lay a strong foundation for reasoning and sense making upon which later success in mathematics crucially depends (CCSSI, 2010; NCTM, 1991, 2000). Providing additional opportunities for students to become more proficient at creating and investigating conjectures, evaluating mathematical arguments, and employing various methods of proof in everyday in mathematics classrooms is an important step in addressing concerns about students' limited experiences with proof. To do so, however, places significant demands on teachers' knowledge and beliefs about proof. As Stylianides and Ball (2008) explain, "unless teachers have a good understanding of proof, we cannot expect that they will be able to effectively promote proving among their students" (p. 309). As future elementary teachers, it is crucial, then, for PTEs to develop substantial understanding of RP to provide meaningful opportunities for their students to engage with reasoning and proof (Stylianides, 2007a, 2007b; Stylianides & Ball, 2008).

Mathematics for elementary teachers courses are the primary site for supporting the development of PTEs' understanding of mathematical content and processes. In universities that offer elementary certification, PTEs are typically required to take at least one semester-long course covering mathematical content PTEs will likely teach in their future elementary-grade (K-8) classrooms (Cannata & McCrory, 2007; Lutzer, Rodi, Kirkman, & Maxwell, 2007, p. 53). Geometry and Measurement is a common content slice for these courses, with explicit attention to the work of proof (Cannata & McCrory, 2007; McCrory et al., 2008). This attention on RP is important for developing PTEs' ability to explain why mathematical relationships are true, as well as for preparing them to help their students develop their RP-abilities.

1.1 Focus of this Study

In many cases, mathematics for elementary teachers courses are taught by graduate teaching assistants (TAs), which could have important implications for the development of

PTEs' understanding of RP. As instructors for these courses, the TAs represent a central feature of PTEs' experiences learning about reasoning and proving (Philipp, 2007) and teachers' conceptions are implicated as a major influence on teachers' practices (Putnam & Borko, 1997; Thompson, 1992). Further, little research directly explores the role of mathematics TAs in creating and teaching concepts of RP (Speer, Gutmann, & Murphy, 2005).

In light of these teaching, learning, and research needs, this study examines the nature of RP opportunities in PTEs' mathematics courses by focusing on a geometry and measurement course at a large, public university. At this university, the mathematics-content courses are taught in the mathematics department with TAs as the sole instructors. To understand the nature of PTEs' opportunities for engaging in RP in this course, I investigate TAs' instructional decisions when teaching about RP by observing their classroom practices and interviewing them about their knowledge and beliefs related to the teaching and learning of RP.

The study of TAs' instructional decisions when teaching about RP is complex due to a large number of variables that affect teachers' classroom decisions. To bound this study and aid in anchoring the examination of TAs' classroom practices when they are teaching about RP, I focused on a specific classroom practice, the enactment of mathematical tasks related to RP, and investigated a subset of factors that influence teachers' decisions, their knowledge and beliefs about the content and the teaching and learning of the content (Figure 1). Since teachers' knowledge and beliefs are a major factor influencing their instructional decisions (Putnam & Borko, 1997; Schoenfeld, 2010), TAs' knowledge and beliefs about RP in mathematics, teaching, and learning were investigated through the use of semi-structured interviews. The focus on the enactment of RP tasks was motivated by the fact that enacting mathematical tasks in ways that provide opportunities for students to engage in *doing* mathematics (including creating and evaluating mathematical generalizations and arguments) is a *high-leverage* (Ball, Sleep, Boerst, & Bass, 2009) teaching practice in secondary mathematics classrooms (Stein & Lane, 1996). Therefore, investigating the enactment of RP tasks in college classrooms presents an opportunity to understand how PTEs are given opportunities to participate in the processes of RP in the classroom community.

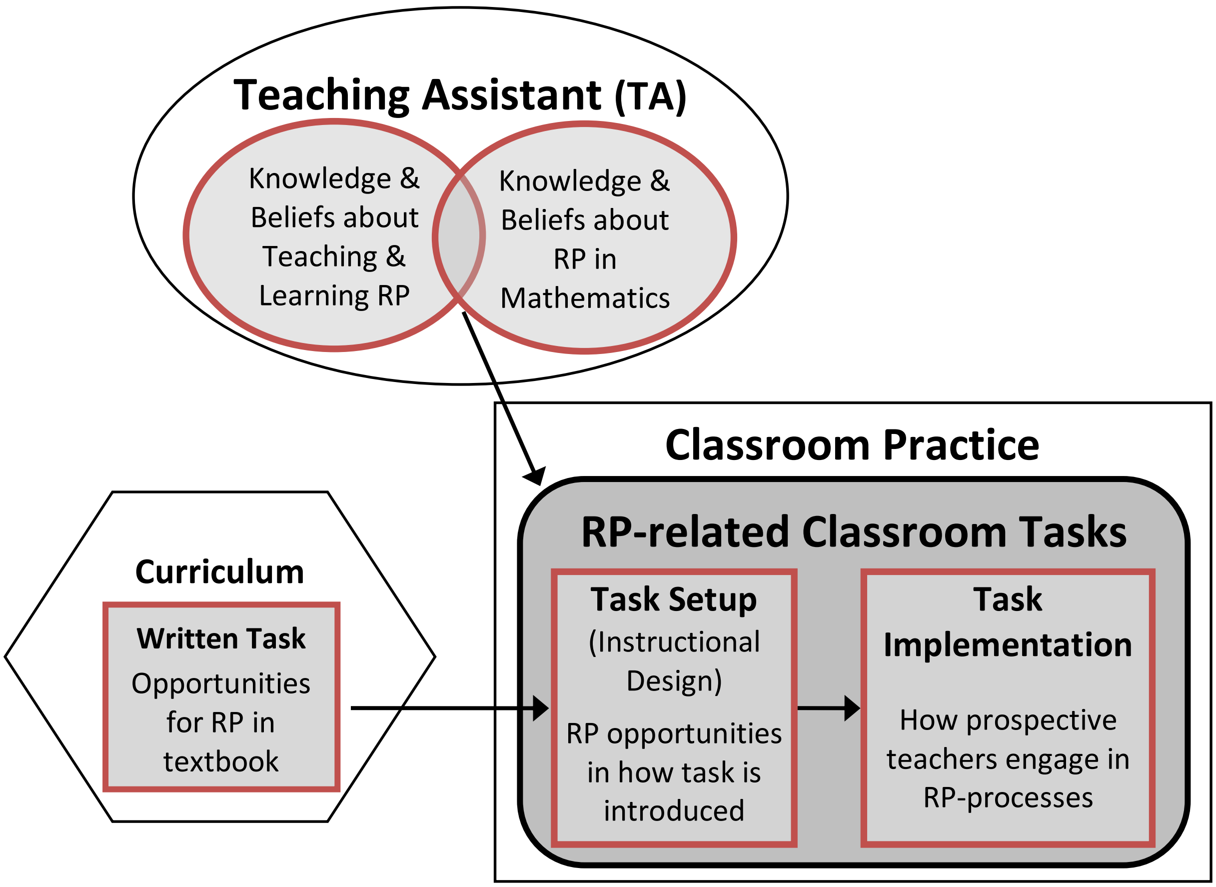


Figure 1. Model of major components of the study
(For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation)

The enactment of RP tasks spans from how tasks are designed in curriculum materials to how teachers and students interact with tasks in the classroom (Henningsen & Stein, 1997). The opportunities for PTEs to experience RP processes were examined in the textbook and then in the task set up and implementation during classroom instruction (Figure 1). The extent to which TAs provided a range of RP opportunities for PTEs to experience and how and why changes in RP opportunities occurred during classroom instruction was analyzed.

1.2 Contributions to the Field

This study refines the knowledge base about TAs' classroom practices when teaching RP, an area that has been under-researched. This study also adds to the understandings of TAs' knowledge and beliefs about RP, specifically as it relates to teaching prospective teachers about RP so they understand the processes of RP in particularly robust ways. Furthermore, this examination of TAs' instructional decisions when teaching about RP and the enactment of RP tasks during classroom instruction provides information about professional development needs of collegiate mathematics instructors. This contribution is particularly timely because the Conference Board of the Mathematical Sciences (2012, p. 17) recently re-emphasized the current lack of and need for professional development and support for collegiate mathematics instructors who are teaching mathematics courses for prospective elementary teachers.

1.3 Limitations

Although this research was carefully designed and executed, it is important to acknowledge some limitations. First, the study is based on a convenience sample consisting of graduate students assigned to teach a geometry and measurement course for prospective elementary teachers at one university. This sample may not be representative of collegiate mathematics instructors in the United States in general. Second, TAs' teaching practices were studied by observing each TA's classroom for at least six hours of class time to see TAs teach different types of RP-related tasks. This study does not claim to characterize each TA's teaching practices in general because the focus was on the enactment of RP tasks. This analysis also describes patterns in how each TA enacted the observed RP tasks, constituting a subset of the total number of RP tasks enacted across the semester. Third, this study did not observe TAs during weekly course meetings to assess any potential effects of this professional development opportunity on their classroom decisions. This avenue warrants investigation but was beyond the scope of this study. Finally, the study of classroom practices considered the nature of PTEs' opportunities to engage in RP processes, but did not include measures of student learning to assess potential gains in PTEs' knowledge and abilities related to RP processes. This is a line of inquiry that also merits further study but was outside the scope of this investigation.

1.4 Organization of the Dissertation

This dissertation is organized into eight chapters. Following this overview chapter, Chapter 2 reviews relevant literature about studying teachers' knowledge and beliefs and classroom practices especially related to the teaching of RP, lays out the frameworks used in this study, and outlines the research questions. Chapter 3 describes the methods used and the data collected in this study. Chapters 4 through 7 detail the results: Chapter 4 includes the results from analyzing TAs' knowledge and beliefs about RP to outline TAs' conceptions before unpacking the classroom observations. Chapter 5 describes the results from analyzing the textbook for the course, which guided the schedule for the classroom observations. The results of the analysis of those classroom observations are presented in Chapter 6. Chapter 7 presents the results from analyzing how TAs talked about their instructional decisions around RP. Finally, Chapter 8 discusses implications of the study and suggests avenues for future research.

CHAPTER 2

LITERATURE REVIEW

This study investigates TAs' conceptions and teaching practices when they are teaching PTEs about RP. To frame this examination, I first describe a framework for considering TAs' decision-making processes in enacting RP tasks. Next, I detail theories of teachers' knowledge and beliefs that inform the design and analysis of the study. Then, I describe frameworks for examining teaching practices, focused on teaching about RP, on which I draw throughout the study. Finally, I list my research questions.

2.1 Examining Teachers' Classroom Decisions

Classroom teaching has been depicted as a more demanding and complicated task than that of a physician determining a diagnosis (Fogarty, Wang, & Creek, 1983; Shulman, 1984). In contrast to clinical diagnostic settings that usually involve relatively short, one-on-one interactions between physicians and patients, teaching typically demands daily interactions with multiple students, simultaneously. During planning and in the midst of instruction, teachers work through a myriad of interconnected and conflicting decision situations. The interactive, dynamic nature of teaching gives credence to the importance of studying teachers' decision making because teachers' decisions directly impact classroom events which in turn influence student learning (Munby, 1982). Therefore, examining teachers' decision making could shed light on how and why teachers structure lessons and provide learning opportunities for students, contributing to a better understanding of effective teaching practices.

Mathematics education researchers and policy makers also note the importance of studying mathematics teachers' decision making because effective teaching requires teachers not only to have a rich and extensive knowledge base for teaching mathematics, but to establish challenging and supportive classroom environments and continue to develop their instructional practices and mathematical knowledge to support student learning (Ball, 1988a; NCTM, 2000). In the classroom, learning to listen to what students say and then construct appropriate responses on a moment-to-moment basis places special demands on teachers (Ball, 1993; NCTM, 2000; Smith III, 1996). In these classroom settings, teachers juggle a variety of responsibilities, as Ball (1988b) explains:

> Teaching mathematics means helping to build bridges between the subject matter and learners in ways that respect the integrity of both. To do this, teachers explain, ask questions, respond to students, develop and select tasks, and assess what students understand. These activities emerge from a bifocal consideration of mathematics and students, framed by the teacher's own understandings and beliefs about each, and shaped further by her ideas about learning and her role in promoting learning. (p. 297-298)

During teaching, decision situations incorporate a host of factors, including: students' behaviors, needs, and experiences; teachers' goals/objectives and lesson plan; teachers' knowledge and beliefs of the subject matter, the students, and teaching strategies; and time constraints.

Since teachers' knowledge and beliefs have been shown to be an important factor influencing their instructional decisions (Ball, 1988b; Putnam & Borko, 1997; Schoenfeld, 2010), examining teachers' knowledge and beliefs about the mathematics content, students, and pedagogy can help us understand their instructional decisions and potential ways to help them improve their teaching practices. Thus, this study investigates teachers' knowledge and beliefs about mathematics and teaching to understand their instructional decisions made during classroom teaching. Considering the complexity of studying teachers' classroom decisions, this study focused on a particular content area, RP, and a specific context, studying TAs when they are teaching PTEs. The sections that follow, therefore, describe related frameworks and literature that this study draws upon for examining TAs' knowledge and beliefs about RP and their classroom practices around RP.

2.2 Examining Teachers' Knowledge and Beliefs

Examining teachers' knowledge and beliefs about mathematics, teaching, and learning can help one better understand teachers' instructional decisions within their classroom practices. Psychological and educational literatures present a range of definitions and labels (e.g., attributed, professed, goals, conceptions) for beliefs, which is why Pajares (1992) described beliefs as a "messy construct." Some of the confusion stemmed from researchers' attempts to differentiate between knowledge and beliefs (Calderhead, 1996). Although differentiating between the two domains can be helpful in theory, they are less distinct in practice (Grossman, 1990). Therefore, in this study the term *conception* (also used interchangeably with the term *view*) is used to refer collectively to teachers' knowledge and beliefs. Teachers' conceptions are defined as how they "think about the nature of mathematics, teaching, and learning" (Aguirre & Speer, 1999, p. 328).

The influence of teachers' conceptions of mathematics, teaching, and learning on how teachers refine mathematical tasks and problems and make other decisions in their instructional approaches is complex (Calderhead, 1996; Cohen, 1990; Kagan, 1992; Leder, Pehkonen, & Törner, 2002; Pajares, 1992; Thompson, 1992). Overall, research indicates that knowledge and beliefs are major factors that shape "teachers' decisions about what knowledge is relevant, what teaching routines are appropriate, what goals should be accomplished, and what the important features are of the social context of the classroom" (Speer, 2005, p. 365). The mathematics education research about teachers' conceptions and practices, however, has focused primarily on practicing K-12 teachers (see Leder et al., 2002) or prospective teachers (e.g., Timmerman, 2004), with only two peer-reviewed examinations of tertiary instructors' conceptions in mathematics teaching (Gutmann, 2009; Speer, 2008). Gutmann offered insights into seven beginning mathematics graduate students' views about the nature of mathematics, teaching mathematics, themselves as learners of mathematics, and their students as learners, but did not investigate how their knowledge and beliefs influenced or were evident in their teaching practices. Speer (2008) conducted an in-depth case study analysis of Calculus TAs' teaching practices, instructional decisions, and the relationship between their conceptions and specific teaching practices. By examining the conceptions and teaching practices of collegiate instructors who are teaching about RP to PTEs, this dissertation study contributes to this growing and needed area of research (Speer, Smith III, & Horvath, 2010).

2.2.1 Teachers' Mathematical Knowledge for Teaching

Although a number of factors affect teachers' classroom practices, this study focuses on teachers' conceptions because it is widely acknowledged that the work of mathematics teaching draws upon a deep and broad knowledge base, including knowledge of mathematics, pedagogy, and student learning (Fennema & Franke, 1992; NCTM, 1991; Shulman, 1986). Several studies suggest that the nature, depth, and organization of teacher knowledge influence teachers' presentation of ideas, flexibility in responding to students' questions, and capacity for helping students connect mathematical ideas (Ball, 1988a; Stein, Baxter, & Leinhardt, 1990). Mathematics teachers are a special class of users of mathematics; the knowledge they need for teaching mathematics goes beyond what is needed by other well-educated adults, including mathematicians (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008). As described by Hill, Rowan, and Ball (2005):

> Mathematical knowledge for teaching [MKT] goes beyond that captured in measures of mathematics courses taken or basic mathematical skills. For example, teachers of mathematics not only need to calculate correctly but also need to know how to use pictures or diagrams to represent mathematics concepts and procedures to students, provide students with explanations for common rules and mathematical procedures, and analyze students' solutions and explanations. (p. 372)

Further, Hill and colleagues link higher levels of MKT to greater student learning gains, underscoring the importance of teachers' MKT and its impact on what students learn in the classroom (Hill et al. 2005).

To ground this investigation of collegiate mathematics instructors' MKT, this study focuses on their mathematical knowledge for teaching about the activity of RP. The selection of RP as a focus was based on the fact that the process of RP is fundamental to the work of doing authentic mathematics, both in the world of professional mathematicians and in the mathematics classroom. To be prepared to teach about RP, teachers need a specialized content knowledge about RP and pedagogical knowledge about the teaching and learning of RP. Teachers' MKT about RP would include a robust understanding of these processes of proving and pedagogical tools for engaging students in these processes (Steele & Rogers, 2012).

Within the context of this study, there are two types of teachers for whom this notion of MKT about RP applies: (a) the undergraduate students enrolled in mathematics courses for elementary teachers who are prospective elementary teachers and (b) their collegiate instructors who are graduate students in mathematics or mathematics education programs. The notion of MKT in these two settings is elaborated further, next.

2.2.2 Teaching about Reasoning-and-Proving in the Elementary Grades

Since the undergraduate students enrolled in the course that is the focus of this study are PTEs, it is important to consider the knowledge elementary teachers need to teach about RP in the elementary grades. Despite widespread agreement that RP should be a central feature of all students' mathematical experiences, a growing body of research shows that elementary students and their teachers have difficulties and limited experiences with RP (Healy & Hoyles, 2000; Stylianides & Ball, 2004; Stylianides, 2005). Teachers often favor empirical arguments over deductive proofs, finding them more convincing or easier to follow, and are more likely to choose an empirical argument for classroom use (Steele, 2006; Stylianides & Stylianides, 2009). In many PTEs' prior experiences as K-12 students learning about RP, they were likely often asked to prove fairly obvious statements, and rarely given opportunities to explore, conjecture, and justify their own conjectures (Herbst, 2002; Usiskin, 1980). These experiences with proving contrast with the recommendations from the NCTM (2000), the Common Core State Standards (2010), and the nature of proving in the mathematics discipline.

Preparing PTEs to teach about RP in the elementary context also begs the question 'what should teachers expect students to learn about proof and proving in elementary grades?' Research literature is rather sparse concerning proving for young children, with the exception of the work done by Maher and colleagues at Rutgers (e.g., Maher & Martino, 1996), Lampert (1990), and A. J. Stylianides (2007c). These works provide evidence that that students are developmentally capable of generating proof even at the elementary school level, and support the notion that teachers should create an environment that nourishes this capability. The field has yet to develop a learning trajectory or more formalized conceptualization of *proving* in the elementary grades, however, which limits our understanding of what preparing elementary-grade students for RP in middle and high school and beyond entails.

This study makes steps towards such a conceptualization by considering the processes of RP (including generating and evaluating generalizations and arguments) as key mathematical habits of mind that are important for all students, starting in the elementary grades. In an elementary classroom, what would engaging in RP include? In the classroom, proof, as defined by Stylianides (2007c), emerges as a product of discussions between the teacher and students where students learn that proof goes beyond convincing oneself and convincing a friend to convincing the classroom community as a broader audience. By engaging in RP classroom tasks, students learn to analyze patterns, articulate how they are thinking about mathematics, and evaluate others' mathematical ideas and arguments. As elementary children solve open ended problems and reason mathematically, "they learn to analyze their own, their classmates, and their teacher's thinking. . . . It is during these early years that young students lay down those habits of reasoning upon which later achievement in mathematics will depend (CBMS, 2001, chap. 7). In alignment with the processes of RP, this study emphasizes the importance of providing opportunities for elementary students to engage in analyzing mathematical situations, determining general relationships they think are true, and generating and evaluating mathematical arguments about these relationships.

To provide such opportunities for elementary students PTEs need a robust understanding of what constitutes a proof, how to generate and evaluate valid mathematical arguments, and the various roles of proof in mathematics and in teaching (Steele & Rogers, 2012). As the primary site for developing PTEs' mathematical knowledge of content and processes, it is important for PTEs' undergraduate coursework to help them to not only know and understand mathematics, but also to know and experience the processes of RP themselves as learners (Copes, 1996; Ernest, 1994; National Research Council, 1996; NCTM, 1991). This study, therefore, examines the nature of the opportunities to learn about RP in a geometry course for PTEs. Although TAs assigned to teach courses for PTEs may have robust conceptions of proof, the lack of clear direction as to what elementary students should be expected to learn and do with regards to RP may influence how TAs engage PTEs in RP in the classroom.

2.2.3 TAs Teaching Prospective Elementary Teachers about Reasoning-and-Proving

Preparing PTEs with a robust understanding of RP is an important aspect of the mathematical courses PTEs take and it is not easy for collegiate mathematics instructors (in this case graduate TAs) to structure a course that accomplishes that goal (CBMS, 2001; National Research Council, 1996). Structuring a course for PTEs around such a goal could be an overwhelming endeavor for any mathematics instructor, and the work of teaching these courses would entail a specialized knowledge for teaching. TAs for these courses will be expected to make sense of curriculum, design lessons, create and administer assessments, and facilitate mathematical discussions in ways that encourage PTEs to investigate, conjecture, evaluate claims and arguments, and produce mathematical justifications. This instructional setting presents a challenge for TAs teaching PTEs where PTEs need to understand the mathematical content (as current undergraduate students) and develop the ability to coherently explain the mathematical ideas to others (as future teachers).

Furthermore, although the elementary mathematics content may seem simple and straightforward to most graduate students assigned to teach this course, the notion of preparing PTEs with the mathematical knowledge they need to teach elementary mathematics involves more than just teaching them content. That is, Ball and Bass (2000, p. 98) contend that in teaching elementary mathematics

> one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. We refer to this as *decompression*. Paradoxically, most personal knowledge of subject matter knowledge, which is desirably and usefully compressed, can be ironically inadequate for teaching. In fact, mathematics is a discipline in which compression is central. Indeed, its polished, compressed form can obscure one's ability to discern how learners are thinking at the roots of that knowledge. Because teachers must be able to work with content for students in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements.

This description of the challenges elementary teachers face in connecting to the mathematical world of their students also holds true for mathematics instructors who are teaching PTEs. Therefore, "in a college course, prospective teachers' ingenuous questions will require instructors to 'decompress' *their* mathematical knowledge to find responses satisfying to both mathematician and teacher" (CBMS, 2001, chap. 7). Teaching PTEs about RP, therefore, calls for TAs to work from what PTEs already know, understand how PTEs think and misunderstand RP, and use pedagogical strategies that engage PTEs in a range of RP processes. It is important, then, to investigate what TAs know and believe about RP.

2.2.3.1 Studying TAs' Conceptions of Reasoning-and-Proving

An examination of literature related to studying TAs' conceptions of RP when teaching mathematics reveals that there are no peer-reviewed empirical studies of TAs' beliefs about RP or of teaching about RP to prospective teachers (Speer et al., 2005). Investigating TAs' beliefs about mathematics, teaching, and learning without a focus on content, Gutmann (2009) interviewed seven beginning mathematics graduate students. He found commonalities among his participants in their positive attitudes toward mathematics, views of teachers as role models, and descriptions of mathematics as having different levels where upper levels were inaccessible to some students. Studying Calculus TAs' beliefs of mathematics, teaching, and learning and their relationship to TAs' instructional practices, Speer (2008) observed and interviewed one TA. She found a connection between the TA's collection of beliefs and his in-the-moment instructional decisions. A common theme in these two studies was the examination of TAs' conceptions of mathematics, teaching, and learning. Since TAs are knowers of mathematics as well as teachers of mathematics, investigating their conceptions along these dimensions provides a more comprehensive view of TAs' conceptions that influence their teaching practices.

Knuth (2002a, 2002b) supports this notion of investigating teachers' conceptions of mathematics, teaching, and learning; his work focused on teachers' views of proof at the secondary level. More specifically, Knuth examined secondary teachers' conceptions of the nature and purposes of proof in mathematics, teaching, and learning. The purposes of proof he emphasized were drawn from some of the primary functions of proof in mathematics (Table 1) that have been identified by mathematics education researchers (Balacheff, 1988; Hanna, 1991; de Villiers, 1999). In this study the terms *purpose*, *role*, and *function* of proof are used interchangeably to refer to this notion that there are purposes proof can serve in the contexts of mathematics and mathematics teaching.

These roles are particularly important to note in the K-12 setting because within K-12 school mathematics the primary purpose of proof emphasized has been *verification* (Herbst, 2002). There have been calls from researchers, however, for more emphasis on proofs that help *explain why* a mathematical statement is true (de Villiers, 1998). Knuth argues, therefore, for the importance of teachers having robust views of the purposes of proof in mathematics and in teaching to provide meaningful opportunities for their students to experience proof in the classroom. Extending this work to the collegiate context presents an opportunity to investigate TAs' salient views of the purposes of RP in mathematics, teaching, and learning.

Table 1. Purposes of Proof in Mathematics and Teaching
(de Villiers, 1998, 1999; Hanna, 1995, 2000; Hanna & Barbeau, 2008; Knuth, 2002a; Steele, 2006; Stylianides, 2009)

1. Verification of truth	*Establishing, checking, or confirming the truth of a known idea or given claim* [Note: Direct methods of proof and proof by contradiction (supposing a claim is false, subsequently drawing a conclusion that contradicts a claim, thereby verifying the truth of the given claim) also fit under the verification purpose of proof]
2. Falsification	*Establishing, checking, or confirming the falseness of an idea or given claim.* Considering proof as an argument against a claim, proof by counterexample and "reduction ad absurdum" (stating a claim and showing that it results in a contradiction—demonstrating the claim to be false) are captured under this purpose. (Stylianides, 2009, p. 269)
3. Explanation	*Providing insights into* why *a mathematical claim is valid.*
4. Discovery/Creation of new mathematical knowledge	*Discovering or developing new mathematical ideas, confirming conjectures, or building mathematical ideas*
5. Communication of mathematical knowledge	*Helping others understand a mathematical idea; disseminating knowledge to other doers of mathematics*
6. Systematization of the domain	*Imposing a logical structure on the mathematical domain; organizing and cataloging results with respect to axioms and other prior knowledge*

2.2.3.2 What is Known about TAs' Potential Conceptions of Reasoning-and-Proving

Although mathematics education literature does not yet provide insights into TAs' conceptions of RP in mathematics, teaching, and learning, based on how mathematicians engage in RP, I outline an informed hypothesis of how TAs' may conceive of RP and how that could compare with PTEs' possible conceptions. Comparing PTEs and TAs' experiences learning about RP suggests potentially different predispositions about RP: PTEs' may have had prior experiences learning about RP as a product that should be produced to verify the truth of a claim, rather than a process of RP in which one engages to make meaning in mathematics (Herbst, 2002; Usiskin, 1980). These prior experiences may predispose them to viewing proof more procedurally and with a different purpose than that recommended by the *Standards* documents. In a process that is more in line with the *Standards*, however, mathematicians typically construct a proof after extensive explorations of mathematical relationships, determinations of meaningful patterns, formulations of conjectures based on the patterns, tests and revisions of their conjectures, and the production of informal arguments that convince them of the validity of the conjectures (de Villiers, 1998; Lakatos, 1976; Stylianides, 2009; Usiskin, 1980). In mathematical research these RP activities (e.g., exploring and conjecturing) may be more educational than the proof itself.

TAs may experience these process-oriented aspects of proving as they engage in mathematical research or work with others solving mathematics problems, but these experiences typically occur outside of the mathematics classroom (Raman, 2003). TAs' previous and current experiences as mathematics learners are primarily in classrooms taught in a traditional, teacher-centered, lecture format (Speer et al., 2010). Based on their *apprenticeship* *of observation* (Lortie, 1975), therefore, TAs are more likely to decide to implement more teacher-centered instructional strategies (e.g., a model that may use primarily lecture-based instruction with little-to-no student-to-student interactions) because they have experienced success learning in classrooms that were more teacher-centered (Grossman, 1989). These potential views of mathematics teaching and learning are hypotheses. To better understand TAs' classroom practices, this study uses interviews to investigate how TAs talked about their conceptions of RP in mathematics, teaching, and learning; thereby refining these theorized views of TAs' conceptions.

2.3 Frameworks for Analyzing Teachers' Classroom Practices

To study TAs' instructional decisions, this study investigates TAs' classroom practices when teaching PTEs about RP. This context is important because many colleges and universities require PTEs to pass one or more mathematics courses designed for them, but there have been few empirical studies of collegiate mathematics teachers' conceptions and teaching practices when teaching such courses (Cannata & McCrory, 2007; Lutzer et al., 2007, p. 53). In fact, little research of collegiate mathematics teaching "has focused directly on *teaching practice*—what teachers do and think daily, in class and out, as they perform their teaching work" (Speer et al., 2010, p. 99). The absence of such research restricts the field's understanding of collegiate mathematics teaching and the nature of the opportunities for undergraduate students to learn about mathematics in these courses; a research need this study addresses.

This study examined TAs' classroom practices by investigating what opportunities they provided for PTEs to experience RP and how PTEs actually engaged in RP processes during instruction. Consistent with a learning-as-participating perspective on learning (Sfard, 1998), the examination of TAs' teaching practices around RP focused on the extent to which PTEs *participated* in the creation and evaluation of generalizations and mathematical arguments. From this perspective on learning, mathematical knowledge is seen as fluid and malleable, a continually negotiated body of ideas that are each dependent on their context. Knowledge is not seen as being acquired as personal property, but as the ongoing product of intellectual doing (Sfard, 1998). In terms of a contemporary mathematics classroom, mathematical knowledge is constructed and negotiated within the complex classroom space and shared by students and teachers. Doing and learning mathematics, then, is not the quest for private ownership; it is the request to participate in an evolving *community of mathematical practice* (Lave & Wenger, 1991). To study how PTEs participated in practices of RP during classroom instruction, the extent to which PTEs were involved with generating and critiquing generalizations and justifications was examined. The Mathematical Task Framework and then the RP-framework, which guide the methodological approach, are described next.

2.3.1 Focusing on RP-related Classroom Tasks

Mathematics learners "develop their sense of what it means to 'do mathematics' from their actual experiences with mathematics, and their primary opportunities to experience mathematics as a discipline are seated in the classroom activities in which they engage" (Henningsen & Stein, 1997, p. 525, citing Schoenfeld, 1992, 1994). Henningsen and Stein describe such classroom activity oriented toward the development of a particular mathematical idea as a *mathematical task*. Rich, mathematical instructional tasks are central to students' learning because "tasks convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). The nature of tasks in which mathematics learners engage can shape how they think and can serve to limit or broaden their views of mathematics (Doyle, 1983).The investigation of PTEs' opportunities to learn about and experience RP in this study is thus framed by the concept of mathematical tasks.

Drawing upon Henningsen and Stein's (1997) conceptualization of mathematical tasks, a segment of classroom activity oriented toward the development of a specific mathematical idea is considered as a mathematical task. In the textbook used for the course in this study (Beckmann, 2008), for instance, a question asked students to generate a conjecture about the measure of vertical angles formed where two lines intersect. The next question in that section asked students to justify their conjecture by proving the relationship always has to be true. These questions provided opportunities for students to generate and justify conjectures about angles formed by two lines, and constitute one task.

Henningsen and Stein's (1997) framework (Figure 2) depicts the progression of mathematical tasks from the written textbook or lesson plan, to the way the teacher introduces the task in the classroom (task set up), and how students and teachers act and interact with the task (task implementation). Henningsen and Stein focused on how the cognitive demand of tasks was maintained or changed in how the teacher introduced and interacted with students around the task. High-level cognitive tasks were typically those that required students to think conceptually and make connections or *do* mathematics. During the implementation of such tasks, however, various factors were identified that contributed to the teachers lowering the cognitive demand of these tasks, such as time constraints and the participants shifting the focus onto correct answers instead of process-oriented solutions.

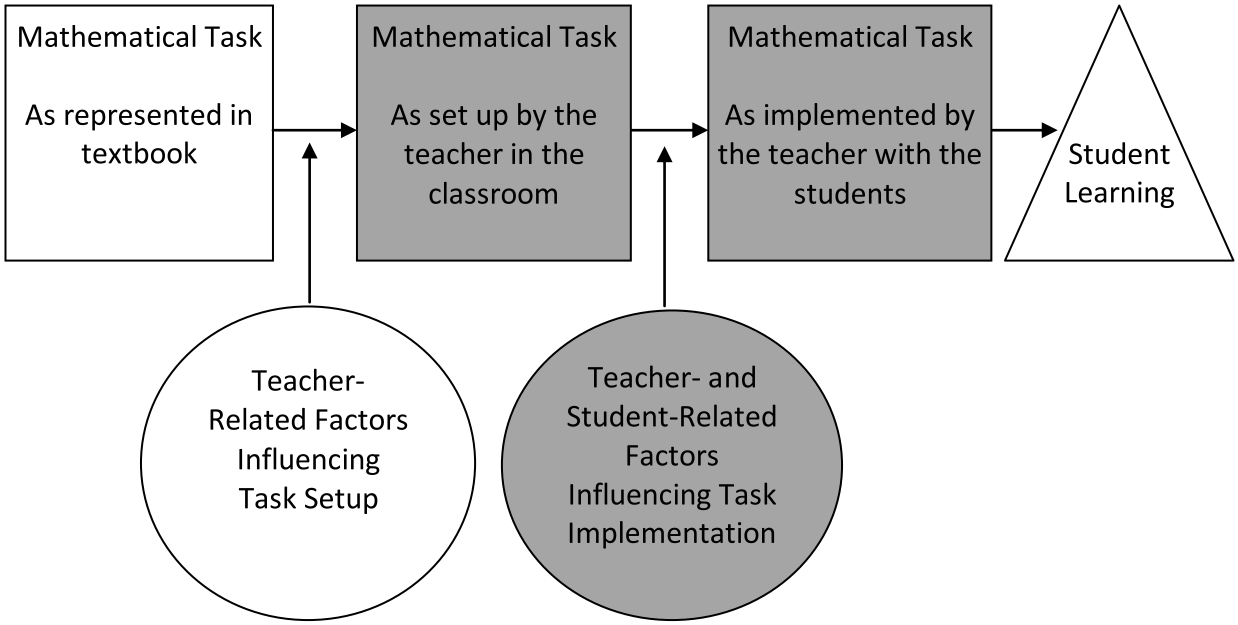


Figure 2. Framework for studying TAs' implementation of proof-related tasks (Modified from Bieda, 2012; Henningsen & Stein, 1997)

Henningsen and Stein characterized the cognitive activity of *doing mathematics* as "including complex mathematical thinking and reasoning activities" (1997, p. 532) such as creating and evaluating conjectures, looking for patterns, justifying, explaining, and solving non-routine problems. Since the processes of RP include exploring mathematical relationships, generating conjectures, revising conjectures, and producing mathematical justifications, tasks that engage students in this work of RP would constitute a specific subset of the doing-mathematics tasks.

Bieda (2010) applied this notion of proof-related tasks as a subset of the doing-mathematics tasks when she extended Henningsen and Stein's (1997) work. Bieda studied the nature of teacher-student interactions when middle school teachers enacted proof-related tasks by observing their classroom practices. Focusing on the shaded portions of Figure 2, her study depicted the opportunities to prove and characteristics of the pedagogy of proving in the middle school classrooms observed. Bieda found that time constraints and insufficient feedback from teachers regarding students' conjectures and justifications were major factors in lowering the cognitive demand of proof-related tasks. Bieda inferred that these results indicated the participants needed professional development and curricular support to help them develop their students' understandings of justification and proof in more mathematically robust ways.

I extended Bieda's work by applying the Mathematical Task Framework to examine the nature of opportunities for RP in a mathematics course for PTEs. To study the classroom practices, I specifically examined the opportunities for PTEs to engage in RP in the tasks as written (in the textbook), as set up (that is, as introduced by the TA), and as implemented (how PTEs engaged in RP as they worked on the task).

2.3.2 Reasoning-and-Proving Framework

Investigating the types of RP opportunities in written, set up, and implemented tasks, required a framework for studying RP opportunities in a textbook and in classrooms. Gabriel Stylianides's (2009) analytic framework (Table 2) was used because he designed this framework for analyzing RP opportunities in curricula, and suggested its utility for analyzing classroom practices. This framework was particularly helpful because it identified how students were asked to engage in RP by considering processes of RP employed in the mathematics discipline (i.e., a subset of *doing mathematics* from the Mathematical Task Framework). Since the activity of RP involves more than generating a proof, examining a textbook by looking for where in the table of context or index words like proof or prove yielded a narrow view of the opportunities for students to engage in RP in previous textbook analyses (McCrory et al., 2008). This framework, contrarily, offered a systematic method for understanding students' opportunities to explore mathematical relations, generate conjectures, and produce justifications.

Table 2. Reasoning-and-Proving Framework (adapted from Stylianides, 2009)

	Reasoning-and-Proving					
	Mathematical Generalizations			*Mathematical Arguments*		
	Investigate Mathematical Relations	*Conjecture*	*Evaluate Claim*	*Evaluate Argument*	*Provide a Non-proof Argument*	*Provide a Proof*
Dimension 1 Components & Subcomponents of RP	• Explore Provided Examples • Generate Examples to Explore	• Make a Conjecture	• Evaluate Mathematical Claim	• Evaluate a Mathematical Argument	• Empirical Argument • Rationale	• Proof
Dimension 2 Purposes of Investigations, Conjectures, Evaluations, & Proofs	• Conjecture Precursor • Conjecture Non-Precursor	• Proof Precursor • Proof Non-Precursor	• Conjecture Precursor • Proof Precursor • Proof Non-Precursor	• Refine a proof • Refine an argument • Justifications for validity of an argument or proof		• Explanation • Verification • Generation of New Knowledge • Falsification
Dimension 3: Meta-level: Reflecting on proof				• Defining Proof • Identifying Proof		

Applying the RP-framework (Table 2) provided a language by which one could talk about what specific RP opportunities were in the textbook and in the task as set up in the classroom, as well as how students engaged in these RP processes when the task was implemented. The advantage of using this framework was that it integrated the types of RP processes in which students may engage (Dimension 1) and how students may engage in a range of RP processes (Dimension 2). By considering the work students would have to do to successfully complete a task, this framework identified a task as related to RP if it provided opportunities for students to create mathematical generalizations or mathematical arguments (Dimension 1). That is, the activities of investigating mathematical relations (through which students may identify patterns for a given set of data or examples) and generating conjectures were considered under the overarching category of "making mathematical generalizations"—the transporting of mathematical relations from given sets to new sets for which the original sets are subsets (Polya, 1954). The activities of providing nonproof arguments (such as rationales or empirical arguments) and generating proofs were considered under the overarching category of "generating mathematical arguments" though which students could have opportunities to provide support to mathematical claims.

Dimension 2 considered the purposes that investigations, conjectures, and proofs may serve as students engaged in a particular task, depicting situations where, within a particular task, a mathematical generalization could give rise to a proof and what roles of proof may be emphasized. For example, in mathematics, investigating mathematical relations can result in conjectures, which could lead to the production of proofs (e.g., Polya, 1954). This framework provided a means for understanding to what extent this structure was found in mathematics textbooks and in classroom instruction. Furthermore, the roles of proof in the framework were drawn from the same set of primary functions of proof in mathematics (Table 1) upon which Knuth (2002a, 2002b) relied when studying teachers' conceptions of proof. Incorporating these same roles of proof into this framework provided a means of examining how PTEs experienced different roles of proof in their classrooms, while paralleling the examination of TAs' conceptions of the nature and purposes of RP.

Since the study's context focused on TAs teaching PTEs, to analyze the opportunities for PTEs to engage in RP in the textbook it was important to also consider the specialized content knowledge PTEs need to develop to be prepared to teach about RP. This specialized content knowledge included additional aspects about RP to which Stylianides's (2009) original framework, which focused on what K-12 students had opportunities to learn about RP, did not attend. Furthermore, Stylianides (p. 283) suggested that the framework could be used to analyze classroom instruction and the alignment between the written and implemented tasks through the lens of RP. Presently there are no peer-reviewed studies that have done so.

In light of using this framework in a different context (i.e., at the collegiate level with a textbook for PTEs) and extending its use by also examining the task set up and implementation during classroom practice, the RP-framework used in this study was modified in three ways: First, the framework was modified to include two additional RP processes (grey columns, Table 2) to be more consistent with the Mathematical Task Framework and with the work around RP that PTEs would be expected to do as teachers in the future. The activity of evaluating conjectures needed to be incorporated because it can be part of the RP process and it was one of the criteria for *doing-mathematics* in the Mathematical Task Framework. Opportunities to evaluate mathematical claims or arguments could also lead to opportunities for students to refine or produce conjectures or justifications. Moreover, in their future classrooms, PTEs will be expected to routinely, often in real time, evaluate mathematical claims or arguments proposed by students, textbooks, or other outside authorities (Ball et al., 2008). Therefore categories for evaluating mathematical claims and evaluating mathematical arguments, which were not explicit in the original framework, were added to the RP-framework.

Second, to be able to apply the framework both in analyzing written mathematical tasks and their implementation, it was important to focus on how students might engage with the mathematics of the task when considering the subcomponents of Dimension 1 of the framework. Specifically, the subcomponents of one of the categories, *investigating mathematical relations*, were modified. According to Stylianides (2009), students *investigate* when they explore a mathematical situation, example, or data to examine mathematical relationships. Through such investigations, they determine mathematical relations they think hold for that situation. Stylianides, however, distinguished between *plausible* relations and *definite* relations as the subcomponents for this category. He coded for questions that represented situations for which it was not mathematically possible to provide conclusive evidence for a specific mathematical relation over other relations (plausible) and those for which it was possible to determine mathematical relations that fit the situation or data given the information in the problem (definite). Since the focus of the study was on the implementation of RP tasks, it was helpful to consider how students may engage in these explorations rather than the type of mathematical relationship they could determine. Therefore, the RP-framework was modified to distinguish between two ways students may engage in these explorations of mathematical relationships: either (a) investigating provided examples/data or (b) generating their own examples/data, to identify mathematical relations.

Third, the original framework focused on the production of generalizations and arguments, but Hanna and de Bruyn (1999) noted that problems or statements can also *discuss* characteristics or features of proof and proving instead of asking students to do proof. I added a third dimension to the framework to identify such opportunities for students to *reflect on what constitutes a proof*. These opportunities could occur after students evaluate or provide a mathematical argument, or they could occur separate from these other dimensions of the framework. This third dimension added a meta-level layer that was not included in the original framework, which pointed out opportunities where students could articulate how they know an argument is or is not a proof. It is important for students and teachers at all levels to be able to define and identify proofs and nonproofs (Stylianou, Blanton, & Knuth, 2009). The two subcategories for this third dimension incorporated these notions of characterizing and identifying proofs and nonproofs:

- *Characterizing Proof*: tasks that focus on discussing the characteristics of proofs, which could include some or all of the italicized portions of the following characterization—a proof is *a mathematical argument* that is *general* for a class of mathematical ideas and *establishes the truth* of a mathematical statement *based on mathematical facts* that are accepted or have been previously proven (Knuth, 2002a; Steele & Rogers, 2012; Stylianides, 2007c).
- *Identifying Proof*: tasks that focus on talking about how one can distinguish proofs from nonproofs, which could include discussions about features of an argument that help students recognize proof across a variety of representational forms based on mathematical criteria (Knuth, 2002a; Steele & Rogers, 2012).

This notion of identifying proof was unique from RP processes of evaluating proofs in the sense that the later was about doing the evaluating and the former was about recognizing and discussing how to evaluate or distinguish between proofs and nonproofs. This third dimension offered an opportunity for students to step back from the process of creating generalizations and arguments so they could think about these proving processes. Providing opportunities for PTEs to articulate what makes an argument a proof or to reflect on the proving process could help prepare them for helping others understand what constitutes a proof. Furthermore, since literature indicated that PTEs are often more convinced by empirical arguments than by proofs (Martin & Harel, 1989), emphasizing the generality of proof in situations where PTEs were experiencing the proving process could provide opportunities to counteract this viewpoint. This dimension, therefore, added a way to categorize if/how the task in the textbook or classroom instruction provided opportunities for PTEs to reflect on characteristics of proof and proving instead of to do proof.

This modified RP-framework (henceforth, referred to as the RP-framework) took into consideration RP processes and ways of thinking about proof that are particularly important to emphasize with PTEs. By using this framework to study tasks as written and enacted from a textbook for PTEs, this work contributes to mathematics education research by applying and refining this framework for use in studying collegiate mathematics teaching.

2.4 Research Questions

This study aims to understand TAs' teaching practices and conceptions when they are teaching PTEs about RP. This investigation supports both deeper understandings of the work of teaching at the collegiate level and the possibilities for improving how PTEs are taught about RP in their mathematics content courses. One way to approach such a study is to consider a best-case scenario. That is, where the PTEs' course is using a textbook with a wide-range of proving opportunities, taught by TAs who have broad conceptions about RP who are also participating in concurrent professional development. In such a scenario, the instructors for the PTEs' course are positioned to potentially make a positive impact on the mathematical preparation of PTEs because of TAs' strong knowledge base about RP. The use of a textbook with a range of RP opportunities and professional development could also help support TAs in implementing instructional strategies that engage students in the learning process (Wyse, 2010). Within such a scenario, the aim of this research is to examine how RP-related tasks are enacted in PTEs' mathematics classrooms and the nature of the RP conceptions that TAs emphasize when they talk about their enactment of RP-related tasks by investigating the following questions:

1. What are TAs' conceptions about the nature and purposes of RP in mathematics, teaching, and learning?
2. How do TAs enact RP-related tasks in a mathematics content course for PTEs?
3. In what ways, if at all, does understanding TAs' conceptions about RP in mathematics, teaching, and learning illuminate their instructional decisions regarding the enactment of RP-related tasks?

Since teachers' beliefs have been implicated as a major factor influencing teachers' practices, research question one (RQ1) investigates TAs' conceptions about RP to PTEs. To understand more about PTEs' experiences learning RP, RQ2 examines how PTEs' opportunities to engage in RP were or were not maintained during instruction. RQ3 then investigates potential relationships between TAs' conceptions and their teaching practices around RP. This research informs the pedagogy of collegiate mathematics instruction for PTEs at a time when improving the RP-related preparation of PTEs is crucial.

CHAPTER 3

METHOD

This study focuses on TAs' conceptions of RP and how they engaged PTEs in RP during classroom instruction. This mixed-methods study features a textbook analysis and a case study design, using observations and semi-structured interviews. The multiple case study design allows for investigation of descriptive research questions where the collective cases are selected because, together, they help illuminate a more general phenomenon (Stake, 2005).

3.1 Context

TAs in this study were teaching a geometry and measurement content course (referred to as *MTH 2*) for PTEs during one semester at a public university. MTH 2 plays an important role in the elementary teacher preparation program at this university. Approximately 350 students enroll in MTH 2 every school year, and (in addition to other programmatic requirements) PTEs must pass MTH 2 to graduate. Therefore, this course represents a significant milestone for PTEs and an opportunity for TAs to positively impact the mathematical perspectives and knowledge of numerous PTEs. There are also important features of MTH 2 that support the purposes of this exploration into TAs' conceptions and teaching practices: the goals and design of the course, as well as who teaches the course and how they are supported.

3.1.1 The Design and Goals of the Course

Stated on the course syllabus, one of the three primary goals for the course is to help PTEs "reason mathematically and do proofs in the context of geometry and measurement" (MTH 2 Syllabus, p. 1). Moreover, the textbook (Beckmann, 2008) aims to promote the development of mathematical thinking and reasoning skills in PTEs. It is organized around activities intended to promote engagement, exploration, and discussion. Furthermore, Beckmann explicitly describes goals of the textbook as intending to help PTEs to "*explain why* mathematics works the way it does" (p. xix, italics added), make sense of mathematics, and carry those abilities into their future classroom. Therefore, the course goals and textbook goals align with national recommendations for RP and support the focus of this study.

3.1.2 The TAs

At this university, graduate students who are pursuing mathematics or mathematics education degrees are appointed to teach the course as sole instructors (TAs) for MTH 2. A mathematics-education faculty member who also coordinates regular professional development meetings with all the MTH 2 TAs supervises them. These course meetings include opportunities for TAs to discuss administrative issues, instructional strategies, mathematical content, and student learning and they are encouraged to collaborate with their colleagues when designing assessments.

Participating in this study were the six TAs who were assigned to teach the course during one semester: Ace, Laura, Isaac, Peter, Evan, and Kelly.[1] They were assigned to teach classes that met either for 80-minutes twice a week (Ace, Laura, and Isaac) or for 50-minutes three times a week (Peter, Evan, and Kelly), a total of three contact hours per week. The TAs' backgrounds and prior experiences teaching MTH 2 are summarized in Table 3. By investigating TAs pursuing mathematics and mathematics education degrees, these data may be fairly representative of the different TAs assigned to teach courses like MTH 2.

[1] All names are pseudonyms to protect confidentiality

Table 3. Participants' Background Information

	Ace	Laura	Isaac	Peter	Evan	Kelly
Degree Pursuing	Math PhD	Applied Math Masters	Math PhD	Math PhD	Math-Education PhD	Math-Education PhD
Grad School Year	1	2	3	6	3	3
International /Domestic	Domestic	Domestic	International	International	Domestic	International
Prior Experience Teaching MTH 2 (semesters)	0	0	0	0	1	1

This course setup and textbook presented an opportunity to study a scenario where these college mathematics instructors are more likely to have opportunities to use innovative RP tasks in their teaching. Since these TAs are responsible for designing their classroom instruction (with some guidance from a course supervisor), this course also provides an opportunity to examine TAs' classroom practices and conceptions about teaching.

3.2 Instruments and Data Sources

To address the research questions about TAs' conceptions and teaching practices, qualitative data were gathered from two series of interviews, six hours of classroom observations of each TA, and a preobservation survey. A textbook analysis was also conducted, which guided the selection of classroom observations. The sections that follow describe the design of the instruments and link the data sources to the research questions.

3.2.1 Data Sources for Studying TAs' Conceptions of Reasoning-and-Proving

The primary data source for investigating TAs' conceptions of the nature and purposes of RP in mathematics, teaching, and learning (RQ1) was a series of interviews, consisting of a pre-semester interview, beginning-of-the-semester interview, and end-of-semester interview (Table 4). These interviews provided TAs with opportunities to talk about RP in different contexts. The pre-semester interview protocol (Appendix A.3) was adapted from interviews used to study secondary teachers' views of the roles of proofs in mathematics by Knuth (Knuth, 2002a; Steele, 2006; Steele & Rogers, 2012). It was designed to elicit responses about how TAs learned about RP (Part I) and conceptualized RP in the mathematics discipline (Part II).

Table 4. Data Sources for Research Question 1: What are TAs' Conceptions of the Nature and Purposes of RP in Mathematics, Teaching, and Learning?

TAs' Conceptions of:	*Pre-Semester Interview*	*Beginning-of-Semester Interview*	*End-of-Semester Interview*
RP in Mathematics	✓		✓
RP in Teaching		✓	✓
RP in Learning	✓	✓	✓

The beginning-of-the-semester interview (Appendix A.4) focused on TAs' conceptions of RP in mathematics teaching. The protocol was adapted from interviews used by researchers examining secondary teachers' views of the roles of proofs in teaching and learning (Knuth, 2002b; Steele & Rogers, 2012) and tertiary teachers' beliefs about teaching and learning (Gutmann, 2009; Wyse, 2010). The questions elicited TAs' conceptions about the purposes of proof in teaching and their views of how and why PTEs learn about RP (Table 4).

The end-of-semester interview (Appendix A.5) provided additional opportunities for TAs to talk about their conceptions of RP in mathematics, teaching, and learning for two reasons. First, after teaching MTH 2, TAs could have more examples to draw upon to further articulate their views of RP in mathematics, teaching, and learning than they did during the earlier interviews. Therefore, a majority of the interview protocol asked questions similar to those from earlier interviews or asked about TAs' experiences teaching RP in MTH 2 to elicit responses from TAs about their conceptions of RP. Second, the last two questions of the interview were included to inquire about how TAs' views of RP in mathematics, teaching, and learning relate to the prevalent views of RP from mathematics education literature that underscore this study. Specifically, TAs described their familiarity with and responded to the NCTM's (2000) recommendations about the role of RP in school mathematics, and they identified which roles of proof they most resonated with, and why. These questions were included only at the end of the end-of-the-semester interview to reduce the chance that introducing constructs from mathematics education literature could influence TAs' decisions or responses during the study. TAs' responses to these questions corroborate and further enhance the data about their views of RP from earlier interviews.

3.2.2 Data Sources for Studying How TAs Enacted Reasoning-and-Proving Tasks

The primary data source for investigating how TAs enacted RP tasks in MTH 2 (RQ2) was six hours of classroom observations for each TA across the semester. These observations aimed to understand how TAs set up RP tasks and in what RP processes PTEs engaged during the implementation of tasks (Figure 3, shaded portions). Additional data sources included an analysis of the written tasks in the textbook and a preobservation survey.

Applying the RP-framework (Table 2), the textbook analysis characterized the RP opportunities afforded by the tasks in the Mathematics for Elementary Teachers' text (Beckmann, 2008). The main goal of this analysis was to identify tasks that provided opportunities for PTEs to engage in different aspects of RP processes so that classroom observations of TAs implementing a range of RP opportunities could be planned. This textbook analysis also provided groundwork for understanding the enactment of those tasks.

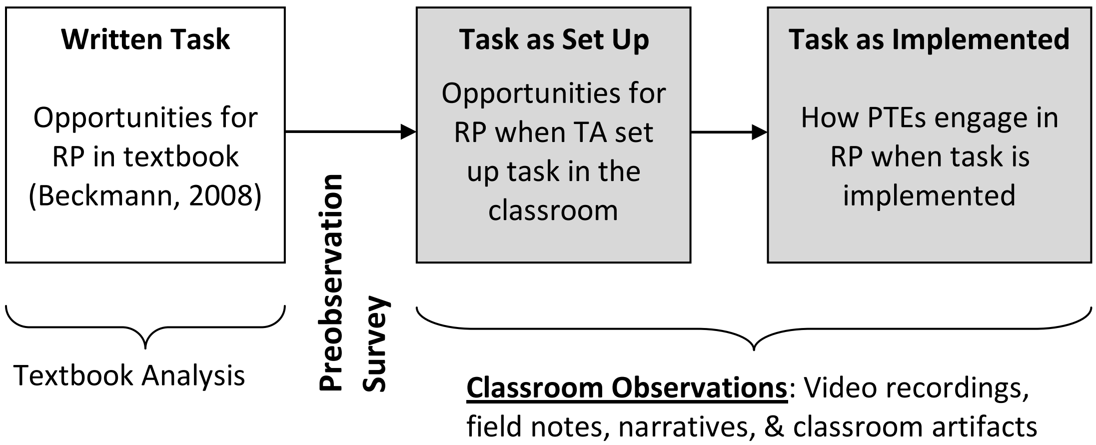


Figure 3. Data source for Research Question 2: How do TAs enact RP tasks in a mathematics content course for PTEs?

Prior to each classroom observation, TAs responded to preobservation survey questions (Appendix A.2). These questions were guided by a preobservation survey modified from Bieda (2010) to collect data about TAs' goals and expectations for the lesson and the mathematical tasks they identified as central for the lesson. This measure was designed to provide context to the observed lesson and document the extent to which RP tasks were considered by the TA to be essential components of the observed lesson.

Each classroom observation was video recorded and field notes were taken by attending to TAs' classroom teaching related to RP. The field notes were guided by an observation protocol modified from Bieda (2010) to focus on the TA and PTEs' actions and interactions around RP tasks. This protocol (Appendix A.1) documented how each task was implemented along the dimensions of the Mathematical Task Framework (Stein, Grover, & Henningsen, 1996): task features in the written text, task features as set up by the TA, cognitive demands as set up by the TA (e.g., how TAs' instructions compared with the actual problem instructions), classroom norms evident during implementation, TAs' and PTEs' instructional habits during implementation, task features as enacted, and cognitive demands of the task as enacted.

Drawing from Ethnographic research (Erickson, 1986; Goetz & LeCompte, 1981), after each observation, a narrative account was created to describe what happened during the enactment by answering the observation protocol questions and referring to the field notes and classroom video. The narratives documented the enactment of RP tasks as a first step in organizing the observational data by focusing on TAs and PTEs' interactions. As outlined in the observation protocol (Appendix A.1), aligned with the Mathematical Task Framework, specific questions focused on documenting how TAs set up the task (Questions 2, 2a, 2b, and 2c) and how they implemented RP tasks (Questions 3 – 10). The narrative attended to how TAs set up RP tasks and how PTEs engaged in RP processes during implementation.

3.2.3 Data Sources for Studying How TAs' Conceptions Illuminate Their Instructional Decisions

The primary data for investigating how TAs' conceptions of RP illuminate their instructional decisions around RP (RQ3) were a series of interviews conducted after the classroom observations (Figure 4).

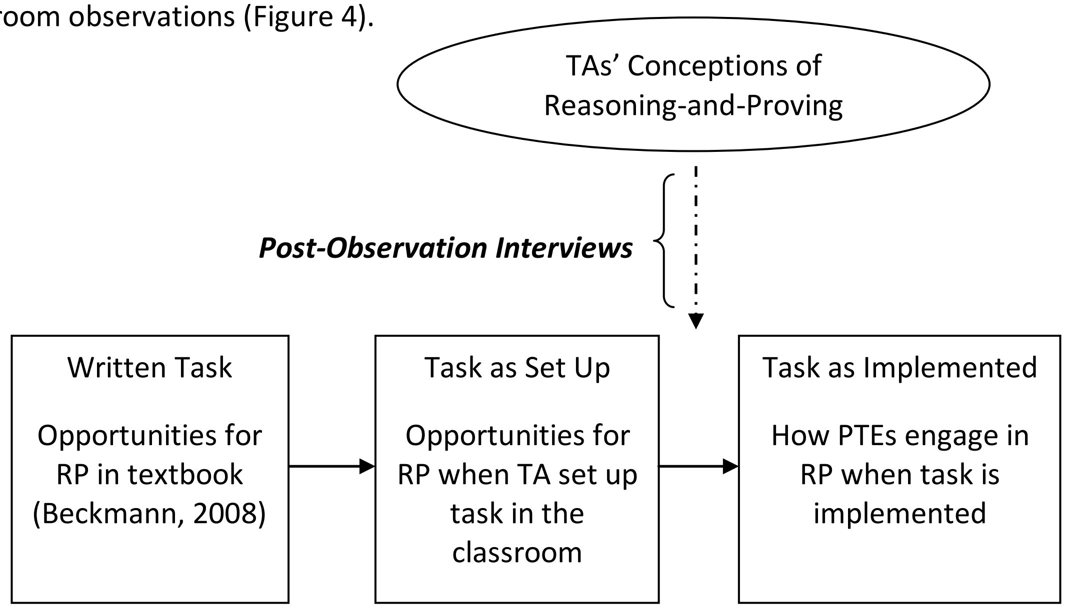


Figure 4. Data source for Research Question 3: In what ways, if at all, does understanding TAs' conceptions of RP illuminate their instructional decisions regarding the enactment of RP tasks?

The semi-structured post-observation interview protocol (Appendix A.6) was adapted from an interview used by Bieda (2010). It asked TAs to reflect on the lesson, articulating why they decided to carry out certain pedagogical moves during the implementation of RP tasks, their learning goals for choosing to implement RP tasks, and how they view RP content from the lesson as relevant for their students as future teachers.

3.3 Data Collection

Data from the classroom observations, preobservation surveys, and two series of interviews were gathered at specific points throughout the semester, with the textbook analysis being completed before data collection. In the sections that follow, I detail the process for analyzing the textbook, procedures for the preobservation survey and classroom observations, and procedure for collecting and administering the interviews.

3.3.1 Textbook Analysis

The Geometry and Measurement chapters of Beckmann's (2008) textbook (Chapters 8, 9, 10, and 11) were analyzed for RP opportunities for PTEs. These chapters addressed geometry and measurement content, including visualization, angles, two-dimensional shapes, three-dimensional shapes, constructions with a straightedge and compass, the geometry of motion and change, measurement, and understanding relationships for area and volume.

3.3.1.1 Textbook Analysis: Unit of Analysis

Within these four chapters, each separate question was analyzed using the RP-framework (Table 2). The term *question* means that each problem, written question, or "parts thereof that have a separate marker" (Stylianides, 2009, p. 270) in the textbook was coded, with the exception of homework problems. Since this study was ultimately concerned with what happened during classroom instruction, the questions listed as practice problems and questions in the activity manual were analyzed because these questions formed the basis of the instructors' classroom teaching. Using the RP-framework, questions were identified that were designed to offer PTEs at least one opportunity for RP; these questions are referred to as RP questions (RPQs).

Since a set of questions oriented toward the development of a mathematical idea constitutes a mathematical task, the questions, by themselves, were not always what one would consider to be a mathematical task (Stein & Lane, 1996). Therefore, the coding was aggregated to take this into account. The textbook contained 115 activities in the activity manual related to the four chapters addressed by MTH 2. The number of questions within each activity ranged from one to eight, but each activity was oriented toward the development of a particular mathematical idea and was regarded as a separate mathematical instructional task. Therefore, if an activity had at least one RPQ then it was regarded as a RP-related task (RPT).

3.3.1.2 Textbook Analysis: Coding Procedure

Questions were analyzed in three stages. First, questions were identified that had RP potential based on a search for the following words: *prove*, *deduce*, *conjecture*, *predict*, *justify*, *verify*, *show*, *apply* (or *use* a mathematical fact), *convince*, and *explain why* (Bieda, 2010; Stylianides, 2009). This process was not intended to identify all possible RP-related questions in the text, but questions where RP was more likely to be a primary mathematical component.

Second, potential RPQs were coded using the categories and subcategories in the RP-framework (Table 2). Definitions for each subcategory (detailed further in Stylianides, 2009) guided the coding, and a single question could be coded with multiple categories from each dimension. Dimension 1 included the activities of investigating mathematical relations, generating or evaluating conjectures, evaluating arguments, and generating proofs and nonproofs. Dimension 2 noted situations where explorations into mathematical relations may serve as precursors for a mathematical generalization, which may then serve as a precursor for a proof. In Dimension 3, instances for PTEs to reflect on the characteristics and features of proofs were identified. If a question did not fit the criteria for these dimensions, it was rejected and coded as not RP-related. For questions that provided opportunities for PTEs to engage in RP, they were typically coded in order of the dimensions. However, a question could be identified as providing opportunities for solvers to engage in Dimension 3 only, though this occurred infrequently. The aggregation of questions by task also identified instances where RP components served as precursors for other generalizations or arguments within tasks.

Since this was an analysis of tasks that can have various solution paths, the coding scheme analyzed the questions using reasonable inferences about the expected solution path for which each problem may have been designed. Factors useful in identifying expected solution paths included: (a) any approach suggested by the wording of the question in the PTEs' text, (b) the approach or solution provided in instructors' supplemental materials, and (c) what PTEs may be expected to know and understand when encountering the question based on where it is positioned in the textbook.

In the third phase of the textbook analysis, the activities that were strongly recommended or encouraged by the course supervisor were used as a reference to identify any additional RP opportunities in tasks TAs were likely to implement. These recommended activities were analyzed using the same procedure outlined in the description of phase two to identify any additional RPQs. Throughout Phases 2 and 3, two trained raters conferred to refine the descriptions and procedure for applying the RP-framework. Both raters coded a random sample of 31% of the RPQs; agreement was reached on 81% of these codes with all disagreements resolved through discussion.

3.3.2 Collecting Data about TAs' Teaching Practices

The textbook analysis identified tasks that embodied different aspects of RP; these tasks were then located in the course syllabus to identify opportunities to observe TAs enact RP tasks. The observation schedule was designed to observe each TA's classroom practices for at least six hours across the semester, and to see that TA implement a variety of RP tasks.

Table 5. Classroom Observation Details for each TA

TA	Class Time Observed	Class Periods Observed	Sections of Textbook Observed	RP Tasks Implemented
Ace	6 hours 40 minutes	5	8	13
Laura	8 hours	6	9	13
Isaac	8 hours 30 minutes	7	9	14
Peter	8 hours 20 minutes	10	8	16
Evan	6 hours	7	9	13
Kelly	6 hours	7	6	13

Due to the different time configurations of TAs' sections, the number of observations of each TA varied (Table 5) to see their classroom for a minimum amount of time and to observe them implement at least 13 RP tasks. Prior to each classroom observation, TAs responded to preobservation survey questions.

3.3.2.1 Preobservation Survey Procedure

Prior to each classroom observation, TAs received a short preobservation survey via email to elicit information about the extent to which RP was central to their goals and expectations for the lesson (Appendix A.2). TAs responded in writing or orally. TAs were asked to complete the survey before teaching the lesson in an attempt to more accurately capture their anticipations for the lesson than would be likely if they were asked these questions after teaching. Since TAs at this institution often plan right before class (Cervello & Steele, 2009), asking these questions in an interview setting before class was not feasible.

3.3.2.2 Classroom Observation Procedure

Each classroom observation was video recorded and field notes were taken to document the enactment of RPTs. TA's and PTEs' actions and interactions around RP were the focus of the observation. After each observation, a narrative account was created to describe what happened during the implementation by first detailing chronologically how the class time was used and what mathematical tasks and topics were introduced, and then answering the observation protocol questions using the field notes and classroom video as a guide.

Consistent with a learning-as-participating perspective (Sfard, 1998), the responses to the observation protocol questions about implementation of RP tasks considered how PTEs participated in the creation and evaluation of generalizations and justifications. A PTE making RP information public to the entire class or a majority of PTEs participating in RP activities was regarded as evidence of PTEs engaging in RP processes. The observation focused on how the whole class experienced RP processes and how these thought processes and ideas were shared among the class. The narrative accounts, therefore, noted what information or aspects of RP was made available to the entire class and how it was made available to them.

3.3.3 Collecting Interview Data

The procedure for conducting the two series of semi-structured interviews was the following: interviews were conducted with individual TAs, audio-recorded, transcribed, and verified. The first series of interviews inquired about TAs' conceptions of RP, and they were not associated with a specific classroom observation. The three interviews associated with this first series were named relative to when they were given: Pre-semester (during the days leading up to the start of the semester), beginning-of-the-semester (during the first three weeks of classes), and end-of-semester (during finals week or the week following finals week). The first two interviews were administered before TAs explicitly taught about proof to study their conceptions of RP before they designed and implemented RP lessons in MTH 2. The end-of-semester interview was given after the TAs' primary teaching responsibilities were concluded for the semester. The first two interviews each lasted an average of 45 minutes and the end-of-semester interview took an average of 75 minutes with each TA.

The second series of interviews, conducted after classroom observations, investigated TAs' instructional decisions related to RP. The variance in number of observations for each TA led to some variation in the number of post-observation interviews. After observing lessons where RP tasks were implemented, I conducted a post-observation interview the same day as the observation (immediately following class, when possible). After consecutive observations of the same TA, a post-observation interview was conducted following the last observation. Thus, I conducted five post-observation interviews with both Ace and Laura, and six with each of the other TAs. The interviews ranged from 22 to 64 minutes, with an average of 44 minutes.

3.4 Data Analysis

The descriptions of the data analysis mirror the structure of the previous section: The analytical approaches for understanding TAs' enactment of RP tasks, studying TAs' conceptions of RP, and then examining how those conceptions illuminate their instructional decisions.

3.4.1 Analyzing TAs' Enactment of RP Tasks

The narrative accounts written after each classroom observation were compiled and analyzed to examine TAs' enactment of RPTs. Enactment of RPTs considered the RP opportunities for PTEs during classroom instruction of the task by examining both the setup and implementation phases. RP opportunities were defined as written or verbal tasks that asked PTEs to engage in one of the RP processes from the framework. The examination of the enactment of these tasks was guided by the Mathematical Task Framework and RP-framework as the narrative accounts were analyzed for what mathematical tasks were included in the lesson and how TAs and PTEs engaged in the processes of RP during the set up and implementation. The interpretation of the patterns observed in TAs' classroom teaching was also informed by considering their preobservation responses.

The analysis of the classroom narratives was modeled after the IQA protocol (Matsumura et al., 2006), which has established reliability and validity for examining the set up and implementation of tasks. For each observed lesson, the segments of the lesson that were considered as the setup phase and the implementation phase for each task were identified. Then, RP processes for the *task-as-written*, *task-as-set-up*, and *task-as-implemented* were coded and analyzed. Finally, the enactment of each task was determined by taking into account the analysis of the setup and implementation phases. In the four subsections that follow, analyses of the task-as-written, task-as-set-up, and task-as-implemented are first described, and then the procedure for analyzing the enactment of the task (to address the second research question) is detailed.

3.4.1.1 Analysis of the Task-as-Written

The task-as-written consists of the words, instructions, problems, and diagrams provided in the textbook or generated by the TA. When the task was taken from the textbook, the opportunities for PTEs to engage in RP identified in the textbook analysis served as the written potential for RP for the task. When the task was generated by the TA, the analysis of the task occurred after the observation but used the same analytical approach as described for the textbook analysis to determine the RP potential for the task.

3.4.1.2 Analysis of the Task-as-Set-up

The task-as-set-up, describes how the TA introduced the task. The setup phase ended when PTEs put their pencils to paper to begin engaging with the questions in the task. This analysis (see Appendix B.4) focused on what TAs said and did when introducing a task and telling PTEs to begin working on a particular task; it also noted any background information or examples the teacher may have provided to introduce the content to determine whether the RP opportunities for a task were modified during the setup. Modifications in the RP opportunities of a task could occur during setup when a TA increased or decreased the types of opportunities for PTEs to work on RP. An increase in RP opportunities occurred when students were expected to engage in an RP category not included in task-as-written. For example, a TA might explicitly ask PTEs to provide rationales for their answers even though this request was not included in the written task. A decrease in RP opportunities occurred when RP opportunities in the task-as-written were omitted during the setup. Consider, for instance, a task with four questions with only the first two relating to RP. A TA might ask PTEs to work on the last two questions, thus removing all RP-potential opportunities during the task setup.

3.4.1.3 Analysis of the Implementation of the Task

The implementation of the task considered how TAs engaged in RP in their work on the task. As PTEs engaged with the task, they could work individually, in small groups, as a whole class, or some combination of these participatory structures. Following the IQA structure, PTEs' engagement in RP both during individual or group-work time and during whole-class discussions was analyzed separately, if both participatory structures occurred during the enactment of a task. The overall analysis of the implementation of the RP task then aggregated these two analyses to note what RP processes PTEs generated or evaluated across the participatory structures. The coding schemes for the analysis of the work time and the whole-class discussions are included in Appendix B (see Appendix B.5 and B.6). To study how PTEs engaged in RP during individual- or group-work times, how most PTEs worked on RP for most of the work time was analyzed. Also, how TAs talked with and guided PTEs in their work on RP in the task was examined to understand how TAs modified or maintained RP demands of tasks. To study PTEs' engagement in RP during whole-class discussions (which could include teacher-led or more discussion-based whole-class participation), how RP processes were discussed and highlighted by the TA and PTEs was analyzed. This analysis considered how RP processes were made public for the whole class and who provided the generalizations or justifications (the TA or PTEs) to indicate how PTEs did or did not engage in the RP processes. The analysis of the implementation of the task indicated how PTEs did or did not participate in generating or evaluating mathematical claims or arguments during the class.

3.4.1.4 Analysis of the Enactment of the Task

The class activity around the task (i.e., the setup and implementation phases) was regarded as the enactment of the task. Therefore, for each task the setup and implementation analyses were considered to see whether or not PTEs engaged in the RP processes from the task-as-written. This analysis of the enactment of the task could result in (a) an *increase*, (b) a *decrease*, (c) a *shift*, or (d) *no change* in RP demands.

First, an increase in RP demands meant that PTEs generated or evaluated an RP process not included in the RP potential of written tasks, indicated in one of two ways: (a) during the set up there was an increase in RP opportunities and PTEs were held accountable for producing that additional RP-category during the implementation; or (b) there was no modification to the RP opportunities during setup, but during implementation PTEs produced or evaluated a generalization or argument that went beyond the RP opportunities of the task-as-written.

Second, a decrease in RP demands referred to instances where PTEs did not participate in one of the six major RP processes elicited in the task-as-written. A decrease in RP demands occurred as a result of the following two scenarios: One, PTEs had fewer opportunities to engage in RP than those identified in the task-as-written. In such a situation, during set up or implementation, RP processes were excluded by a TA or by PTEs. Two, how PTEs participated in the RP processes differed from the task-as-written because the TA generated RP processes elicited in the task. In these situations, when introducing the task or during implementation the TA generated one or more aspects of RP processes called for in the task instead of PTEs.

Third, a shift in RP demands details situations that were not an increase or decrease. This shift in RP demands referred to instances where during set up or implementation PTEs were asked to or actually did generate generalizations or arguments that were different from the task-as-written, but the overall RP processes that PTEs engaged in were not different from the task-as-written. This shift was noted in two types of scenarios: One, during the set up, an increase or decrease in RP opportunities occurred but during implementation PTEs ended up engaging in the RP processes expected in the written task. Therefore, the task was shifted in regards to what PTEs were expected to do related to RP, but during class activity or discussion PTEs produced the generalizations or arguments elicited in the task-as-written. Two, during implementation PTEs engage in the same major RP processes identified in the written task but they did so in a manner that was qualitatively different from what was expected in the written task. That is, the types of RP processes PTEs engaged in was not increased or decreased, but the mathematical content or level to which PTEs engaged in RP processes differed from the task-as-written. For example, consider a task with two questions about properties of prisms. In the first question, PTEs were expected to generate a conjecture about the shape of the faces of prisms and provide a rationale for their conjecture. In the second question, they were asked to generate a conjecture about the relationship between number of vertices, edges, and faces of any n-sided prism. A shift of RP demands would occur if, during enactment, PTEs did not provide a rationale for their conjecture in Question 1 but they provided a rationale for their conjecture in Question 2. In this example, the RP demands were shifted but not in a way that increased or decreased the aspects of RP in which PTEs engaged.

In the fourth case, no change in RP demands meant that there was no modifications in the RP opportunities for the task and PTEs engaged in the RP processes that were elicited by the task. Therefore, when none of the changes that were described in the first three cases occurred, PTEs participated in the activities of RP that the written task called for and the enactment resulted in no change in RP demands.

3.4.2 Analyzing TAs' Conceptions of RP and How their Conceptions Illuminate their Instructional Decisions

To address RQ1 and RQ3, interviews were transcribed and analyzed. To study the measures for answering RQ1, focusing on TAs' conceptions of RP in mathematics, teaching, and learning, I analyzed the first series of interviews (pre-semester, beginning-of-the-semester, and post-semester). To analyze the data for answering RQ3, considering how TAs' conceptions illuminated their instructional decisions, I examined each TA's responses to the second series of interviews (post-observation interviews). In light of the exploratory nature of the study, analysis of interview transcripts followed a constant comparative method (Glaser & Strauss, 1967). This inductive analytical approach was used as the study sought to promote understanding of TAs' conceptions, not prove a preconceived theory. The analytical approach is detailed further for the first series of interviews, then for the analysis of the post-observation interviews.

3.4.2.1 Analysis of TAs' Conceptions of RP in Mathematics, Teaching, and Learning

The purpose of this analysis was to understand TAs' conceptions of the purposes of proof, of methods for teaching about RP, of why it is important to teach about RP, and of what PTEs should learn about RP in MTH 2. The analysis of the first series of interviews began by first identifying utterances where TAs talked about their views of RP within mathematics, teaching, or learning (Appendix C). The goal in this first coding level was to note places where TAs talked about RP in these general domains to then investigate how TAs talked about the nature and purposes of RP within the domains. For example, Evan explained proof's role in mathematics:

> To establish new results and grow the field. . . . The field's constantly growing because of new results that are proven and new objects that can be formed with the results that we have. And then things can be proven about those. . . . Role of proof in mathematics so just establish new results and to build onto the field. (pre-semester, para. 72)

This utterance was coded as "mathematics" in the first level of analysis, but TAs did not always explicitly distinguish between the three general domains. As previous studies of TAs' beliefs note (Speer, 2001), TAs made some statements that appeared to reflect their views of teaching and learning. The intension was to identify these broad areas and faithfully capture TAs' conceptions, so if a TA talked about teaching and learning together it was coded both as "teaching" and "learning."

Within these broad categories (as detailed further in Appendix C) the three transcripts for each TA were coded for their views of (a) the purposes of proof in mathematics, (b) the purposes of proof in mathematics classrooms, (c) the educational goal (that is why it is important) for teaching about RP, (d) the learning goal (that is, what do people need to know) about RP, and (e) how to teach about RP. The initial categories for items (a) and (b) were informed by previous literature about the purposes of proof in mathematics and teaching (Knuth, 2002a, 2002b; Steele & Rogers, 2012), introduced in Chapter 2 (Table 1). The other aspects of the coding scheme, however, emerged from the data. That is, after initially coding for these broad-level categories, a constant comparative method (Glaser & Strauss, 1967) was used as follows: Within statements about RP in mathematics, how TAs talked about the role of proof in mathematics was coded. The quote above where Evan described a role of proof in mathematics received a sub-code of "discovery or creating new mathematical knowledge."

Similarly, statements about RP in teaching were coded for how TAs talked about the role of proof in teaching, why it was important to teach about RP (educational goal), and methods for teaching about RP. Statements about learning were coded for how TAs talked about their learning goals for RP (what they thought learners needed to know about RP). Within each of these categories, utterances where TAs specified who the doer, teacher, or learner of RP was to which they were referring were coded, as well. Noting the *who* was helpful because TAs were teaching current undergraduate students who were also PTEs. This analysis noted to what extent TAs attended to their students as undergraduate learners or future teachers.

Through an iterative process, the codes and subcodes were applied and refined. Through this process, when transcript about the nature and purpose of RP could not be categorized with existing codes and sub-codes, previously analyzed data was re-examined and sub-codes were modified to reflect the data more accurately. There were a category of responses that did not fit the original six codes, and as a result, a seventh category was added. This category was a code about the *general* purpose of proof where TAs talked about proof's foundational role in mathematics or in mathematics teaching; it served as an important feature of mathematics that PTEs need to understand. Evan, for example, answered questions about the purposes of proof in mathematics by explaining that:

> Proof is doing mathematics. I mean . . . you can't have mathematics without proof as a discipline. I mean that would be like you're gonna go to a culinary arts school and get a cooking degree and never use food. It's an integral part of mathematics. (pre-semester, para. 70)

All the transcripts were coded using the seven sub-categories listed in the coding scheme in Appendix C. The other sub-codes within the teaching and learning categories emerged from the cyclic analysis of the data described. The analysis of this first series of interviews identified prevalent themes (represented in frequencies of the coded categories) detailing TAs' conceptions of the RP in mathematics, teaching, and learning to describe how each TA emphasized particular roles of or aspects of proof.

3.4.2.2 Analysis of How TAs' Conceptions Illuminate their Instructional Decisions

The purpose of this analysis was to examine the relationship between TAs' instructional decisions around RP and their conceptions about RP. Using the constant comparative method (Glaser & Strauss, 1967), TAs' responses to post-observation interviews were analyzed for how they talked about their pedagogical choices after implementing RPTs in their classroom. This analysis focused on TAs' instructional decisions that directly impact PTEs' opportunities to learn about RP: (a) TAs' selection of RPTs to teach and (b) TAs' in-the-moment decisions around RP during classroom instruction.

TAs' responses about their goals for selecting RPTs for the lesson and their enactment of RPTs were coded for themes about the extent to which and how RP was emphasized when talking about their instructional decisions and goals during the enactment of the task. The first level of the analysis identified where in the post-observation interviews TAs talked about their selection of RPTs or about their in-the-moment decisions from the class. Within these sections of the transcripts, the constant comparative method was used to investigate how TAs talked about learning goals and contextual factors that contributed to their instructional decisions. Some of the major themes that emerged through this cyclic, iterative coding process were (a) RP content-related learning goals, (b) RP process-related learning goals, and (c) contextual factors that informed their instructional decisions. The iterative process of coding, refining codes, and re-coding was followed through this analysis to reveal patterns in how TAs' conceptions about RP were related to their instructional decisions.

CHAPTER 4

TEACHING ASSISTANTS' CONCEPTIONS OF REASONING-AND-PROVING

This chapter focuses on TAs' conceptions of RP, important ultimately for informing professional development strategies for supporting mathematics instructors of future teachers. Using the constant comparative method (Glaser & Strauss, 1967), TAs' interview responses were analyzed to determine their conceptions of RP in the contexts of mathematics, teaching, and learning (RQ1).

There are two purposes for analyzing TAs' conceptions of RP: First, although these courses are a primary means through which PTEs' knowledge for teaching mathematics is developed (AMS & MAA, 2012), very little is known about the knowledge and beliefs of the instructors assigned to these courses (Speer et al., 2010). This analysis describes the range of conceptions these six TAs have about teaching and learning RP, thus addressing the research need and first research question. Second, given the major role of teachers' knowledge and beliefs in influencing teachers' instructional decisions (Schoenfeld, 2010), this analysis provides an important starting point for understanding TAs' instructional decisions around RP (RQ3); explored further in Chapter 7. To present these results, this chapter first describes TAs' views of RP from the analysis of their responses to the first series of interviews. Then, prominent aspects of individual TA's conceptions of teaching RP are presented.

4.1 Analysis of All the TAs' Conceptions of Reasoning-and-Proving

Based on the influence teachers' conceptions have on classroom instruction (Putnam & Borko, 1997), TAs' success in providing opportunities for PTEs to understand RP during MTH 2 relies in large part on the nature of the TAs' own conceptions of RP. To understand TAs' conceptions of RP in mathematics, teaching, and learning (RQ1), I examined three aspects that have particular implications in the context of teaching PTEs about RP: TAs' conceptions of (a) the purposes of proof both in mathematics and in teaching, (b) why and what to teach about RP in MTH 2, and (c) how to teach about RP.

4.1.1 TAs' Conceptions of the Purposes of Proof in Mathematics and in Mathematics Teaching

I studied TAs' conceptions of the purposes of proof in both mathematics and teaching to understand to what extent TAs have conceptions of proof that incorporate a variety of roles of proof. Such conceptions are important so TAs can help PTEs develop a strong understanding of multiple purposes of proof; thereby, preparing PTEs to bridge the gap between these roles of proof and their manifestation in K-12 mathematics classrooms (Balacheff, 1991).

To understand the extent to which TAs articulate a variety of purposes of proof, I analyzed their interview responses (from pre-semester, beginning-of-semester, and end-of-semester) for how they talked about purposes of proof. Seven codes were applied (Table 6), one of which was a general purpose emerging from the data and the others were based on previous literature detailing purposes of proof (Knuth, 2002a, 2002b; Steele & Rogers, 2012). For the coding process I attributed a code to phrases, sentences, paragraphs, or sections of the transcript and TAs' statements were given no code, one code, or multiple codes. Statements coded as evidence of purposes of proof were aggregated to portray the degree to which each TA attended to different purposes of proof. This method assessed how TAs talked about different purposes of proof in mathematics and mathematics teaching. TAs described proof as *verification, falsification, explanation, communication, discovery, systemization,* and in *general* fundamental to mathematics (Table 6).

Table 6. TAs' Statements about Purposes of Proof in Mathematics and in Mathematics Teaching

Description	Response about Mathematics / Response about Teaching
Verification: confirming the truth of mathematical statements	"If we can prove that something's true in a general case, we don't have to go through that every single time." (Laura, pre-semester, para. 54)
	"I want [the students] to take away . . . that proof's a way to be convinced about something mathematically that is valid." (Evan, beginning-of-semester, para. 134)
Falsification: establishing when a claim is not true	Proof is used "to prove [a claim] is not true or prove that it is false." (Evan, pre-semester, para. 56)
	"If they are given a true or false question that they can be like, 'No, this really can't happen.'" (Laura, end-of-semester, para. 276)
Explanation: showing *why* a claim is true (different from verifying its truth)	"Proof can explain something—Why something is true, NOT just a confirmation." (Kelly, pre-semester, para. 212)
	"The students get a feeling for what math is about and it's not just about guessing or assuming things; they should provide arguments for WHY it's true." (Isaac, beginning-of-semester, para. 106)
Communication: sharing mathematical ideas and knowledge with others	"If I have an idea of mathematics and I want to show this to other people, to do that . . . we need proof. . . . We need a proof to communicate to each other." (Peter, pre-semester, para. 179)
	"As future teachers . . . they should know the role of proof too . . . [to] explain some facts to other people without any logical jumps or gaps." (Peter, beginning-of-semester, para. 186)
Discovery: developing new mathematical ideas (no sense of communicating the ideas needed)	Proof "is the process of building new knowledge." (Kelly, end-of-semester, para. 491)
	Students should "learn that you can take things you're given or things you've already proven and establish new results." (Evan, end-of-semester, para. 136)
Systematization: imposing a logical structure (e.g., utilizing and classifying axioms or theorems) on the mathematical domain	Think of it "like a tree; each time you are adding new knowledge, how that knowledge be (*sic*) qualified to adding (*sic*) to the original tree is proof. [Proof] qualifies that it goes into some branch [of the tree]." (Kelly, pre-semester, para. 226)
	"I still feel [it] is important for the teacher: How they are going organize the knowledge for their students." (Kelly, end-of-semester, para. 513)
General: reinforcing the nature of proof as central in mathematics or learning	Proof "is a fundamental part of mathematics and . . . in a lot of ways it's synonymous with what math is." (Evan, pre-semester, para. 48)
	Students "need to know the idea that proofs are important." (Ace, end-of-semester, para. 64)

These examples provide a means of understanding how TAs generally talked about purposes of proof in mathematics and in teaching. In both contexts, they talked about the same types of characteristics for each purpose, but focused on general users of mathematics in the mathematics discipline or on students in a mathematics classroom setting.

Regarding these seven purposes of proof, mathematics education researchers conceptualize “an *informed conception of proof*—one that reflects the essence of proving in mathematical practice—must include a consideration of proof in each of these roles” (Knuth, 2002a, p. 381, italics added). Like Knuth, I considered a robust or informed conception of proof to be evidenced by TAs articulating multiple purposes of proof and I studied the TAs’ ability to articulate informed conceptions of proof. The extent to which each TA talked about the purposes of proof in mathematics is displayed in Table 7, with the purposes ordered from greatest to least frequency.

Table 7. Frequency of Purposes of Proof in Mathematics Articulated by TAs

TA (total coded statements)	**Verify**[a]	**Falsify**	**Discover**	**Communicate**	**Systematize**	**Explain Why**	**General**
Ace (12)	66.7	25.0	8.3				
Laura (5)	60.0	20.0	20.0				
Isaac (8)	12.5	12.5		50.0	25.0		
Peter (11)	9.1	9.1	9.1	63.6		9.1	
Evan (22)	18.2	18.2	22.7	13.6	9.1	9.1	9.1
Kelly (16)	12.5	6.2	25.0		31.3	18.8	6.2

[a] Values are percentages out of the total number of coded statements for each TA

The TAs all talked about more than one purpose of proof in mathematics, with some noticeable differences in their salient views of purposes of proof. The mathematics education graduate students, Kelly and Evan, articulated the most robust conceptions of proof, with Kelly describing six and Evan describing seven of the seven roles of proof. They also did not emphasize, in term of a majority of statements, only one role, and talked about the importance of these various roles in mathematics. Their informed conceptions of proof incorporated all (or almost all) the purposes of proof, suggesting these two TAs seem more equipped to help PTEs learn how to talk about and understand each of these roles in mathematics.

The mathematics graduate students, Ace, Laura, Isaac, and Peter, talked about fewer roles of proof and a majority of their statements were about one of two purposes: verification or communication. By privileging these purposes of proof in mathematics, these TAs could be more likely to reinforce PTEs' narrow conceptions of proof and less likely to articulate and help PTEs to explain connections between all the roles of proof in mathematics. Among these TAs, however, Peter was unique in how he explicitly connected a salient purpose to others. Peter explicitly described communication in a way that encompassed other roles: "Communication includes everything. Verification of truth, why do you need this? To communicate. Falsification, why do you need this? To communicate. Explanation? To communicate. Discovery . . . why? Need to communicate" (Peter, end-of-semester, para. 587)[2]. His conception of proof in mathematics included other purposes of proof that were couched under the primary purpose of communicating to other doers and users of mathematics. These results do not necessarily mean that the mathematics TAs do not know or understand all seven roles of proof, rather, certain roles of proof, or relationships among the purposes of proof, may remain more implicit than others in their coursework and research.

[2] Interview citations include TA's name, interview name, and transcript paragraph number. A new paragraph started when there was a change between interviewer and TA talk.

Comparing these views about proof in mathematics with the salient views when talking about teaching, the TAs emphasized different roles and articulated more robust conceptions of purposes of proof in mathematics teaching. This contrast suggests that TAs conceptualize purposes of proof in the two contexts differently, supporting the notion that teaching about proof necessitates different knowledge on the part of the teacher than it takes to use and do proof in the mathematics discipline. These results are displayed in Table 8, with the purposes of proof ordered from greatest to least frequency.

Table 8. Frequency of Purposes of Proof in Mathematics Teaching Articulated by TAs

TA (total coded statements)	**Explain Why**[a]	**Communicate**	**General**	**Verify**	**Systematize**	**Discover**	**Falsify**
Ace (13)	38.4	30.8	30.8				
Laura (15)	46.6	26.7			6.7	6.7	13.3
Isaac (11)	27.2	9.1	36.4	18.2	9.1		
Peter (11)	45.4	36.4	9.1	9.1			
Evan (18)	50.0		11.1	22.2	5.6	5.6	5.6
Kelly (20)	15.0	25.0	10.0	10.0	20.0	20.0	

[a] Values are percentages out of the total number of coded statements for each TA

TAs' responses about purposes of proof in teaching differed from how they talked about proof in mathematics; they included at least one of the three roles, explaining why, communicating, and the general, fundamental nature of proof, contrasting with the salient purposes of proof talked about in mathematics (verifying, falsifying, and discovering). The three roles emphasized in the context of teaching are particularly relevant to the work of preparing PTEs to explain to others about the fundamental nature of proof. The contrasts noted between mathematics education TAs' conceptions of proof and the mathematics TAs' conceptions were also less apparent in these statements about teaching. Although Evan and Kelly continued to articulate the most robust conceptions of purposes of proof, most of the other TAs talked about more purposes in this context than when considering the mathematics discipline. Also, the way TAs talked about the purposes tended to privilege two or three purposes more often in this context. The way TAs talked more frequently about more purposes of proof in teaching could suggest a greater likelihood that they will talk about more than one purpose of proof in the classroom. TAs' responses about these purposes of proof in mathematics teaching suggest that their conceptions of proof include multiple purposes that are particularly important in the context of teaching MTH 2.

4.1.2 TAs' Conceptions of Why and What PTEs Should Learn about Reasoning-and-Proving

I also studied TAs' conceptions of why and what to teach about RP in MTH 2 to understanding how their goals aligned with the RP goals of the course. The course goals and textbook aim to develop PTEs' understanding of both RP content and processes so they are successful in MTH 2 and able to teach others about RP. TAs with conceptions that strongly align with these course goals could be more likely to develop PTEs' knowledge of and ability to articulate RP. To investigate TAs' goals for teaching RP in MTH 2, I examined TAs' talk about why it was important to teach RP and what they wanted PTEs to learn about RP.

To prepare PTEs to understand mathematics in MTH 2 and how their work in MTH 2 relates to their roles as elementary teachers, the course should ideally include learning goals that attend to both contexts (horizontal axis, Figure 5). To explain why it was important to teach RP in MTH 2, however, TAs did not all express such a balanced perspective, with two TAs attending to mathematical learning goals applicable to the MTH 2 context more so than how those goals related to PTEs roles as future elementary teachers. The mathematical aspects that

TAs commonly identified were either *RP content* or *RP processes* (vertical axis, Figure 5). Since TAs talked about RP content or processes goals with respect to the undergrad mathematics or elementary classroom perspectives, exemplars from TAs along these dimensions are displayed in Figure 5.

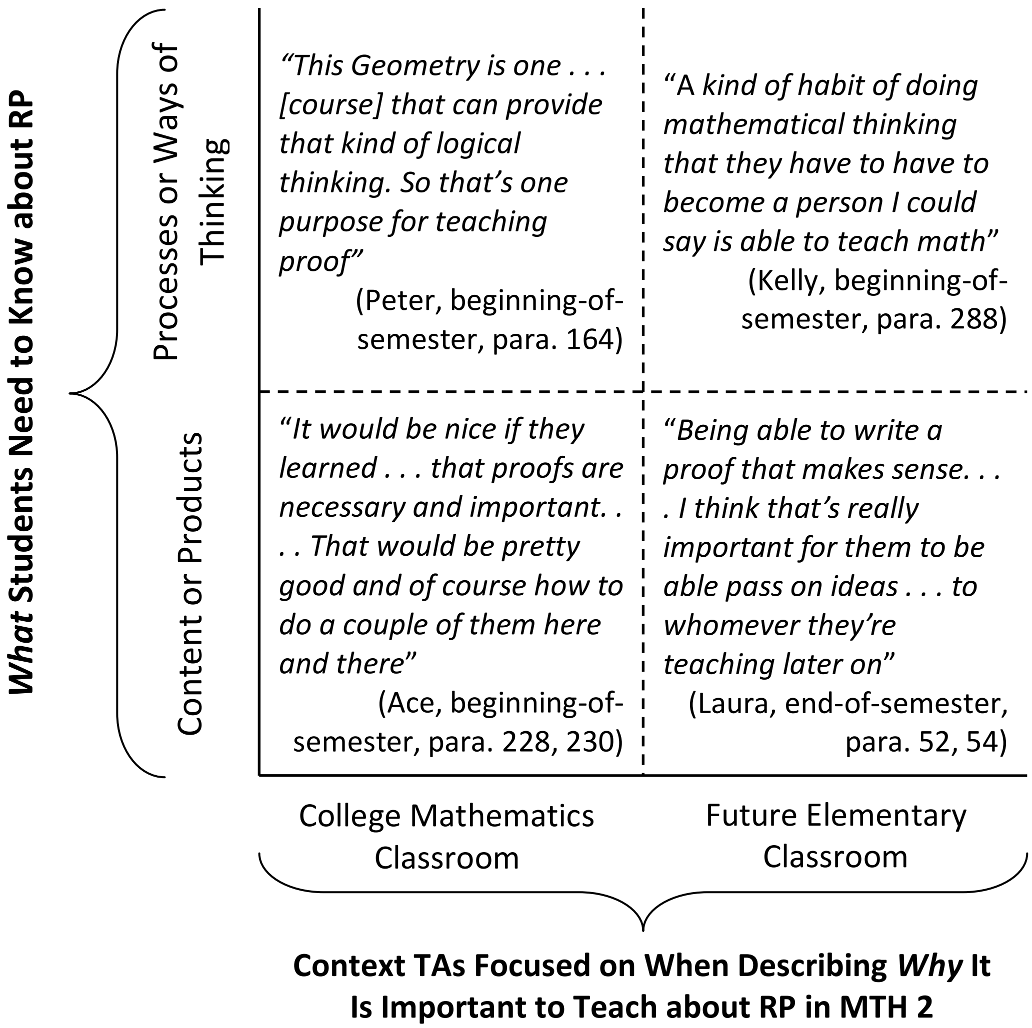


Figure 5. Examples of how TAs talked about what and why they wanted students to learn about reasoning-and-proving

All of the TAs made some statements about end products or content related to RP they wanted MTH 2 students to understand. The quotes from Laura and Ace (Figure 5) depict some of the common ways RP content was emphasized, including: knowing what a proof is, producing proofs or concise explanations, supporting each statement with a reason, using logical arguments in writing proofs, and knowing a proof exists for mathematical statements that could be taught in elementary grades. Most of the TAs also talked about processes of proving they aimed to teach PTEs. Kelly and Peter, for example, talked about developing PTEs' ability to think mathematically or logically, a common RP process TAs described. TAs also talked about providing opportunities for PTEs to "make and investigate mathematical conjectures" (Laura, end-of-semester, para. 238) and "evaluate the arguments that [their] students [could] make" (Evan, end-of-semester, para 51).

To prepare PTEs to teach others about both RP content and processes, it is important for TAs to move back and forth between RP content and RP process goals, as well as to address both the MTH 2 context and future elementary classroom setting in planning and teaching MTH 2. The patterns in TAs' conceptions along these dimensions are depicted in Figure 6. The frequency of statements along the four dimensions guided the placement of the circle in the figure. The circles were placed to represent each TA's tendency to focus on particular views of teaching and learning RP along these dimensions. The intersection of the two axes represents equally talking about all four dimensions, which would suggest conceptions aligned with the design and goals of MTH 2.

To understand how data were represented in Figure 6, consider Ace as an example. Ace's circle only touches the line between the two contexts along the horizontal dimension. This placement indicates how Ace rarely talked about why teaching RP was important for PTEs to be prepared to teach others and predominantly talked about what he wanted them to learn and be able to understand in MTH 2. Along the vertical dimension, he only articulated RP content and did not talk about RP processes.

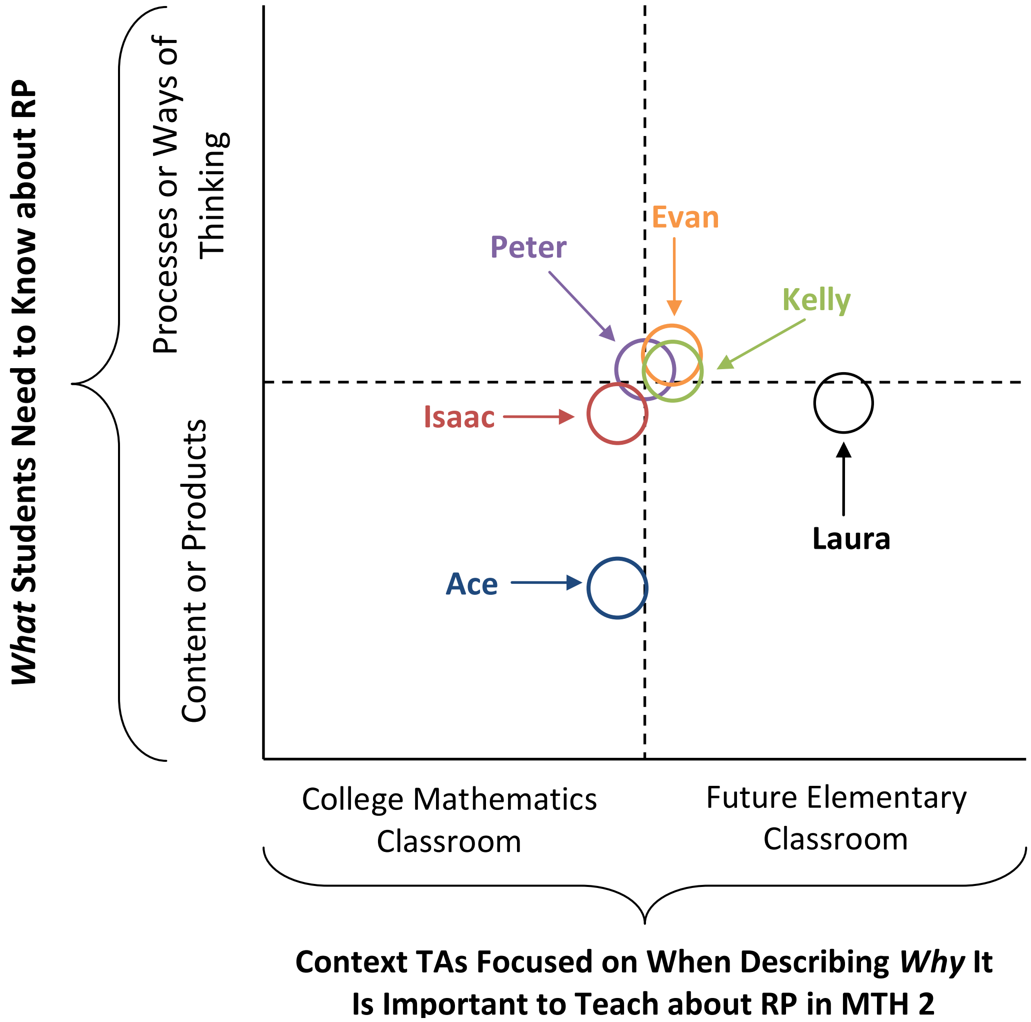


Figure 6. TAs' conceptions of teaching and learning reasoning-and-proving in MTH 2

The patterns in TAs' responses suggest that they possess a range of conceptions about why and what they are teaching about RP. This range of conceptions supports the notion that teaching about RP to college students who are prospective teachings represents a complicated context: the TAs are teaching PTEs who need to develop rich understanding of RP to be prepared to teach others about both RP content and RP processes. To navigate that context, Peter, Kelly, and Evan articulated conceptions of teaching and learning RP that seemed to align with the course goals and design. They focused more on a balance of teaching RP content and processes and attended both to what they wanted PTEs to understand in the college mathematics classroom and to prepare them for their future teaching. TAs with these conceptions could be more likely to engage PTEs in a range of RP processes during classroom instruction than TAs without such views. The other TAs talked infrequently (Laura and Isaac) or never (Ace) about RP processes; they focused instead on developing PTEs' understanding of mathematical content. Since these conceptions of teaching RP do not align with the textbook and course goals, they may be more likely to provide opportunities for PTEs to learn RP content more than processes.

4.1.3 TAs' Conceptions of How to Teach about Reasoning-and-Proving

I also studied TAs' conceptions of methods for teaching RP to understand how their views aligned with the structure of the course. The textbook, course structure, and course supervisor encourage a student-centered pedagogical approach consistent with how PTEs are increasingly expected to engage future elementary students in RP (CCSSI, 2010; NCTM, 2000). Alignment between TAs' conceptions and this course design could lead to more authentic RP learning experiences for PTEs.

Based on the stance embodied by the textbook and instructional supports, one might anticipate a student-centered philosophy to be most resonant for TAs. Their experiences in K-16 mathematics classrooms, as learners and instructors, however, are not likely to reflect those views of teaching. In fact, teaching MTH 2 represented the first experience for all the TAs to teach mathematics using more student-centered techniques. Isaac explained, for instance:

> "I mean everything I taught before this course . . . [has] been a teacher standing at the blackboard writing and talking for 50 minutes. And now I can see much more how you

> can interact with the students and let the students work instead of [students] just listening or not listening. . . . I definitely think this is a better way of teaching." (end-of-semester, para. 52)

All the TAs had no experience teaching using non-lecture-based pedagogical approaches or learning mathematics in these ways, but five of the six TAs talked positively about using strategies to develop PTEs' conceptual understanding of RP by engaging PTEs in tasks and discussions in class (Figure 7); strategies encouraged by the text and course supervisor. TAs also described these strategies as important for helping PTEs understand MTH 2 material better and/or learn mathematics in ways they will likely be expected to teach it. These results suggest that all the TAs, except Ace, articulated perspectives that value the pedagogy called for in the design and structure of the course, talking about it in a way that suggests they aimed to teach MTH 2 using these strategies to engage PTEs in RP tasks and develop PTEs' understanding of RP.

The pedagogical strategies these five TAs positively talked about that seemed to align with the activity-based structure of the course, included: TAs working as discussion facilitators, Socratic tutors, and monitors of students' progress; and students working on tasks, working in small groups, articulating mathematical ways of thinking, and reasoning through mathematics themselves. Examples of statements about two of these areas are provided to depict how TAs attended to the importance of PTEs engaging in RP themselves (or with classmates) to develop a conceptual understanding of the mathematics.

Pedagogical Approaches	TAs					
	Ace	**Laura**	**Isaac**	**Peter**	**Evan**	**Kelly**
Work TAs do:						
Facilitate Class Discussions		•	+	•	+	+
Socratic Tutoring				+		
Monitor Individual/Small Group Progress		+	+	•	+	+
Provide Notes or Example Problems	+					•
Lecture	+	•	•	•	-	•
Work PTEs do:						
Work on Interactive Mathematical Tasks	-	+	+	+	+	+
Work in Small Groups	-	+	+	+	+	+
Articulate Mathematical Ideas & Ways of Thinking		•	•	+	+	+
Reason through Mathematics Themselves				+	+	
Work on Practice Problems	+	•				+

Symbols indicate that TAs:
+ Frequently mentioned a pedagogical approach positively
• Mentioned a pedagogical approach positively
- Frequently mentioned a pedagogical approach negatively

Figure 7. How TAs talked about approaches for teaching reasoning-and-proving

Evan, for instance, explained the importance of facilitating class discussions:

> By teaching the class the way I do, leading discussions [and] having students come up to the board, these students are participating in a mathematical discussion in class, which can get them out of the traditional mindset of a mathematics classroom. So they could possibly teach in this way in the future. . . . I'm trying to get out of the mode of just standing at the board and writing things on it for 50 minutes and collecting homework once a week after I've written stuff on the board for two and a half hours over the course of the week. I'm trying to get people more involved in thinking and sharing their thoughts. (end-of-semester, para. 74)

Evan described facilitating discussions instead of lecturing to get PTEs "involved in thinking and sharing their thoughts;" a teaching approach like that could support the development of PTEs' ability to articulate RP, as well as the ability to hear and critique others' justifications.

Peter also talked about PTEs learning how to reason mathematically and share ideas during class by leading what he called *question-and-answer* dialogues. Peter described these dialogues as involving a whole class discussion he lead by asking questions based on PTEs' answers to help them think critically and construct arguments. This approach was called *Socratic tutoring* because Peter related it to books he read as an undergraduate mathematics education student: "for example, like, Socrates—the question-answer stuff is from Socrates" (end-of-semester, para. 312). From Peter's perspective, he intended to provide opportunities for PTEs to reason and think mathematically as he questioned their answers and asked them to think about mathematical ideas from multiple perspectives. He explained that to learn to reason and think mathematically PTEs "have to experience that question-answer [dialogue]. . . . You cannot learn that from your teacher. You have to experience that. Your teacher can encourage that or make them to experience more" (end-of-semester, para. 316-318). The Socratic tutoring method was a particular approach Peter endorsed for thinking through mathematical ideas as a class that is similar to the way Evan talked about mathematical discussions because PTEs have opportunities to articulate, hear, and critique mathematical justifications and reasoning. Although Peter did not make an explicit connection to PTEs' roles as future teachers in his statements about Socratic tutoring (as Evan did about facilitating discussions) the pedagogical approach and nature of PTEs' involvement during that strategy still aligns with the goals and structure of the course.

The pedagogical approaches about which Ace talked positively contrasted with the rest of the TAs, where he focused on providing notes, lecturing, and asking PTEs to do practice problems; perspectives that tend to be more teacher-centered and encourage passivity on the part of the students, which is not aligned with the goals of this activity-based course.

Comparing how Ace talked about providing notes for PTEs with the way it was mentioned by Kelly highlights how Ace's conceptions of teaching RP in MTH 2 differed from the other TAs and from the course structure. When talking about providing notes for PTEs, Ace explained why he often elected to provide proofs as worked out examples, saying that these proofs are "really just something that [students] hold [their] breath through to get the tools [they] need to complete the problems at hand" (Ace, beginning-of-semester, para. 204). This perspective focuses on proof as a product to be memorized or used and does not seem to value PTEs' participation in proving processes. Kelly, on the contrary, explained why she provided notes by writing some main ideas on the board after a verbal discussion of a proof, saying that the notes could help

> some students who lag behind or don't get what I'm talking about. So leave them a good notes [*sic*] that they can refer to after class may be a help for them. . . . I would rather write more just to make sure that they get the answer but I don't like to write like a sentence-by-sentence. That's kind of stupid. I just write more mathematical expression and they can reconstruct their sentences from those expressions but I try to keep the big important steps on the board. . . . So they can use that to recall . . . the class. (end-of-semester, para. 281)

Kelly used notes to provide some main ideas that she expected PTEs to use to reconstruct the class discussions, suggesting that (unlike Ace) Kelly endeavored to encourage PTEs' active participation in their learning. These examples also highlight how a pedagogical strategy is not inherently student-centered or teacher-centered. All strategies TAs discussed, from facilitating class discussions to providing PTEs with notes, may be conceptualized and implemented in ways that encourage or discourage student engagement in the learning process.

In MTH 2, the textbook endorsed student-centered instructional strategies where PTEs develop their own understanding of and explanations for mathematical ideas by engaging in mathematical tasks. Most TAs talked about teaching MTH 2 in ways that aligned with the design of the course and textbook. This alignment suggests that almost all the TAs viewed the use of mathematical tasks favorably, contributing to their likelihood to enact RPTs during instruction, which I was then able to examine for the nature of task enactment (task enactment is explored in Chapter 6). More details about Ace's contrasting conceptions of teaching RP in MTH 2 are provided in the next section.

4.2 Case Studies of TA's Conceptions of Reasoning-and-Proving

A feature of MTH 2 that distinguishes it from general mathematics courses is the student population: PTEs in the course are current undergraduate mathematics learners who also need to learn mathematics so they are prepared to explain it to others. Since TAs differed in how they attended to this dual nature of the course in their conceptions of teaching and learning RP in MTH 2, four case studies are considered. The goal of these cases is to provide background information that informs our understanding of TAs' instructional decisions (presented in Chapter 6, with relationships discussed in Chapter 7). The first pair of cases consists of two groups of TAs who attended to the elementary perspective, but have different conceptions of what RP is important for PTEs to learn. The second pair consists of individual TAs who attended primarily to the mathematics that MTH 2 students needed to learn, but have different reasons for holding that perspective.

Recalling general information about the TAs' backgrounds and prior teaching experiences provides context for understanding their case descriptions (refer to Table 3). In particular, none of the TAs had experience teaching K-12 mathematics in the US, but they were expected to teach MTH 2 with the goal of preparing PTEs to engage future elementary students in RP processes and content. Therefore, the TAs did not have personal teaching experience or training upon which to base their perspectives of how RP in MTH 2 could relate to PTEs as future elementary teachers. Furthermore, half of the TAs (Isaac, Peter, and Kelly) did not attend elementary school in the US. These international TAs, therefore, also approached MTH 2 without any personal experiences upon which to draw to understand the context they were expected to prepare PTEs to enter. In the first case, however, the two mathematics education TAs, Evan and Kelly, along with the mathematics graduate student in his sixth year, Peter, expressed similar conceptions of RP process goals that incorporated the elementary perspective. In the second case, Laura, a second-year applied mathematics graduate student, also attended to the preparation of PTEs, but in a way that contrasted with the first case and the course goals by focusing on RP content. The investigation into these two cases suggests a commonality among the TAs in the first case that may contribute to the conceptions observed. The third and fourth cases focus on Isaac's and Ace's conceptions of teaching RP in MTH 2, respectively, as they do not attend to the elementary perspective but for different reasons. These cases provide an opportunity to understand how Isaac's background and Ace's perspective on pedagogical approaches seem to be significant factors influencing their conceptions of teaching RP in MTH 2.

4.2.1 Preparing Prospective Teachers to Teach Others about RP Processes

> MTH 2 has some unique characteristics in it because you need to build some mathematical knowledge for undergraduate students and also you need to think about the content knowledge for future teachers. (Peter, beginning-of-semester, para. 180)

> It's the way of thinking mathematics that [the PTEs] should learn, which is more important than the content that they can directly transfer to their students. (Kelly, end-of-semester, para. 85)

> Taking what you're given, using logic, being able to reason and make an argument about other stuff that has to be true because of what you're given and relationships that you have already established. . . . It's gonna be important for them . . . when they're going out as teachers being able to do that. . . . We just want to establish that thought process. (Evan, beginning-of-semester, para. 128)

Peter, Kelly, and Evan attended to the dual nature of MTH 2, where the students need to not only learn mathematics but also be prepared to teach others about it, by talking extensively about engaging PTEs in RP processes. They specified mathematical processes in their own ways but incorporated the notion of preparing PTEs to talk about mathematics and about RP with their future students by developing PTEs' ability to think and reason mathematically in MTH 2. Understanding how they articulated these RP processes underscores the importance for incorporating processes into instructional practices in MTH 2 and suggests a common experience these TAs share that may have informed their conceptions of teaching RP.

Peter talked about helping students "study proof as a proof in Mathematics . . . [and] experience more about the role of proof" (beginning-of-semester, para. 186) by engaging them in the process of proving. To help PTEs experience the proving process, he talked about providing opportunities for them to construct and refine mathematical arguments in a way that was similar to the work in which mathematicians engage. That is, he explained how "false starts," "logical gaps," and going the "wrong way" can help mathematicians refine their mathematical arguments, understand mathematical ideas better through proof, and communicate those proofs to others (Peter, end-of-semester, para. 226-236). He saw MTH 2 as an opportunity to help PTEs develop "that kind of logical thinking" (beginning-of-semester, para. 164). And he thought it was important for PTEs to experience the proving process in MTH

2 because "as a teacher if they can experience those things before as their own experience then they can easily figure out and help their own students" (end-of-semester, para. 250). Peter talked about how this proving process could help PTEs understand proof in mathematics, prepare them to engage their future students RP processes, and make sense of their future students' thought processes.

Kelly and Evan talked about helping PTEs understand RP as a mathematical way of thinking as well, specifying the importance of preparing PTEs to engage their future students in RP. For Kelly, learning about RP helped PTEs develop the ability to reason mathematically, as she explained:

> It's the idea of thinking, even though they don't teach those things literally to their students, they still have that idea that they need to connect the unknown to known . . . by using . . . valid evidence. Those ideas you have to have them to become a good elementary teacher. . . . So I think it's the way of thinking mathematics that they should learn, which is more important that the content that they can directly transfer to their students. (Kelly, end-of-semester, para. 85)

Instead of specifying RP content that PTEs might need to rely on in their future teaching, Kelly viewed the development of "the way of thinking mathematics" as the primary reason for teaching PTEs about RP. Evan expressed similar sentiments by talking about the importance of engaging PTEs in thought processes in which they would then engage their future students:

> They're not gonna be teaching about the exact same stuff in their classrooms. Maybe if they have an eighth-grade classroom sure . . . but if they have a second-grade classroom they're probably not. We just want to establish that thought process. If . . . in MTH 2 we look at proving stuff that you might prove with a second-grade class, . . . [the PTEs] are gonna understand it but it's not gonna engage them- like- get them to think. . . . So we spend most of our time in upper elementary maybe a little bit in high school [content] just to engage them . . . have them practice building that line of reasoning we want them to build with their students, even though [in elementary grades] they might be looking at little bit simpler stuff than what we're doing. (Evan, beginning-of-semester, para. 128)

These three TAs made explicit connections between RP in MTH 2 and what they wanted PTEs to understand about RP as future teachers. All three of them also have experience taking courses as undergraduate and graduate students in mathematics education. Evan and Kelly used phrases such as "our field of mathematics education" (Kelly, beginning-of-semester, para. 292) and "in a particular mathematics education . . . class" (Evan, end-of-semester, para. 78) to which they related their conceptions of RP content and processes. Although Peter was not in the doctoral program for mathematics education, his undergraduate background included certification to teach secondary mathematics and he was enrolled in a graduate-level mathematics education course at the time of the study. Peter identified mathematics education books he read as an undergraduate as prominent influences on his conceptions of teaching PTEs RP processes (end-of-semester, para. 312). When commenting on the influence of their coursework or readings, there was not a single, specific class or book to which they all referred except for the fact that they all expressed familiarity with the National Council of Teachers of Mathematics' (2000) recommendations for RP. Therefore, these three TAs identified mathematics education coursework and the NCTM's recommendations as helpful for framing their views of how RP in MTH 2 relates to PTEs' roles in elementary classrooms. Their views suggest these TAs have rich conceptions of how the mathematical content and processes in MTH 2 relate to the PTEs as future teachers that align with the design and goals of the course and were shaped, at least in part, by their experiences reading and talking about mathematics education policy and research.

4.2.2 Preparing Prospective Teachers to Produce Mathematical Explanations

> I think . . . what [the PTEs] are gonna be doing is explanations of things for students who are younger and not necessarily doing rigorous proofs. (Laura, end-of-semester, para. 10)

Laura also attended to the preparation of PTEs to teach others about RP in her goal statements, but her focus on content over process contrasted with the conceptions in the first case. This case study helps us understand how and why Laura's particular focus on an aspect of RP content contrasts with the goals of the course. Laura talked about helping PTEs produce concise nonproof arguments, as she explains,

> I think the explanations are really probably the most key for [the PTEs] since they're gonna be teaching elementary students and really most of them are . . . not gonna be doing higher-level, like the Grade 8 sort of stuff, where they would actually need to be using . . . proofs. . . . I think these explanations are really important for them and to be concise so that that their students can understand . . . this intuitive explanation. (end-of-semester, para. 162, 164)

Across the interviews, Laura articulated a goal of helping PTEs generate clear, concise, but informal (or intuitive) explanations for the validity of claims about general mathematical relationships based on how she remembered learning about RP. Laura also focused on explanations because (as expressed in the above quotation) she considered formal proofs as content that only a select few PTEs (possibly "like the Grade 8" future teachers) may ever use again. Laura's conceptualized teaching RP, therefore, differently than the NCTM, which recommended that in all grade levels mathematics classrooms RP should be "a natural, ongoing part of classroom discussions, no matter what topic is being studied" (NCTM, 2000, p. 342). Laura, however, seemed unaware of any misalignment between her conceptions and these recommendations as she expressed unfamiliarity with NCTM's (2000) recommendations for RP in K-12 mathematics (end-of-semester, para. 216). Her conception of teaching RP in MTH 2 focused on PTEs producing concise explanations of solutions and informal justifications for mathematical relationships, which could reinforce PTEs' misconceptions about what constitutes a proof if the informal justifications are accepted as proofs. In light of the previous case and this one, TAs like Laura, who aim to prepare PTEs to teach RP but may unknowingly reinforce PTEs' misconceptions, could potentially benefit from support that incorporates some applicable mathematics education literature, such as the NCTM's (2000) recommendations for RP.

4.2.3 Helping MTH 2 Students Understand Formal Proofs

> My students . . . should get a feeling for how math works and the best thing is to do proofs in some kind of formal way. (Isaac, beginning-of-semester, para. 110)

Isaac's conceptions of teaching RP are diametrically opposed to the previous case because of his focus on helping MTH 2 students learn how to construct formal proofs and his difficulty articulating an elementary classroom perspective. Although Isaac's conceptions of teaching RP are like Ace's in how he privileged content, these TAs' conceptions are presented as two separate cases because Isaac's background, as an international graduate student, seems to be a significant factor in his perspective.

Since Isaac did not grow up in the US, he was unfamiliar with the teaching and learning of mathematics in the elementary grades. He only talked about how the material about RP related to his students as future teachers when he was specifically asked about that potential connection during some interviews. In response to these prompts he frequently asked clarifying questions about what grade levels PTEs would be certified to teach. He would, then, often follow up those questions with a statement such as: "*I don't know what they do in grade eight in US* but . . . maybe they would talk about the Pythagorean Theorem for example and maybe do the proof that was in the last part [referring to a picture proof of the Pythagorean Theorem in the MTH 2 text]" (Isaac, beginning-of-semester, para. 140, emphasis added). When prompted, therefore, Isaac tended to specify RP content from MTH 2 that PTEs could potentially use in their future teaching. He also expressed some uncertainty about how the material from MTH 2 prepared PTEs for elementary teaching as he used phrases like *maybe* and *I don't know*.

Isaac readily attended to the undergraduate mathematics perspective, talking about RP content he wanted MTH 2 students to know and understand. He articulated purposes of proof in mathematics teaching by explaining how learning about proof was important content in his MTH 2 classroom (*Int* indicates the interviewer, beginning-of-semester, para. 106, 110-112).

106	Isaac:	I think [proof] serves the purpose that the students get a feeling for what math is about and it's not just about guessing or assuming things; they should provide arguments for why it's true.
. . .		
110	Isaac:	My students . . . should get a feeling for how math works and the best thing is to do proofs in some kind of formal way.
111	Int:	When you said . . . "do proofs in a formal way", what does that mean, "formal"? Can you explain that a little bit?
112	Isaac:	Formal is like, you talk about things like axioms, theorems, definitions and write things down not just give some kind of talking.

Isaac said that students should understand that proofs are necessary for explaining why mathematical statements are true (para. 106). He also described the importance of teaching about proof to help his students understand "what math is about" and "how math works" (para. 106, 110). He talked about how students needed to understand that math was "not just about guessing and assuming things," and that doing proofs is "the best" means to do so (para. 106, 110). This focus on formal proof as a primary content learning goal may also have contributed to Isaac's difficulty articulating a connection between the RP work in MTH 2 and PTEs' future teaching because formal proofs do not often play a prominent role in elementary mathematics classrooms. Isaac's focus on formal proof differs from the goals of MTH 2, which emphasize PTEs learning to reason mathematically in addition to doing proofs.

4.2.4 Helping MTH 2 Students Learn RP Content Using Lectures

> You mention the proof and [the students] hold their breath while you go through it. . . . You hope that the students take something away from this and hopefully should be able to regurgitate it to their students one day. (Ace, beginning-of-semester, para. 220)

Like Isaac, Ace also focuses on helping MTH 2 students understand formal proofs, but his perspective on methods for teaching RP does not align with the structure of the course, setting him apart from the other cases. Of all the TAs in this study, Ace also has the least amount of teaching experience (one semester teaching college algebra prior to teaching MTH 2) and seems to explicitly rely on his experiences as a mathematics learner. Ace 's perspective on how to teach RP in MTH 2, by using lecture-based instruction, is unique among these TAs, and is a significant factor related to his goals for teaching PTEs about RP.

Ace conceptualized his role as the one who "gives [the PTEs] knowledge" (Ace, end-of-semester, para. 30) in MTH 2. The RP-related knowledge he wanted PTEs to learn included: what constitutes a proof, how to write proofs, and that proofs exist for elementary school mathematics content, to ultimately be prepared to respond if elementary students asked a *why*-question. The following dialogue typifies how Ace articulated his goals for teaching about proof in MTH 2 (*Int* indicates the interviewer, beginning-of-semester, para. 215, 218-222):

215	Int:	Thinking about MTH 2, can you say a little bit about the purpose proof and proving serves in that classroom?
. . .		
218	Ace:	It falls under this category: You mention the proof and they hold their breath while you go through it. And then at the end of the day, they use the tools at the end of it.
219	Int:	Ok.
220	Ace:	You hope that the students take something away from this and hopefully should be able to regurgitate it to their students one day, and say, "yeah I did this in class. It comes from so and so." And you really hope that's the case. . . . I'd have go and visit K-8 . . . to really determine how useful it is gonna be to them in their . . . day-to-day lives which is probably not very.

221 Int: And you said you "hope they regurgitate this later," can you say a little bit about what you were picturing there, what did you mean by "regurgitate this later"?

222 Ace: Regurgitate is the word I use for memorize- or something like this. It's not that you truly learn [the proof] or understand it and you know actually all the parts of how it fits together and you've really explored the idea completely. But it's the idea that you believe it and you've memorized at least 90 percent of it so that you can spit it out at someone else.

Similar to Isaac, Ace expressed uncertainty about how proof in MTH 2 related to the K-8 classroom setting (para. 220), but Ace and Isaac differed in their responses to that uncertainty. Isaac either stated he did not know or (usually after additional presses from the interviewer) he ventured a guess. In his guess, he often chose a RP task from MTH 2 and described how PTEs might use it in an upper-elementary-grade classroom (e.g., Isaac, beginning-of-semester, para. 140). Contrarily, as Ace expressed in the statement above (para. 220), without any additional promptings during the interview Ace quickly evaluated the likelihood that visiting an elementary classroom would reveal any RP insights as unlikely. The contrast in their responses is apparent: Isaac seemed open to considering the possibility that MTH 2 proofs can relate to the elementary context, and Ace seemed certain that they do not relate.

The above dialogue from Ace also indicates another difference between the third case and this case considering their conceptions how to teach PTEs about RP. To emphasize the importance of students participating in the construction of proofs, Isaac explained that "if you never . . . do proofs yourself, you don't really get a feeling for how . . . to prove things"(Isaac, end-of-semester, para. 64). A prominent feature of Ace's conception of teaching RP, however, was using lectures to teach about RP. He described PTEs' response during the teaching of a proof as "they hold their breath while you go through it" (para. 218), suggesting PTEs did not participate in the proof's construction while the TA presented it to them. Earlier in the same interview, Ace explained his preference for this style of instruction, saying that presenting the material for PTEs

> is much more of a time saving method . . . however it's a trade off because then they don't understand the material as much. So I mean obviously if they sit there and really think through these ideas and do all these activities and stuff I think their understanding is greater however it also takes quite a bit more time. (beginning-of-semester, para. 88)

When describing the "trade off," Ace said using mathematical tasks (activities from the textbook) can provide opportunities for PTEs to understand material better. He consistently talked about the efficiency, however, of using a lecture-style of instruction.

Despite this view, Ace did not ignore the textbook's tasks and he talked about why he still asked PTEs to work in groups on tasks, as follows (beginning-of-semester, para. 127-134):

127	Int:	Before teaching MTH 2, had you ever taught using group work and activities before?
128	Ace:	No. And . . . I would never do it again.
129	Int:	Why do you say that?
130	Ace:	I don't know. It's not math. It's not math to me. I don't know.
131	Int:	What does that mean? Can you help me understand that?
132	Ace:	Every single class I've had is always lecture-based. And with the exception of maybe like elementary school or something like this. So the concept of class work or group work or something like this to help get a problem solved is unreal. I don't know. It's too far out there for me. I'm a lost cause.
133	Int:	So how did you decide to even use group work then for MTH 2?
134	Ace:	I was confused at the beginning. I didn't understand what was going on for this course and [the course supervisor] said "these are group work activities" and stuff like this so then I went with it the first day and have gone with it ever since.

Although Ace made a statement about how the elementary grades could potentially include more activity-based instructional strategies (para. 132), it did not seem to influence his perspective on the utility of this pedagogical approach for PTEs in MTH 2. Ace's prior experiences learning mathematics shaped his less-than-favorable view of activity-based learning in mathematics teaching. He talked about lecture-based instruction as time saving and preferable. Ace explicitly devalued the pedagogy called for in the design and goals of MTH 2.

4.3 Conclusions about the Analysis of TAs' Conceptions of RP

Through this examination of TAs' conceptions of RP it is evident that TAs' conceptions of purposes of proof in mathematics contrast with their conceptions in teaching. The context of teaching MTH 2, therefore, elicits TAs' talk about purposes of proof specific to the work of teaching PTEs about proof, supporting the notion that teachers need a specialized understanding of mathematical content to teach. TAs may complete their mathematical work in their courses and research without explicitly articulating the purposes of proof, but, as an instructor for PTEs, TAs need to know and incorporate the purposes of proof into the classroom to develop an informed conception of proof in PTEs.

In this context, it is also important for TAs to develop PTEs' ability to articulate both RP content and RP processes, and attend to goals specific to the MTH 2 context as well as to more long-term goals about the future elementary classroom setting. To establish these goals TAs should take into consideration the mathematical content along with PTEs' learning needs for both the undergraduate and elementary contexts; highlighting an aspect of pedagogical content knowledge called for in this context, described by other researchers as knowledge of *content and students* (Ball et al., 2008). To navigate that setting, some TAs (Peter, Kelly, and Evan) articulated a balance between RP content and RP processes PTEs needed to learn, while others emphasized RP content (Laura, Isaac, and Ace), varying in their attention to the elementary context. Their goals for teaching RP in MTH 2 were influenced, in part, by differences among the TAs regarding their familiarity with and understanding of mathematics education theory, national recommendations for teaching and learning RP, and the nature of teaching and learning of RP in elementary classrooms in the US.

A commonality among the TAs was their lack of experience teaching mathematics in the K-12 setting. This study did not collect data from any TAs with K-12 teaching experience, and such an investigation could compare how the RP conceptions of TAs who have experience teaching in those settings might compare with those in this study. One might expect TAs with that experience to be more aware of the knowledge and pedagogical skills PTEs need to succeed in their future classrooms and, therefore, be more explicit about connections to the elementary context. Research indicates, on the contrary, that it is not automatic for teachers to learn to teach from their own teaching experiences and they often need outside help to learn to anticipate students' potential misconceptions (Grossman, 1989). The study of TAs with K-12 teaching experience could inform our understanding of prominent factors influencing TAs' conceptions of teaching and learning RP, and whether or not it is important for colleges and universities to consider assigning TAs to teach courses like MTH 2 who have a variety of teaching experiences.

The results from this examination of TAs' conceptions of RP in mathematics, teaching, and learning have implications for the potential opportunities PTEs will have to learn RP content and processes in MTH 2. Those who were more aligned with the course goals were expected to design lessons around these goals, thereby aiming to engage PTEs in RPTs. How TAs enacted RP tasks is studied in Chapter 6, after first exploring the different types of RPTs in the textbook, in Chapter 5.

CHAPTER 5

REASONING-AND-PROVING OPPORTUNITIES IN THE TEXTBOOK

This chapter presents the results of the analysis of the textbook (Beckmann, 2008) used in MTH 2. The Geometry and Measurement chapters were analyzed to understand the types of RP opportunities afforded, ultimately guiding the selection of lessons to observe. This identification of textbook sections wherein PTEs may have different types of opportunities for RP serves as a resource for understanding how one might structure mathematics courses for PTEs to provide systematic and varied opportunities to engage in RP.

The RP-framework guided this analysis: questions were coded for the processes of RP outlined in the framework (Table 2), and questions that were designed to offer PTEs at least one opportunity for RP were referred to as RP-related questions (RPQs). The questions that constituted a mathematical task (Stein & Lane, 1996) were then grouped together and a task with at least one RPQ was regarded as a RP-related task (RPT).

The sections following describe the results along the RP-framework's dimensions: First, the overall distribution of opportunities for PTEs to engage in RP is described. Second, the types of RP opportunities are presented along Dimension 1 to understand how tasks present opportunities for PTEs to experience the six RP processes. Third, results are detailed from the Dimension 2 analysis, examining how tasks were designed for PTEs to experience the process through which generalizations are formed and then justified. Fourth, opportunities for PTEs to understand features of proof and proving instead of doing proofs are described. Finally, a summary of these results outlines implications for studying mathematics instructors teaching PTEs RP from such a textbook.

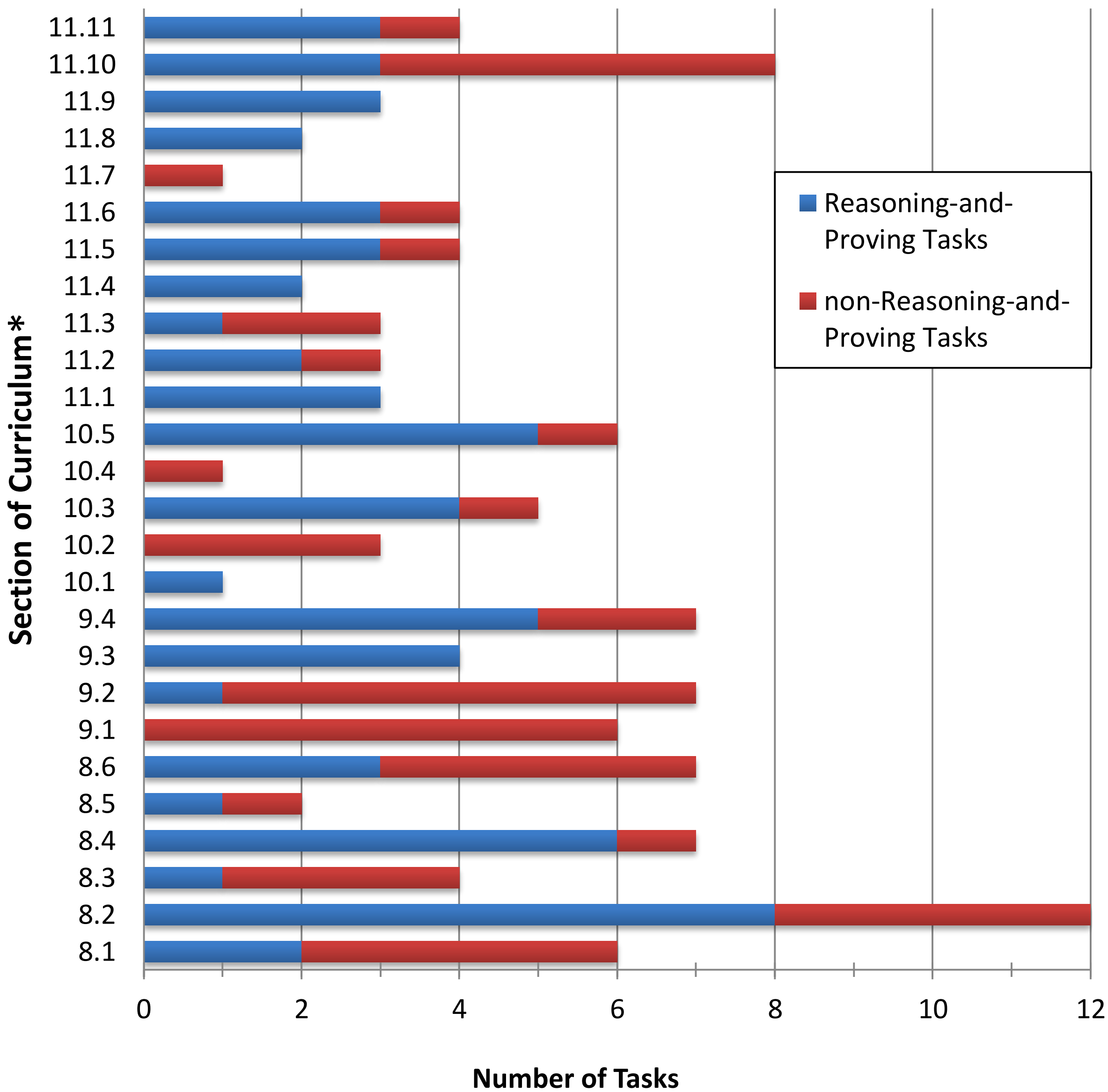
5.1 Distribution of Opportunities for Reasoning-and-Proving Across Four Textbook Chapters

Opportunities for PTEs to engage in RP are indicated by how the RPQs and RPTs were distributed across these chapters of the textbook. The analysis included 565 questions of which the initial key word search identified 242 as potentially RP-related. Examining the questions recommended by the course supervisor identified 16 additional questions that were potentially related to RP. These 16 questions were missed by the key-word search because, although they focused on investigating mathematical relations, they did not specifically ask PTEs to generate conjectures or provide justifications. These 258 questions were analyzed for substantive opportunities for RP. Of the 258 potentially RP-related questions, 77% (198 questions) represented opportunities for PTEs to engage in at least one RP process from the RP framework. Therefore, these 198 questions were regarded as RPQs because they were designed to offer PTEs at least one opportunity for RP. The remaining 60 questions did not offer any opportunities for PTEs to engage in RP.

Table 9. Distribution of Reasoning-and-Proving-related Questions and Tasks

Textbook Chapter: Topic of Study		Number of RPQs in Chapter	Percent of RPQs in Chapter	Number of RPTs in Chapter	Percent of RPTs in Chapter
8:	Geometry	86	44.3%	21	55.3%
9:	Motion & Change	17	15.7%	10	41.7%
10:	Measurement	24	27.2%	10	62.5%
11:	More about Area & Volume	71	40.6%	25	67.6%
	Total Across these Chapters:	198	35%	66	57.4%

The distribution of these 198 RPQs is displayed in Table 9, which indicates that RPQs are more prevalent in the first and fourth geometry and measurement chapters, but at least 15% of the questions provide opportunities for PTEs to engage in RP within the middle two chapters, as well. As described in Chapter 3 (Section 3.3.1.2), the textbook contained 115 tasks, and if a task had at least one question that afforded opportunities for PTEs to engage in one or more dimensions of the analytic framework, it was identified as an RPT. Of these 115 tasks, 57% (66) contained at least one RPQ. This distribution of RPQs and RPTs (see Table 9) indicates numerous RP opportunities across these chapters.



* The names of these sections of the text are provided in Appendix B.3

Figure 8. Breakdown of RPTs and non-RPTs in the textbook

Since I observed TAs teaching individual lessons, I also considered the RP opportunities at a smaller grain size by considering sections of the chapters (Figure 8) to examine the distribution of RPTs and guide the observation schedule. For sixteen of the 26 sections in the textbook, at least 50% of the tasks were RPTs, suggesting a number of sections for which classroom instruction would potentially involve tasks with potential to engage PTEs in RP. It is also important to note that the later sections in some chapters (for Chapters 8, 9, and 10) tended to have more RP tasks than earlier sections or the chapter had an even spread among subsections (as noticeable in Chapter 11). It could be problematic if the chapter frontloaded RPTs in the beginning sections. The sections of the textbook with fewer RPTs were also spaced out in the textbook so that PTEs could arguably have opportunities to encounter RPTs at various points throughout a course using this textbook.

5.2 Types of RP opportunities Afforded in the RP Tasks

This analysis also identified the nature of RP opportunities the textbook afforded by investigating the types of RP processes the RPTs elicited. The distribution of tasks with respect to the six types of RP processes (Dimension 1 of RP-framework, Table 2) is displayed in Figure 9. Individual tasks could be coded for more than one RP process because a task fell within one of these categories if it contained at least one RPQ that elicited the RP process. Two categories in the framework, *evaluating mathematical claims* and *evaluating mathematical arguments*, accounted for the fewest number of RPTs (7 and 6, respectively). Since these processes were added to this framework as particularly relevant for PTEs as future teachers, TAs may need to augment the text or modify tasks to provide additional opportunities for these processes.

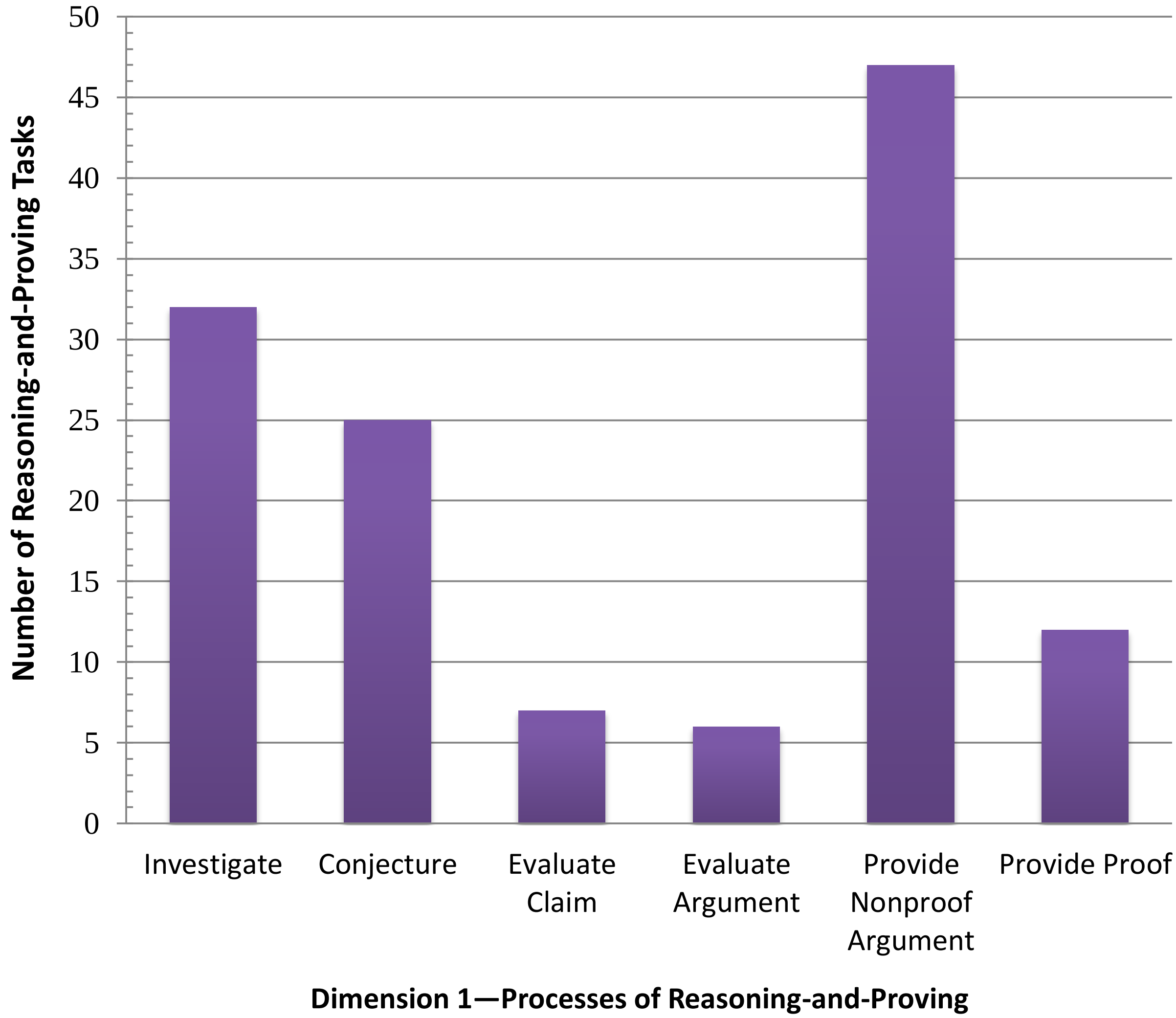


Figure 9. Number of tasks addressing RP processes in the textbook

The most common category, *providing a nonproof argument*, indicates that for most of the 66 RPTs PTEs were asked to provide mathematical explanations (i.e., rationales or empirical arguments) that were not necessarily proofs (Figure 9). Such tasks provide important opportunities for PTEs to create mathematical justifications, but also have implications for the implementation of tasks and the extent to which the TA and PTEs may need to talk about whether or not the justifications produced were or were not proofs and why.

5.3 Purposes of Explorations of Mathematical Relations, Conjectures, Evaluations, and Proofs

The exploration of mathematical relationships can lead to the generation of conjectures, which can then be evaluated, revised, or proven. Thus, the RPQs and RPTs were also analyzed for these purposes (Dimension 2). This analysis investigated how PTEs were asked to engage in sequences of related RP processes and what roles of proof were emphasized.

5.3.1 Purposes of Investigating, Conjecturing, and Evaluating

Providing PTEs with opportunities to engage in sequences of related RP supports the development of a richer understanding of RP processes and their classroom use. A major role that investigating patterns plays in mathematics is they can lead to the development of conjectures. If such an investigation served the purpose of a conjecture precursor, it was counted as an opportunity for PTEs to understand the relationship between investigating mathematical relations and generating conjectures. Conjectures can lead to the development of proofs, supporting the distinction of whether PTEs were asked to make conjectures that forerun the development of proofs associated with these conjectures (proof precursors) or not (proof non-precursors). Similarly, an evaluation of a provided mathematical claim could serve the purpose of a conjecture precursor or proof precursor and an evaluation of a mathematical argument could serve the purpose of refining proofs or providing nonproof arguments for why an argument was or was not valid. These situations were considered as opportunities for PTEs to understand the trial-and-error and revision processes of generating and refining proofs.

5.3.1.1 Considering the Analysis of the RPQs

The RPQs identified as providing opportunities to investigate mathematical relations led to opportunities for PTEs to generate conjectures 50% of the time and did not press PTEs to generate a conjecture the other 50% of the time. When a question asked for a conjecture, 89% of the time there was no press to provide a related proof. All instances of evaluating mathematical claims and arguments did not serve as a precursor for a new claim or refined argument: PTEs evaluated claims or arguments as valid or not, but were not asked to produce a revised claim or argument, which could have provided opportunities for them to engage in the iterative process of generating and refining arguments.

5.3.1.2 Considering the Analysis of the RPTs

At the task level, however, there were additional opportunities for PTEs to experience these processes of proving that were not captured at the question level. For example, at the question level, every instance of evaluating mathematical arguments either stopped after something was identified that was correct or incorrect about the argument or asked PTEs to provide a rationale (a nonproof argument). There was a task, however, Activity 11H, where PTEs would evaluate a mathematical argument that served as a proof precursor for the next question in the task. That is, PTEs were asked to evaluate an argument that incorrectly tried to prove the formula for the area of a triangle when the height was drawn outside of the triangle (in 11H.2). Their evaluation of the provided mathematical argument was a precursor for the development of a proof in the next question in the task (11H.3), asking PTEs to prove the formula for the area of a triangle for any such triangle.

To capture such opportunities for PTEs to progress through RP processes, the 66 tasks were grouped based on the types of RP opportunities in tasks, as described in Table 10:

Table 10. Categories for Types of Reasoning-and-Proving Opportunities in Tasks

Category	Description
Mathematical Generalizations (MG)	Tasks that focus on investigating mathematical relationships, generating conjectures, or evaluating mathematical claims (the generalization side of the RP-framework), but do not prompt PTEs to generate or evaluate mathematical arguments.
Mathematical Arguments-Nonproofs (MA-N)	Tasks that focus on evaluating arguments or generating nonproof arguments (the mathematical argument side of the RP-framework), but do not press PTEs to provide a proof.
Mathematical Arguments-Proofs (MA-P)	Tasks that only include questions from the mathematical argument side of the framework, and must contain at least one prompt for PTEs to generate a proof.
Generalizations and Arguments-Nonproofs (GA-N)	Tasks that include at least one mathematical generalization process and at least one mathematical argument process, but there are no explicit prompts for a PTEs to generate proofs.
Generalizations and Arguments-Proofs (GA-P)	Tasks that include at least one mathematical generalization process and at least one prompt for providing a proof.

Grouping the tasks into these five categories provided a way to examine the nature of the types of RP processes elicited in written tasks (Table 11).

Table 11. Opportunities for PTEs to Engage in RP processes within the 66 RP Tasks

Type of RPT	Number of RPTs of this type (% of 66 RPTs)
Generalizations & Arguments: Proofs	7 (10.6%)
Generalizations & Arguments: Nonproofs	17 (25.8%)
Mathematical Arguments: Proofs	5 (7.6%)
Mathematical Arguments: Nonproofs	22 (33.3%)
Mathematical Generalizations	15 (22.7%)

Considering these last two types of RPTs, approximately 36.4% of the 66 RPTs provided opportunities for PTEs to generate or evaluate conjectures and then provide some mathematical arguments for their generalizations. In 17 of those 24 tasks, however, PTEs were asked to generalize and provide arguments that did not call for a proof, suggesting a prevalence of non-proof arguments elicited in the text.

5.3.2 Purposes of Proofs

Aligned with the purposes of proof examined in the interviews, contextualized for a textbook examination, questions that elicited proofs were analyzed to note if the proof would serve one or more of the following purposes: explanation, verification, falsification, and generation of new knowledge. Most of the 27 RPQs that provided opportunities for PTEs to produce proofs served multiple purposes. A majority of these RPQs served the purpose of explanation (81%), and about 70% were coded as verification. Approximately one out of ten (11%) contributed to the generation of new knowledge. Only 7% were coded as falsifications. These results suggest that PTEs may experience some purposes of proof so infrequently (i.e., generation of new knowledge and falsification) that they do not develop a robust understanding of all the roles of proof in mathematics.

At the task level, 12 of the 66 RPTs provided opportunities for PTEs to produce proofs and these proofs served multiple purposes. The frequencies of the four codes at the task level were fairly comparable to those at the question level (83%, 58%, 17%, and 8%, respectively). The most salient purpose of proof was for PTEs to experience the explanatory power of proof, which is a role that many researchers have called for more emphasis on in school mathematics (de Villiers, 1999; Hanna, 2000).

The utility in considering the purposes of proofs at the task-level was particularly apparent when considering how RP opportunities from one question in a task led to particular RP opportunities in another question in the same task in a way that represented how mathematicians make meaning in mathematics. That is, when examining one activity (8J) a mathematical chain was noticed where the completion of a proof could lead to a conjecture about another mathematical relationship. Specifically, 8J.1 (question 1 of Activity 8J) asked PTEs to generate a conjecture about the measures of alternate interior angles of two parallel lines and prove their proposed relationship was true. Next, 8J.2 asked PTEs to generate a conjecture about the sum of the measures of the interior angles of a triangle based on what they discovered in 8J.1. PTEs were provided with an opportunity to engage in a mathematical process in which mathematicians frequently partake—stating a conjecture and proving that it is true (in 8J.1), then using that proven result to generate a new conjecture (in 8J.2).

5.4 Opportunities to Reflect on Characteristics of Proof and Proving Rather than to Do Proof

In the third dimension of the RP-framework, opportunities for PTEs to reflect on features of proofs and proving instead of doing proofs were examined. Providing opportunities for PTEs to articulate what makes an argument a proof or to reflect on the proving process could help prepare them for teaching about what constitutes a proof. In this analysis, however, only 7 questions were coded as providing opportunities for PTEs to reflect on characteristics of proofs and proving. The seven questions asked PTEs to consider the importance of the generality of proofs (4 questions) or how proofs should be logical arguments that were based on mathematical facts (3 questions), coded as *characterizing proofs*. Three of the questions about the generality of proofs also asked PTEs to compare the argumentation form and discuss whether or not that form of argumentation was a proof and why. These instances talked about how one can distinguish proofs from nonproofs, and in these particular cases emphasized whether or not the argument was general in making the distinction. This small number of opportunities for PTEs to talk about the characteristics of proofs and talk about how to identify proofs and nonproofs suggests an aspect of RP that was not readily accessible in these geometry and measurement chapters of the textbook.

5.5 Conclusions about the Textbook Analysis

This examination of the geometry and measurement questions and tasks indicates Beckmann's (2008) text provides a range of opportunities for PTEs to engage in RP. Throughout these chapters, PTEs are asked to investigate patterns, generate conjectures, and justify mathematical claims with rationales or with proofs. There are also opportunities for PTEs to engage in many RP processes within one task, meaning they may experience the iterative process of generating and refining arguments. This analysis provides a more detailed view of RP opportunities in Beckmann's text than previous analyses (McCrory et al., 2008) by identifying how PTEs could have opportunities to engage in specific aspects of the activity of RP.

These results have a number of implications for how this textbook might be used. First, the small number of questions addressing Dimension 3 of the framework coupled with the infrequent opportunities for PTEs to evaluate claims and arguments suggests a potential lack of opportunities to discuss features of proofs and proving. It was not initially surprising that there were a small number of questions coded for Dimension 3 (Meta-Level: Reflecting on Proof) because one might anticipate such discussions being more evident in classroom practices than in a textbook. For such discussions to occur, however, instructors need to recognize the importance of having such discussions and these discussions could potentially flow out of questions designed for PTEs to evaluate claims and arguments. With so few questions about evaluating claims or arguments, there would not necessarily be opportunities to discuss norms of proving or to reflect on different types of reasoning explicitly during classroom instruction.

Second, only seven of the 66 RP tasks provided opportunities for PTEs to prove generalizations that they themselves had generated (Table 11). This result contrasts with how mathematicians engage in RP; they typically explore patterns, which can lead to new generalizations, which can then be tested and revised or proven (Lakatos, 1976; NCTM, 2009). This finding suggests that when asked to prove mathematical statements in the textbook, PTEs would more often prove statements that were provided to them and not ones that they had generated. Admittedly, in mathematics teaching, there is always a tension between how much to let students discover on their own and when to tell them something (i.e., provide a mathematical statement) and ask them to do something with that information (i.e., prove that the statement is valid). In the context of MTH 2, however, PTEs need to develop the ability to engage in proving processes to also teach others. Providing more opportunities for PTEs to generate their own conjectures and work through proving and refining their conjectures could potentially motivate them to explore mathematics and help them experience the process of RP.

Third, as was particularly apparent in Figure 9 and Table 11, a majority of the RP opportunities were about generating nonproof arguments. This prevalence of nonproof arguments suggests that the textbook would likely be implemented to provide opportunities for PTEs to explain why mathematical statements were valid, particularly in the form of informal arguments. There are positive and negative potential outcomes of the prevalence of that type of RPT in the textbook: Such a focus could provide helpful opportunities for PTEs to practice communicating their mathematical reasoning more clearly, important for demonstrating their mathematical understanding and being prepared to explain ideas to their future students.

Since many prospective and practicing teachers maintain the misconception that empirical arguments are proofs (Knuth, 2002b; Steele, 2006), however, when PTEs are taking mathematics courses for teachers it is important to help them transition from this way of thinking to a more robust view of the role of proof in mathematics. Although the inclusion of rationales, explanations, and empirical arguments could help PTEs articulate how they are thinking about mathematical ideas, without some contextualization and thoughtful implementation on the part of the instructor, PTEs may walk away from a course using this textbook with that conception maintained or reinforced. To design instruction without reinforcing this misconception, for example, an instructor could choose to modify some tasks by pressing PTEs to provide a proof even when not called for in tasks. An instructor may instead choose to contextualize the fact that the argument PTEs provided was not a proof and facilitate a class discussion about features that make arguments proofs and nonproofs. Modifying and contextualizing tasks from the textbook is an important aspect of instructors' teaching practice that needs to happen to broaden PTEs' experiences with and conceptions of RP, and how RP tasks from this textbook were enacted is described in the next chapter.

CHAPTER 6

ENACTMENT OF REASONING-AND-PROVING TASKS

I observed each TA's classroom instruction for at least six hours to see them enact RPTs that afforded various opportunities for PTEs to engage in RP processes. I applied the Mathematical Task Framework to examine how TAs set up and implemented RPTs by maintaining or modifying RP demands of tasks during classroom instruction. This analysis, therefore, shows how RPTs were enacted to provide a range of opportunities for PTEs to work on RP processes, reveals patterns of enactment across different types of tasks and TAs, and suggests some factors that contributed to modifications of the RP demands of RPTs.

To understand the opportunities for PTEs to engage in RP in MTH 2, the analytical approach is summarized next. Then, the results are presented by first considering the enactment of all RPTs observed across all TAs. These results are then broken down by type of RPT to understand the enactment of tasks that afforded opportunities for PTEs to engage in different RP processes. Next, the results from individual TA's classroom practices are presented to look for patterns in opportunities for PTEs to engage in RP in the individual classrooms. The final section summarizes the nature of opportunities for PTEs to learn about RP observed in this study and outlines implications for future research and professional development.

6.1 Analytical Approach for Examining Enactment of Reasoning-and-Proving Tasks

The goal of this analysis was to address the second research question about how TAs enacted RPTs in MTH 2. I organized the observation schedule by considering the tentative pacing guide for the course and the opportunity to see TAs implement different types of RPTs. Based on this observation schedule, I observed the enactment of 35 RPTs, accounting for over half of the 66 identified RPTs in the textbook (Table 9). Because TAs occasionally relied on resources other than the textbook (Beckmann, 2008), I also saw TAs enact 11 RPTs from other resources. Whenever schedules allowed, I observed the enactment of these 46 RPTs in more than one classroom. Altogether, I observed 82 enactments of RPTs during the semester.

The 82 observed RPTs entailed engaging PTEs in one or more of the RP processes detailed in the first dimension of the RP-framework (Table 2). The purpose of this analysis was to understand how RP demands of these tasks changed or did not change during enactment of the tasks. The enactment was analyzed by comparing the potential for PTEs to engage in RP in the task-as-written with the opportunities for PTEs to engage in RP during classroom instruction (explained in more detail in Section 3.4.1), indicating when RP demands were (a) *increased*, (b) *decreased*, (c) *shifted*, or (d) *not changed*. An increase in RP demands means that PTEs generated or evaluated a process of RP that was not included in the potential RP opportunities in the written task. A decrease in RP demands refers to instances where PTEs did not participate in one of the major RP processes that were elicited in the task-as-written. A shift in RP demands means that during enactment PTEs had opportunities to generate generalizations or arguments that were different from the task-as-written, but the overall types of RP processes that PTEs engaged in were not different from the task-as-written. Finally, no change in RP demands refers to instances where there were no modifications in the RP opportunities for the task and PTEs engaged in the RP processes that were elicited by the task.

6.2 Overall Results from Analyzing the Enactment of Reasoning-and-Proving Tasks

The analysis of enactments of 82 RPTs compared the RP opportunities in the tasks-as-written with the RP processes PTEs engaged in during enactment. The enactment of seven tasks and 27 tasks resulted in increases and no changes in RP demands, respectively (Figure 10). That is, in approximately 41% of the instances when RPTs were observed, PTEs either engaged in more RP processes than were anticipated for a given task (when RP demands increased) or engaged in the RP processes that were elicited in these tasks (when there was no change). In these instances, PTEs worked with their classmates in small groups and engaged in whole-class discussions by generating and evaluating conjectures and justifications.

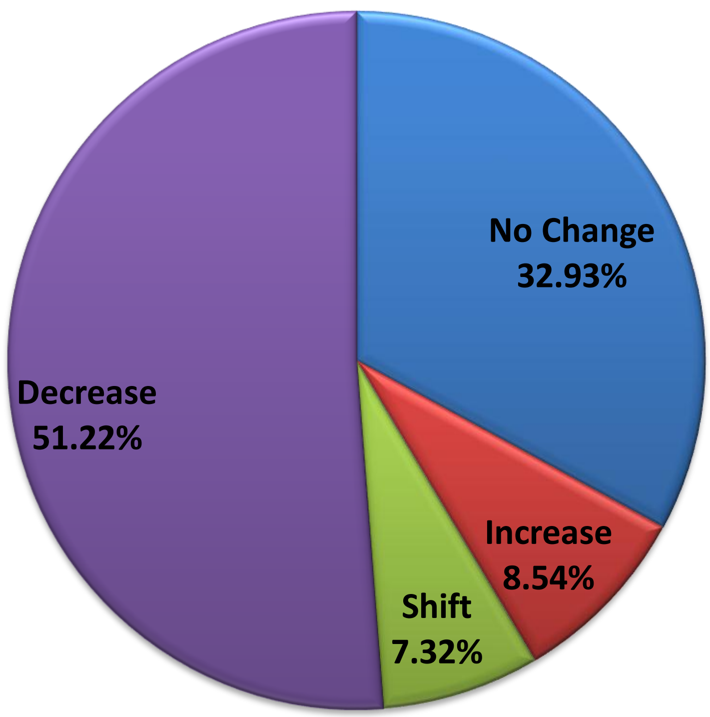


Figure 10. Enactment of RP tasks, results from observing eighty-two tasks enacted

Moreover, for six of the observed RPTs, TAs shifted RP demands without changing the overall RP processes in which PTEs engaged (Figure 10). That is, TAs changed the order in which questions were asked or added additional questions to tasks, but PTEs investigated relationships, generated and evaluated conjectures, or generated and evaluated mathematical arguments that were expected. During the enactment of 40 tasks, therefore, PTEs had opportunities to work on RP processes that were called for, or went beyond, the written task.

When during the enactment of the other 42 tasks, however, the RP demands decreased. Thus, approximately 51% of the time PTEs did not engage in at least one RP process that was elicited in an RP task-as-written. Two different types of decreases were observed: (1) RP aspects from the task were *excluded* during enactment and (2) TAs *generated* RP processes instead of PTEs. These types of decreases are conceptualized in the following way: for type (1), PTEs were not held accountable for engaging in specific RP processes from the task. For type (2), during the class the TA, instead of PTEs, generated one (or more) aspect of RP that the task asked PTEs to determine. The results from observing the enactment of 82 RPTs are displayed in Figure 11, including the prevalence of these two types of decreases. Accounting for 16 of the 42 RPTs where RP demands were decreased, decrease type (1), where TAs excluded or did not emphasize RP processes in tasks, was less frequent than type (2).

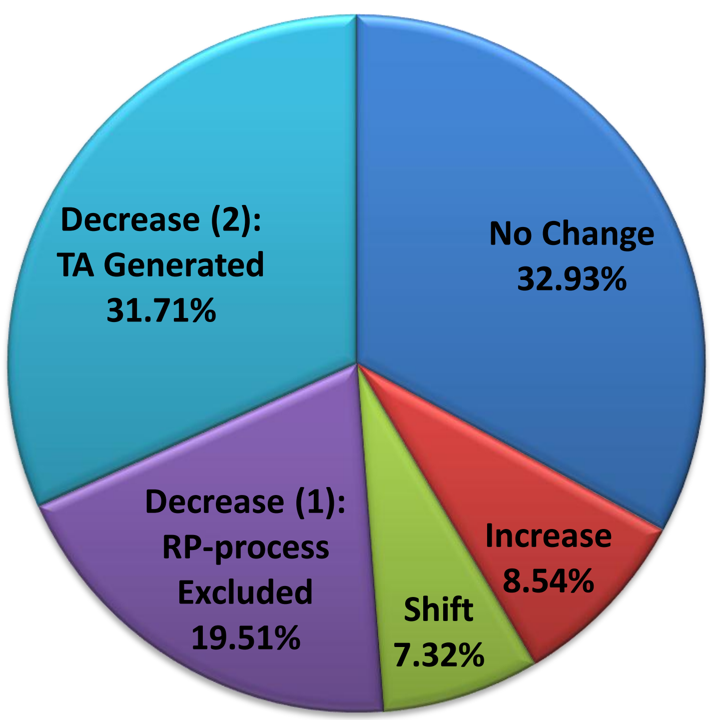


Figure 11. Enactment of RP tasks including different types of decreases in RP demands

To understand the nature of the classroom interactions when TAs excluded RP processes in tasks (decrease type 1), consider the two primary situations when this occurred. First, during the task set up, PTEs were told to only work on specific questions in a task or to not focus on parts of questions that provided opportunities for PTEs to engage in RP processes. This set up decreased what PTEs were expected to do in regards to RP during the task. Second, during implementation PTEs were not held accountable for producing RP processes elicited in the task. In these instances, individual work, group work, or whole-class discussions ended without addressing all RP processes in the task. For example, a task that asked PTEs to generate a conjecture and explain why it was valid resulted in a decrease of this type when Peter asked PTEs to move on to the next task after generating the conjecture. Furthermore, these types of decreases often occurred in Laura's class when PTEs offered explanations that were not mathematical or did not explain mathematically why, but Laura and the class accepted these non-mathematical arguments as justifications and moved on. In these examples, the conjectures created were the focus and not how to explain its validity or what constitutes a valid mathematical justification for the conjecture. In general, when this type of decrease occurred during implementation it was often because TAs affirmed PTEs' responses even when they did not fully address all RP processes called for in the task.

For the more common type of decrease, accounting for about 62% of the decreases observed (i.e., 26 of the 42 RPTs where decreases occurred), TAs generated conjectures, rationales, or proofs for PTEs. The prevalence of this decrease type 2 (Figure 11) implies a particular pedagogical choice TAs made frequently. In four of these 26 instances, when setting up tasks, TAs told PTEs a conjecture (twice), empirical argument (once), or proof (once) before

PTEs began working on tasks that included opportunities for them to generate these conjectures and arguments themselves. During the other 22 instances, TAs generated the RP processes during implementation of the tasks.

There were three prominent ways TAs and PTEs interacted during implementation that characterized these instances: First, during group- or individual-work time PTEs stopped working on a task or asked TAs for guidance in generating conjectures or arguments when they experienced a struggle or confusion. During these interactions TAs were unsuccessful at guiding PTEs through their struggle without generating the RP process for PTEs. Sometimes TAs' responses to PTEs' questions were general (e.g., "just keep trying", Ace, observation 2, 53mins) and forward progress on the task did not continue. More often, however, TAs would tell PTEs an answer, where to find an example explanation in the textbook, or that they (the TA) would provide an answer for the class in a few minutes. Then, PTEs would read in the textbook or hear from the TA what an appropriate conjecture, proof, or rationale could be for the task, but PTEs did not experience the process of generating or evaluating the RP aspects themselves.

Second, TAs led the class through tasks by telling and showing PTEs how to complete them. In these situations, PTEs often did not have time to think about tasks (individually or with others) prior to this instruction. During such TA-led discussions, PTEs sometimes provided brief answers about why approaches shown made sense, but the overall thinking and reasoning was generated by TAs. PTEs primarily listened, watched, and took notes as TAs generated and explained mathematical relationships, conjectures, and arguments. Here, again, PTEs were exposed to ideas about RP and the content in the tasks, but this was regarded as a decrease in RP demands because PTEs did not participate in the processes of RP the task elicited.

Third, TAs attempted to elicit PTEs' ideas during class discussions of tasks but PTEs did not readily participate and TAs generated the RP processes. These discussions sometimes occurred after PTEs spent some time working on the task, but not always. To clarify, a class discussion was not considered to be a decrease in RP processes only because a TA generated a RP process without input from PTEs. If a majority of PTEs generated RP processes elicited in the task during individual- or group-work time but they did not share their ideas during a class discussion that followed the task, the implementation was still regarded as no change in RP demands. In those discussions, PTEs may not have voiced their ideas because they already understood them or TAs may have shown examples of the types of answers they had observed from PTEs' work. The class discussions that decreased RP demands, however, occurred when the work of figuring out RP processes associated with the task was still in progress. During these discussions, TAs often asked for volunteers to share their ideas for specific parts of the task. Many TAs also used substantial wait time to elicit these volunteers. When TAs were unable to elicit student participation, or if class time was running out during the discussion, TAs often chose to generate the RP processes. TAs initiated these class discussions in ways that suggested they wanted to facilitate a discussion and encourage PTEs to engage in the generation of RP processes. TAs' tendencies to eventually lead discussions and difficulty involving PTEs suggest a need for professional development addressing specific practices (Stein, Engle, Smith, & Hughes, 2008) for facilitating mathematical discussions around high cognitive demand tasks.

6.3 The Types of Reasoning-and-Proving Tasks Observed and How they were Enacted

Parsing these results to see how different types of RPTs were enacted provided insights into the nature of PTEs' opportunities to engage in RP in MTH 2. As introduced in the textbook analysis, RPTs were grouped according to the type of RP processes in which PTEs could engage. This grouping provided a means for examining what RP processes were elicited in written tasks (in Chapter 5) and whether PTEs engaged in those RP processes in class. The five general types of RPTs, introduced in Chapter 5 (Table 10), were: Mathematical Generalizations (MG), Mathematical Arguments-Nonproofs (MA-N), Mathematical Arguments-Proofs (MA-P), Generalizations and Arguments-Nonproofs (GA-N), and Generalizations and Arguments-Proofs (GA-P). The number of observed RPTs of each type is displayed in Table 12.

Table 12. Types of Reasoning-and-Proving Tasks Observed: Opportunities to Prove in the Tasks-as-Written

			Number of each Type of RPT Observed for each TA					
Type of RPT Task-As-Written	**Number of RPTs Observed**	**% of 82 RPTs**	**Ace**	**Laura**	**Isaac**	**Peter**	**Evan**	**Kelly**
MG	14	17.1%	2	3	3	1	3	2
MA-N	19	23.2%	3	5	4	4	2	1
MA-P	24	29.3%	3	3	5	4	2	7
GA-N	14	17.1%	2	1	2	4	3	2
GA-P	11	13.4%	3	1	0	3	3	1
TOTALS	82	100%	13	13	14	16	13	13

The type of tasks observed most frequently (MA-P) provided opportunities for PTEs to generate mathematical arguments including at least one question that expected PTEs to generate a proof (Table 12). Considering the GA-P tasks along with MA-P tasks, approximately 43% of RPTs observed explicitly asked PTEs to generate proofs. Examining how these RPTs were enacted could reveal whether or not TAs were more likely to generate RP processes for PTEs when the task called for a proof. Furthermore, approximately 40% of the RPTs observed did not ask PTEs to generate a proof, eliciting nonproof (or informal) arguments instead (in MA-N and GA-N tasks). For these tasks, did TAs press PTEs to produce a proof, thereby increasing the RP demands and emphasizing what constitutes a proof? The examination of the enactment of the 14 MG tasks observed could also provide insights into if TAs often pressed PTEs to generate a rationale or proof when the written task did not.

Table 13. Enactment of Different Types of Reasoning-and-Proving Tasks

	Enactment of Tasks				
Type of RPT	**No Change**	**Increase**	**Shift**	**Decrease (2): TA Generated**	**Decrease (1): RP process Excluded**
MG (14)	4	3	1	3	3
MA-N (19)	7	0	0	4	8
MA-P (24)	9	2	3	10	0
GA-N (14)	4	2	1	5	2
GA-P (11)	3	0	1	4	3
Totals (82)	27	7	6	26	16

Examining how often TAs maintained or changed the RP demands of RPTs provides insights into the nature of opportunities for PTEs to engage in RP in MTH 2. These results are presented (Table 13), from which three notable patterns are highlighted: First, consider the 27 RPTs enacted with no change to RP demands and six RPTs that were shifted during enactment. These enactments were reasonably spread out among the five different types of tasks. Tasks that were shifted meant that PTEs engaged in the RP processes that were elicited in the task, but in different ways or around slightly different content than what was expected from the written task.

Consider, for instance, a GA-N task with three questions about determining volumes of prisms and cylinders. In the first two questions, PTEs were asked to investigate mathematical relationships and generate a conjecture about a formula for the volume of triangular prisms and parallelogram-base prisms. The third question provided similar prompts for a formula for the volume of cylinders but also asked PTEs to justify (informally) why their formula was valid. During enactment, however, Evan asked PTEs to explain why the formula for triangular prisms had to be true, and he did not ask them to justify the formula for cylinders. PTEs still investigated, generated conjectures, and generated a rationale, but the rationale was for a different conjecture. In general, the way that these shifts and instances of no change in RP demands were spread out across different types of RPTs suggests that PTEs engaged in a range of RP processes during MTH 2. For various types of tasks (eliciting generalizations, informal arguments, or formal arguments) there were a number of observations during which PTEs were asked to engage in these RP processes and they did so during implementation of these tasks.

Second, TAs did not increase RP demands of MA-N tasks during classroom instruction (Table 13). These tasks typically asked PTEs to provide nonproof justifications for mathematical claims, such as explaining why a right pyramid that is 1 unit high and has a 1-unit-by-1-unit square base has volume 1/3 cubic units. In some MA-N tasks, the textbook specifically stated that a general proof for the mathematical relationship being explored was outside the scope of the class, but encouraged PTEs to provide nonproof arguments or demonstrations to develop intuitive notions about mathematical relationships without providing a formal proof. In the textbook there were also statements about a proof being outside the scope of the class for some of these types of tasks. When these tasks were observed, TAs did not press PTEs to provide a proof. This instructional decision could affect PTEs' understanding of what constitutes a proof and perpetuate potential misconceptions that favor empirical arguments over proofs.

Third, when RP demands for MA-P tasks were decreased, it was only because TAs generated the proofs for PTEs (Table 13). This type of decrease in RP demands was prevalent across most of the types of tasks, but it is worth noting that it was most prominent for the MA-P tasks. For these tasks, the primary RP focus was on eliciting proofs from PTEs, but during implementation when there was a decrease it was because the structure, main ideas, and overall creation came from TAs with little or no student input. There were also no instances observed where a TA generated the proof and the class had an open discussion to evaluate aspects of the provided argument. Instead, the provided proof was regarded as a product to remember and PTEs occasionally asked clarifying questions about why TAs included particular steps or reasons. This choice could affect PTEs' understanding of the processes of RP and the development of their own abilities to construct and evaluate mathematical arguments.

6.4 Examining each TA's Enactment of Reasoning-and-Proving Tasks

Since RPTs were observed in six classrooms, individual TA's classroom practices were studied for patterns in how RP demands were modified during enactment. In Figure 12 and Figure 13 the results are presented for the observations of each TA's classroom across the semester. Each circle in the figures represents one RPT observed. The placement of the circle indicates when it was observed during the semester (the week of the semester) and how it was enacted (RP demands decreased, not changed, shifted, or increased). The opportunities for RP elicited by the task-as-written are indicated by the color. As a reminder, no change or shifted means PTEs engaged in the types of RP processes elicited by that task. Increased means PTEs engaged in more RP processes than what was called for in the written task. For either type of decrease, PTEs did not engage in at least one RP process elicited by that task. From these figures, I note patterns in individual TA's enactments of RPTs across the semester (esp., Ace and Isaac) and of specific types of RPTs (esp., Kelly and Peter).

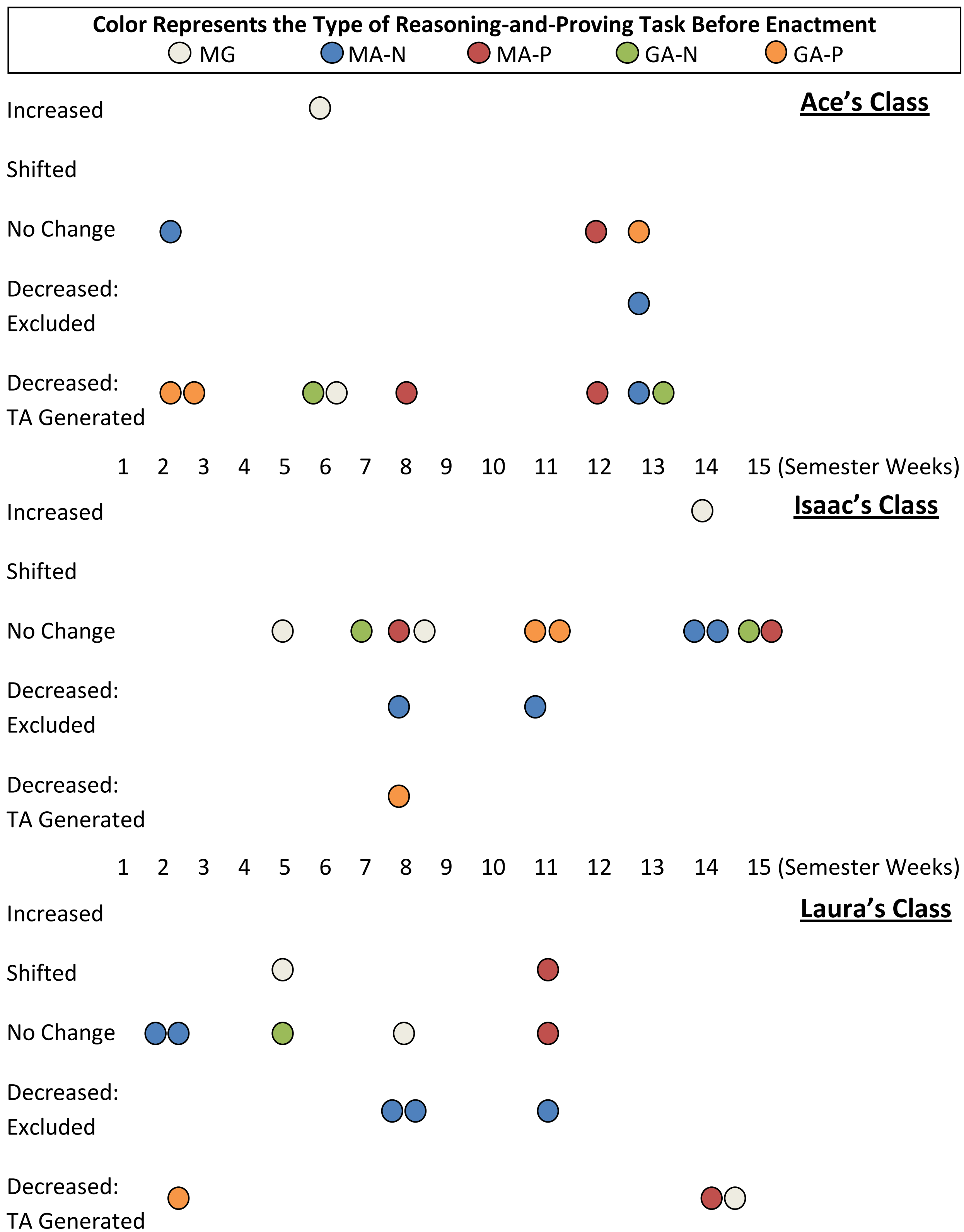


Figure 12. Individual TA's enactments of RPTs. Each circle represents one task observed during a specific week of the semester: Ace, Isaac, and Laura

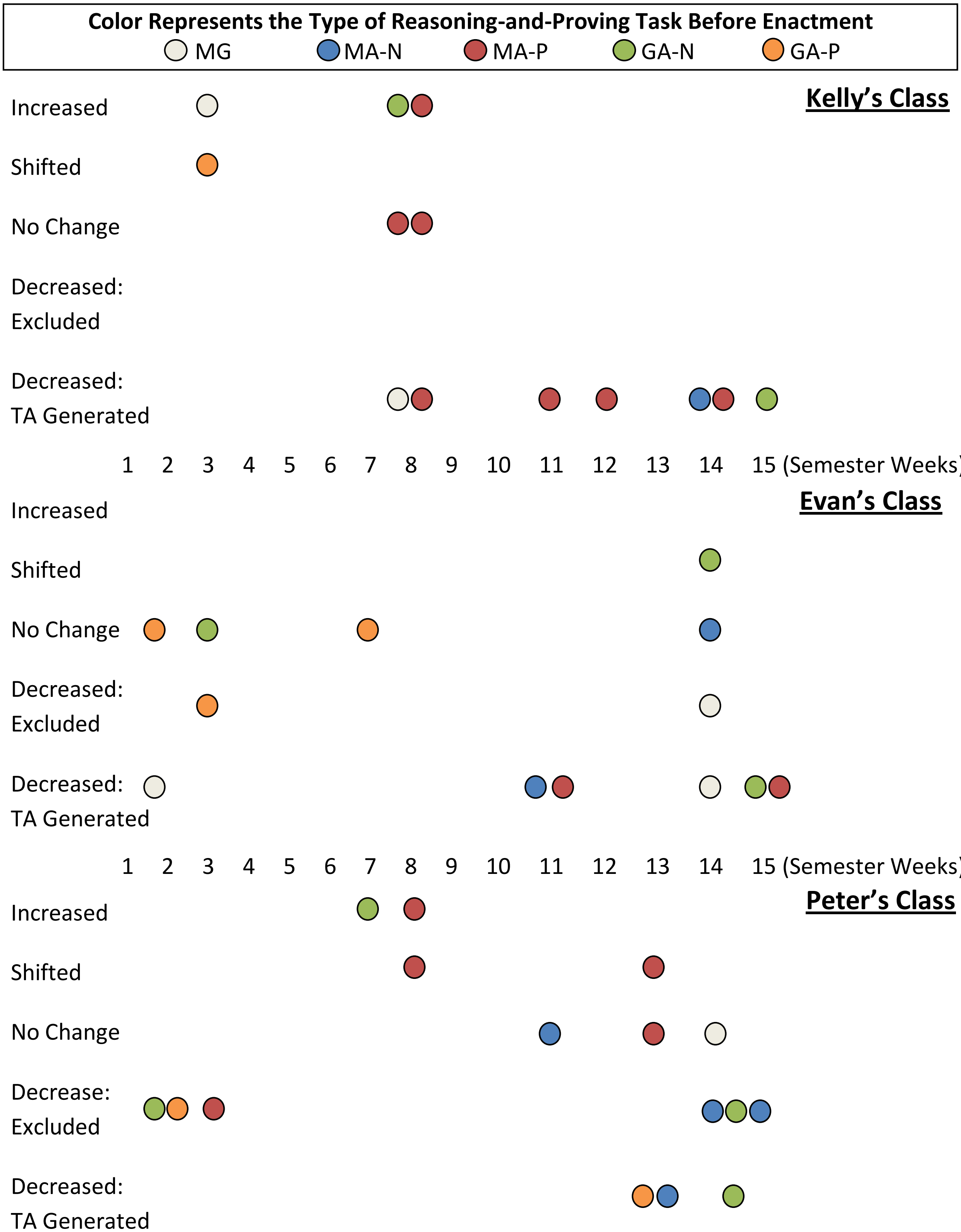


Figure 13. Individual TA's enactments of RPTs. Each circle represents one task observed during a specific week of the semester: Kelly, Evan, and Peter

6.4.1 Individual TA's Enactment of Reasoning-and-Proving Tasks across the Semester

Considering each TA's enactment of RPTs across the semester indicates some noticeable trends in the enactments of RPTs. As presented in Figure 12, Ace and Isaac's patterns of enactment across the semester were quite different. Ace had a tendency to decrease the RP demands of all types of RPTs throughout the semester. As PTEs worked on tasks, Ace would say things to encourage them to keep working, but in the same sentence remind them that in a few minutes he was planning on pulling the class back together to tell them what they should have figured out. PTEs often responded to these comments by slowing or stopping their own progress on the task. Then, during whole-class discussions, Ace would often generate RP processes for PTEs with little or no student input. For many of the observed RPTs, therefore, PTEs in Ace's class did not work on the RP processes themselves. In Isaac's class, however, he tended to enact most of the RPTs without changing the RP demands throughout the semester. PTEs in Isaac's class were observed engaging in RP processes across all types of RPTs throughout the semester. These differences in enactment patterns show variation between TAs that has implications for PTEs' opportunities for learning about RP.

The noticeable patterns from Laura's class (Figure 12) are similar to Isaac's toward the beginning of the semester because she tended to shift or not change the RP demands of tasks. Toward the end of the semester, however, Laura began excluding or generating RP processes for PTEs more often. Kelly's and Evan's enactments of RPTs across the semester (Figure 13) show this type of trend in a little more pronounced way, as they tended to generate RP processes for PTEs more from the middle through the end of the semester. These patterns suggest that there may be some relationship between the week of the semester and some TAs' tendencies to decrease RP demands of tasks. Providing instructional support for TAs' teaching throughout a semester is therefore important because at different points there could be certain content, external time constraints (i.e., staying on pace with the syllabus or demands on TAs' time in their own courses and research), or classroom interactions and dynamics, through which TAs may need help navigating so they do not decrease the opportunities for PTEs to learn about RP.

Peter's enactments of RPTs do not seem to be strongly correlated with time, but a majority of the tasks observed implemented by Peter were decreased by excluding RP processes. A tendency for this type of decrease was not realized in the other TAs' data, and could suggest that Peter was comfortable modifying tasks and that he often did so in a way that decreased the RP processes with which PTEs were expected to engage.

Considering these general patterns for the six TAs suggests avenues for continued research into teachers' instructional decisions about RP. In particular, all six TAs used the same textbook, which emphasized nonproof arguments and provided a range of opportunities for PTEs to engage in RP processes. Despite this commonality, TAs enacted RPTs differently. This variance suggests the importance of understanding how teachers make instructional decisions and helping them develop abilities to understand and modify tasks and instructional resources. Investigating what factors contributed to TAs' enactment of RPTs could inform the design of courses and support of instructors for such courses, and is explored further in the next chapter.

6.4.2 How Individual TAs Enacted Different Types of Reasoning-and-Proving Tasks

Another pattern evident in individual TA's enactments of RPTs was how they enacted different types of RPTs. Since different numbers of the types of tasks were observed for TAs, the data did not indicate specific patterns along this dimension for each participant (Figures 12 & 13). Kelly's enactment of MA-P tasks (Figure 13), Laura's enactment of MA-N tasks (Figure 12), and Peter's enactment of both of these types of tasks (Figure 13), in particular, provide some additional insights into PTEs' opportunities to learn about RP in these classes.

In Kelly's class, tasks that provided opportunities for PTEs to generate proofs for provided mathematical statements (MA-P tasks) were observed most frequently. During week eight, Kelly enacted four MA-P tasks: for two of these tasks PTEs generated proofs as elicited in the task; for the third task Kelly pushed PTEs farther in RP by asking them to investigate mathematical relationships, generate conjectures, and then prove them instead of proving a provided statement; and for the fourth task Kelly generated a proof that the opposite sides of a parallelogram were congruent when PTEs struggled to generate the proof. After week eight, Kelly enacted MA-P tasks by generating proofs for PTEs during class discussions. These class discussions occurred after PTEs had worked on the tasks in small groups, but had made little progress on the task beyond listing some given information or definitions. Kelly often encouraged the class to work together during the discussion to construct the proof, but PTEs were not vocal during the creation of these proofs. As the semester continued they explicitly asked Kelly to just show them the proof because they wanted to see how she would write it instead of trying to generate the proof themselves. The PTEs seemed to value the end result of these tasks, the proof that they asked Kelly to generate for them, instead of the process of reasoning and proving.

In Laura's class, as the semester continued, she tended to decrease the RP demands of MA-N tasks by excluding the rationales the tasks elicited (Figure 12). In particular, MA-N tasks observed during weeks eight and eleven were designed to elicit rationales. In this study, rationales were conceptualized as valid arguments for or against mathematical claims that did not constitute formal proofs. When Laura enacted these tasks, however, PTEs tended to restate what mathematical claims meant or state theorems or facts they thought might be related to mathematical claims without explaining why claims were valid. For example, a task asked PTEs to evaluate an argument and then explain why the provided argument was or was not valid. The PTEs quickly noticed that a statement in the argument (that one could scale a poster additively instead of multiplicatively) was mathematically incorrect, but they did not provide justifications for why other than saying “that’s not the right way to think about scaling.” Laura did not press PTEs to explain why the additive method was incorrect or note any other features of the argument that were problematic. In this manner, Laura tended to not help PTEs refine their understanding of rationales when she enacted MA-N tasks, which could reinforce PTEs’ misconceptions about what constitutes a proof, nonproof, and valid argument in mathematics and in teaching.

Peter displayed an overall pattern dissimilar from Kelly because he increased or did not change RP demands of MA-P tasks, but similar to Laura because he decreased RP demands of MA-N tasks (Figure 13). For the MA-N tasks where he decreased RP demands, however, he did not emphasize informal explanations as Laura did. Instead, Peter excluded parts of tasks in such a way that he took away opportunities for PTEs to provide rationales. That is, when Peter excluded RP processes from MA-N tasks he noted that they had worked on some of the concepts for the tasks earlier in the week (when I was not observing) and he focused their attention on the application questions in the task. He therefore diverted PTEs’ attention away from the rationale-related questions, but contended that PTEs had talked about these ideas in previous class periods. Since Peter did not make this pedagogical choice explicit to PTEs, the way RP aspects of these tasks were excluded could affect how PTEs think about tasks that ask for informal justifications.

Peter enacted MA-P tasks differently from Kelly in that he enacted all but one of the observed MA-P tasks by engaging PTEs in the RP demands of the tasks or increasing the RP demands of the tasks. When Peter increased the RP demands of an MA-P task, he prompted PTEs to generate conjectures and refine their conjectures by learning from failed attempts to prove they were true. Since Peter enacted these types of tasks in these ways, PTEs in his class often generated proofs and engaged a proving process through which they tested and refined conjectures and arguments during observations of his teaching. By examining how Peter enacted specific types of tasks, an emphasis on engaging PTEs in generating and refining proofs emerges as a prevalent theme observed in his practice.

6.5 Conclusions about the Enactment of Reasoning-and-Proving Tasks

These results from analyzing six TAs' enactment of 82 RPTs during the semester indicate that PTEs engaged in a range of RP processes for about 49% of the observed tasks because TAs maintained, increased, or shifted RP demands. Increases in RP demands occurred for three types of tasks: (1) MG tasks, where PTEs provided justifications for generalizations when the written task did not call for justifications; (2) MA-P tasks, where PTEs investigated mathematical relationships and generated conjectures for tasks that only asked them to generate mathematical arguments, including a proof; and (3) GA-N tasks, where PTEs generated proofs when the task-as-written only pressed for rationales. Although these were the only types of tasks where increases in RP demands were observed, other types of increases were possible. For MA-N tasks, the task-as-written provided a statement for which PTEs were asked to generate informal justifications. To increase RP demands of MA-N tasks, for instance, one could ask PTEs to investigate mathematical relationships and generate conjectures originally provided in tasks. Such modifications could help PTEs generate justifications based on their investigations into mathematical relationships and experience the process of generating and refining conjectures. The lack of increases to RP demands for MA-N tasks suggests the need for helping TAs understand why and how to modify tasks to help PTEs engage in RP processes.

For the other 51% of observed tasks, TAs decreased RP demands by either excluding aspects of tasks that elicited some RP processes or generating RP processes for PTEs. These different types of decreases in RP demands suggest opportunities for supporting instructors who are teaching about RP. When TAs did not consistently hold PTEs accountable for RP processes in tasks (as was described with Laura's enactment of MA-N tasks), PTEs' misconceptions of RP (e.g., what constitutes a mathematical argument) may have been reinforced. Reinforcing PTEs' misconceptions about RP in MTH 2 could lead to less authentic RP experiences in their future elementary classrooms. TAs who tend to decrease RP demands in this manner may need help understanding how their instruction around tasks can affect PTEs' opportunities to learn about RP. A potentially applicable professional development strategy could include specific supports about designing and implementing instruction in ways that hold students accountable for RP processes in tasks.

For the other type of decrease, when TAs generated RP processes for PTEs, they did so as PTEs worked on tasks individually, in groups, or during whole-group discussions. Their interactions with PTEs during individual- or group-work did not help PTEs work through difficulties to experiences success with RP processes. These interactions suggest a need for helping TAs develop specific teaching practices to monitor PTEs' work to help them to continue experiencing RP processes. TAs also decreased RP demands by generating RP processes for PTEs during whole-group discussions. In these situations TAs often had difficulties facilitating class discussions around RPTs in ways that encouraged PTEs to participate and develop their own understanding of RP processes. One noticeable factor common to both types of interactions (TA-to-small group and TA-to-class) was a sense that PTEs tended to be frustrated or unmotivated. They either did not stopped making forward progress on RPTs or they did not readily participate and engage with RP during class discussions.

A better understanding of why PTEs responded in these ways is outside the scope of this study. It could be motivating to PTEs, however, if TAs explicitly articulated the RP aspects they wanted PTEs to experience in working through tasks and how the RP processes could prepare them as future teachers. The RP-framework is one means for helping TAs identify RPTs and develop TAs' language for talking about RP processes during instruction so they can communicate the importance of PTEs engaging in and experiencing these processes in MTH 2.

These results also highlighted how TAs implemented RPTs differently. This variation suggests that using a common textbook with different opportunities for PTEs to engage in RP processes does not mean that PTEs across the course sections will have the same opportunities to experience RP. The different course sections have PTEs with their own experiences with and views of RP, and TAs approach the course, interpret the text, and plan and implement instruction based on their own conceptions of RP, contributing to the variation.

CHAPTER 7

TEACHING ASSISTANTS' INSTRUCTIONAL DECISIONS

This chapter presents the results from examining the relationship between TAs' conceptions of RP and their instructional decisions when teaching PTEs about RP. As introduced in Chapter 1, the study of TAs' instructional decisions is complex due to a large number of variables affecting their decisions. In the context of this study, I examined a subset of these variables: To focus on TAs' RP classroom practices and conceptions, I examined their enactment of RP tasks, conceptions of RP, selection of RPTs, and classroom interactions (Figure 14).

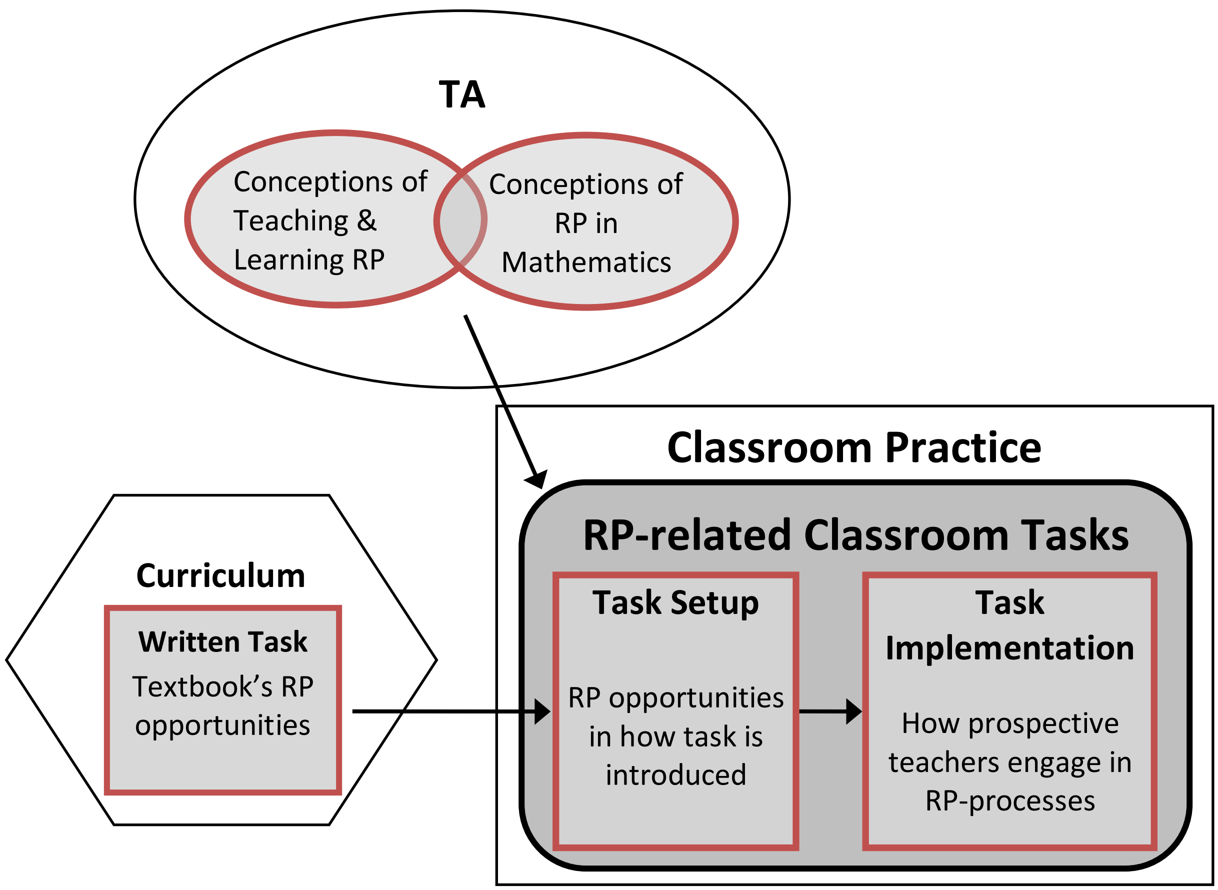


Figure 14. Major components of the study

Considering TAs' decisions directly impacting PTEs' opportunities to learn about RP, I studied two types of decisions: (a) TAs' selection of RPTs and (b) TAs' in-the-moment decisions around

RP during instruction. Through this analysis, I investigated how TAs talked about prominent factors that influenced these decisions and examined the relationship between TAs' conceptions of RP and their instructional decisions. These results inform specific professional development supports for TAs teaching about RP.

Using the constant comparative method (Glaser & Strauss, 1967), I analyzed TAs' post-observation interviews for how they talked about their pedagogical choices after enacting RPTs. I examined two broad aspects: First, how did TAs explain their choices to select observed RPTs for classroom instruction? Second, how did TAs talk about their decisions to implement RPTs in the ways observed? That is, after classroom observations, TAs responded to questions about why they enacted particular RPTs and how they made classroom decisions during enactment. Results indicate how TAs' conceptions of teaching RP and other contextual factors informed their selection and enactment of tasks. The case studies of TAs' conceptions of RP (Chapter 4) and patterns in TAs' enactment of RPTs (Chapter 6) provide a lens for interpreting these results and suggest specific professional development implications. The results about TAs' selection of RPTs are described first, followed by the analysis of their in-the-moment instructional decisions.

7.1 TAs' Decisions to Teach Reasoning-and-Proving Tasks

To understand how TAs talked about their RPT selection and how their conceptions of RP relate to these decisions, the types of responses are described first, followed by the results.

7.1.1 How TAs Articulated their Selection of Reasoning-and-Proving Tasks

TAs talked about their choices for teaching PTEs RPTs by articulating different learning goals related to RP and other contextual factors (Table 14). Their statements detailed how they aimed to use tasks to help PTEs: (a) learn what proof is and how to write proofs, (b) understand mathematics content through their use of proof and proving, or (c) understand RP processes by experiencing them in class. They also talked about contextual factors (i.e., other people, resources, or assessments) as exerting some authority over their selection of tasks.

Table 14. Categories for How TAs Explained Why they Used Reasoning-and-Proving Tasks

Learning Goals Related to Content	**A) <u>Learning about the Nature of Proof and Proving</u>** • Know *what a proof is* • Produce *clear, concise explanations* and *proofs* • Understand *proving conventions* • Learn a *specific proof* or *way to prove* a mathematical statement • Understand the *generality* of proofs **B) <u>Understanding Mathematics Content:</u>** The mathematical task helps PTEs: • Understand mathematical relationships in the task • Make real world connections or solve application problems • Understand mathematics content better o Recognize the importance of proof in mathematics o Understand content needed for teaching o Prepare PTEs to respond to future students if they ask "why"
Learning Goals Related to Processes	**C) <u>Engaging in and Developing Understanding of RP processes:</u>** The mathematical task helps PTEs • Generate conjectures • Evaluate conjectures • Evaluate provided arguments • *Reason through* how to construct proofs • *Explain why* mathematical relationships are true • Develop *mathematical ways of thinking* (e.g., critical reasoning, critical thinking, mathematical thinking)
Contextual Factors	**D) <u>Outside Authorities</u>** • Course supervisor, other TAs for MTH 2, or course meetings • Textbook, syllabus, or pacing guide • Assessments (e.g., in-class tests, homework, quizzes, and final exam)

These categories emerged from the data and were characterized in four ways: First, TAs described features or characteristics of proof they wanted PTEs to learn through their work on the task (Category A). Ace, for example, talked about his selection of a task that asked PTEs to prove the Pythagorean Theorem by explaining that he wanted them to learn "what a proof is or what a proof should be" (post-observation 4, para. 86). Laura also described why she asked PTEs to work on the same task, as follows:

> In general, I want them to be concise. . . . They always write much more than they need to and I have told them several times, "I just need the information that I need. Like, I don't need this whole paragraph about all this other stuff." And so I think that it's important to emphasize being concise because I think by nature mathematicians are more concise . . . and so I think that it's a good thing for them to keep in mind when they are trying to do a proof. (post-observation 4, para. 54)

In these ways, TAs talked about RP content they wanted PTEs to learn (e.g., a specific proof or what constitutes a proof) to explain why they used an RPT in their teaching.

Second, instead of specifying RP as a content learning goal for RPTs TAs selected, they described how RP in tasks helped PTEs understand other mathematical content (Category B). Isaac, for example, chose a task where PTEs explored and justified conjectures about the area of objects with a fixed perimeter and explained that by working through the task PTEs should "remember that . . . given a certain perimeter you can't say what the area is. . . . And . . . rectangles with the biggest area is (*sic*) a square and for arbitrary shapes the biggest area is a circle" (post-observation 5, para. 24). Isaac's goal was for PTEs to understand and retain information about relationships about perimeter and area by engaging in RP in the task. In this category, TAs sometimes also explained how RPTs could help prepare PTEs with content knowledge they would need as future elementary teachers. Peter, for instance, said PTEs need to generate rationales for the area formula for a parallelogram,

> because they will be teachers and they will show their students to do that kind of thing; not just memorize the formula but they can do a lot of different things related with that formula. That's what I wanted to teach today. (post-observation 5, para. 10)

As these examples depict, TAs' statements coded as Category B described content PTEs should learn as mathematics learners or explicitly as future teachers. In either case, however, they articulated RP opportunities in tasks as helping PTEs learn content.

Third, TAs talked less about content and more about RP processes or ways of thinking the task could help PTEs develop (Category C). TAs made general statements about RP processes they wanted PTEs to learn as important in mathematics, and sometimes specified how PTEs could use certain ways of thinking in their future classrooms. For a task where students investigated the validity of and generated justifications for triangle congruence criteria, for example, Peter explained, in general, that "this activity is important for all students in mathematics, not just for . . . future elementary school teachers because this . . . justification or questioning ability is kind of core idea of mathematics" (post-observation 3, para. 136). Building off of this idea, that the ability to think critically was important for his students to develop, Peter also specified how PTEs may apply this ability as future teachers, saying,

> as a teacher . . . if [they] have critical thinking ability then [they] can actually . . . question definitions and information in the textbook and . . . help . . . their future students to question and think mathematics, too. (post-observation 3, para. 140)

In these types of responses, TAs talked about how RPTs prepared PTEs to understand RP processes and develop mathematical ways of thinking.

Fourth, TAs talked about contextual factors by noting other people, resources, or assessments that motivated their decisions to enact tasks (Category D). Statements about contextual factors included: the course supervisor "told us actually we have to have this one" (Kelly, post-observation 2, para. 88), "I have a piece of paper [referring to the pacing guide] that says, 'Cover these activities'" (Ace, post-observation 5, para. 33), and "so that they can know . .

. for the exam . . . how to do this or . . . a similar proof" (Laura, post-observation 4, para. 24). TAs referred to interactions with the course supervisor, use of the text and syllabus, or the importance of preparing PTEs for assessments as contextual factors guiding task selection.

7.1.2 Analysis of TAs' Responses about Selecting Reasoning-and-Proving Tasks

This analysis aimed to understand how TAs' conceptions of RP illuminated their decision to enact RPTs. The analysis of TAs' post-observation interview responses yielded the four aforementioned response types, displayed for individual TAs in Figure 15; the percentage indicates the frequency for each type of response out of the TA's total number of coded statements.

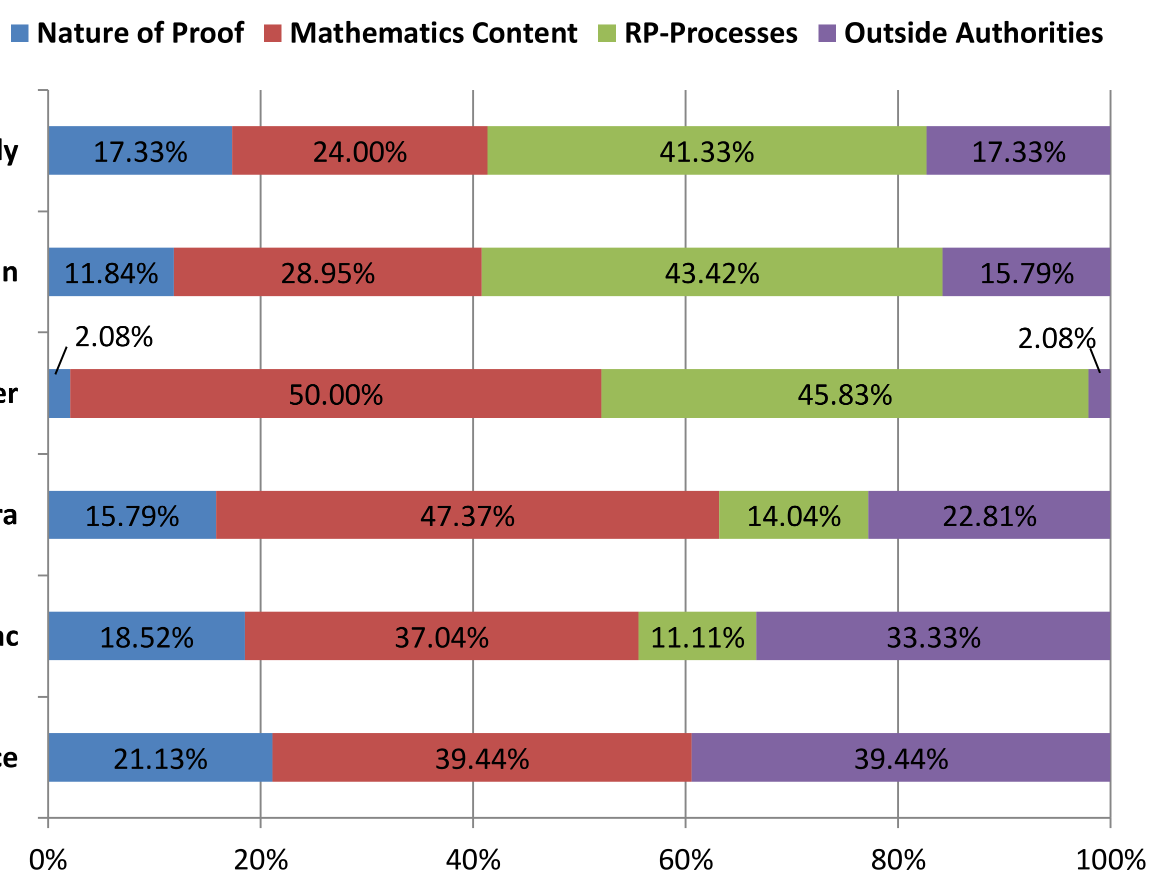


Figure 15. How TAs talked about their decisions to enact reasoning-and-proving tasks For each TA, percentages are out of their total number of coded statements

Kelly's, Evan's, and Peter's talk about their selections of RPTs aligned with their conceptions of teaching RP (Chapter 4) because they articulated RP process learning goals more frequently than the other three TAs (Figure 15). Although they also talked about mathematical content as important to teach PTEs, they articulated RP processes as reasons why they selected RPTs more often than the other TAs. These three TAs talked about RP processes in their own ways and in manners similar to how they articulated their conceptions of teaching RP: Peter, for example, talked about selecting tasks that would provide opportunities for PTEs to construct and refine mathematical arguments as mathematicians do; explaining,

> That's how . . . you do proofs in mathematics actually. . . . So you have some statement you want to prove and out of nothing you have to construct step-by-step proof. . . . You can go into the wrong way and then maybe you can figure out what's wrong and you can come back to the original idea or through a different thinking you can take a totally different route but always you should have a bunch of these different thinking in your mind together and . . . you maybe connect different segments together to get the whole idea. So that's how you build your proof mathematically. . . . So . . . if you really wanted to study proof then you need to experience that kind of knowledge not just memorizing the step-by-step procedures. You need to actually do this fixing and correcting type of activity in your mind to learn proof. (post-observation 3, para. 82, 84, 86)

Kelly and Evan continued to focus on helping PTEs develop their ability to think mathematically, beyond the specific content in MTH 2. Evan explained his selection of tasks to help PTEs work

> on that line of mathematical thinking—coming up with proofs for something; taking knowledge you've applied out of the book and theorems and applying them to a novel situation where you're trying to prove something else and build on the mathematics you already know. (post-observation 3, para. 30)

Kelly explained that "it's not the content . . . I feel the most important thing is that [PTEs] need to develop their ability to justify things mathematically" (post-observation 1, para. 122) and

> I want to accomplish this goal: "how to build on the old knowledge and develop the new knowledge." This kind of thinking is similar to the mathematical- like- we use the known to prove the unknown. As a teacher [they] have to have a good idea of what the

> students know and what [they] want to build around those things. So . . . the kind of mathematical thinking may transfer to their teaching practice. (Kelly, post-observation 2, para. 136)

Since PTEs will be expected to teach future elementary students how to think and reason mathematically, these three TAs considered instruction about RP processes as an important factor in their decision to teach RPTs. This conception about teaching PTEs about RP processes was apparent both in their conceptions of teaching RP (Chapter 4) and their responses about their selection of RPTs. In these responses, Peter, Kelly, and Evan were able to articulate how certain tasks could help PTEs learn RP processes, suggesting strong pedagogical knowledge for teaching to be able to analyze textbook tasks for particular instructional goals.

In the earlier analyses of Laura's and Isaac's conceptions of teaching RP and this analysis of their selection of RPTs, they sometimes articulated RP processes, but focused on mathematical content most often (Figure 15). Isaac occasionally talked about selecting tasks that provided opportunities for PTEs to generate and evaluate conjectures (post-observation 3 and 5). Laura sometimes explained how a task "challenges [PTEs] to think critically" (post-observation 4, para. 24). Most of their talk about the selection of RPTs, however, attended to mathematical content (including concise explanations for Laura and understanding specific mathematical relationships for Isaac) they wanted PTEs to understand. Their articulation of a range of learning goals suggests they have a foundation of understanding how tasks can support PTEs' learning of the nature of proof and mathematical content, but only occasionally made explicit connections to RP processes.

Ace, however, never talked about RP processes he wanted PTEs to learn from RPTs he chose to enact (Figure 15), which was also reflected in his conceptions of teaching RP (see

Figure 6). Ace also talked about outside authorities as often as he talked about mathematical content as influencing his selection of RPTs. When asked why he enacted a task where PTEs were expected to explain why there are only five platonic solids, for example, Ace said, “I think [the course supervisor] really likes this one. So it’s very possible that [the course supervisor] will like to see a question like this on both the exam and the final” (post-observation 2, para. 148). In addition to this idea that content in tasks could appear on assessments, Ace also explained how PTEs needed to know tasks’ content to be “smarter than [their] students and . . . a little bit ahead of the material that [they’re] actually teaching them” (post-observation 2, para. 162). Ace’s consistent talk about RP content and contextual factors suggests he has a strong content knowledge of mathematics and his enactment was heavily influenced by outside authorities.

This analysis shows alignment between TAs’ conceptions of teaching and learning RP and how they talked about task selection, suggesting a relationship between TAs’ conceptions of RP content and processes and their ability to select tasks for specific instructional goals. In addition, TAs who talked less often about RP process-related goals, talked more often about outside authorities influencing task selection. This relationship implies that TAs who did not explicitly articulate process-related goals for RPTs still enacted RPTs because they viewed the course supervisor, pacing guide, textbook, and assessments as supporting the decision to incorporate RPTs into their teaching. Their tendency to decrease RP demands during enactment, however, suggests that these outside factors influenced what was taught more than how it was taught. The selection of curriculum and course supervisor’s recommendations about what to teach seems to have encouraged the use of mathematically rich tasks, but more is needed to address TAs’ pedagogical needs, helping them develop PTEs’ understanding of RP.

7.2 TAs' In-the-Moment Instructional Decisions

TAs made decisions when orchestrating the enactment of RPTs in the classroom that maintained, shifted, increased, or decreased RP demands of tasks. As presented in Chapter 6, TAs occasionally increased RP demands by asking PTEs to generate conjectures, rationales, or proofs that were not explicitly called for in written tasks; sometimes maintained or shifted RP demands by holding PTEs accountable and pressing PTEs to engage with RP-aspects of the tasks; and often decreased RP demands by either excluding RP-aspects of tasks or generating RP processes for PTEs. To understand TAs' decisions to enact RPTs in these ways, during post-observation interviews, I asked TAs to talk about how or why they decided to respond to PTEs and PTEs' ideas about RP during class. The analysis of their responses is presented next, and it serves to investigate how TAs talked about their in-the-moment, pedagogical decisions. Then, to understand TAs' instructional decisions in light of their conceptions of RP, the cases introduced in Chapter 4 are revisited and used to interpret these results.

7.2.1 Analysis of TAs' Responses about In-the-Moment Instructional Decisions

Studying TAs' responses about their decisions to react to PTEs and PTEs' ideas about RP as observed provides insights into these (often quick) decisions TAs made throughout the lesson. TAs' talk about their interactions with PTEs focused on their responses to PTEs' questions and how they evaluated PTEs' their claims. TAs rely on components of *pedagogical content knowledge* (Shulman, 1986) which TAs' responses illuminate further.

There were two general themes particularly noticeable when TAs talked about instances where RP demands decreased: TAs talked about time constraints (how class was running out of time or that there was so much material to cover in so little time) and difficulties eliciting student participation in mathematical discussions. When talking about how time constraints influenced their decision to generate RP processes for PTEs, TAs explained "maybe if there was more time . . . for them to actually work on it they would have gotten it on their own" (Ace, post-observation 4, para. 24), "the time limitation is an important factor I consider" (Kelly, post-observation 3, para. 96), or "I knew we didn't have enough time" (Peter, post-observation 2, para. 50). All TAs talked about how time constraints drove their decisions, which then decreased RP opportunities for PTEs.

The TAs also all mentioned student participation during the enactment of RPTs either as something they did not feel skillful at eliciting or something they chose not to pursue because they were uncomfortable responding to PTEs' ideas that could be wrong or not fully formed. On the one hand, many TAs talked about wanting to encourage more student participation during mathematical discussions, but were dissatisfied with how that did not happen during the semester. Kelly and Evan were particularly vocal about a lack of student participation during discussions; they explained that they wanted PTEs to hear and critique their classmate's ideas, but that "people rarely volunteer in this class" (Evan, post-observation 2, para. 44) and they struggled to encourage "students to participate more" (Kelly, post-observation 4, para. 66).

To explain why he often chose not to engage the class in discussions, on the other hand, Ace said asking for PTEs' ideas presents the opportunity for

> someone [to get] it wrong. . . . And I don't wanna be like, "Well, no." I'm not very good at telling people, "Well, no maybe we could look for another idea." . . . It's a lot easier for me to . . . take charge of the class more. (post-observation 2, para. 84 - 90)

Evan and Peter also talked about the challenges associated with eliciting PTEs' ideas during mathematical discussions that may be incorrect or only partially formed. These TAs talked uneasily about “trying to decide on the spot if [a PTE’s justification] was a proof or not” (Evan, post-observation 3, para. 52). When faced with this challenge of understanding PTEs’ alternative ways of thinking on the spot, they often chose to generate a RP process for the class instead of figuring out if the idea raised by the student could potentially be correct. This notion, of responding to PTEs’ partially formed ideas is a component of pedagogical content knowledge (Shulman, 1986). Although these TAs understand the content taught in MTH 2 and understand what RP is in mathematics, their work as teachers also requires pedagogical skills and knowledge to develop mathematical concepts as they arise from PTEs’ statements. These TAs’ responses imply a need for support that builds their teaching practices, helping them facilitate mathematical discussions (Stein et al., 2008) in ways that engage PTEs in RP and develop PTEs’ understanding of mathematical RP.

7.2.2 TAs’ Conceptions of RP and their In-the-Moment Classroom Decisions

TAs’ interactive classroom decisions can be influenced by their conceptions of RP in mathematics, teaching, and learning, their selection of RPTs, interactions and constraints during enactment, and their interpretation of these classroom events (Figure 14). By considering the results discussed in the previous chapters, and earlier in this chapter, potential relationships between TAs’ conceptions and decisions are explored.

7.2.2.1 How TAs who Aimed to Prepare PTEs to Teach others about RP Processes Enacted RPTs

Attending to the dual nature of MTH 2, where PTEs need to both learn mathematics and be prepared to teach others about it, some TAs described learning goals about engaging PTEs in RP processes (Figure 6). Peter’s, Kelly’s, and Evan’s conceptions of teaching RP aligned with the goals and purposes of the course. They also talked about developing PTEs’ ability to think and reason mathematically when articulating their task selection. Their conceptions of teaching RP and learning goals suggest that they wanted PTEs to experience the process of RP and be prepared to help future elementary students engage in these processes, as well. Their enactment of RPTs in MTH 2, however, frequently resulted in decreases in RP demands (Figure 13). Peter decreased RP demands of tasks most often by excluding questions or parts of tasks related to RP, and he sometimes generated RP processes for PTEs. From the middle of the semester onward, Kelly and Evan tended to generate RP processes for PTEs. During post-observation interviews, after Peter excluded RP processes he often talked about time constraints and modifying the tasks to try to get back on pace with the course schedule. Although Peter articulated rich conceptions of RP and how RPTs support process-related goals, his enactment patterns suggest a need for support about modifying tasks in ways that do not decrease PTEs' opportunities to learn RP processes he intended to teach.

When generating RP processes for PTEs, these three TAs did so during class discussions, typically after PTEs had worked on RPTs, but a majority of PTEs had not generated or evaluated RP aspects of tasks. The TAs explained difficulties they experienced facilitating mathematical discussions in a way that would allow PTEs to talk about mathematics and engage in RP processes. Evan, for example, said why he wanted to have class discussions of RPTs and the challenges he experienced facilitating these discussions:

> I wanted them to explain it just so they'd get experience explaining it. They'd get experience hearing it from somebody else; seeing how it lines up with what they did. And the people who don't have it yet, they can see how it was done. . . . People tell me to just call on people and I hate cold calling on people because I tend to be a passive learner and I learn by hearing more than by speaking. And for those people, I appreciate that. . . . I've never BEEN in that situation before where they just- nobody wants to take responsibility and do anything during class discussion. (end-of-semester, para. 89, 91)

Kelly also talked about wanting PTEs to talk to one another about mathematics, but during the eleventh week of the semester PTEs specifically asked Kelly to tell them the proof because they wanted to see how she would write it instead of generating the proof themselves. As detailed in the previous section, Peter, Kelly, and Evan articulated their challenges facilitating class discussions including difficulties eliciting student participation and feeling unprepared to negotiate class discussions when PTEs shared informal or partially formed ideas. Mathematics education literature agrees that managing mathematical discussions in the classroom is challenging work for mathematics teachers (Speer & Wagner, 2009; Stein et al., 2008), which suggests a need for specific supports for TAs' pedagogical knowledge for teaching.

7.2.2.2 How a TA who Aimed to Prepare PTEs to Produce Explanations Enacted RPTs

A prominent aspect of many of the TAs' conceptions of the purposes of proof in teaching was *explanation*; they talked about proofs as helping students understand why mathematical statements are true instead of simply to verify (Table 8). Explaining why was also stressed in the goals of MTH 2's textbook and noticeable in the prevalence of tasks that elicited rationales (Figure 9). In line with this focus, explanation was a salient purpose of proof in teaching Laura talked about. She also articulated a goal of helping PTEs generate clear, concise, informal explanations for the validity of claims about general mathematical relationships. Laura's talk about informal explanations seemed aligned with how rationales were conceptualized in this study as valid arguments for or against mathematical claims that did not constitute formal proofs. In Laura's class, however, she tended to enact tasks that called for rationales in ways that did not hold PTEs accountable for generating rationales. During enactment, Laura accepted PTEs' responses that included restatements of mathematical claims or stated theorems or facts PTEs thought might be related to mathematical claims without pressing for rationales for why mathematical claims were valid.

Given the fact that many prospective and practicing teachers consider empirical arguments as proofs (Knuth, 2002a; Steele, 2006), PTEs need to understand the generality of proof and develop robust understandings of the roles of proof. Although developing PTEs' abilities to produce concise rationales could be one aspect of helping them understand the purposes and processes of RP, there is the potential to reinforce misconceptions if the instruction does not go beyond nonproof arguments. Furthermore, Laura's tendency to accept informal, invalid arguments as rationales could reinforce misconceptions PTEs may have about what constitutes a proof, nonproof, and valid arguments in mathematics and in mathematics teaching. Since Laura consistently expressed concern for preparing PTEs to teach others, but she did not talk about how her enactment of RPTs could affect PTEs' opportunities to learn RP, it is likely that she was not aware how these instructional decisions alter opportunities to learn. These results suggest that Laura had difficulty *noticing* (van Es & Sherin, 2002) classroom situations that affect PTEs' opportunities to learn (e.g., recognizing when her decisions to highlight particular responses can affect RP demands of tasks).

7.2.2.3 How a TA who Aimed to Help MTH 2 Students Understand Formal Proofs Enacted RPTs

Isaac tended to enact most of the observed RPTs without changing the RP demands throughout the semester, but interpreting this enactment pattern in light of Isaac's learning goals and reasons behind his instructional decisions has important pedagogical implications. Isaac's learning goals primarily stressed content MTH 2 students should know and understand as undergraduate mathematics learners; especially helping them understand that proofs are necessary in mathematics. He talked about how students needed to understand that math was "not just about guessing and assuming things," and that doing proofs is "the best" means to do so (beginning-of-semester, para. 106, 110). When articulating his selection of RPTs, Isaac also talked frequently about outside authorities, especially the textbook, explaining,

> I have some idea that we should follow the book. . . . I can't really choose how I want to do things. The only thing I can do is to follow the book otherwise maybe I just talk about things that they're not supposed to learn or something. (post-observation 2, para. 163)

Isaac's tendency to enact RPTs without changing RP demands aligns well with his focus on remaining faithful to the text. When Isaac increased RP demands for one observed RPT, he asked PTEs to justify a conjecture they generated even though the task did not elicit rationales or proofs, aligning with Isaac's focus on helping PTEs understand the importance of proofs. When asked about how he made his instructional decisions, however, Isaac had the most difficulty out of all the TAs articulating his pedagogical choices. He made statements about how he made in-the-moment decisions, saying they were "just something spontaneous- I mean- I didn't think about it too much" (post-observation 3, para. 72). He expressed uncertainty about how the material from MTH 2 prepared PTEs to teach others about RP. That is, Isaac only talked about the relationship between PTEs' future teaching and what they learn in MTH 2 when explicitly asked in interviews. In his responses, he also used phrases such as *maybe* and *I don't know* while articulating RP content PTEs could use in their teaching.

Isaac's responses about his instructional decisions have particular implications for the opportunities for PTEs to develop the ability to articulate RP processes to teach others. Although RPTs in Isaac's class were enacted in ways that maintained RP demands more often than in other TAs' classes, rationales behind Isaac's instructional decisions were not readily expressed during interviews; suggesting that they also likely remained implicit to PTEs during instruction. Isaac's hesitancy in responding about how RP instruction in MTH 2 prepares PTEs to teach about RP also suggests that these connections remained implicit during class. Although PTEs in Isaac's class engaged in RP processes during many of the observed tasks, it is unclear to what extent they could recognize and articulate the work they did as RP. Due to the lack of explicitness regarding RP, PTEs may be less likely to associate RP in MTH 2 with the elementary classroom context; thereby, reducing the probability that PTEs will intentionally incorporate RP processes into their future teaching.

7.2.2.4 How a TA who Aimed to Help MTH 2 Students Learn RP Content Using Lectures Enacted RPTs

The importance of encouraging TAs to become critically reflective practitioners is particularly apparent when considering Ace's conceptions of RP and instructional practices around RP. Ace's conceptions of teaching and learning RP differed from the other TAs in how he focused on the efficiency of using lectures and did not talk about RP processes. Ace's assertion that lecture-based instruction can be more efficient than activity-based is valid. His focus on lecture-based instruction and RP content, however, differed from the particular pedagogical stance supported by the design and goals of MTH 2. Thus, for a majority of RPTs observed in Ace's classroom, RP demands were decreased. He frequently generated RP processes for PTEs, sometimes while PTEs worked on tasks in small groups or other times before they worked on a task by first providing a lecture about the material.

In contrast with Ace's conceptions of teaching RP, he still chose to implement RPTs by often asking PTEs to work in groups and talk with one another about the mathematics. He explained this decision by referring to how the structure of the course and textbook were designed for the use of mathematical tasks, which he called *activities*, *group activities*, or *class work*, saying,

> In this class there was obviously time built in so that they can do these group activities. Otherwise I'd be finishing way earlier than everyone else [the other TAs] and I'm sure [the course supervisor] would yell at me and say, "You should have them do activities every once in a while." So I mean if they're going into teaching this material then it's important that they actually understand it. So of course that's the reason why they built in time to this so that you have a better understanding so that when you present it to your students in the future that you will hopefully actually know it in comparison to something that you've memorized. (post-observation 2, para. 176)

When Ace interacted with PTEs as they worked, however, he would often tell PTEs an answer or remind PTEs he would provide them with the final answer in a few minutes. During class discussions of tasks after PTEs worked on them, Ace primarily lead the class in a lecture-style format through the major aspects of tasks. In these ways, Ace interacted with PTEs as the one who "gives [the PTEs] knowledge" (Ace, end-of-semester, para. 30).

Later in the semester, Ace reflected on these interactions with PTEs, explaining that he wanted to learn how to become better at engaging PTEs in mathematical tasks. During a post-observation interview in week 12, Ace expressed dissatisfaction with PTEs' interactions with mathematical tasks, explaining,

> For a lot of their activities they just wait for me to give the answers at the end of them. . . . As soon as activities come out they get out their phones . . . and they just [think], "well, who cares he always gives us the answers to the activities. I don't have to work through it. I can just hear the answers at the end. There's no sense in me trying." . . . Today, I kind of wanted to . . . motivate them to actually try on the activities . . . rather than just to sit back and wait for me to give them the answers. (post-observation 4, para. 6, 8)

Ace was, therefore, aware that PTEs were not engaging in tasks, and he addressed this concern during that class by telling the class they would be expected to do a proof of the Pythagorean

Theorem on the next exam and on the final before they started to work on the task. Ace did not maintain this reflective posture or express further dissatisfaction later in the semester, however. His tendency to telegraph what was going to happen and focus on PTEs learning RP content seemed to work against the course goals for PTEs to engage in RP and explain why mathematical ideas make sense (goals which Ace also pontificated occasionally later in the semester). Ace talked about his instructional decisions by focusing on content objectives with little consideration of potential mathematical connections between different tasks or connections to what PTEs already knew or would need to understand as future teachers. These characteristics are similar to the agendas and routines mathematics education researchers have described in novice K-12 mathematics teachers (Leinhardt, 1989; Leinhardt, Weidman, & Hammond, 1987).

Implications of the results from this study are detailed further in the next chapter.

CHAPTER 8

CONCLUSIONS AND IMPLICATIONS

This study examined TAs' conceptions of RP and how they engaged PTEs in RP during classroom instruction. The MTH 2 textbook included tasks providing various opportunities for PTEs to experiences RP processes, including generating and evaluating conjectures and mathematical arguments. Results indicate that TAs engaged PTEs in a range of RP processes. For a plurality of observed tasks, however, opportunities for PTEs to engage in RP were decreased. A decrease in RP opportunities typically occurred either when TAs generated a conjecture or justification instead of allowing PTEs to generate them or when RP-related questions in a task were excluded during classroom instruction.

TAs articulated different conceptions of the goals and purposes of RP in mathematics, teaching, and learning. The depth and breadth of purposes of proof TAs articulated indicated that they talked about purposes of proof in mathematics differently than in mathematics teaching, focusing on different roles of proof in the two contexts. In the context of teaching, TAs focused on explanation, communication, and the general, fundamental purposes of proof, suggesting that their conceptions of proof include multiple purposes important in the context of teaching MTH 2. When articulating learning goals, it was also important in this context to consider the fact that TAs were teaching PTEs who were current undergraduate students and future teachers: On the one hand, TAs tended to specify RP content (i.e., what constitutes a proof) they wanted students to understand as short-term goals focused on the context of the MTH 2 classroom. On the other hand, TAs' long-term goals often included talk about their students as future teachers, identifying both RP content (i.e., generating concise mathematical explanations) and RP processes (i.e., developing critical thinking skills) important for preparing PTEs to teach others. This range of conceptions supports the notion that teaching about RP to college students who are prospective teachings represents a complicated context, illuminating some specific professional development needs.

The analysis of TAs' instructional decisions examined TAs' selection of RPTs to teach and their in-the-moment decisions to respond to PTEs about RP as observed. The results indicated that TAs' conceptions of teaching and learning RP in MTH 2 aligned strongly with how they talked about their goals for selecting RPTs. This alignment suggests that there is a relationship between TAs' conceptions of RP content and processes and how they select and analyze tasks for specific instructional goals. When talking about their classroom decisions around RP, TAs attended to surface level characteristics of classroom interactions (e.g., how to encourage student participation) and time constraints. In light of these results, implications for supporting TAs who are teaching mathematics courses for prospective teachers are detailed.

8.1 Implications and Recommendations for the Field

In this study, I considered small groups of TAs and individual TAs as cases because understanding the conceptions and practices in these cases implies ways to support TAs' knowledge and pedagogy development. A unique characteristic of the institution at which this study was conducted is that both mathematics and mathematics education TAs are assigned to teach mathematics courses for future teachers. This setup provides the opportunity to examine and learn from case studies of TAs with different backgrounds that could inform collegiate mathematics teaching in a variety of settings. The four cases in this study were grouped based on TAs' conceptions of RP in mathematics, teaching, and learning and how their conceptions aligned with the goals and purposes of MTH 2. The ways TAs' conceptions or classroom practices did and did not align with aspects of the course's purposes for RP have particular implications for PTEs' opportunities to learn about RP. These four cases inform the field's understanding of knowledge needed to teach mathematics at the collegiate level. Recommendations for supporting TAs who are teaching PTEs about RP are extrapolated from the study's results about TAs' conceptions of RP and their instructional decisions around RP.

8.1.1 Supporting TAs' Enactment of Reasoning-and-Proving Tasks

Mathematics education researchers have shown that teachers have a tendency to lower the cognitive demand of mathematical tasks during implementation (Stein et al., 1996), and a similar tendency was apparent for TAs implementing RPTs in MTH 2. Since maintaining RP demands of tasks implies there are more opportunities for PTEs to experience RP processes, it is important to provide pedagogical support that helps TAs maintain RP demands during classroom instruction. Understanding the two types of decreases observed, and reasons behind these instructional decisions, informs two primary ways to support TAs' classroom practices around RP.

The two types of decreases occurred when RP aspects were excluded or the TAs generated RP aspects (often proofs) for PTEs. For the first type of decrease, when Peter, for example, talked about his instructional decisions he was often concerned with time constraints, without attending to the way his modifications affected RP opportunities. Since Peter articulated fairly robust conceptions of RP and conceptions of teaching and learning RP that aligned with the course goals, he seems to know and articulate RP content and processes well. To support his instruction of RPTs, therefore, I suggest focusing on pedagogical skills and tools that could help TAs plan and implement RPTs. These pedagogical supports could include helping TAs learn how to modify tasks in ways without decreasing PTEs' opportunities to learn the RP processes they aimed to teach. The RP-framework could help in this endeavor by providing a means for TAs to analyze tasks for RP processes, before and after modifying tasks. Then, TAs could be more aware of the RP opportunities tasks afford before modification and how modifying tasks affect how PTEs experience RP. Professional development that helps TAs apply this framework to learn how to analyze tasks could help them plan lessons around RP processes in tasks and learn how to modify tasks based on RP-learning goals.

Second, the primary type of decrease observed, where TAs generated RP aspects for PTEs, highlights challenges TAs faced when facilitating mathematical discussions about RPTs. When describing these enactments, for example, Kelly, Peter, and Evan articulated difficulties facilitating these discussions in ways that allowed PTEs to talk about mathematics and engage in RP processes. They also expressed some apprehension about responding to PTEs' partially formed or potentially incorrect ideas. Drawing on secondary education mathematics research (Stein et al., 2008) that describes specific practices for facilitating mathematical discussions around high cognitive demand tasks is a way to address these needs. Stein et al. suggest it is particularly important to help *novice teachers*, those learning how to facilitate mathematical discussions, to use a set of practices that can prepare them to plan for students' ways of thinking or misconceptions. This set of five practices can also "help them . . . learn how to become better discussion facilitators over time" (p. 321). By anticipating and planning responses to PTEs' misconceptions, TAs can feel better prepared "to decide on the spot if [a PTE's justification is] a proof or not" (Evan, post-observation 3, para. 52). By learning how to monitor, select, sequence, and connect PTEs' thinking in the classroom, TAs could develop specific practices for orchestrating rich mathematical discussions that incorporate PTEs' ideas and ways of thinking.

As Evan said, "trying to make discovery learning work . . . and get through what you need to get through- that's tough. That's not a simple thing to do" (post-observation 3, para. 140). Incorporating the five practices (Stein et al.) into TAs' professional development would not make facilitating mathematical discussions simple, but could provide instructional support to develop TAs' abilities to engage PTEs in RP processes and respond to PTEs' thinking in the classroom. This case underscores the notion that mathematics instructors need to not only understand the content and students, but also develop a repertoire of pedagogical tools to use during instruction to help students develop rich conceptions of RP.

8.1.2 When is an Explanation a Justification? Developing a Reasoning-and-Proving Language

The enactment of RPTs involves TAs interacting with PTEs around RP, and from these interactions PTEs need to develop an understanding of RP to be prepared to articulate these processes to their future students. In this complex classroom setting, the way TAs talk about and design instruction around RP will influence how PTEs experience RP, and ultimately affect how PTEs will likely articulate and design instruction in their future elementary classrooms. The study of TAs' enactment of RPTs and responses about their instructional decisions suggests the importance for developing TAs' ability to notice (van Es & Sherin, 2002) instructional situations that affect PTEs' RP understanding, as well as developing TAs' use of a common language for talking about RP in teaching.

Laura, for example, often decreased RP demands by accepting PTEs' responses that constituted informal explanations (responses that lacked reasons about why mathematical claims were valid) when the task elicited rationales or proofs. During interviews, Laura could accurately articulate differences between these types of informal explanations, nonproof justifications, and formal proofs, indicating that her understanding of the mathematical content was strong. She did not, however, indicate an awareness of how PTEs may not yet comprehend the differences between different mathematical explanations and how her instructional decisions affected PTEs' opportunities to learn these distinctions. Imprecision about what constitutes a proof versus a rationale versus an informal explanation in mathematics classrooms for PTEs is particularly worrisome because such imprecision perpetuates misconceptions that PTEs could bring into teaching. TAs, like Laura, with strong knowledge of RP content, may benefit from support that helps them notice or recognize when decisions to highlight particular responses can affect RP demands of tasks or potentially reinforce PTEs' misconceptions.

One means for addressing this need is helping TAs use a precise language for talking about RP content and processes and teaching RP. Although mathematics uses precise language, K-12 teachers and teacher educators typically talk about teaching mathematics using imprecise language. Mathematics education researchers have recently stressed the importance of developing a precise, common professional language to talk about teaching practices (Ball, 2012). Based on the range of conceptions of purposes of proof, RP content, and RP processes TAs articulated in this study, they did not appear to have a common language for talking about these purposes and processes. Research literature presents clear conceptions of purposes of proof (Knuth, 2002b; Steele, 2006), but it is not clear how well these are shared with TAs.

Providing instruction and support for TAs to use the RP-framework in planning and teaching is suggested as one way to address this concern. In this study, this framework provided a lens through which tasks, classroom activity, and TAs' interview statements could be analyzed as related to specific aspects of RP or not related to RP. The framework could help TAs develop a common language for talking about processes of RP (Dimension 1) and purposes of RP (Dimension 2) in mathematics teaching. This common language is particularly important in the context of teaching about RP. That is, as users of mathematics, TAs could potentially complete coursework and do mathematics without the need to articulate various purposes of proof and aspects of RP processes. In teaching, however, TAs have to articulate RP content and processes in the classroom, explaining about RP using consistent language to develop PTEs' understanding of RP. Incorporating consistent RP processes language into MTH 2 could help TAs model and facilitate ways of talking about proof and proving during instruction that PTEs could apply in teaching.

8.1.3 Developing TAs' Knowledge about PTEs and How RP Processes Prepare PTEs to Teach

TAs who have never taken mathematics education courses, who have never taught mathematics courses for future teachers, who have no K-12 teaching experience, or who attended primary or secondary school outside the United States are likely to have little or no basis for knowing how the subject matter for MTH 2 relates to their PTEs as future teachers. It is well known that in mathematics teaching, it is helpful to make explicit connections to prior or future courses or subject matter to develop students' conceptual understanding of mathematics. In courses for PTEs, in particular, future connections include, ultimately, the elementary classroom setting in which PTEs will teach. In MTH 2, it is important to help PTEs understand how the subject matter (in this case RP content and processes) relates to their role as future elementary teachers. For TAs to help make these connections explicit for PTEs, they need a specialized knowledge of RP and of PTEs, calling for outside resources or support. Isaac's conceptions of teaching and learning RP help contextualize this aspect of specialized content knowledge in this context.

In K-12 mathematics, the NCTM calls for teachers to "hold all students in every . . . mathematics classroom accountable for personally engaging in reasoning and sense making, and thus lead students to experience reasoning for themselves rather than merely observe it" (NCTM, 2009, pp. 5–6). To ultimately engage future students in RP, therefore, PTEs need to articulate the purposes and processes of proof themselves. Isaac, however, focused on RP content in interviews, describing learning goals that did not connect to the preparation of PTEs as future teachers. Although PTEs engaged in many RPTs by experiencing RP processes in Isaac's class, neither they nor Isaac made explicit the RP processes in which they engaged and how their work on RPTs prepared them to explain about RP in the future. Engaging in and doing proofs and proving is not sufficient for preparing PTEs who need to be aware of and able to teach others about the RP processes in which they engaged. Without explicit connections, PTEs are less likely to associate the work around RP they do in MTH 2 with the elementary classroom context.

To develop TAs' understanding of how RP in MTH 2 relates to PTEs' future teaching, additional support materials from textbook authors and mathematics education documents (e.g., NCTM, 2009) explaining these connections are one means for addressing this need. TAs may also better understand what RP entails in K-8 mathematics classrooms if they have opportunities to watch video or read examples (e.g., Carpenter, Franke, & Levi, 2003) of elementary students reasoning and generating conjectures. Informing TAs' understanding of the context in which PTEs will teach can help TAs contextualize RP instruction, thereby preparing PTEs to engage elementary students in reasoning and sense making.

8.1.4 Developing Critically Reflective TAs

The first three implications described how the study's results inform our understanding of mathematical knowledge for teaching RP at the college level and recommendations for supporting TAs' development of aspects of this knowledge. Before attempting to work with TAs in these ways, however, it is important to develop TAs' ability to be *critically reflective practitioners*—open to identifying and examining their assumptions that influence how they conduct and perceive their teaching (Brookfield, 1995). Without such an awareness, TAs are less likely to desire change or input on their teaching practices or conceptions about teaching; attempts at learning new pedagogical strategies would likely be short-lived if TAs do not understand their current perspectives and how new strategies or ideas relate to their existing framework.

The results from examining Ace's conceptions of RP and teaching practices support this notion of the importance of developing critically reflective TAs. Throughout the semester, Ace maintained conceptions of teaching and learning RP that were contrary to the course design. He taught MTH 2 by combining the style of instruction with which he was familiar, lecturing, with the instructional strategy endorsed by the course supervisor and textbook, group work on mathematical tasks. To do so, he typically lectured about mathematical content related to RPTs before asking PTEs to work with others on the tasks, usually decreasing RP demands either by generating RP aspects for PTEs in the notes before they worked on tasks or during implementation. Trying out new instructional strategies that may be different from one's primary views of teaching and learning is not inherently problematic, but calls for thoughtful integration of new strategies with the intention of continued reassessment and modification to help students achieve desired learning outcomes. An unexamined, contradictory teaching practice, however, lends itself to potentially confusing students who receive mixed messages regarding the purposes of instruction and RP.

There was little evidence that Ace critically examined his teaching practice; at the beginning of the semester, he said he started incorporating tasks as group work because he followed the course supervisor's recommendations, saying, he "went with it the first day and [has] gone with it ever since" (Ace, beginning of semester, para. 134). A notable instance when Ace made some inroads reflecting on his teaching practice, however, was during the semester's twelfth week: Ace expressed dissatisfaction with how PTEs waited for him to tell them the answer to tasks instead of working on tasks during group-work time (post-observation 4, para. 6, 8). In response to this dissatisfaction, Ace reminded PTEs about an upcoming exam to try to motivate them to work on tasks more themselves, without making significant changes to or assessments of his instructional style. This pedagogical move suggests Ace viewed PTEs' lack of engagement as primarily a result of a lack of student motivation, rather than considering any potential additional factors, such as his instructional decisions or classroom norms. After that instance, Ace made no other statements that significantly reflected on his teaching practices.

Continuing to provoke these types of reactions, where TAs experience cognitive dissonance among their conceptions of teaching RP, how they conducted teaching, and students' reactions, could develop a better understanding of their personal teaching perspectives. Brookfield (1995) encourages this process of discovering one's assumptions by viewing one's practice from four perspectives: (a) writing one's autobiography as a teacher and learner, using personal self-reflection and collecting insights into teaching; (b) making an assessment of one's self from the students' perspective by seeking their input, considering the classroom structure, and learning from their viewpoint; (c) learning from peer review of one's teaching, which could include talking regularly with colleagues about teaching experiences and being open to have colleagues observe and provide feedback about one's teaching; and (d) frequently referring to the theoretical literature that may provide an alternative interpretive framework for a situation. The intention is not that TAs need to apply all four recommended critical reflection practices before other, potentially more content- or course-specific, professional development strategies are also incorporated. These are some strategies, rather, through which TAs could begin to become more self-aware and therefore open to input about their teaching.

8.2 Future Research

Based on the results of this study, additional studies are recommended to grow the knowledge base of the teaching and learning of RP, especially at the collegiate level. The following sections detail next steps in this line of inquiry for the researcher.

8.2.1 What about the Prospective Elementary Teachers' Learning?

This study assessed PTEs' opportunities to learn about RP in MTH 2 by considering how they engaged in RP processes in the classroom, but did not include measures of student learning about RP in MTH 2 or potential long-term effects of RP instruction on future teachers' classroom teaching. Such investigations were outside the scope of this study, but are important for understanding how PTEs think about and understand RP in college and how they apply that understanding in their future teaching.

This study's investigation of TAs' enactment of RPTs raises the question of how these different opportunities to learn RP affected PTEs' understanding of RP. That is, the greatest number of instances of RP opportunities being maintained was during Isaac's class, which contrasts with the tendency to decrease RP demands by generating RP processes for PTEs observed in Ace's class. This study was not designed to compare and contrast student learning between classes, but these different enactment patterns beg the question: How do different instructional strategies and opportunities to learn about RP in the classroom affect PTEs' understanding of and retention of RP? Future studies could examine this relationship and may include test or interview data from PTEs to assess PTEs' knowledge and abilities related to RP.

Since mathematics courses for PTEs aim to develop PTEs' knowledge of mathematical content and processes, preparing them to teach others about mathematics, examining long-term effects of instruction about RP is another potential line of inquiry. In this study, many TAs talked about how the RP instruction MTH 2 was intended to prepare PTEs to teach. For example, Evan explained that he included RPTs in instruction to "prepare [PTEs] to evaluate arguments in their own teaching and make arguments in their own teaching" (end-of-semester, para. 51). He endeavored to help PTEs understand RP and develop their ability to teach others about RP. The study of how, if at all, such instruction affects prospective teachers' future teaching is one avenue for understanding long-term effects of RP instruction in college preparatory programs.

8.2.2 Supporting TAs' Teaching and Learning to Teach

This study did not collect data from TAs' weekly course meetings to assess their potential effects on TAs' classroom decisions or conceptions about RP, as it was outside the scope. Understanding whether current support systems are helping TAs grow in their abilities to teach about RP, represents an important line of inquiry to then determine any necessary changes to support systems. Professional development strategies to support TAs who are teaching PTEs about RP were proposed based on TAs' conceptions and teaching practices in this study and draw upon mathematics education literature. These recommendations are purposed to develop TAs as critically reflective practitioners and support their efforts to engage PTEs in RP in classrooms. Next steps in this research could include examinations of the effects of providing such supports for TAs teaching PTEs about RP in courses like MTH 2. Such studies could investigate how professional development affects TAs' conceptions and instructional decisions, suggesting strategies that are particularly helpful for developing TAs' knowledge and ability to teach.

8.2.3 Collegiate Mathematics Teaching and Learning

Although this study focused on TAs who were teaching about RP in a specific context, there are recommendations for future research that extend into other collegiate teaching and learning settings. Collegiate mathematics courses serve to prepare general mathematics learners, mathematics majors, and prospective teachers to be able to understand mathematical ideas. The CBMS (2012) suggests that collegiate mathematics instruction should help learners "develop not only knowledge of content but also the ability to work in ways characteristic of the discipline" (p. 8). This notion of preparing learners to "work in ways characteristic of the discipline" could look different, however, depending on the students to whom that statement refers. That is, mathematics majors need to have authentic mathematics learning experiences to be prepared to think like a mathematician. That is, mathematics majors need to have authentic mathematics learning experiences to be prepared to think like a mathematician. The work that mathematicians do, however, remains largely behind the scenes because pure mathematicians typically disseminate a "recorded version of their mathematical work that differs from their actual thinking, having much greater emphasis on language, symbols, logic, and formalism" (Thurston, 1994, p. 167). They do not usually need to articulate or make visible the process through which they did the mathematical work beyond sharing the streamlined proof (Thurston, 1994). It is important, however, for prospective teachers to have mathematical learning experiences that prepare them to teach others about mathematics (developing their specialized content knowledge). To help develop prospective teachers' specialized content knowledge, therefore, mathematics courses should provide opportunities for prospective teachers to talk about how to do the mathematical work, preparing them to explain mathematics to future students.

The notion of preparing prospective teachers, therefore, to "work in ways characteristic of the discipline" should look different from that of preparing mathematics majors. Consider, for example, two mathematics courses both focused on introducing students to proof and proving but geared toward different student populations: (a) general mathematics learners and (b) prospective teachers. One goal in the class for general mathematic learners might be to "help students learn how to do proofs and important features of proofs." Instruction that includes examples of inductive arguments could then help students know what a proof is and some features of proofs. In the other class, however, prospective teachers need to learn how to articulate and engage in processes related to RP to teach others about mathematics. They may need to know "why is induction an acceptable thing to do in the discipline?" The extent to which the different student populations are expected to explicitly talk about the processes of RP in the discipline is different and the mathematics instruction should attend to that difference.

When talking about learning goals for MTH 2, the results indicated that Kelly, Peter, and Evan talked about this notion of developing PTEs' understanding of both content and processes to prepare PTEs to talk about RP with others in their future classroom. Since prospective teachers may enroll in courses other than MTH 2, how does this notion extend into other mathematics courses for prospective teachers, into other mathematics courses prospective teachers may choose to take that also include mathematics majors and general education students, or to other university settings beyond this study? Future lines of inquiry could investigate how instructors who teach courses that include prospective teachers do (or do not) design instruction around learning goals focused on helping prospective teachers "develop not only knowledge of content but also the ability to" (CBMS, 2012, p. 8) teach the content to others. TAs who specified RP process-related learning goals for preparing PTEs as future teachers did not consistently maintain RP demands of RPTs in this study, suggesting that preparing prospective teachers is challenging work. Future research could look for exemplary cases of collegiate mathematics instructors engaging prospective teachers in mathematical processes. Other collegiate mathematics instructors could learn from these exemplary cases. Are there *best practices* for working towards these learning goals in these types of courses? What resources and supports are needed to help collegiate instructors prepare prospective teachers to articulate and engage in mathematical processes?

8.3 Conclusion

As current teachers of mathematics and potential future faculty, TAs represent an important and under-researched group (Speer et al., 2010). By examining TAs' conceptions and teaching practices, this study contributes to this growing and needed area of research, addressing a specific research need by studying TAs who were teaching PTEs about RP. In this context, the Conference Board of the Mathematical Sciences (2012, pp. 7–8) argues for mathematical reasoning and thinking to play a more prominent role in mathematics courses for PTEs. Their recommendations fall short, however, in helping the field understand more about who is currently teaching these types of courses and how PTEs' RP abilities are currently developed.

The TAs in this study articulated fairly robust conceptions of purposes of proof in teaching and designed lessons around mathematically rich RPTs intending to provide opportunities for developing PTEs' understandings of RP concepts or processes. During classroom instruction around these tasks, however, TAs tended to decrease RP demands by generating RP processes for PTEs. This type of decrease in RP demands also lowered PTEs' opportunities to experience proving processes for themselves. The examination of TAs' conceptions of RP, their classroom practices, and how they talked their instructional decisions, uncovered some ways that TAs could benefit from different types of pedagogical supports. TAs, and other instructors, who desire to engage students in authentic RP experiences could need support that helps develop (a) pedagogical strategies for anticipating, eliciting, and responding to students' thinking during instruction, (b) a consistent and precise language for RP, (c) an informed perspective on the relationship between the mathematical content and the students' learning needs, or (d) critically reflective teaching practices. Such supports are particularly important for instructors teaching prospective teachers, as prospective teachers' experiences with and understanding of RP in college courses can ultimately affect the quality of instruction their future students receive. In light of increasing demands on K-12 teachers' knowledge for teaching about RP (CCSSI, 2010; NCTM, 2009), continuing to study and support the mathematics teaching and learning of prospective teachers in college mathematics classrooms is vital.

APPENDICES

APPENDIX A

PROTOCOLS AND SURVEY INSTRUMENTS

A.1 Observation Protocol

Observation Protocol Dimensions

1.1 Task as designed in the textbook
2.1 Task features as set up by the TA
2.2 Cognitive demands as set up by the TA
3.1 Classroom norms evident during implementation
3.2 TA's and PTEs' instructional habits during implementation
3.3 Task features as enacted
3.4 Cognitive demands as enacted

Observation Protocol Questions

1. What was the potential for engaging in reasoning-and-proving for each enacted task? **(1.1)**
    a. List the potential rating for each task by question, also indicating which questions from the task were specifically addressed during the observation.
    b. If an enacted task was not already rated (e.g., a worksheet or TA generated task), code it for RP potential using the RP-framework after the observation and note those ratings here.
    c. If an enacted task was coded as not related to RP, also record that information here.

2. How did the TA introduce each task? **(2.1, 2.2)**
    a. What instructions were given?
    b. What instructions, if any, were given about how PTEs will be expected to communicate their work?
    c. How did the instructions compare with the actual written instructions?

3. What were the PTEs' responses to the lesson introductions/instructions? **(3.2)**
    a. What questions did PTEs have and what were the TA's responses?
    b. In what ways, if any, did PTEs not follow instructions?

4. How did the TA interact with PTEs during the time when PTEs worked on the task? **(3.2)**
    a. How did the TA assess student progress?
    b. How did PTEs respond to the TA's guidance during task?
    c. What changes did the TA make to the lesson during the task?
    d. What questions/responses from PTEs prompted changes?

5. What kinds of generalizations or mathematical arguments did PTEs generate? **(3.4)**
    a. Describe the kinds of generalizations or arguments PTEs generated by question.

b. How were they presented (on the board, verbally)?
c. Were the ideas or explanations discussed, compared, evaluated, etc.? In what way(s)?

6. How were PTEs' generalizations or mathematical arguments elicited during the lesson? **(3.3)**
   a. Did the TA explicitly ask for generalizations or justifications?
   b. Did PTEs explicitly ask for other PTEs' generalizations or justifications?

7. How did the TA respond to PTEs' generalizations or mathematical arguments? **(3.1, 3.2)**
   a. Did the TA restate, ask for feedback from the class, or ask for other ideas or arguments?
   b. When did the TA ask PTEs to explain their generalization or justification further?
   c. What kinds of information were assumed/"taken-as-shared" during the discussion?
   d. Were any responses or explanations highlighted and in what way(s)?

8. Who was responsible for judging the worth of generalizations, justifications, or proofs? **(3.1)**
   a. What elements of PTEs' justifications were deemed to be good? Who determined this? How was this established?
   b. What aspects of justifying were determined to be not as good as other aspects? Who determined this? How was this established?

9. What aspects of the lesson discussion seemed difficult for the TA to facilitate? **(3.2)**

10. What aspects of the lesson discussion seemed confusing for PTEs? **(3.3, 3.4)**
   a. Did the TA address PTEs' confusion? How?

A.2 Preobservation Survey

1. What are your goals for the class that I will observe?

2. How will you determine if this class period accomplishes your goals?

3. What problem(s) from today's class do you think is (are) particularly important?

    a. How do you hope students will respond to this (these) problem(s)?

A.3 Pre-Semester Interview Protocol

TAs' conceptions about the roles proof in mathematics.

This is (YOUR NAME) interviewing (SUBJECT'S NAME) on (DATE). The purpose of this interview is to better understand your math background and current thinking on mathematical ideas related to Geometry.

Part I: Initial Biographical information: **To start, I'd like to ask you a few background questions.**

- **What is your degree program and what stage are you at in your degree?**
- **Please tell me about your schooling background and the path that led you to your current status as a mathematics/mathematics-education graduate student?**
  - **i) Were there any math classes or concepts with which you struggled as a learner?**
  - **ii) Were there any math classes or concepts with which you really excelled?**
- **Since you're teaching a Geometry course, what Geometry-related classes have you taken?**
  - **i) And what were your learning experiences like in Geometry?**

Part II: TAs' Conceptions of Proof in the Domain: **Since reasoning and proof is one of the goals of MTH 202, I have some questions about your views of proof**

1. **How do you think about proof in mathematics?**
   a) *If the TA uses particular language (e.g., formal or informal proof) or refers to students probe on what the terms mean and what the student population is.*
   b) If they focus on doing proof, ask *can you tell me what a proof actually is?*

   Use question 2 as follow up if it hasn't been addressed, yet.
2. **What does it mean to prove something in mathematics?**
   a) *Probe any mathematical language*
   b) *If it hasn't been addressed yet, follow up with,* **What does it mean to disprove something?**
3. **What makes an argument a proof?**
   a) *Intends to get at the social aspects of proof, but do not lead in that direction if they do not offer that idea. If they use language suggesting a social or community (e.g., mathematical community or mathematicians) aspect to proof, probe that language.*
   b) *If it hasn't been addressed yet, then follow up with,* **What makes an argument not a proof?**
4. **Once a proof is completed, does that proof ever become invalid?**
   a) *Probe for how TAs see proofs as becoming invalid if they do not offer an explanation*
5. **Considering the discipline of mathematics, and not a mathematics classroom, what purpose does proof serve in mathematics?**
   a) *Probe any mathematical language.*
6. **How do you remember learning about proof?**
   a) Provide some examples of classroom settings to explain
   b) *Have you ever learned about proof in classrooms that used more hands on or interactive methods?*

A.4 Beginning-of-the-Semester Interview

TAs' conceptions of the roles of proof in teaching and learning

This is (YOUR NAME) interviewing (SUBJECT'S NAME) on (DATE). The purpose of this interview is to better understand your current thinking on ideas related to the teaching and learning of reasoning and proving.

<u>The Teaching and Learning of Proof in MTH 2</u>: **Please consider your role as an instructor for MTH 2, as you respond to these questions:**

1. **Considering your role as an instructor of MTH 2, how do you think about proof?**
    a) *If the TA uses particular language (e.g., <u>formal</u> or <u>informal proof</u>) or refers to students probe on what the term means and what the student population is.*
2. **What purpose does proof and proving serve in mathematics classroom?**
    a) Purpose of proof in K-12 math classes. *And probe any mathematical language and ask for examples.*
    b) ***Thinking about MTH 2, can you say a little bit about what purpose proof and proving serves in that classroom?***
3. **Considering your current MTH 2 students, what do you want them to learn about proof?**
    a) Probe any mathematical language.
    b) *Follow up with:* **What challenges do you anticipate in supporting their learning about proof?**
        i) *If it hasn't been addressed yet:* ***Do you think proof and proving is content they*** **[your MTH 2 students]** ***will use again?***
4. **In your opinion, why do you think these students [future elementary teachers] are required to take this course?**
    a) Intends to probe for how the TA has a pedagogical focus when teaching this course. If the TA talks about students as math learners probe any mathematical language there, if they talk about the students as future teachers, probe any pedagogical language there.
    b) Follow up with: **Why do you think it is important to teach these students about proof and proving in this course?**

A.5 End-of-Semester Interview Protocol

This is Kim interviewing (TA's Name) on (Date). The purpose of this interview is to better understand your ideas related to reasoning and proving in mathematics and in teaching mathematics, at the end of this semester.

1. **How do you think about proof?**
    a) *If the TA uses particular language (e.g., formal or informal proof) or refers to students probe on what the terms mean and what the student population is.*
    b) If they focus on doing proof, ask *"can you tell me what a proof actually is?"*
2. **In the context of teaching MTH 2, how do you think about proof?**
3. **In your view, how do the students in your section of MTH 2 best learn about proof and proving?**
    a) **What purpose does proof and proving serve in the MTH 2 classroom?**
    b) **Do you think proof and proving is content that your MTH 2 students will use again?**
        i) *If so how. If not why not.*
4. **In your view, what is your role as the mathematics teacher in MTH 2?**
5. **Have your views about teaching and student learning changed over the course of the semester?**
    a) If so, how? If not, why not?
    b) Insert a few questions specifically for each TA by providing quotes from earlier interviews where they expressed a strong view of the nature or purposes of RP in mathematics, teaching, or learning. Read the quote from their earlier interview and ask: **After your experiences this semester teaching MTH 2, I'm wondering if you have any reflections on this idea/statement.**
6. **What people or experiences have been the most prominent influences in your teaching practice in MTH 2?**
    a) *Probe for details about how they influenced their practice AND for additional influences.*
    b) ***Can you describe for me how you typically prepared to teach MTH 2? Do you write a lesson plan or notes to yourself?*** *If yes, what do they generally include, if not, why not? Can you tell me what (if anything) you changed about how you prepared to teach MTH 2 and why?*
7. **If you were to teach this course again, with more autonomy to choose the textbook and all aspects of the course, what would you do differently and what would you do the same?**
    a) Probe for both similarities and differences
    b) *In this more autonomous situation, please describe your goals for the course.*
        i) Possibly, probe for student learning goals: *At the end of the course, what would you want your students to be able to do?*
    c) One of the goals stated on the MTH 2 syllabus is for students to be able to "reason mathematically and do proofs in the context of geometry and measurement." *In what ways did your work in MTH 2 address this goal, if at all?*
        i) Would you include this goal in the MTH 2 course you designed? If not, why not? If so, why?

8. **Have you ever seen the NCTM *Principles and Standards for School Mathematics* document?**
   a) *If yes,* ask: Can you describe in your own words how the authors of the document describe the role of reasoning and proof in K-12 mathematics.
   b) *If no,* show them the quote from NCTM (2000) that is on a separate page and says the following:

In the section on reasoning and proof, in the National Council of Teachers of Mathematics Principles and Standards for School Mathematics (2000) Document, the authors advocate that students should be able to

- *recognize reasoning and proof as fundamental aspects of mathematics;*
- *make and investigate mathematical conjectures;*
- *develop and evaluate mathematical arguments and proofs;*
- *select and use various types of reasoning and methods of proof.*

They also contend that in K-12 mathematics classrooms, reasoning and proof should be "a natural, ongoing part of classroom discussions, no matter what topic is being studied" (p. 342).

   c) ***What do you think about the recommendations for reasoning & proof set forth by the NCTM?***
      i) *Possibly probe:* In what ways do you think this idea for reasoning & proof to be "a natural, ongoing part of classroom discussions" is or is not applicable in your MTH 2 class?

9. Show them and ask them to read the descriptions of the roles of proof in literature (same role of proof in Table 1). **Of this set, which roles of proof resonate with you and why?**
   a) Do you have any questions about this list or the descriptions
   b) If it is brought up, possibly encourage the TA to answer the question considering their own uses of proof and proving [General] and then in the context of MTH 2 [Specific].
   c) *STRESS the WHY!*

A.6 Post-Observation Interview Protocol

This is (YOUR NAME) interviewing (SUBJECT'S NAME) on (DATE) and this is an interview about the lesson from today. As a reminder, the point of this interview is not to evaluate you or your teaching. I want to hear your thoughts and ideas about your teaching.

a) **What do you think about how the lesson went today?**
   - **i.** If TA asks interviewer for an opinion about how the lesson went say, *"I'm trying to learn about how YOU think about the lesson."* This question could go down a rabbit hole, don't stay too long here and consider asking this later in the interview sometimes.

b) [From preobservation survey] **You identified Activity ___ as important for today's lesson.**
   - **i. Can you tell me a little bit more about why you considered it particularly important for today's lesson?**
   - **ii. What do you think about how Activity ___ went today?**
     - *What went well?*
     - *What did not go well?*
     - *Describe any difficulties you encountered and what you did to try to overcome them.*

   Repeat part (b) for any other activities identified as important.

c) **Rereading the goals you wrote before class, in your opinion, did the lesson accomplish your goals?**
   - i. Why or why not?
     - *Provide explicit examples of how you came to this conclusion*

d) **Did anything surprise you about how students responded to today's lesson?**
   - **i.** *Please provide examples from the lesson if you can think of any that illustrate your point*

e) Based on specific student responses or questions TAs asked during class: **Do you remember when . . . can you say a little bit more about what you were thinking when that happened?**

   Can you say a little bit more about what you were trying to do (or why you asked that question) in that situation? And why were you trying to do that?

   Can you tell me a little bit about why you responded that way?

   Why did you decide (not) to write on the board/restate/have someone else respond to {student's} explanation?

   What kinds of responses were you expecting students to give? Would that be what you would consider a proof?

   What did you think of [student's name]'s explanation?

f) Additional questions not used every time, but woven in sometimes:
   - **i. How does the work from Activity/Activities ___ relate to your students as future elementary teachers?**

- TAs may focus on the PTEs as math learners or as future math teachers
- Follow up with, *in what ways do you see this content as content that these future teachers will use again in their teaching?* How might the content from these activities impact what the future teachers do in the classroom? What would you envision their future elementary students doing around this content in the class?

ii. **Why was it important to teach this lesson to these particular students?**

iii. In earlier interviews, you mentioned you learned about proof ______, and/but in class today you used _______. Can you tell me a little bit about why you decided to use this instructional strategy?

- *What do you think about how the use of small groups went today?*
- *Have you previously ever taught using these types of hands-on strategies? (please describe)*

APPENDIX B

TEXTBOOK ANALYSIS AND CLASSROOM OBSERVATION CODING SCHEME

B.1 Coding the Task-as-Written

Segment of Task Implementation	**Top-Level Code** (with description)	**Second-Level Code** (with description)
Task as Written	**MG** (Task provides opportunities for students to create *Mathematical Generalizations*)	**IP** (*Investigate* mathematical situation with *provided* examples or data)
		IG (*Investigate* mathematical situation by *generating* examples or data)
		C (Generate a *Conjecture*)
		EvC (Evaluate a given mathematical claim)
	MA (Task provides opportunities for students to generate *Mathematical Arguments*)	**EvA** (Evaluate a given mathematical argument or method)
		R (Provide a *Rationale*—a well-reasoned mathematical argument that is not a proof)
		Ex (Provide an empirical argument or example)
		Pf (Generate a *proof*)
		Meta_CPf (Discussing or reflecting on a *Meta-Level* aspect of proof and proving—in this case *Characterizing Proof*)
		Meta_iPf (Discussing or reflecting on a *Meta-Level* aspect of proof and proving—in this case *Identifying Proof*)
	GA (Task provides opportunities for students to generate *Generalizations and Arguments*)	**GA-P** (The type of argument the task provides the opportunity for students to generate is specifically a *Proof*)
		GA-N (The type of argument the task provides the opportunity for students to generate is a *nonproof argument*—an Ex or an R)
	NoRP (Based on the Reasoning-and-Proof framework, this task does *not* provide opportunities for students to engage in specific RP processes)	

Definitions for some of the second-level codes are included on the next page.

B.2 Definitions for Components of Reasoning-and-Proving Framework Used in Task Analysis

Investigate: explore a mathematical situation, example, or data to determine mathematical relations that fit the given or generated data or examples.

Explore Provided: explore and determine mathematical relations that fit a given set of data or examples

Generate Examples: involved in constructing, creating, or providing examples or data that fit the mathematical situation in order to then explore mathematical relations therein.

Conjecture: a reasoned hypothesis about general mathematical relation based on incomplete evidence.

Nonproof Argument: an argument for or against a mathematical claim that does not qualify as a proof

Empirical Argument: an argument that purports to show the truth of a mathematical claim by validating the claim in a proper subset of all possible cases covered by the claim.

Rationale: valid arguments for or against mathematical claims that do not qualify as proofs—it may not make explicit reference to key accepted truths that it uses, or it uses statements that do not belong to the set of accepted truths of a particular community.

Proof: a valid argument based on accepted truths for or against a mathematical claim that makes explicit reference to key accepted truths that it uses.

B.3 Sections of Textbook Analyzed

Curriculum Section:	Title of Topic of Study
8.1:	Visualization
8.2:	Angles
8.3:	Circles & Spheres
8.4:	Triangles, Quadrilaterals, & Other Polygons
8.5:	Constructions with Straightedge & Compass
8.6:	Polyhedra & Other Solid Shapes
9.1:	Reflections, Translations, & Rotations
9.2:	Symmetry
9.3:	Congruence
9.4:	Similarity
10.1:	Fundamentals of Measurement
10.2:	Length, Area, Volume, & Dimension
10.3:	Calculating Perimeters of Polygons, Areas of Rectangles, & Volumes of Boxes
10.4:	Error and Accuracy in Measurements
10.5:	Converting from One Unit of Measurement to Another
11.1:	The Moving & Additivity Principles about Area
11.2:	Using the Moving & Additivity Principles to Prove the Pythagorean Theorem
11.3:	Areas of Triangles
11.4:	Areas of Parallelograms
11.5:	Cavalieri’s Principle about Shearing & Area
11.6:	Areas of Circles & the Number Pi
11.7:	Approximating Areas of Irregular Shapes
11.8:	Relating the Perimeter & Area of a Shape
11.9:	Principles for Determining Volumes
11.10:	Volumes of Prisms, Cylinders, Pyramids, & Cones
11.11:	Areas, Volumes, & Scaling

B.4 Coding the Task-as-Set-Up

Segment of Task Implementation	Who	Code	Description
Task as Set Up	TA Moves	**T C Push**	TA asks students to develop a conjecture
		T R Push	TA asks students to provide an explanation about why their answer makes mathematical sense without explicitly asking for a proof
		T Pf Push	TA asks students to develop a proof (general argument)
		T InvG Push	TA asks students to generate examples to investigate a mathematical situation
		T InvP Push	TA asks students to investigate a mathematical situation using the provided data, examples, or diagrams/models
		T EvA Push	TA asks students to evaluate a given argument
		T EvC Push	TA asks students to evaluate a given mathematical claim
		T Proc Q	TA asks students to explain how or why they arrived at a particular answer
		T ExR S	TA provides an example of the reasoning why a mathematical claim associated with the activity makes sense (with **S input** or not)
		T UsePK	TA asks students to use prior knowledge for statements
		T Lead Diss	TA leads a discussion about the activity instead of asking students to work through it
		T Group	TA asks students to work in small groups
		T Ind	TA asks students to work individually
		T WorkIns	TA does not specify if students should work individually or in small groups
		T Ch	TA says, demonstrates, or does something to change the RP-potential of the task
		T NoCh	TA does not say, demonstrate, or do something that noticeably changed the RP-potential of the task
	Student Moves	**S Ins Q**	Students ask questions about instructions
		S NoIns Q	Students do not ask questions about instructions
		S Follow	Students, in general, follow TA's instructions
		S NoF	Students, in general, do not follow TA's instructions

B.5 Coding the Task as Implemented, Focusing on Individual or Group Work

Enactment	Who	Code	Description
Task Implementation (Group-Work or Individual Work Time)	TA Moves	**T Circ**	TA circulates among classroom for a majority of the group/individual-work time
		T Front	TA stays near the front of the room for a majority of the group/individual-work time
		T CkIn	TA asks small groups or individual students general-progress questions, such as "Are you done? Did you get the answers? Any questions?"
		T Proc Q	TA asks students to explain how or why they arrived as a particular answer
		T R Push	TA specifically asks students to explain their mathematical reasoning and/or why their answer makes mathematical sense (though they do not specifically ask for a proof)
		T C Push	TA asks students to develop a conjecture—will this work for all such objects?
		T Pf Push	TA asks students to develop a proof (general argument) or if the students know how to prove their claim or answer
		T Meta Push	TA asks students to consider Meta-Level aspects of proof and proving
		T UsePK	TA asks students to use prior knowledge in developing statements
		T ExA	TA provides the answer to small groups or individual students (sub-codes: **T ExC**, **T ExR**, **T ExPf** for TA providing an example of the desired Conjecture, Rationale, or Proof, respectively)
		T Exs	TA provides empirical examples to support a mathematical claim associated with the task
		T Und Push	TA encourages students to understand the reasoning or proof for themselves, or reword it in their own words
	Student Moves	**S Ask**	Students ask TA to evaluate their work (sub-codes: **S AskEv R**, **S AskEv Pf**, for evaluating Reasoning or Proof, respectively)
		S Ask Pf	Students ask TA *how* to prove a statement
		S InvP	Most students investigate mathematical situation (**InvP** and **InvG** for provided or generated examples/data)
		S Conj	Most students generate conjectures
		S Exs	Most students explain claims with empirical examples
		S R Gen	Most students explain why their answer makes mathematical sense that is not a formal proof
		S Pf Gen	Most students develop a proof (general argument)
	NA		No time was provided for group or individual student work

B.6 Coding the Task-as-Implemented, Focusing on Whole-class Discussions

Enactment	Who	Code	Description
Task Implementation (Whole-class Discussion of Task)	TA Moves	**T S to S Push**	TA asks students to evaluate other students' reasoning
		T R Push	TA asks students to explain why their answer makes mathematical sense (does not specifically ask for a proof)
		T C Push	TA asks students to tell the class a conjecture
		T Pf Push	TA asks students to tell the class their proof
		T Meta Push	TA asks students to consider Meta-Level aspects of proof and proving
		T Exs	TA generates empirical examples to support a mathematical claim associate with the task
		T ExC	TA generates example of the conjecture expected (Sub-code: attributing to **Unnamed S** or **Specified S**)
		T ExR	TA generates example of the rationale expected (Sub-code: attributing to **Unnamed S** or **Specified S**)
		T ExPf	TA generates an example of the proof expected (Sub-code: attributing to **Unnamed S** or **Specified S**)
		T OutPf	TA provides an outline of key points or statements-and-reasons needed in a proof for a mathematical claim
	Student Moves	**S Stmt**	Students provide statements to use in the construction of a statement-and-reason type proof
		S Reas	Students provide reasons why a provided statement is true in the construction of a statement-and-reason type proof
		S RepAns	Students report out answers or results (one-word/short ans)
		S Inv	Students investigate mathematical situation (sub-codes: **S InvP** and **S InvG** for provided or generated examples/data)
		S Conj	Students provide conjectures
		S EvA	Students evaluate given argument or method
		S EvC	Students evaluate given mathematical claim
		S Exs	Students provide empirical examples as reasons why a mathematical claim or conjecture is true
		S R Gen	Students provide an explanation about why their answer makes mathematical sense, not a proof
		S Pf Gen	Students develop a proof (general argument)
		S Meta	Students discuss meta-level aspects of proof and proving
		S Non	Students non-responsive to TA's request to offer thinking, reasoning, and ideas
	NoDiss		There was no discussion of the task

APPENDIX C

INTERVIEW CODING SCHEMES

C.1 Coding Interviews for Conceptions of RP in Mathematics

Initial Coding Level	**Second Level: Specific Math-Related Categories**	**Third Level: Sub-codes within the level two categories**	**Descriptions of Sub-codes**
Mathematics	*For what population /group to know*	Teachers	Those teaching K-16
		Future teachers	Undergraduates in courses (like MTH 2) who are future teachers
		Students	No grade level specified
		Elementary Students	K-8 students
		Undergrad Students	College-level students
		Mathematicians	Researchers and professors in mathematics
		Ordinary people	For anyone, not necessarily mathematicians, students, or teachers
		No explicit distinction	No distinction noted
	Purpose of Proof in Mathematics	Verification	Establishing, checking, or confirming the validity of a known idea or given mathematical claim
		Falsification	Establishing, checking, or confirming the falseness of an idea or given mathematical claim
		Explanation	Providing insights into why a mathematical claim is valid
		Communication	Disseminating mathematical knowledge to other doers of mathematics
		Discovery/Create Knowledge	Discovering or developing new mathematical ideas, confirming conjectures, or building mathematical ideas
		Systematization	Imposing a logical structure on the mathematical domain; organizing and cataloging results with respect to axioms, major concepts, theories, and other prior knowledge
		Meta-Level: Fundamental	At a broader level, proof is fundamental to mathematics

C.2 Coding Interviews for Conceptions of RP in Teaching

First Level of Coding	Second Level: Specific Teaching-Related Categories	Third Level: Sub codes within level two	Descriptions
Teaching	Who are the teachers?	Self	TA being interviewed is talking about their own teaching of RP
		College-level instructors	People teaching college courses (but the participant is not explicitly referring to themselves and their own teaching)
		Other Teachers	Code for *K-12*, *Secondary*, or *Elementary*, if specified.
		Anyone	Either said, teachers at any level, or did not specify explicitly 'who'
	Purposes of proof in mathematics teaching	Verification	Establishing, checking or confirming the validity of a known idea or claim
		Falsification	Establishing, checking or confirming the falseness of an idea or claim
		Explanation	Providing insights into *why* a mathematical claim is valid
		Communication	Disseminating mathematical knowledge to other doers of mathematics
		Discovery / Create Knowledge	Developing/discovering new mathematical ideas confirming conjectures
		Systematization	Imposing a logical structure on the mathematical domain; organizing & cataloging results with respect to axioms, major concepts, theories, and other prior knowledge
		Meta-Level: Fundamental	Helping students understand that proof is fundamental to mathematics
	HOW to Teach about Reasoning-and-Proving	Methods that Support or Encourage Student Participation	• Getting students to *share their ideas* during class • Facilitating *class discussions* • Importance of *using activities* or *small group work*
		Methods that Do Not Support or Encourage Student Participation	• Using a *lecture*-style to teach about RP • Providing well-planned *notes* and *examples* for students • Assigning *practice problems* for students to complete individually

Teaching (continued)	**Educational Goal:** *WHY* it is important to teach about reasoning and proving	Teach *Undergraduate Mathematics Students*	Focus on RP as important mathematical content knowledge: • need to teach this content because this is a *mathematics class* • *proof is important in mathematics;* therefore it's important to know • students can *develop their own ability* to reason and think mathematically
		Teach *Prospective Teachers*	Focus on preparing PTEs to teach about RP: • *prepare* 202 students to be able to *help their future elementary students* to reason and think mathematically • *202 students need to know more math* than their future students
		Outside Authorities	Other authorities say it's important, including the *course supervisor*, *textbook*, and *syllabus*

C.3 Coding Interviews for Conceptions of RP in Learning

First Level of Coding	Second level: Specific learning-related categories	Third Level: Sub codes within level two	Definitions or descriptions
Learning	Who are the learners	Students	Sub-codes if TA specifies *themselves; elementary*, *undergraduate*, or *graduate* students; or PTEs
		Mathematicians	Research mathematicians
		Anyone	Either learners at any level or did not specify 'who'

	Learning goals: *WHAT* do people need to know about RP	Produce a product related to reasoning-and-proving	Content, structure, or end product students should:
			• Know how to *write* a proof
			• Know *what a proof is*
			• Able to *regurgitate* the same proof later
			• Produce *clear, concise explanations*
			• Support each *statement with a reason*
			• Use *logical arguments* in writing a proof
			• Know a *proof exists* for a mathematical statement
			  ○ This is important math knowledge
			  ○ To respond to future students if they ask "why"
			• Know *Generality*: examples do not constitute proof

		Engage in RP processes	Way students build understanding of RP processes:
			• Determine "given" info & what is meant "to be proved" for a mathematical claim
			• Investigate mathematical situations
			• Generate conjectures
			• Evaluate conjectures; Evaluate provided arguments
			• Reason through how to construct a proof
			  ○ Build argument from end to beginning & beginning to end
			  ○ Try out approaches that may fail & learn from these failed attempts
			• Be convinced a statement is true by seeing a proof; not by examples
			• Understand RP pervasive in mathematics
			• Develop Habit of Mind (e.g., critical thinking)

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