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EFFECT OF VISCOUS DAMPING ON ACCURACY OF DYNAMOMETER MEASUREMENT OF A CYCLIC FLUCTUATING TORQUE

Thesis for the Degree of M. S.

MICHIGAN STATE UNIVERSITY
John Edward Nolan

1960

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This is to certify that the

thesis entitled

"Effect of Damping on the Accuracy of Dynamometer Measurement of a Cyclic Fluctuating Torque"

presented by

John Edward Nolan

has been accepted towards fulfillment of the requirements for

M. S. degree in M. E.

Louis L. Otto
Major professor

Date August 16, 1960

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EFFECT OF VISCOUS DAMPING ON ACCURACY OF DYNAMOMETER MEASUREMENT OF A CYCLIC FLUCTUATING TORQUE

By

JOHN EDWARD NOLAN

AN ABSTRACT

Submitted to the College of Engineering
Michigan State University of Agriculture and
Applied Science in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

Department of Mechanical Engineering

1960

Approved by Louis L. Otto

John Edward Nolan

ABSTRACT

A dynamometer is a device to measure force or torque. When the force or torque being measured fluctuates it becomes necessary to damp the motion of the indicating mechanism to obtain a reading. Damping mechanisms modify the magnitude of the force transmitted to the indicating mechanism, altering the instantaneous indications. This thesis describes the results of an investigation into the accuracy of the indication for a cyclic fluctuating torque impressed on a conventional dynamometer which uses viscous damping to produce a readable indication. A single-cylinder four-stroke-cycle internal-combustion engine, directly coupled to a cradled-field induction-motor electric dynamometer is used to experimentally check the analytical treatment of the variables involved.

The cylic-torque curve impressed on the dynamometer by the engine was measured by a strain-gage rosette on the connecting shaft between the two units, and recorded by photographing the trace on an oscilloscope produced by the amplified strain-gage signal. The average torque cycle thus obtained was treated by harmonic analysis methods to obtain an analytical expression representing this torque function.

The actual damped-scale system of the dynanometer was analytically represented as an equivalent simple damped spring-mass system. Using the analytical torque function developed from the oscilloscope trace as a forcing function, the equation of motion of the scale system was developed.

The solution of this motion equation was obtained, leading to the conclusion that a damped-scale system truly represents the average of a cyclic fluctuating torque, and that this situation is true regardless of the amount of viscous damping involved.

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John E. Nolan

BRIEF AUTOBIOGRAPHY

The author was born in Lansing, Michigan on January 7, 1934. He has resided in Lansing all of his life, and obtained his elementary education in the Lansing public schools. Upon graduating from Lansing Eastern High School in June, 1951, he entered Michigan State University the following September.

In June, 1957, he received a Bachelor of Science degree in Mechanical Engineering. He was employed as a laboratory technician for Ethyl Corporation during the summer of 1957, but returned to Michigan State in the fall to continue his education. Upon completion of requirements for a Master of Science degree, the author intends to remain at the university until he has obtained the degree of Doctor of Philosophy in Mechanical Engineering.

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INTRODUCTION

Throughout the field of engineering, and in other fields in which it is of interest to know the amount of power produced by any type of prime mover, such as an internal-combustion engine, steam engine, steam turbine, etc., a dynamometer is generally employed. While there are a number of different types of dynamometers, the purpose of any one of them is to measure the amount of force or torque which is produced by the power source in question. This measurement, along with a measurement of either angular or linear-displacement rate, can be used to obtain an expression for the horsepower output of the machine being tested. This paper will be concerned only with the type of dynamometer that measures horsepower output of a rotating shaft, this being the variety most commonly encountered.

Included in the types of dynanometers which measure the torque in a rotating shaft are the following: dry-friction dynamometers including water brakes and fan (air) brakes, and electromagnetic dynamometers which include eddy-current dynamometers and cradled-field electric dynamometers. All of these dynamometers can be used for measuring input torque. In all types except the last, the power which is transmitted to them from the source is dissipated in the form of heat energy generated within the dynamometer. The last type mentioned, ie., the cradled-field electric dynamometer, does not directly waste all of the power transmitted to it, but instead converts a large percentage of the power into useful electrical energy. Another feature of this type is that, with proper connections, it can be used as a driving motor to supply power to driven equipment. However, in many cases, the cost of this type of dynamometer is prohibitive.

Mechanical energy is supplied to each type of dynamometer as input torque impressed on the dynamometer shaft. The shaft is coupled either mechanically (by friction), magnetically, or by viscous forces, to an outer shell or field structure, which is mounted so that it is able to rotate in nearly frictionless bearings. If a lever arm of known length is attached to the side of this outer shell, and if the outer end of this arm is made to react on a force-measuring scale, for instance a Toledo scale, the torque that enters the dynamometer will be transmitted through the entire system until it meets an equal and opposite torque at the lever arm. The scale reading can then be multiplied by the effective length of the lever arm to obtain torque.

The amount of fluctuation in the torque vs time curves of various prime movers at a given average rotational speed varies from an almost constant torque for a steam turbine or an electric motor to the highly fluctuating torque of a one-cylinder internal-combustion engine. For an electric motor or steam turbine, probably the dynamometer arrangement as described above would result in an acceptable torque measurement, because the scale reading would remain at nearly the same value. However, with a one-cylinder gasoline engine, or anything else which produces a varying torque curve, the above arrangement would not be sufficient because the scale reading would fluctuate with the torque, and it would be impossible to obtain an accurate reading.

A certain amount of this fluctuation can be eliminated by using a large flywheel, a heavy armature, and a heavy field or outer shell, plus additional heavy weights on the bottom of the lever arm below the scale, with all of these heavy masses serving to increase the inertia of the system. However, the final answer for engines which produce any more than a small variation in torque is some sort of damping, usually in the form of dashpots.

It is the use of dashpots that brings up the main question of this paper. Dashpots are devices which absorb energy and dissipate it as heat, and in this case since the energy must come from the engine being tested, does this energy go to the dashpot at the expense of the desired torque reading and, as a result, is the reading on the scale lower than it should be?

PROCEDURE

An experimental investigation of this question requires that two things be known. The first of these is the torque as a function of time at some location on the connecting shaft between the engine and dynamometer, and the second is the physical characteristics of the entire system from the point at which the torque is measured to the end of the scale needle. This latter includes inertias, dampings, etc.

As a typical source of a cyclic fluctuating torque, a single-cylinder spark-ignition engine operating on the four-stroke cycle was used. The engine in question is better known as a CFR Octane-Rating Test Engine. This engine was directly coupled to a cradled-field three-phase induction motor by a short shaft and two flexible couplings. The shaft was carefully machined to produce known elastic properties. On the machined section a properly-oriented SR-4 strain-gage rosette was mounted. The signals produced by this strain-gage circuit in a Wheatstone bridge could be used to measure with a high degree of accuracy the torque carried by the shaft. The signal from the strain gages was carried through a set of slip rings to an amplifier, and the amplified signal was displayed on a cathode-ray oscilloscope. A Polaroid Land camera was mounted on the oscilloscope so that the trace could be recorded. A wire looped around the secondary ignition wire and plugged into one channel of a dual-beam plug-in on the scope was found to produce sufficient signal so that the scope could be triggered by the engine ignition.

After calibrating the torque-pickup by means of known weights, the engine was started and twenty-three pictures were taken of the torque curve. At the time that each picture was taken, the scale reading and engine rpm were noted. These twenty-three curves were averaged to give a single torque function, and from this point on it was assumed that the torque could be represented by a periodic repetition of this function. This function was called f(t), where the t stands for time.

In order to make use of f(t) in a mathematical equation, it was necessary to obtain an analytical expression for it. Since the function was periodic, the twenty-four ordinate system of an harmonic analysis was used to get an expression for f(t) in terms of a constant term plus ten sine terms and ten cosine terms, as shown here:

$$f(t) = a_0 + \sum_{n=1}^{10} a_n \cos \omega_n t + \sum_{n=1}^{10} b_n \sin \omega_n t .$$

Thus the first requirement was fulfilled.

The dynamometer used was of the cradled-field electric type which could be used as either a driving or absorption unit. The motor was an induction motor and operated above or below a synchronous speed of 1800 rpm. Protruding from one side of the motor housing was the lever arm, which was connected through linkages, and a small stiff spring to a "black box". This box contained a complicated arrangement of lever arms, one of which was connected to a small dashpot. The output link of this black box was connected to the input of a Toledo scale. The mechanism inside the Toledo scale was even more involved than that inside the black box. In addition to being connected to the scale system, the lever arm from the dynamometer had two heavy weights hanging from it at the same lever arm distance. Finally, there was another lever arm extending from the opposite side of the dynamometer housing, this arm being connected to a second and larger dashpot.

Needless to say, any attempt to incorporate the individual effect of each of the many components of this system into an equation would be a very difficult, if not impossible, task; and, even if it could be done, the resulting equation would be more difficult to solve than it was to derive. Therefore it was decided to develop the problem in the form of an equivalent system.

The equivalent system proposed was the one shown here, which is as simple as possible. It consists of an equivalent mass m hanging on a spring of equivalent spring constant k. Any motion of the mass is through an equivalent displacement x, this motion being retarded by a damping with coefficient c. f'(t) is the forcing function on the equivalent system.

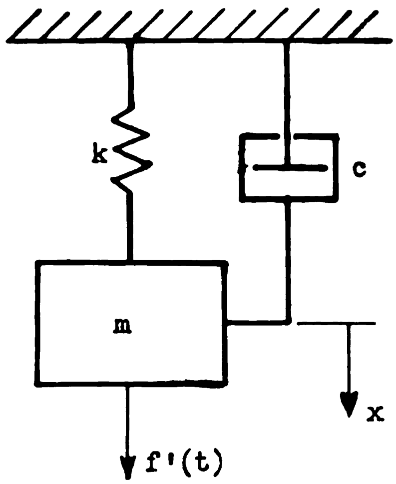


FIGURE 1

Assuming for the moment that k and m are constants, and that the damping is all viscous damping, which is the usual assumption for dashpots, the equation of motion for this system is

$$m\ddot{x} + c\dot{x} + kx = f'(t).$$

If the effect of the small spring between the dynamometer lever arm and the black box was neglected, everything from the dynamometer field to the scale pointer would move together, and could therefore be considered as the single equivalent mass m. This mass included the effects of the dynamometer field, the large weights hung on the lever arm, the weight of the part of the large dashpot that moved, and everything that moved in the black box and in the Toledo scale. It did not include the effect of the mass of the motor armature.

It was arbitrarily chosen that the displacement of this equivalent system be measured at the end of the dynamometer lever arm. In the equivalent system the equivalent mass would move through displacements as measured at this point. To determine the equivalent spring constant of the system, a dial gage was connected so that its input shaft rested on the top of one of the heavy weights below the black box. Since these weights were at the correct torque arm distance, the dial gage measured equivalent displacement. The scale was loaded to various readings and the dial gage reading was taken at each loading. The indicated load force was plotted against displacement at this point, and fortunately a straight-line plot resulted. This meant that the equivalent spring constant k was actually a constant, its value being equal to the slope of this straight line.

As is shown in the appendix the damping coefficient c also turned out to be a constant, at least within the limits of experimental accuracy. The equivalent mass m was found not to be constant over the whole range of displacements, but for the small variations in displacement encountered in operation, the mass was assumed constant also. The value for equivalent mass was picked from a curve once the range of operation on the scale was known.

The only term in the equation of motion for the equivalent system which has not been discussed is the forcing function, $f'(t)$. Since the forcing function must originate at the engine, the only way that it can act on the equivalent system is through the coupling of the armature and field of the motor. This coupling torque is a function of time and is directly proportional to f'(t). Since the forcing function must act at the point where equivalent displacement is measured, ie., at the end of the lever arm, f'(t) is equal to the coupling torque divided by the length of the lever arm.

While f'(t) has now been defined, an expression for it which can be used in the equation must still be found. For purposes of discussion, it is convenient to give the coupling torque between the armature and field a name; let it be called f"(t).

The armature of the motor experiences a varying torque f(t) on its input shaft, and another varying torque f"(t) in the opposite direction produced by the field coupling. Neglecting any friction effects, application of Newton's second law of motion to the armature results in the equation

$$f''(t) = f(t) - I\alpha,$$

where I is the moment of inertia of the armature, and α is its angular acceleration. Since f(t) is now known, and I can be determined experimentally, an expression for α, which would also be expected to be a function of time, would facilitate finding an expression for f"(t). Intuitively, one would expect f"(t) to be a periodic function with the same frequency as f(t), but with much lower extremes of amplitude. The average values of both f(t) and f"(t) would be expected to be the same.

As is discussed in the appendix, it was found that it was impossible to determine an expression for α, as the angular velocity over the cycle was constant within the limits of experimental accuracy. Therefore, since the equation of motion for the equivalent system was still lacking a forcing function, it was decided to assume that $f''(t) = f(t)$. While it is obvious that this assumption is not valid, it should not affect the answer to the main question of the paper, ie., the one concerning the dashpots. It is in fact conceivable that a function could be produced at the strain gages which would result in a function at the field-armature coupling that would exactly equal $f(t)$. Actually, all that is required here is a representative forcing function, and $f(t)$ is certainly that. $f'(t)$ is now equal to $f(t)$ divided by the lever-arm length L.

Now everything in the equivalent system equation has been accounted for, and the equation is again,

$$m\ddot{x} + c\dot{x} + kx = f'(t) = \frac{f(t)}{L}$$

$f'(t)$ can be written, as was $f(t)$, as a constant plus ten sine terms and ten cosine terms, so the equation becomes

$$m\ddot{x} + c\dot{x} + kx = a_0' + \sum_{n=1}^{10} a_n' \cos \omega_n t + \sum_{n=1}^{10} b_n' \sin \omega_n t \, .$$

This equation is solved in the appendix for the particular values which applied when the data was taken. The resulting expression for x is of the form

$$x = \frac{a_0'}{k} + \sum_{n=1}^{10} M_n \sin \omega_n t + \sum_{n=1}^{10} N_n \cos \omega_n t \, .$$

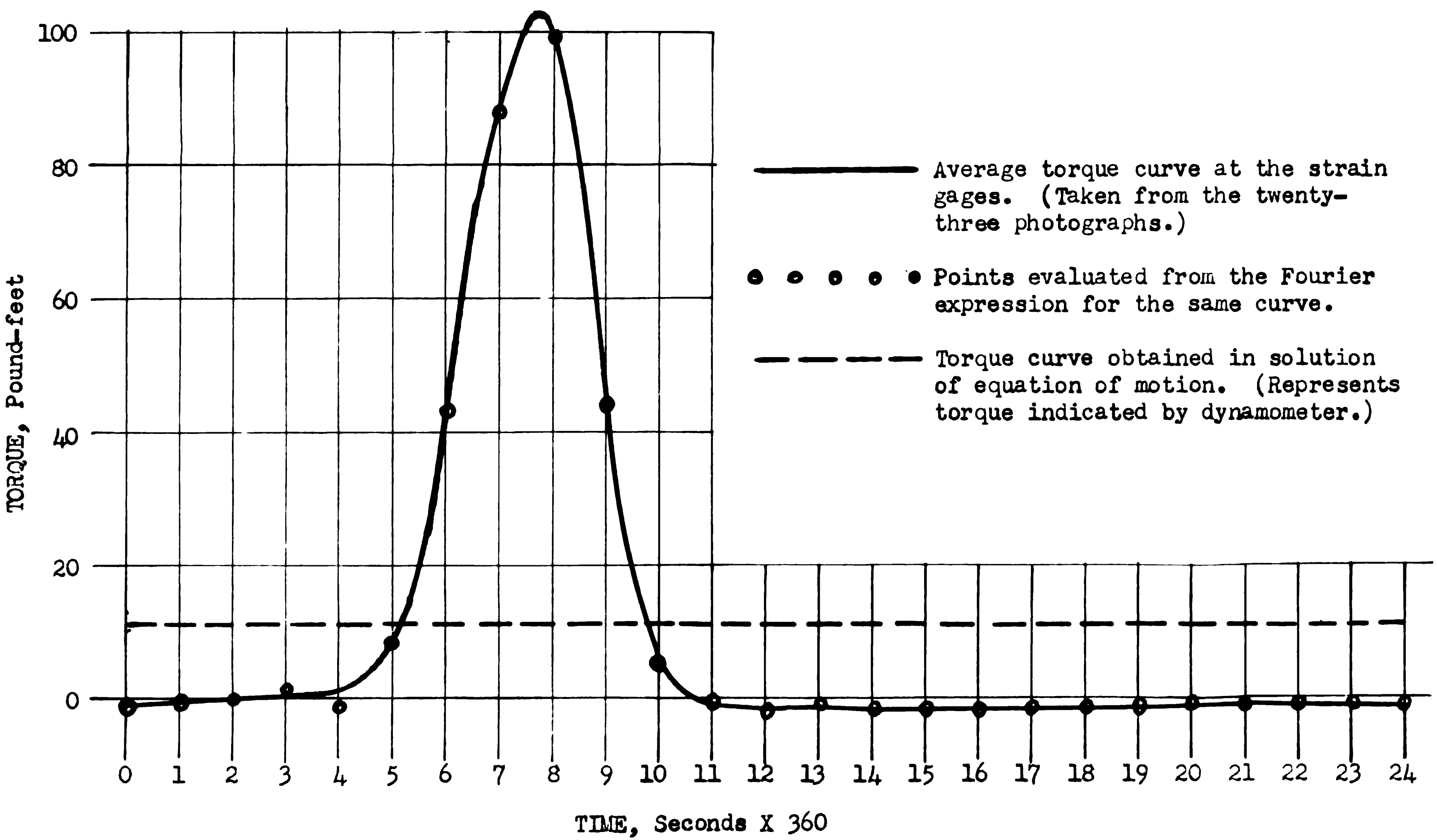
Each sine term and each cosine term in this expression will, over the cycle, average out to zero. Therefore the average displacement is given by

$$x_{avg.} = \frac{a_0'}{k} \, .$$

The force associated with this displacement is just a_0'. The torque associated with this force is $a_0' L$, and $a_0' L = a'$ which is the constant term of f(t). As in the case of x, the sine and cosine terms of f(t) also average out to zero, which means that the average value of the torque is not affected by the dashpots. Therefore, the value obtained by multiplication of the scale reading by the lever-arm length should be a true representation of average torque.

Although a conclusion has already been reached, without the use of any numerical data, it is interesting to see the results that were obtained for a special case. While the details of the calculations for this special case are left for the appendix, the results of interest are contained in the two torque curves on the next page. The curve with the large hump represents the torque curve as picked up at the strain gages, while the other more-level curve (which to the scale of this graph is practically a straight line) represents the torque function which results from the solution of the equation. As predicted, these two curves fluctuate about the same average torque value, although there is considerable difference in the amounts of fluctuation. During the time that the data was taken, it was noted that the usual fluctuation of the scale needle was less than 0.2 pounds, except for occasional small unpredictable jumps. Since the lever arm was very nearly one foot long, the torque represented by the scale reading varied less than 0.2 pound-feet. In looking at the data for the more-level curve in figure 2, it was found that the maximum variation was 0.154 pound-feet. This is very good agreement considering that 0.2 pound-feet was the smallest graduation on the dynamometer scale.

FIGURE 2 — Average torque curves over the cycle.



DISCUSSION OF RESULTS OBTAINED

The results of the problem as considered up to this point indicate that the torque as determined from the dynamometer is a true representation of average torque output of the engine. However, there were a few simplifying assumptions made during the procedure, the effects of which should be considered. The first of these assumptions was that the damping coefficient c is a viscous damping coefficient only. It is more than likely that there are other types of damping in the system, and there should be some justification for neglecting their effect. The main argument to this point is that their effect is small compared to the viscous damping. That this is true is made quite apparent by allowing the system to vibrate with and without the dashpots being connected. With the dashpots connected, the system, upon being released from a position displaced from its equilibrium position, tends to return to equilibrium directly with no overtravel. With the dashpots disconnected, the system, when subjected to the same displacement, vibrated for many cycles before coming to rest. This reasoning implies of course that the damping in the dashpots is truly viscous. While this assumption might not be exactly correct either, it is usually made in dealing with dashpots, and chances are that the majority of their damping is viscous. Two more reasons for assuming viscous damping only is that the magnitudes of other types of damping are difficult to determine, and, even if they could be determined, the resulting equation of motion would be non-linear and consequently much more difficult to solve.

Another assumption made was that the dynamometer armature rotated at a constant angular velocity. While it is shown in the appendix that the variation is small, it is of interest to determine if the final results of the problem would have been changed had this fluctuation been included in the analysis. Including it would have produced the same equation of motion to be solved except for the forcing function. The forcing function would have had the same constant term, but different coefficients for the sine and cosine terms. As with the forcing function that was used, the only term which would have had any effect on the average value of displacement would have been this same constant term, so the conclusion is that this assumption had absolutely no effect on the outcome of the problem.

The third assumption to be considered is the one that the forcing function is periodic. It was apparent by examining the pictures taken of the torque function that there were no two cycles exactly alike, but still twenty-three representative possibilities were averaged and the torque function used as f(t) was assumed to be this average curve repeated periodically. The only justification for this assumption was that it did allow the equation of motion to be solved. If an attempt had been made to include variations in each different cycle in the analysis, probably the first big problem would have been to determine an expression for the forcing function, since it would no longer have been periodic and harmonic analysis would no longer apply. If an expression were found for f(t), it would not likely be a simple sine and cosine relation, and the solution to the equation would be much more difficult. Each new cycle, being different from the one preceding, would start a new transient component in the response curve; and before sufficient time could elapse for this transient to die out, another different one would appear. In short, solution of the equation by manual methods would be an impossibility. It is, however, conceivable that a computer could be used to analyze the equation if the $m\ddot{x} + c\dot{x} + kx$ part were set up and if the forcing function were then fed in directly from the amplified strain gage signal. Without the use of a computer, the method chosen was felt to be the best available, and the results produced by it were quite satisfactory.

A slight discrepancy which may be noticed is in the value of beam load during the test. While the pictures were being taken, the average beam load as indicated by the Toledo scale was 11.3 pounds, corresponding to an average torque of 11.865 pound-feet. In averaging the torques from the pictures, the value was found to be 11.197 pound-feet. This is a difference of 0.668 pound-feet. When the signal to the ocilloscope was calibrated, each centimeter on the screen was made to represent only 0.0334 centimeter. The blame for this discrepancy was put on the drift that was encountered in the amplifier and oscilloscope. Even after allowing both instruments to warm up for a number of days the drift was still present, and it occurred sometimes in a very short time, easily in the time required to take twenty-three pictures. When the discrepancy was first noticed it was felt that possibly another set of pictures should be taken, but when the small amount of displacement corresponding to 0.668 pound-feet on the screen was calculated, it was decided that the pictures which had already been taken were probably as good as would be obtained, since the drift had been a problem from the start. Therefore it was decided to use the average torque obtained from the pictures in the calculations so that this drift error would not appear as an error. On this basis, the question of the paper could still be answered.

Possibly the amount of drift trouble could have been decreased if the shaft upon which the strain gages were mounted had been of a smaller diameter. This would have required less amplification from Brush amplifier and oscilloscope and the drift might have been correspondingly lessened.

SUMMARY AND CONCLUSIONS

In considering a damped dynamometer system upon which is impressed a fluctuating torque, intuition leads one to suspect that possibly the smoothing out of the curve for the scale by the damping might be done at the expense of the average torque as indicated by the scale. This paper has shown that the torque value which is determinable by multiplying the beam load by the dynamometer lever-arm length is the true value of average torque if the impressed torque from the engine is periodic, regardless of the amount of viscous damping involved.

APPENDICES

Appendix A -- Detailed Description of Test Equipment

Dynamometer
Induction Type Dynamometer
Type HDM 365 Serial No. 263898
No load rpm 1800 volts 440 cy. 60 ph. 3
As motor delivers 25 hp at 1750 rpm
As generator absorbs 27 hp at 1855 rpm
Full load current as motor 30 amperes
Inrush current 170 amps at full voltage
Manufactured by
Harnischfeger Corporation
Milwaukee, Wisconsin

Scale
Toledo Springless Scale
Model 2081 Serial No. 2504
Factory No. 2081-0-5513712
Capacity 75 pounds
Manufactured by
Toledo Scale Company
Toledo, Ohio

Engine
CFR Octane-Rating Test Engine
No. 807998
Compression ratio variable from 4 to 10:1
Manufactured by
Waukesha Motor Company
Waukesha, Wisconsin

The dynamometer, Scale, and Engine are all included under ME No. 2706.

Stop Clock
Type S-6 Inst. No. 21941
115 volts .036 amps 60 cycle
Speed 10 rpm
Smallest graduation .001 minute
ME No. 3474
Manufactured by
The Standard Electric Company
Springfield, Massachusetts

Electronic Counter
Model 521 A   ME No. 3440
Manufactured by
Hewlett Packard Company
Palo Alto, California

Strain Gages
SR-4 Strain Gages
Four type C-10
Resistance 1000 ± 5 ohms
Gage factor 3.22 ± 2%
Lot No. A
Manufactured by
The Baldwin-Lima-Hamilton Corporation
Waltham 54, Massachusetts

Amplifier
Brush Amplifier
Model RD 561200   Serial No. 134
Amplifier ME No. 3634 with Bridge Balance ME No. 3635
Frequency Response 50 to 2000 cps.
Manufactured by
Brush Instruments
Division of Clevite Corporation
Cleveland 14, Ohio

Oscilloscope
Type 532-57 Oscilloscope ME No. 3714, Serial No. 6325
Type CA Plug-in unit, ME No. 3703, Serial No. 002892
Manufactured by
Tektronix Incorporated
Portland, Oregon

Measuring Microscope
Pye Two-Dimensional Measuring Microscope AM No. 708
Serial No. 39,566   Catalog No. 6147
Minimum graduation .01 mm.
Manufactured by
W. G. Pye and Company Ltd.
Granta Works, Cambridge, England

Appendix B — Description of Test Equipment Assembly

The engine and dynamometer used are a permanent installation, and were modified for this test only to the extent of installing a different driveshaft which would accommodate slip-rings and strain gages. The shaft used was a one and one-half inch diameter steel shaft and it was connected to both the engine and dynamometer by means of gear couplings producing a direct drive.

The slip-rings and brushes were built up from a kit directly on the driveshaft. The four strain gages were mounted on the shaft, and then were connected in a Wheatstone bridge circuit, the corners of the bridge being connected to the slip-ring leads. This method of mounting the gages and connecting them into the circuit eliminates any response in the bridge due to either bending or compressing and pulling the shaft. Any signal from the bridge should then represent torque only. This is discussed more thoroughly in Perry and Lissner (1). The bridge output was taken from the slip-rings through brushes and connected to the bridge balance input on the front of the Brush Amplifier. After amplification the signal was then sent to the oscilloscope. A schematic sketch of the torque pickup circuit is shown in Figure 3.

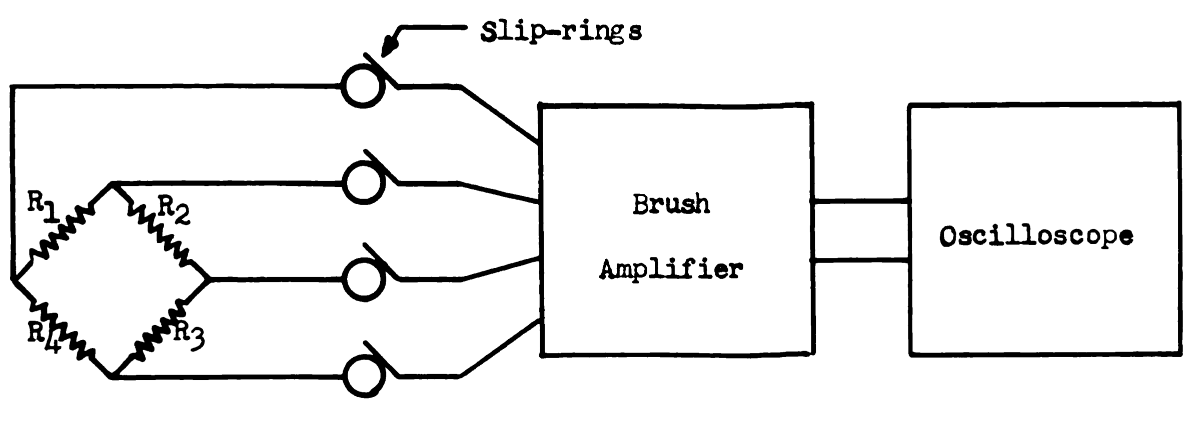


FIGURE 3

In order to show just how the four strain gages were mounted on the shaft, suppose that the outside cylindrical surface of the shaft were unrolled and laid out flat; then the orientation of the gages is easier to see as shown here.

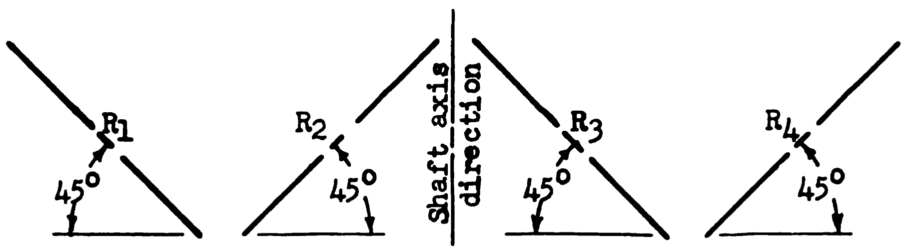


FIGURE 4

The centers of gages R_1 and R_3 were diametrically opposite on the shaft, as were those of R_2 and R_4.

The electronic tachometer received its signal from a sending device which was attached to the shaft on the front of the engine.

To assure that all of the pictures taken started at the same spot in the cycle, the oscilloscope was triggered by the ignition pulse of the engine. It was found that a piece of ordinary wire looped once around either the secondary or the primary ignition wire and plugged into the oscilloscope produced sufficient signal to trigger the sweep circuit. To obtain the trigger signal and the torque signal simultaneously, a dual beam plug-in unit was used in the oscilloscope. The triggering level was set so high that it could not be triggered by the torque signal, and the ignition signal was adjusted in magnitude so that it triggered consistently. When the pictures were taken the triggering signal was raised until it no longer showed on the screen, but only served as a trigger. Only the torque curves were photographed.

Appendix C — Calibration of Measuring Equipment

Calibration of the torque pickup.

The torque signal indicated on the screen of the oscilloscope could be interpreted most easily if the amplifiers in the torque signal circuit were adjusted to yield a known torque level for each centimeter of signal rise. This adjustment could have been accomplished in either of two methods. The first method requires an accurate knowledge of the values of Poisson's ratio and modulus of elasticity of the shaft on which the strain gages were mounted. If these two values are known accurately, they can be used, along with the shaft diameter and the strain gage resistance and gage factor, to compute the amount of torque that would produce the same deflection of the scope trace as would a known resistance shunted across one leg of the strain gage bridge. Then whenever a resistor with this value of resistance is shunted into the proper place in the circuit, the resulting scope deflection could be adjusted in amplitude by means of the amplification controls on either the scope or the Brush amplifier until the torque represented by this deflection was on a convenient scale. When the shunt resistor is removed from the circuit the deflection should return to zero. There was a resistor built into the Brush amplifier which was meant to be used for this purpose. However, its value (390,000 ohms) was too low for this particular application, and when the scope was adjusted to give a satisfactory deflection for the torque picture, the calibration signal produced by this resistor was off the screen. A resistor with about 800,000 ohms resistance would have produced an acceptable deflection; however a precision resistor would have been required, and these are not readily available. This difficulty, along with the improbability that Poisson's ratio and the modulus of elasticity could be determined accurately, resulted in the second method of calibration being used.

This second method was much more direct and depended upon fewer factors than the first. A lever arm was made which could be attached to the free end of the dynamometer shaft. A hole was drilled in this lever arm exactly eighteen inches from the center of the shaft, and a weight hanger suspended in this hole. The system was rotated until this lever arm was horizontal,and the engine flywheel was then clamped in position. With the lever arm and weight hanger on the shaft, the scope trace was brought to the desired zero line. Eighty pounds of weight were placed on the hanger to produce a deflection corresponding to 120 pound-feet of torque. The amplification of the signal was then adjusted to a convenient scale (1 cm. = 20 lb. ft.). Then the weights were removed one at a time and the trace displacement checked between weights in order to assure that the response was linear. The first time that this was done, the scope did not return to zero. After sufficient repetitions, however, the system settled down and finally returned to zero with each unloading. After this calibration was completed, the lever arm and weight hanger were removed. This of course caused the scope trace to drop, but since the displacement vs torque plot was linear, the trace could be brought back to zero with the vertical position control on the oscilloscope with no loss in accuracy.

There was a considerable problem of drift in both the Brush amplifier and in the oscilloscope even over relatively short periods of time. Therefore, once the system is calibrated as described above the data should be taken as soon as possible, or the calibration should be repeated. Calibration by this method is somewhat time consuming, and it was felt that it would be desireable to have a quicker way of bringing the system back into calibration once it had been done accurately with the weights. As was mentioned before, a resistor of about 800,000 ohms was calculated to be the size which would produce a desireable trace deflection when shunted across one leg of the bridge. It was also stated that a precision resistor would be required to calibrate with. However, the system could be calibrated with the weights, and once it is calibrated a non-precision resistor of about 800,000 ohms could be shunted across the bridge to produce a deflection. The amount of this deflection could then be noted, and for subsequent calibrations it would only be necessary to zero the signal, shunt the resistor and adjust its deflection to the noted value, and then check to see that the signal returns to zero when the shunt is removed. This is the method which actually was used.

Other calibrations

The calibration of the torque picture is the only one that requires any discussion. The preliminary calibration and adjustment of the Brush amplifier is described thoroughly in the instruction manual, and adjustments such as the method of zeroing the needle on the Toledo scale are so obvious as to need no instruction.

Appendix D — Details of calculations

Analytical expression for the torque curve.

After the twenty-three pictures of representative torque curves were taken, each curve was divided into twenty-four equal segments. The height of each curve at the left end of each of these segments was then measured accurately with the aid of a two-dimensional measuring microscope. The torques represented by each of these measurements were then computed, and the curves were averaged to obtain the curve for f(t) which is shown in Figure 2.

To determine an analytical expression for f(t) the method of harmonic analysis was used as is outlined in Wylie (2). The expression produced for f(t) by using this method is in the form of a Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{10} a_n \cos \omega_n t + \sum_{n=1}^{10} b_n \sin \omega_n t ,$$

and harmonic analysis is simply the means of computing the coefficients a_n and b_n. The number of terms that will result depends on the particular harmonic analysis system chosen. The one used here was the twenty-four ordinate system.

To use this system, the curve for f(t) is divided into twenty-four equal sections. Then the torque at the left end of each of these sections is determined. This was done when the curves were averaged. Let the torque at the beginning of the first section be called y_0, that at the beginning of the second section y_1, and so on up to y_{23} at the beginning of the last section. These values were, in pound-feet:

$y_0 = -1.020$	$y_6 = 42.805$	$y_{12} = -1.593$	$y_{18} = -1.753$
$y_1 = -0.375$	$y_7 = 88.621$	$y_{13} = -1.568$	$y_{19} = -1.447$
$y_2 = -0.076$	$y_8 = 98.979$	$y_{14} = -1.649$	$y_{20} = -1.012$
$y_3 = 0.167$	$y_9 = 43.835$	$y_{15} = -1.729$	$y_{21} = -0.891$
$y_4 = 0.520$	$y_{10} = 5.341$	$y_{16} = -1.694$	$y_{22} = -0.993$
$y_5 = 8.328$	$y_{11} = -1.192$	$y_{17} = -1.679$	$y_{23} = -1.181$

Once these values were known, the coefficients were found by using the tabular outline described in Wylie. Since the operations in this table are self-explanatory, no additional discussion is necessary. The table for this problem is shown on the following four pages. The resulting a_n and b_n coefficients through $n = 10$ are:

$a_0 = 11.197$	$a_6 = 4.209$	$b_1 = 22.244$	$b_6 = -2.966$
$a_1 = -8.522$	$a_7 = 0.211$	$b_2 = -14.638$	$b_7 = 2.839$
$a_2 = -14.644$	$a_8 = -1.220$	$b_3 = -6.134$	$b_8 = -0.874$
$a_3 = 15.506$	$a_9 = 0.885$	$b_4 = 12.558$	$b_9 = -0.025$
$a_4 = -0.658$	$a_{10} = -0.480$	$b_5 = -3.827$	$b_{10} = 0.547$
$a_5 = -7.576$			

Now ω_1 is equal to $2\pi f$, where f is the frequency at which the cycle is repeated.

$$f = 1800 \text{ rev./min.} \times 1 \text{ cycle/2 rev.} \times 1 \text{ min./60 sec.}$$

$$= 15 \text{ cycles/second}$$

Then $\omega_1 = 30\pi$ and $\omega_n = 30n\pi$, so everything in

$$f(t) = a_0 + \sum_{n=1}^{10} \left[a_n \cos \omega_n t + b_n \sin \omega_n t \right]$$

has been defined. t is time in seconds.

On page 11 is shown the torque function f(t) as plotted from the results of averaging the twenty-three representative pictures. Plotted as circles along this curve are the values of torque that result from evaluating the Fourier expression for f(t) at each of these points. The close proximity of these points to the curve substantiates the accuracy of the method of harmonic analysis.

Determination of k, the equivalent spring constant.

The procedure used to determine k is described in detail in the main part of this paper. A plot of force vs deflection is shown on page 30, and the slope of this curve is k. Its value was found to be

$$k = \underline{172.4} \text{ pounds/inch} .$$

Tabular Determination of Fourier Coefficients

y_0 to y_5	-1.020	- .375	- .076	.167	.520	8.328
y_{23} to y_{19}	. .	-1.181	- .993	-.891	-1.012	-1.447
Sums (c_0-c_5)	-1.020	-1.556	-1.069	-.724	- .492	6.881
Diffs. (d_1-d_5)	. .	.806	.917	1.058	1.532	9.775

y_6 to y_{12}	42.805	88.621	98.979	43.835	5.341	-1.192	-1.593
y_{18} to y_{13}	-1.753	-1.679	-1.694	-1.729	-1.649	-1.568	. .
Sums (c_6-c_{12})	41.052	86.942	97.285	42.106	3.692	-2.760	-1.593
Diffs. (d_6-d_{11})	44.558	90.300	100.673	45.564	6.990	.376	. .

c_0 to c_6	-1.020	-1.556	-1.069	-.724	-.492	6.881	41.052
c_{12} to c_7	-1.593	-2.760	3.692	42.106	97.285	86.942	. .
Sums (e_0-e_6)	-2.613	-4.316	2.623	41.382	96.793	93.823	41.052
Diffs. (f_0-f_5)	.573	1.204	-4.761	-42.830	-97.777	-80.061	. .

d_1 to d_6	.806	.917	1.058	1.532	9.775	44.558
d_{11} to d_7	.376	6.990	45.564	100.673	90.300	. .
Sums (g_1-g_6)	1.182	7.907	46.622	102.205	100.075	44.558
Diffs. (h_1-h_5)	.430	-6.073	-44.506	-99.141	-80.525	. .

e_0 to e_3	-2.613	-4.316	2.623	41.382
e_6 to e_4	41.052	93.823	96.793	. .
Sums (j_0-j_3)	38.439	89.507	99.416	41.382
Diffs. (k_0-k_2)	-43.665	-98.139	-94.170	. .

h_1 to h_3	.430	- 6.073	-44.506
h_5 to h_4	-80.525	-99.141	. .
Sums (l_1-l_3)	-80.095	-105.214	-44.506
Diffs. (m_1-m_2)	80.955	93.068	. .

Determination of Fourier Coefficients—Continued

	j_1= 89.507	j_2= 99.416	k_2= -94.170	l_1= -80.095
X .500	j'_1= 44.754	j'_2= 49.708	k'_2= -47.085	l'_1= -40.048

	k_1= -98.139	l_2= -105.214	m_1= 80.955	m_2= 93.068
X .866	k'_1= -84.988	l'_2= - 91.115	m'_1= 70.107	m'_2= 80.597

	j_0= 38.439 j_1= 89.507 j_2= 99.416 j_3= 41.382	k_0= -43.665 k'_1= -84.988 k'_2= -47.085
Sum col. 1 Sum col. 2	137.855 130.889	- 90.750 - 84.988
Sum Difference	268.744 = $24a_0$ 6.966 = $24a_{12}$	-175.738 = $12a_2$ - 5.762 = $12a_{10}$

	j_0= 38.439 j'_1= 44.754 $-j'_2$= -49.708 $-j_3$= -41.382	k_0= -43.665 k_2= -94.170
Sum col. 1 Sum col. 2	-11.269 3.372	. . . .
Sum Difference	- 7.897 = $12a_4$ -14.641 = $12a_8$	. . 50.505 = $12a_6$

	l'_1= -40.048 l'_2= -91.115 l_3= -44.506	m'_1= 70.107 m'_2= 80.597	l_1= -80.095 l_3= -44.506
Sum col. 1 Sum col. 2	- 84.554 - 91.115	. . . .	. . . .
Sum Difference	-175.669 = $12b_2$ 6.561 = $12b_{10}$	150.704 = $12b_4$ -10.490 = $12b_8$	. . -35.589 = $12b_6$

	f_1= 1.204	f_3= -42.830	f_5= -80.061
X .707	f'_1= .851	f'_3= -30.281	f'_5= -56.603

Determination of Fourier Coefficients—Continued

	$g_1 = 1.182$	$g_3 = 46.622$	$g_5 = 100.075$
X .707	$g_1' = .836$	$g_3' = 32.962$	$g_5' = 70.753$

	$f_1' = .851$	$g_1' = .836$
	$f_5' = -56.603$	$g_5' = 70.753$
Sum	$-55.752 = p_1$	$71.589 = s_1$
Difference	$57.454 = p_2$	$-69.917 = s_2$

	$f_2 = -4.761$	$p_1 = -55.752$	$g_4 = 102.205$	$s_1 = 71.589$
X .866	$f_2' = -4.123$	$p_1' = -48.281$	$g_4' = 88.510$	$s_1' = 61.996$

	$f_4 = -97.777$	$p_2 = 57.454$	$g_2 = 7.907$	$s_2 = -69.917$
X .500	$f_4' = -48.889$	$p_2' = 28.727$	$g_2' = 3.954$	$s_2' = -34.959$

	$f_4' = -48.889$	$p_1' = -48.281$	$g_3' = 32.962$	$g_6 = 44.558$
	$f_2' = -4.123$	$p_2' = 28.727$	$s_2' = -34.959$	$g_2' = 3.954$
Sum	$q_1 = -53.012$	$r_1 = -19.554$	. .	$u = 48.512$
Difference	$q_2 = -44.766$	$r_2 = -77.008$	$t = 67.921$	. .

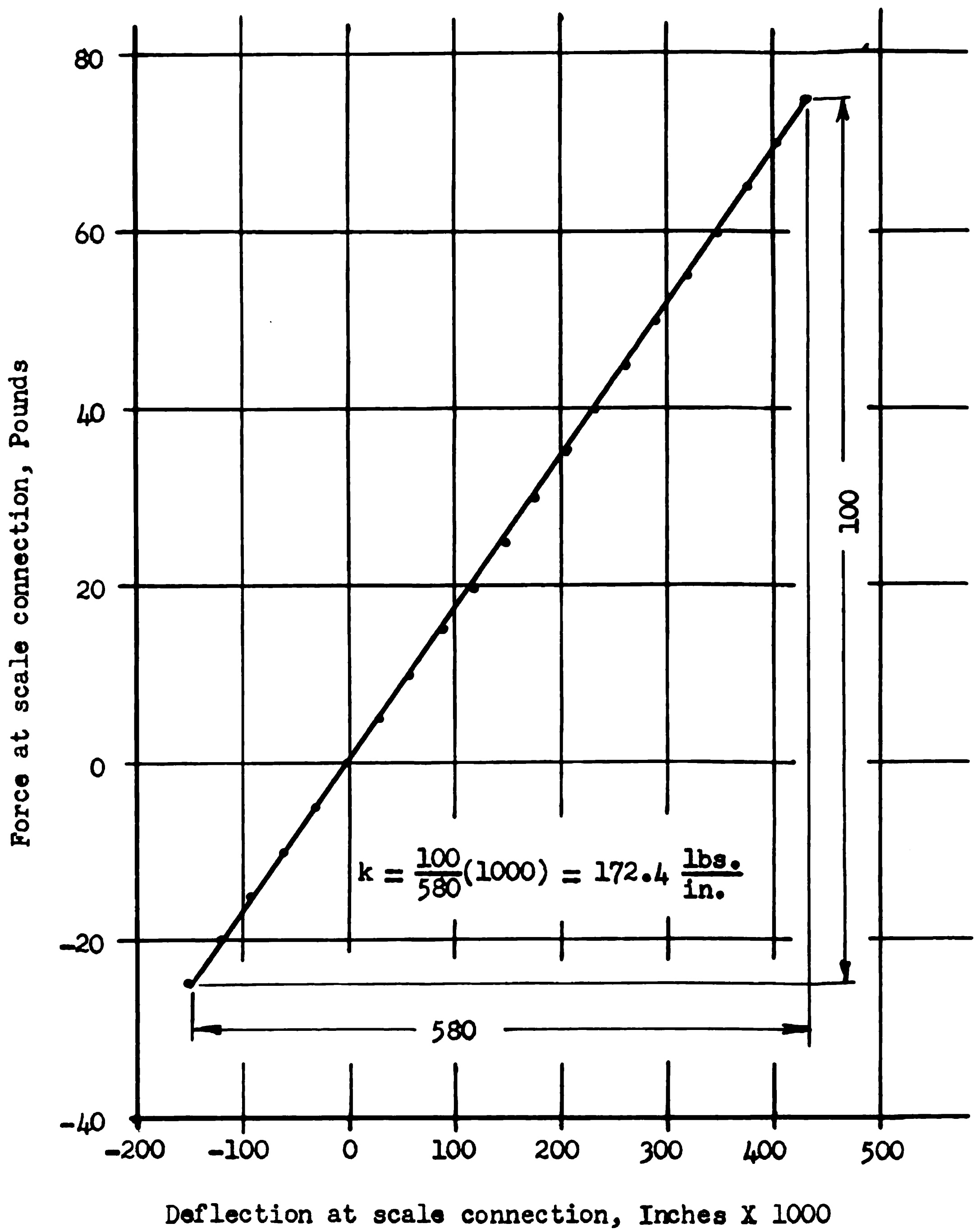
	$f_0 = .573$	$f_3' = -30.281$	$f_0 = .573$	$p_2 = 57.454$
	$q_1 = -53.012$	$r_1 = -19.554$	$-f_4 = 97.777$	$-f_3' = 30.281$
Sum col. 1	-52.439		98.350	
Sum col. 2	-49.835		87.735	
Sum	$-102.274 = 12a_1$		$186.085 = 12a_3$	
Difference	$-2.604 = 12a_{11}$		$10.615 = 12a_9$	

Determination of Fourier Coefficients—Continued

	$f_0 =$ .573 $r_2 =$ -77.008 $q_2 =$ -44.766 $-f_3' =$ 30.281	$s_1' =$ 61.996 $u =$ 48.512 $t =$ 67.921 $g_4' =$ 88.510
Sum col. 1 Sum col. 2	-44.193 -46.727	129.917 137.022
Sum Difference	$-90.920 = 12a_5$ $2.534 = 12a_7$	$266.939 = 12b_1$ $7.105 = 12b_{11}$

	$s_2 =$ -69.917 $g_2 =$ 7.907 $g_3' =$ 32.962 $-g_6 =$ -44.558	$s_1' =$ 61.996 $u =$ 48.512 $-t =$ -67.921 $-g_4' =$ -88.510
Sum col. 1 Sum col. 2	-36.955 -36.651	- 5.925 -39.998
Sum Difference	$-73.606 = 12b_3$ $- .304 = 12b_9$	$-45.923 = 12b_5$ $34.073 = 12b_7$

FIGURE 5 -- Curve used for determination of the spring constant, k.



Determination of m, equivalent mass.

The equivalent system which was chosen to represent the actual system from the dynamometer field to the scale pointer is shown in Figure 1 on page 6. If the forcing function is removed, the damping disconnected, and the remaining parts of the system allowed to vibrate freely, the frequency of this vibration is, from elementary vibrations,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad .$$

Solving this for m,

$$m = \frac{k}{(2\pi f)^2}$$

The equivalent weight of this mass would be

$$W = mg = \frac{kg}{(2\pi f)^2} = \frac{(172.4)(386)}{(2\pi)^2 f^2} = \frac{1688}{f^2} \quad .$$

Since the character of the moving parts which were included in the equivalent mass was quite complicated, it was felt that probably the equivalent mass would not be the same for all ranges of the scale. Therefore it was decided to determine the equivalent weight at intervals of five pounds from zero to thirty pounds, and to plot a curve of equivalent weight versus range on the scale. Then, after the range of operation during the experiment was known, the value of equivalent weight to be used in the differential equation could be picked from this curve.

For oscillations about the zero position on the scale, the equation $W = \frac{1688}{f^2}$ is valid as it stands. However, in order to make the system oscillate about other load levels, it was necessary to add weights so that the scale when at rest indicated the load which was to be oscillated about. These additional weights were placed on top of the large weights which were underneath the "black box". Since displacements of these large weights were the same value as equivalent displacements in the equivalent system, adding the extra weights at this point served to increase equivalent weight by the same amount added. These additional weights should not be included in the value of equivalent weight which is finally used in the equation. To compensate for them, let W" represent the amount added, and let W' represent the total of effective weight plus added weights, or W' = W + W". When the system is allowed to oscillate freely about some equilibrium position other than zero, the equation to be used for determining W is

$$W' = W + W'' = \frac{1688}{f^2} ,$$

or

$$W = \frac{1688}{f^2} - W'' ,$$

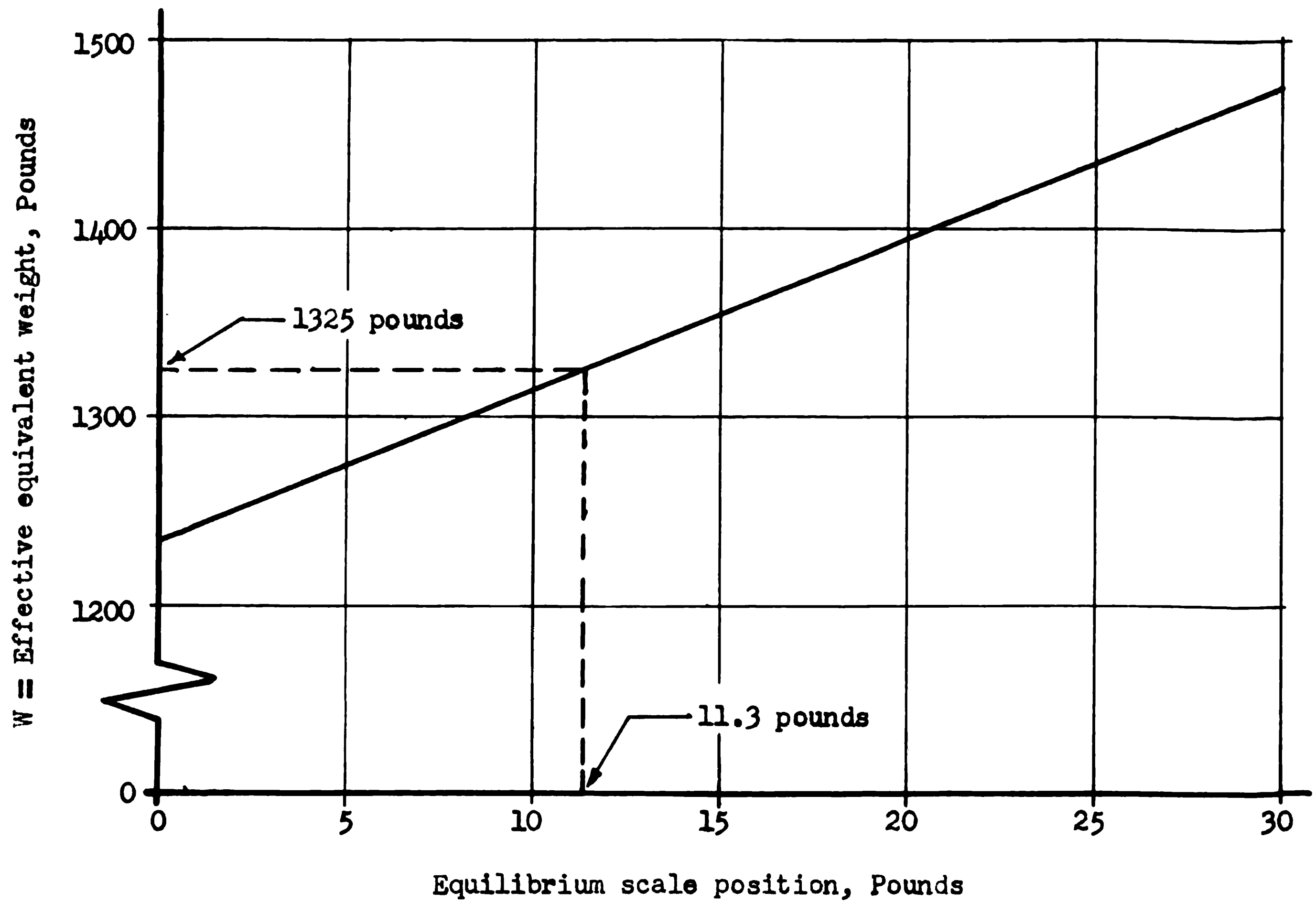
or the equivalent weight to be used in the equation is determined by dividing the square of the observed frequency into 1688 and subtracting the amount of additional weight which was used.

In order to obtain an exact value of frequency for a given equilibrium position of the scale, it would be necessary to let the system oscillate through very small amplitudes about this position, since mass does vary with displacement. However for very small amplitudes, even with the dashpots disconnected, the amount of damping present in the system due to friction is sufficient to practically stop the motion completely before enough cycles are completed to obtain data which would result in an accurate value of frequency. Theoretically, if everything involved in the method of taking data is working perfectly, one cycle of oscillation should be sufficient. However, the clock that was used to time the oscillations was accurate to only .001 minute, and the order of magnitude of the frequencies was about one cycle per second, or one oscillation would take place in about .017 minute. For time intervals of this small value there is chance for error not only in the reading of the clock but also in the human reaction time involved in both starting and stopping the clock. For this reason then it was felt that the time for a number of oscillations should be used instead of the time for just one.

To overcome the problem of varying mass about a point, the procedure was as follows. The system was allowed to oscillate about each point with starting amplitudes of five, four, three and two pounds scale deflection. For each starting amplitude the frequency was then plotted versus original amplitude for each range. These curves were then extrapolated to the point where starting position is zero. The resulting frequencies were the ones assumed to be correct about their respective equilibrium positions. These resulting frequencies were then plotted against their respective equilibrium scale locations. Because this plot of points was somewhat scattered, it was decided that a straight line would be as good a representation of the general trend as could be chosen. After this line was drawn, it was used as being accurate, and the value of frequencies used in subsequent calculations were those picked from this curve.

To determine the variation in equivalent weight W with scale position, the formula $W = \frac{1688}{f^2} - W''$ was used, the values of f^2 being determined from the plot of f^2 versus scale position, W" being the scale position. This plot of W versus scale position was the one used to determine the value of W to put into the differential equation, and is presented on the following page. Since the average beam load during

FIGURE 6 -- Curve used for determination of effective weight, W.



the test run was 11.3 pounds, the corresponding equivalent weight from this curve is found to be $W = 1325$ pounds. This is the value of equivalent weight to be used in the equation. Since the equation calls for mass instead of weight, the value of mass corresponding to this weight is

$$m = \frac{W}{g} = \frac{1325 \text{ lb. sec.}^2}{386 \text{ in.}} = 3.43 \frac{\text{lb. sec.}^2}{\text{in.}} .$$

Determination of c, coefficient of damping for the equivalent system.

With the dashpots connected, when the system was displaced from equilibrium and released, it tended to return to equilibrium; however the length of time that was required for its return indicated that the equivalent damping coefficient was greater than critical. Jacobsen and Ayre (3) give the following relation for systems with damping coefficients appreciably greater than the critical:

$$t_\epsilon = \frac{2\gamma}{p} \ln \frac{1}{\epsilon}$$

where: γ = damping factor = ratio of damping coefficient to critical damping coefficient.

p = natural frequency of the system in radians per second.

ϵ = fraction of starting displacement at time t_ϵ after start.

t_ϵ = time for the system to move from its starting displacement to ϵ X starting displacement.

This formula is meant for use in cases of large γ and small ϵ. Since damping coefficient c is the unknown in this case, the above equation can be rearranged using the following identities:

$$\gamma = \frac{c}{c_c}$$

$$c_c = 2mp$$

$$p^2 = \frac{k}{m}$$

$$t_\epsilon = \frac{2\gamma}{p} \ln \frac{1}{\epsilon} = \frac{2c}{c_c p} \ln \frac{1}{\epsilon} = \frac{2c}{2mp^2} \ln \frac{1}{\epsilon} = \frac{cm}{mk} \ln \frac{1}{\epsilon} = \frac{c}{k} \ln \frac{1}{\epsilon}$$

So:

$$c = \frac{k}{\ln \frac{1}{\epsilon}} t_\epsilon .$$

It has already been shown that $k = 172.4$ lbs./in., and to simplify the above relation it was decided to make ϵ a constant, that is to let the amplitude decay to the same fraction of its initial value for every run. The value of this fraction chosen was 1/5, so $\ln \frac{1}{\epsilon} = \ln 5 = 1.161$. Then $c = \frac{172.4}{1.61} t_\epsilon = 107.1\, t_\epsilon$ lb. sec./in. if t_ϵ is measured in seconds. However, since the clock was graduated in minutes, it was found more convenient to use

$$c = 6426\, t_\epsilon \text{ lb.sec./in.}$$

where t_ϵ is measured in minutes.

This equation indicates that the damping coefficient should be independent of mass and of range of operation on the scale. Since c is a characteristic of the dashpots, this should have been expected to be the case. As a check, however, data was taken around the scale position of zero pounds as well as around ten, twenty, and thirty pounds by adding weights as was done in determining equivalent mass. For each run, the system was displaced and released at the instant that the clock was started. The clock was stopped when the displacement dropped to one-fifth of its initial value. A number of starting amplitudes were used at each range, and it was hoped that a plot of time versus starting amplitude could be made for each range, and that each of these curves could have been extrapolated to zero starting amplitude, as was done in determining m. However, since, as the above equation predicted, c did not depend on scale range or mass, there was not sufficient spread in the values of t_ϵ for various starting positions to gain anything by doing this. Therefore, it was decided to simply average all of the values of t_ϵ and to use this average value in computing c. The average time for 179 trials was $t_\epsilon = 0.0978$ minutes. Putting this into the equation for c gives

$$c = 6426(0.0978) = 628.2 \text{ lb.sec./in.}$$

This is the value of c that was used in the equation.

For a damped spring-mass system of the type chosen as the equivalent system, the critical damping coefficient c_c is given by

$$c_c = 2m\omega_n = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk} = 2\sqrt{\frac{Wk}{g}}\,.$$

At the time that the test run was made, the equivalent weight was, from the preceding section, 1325 pounds, and k equals 172.4 lbs./in. Then,

$$c_c = 2\sqrt{\frac{(1325)(172.4)}{386}} = 48.66 \text{ lb.sec./in.}$$

The damping factor during the run was then

$$\gamma = \frac{c}{c_c} = \frac{628.2}{48.66} = 12.9.$$

Therefore, the assumption made that γ was large was valid.

Determination of the forcing function for the equation of motion.

In the main body of this paper it was pointed out that the forcing function $f'(t)$ which was finally used in the equation of motion was the function $f(t)$ which was picked up by the strain gages divided by the length of the dynamometer lever arm. It was also mentioned that the reason for using this as the forcing function was that it was impossible to obtain a function which would accurately represent the angular acceleration of the dynamometer armature over the cycle, at least with the instrumentation which was available. This section is intended to describe the procedure which was attempted to determine this acceleration.

A magnetic pickup was mounted so that it pointed at the outside cylindrical surface of the engine flywheel. The engine was turned by hand to its top dead center position and a piece of steel wire was then taped on the flywheel opposite the magnetic pickup. Then nine other pieces of smaller diameter steel wire were also taped on the flywheel at 36 degree intervals so that they all would pass underneath the pickup as the engine rotated. The magnetic pickup was connected to the vertical input of a cathode ray oscilloscope. The engine was then started and the resulting trace on the screen was a horizontal line except for a number of pips which were created whenever one of the pieces of steel wire passed underneath the magnetic pickup. Since the space between each consecutive pair of pips represented 36 degrees of crank rotation, and, assuming that the horizontal internal sweep of the oscilloscope was linear with time, it should have been possible to determine an average $\Delta\theta/\Delta t$ over twenty intervals of the cycle of the engine. Then, since $d\theta/dt = \omega$, plotting these values of $\Delta\theta/\Delta t$ against t should have approximated a function which would represent ω over the cycle. This curve could then have been analyzed by harmonic analysis as was the one for f(t). The resulting analytical expression for ω could then have been differentiated once with respect to time to give an analytical expression for α. When this method was tried, the pips produced were all separated by the same amount as closely as could be determined, and even after being magnified five times by using the 5X multiplier on the oscilloscope, there were no apparent consistent differences in separations. This meant that, within the accuracy of the method, ω was constant, and that consequently α was equal to zero over the whole cycle.

Because it was obvious that the above conclusion was not exactly correct, it was decided to try to determine just how much ω did vary from peak to peak over the cycle. A standard method of flywheel analysis was used to determine this variation in ω. The torque function f(t) was used and it was supposed that this amount of fluctuation was being impressed upon the dynamometer armature and that the field was disconnected so that the armature acted only as a flywheel. If this condition could be set up, the maximum variation in ω from its minimum to its maximum would be 1.15 percent of its average. Therefore, it is not surprising that the pip method did not result in a function for α. Actually, if the armature were free-wheeling as a flywheel, the impressed torque from the engine required to keep it turning at 1800 rpm would not be nearly as rough as the curve represented by f(t). The fluctuation of ω in this case would even be less than 1.15 percent of ω_{avg}. Also if the field was connected and if f(t) was the forcing function, it would seem likely that the fluctuation in ω would again be less than .0115 ω_{avg} because the field reaction would probably tend to dampen out the fluctuation rather than to amplify it. Therefore, it was decided that the analysis made was on the safe side.

Because of this small fluctuation in ω, it was concluded that it would be very difficult to obtain an accurate expression for α even with more refined equipment, and that the only alternative was to assume $\alpha = 0$, and to use f(t)/L for the forcing function f'(t).

The equation of motion and its solution.

It was shown earlier that the form of the equation of motion for the equivalent system is

$$m\ddot{x} + c\dot{x} + kx = f'(t) = f(t)/L \quad .$$

In the few sections preceding this one, values of m, c, k, and f(t) which applied during the test were determined. Using these values and recalling that L = 1.050 feet, the above equation becomes:

$$3.43\,\ddot{x} + 628.2\,\dot{x} + 172.4\,x = 10.664 - 8.116\cos(30\pi t)$$
$$- 13.947\cos(60\pi t) + 14.763\cos(90\pi t) - 0.627\cos(120\pi t)$$
$$- 7.215\cos(150\pi t) + 4.009\cos(180\pi t) + 0.201\cos(210\pi t)$$
$$- 1.162\cos(240\pi t) + 0.843\cos(270\pi t) - 0.457\cos(300\pi t)$$
$$+ 21.185\sin(30\pi t) - 13.941\sin(60\pi t) - 5.842\sin(90\pi t)$$
$$+ 11.960\sin(120\pi t) - 3.645\sin(150\pi t) - 2.8\text{[illegible]}5\sin(180\pi t)$$
$$+ 2.704\sin(210\pi t) - 0.83\text{[illegible]}\sin(240\pi t) - 0.024\sin(270\pi t)$$
$$+ 0.521\sin(300\pi t) \quad .$$

At first this equation appears quite unwieldy, but since it is a linear equation it can be solved in parts and all of these parts can then be superimposed to give the total solution. In other words, twenty-one equations can be set up, all of them the same as the big equation on the left side of the equals sign, but each with a different term of f'(t) as its forcing function. Each of these twenty-one equations can then be solved for x, and the solutions can then all be superimposed to obtain the solution to the big equation.

There are two different types of small equations to be solved; the ones with sine term forcing functions, and those with cosine term forcing functions The one with the constant forcing function is a special case of the cosine type with $\omega = 0$. Consider the cosine function type

$$m\ddot{x} + c\dot{x} + kx = a'_n \cos \omega_n t \quad .$$

The general solution of this equation would consist of a solution to the homogeneous equation $m\ddot{x} + c\dot{x} + kx = 0$, plus a particular solution which would satisfy the equation with the forcing function. The solution to the homogeneous equation would, however, represent the transient part of the response, and since it is supposed that the system had settled down to steady state when the data was taken, only the particular solution is of interest here. Assume then that the particular solution is of the form

$$x = A_n \sin \omega_n t + B_n \cos \omega_n t$$

Then

$$\dot{x} = A_n \omega_n \cos \omega_n t - B_n \omega_n \sin \omega_n t \ , \qquad \text{and}$$

$$\ddot{x} = -A_n \omega_n^2 \sin \omega_n t - B_n \omega_n^2 \cos \omega_n t \ .$$

Substituting these values into the equation $m\ddot{x} + c\dot{x} + kx = a_n' \cos \omega_n t$ gives

$$\left[(k - m\omega_n^2) A_n - c\omega_n B_n\right] \sin \omega_n t + \left[c\omega_n A_n + (k - m\omega_n^2) B_n\right] \cdot \cos \omega_n t = a_n' \cos \omega_n t \ .$$

Equating coefficients of $\sin \omega_n t$ and $\cos \omega_n t$ gives

$$(k - m\omega_n^2) A_n - c\omega_n B_n = 0 \ ,$$

and

$$c\omega_n A_n + (k - m\omega_n^2) B_n = a_n' \ .$$

These equations can be solved simultaneously for A_n and B_n to give:

$$A_n = \frac{c\omega_n a_n'}{(k - m\omega_n^2)^2 + c^2\omega_n^2}$$

and

$$B_n = \frac{a_n'(k - m\omega_n^2)}{(k - m\omega_n^2)^2 + c^2\omega_n^2}$$

The particular solution of a typical equation with a cosine forcing function is then

$$x = \frac{c\omega_n a_n'}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \sin \omega_n t + \frac{a_n'(k - m\omega_n^2)}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \cos \omega_n t$$

For the special case of the constant forcing function, $\omega_n = 0$ and the above expression for x reduces to $x = \frac{a_0'}{k}$. The beam load which corresponds to this displacement is just $F = kx = a_0' = 10.664$ pounds. It will be shown that solutions to the remaining equations will be combinations of sine and cosine functions which average out to zero over the period of one cycle. This should already be obvious for the equations with cosine forcing functions (except $\omega_n = 0$) through an inspection of the general particular solution above for these equations.

The form of the equations with sine forcing functions is

$$m\ddot{x} + c\dot{x} + kx = b_n' \sin \omega_n t \quad .$$

The procedure for finding the required particular solution of this equation is the same as used before. First assume a solution of the form

$$x = C_n \sin \omega_n t + D_n \cos \omega_n t \ .$$

Then

$$\dot{x} = C_n\omega_n \cos \omega_n t - D_n\omega_n \sin \omega_n t \ , \quad \text{and}$$

$$\ddot{x} = - C_n\omega_n^2 \sin \omega_n t - D_n\omega_n^2 \cos \omega_n t \ .$$

Substitute these into the above equation as before to obtain

$$\left[(k - m\omega_n^2)\ C_n - c\omega_n D_n\right] \sin \omega_n t + \left[c\omega_n C_n + (k - m\omega_n^2)\ D_n\right] \cdot \cos \omega_n t = b_n' \sin \omega_n t \ .$$

Equate coefficients of sine and cosine terms;

$$(k - m\omega_n^2)\ C_n - c\omega_n D_n = b_n'$$

$$c\omega_n C_n + (k - m\omega_n^2)\ D_n = 0$$

Solve for C_n and D_n:

$$C_n = \frac{b_n'(k - m\omega_n^2)}{(k - m\omega_n^2)^2 + c^2\omega_n^2}$$

$$D_n = \frac{-b_n'\omega_n c}{(k - m\omega_n^2)^2 + c^2\omega_n^2}$$

Then the general form of the solution to the equations with sine term forcing functions is

$$x = \frac{b_n'(k - m\omega_n^2)}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \sin \omega_n t - \frac{b_n'\omega_n c}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \cos \omega_n t \ .$$

The combined solution of all twenty-one equations is:

$$\begin{aligned} x = \Big[& 61860 - 227.887 \cos (30\pi t) + 116.081 \cos (60\pi t) \\ & - 28.233 \cos (90\pi t) + 94.413 \cos (120\pi t) + 9.842 \cos (150\pi t) \\ & - 2.552 \cos (180\pi t) - 0.591 \cos (210\pi t) + 0.660 \cos (240\pi t) \\ & - 0.324 \cos (270\pi t) + 0.112 \cos (300\pi t) - 253.729 \sin (30\pi t) \\ & + 1.598 \sin (60\pi t) + 39.582 \sin (90\pi t) - 20.352 \sin (120\pi t) \\ & + 0.956 \sin (150\pi t) + 3.402 \sin (180\pi t) - 1.646 \sin (210\pi t) \\ & + 0.226 \sin (240\pi t) + 0.079 \sin (270\pi t) - 0.193 \sin (320\pi t) \Big] \\ & \qquad \qquad \times 10^{-6} \text{ inches.} \end{aligned}$$

t is time in seconds.

It should be remembered that this displacement is measured at the end of the dynamometer lever arm. Multiplying this expression for x times k gives an expression which represents the fluctuation of the Toledo scale needle, and multiplying it again by the dynamometer lever arm length L gives an expression for the corresponding torque fluctuation as indicated by the dynamometer. If both of these steps are made, and if the resulting expression for torque is evaluated at a number of different values of time over the period of one cycle, the resulting curve is the smoother one shown on page eleven. The other curve shown on this page is the input torque f(t) as measured at the strain gages.

Returning to the above long expression for x, in each of the cos ω_n t terms and in each of the sin ω_n t terms, ω_n is an integral multiple of 30π. Therefore in every 1/15 second, each sine and cosine term completes an integral number of complete cycles. The average value of a sine or a cosine over a complete cycle is equal to zero, so all of these sine and cosine terms in the expression for x average out to zero. This leaves only the constant term to represent average displacement,

$$x_{avg.} = 0.06186 \text{ inches.}$$

The average beam load corresponding to this displacement is

$$F_{avg.} = kx_{avg.} = 10.664 \text{ pounds,}$$

and the average torque is

$$T_{avg.} = 1.05\ F_{avg.} = 11.197 \text{ pound feet.}$$

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