STABILITY ESTIMATES FOR ELECTROMAGNETIC SCATTERING FROM OPEN CAVITY

By

Qiong Zheng

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Applied Mathematics – Doctor of Philosophy

2013

ABSTRACT

STABILITY ESTIMATES FOR ELECTROMAGNETIC SCATTERING FROM OPEN CAVITY

By

Qiong Zheng

The electromagnetic scattering from an open large cavity embedded in an infinite plane is of practical importance due to its significant industrial and military applications. Examples of cavities include jet engine inlet ducts, exhaust nozzles and cavity-backed antennas [3, 4]. In many practical applications, one is interested in the cavity problem with either a large wave number k or a large diameter a, in which case the solution has a highly oscillatory nature [7]. While the original time-harmonic problem is modeled by Helmholtz equation in the unbounded domain, the reformulated model through Fourier transform is essentially a Helmholtz equation in the bounded domain with mixed nonlocal boundary condition. Deriving an explicit dependency between the wave energy and the wave number is mathematically interesting and challenging. The stability estimate is also important as it defines relations between the wave number and the discretization parameters in the error analysis [16]. For the open cavity problem, while the stability analysis for the rectangular cavity was derived recently [8] as described , the stability results for more general shapes of cavities are to be explored. The objective of this thesis work is to partially answer this question by imposing some geometric assumptions.

We first start from considering a class of cavity with a strong geometric constraint. The energy stability is established by careful choices of the parameters, and test functions, which take full advantage of geometric properties. The arguments are based on the appropriate usage of the real and imaginary part of the weak formulation of the problem, the separation of

lower frequency and higher frequency part, and connections between frequency components and spatial components. The energy in cavity is bounded by the energy of incoming field with coefficient in terms of powers of wave number. Next, we investigate the case where a weaker geometric constraint is imposed. A new auxiliary function with compact support near the boundary of the cavity is carefully constructed to reformulate the problem. However, the original homogeneous Helmholtz equation is changed to a non-homogeneous one, all previous work in homogeneous equation must be suitably modified, and the estimate in terms of wave number k is obtained from detailed analysis of this auxiliary function. The energy norm is proved to be at most in the order of $k^{\frac{7}{10}}$, which is the same in terms of the power of wave number k as the case with strong geometric conditions but with other additional terms. Furthermore, we studied the case where the cavity domain is of rectangular-like shape, where new test function is introduced and new inequalities are established to derive the energy estimate.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my advisor Dr. Zhengfang Zhou, who provides exceptional guidance and tremendous support during my graduate study. Without his guidance, this thesis would not have been possible. I feel very lucky to have the opportunity to study under the supervision of such a knowledgable and smart mathematician with great personality. He always generously offers time to meet for questions and discussions, he is very inspiring and patient in enhancing my mathematical understanding, and his view on mathematics helps me to shape my own understanding. I am also very thankful for his encouragements and suggestions when I meet with difficulties. His elegant personality and persistent passion in mathematics & life will have a long time influence on me.

I would like to thank Dr. Gang Bao, Dr. Andrew Christlieb, Dr. Keith Promislow and Dr. Baisheng Yan, for serving as members of my doctoral committee and for their invaluable suggestions. In particular I want to thank Dr. Baisheng Yan for his consistent support and great help during my graduate study.

I am grateful to the entire faculty and staff in the Department of Mathematics who have taught and assisted me in one way or another. My special thanks go to Dr. Guowei Wei for offering opportunities and research assistantship in research projects, as well as providing valuable discussions, suggestions and assistance. And I would like to thank Dr. Casim Abbas, Dr. Gabor Francsics, Dr. Jeanne Wald for their help and great support. Many thanks to Dr. Clifford Weil for his help in providing useful information for formatting the thesis. And I am grateful to Ms. Barbara Miller and Ms. Leslie Aitcheson for their assistance in providing information and help.

Also, I want take this opportunity to thank my graduate fellows and friends for providing kind help, sharing with me your expertise knowledge, and sharing the challenges and happiness together. Particular thanks go to Cheryl Balm, Liangzhang Bao, Liping Chen, Yuqi Hong, Jun Lai, Junshan Lin, Shu-man Lin, Xun Wang, Daniel Smith, Li Yang, Xin Yang, Hai Zhang, Liangming Zhou.

Finally I would like to thank my parents and my husband for their endless love and always standing besides me over the years.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Electromagnetic Scattering from Open Cavity

One of the important subjects studied in electromagnetics is Radar Cross Section (RCS), which measures how detectable an object is with a radar. Accurate predictions of RCS of complex objects are of great interest to the designers [4], for instance, a stealth aircraft is expected to have low detectability such that it is less observable (or even invisible) while a passenger airliner is designed to have a high RCS such that it is more visible. Furthermore, it has been studied that the RCS of a B-1A bomber is two orders of magnitude less than that of a B-52 [30, 21]. This is mainly achieved by two approaches: shaping the body of the airplane and using radar absorbing material; meanwhile, the jet intake plays a significant role among the remaining contributors to the total RCS of the airplane. The jet intake is modeled as a cavity structure which has one open end and the other end such as the blade could be modeled as the perfect electric conductor. Other examples of cavities include exhaust nozzles and cavity-backed antennas [3, 4].

In many practical applications, one is interested in the cavity problem with either a large wavenumber k or a large diameter a of the computational domain, which leads to the large ka numbers [7]. It is challenging to solve for a large wavenumber or a large diameter cavity problem due to the high oscillation of the solution. Over the past decade, many computational approaches have been developed in the literatures to solve the open cavity problems. If the cavity has an opening with length less than one wavelength, i.e, the low frequency end, the integral equation formulation could be used for the computation. However, if the aperture of the cavity has a length on the order of several wavelengths, then high-frequency computational techniques are expected [24]. One of the approaches proposed is waveguide modal analysis, which has been applied in early literatures [26, 21], and still provides reference solutions to compare with results obtained from more approximate methods. In waveguide analysis, the field inside the cavity is expressed in terms of the known waveguide modes. The unknown modal coefficients are obtained through using the reciprocity relationship and Kirchhoff's approximation. Another popular approach is the shooting and bouncing ray (SBR) method as introduced in [23]. This method involves tracing a dense grid of geometrical optics rays originating from the incident wave into the cavity through the front aperture. After multiple bounces in the interior walls of the cavity, the rays eventually return to the opening of the cavity. A physical optics approximation is used to calculate the scattered field from each exit ray, then the total scattered field results from summing the scattered field due to individual rays. In [24], the two aforementioned approaches are investigated for cavities with rectangular and circular cross sections. For an aperture opening on the order of ten wavelengths, the computational results agree with each other fairly well. At lower frequencies, the modal analysis results is accurate than the SBR results ; however, the modal approach is limited only to cavities with uniform cross sections whereas the SBR approach can be applied with much greater flexibility in geometrical modeling. However, these methods are restricted to relatively shallow cavities, due to the computational cost and the ray and beam distortion problems associated with deep cavities, where a large number of internal reflections need to be considered. Another ray-based method is the Generalized Ray Expansion (GRE) method [29]. The major difference between SBR and GRE methods, as mentioned in [11], lies in that the SBR method includes only the incident geometrical optics field that enters the cavity, while the GRE method intrinsically includes the fields diffracted into the cavity by the edges at the end of the cavity aperture. Furthermore, for each incidence angle SBR method needs a new set of rays to be tracked, while in the GRE method only one set of rays needs to be tracked regardless of the incidence angles. Other computational approaches include the Iterative Physical Optics (IPO)[28] method and the Progressive Physical Optics (PPO) method [27], where in both methods the magnetic field integral equation (MFIE) is obtained for the equivalent currents in the interior cavity walls, and solved by different algorithms. Moreover, many hybrid methods have been proposed by combining one of the preceding aysmptotic approaches and some other more accurate methods, such as the Method of Moments (MoM), the Finite Element Method (FEM) or the Finite-Difference Time-Domain method (FDTD) [10, 13, 20, 32, 33]. Anastassiu [5] presented a comprehensive review on the methods for the related electromagnetic scattering problems. However, theoretical analysis is quite limited, especially we hope to establish relations between the energy in cavity, high wave number and incoming field.

While the original time-harmonic problem is modeled by Helmholtz equations in the unbounded domain, the reformulated model through Fourier analysis in the upper half plane is essentially a Helmholtz equation in the bounded domain with nonlocal boundary condition on the aperture of the cavity. Deriving an explicit dependence between the wave energy and the wave number is mathematically interesting and challenging. The stability estimate is also important as it defines relations between the wave number and the discretization parameters in the error analysis [16]. Also, using the stability of the continuous problem with perturbation argument, the stability result of a numerical method can be obtained

[15]. It turns out for the stability analysis on Helmholtz equations, the bounds on k highly depends on the geometry of the domain and the type of boundary conditions [15]. Ihlenburg and Babuška [18] considered the one dimensional Helmholtz equation on $D = (0, 1)$ with Dirichlet and nonreflecting boundary condition:

$$\begin{cases} u'' + k^2 u = -f \quad \text{in } D, \\ u(0) = 0, \\ u'(1) - iku(1) = 0. \end{cases}$$

They proved the stability estimate $|u|_1 \leq Ck|f|_{-1}$ for the solution u in H^1 norm. Moreover, they generalized the stability result under higher regularity assumptions in [19]. For $l > 1$, $f(x) \in H^{l-1}(0, 1)$, then $u \in H^{l+1}(0, 1)$ and the estimate $|u|_{l+1} \leq Ck^{l-1}\|f\|_{l-1}$. Melenk [25] studied the two dimensional problem with the Robin boundary condition:

$$\begin{cases} -\Delta u - k^2 u = f \quad \text{in } D, \\ \partial_{\mathbf{n}} u + iku = g \qquad \text{on } \partial D. \end{cases}$$

The geometric assumption is that the domain D is a bounded star-shaped domain with smooth boundary or a bounded convex domain; then for any $f \in L^2(D)$, $g \in H^{\frac{1}{2}}(\partial D)$,

$$|\nabla u|_{L^2(D)} + |k||u|_{L^2(D)} \leq C(D) \left[|f|_{L^2(D)} + |g|_{L^2(\partial D)} \right],$$

where the constant C depends only on the domain. This was extended to the three-dimensional case by Cummings and Feng [14]. When $g = 0$, a sharp estimate is established as below, $\|u\|_{H^j(D)} \leq C_j(k)\|f\|_{L^2(D)}$, where $C_j(k) = O(k^{j-1}) + \frac{1}{k^2}, j = 0, 1, 2$

for $N \geq 2$ under the assumption that D is star-shaped when $j = 0, 1$, and D is convex polygons or smooth domains when $j = 2$. Moreover, Melenk [15] showed that for any $f \in L^2(D)$,$g \in H^{\frac{1}{2}}(\partial D)$, assume D is a bounded Lipschitz domain, there exists a constant $C > 0$ (independent of k) such that

$$|\nabla u|_{L^2(D)} + |k||u|_{L^2(D)} \leq C(D) \left[k^{\frac{5}{2}} |f|_{L^2(D)} + k^2 |g|_{L^2(\partial D)} \right].$$

Hetmaniuk [16] presented the stability analysis for two dimensional and three dimensional mixed boundary problems:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } D, \\ u = 0 & \text{on } \Gamma_d, \\ \partial_n u = 0 & \text{on } \Gamma_n, \\ \partial_n u = (i\beta - \alpha)u + g & \text{on } \Gamma_r. \end{cases}$$

Based on the following geometric assumption: there exists a point $\mathbf{x}_0$ and a constant γ such that

$$\begin{cases} (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \leq 0 & \text{for } \forall \mathbf{x} \in \Gamma_d, \\ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) = 0 & \text{for } \forall \mathbf{x} \in \Gamma_n, \\ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \geq \gamma & \text{for } \forall \mathbf{x} \in \Gamma_r; \end{cases}$$

the stability estimate is given as

$$|\nabla u|_{L^2(D)} + |k||u|_{L^2(D)} \leq C(D) \left[|f|_{L^2(D)} + |g|_{L^2(\partial D)} \right].$$

A major difference between the open cavity scattering problem and the results mentioned above is the boundary condition, where the cavity problem in the bounded domain formulation involves the nonlocal boundary condition as indicated in the following section. More specifically, part of the boundary condition is associated with Fourier transform. The first stability result for the cavity problem was derived recently [8] for rectangular cavity. Because of this particular shape of the cavity, the solution u can be expressed as the Fourier expansion. Two special norms, which behave like $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ on Γ, are introduced for the technical analysis. And the stability estimate under the rectangular cavity of depth y_0 in the y direction, is given by

$$|\nabla u|_{L^2(D)} + |k||u|_{L^2(D)} \leq C \left[k^{\frac{15}{4}} y_0^{\frac{7}{2}} (\log k)^{\frac{1}{4}} + k^{\frac{13}{4}} y_0^4 \right] |g|_{L^2(\Gamma)}$$

when $f = 0$, where Γ represents the aperture part of the cavity. In this thesis, we study the stability estimates for open cavity embedded in an infinite ground plane in a more general framework, where the shape of the cavity is under certain geometric constraints.

1.2 Mathematical Formulation

In this section, the formulation of the problem is reviewed and notations are introduced. Let D be the cavity region, Γ denote the aperture boundary part of the cavity and $\Gamma := [a, b] \times \{y_2\}$, $\Gamma^C = (\mathbb{R} \backslash [a, b]) \times \{y_2\}$ and $S = \partial D \backslash \Gamma$. Assume $X^* = (x_0, y_0)$ represents a reference point on the plane for the convenience of introducing geometric assumptions in the following chapters.

For the TM (transverse magnetic) polarization, the cavity scattering problem in the

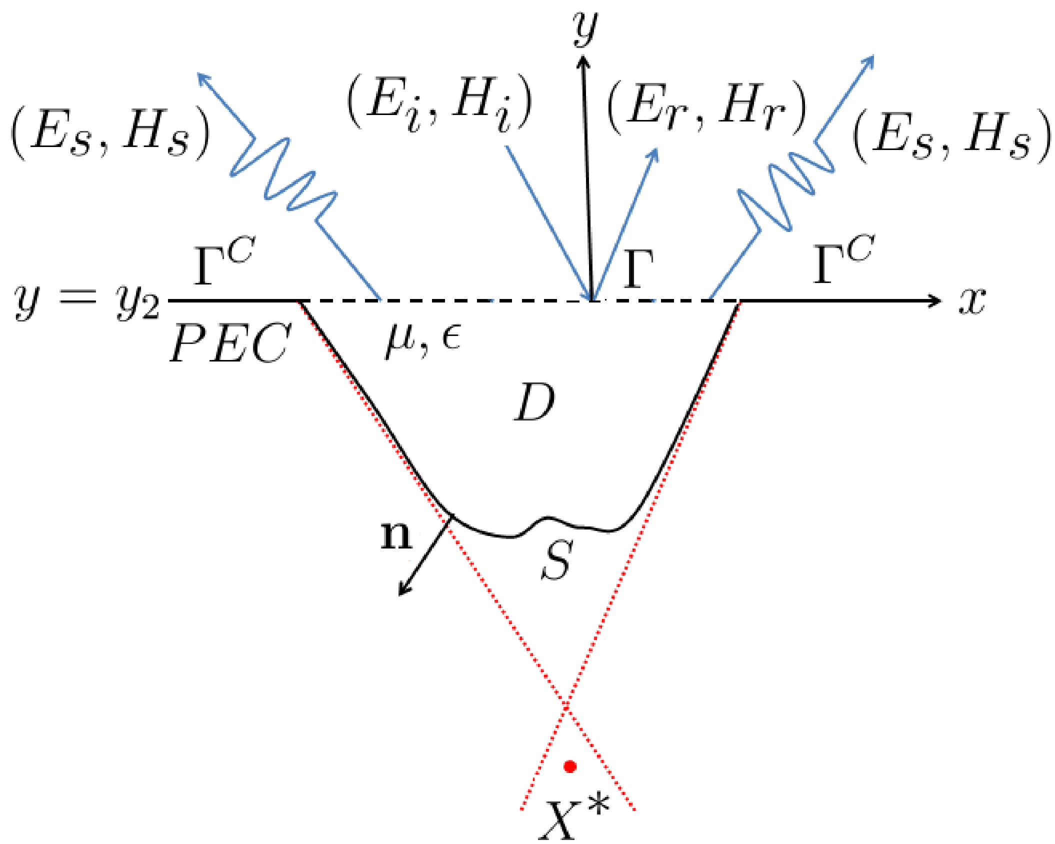


Figure 1.1: Geometry of the problem. (For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.)

unbounded domain could be reduced to the bounded domain (the cavity) problem through a Fourier transform and radiation condition [3], which is briefly reviewed here. Let E and H denote the total electric and magnetic fields, then the following time harmonic ($E(X,t) = e^{-i\omega t}E(X)$) Maxwell equations are satisfied:

$$\nabla \times E - i\omega\mu H = 0,$$

$$\nabla \times H + i\omega\epsilon E = 0,$$

where ω is wave frequency, μ is the magnetic permeability and ϵ is the electric permittivity.

Denote (E_i, H_i), (E_r, H_r),(E_s, H_s) as the incident, reflective and scattering fields respectively. In the TM polarization, the electric filed is $E = (0, 0, u(x, y))$. The total field contains three components, that is, $u = u_i + u_r + u_s$. When $u_i = e^{i\alpha x - i\beta(y - y_2)}$ is a plane wave, $u_r = -e^{i\alpha x + i\beta(y - y_2)}$, where $\alpha = k \sin\theta$, $\beta = k\cos\theta$, θ is the incident angel with respect to y axis, and the wave number $k = \omega\sqrt{\epsilon\mu}$. Moreover, u_s satisfies

$$\begin{cases} \Delta u_s + k^2 u_s = 0 & \text{in } D \cup \{y > y_2\}, \\ u_s = -(u_i + u_r) & \text{on} \qquad S \cup \Gamma^C, \end{cases}$$

with the radiation condition $\lim_{r\to\infty} \sqrt{r}(\partial_r u_s - iku_s) = 0$.

By taking the Fourier transform with respect to x, and defining

$$\widehat{u}_s(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x, y) e^{-ix\xi} dx,$$

we have

$$(\partial_{yy} + (k^2 - \xi^2))\widehat{u}_s = 0 \quad \text{for } y > y_2. \tag{1.1}$$

By solving Eq. (1.1) with the radiation condition and taking the inverse Fourier transform, we obtain

$$u_s = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(y - y_2)\sqrt{k^2 - \xi^2}} \widehat{u}_s(\xi, y_2) e^{i\xi x} d\xi.$$

Therefore, on $\mathbb{R} \times \{y_2\}$,

$$\partial_{\mathbf{n}} u_s = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - \xi^2} \widehat{u}_s(\xi, y_2) e^{i\xi x} d\xi.$$

To simplify notations, define an operator T on Γ as

$$T(\varphi_\Gamma) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - \xi^2} \widehat{\varphi_\Gamma} e^{i\xi x} d\xi$$

for any φ with $\varphi \in H^{\frac{1}{2}}(\mathbb{R})$, and

$$\varphi_\Gamma := \begin{cases} \varphi & \text{on } \Gamma, \\ 0 & \text{on } \Gamma^C. \end{cases}$$

Note that $u_i + u_r = 0$ when $y = y_2$, we have

$$\partial_{\mathbf{n}} u = T(u) + g,$$

where $g = \dfrac{\partial}{\partial \mathbf{n}}(u_i + u_r)|_\Gamma$. Therefore, the original problem in the unbounded domain (the cavity and the upper half plane, that is, $D \cup \{y > y_2\}$) can be reduced to the following problem in the bounded domain:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } S, \\ \partial_{\mathbf{n}} u = T(u) + g & \text{on } \Gamma. \end{cases} \tag{1.2}$$

Note that the reformulated model as shown in Equation (1.2) is essentially a Helmholtz equation with mixed nonlocal boundary condition; in particular, part of the boundary condition is associated with Fourier transform. There were many papers studying the equation with boundary condition $\partial_{\mathbf{n}} u = iku + g$, which is a much simplified and local form, but the problem is not physical. The problem with nonlocal boundary condition is much harder mathematically, but it is more significant physically. There are very few results in this direction. New methods have to be developed. Based on certain geometric assumptions of the cavity domain, a few stability estimates are established in the following chapters.

The rest of the dissertation is arranged as follows. In Chapter 2, the stability estimate for a class of cavity under a strong geometric constraint is established. In Chapter 3, we derived the stability estimate for a class of cavity under a weaker geometric assumption. In Chapter 4, we assume the cavity structure is of rectangular-like shape, where the geometric assumptions introduced in Chapter 2 and 3 are not satisfied; the corresponding stability estimate is obtained. In Chapter 5, a brief summary of the main contributions of the thesis work is presented, followed by the discussion of future work.

Chapter 2

Stability Estimates under a Strong Geometric Assumption

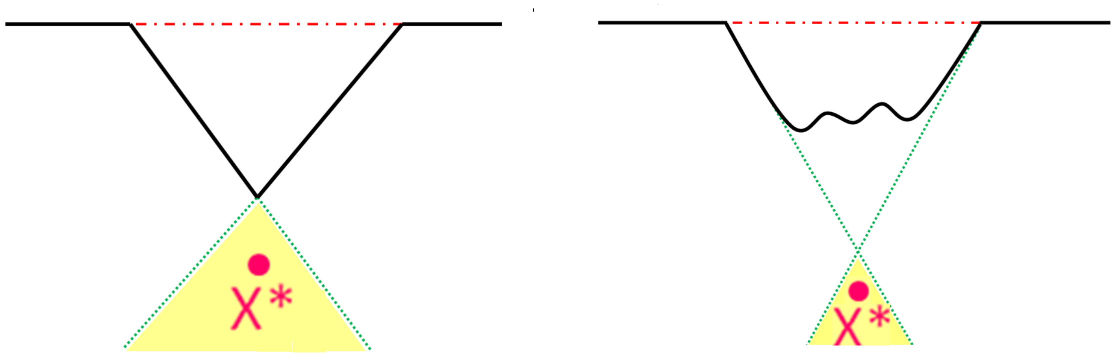
2.1 Main Theorem and Outlines of the Approach

Throughout this chapter, an open cavity that satisfies the following geometric assumption is considered; that is, there is a point $X^* = (x_0, y_0)$ on the plane and positive constant p_1 such that

$$(X - X^*) \cdot \mathbf{n} \leq -p_1 \quad \text{on} \quad S, \tag{2.1}$$

where $\mathbf{n}$ is outnormal. We should remark that if S is given by $y = h(x),\ x \in (a, b)$ and h is C^1, then Equation (**??**) is always satisfied. Without loss of generality, we assume $X^* = (0, 0)$ in this chapter; also for simplicity, we assume that $b - a = \pi$. It should be remarked that if $X^* = (0, 0)$, then $y_2 > y_1 > 0$, where $y_1 = \min\{y|(x, y) \in \overline{D}\}$ from condition (2.1), one example is shown Figure 2.1.

A similar geometric constraint was presented by Hetmaniuk [16] to study a different mixed boundary problem. In their case, p_1 can be zero. We need a slightly stronger geometric constraint for our arguments. If $p_1 = 0$, some additional techniques will be developed as indicated in the next chapter. This condition guarantees the positivity of one term which



(a): Triangular shaped (b): Curly shaped

Figure 2.1: A cavity structure satisfying geometric assumption in (2.1).

lies on the same side as the wave energy terms in an inequality which will be derived later. Furthermore, we assume that the medium for the upper half plane $\{(x, y) \mid y \geq y_2\}$ and the cavity is homogeneous. The main result is Theorem 2.1.1 as stated below, which provides an explicit dependence for the wave energy on the wave number k.

Theorem 2.1.1. *Under the geometric assumption given in Equation (2.1), there exists a positive constant C such that*

$$\|\nabla u\|_{L^2(D)} + |k|\|u\|_{L^2(D)} \leq C \left[k^{\frac{7}{10}} \|g\|_{L^2(\Gamma)} + \frac{1}{\sqrt{k}} \|gx\|_{L^2(\Gamma)} \right]. \tag{2.2}$$

Essentially, our proof of the main theorem involves three major steps.

First, we consider the weak formulation of the problem, and adopt the test functions v used in some of the earlier work [16, 17], which are $v = u$ and $v = X \cdot \nabla u$, then the standard computation yields the following identity,

$$\begin{aligned} &\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{u}|^2 d\xi - \int_S (X \cdot \mathbf{n})|\nabla u|^2 \\ &= \Re \int_\Gamma g\overline{u} dx - \int_\Gamma (X \cdot \mathbf{n})|\nabla u|^2 dx + k^2 \int_\Gamma (X \cdot \mathbf{n})|u|^2 dx + 2\Re \int_\Gamma \partial_{\mathbf{n}} u(X \cdot \nabla \overline{u}) dx. \end{aligned} \tag{2.3}$$

Second, since the boundary condition on Γ is given in a simpler form in terms of the Fourier transform, it is easier to evaluate the last three terms in Equation (2.3) in the frequency domain. So the main focus of our second step is to convert those terms into the frequency domain. After the conversion, the following inequality can be derived:

$$
\begin{aligned}
&\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi - \int_S (X\cdot\mathbf{n})|\nabla u|^2 \\
\leq&\Re\int_\Gamma g\overline{u}dx + y_2\int_{|\xi|\leq k} 2(k^2-\xi^2)|\widehat{u}|^2 d\xi + 2y_2\Re\int_\Gamma T(u)\overline{g} \qquad (2.4)\\
&+2\Re\int_{|\xi|\leq k}\left[-\xi\overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi + 2\Re\int_\Gamma x\overline{u}_x g dx + y_2\int_\Gamma |g|^2 dx.
\end{aligned}
$$

Third, terms on the right hand side of Equation (2.4) will be named by I_1 through I_6, and we have to estimate each term from I_1 to I_5. The goal is to show that each term is controlled by norms of g and the left hand side of Equation (2.4) multiplied by a small coefficient. To illustrate the ideas and simplify notations, let

$$
\begin{aligned}
A_L &= \int_{|\xi|\leq k}\sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi, \quad A_H = \int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi,\\
B_H &= \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi,\\
G &= \int_\Gamma |g|^2 dx, \quad J = \int_S |\nabla u|^2 dS.
\end{aligned}
$$

For I_1, Lemma 2.4.1 will show that $A_L \leq \delta B_H + C(\delta)G$, i.e., the lower frequency part of u is dominated by L_2 norm of g and higher frequency part of u. Consequently the bound of $\|u\|_{L^2(\Gamma)}$ is obtained, which in turn will yield the estimate for I_5 after the integration by parts. Since I_2 involves only lower frequency part, Lemma 2.4.1 can be used to estimate this term. For I_3, a new test function will be constructed to derive that $I_3 \leq \delta\, Energy +$

$\delta J + C(\delta)\|g\|_{H^1}$. This step uses the fact that $X \cdot \mathbf{n} \leq -p_1 < 0$, and the detailed arguments are provided in Lemma 2.4.4. I_4 is also difficult to estimate, it will use Lemma 2.4.1 and cancelation between positive and negative part of an introduced function. This part requires careful choice of parameters and gives the reason why $k^{\frac{7}{10}}$ appears in Theorem 2.1.1. It is proved in Lemma 2.4.5.

2.2 Preliminary Lemmas for the Energy Identity

In this section, we start from the weak formulation of the problem, and utilize appropriate test functions and separate the real and imaginary components. An equality involving the energy terms is derived as shown in Equation (2.16).

Define

$$H_{00}^{1/2}(\Gamma) = \{\varphi \in H^{1/2}(\Gamma) : \exists \widetilde{\varphi} \in H^{1/2}(\mathbb{R}) \text{ such that } \widetilde{\varphi} = 0 \text{ on } \mathbb{R}\backslash\Gamma \text{ and } \varphi = \widetilde{\varphi}|_{\Gamma}\},$$

where $\widetilde{\varphi}$ is called an extension of φ to $H^{1/2}(\mathbb{R})$. The weak formulation of the scattering problem shown in Equation (1.2) is as follows:

Find $u \in H_S^1(D) = \{\phi \in H^1(D), \phi = 0 \text{ on } S, \phi \in H_{00}^{1/2}(\Gamma)\}$ such that

$$a(u, \phi) = (g, \phi) \quad \forall \phi \in H_S^1(D),$$

where

$$a(u, \phi) = \int_D \nabla u \cdot \nabla \overline{\phi} - \int_D k^2 u \overline{\phi} - \int_\Gamma T(u)\overline{\phi}. \tag{2.5}$$

If in particular, by choosing $\phi = u$, the following lemma is obtained.

Lemma 2.2.1.

$$\int_{|\xi|\le k} \sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi = -\Im \int_\Gamma g\overline{u}dx, \tag{2.6}$$

$$\|\nabla u\|^2_{L^2(D)} - k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi = \Re \int_\Gamma g\overline{u}dx. \tag{2.7}$$

Proof. Since $T(u) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2-\xi^2}\widehat{u}e^{i\xi x}d\xi$, then $\widehat{T(u)} = i\sqrt{k^2-\xi^2}\widehat{u}$, and by Parseval's Theorem,

$$\int_\Gamma T(u)\overline{u} = \int_{\mathbb{R}} (\widehat{T(u)})\,\overline{\widehat{u}}d\xi = \int_{\mathbb{R}} i\sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi. \tag{2.8}$$

Now choosing $\phi = u$ in Equation (2.5) and using the identity in (2.8) yield that

$$\int_D |\nabla u|^2 - \int_D k^2|u|^2 - \int_{|\xi|\le k} i\sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi + \int_{|\xi|>k} \sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi = \int_\Gamma g\overline{u}.$$

The identities (2.6) and (2.7) are obtained by taking the imaginary part and real part of the equation above respectively. □

While the identity Equation (2.7) extracted from the real part will be combined with the result in the next lemma to form the energy equality, the imaginary part Equation (2.6) will be used later for the estimate on the L_2 norm of u as shown in Lemma 2.4.1. Note that the sign for $\int_D |u|^2$ is $-k^2$, not k^2 for the energy of u. This is natural from the partial differential equation. Next, we try to get a formula with a "right" sign for $\int_D |u|^2$ by using the test function $X\cdot\nabla u$, which was used in the literatures [16, 25] for the stability estimates, and in the well-known Pohozaev identity for nonlinear elliptic equations.

Lemma 2.2.2. *For all* $u \in H^1_S(D) \bigcap H^{\frac{3}{2}+\epsilon}(D), \epsilon > 0$, *we have*

$$\begin{aligned} & -\int_S (X \cdot \mathbf{n})|\nabla u|^2 + \int_\Gamma (X \cdot \mathbf{n})|\nabla u|^2 + 2k^2 \int_D |u|^2 \\ = \; & k^2 \int_{\partial D} (X \cdot \mathbf{n})|u|^2 + 2\Re \int_{\partial D} \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}). \end{aligned} \tag{2.9}$$

Proof. Note that for any test function v, $\int_D \nabla u \cdot \nabla \overline{v} - \int_D k^2 u\overline{v} - \int_{\partial D} \frac{\partial u}{\partial \mathbf{n}} \overline{v} = 0$. Choose $v = X \cdot \nabla u$ and plug into the above identity, it follows that

$$\int_D \nabla u \cdot \nabla(X \cdot \nabla \overline{u}) - \int_D k^2 u(X \cdot \nabla \overline{u}) - \int_{\partial D} \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}) = 0. \tag{2.10}$$

Two technical identities are used here to convert Equation (2.10) to an equation associated with the energy terms, that is, $\int_D |\nabla u|^2$ and $k^2 \int_D |u|^2$.

$$2\Re \int_D \nabla u \cdot \nabla(X \cdot \nabla \overline{u}) = -(d-2)\int_D |\nabla u|^2 + \int_{\partial D} (X \cdot \mathbf{n})|\nabla u|^2. \tag{2.11}$$

and

$$2\Re \int_D k^2 u(X \cdot \nabla \overline{u}) = k^2 \int_{\partial D} (X \cdot \mathbf{n})|u|^2 - dk^2 \int_D |u|^2. \tag{2.12}$$

where $d = div(X)$ is the dimension. These two identities are the direct consequences from Lemma 3.1 and 3.2 in [16], which also could be derived through the following standard computation. We see that $div(|\nabla u|^2 X) = div(X)|\nabla u|^2 + \nabla(|\nabla u|^2) \cdot X$, and

$$\nabla(|\nabla u|^2) \cdot X = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i}(|\nabla u|^2) = \sum_{i=1}^d x_i 2\Re(\sum_{j=1}^d u_{x_j} \overline{u}_{x_j x_i}) = 2\Re \sum_{j=1}^d u_{x_j}(\sum_{i=1}^d x_i \overline{u}_{x_j x_i})$$

$$= 2\Re \sum_{j=1}^d u_{x_j}(\sum_{i=1}^d (x_i \overline{u}_{x_i})_{x_j} - \overline{u}_{x_j}) = 2\Re \nabla u \cdot \nabla(X \cdot \nabla \overline{u}) - 2|\nabla u|^2.$$

It follows that

$$
\begin{aligned}
div(|\nabla u|^2 X) &= div(X)|\nabla u|^2 + 2\Re \nabla u \cdot \nabla(X \cdot \nabla \overline{u}) - 2|\nabla u|^2, \\
2\Re \nabla u \cdot \nabla(X \cdot \nabla \overline{u}) &= -(d-2)|\nabla u|^2 + div(|\nabla u|^2 X). \qquad (2.13)
\end{aligned}
$$

Integrate Equation (2.13) over D and use the divergence theorem, then Equation (2.11) can be derived. Equation (2.12) can be obtained by the following direct computation.

$$
\begin{aligned}
div(|u|^2 X) = div(X)|u|^2 + \nabla |u|^2 \cdot X &= div(X)|u|^2 + 2\Re(u \nabla \overline{u}) \cdot X, \\
2\Re\, u(X \cdot \nabla \overline{u}) &= div(|u|^2 X) - div(X)|u|^2. \qquad (2.14)
\end{aligned}
$$

Similarly, integrating Equation (2.14) over D and using the divergence theorem, we have Equation (2.12). Then through multiplying Equation (2.10) by 2 and taking the real part, applying Equation (2.11) and Equation (2.12), it follows that

$$
\begin{aligned}
& \int_{\partial D} (X \cdot \mathbf{n})|\nabla u|^2 + dk^2 \int_D |u|^2 \\
= \ & -(d-2)\int_D |\nabla u|^2 + k^2 \int_{\partial D} (X \cdot \mathbf{n})|u|^2 + 2\Re \int_{\partial D} \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}). \qquad (2.15)
\end{aligned}
$$

Note that $\nabla u = \dfrac{\partial u}{\partial \mathbf{n}}\mathbf{n}$ on S, it implies that

$$
2\Re \int_S \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}) = 2\Re \int_S \frac{\partial u}{\partial \mathbf{n}}(X \cdot (\frac{\partial \overline{u}}{\partial \mathbf{n}}\mathbf{n})) = 2\int_S (X \cdot \mathbf{n})|\frac{\partial u}{\partial \mathbf{n}}|^2 = 2\int_S (X \cdot \mathbf{n})|\nabla u|^2.
$$

Therefore,

$$
\int_S (X \cdot \mathbf{n})|\nabla u|^2 - 2\Re \int_S \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}) = -\int_S (X \cdot \mathbf{n})|\nabla u|^2.
$$

Accordingly, Equation (2.15) can be written as Equation (2.9) by using the identity above and the fact that $u = 0$ on S and $d = 2$, the lemma is proved. $\square$

Add Equation (2.7) and Equation (2.9) together, the following expression for the energy of u is obtained, which is named the energy identity.

$$\begin{aligned}&\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{u}|^2 d\xi - \int_S (X \cdot \mathbf{n})|\nabla u|^2 \\ =&\Re \int_\Gamma g\overline{u} - \int_\Gamma (X \cdot \mathbf{n})|\nabla u|^2 + k^2 \int_\Gamma (X \cdot \mathbf{n})|u|^2 + 2\Re \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}).\end{aligned} \tag{2.16}$$

2.3 Conversion through Fourier Transform

Notice that in Equation (2.16), the last term on the Left Hand side, that is $-\int_S (X \cdot \mathbf{n})|\nabla u|^2$ will help us in the energy estimate, since it is positive under the geometric assumption $X \cdot \mathbf{n} \leq -p_1 < 0$ on S. Note that ∇u on ∂D can not be controlled by energy directly, it needs some further analysis. Since the operator T is simply a multiplication operator in terms of Fourier transform, in this step, the main issue is how to convert the last three terms in the right hand side of Equation (2.16) into the frequency domain. While the second and third terms can be easily handled, the last term is not so trivial. So we first see how to express the last term by Fourier transform, which is shown in the following lemma.

Lemma 2.3.1. *In terms of Fourier transform, we have*

$$\begin{aligned}2\Re \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}) = 2\Re \int_{|\xi|\leq k} &\left[-\xi \overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2 - \xi^2}\,\widehat{u}(\xi) d\xi \\ &- \int_{|\xi|>k} \frac{k^2}{\sqrt{\xi^2 - k^2}}|\widehat{u}|^2 d\xi + 2\Re \int_\Gamma x\overline{u}_x g dx + 2y_2 \int_\Gamma |u_y|^2\end{aligned} \tag{2.17}$$

Proof. Note that on Γ, $X \cdot \mathbf{n} = (x, y_2) \cdot (0,1) = y_2$ and $\dfrac{\partial u}{\partial \mathbf{n}} = u_y$, it follows that

$$2\Re \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}(X \cdot \nabla \overline{u}) = 2\Re \int_\Gamma u_y(x\overline{u}_x + y_2\overline{u}_y) = 2\Re \int_\Gamma x\overline{u}_x u_y + 2y_2 \int_\Gamma |u_y|^2. \tag{2.18}$$

Since $u_y = \dfrac{\partial u}{\partial \mathbf{n}} = T(u) + g$ on Γ, then

$$2\Re \int_\Gamma x\overline{u}_x u_y = 2\Re \int_\Gamma x\overline{u}_x(T(u) + g) = 2\Re \int_{\mathbb{R}} \overline{\widehat{xu_x}}\widehat{T(u)} + 2\Re \int_\Gamma x\overline{u}_x g. \tag{2.19}$$

Note that

$$\begin{aligned}
\widehat{xu_x} &= \int_a^b xu_x e^{-i\xi x} dx = \int_a^b \frac{d}{d\xi}(iu_x e^{-i\xi x})dx \\
&= \frac{d}{d\xi}\left\{ iue^{-i\xi x} \mid_a^b -i \int_a^b (-i\xi)ue^{-i\xi x} dx \right\} = \frac{d}{d\xi}\left\{ -\int_a^b \xi ue^{-i\xi x} dx \right\} \\
&= \frac{d}{d\xi}\left[-\xi\widehat{u}(\xi) \right].
\end{aligned}$$

Hence Equation (2.19) becomes

$$2\Re \int_\Gamma x\overline{u}_x u_y = -2\Re \int_{\mathbb{R}} \frac{d}{d\xi}\left[\xi\overline{\widehat{u}}(\xi) \right] i\sqrt{k^2 - \xi^2}\widehat{u}(\xi)d\xi + 2\Re \int_\Gamma x\overline{u}_x g. \tag{2.20}$$

We rewrite the first term in the right hand side of Equation (2.20) in terms of $|\xi| \leq k$ and $|\xi| > k$ respectively. For the low frequency part,

$$\begin{aligned}
&-2\Re \int_{|\xi|\leq k} \frac{d}{d\xi}\left[\xi\overline{\widehat{u}}(\xi) \right] i\sqrt{k^2 - \xi^2}\widehat{u}(\xi)\ d\xi \\
=& -2\Re \left\{ \int_{|\xi|\leq k} i\sqrt{k^2 - \xi^2}|\widehat{u}(\xi)|^2 d\xi + \int_{|\xi|\leq k} \xi\overline{\widehat{u}'}(\xi) i\sqrt{k^2 - \xi^2}\widehat{u}(\xi)d\xi \right\} \\
=& -2\Re \int_{|\xi|\leq k} \xi\overline{\widehat{u}'}(\xi) i\sqrt{k^2 - \xi^2}\widehat{u}(\xi).
\end{aligned}$$

Similarly, for the high frequency part,

$$
\begin{aligned}
&-2\Re\int_{|\xi|>k}\frac{d}{d\xi}\left[\xi\overline{\widehat{u}}(\xi)\right]i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi\\
=&2\Re\int_{|\xi|>k}\left[\overline{\widehat{u}}(\xi)+\xi\overline{\widehat{u}'}(\xi)\right]\sqrt{\xi^2-k^2}\widehat{u}(\xi)d\xi\\
=&2\int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}(\xi)|^2d\xi+2\Re\left\{\int_{|\xi|>k}\xi\sqrt{\xi^2-k^2}\overline{\widehat{u}'}(\xi)\widehat{u}(\xi)d\xi\right\}\\
=&2\int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}(\xi)|^2d\xi+\int_{|\xi|>k}\xi\sqrt{\xi^2-k^2}\frac{d}{d\xi}(|\widehat{u}(\xi)|^2)d\xi\\
=&2\int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}(\xi)|^2d\xi-\int_{|\xi|>k}\frac{d}{d\xi}(\xi\sqrt{\xi^2-k^2})|\widehat{u}(\xi)|^2d\xi\\
=&-\int_{|\xi|>k}\frac{k^2}{\sqrt{\xi^2-k^2}}|\widehat{u}|^2d\xi.
\end{aligned}
$$

Therefore, by applying the two equalities above, Equation (2.20) can be written as

$$
\begin{aligned}
&2\Re\int_\Gamma x\overline{u}_xu_y\\
=&-2\Re\int_{\mathbb{R}}\frac{d}{d\xi}(\xi\overline{\widehat{u}}(\xi))i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi+2\Re\int_\Gamma x\overline{u}_xg\\
=&-2\Re\int_{|\xi|\le k}\xi\overline{\widehat{u}'}(\xi)i\sqrt{k^2-\xi^2}\widehat{u}(\xi)-\int_{|\xi|>k}\frac{k^2}{\sqrt{\xi^2-k^2}}|\widehat{u}|^2d\xi+2\Re\int_\Gamma x\overline{u}_xg.
\end{aligned}
\tag{2.21}
$$

Combing results in Equation (2.18) and Equation (2.21), the lemma is proved. □

From Lemma 2.3.1, we can observe that the term $2\Re\int_\Gamma\frac{\partial u}{\partial\mathbf{n}}(X\cdot\nabla\overline{u})$ actually contains lots of information, essentially the higher frequency components is helpful in the estimate. Now by using the result in Lemma 2.3.1 and through the Fourier transform and some additional calculation, the following lemma can be derived which is in terms of the last three terms in the energy equality Equation (2.16).

Lemma 2.3.2. *For the last three terms of Equation (2.16), we have*

$$
\begin{aligned}
& -\int_\Gamma (X\cdot\mathbf{n})|\nabla u|^2 + k^2\int_{\partial D}(X\cdot\mathbf{n})|u|^2 + 2\Re\int_\Gamma \frac{\partial u}{\partial \mathbf{n}}(X\cdot\nabla\overline{u}) \\
\leq\ & 2y_2\int_{|\xi|\leq k}(k^2-\xi^2)|\widehat{u}|^2 d\xi + 2y_2\Re\int_\Gamma \overline{g}\,T(u) + 2\Re\int_{|\xi|\leq k}\left[-\xi\overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2-\xi^2}\,\widehat{u}(\xi)d\xi \\
& + 2\Re\int_\Gamma x\overline{u}_x g dx + y_2\int_\Gamma |g|^2 dx - \int_{|\xi|>k}\frac{k^2}{\sqrt{\xi^2-k^2}}|\widehat{u}|^2 d\xi. \qquad (2.22)
\end{aligned}
$$

Proof. Since $|u_y|^2 = |T(u)+g|^2 = |T(u)|^2 + |g|^2 + 2Re\left[\overline{g}\,T(u)\right]$ and $X\cdot\mathbf{n} = y_2$, then

$$
\begin{aligned}
& -\int_\Gamma (X\cdot\mathbf{n})|\nabla u|^2 + k^2\int_{\partial D}(X\cdot\mathbf{n})|u|^2 + 2\Re\int_\Gamma \frac{\partial u}{\partial \mathbf{n}}(X\cdot\nabla\overline{u}) \\
=\ & y_2\int_\Gamma(-|u_x|^2 - |u_y|^2 + k^2|u|^2) + 2\Re\int_\Gamma x\overline{u}_x u_y + 2y_2\int_\Gamma |u_y|^2 \\
=\ & y_2\int_\Gamma(-|u_x|^2 + |u_y|^2 + k^2|u|^2) + 2\Re\int_\Gamma x\overline{u}_x u_y.
\end{aligned}
$$

Using the fact that u and u_x are compactly supported on Γ and $\int_\Gamma |u_y|^2 \leq \int_{\mathbb{R}} |u_y|^2$, then

$$
\begin{aligned}
& y_2\int_\Gamma(-|u_x|^2 + |u_y|^2 + k^2|u|^2) + 2\Re\int_\Gamma x\overline{u}_x u_y \\
\leq\ & y_2\int_{\mathbb{R}}(-\xi^2|\widehat{u}|^2 + |i\sqrt{k^2-\xi^2}\widehat{u}|^2 + k^2|\widehat{u}|^2)d\xi \\
& + 2y_2\Re\int_\Gamma \overline{g}\,T(u) + y_2\int_\Gamma |g|^2 + 2\Re\int_\Gamma x\overline{u}_x u_y \\
=\ & y_2\int_{|\xi|\leq k} 2(k^2-\xi^2)|\widehat{u}|^2 d\xi + 2y_2\Re\int_\Gamma \overline{g}\,T(u) + y_2\int_\Gamma |g|^2 + 2\Re\int_\Gamma x\overline{u}_x u_y.
\end{aligned}
$$

Plugging Equation (2.21) into the inequality above, Lemma 2.3.2 is obtained. □

Combining Lemma 2.3.2 and Equation (2.16) yields the following inequality for the energy.

$$
\begin{aligned}
&\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi - \int_S (X\cdot\mathbf{n})|\nabla u|^2 \\
\leq &\Re\int_\Gamma g\overline{u} + y_2\int_{|\xi|\leq k} 2(k^2-\xi^2)|\widehat{u}|^2 d\xi + 2y_2\Re\int_\Gamma T(u)\overline{g} \\
&+ 2\Re\int_{|\xi|\leq k}\left[-\xi\overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi + 2\Re\int_\Gamma x\overline{u}_x g dx + y_2\int_\Gamma |g|^2 dx
\end{aligned}
\tag{2.23}
$$

So far, an inequality involving the energy terms and other additional terms is obtained, and the term ∇u on Γ has been canceled out by using the lemmas shown in this section. In the next section, we will estimate each term on the right hand side of Equation (2.23).

2.4 Detailed Estimates

To estimate each term on the right hand side of Equation (2.23) associated with u or $\hat{u}$, the inequalities for $\|u\|^2_{L^2(\Gamma)}$ and $\int\sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi$ are needed, which is presented in Lemma 2.4.1, this is obtained by using the technique of Lemma 3.5 in [8] and Lemma 2.2.1 . Recall that we denote $A_L = \int_{|\xi|\leq k}\sqrt{k^2-\xi^2}|\widehat{u}|^2 d\xi$, $A_H = \int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi$, $B_H = \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi$, and $G = \int_\Gamma |g|^2$. In essence, A_L, the lower frequency part of u, is relatively small compared to B_H, higher frequency part of u, the term $\frac{k^2}{\sqrt{\xi^2-k^2}}|$ in B_H played an important role.

Lemma 2.4.1. *There exists a positive constant C, such that*

$$
A_L \leq \frac{\delta}{k}B_H + \frac{1}{\delta}G, \tag{2.24}
$$

for any $\delta > 0$ *such that* $C\delta \leq k^{\frac{1}{2}}$. *Furthermore,*

$$\|u\|^2_{L^2(\Gamma)} \leq \frac{2}{k} B_H + \frac{C^2}{k} G. \tag{2.25}$$

Proof. To proof Equation (2.24), the function u_α constructed in Lemma 3.5 in [8] is introduced. Define $u_\alpha = u(x) - u(x - \frac{2\pi}{k}[2k])$, then, using the fact that the support of u is of length at most π, *supp* $u(x) \cap$ *supp* $u(x - \frac{2\pi}{k}[2k]) = \varnothing$,

$$2\|u\|^2_{L^2(\Gamma)} = \|u_\alpha\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |1 - e^{\frac{2\pi}{k}[2k]\xi i}|^2 |\widehat{u}|^2 d\xi.$$

For $|\xi| \leq k$, there exists a positive constant C such that $\sqrt{k}|1 - e^{\frac{2\pi}{k}[2k]\xi i}|^2 \leq C\sqrt{k^2 - \xi^2}$. For $|\xi| > k$, $k|1 - e^{\frac{2\pi}{k}[2k]\xi i}|^2 \leq k \cdot 4 \leq 2(\sqrt{\xi^2 - k^2} + \frac{k^2}{\sqrt{\xi^2 - k^2}})$. It follows that

$$|1 - e^{\frac{2\pi}{k}[2k]\xi i}|^2 \leq \begin{cases} \frac{C}{\sqrt{k}}\sqrt{k^2 - \xi^2} & \text{for } |\xi| \leq k, \\ \frac{2}{k}(\sqrt{\xi^2 - k^2} + \frac{k^2}{\sqrt{\xi^2 - k^2}}) & \text{for } |\xi| > k. \end{cases}$$

$$\text{Therefore,} \quad \|u\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}} A_L + \frac{1}{k} B_H. \tag{2.26}$$

From Equation (2.6) and the inequality above, we have

$$\begin{aligned} A_L &\leq \|g\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)} \leq \frac{\delta}{2}\|u\|^2_{L^2(\Gamma)} + \frac{1}{2\delta}\|g\|^2_{L^2(\Gamma)} \\ &\leq \frac{\delta}{2}(\frac{C}{\sqrt{k}} A_L + \frac{1}{k} B_H) + \frac{1}{2\delta}\|g\|^2_{L^2(\Gamma)} \end{aligned}$$

If $C\delta \leq \sqrt{k}$, then

$$\frac{1}{2}A_L \leq (1 - \frac{C\delta}{2\sqrt{k}})A_L \leq \frac{\delta}{2k}B_H + \frac{1}{2\delta}\|g\|^2_{L^2(\Gamma)}.$$

We finished the proof of Equation (2.24).

Plug Equation (2.24) into Equation (2.26), we have

$$\|u\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}}(\frac{\delta}{k}B_H + \frac{1}{\delta}\|g\|^2_{L^2(\Gamma)}) + \frac{1}{k}B_H.$$

Choose $C\delta = \sqrt{k}$, then Equation (2.25) is obtained. □

This is an important lemma as it determines the power of k in the estimate, and provides means to relate two integrals $\int_\Gamma |u|^2$ and $\int_{|\xi|\leq k} \sqrt{|\xi^2 - k^2|}|\widehat{u}|^2$ to the high frequency part B_H and L_2 norm of g . Now we will estimate each term on the right hand side of Equation (2.23). Note that some terms are easier to estimate, the most difficult ones are: $2y_2\Re\int_\Gamma T(u)\overline{g}$ and $2\Re\int_{|\xi|\leq k}\left[-\xi\widehat{\overline{u}'}(\xi)\right] i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi$. For the term $2y_2\Re\int_\Gamma T(u)\overline{g}$, the direct usage of Schwarz inequality does not work, as after converted to the frequency domain, it would lead to either the integral of $\xi^2|\widehat{u}|^2$ which can not be controlled by the energy or $\xi^2|\widehat{g}|^2$ which may not be integrable as g may not be compactly supported on Γ. Other technique are needed for the estimate as shown in Lemma 2.4.4.

Lemma 2.4.2. *The first term in Equation (2.23) can be estimated as follows.*

$$I_1 = \Re\int_\Gamma g\overline{u} \leq \varepsilon_1 B_H + d_1 G, \tag{2.27}$$

where ε_1 is any positive number, and d_1 is given by $d_1 = \frac{\varepsilon_1 C^2}{2} + \frac{1}{2\varepsilon_1 k}$.

Proof. Using Schwarz inequality and Lemma 2.4.1, we have

$$I_1 = \Re \int_\Gamma g\overline{u} \le \|g\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)} \le \frac{\varepsilon_1 k}{2}\|u\|^2_{L^2(\Gamma)} + \frac{1}{2\varepsilon_1 k}\|g\|^2_{L^2(\Gamma)}$$
$$\le \frac{\varepsilon_1 k}{2}(\frac{2}{k}B_H + \frac{C^2}{k}\|g\|^2_{L^2(\Gamma)}) + \frac{1}{2\varepsilon_1 k}\|g\|^2_{L^2(\Gamma)}$$

then

$$I_1 \le \varepsilon_1 B_H + (\frac{\varepsilon_1 C^2}{2} + \frac{1}{2\varepsilon_1 k})G = \varepsilon_1 B_H + d_1 G.$$

□

Lemma 2.4.3. *For the second term of Equation (2.23), we have*

$$I_2 = y_2 \int_{|\xi|\le k} 2(k^2 - \xi^2)|\widehat{u}|^2 d\xi \le \varepsilon_2 B_H + d_2 kG, \tag{2.28}$$

where ε_2 *is any positive number, and* $d_2 = \frac{4y_2^2}{\varepsilon_2}$.

Proof. Since $|\xi| \le k$, we have $\sqrt{k^2 - \xi^2} \le k$, then

$$I_2 = y_2 \int_{|\xi|\le k} 2(k^2 - \xi^2)|\widehat{u}|^2 d\xi \le y_2\, 2k \cdot \int_{|\xi|\le k} \sqrt{k^2 - \xi^2}|\widehat{u}|^2 d\xi$$
$$\le 2y_2 k(\frac{\delta}{k}B_H + \frac{1}{\delta}G)$$

Choose $\delta = \frac{\varepsilon_2}{2y_2}$, where ε_2 is a small positive number, then Equation (2.28) is obtained. □

Next, we deal with I_3, which is more involved to estimate this term.

Lemma 2.4.4. *For I_3, we have the following estimate.*

$$\begin{aligned} I_3 &= 2y_2\Re\int_\Gamma T(u)\overline{g} \\ &\le \varepsilon_{31}\int_D\left(|\nabla u|^2+k^2|u|^2\right)+\varepsilon_{32}\int_S|\frac{\partial u}{\partial \mathbf{n}}|^2+d_3kG+\frac{e_3}{k}\int_\Gamma|g_x|^2, \end{aligned} \tag{2.29}$$

where $\varepsilon_{31},\varepsilon_{32}$ are any positive numbers, $d_3=\frac{\widetilde{c}}{\varepsilon_{31}}\left(\frac{1}{k^3}+\frac{1}{k}+1\right)+\frac{\widetilde{c}}{\varepsilon_{32}}$ and $e_3=\frac{\widetilde{c}}{\varepsilon_{31}}+\frac{\widetilde{c}}{\varepsilon_{32}k}$, $\widetilde{c}$ is a positive constant depending on the domain D.

Proof. Note that $\int_\Gamma T(u)\overline{g}=\int_{\mathbb{R}}T(u)\overline{g}_\Gamma=\int_{\mathbb{R}}\widehat{T(u)}(\xi)\overline{\widehat{g}}_\Gamma(\xi)=\int_{\mathbb{R}}i\sqrt{k^2-\xi^2}\widehat{u}(\xi)\overline{\widehat{g}}_\Gamma(\xi)$, so the direct usage of the Hölder inequality does not work since $\widehat{g}_\Gamma(\xi)$ decays like $\frac{1}{\xi}$ when $\xi\to\infty$ if g_Γ does not vanish on the boundary of Γ. Here another test function η is constructed for the estimate of I_3.

To define η, first we extend $g(x)$ on Γ to a function $\widetilde{g}(x)$ on the line $y=y_2$ such that $\|\widetilde{g}\|_{H^1(\mathbb{R})}\le\widetilde{c}\|g\|_{H^1(\Gamma)}$ and $\|\widetilde{g}\|_{L^2(\mathbb{R})}\le\widetilde{c}\|g\|_{L^2(\Gamma)}$. Moreover, let

$$h(y):=\begin{cases}1+k(y-y_2) & \text{for } y_2-\frac{1}{k}\le y\le y_2,\\ 0 & \textit{otherwise.}\end{cases}$$

Then set $\eta(x,y)=\widetilde{g}(x)h(y),(x,y)\in D$. For this particular choice of η, we have $\eta=g$ on Γ and $|\eta(x,y)|\le|\widetilde{g}(x)|$.

Adopt η as the test function, then

$$\int_D\nabla u\cdot\nabla\overline{\eta}-\int_D k^2u\overline{\eta}-\int_\Gamma(T(u)+g)\overline{\eta}=\int_S\frac{\partial u}{\partial\mathbf{n}}\overline{\eta},$$

Note that $\eta = g$ on Γ, this leads to

$$\int_\Gamma T(u)\overline{g} = \int_D \nabla u \cdot \nabla \overline{\eta} - \int_D k^2 u \overline{\eta} - \int_S \frac{\partial u}{\partial \mathbf{n}} \overline{\eta} - \int_\Gamma |g|^2. \tag{2.30}$$

To estimate I_3, it suffices to estimate each term on the right hand side of Equation (2.30).

We start from the estimate of $\int_D \nabla u \cdot \nabla \overline{\eta}$. Since

$$\int_D \nabla u \cdot \nabla \overline{\eta} = \int_D \left(u_x \widetilde{g}'(x) h(y) + u_y \widetilde{g}(x) h'(y) \right),$$

the two terms on the right hand side of the identity above are handled separately as follows.

Choosing a rectangular domain $D_1 = [A, B] \times [y_1, y_2]$ such that $D \subset D_1$, considering $h(y)$ in D_1, we have $\int_{y_1}^{y_2} |h(y)|^2 \sim O(\frac{1}{k})$ and $\int_{y_1}^{y_2} |h(y)|^2 \sim O(k)$. Let χ_D denote the characteristic function of D, then

$$\begin{aligned}
\int_D u_x \widetilde{g}'(x) h(y) &= \int_{D_1} u_x \widetilde{g}'(x) h(y) \chi_D dy dx \\
&\le \int_A^B |\widetilde{g}'(x)| \int_{y_1}^{y_2} |u_x||h(y)| \chi_D dy dx \\
&\le \int_A^B |\widetilde{g}'(x)| \left(\int_{y_1}^{y_2} |u_x \chi_D|^2 dy \right)^{\frac{1}{2}} \left(\int_{y_1}^{y_2} |h(y)|^2 dy \right)^{\frac{1}{2}} dx \\
&\le \frac{1}{\sqrt{k}} \int_A^B |\widetilde{g}'(x)| (\int_{y_1}^{y_2} |u_x \chi_D|^2 dy)^{\frac{1}{2}} dy \\
&\le \left[\int_A^B \int_{y_1}^{y_2} |u_x \chi_D|^2 dy dx \right]^{\frac{1}{2}} \left[\int_A^B \left(\frac{1}{\sqrt{k}} |\widetilde{g}'(x)| \right)^2 dx \right]^{\frac{1}{2}} \\
&\le \frac{\varepsilon_{31}}{2} \int_D |\nabla u|^2 + \frac{1}{\varepsilon_{31}} \frac{1}{k} \int_A^B |\widetilde{g}'(x)|^2 dx.
\end{aligned}$$

Therefore,

$$\int_D u_x \widetilde{g}'(x)h(y) \le \frac{\varepsilon_{31}}{2}\int_D |\nabla u|^2 + \frac{\widetilde{c}}{\varepsilon_{31}}\frac{1}{k}\left(\int_\Gamma |g'(x)|^2 + \int_\Gamma |g(x)|^2\right). \tag{2.31}$$

Similarly,

$$\begin{aligned}
\int_D u_y \widetilde{g}(x)h'(y) &= \int_{D_1} u_y \widetilde{g}(x)h'(y)\chi_D dydx \\
&\le \int_A^B |\widetilde{g}(x)| \int_{y_1}^{y_2} |u_y||h'(y)|\chi_D dydx \\
&\le \int_A^B |\widetilde{g}(x)| \left(\int_{y_1}^{y_2} |u_y\chi_D|^2 dy)^{\frac{1}{2}} (\int_{y_1}^{y_2} |h'(y)|^2 dy\right)^{\frac{1}{2}} dx \\
&\le \sqrt{k}\int_A^B |\widetilde{g}(x)|(\int_{y_1}^{y_2} |u_y\chi_D|^2 dy)^{\frac{1}{2}} dx \\
&\le \left[\int_A^B \int_{y_1}^{y_2} |u_y\chi_D|^2 dydx\right]^{\frac{1}{2}} \left[\int_A^B \left(\sqrt{k}|\widetilde{g}(x)|\right)^2 dx\right]^{\frac{1}{2}} \\
&\le \frac{\varepsilon_{31}}{2}\int_D |\nabla u|^2 + \frac{1}{\varepsilon_{31}} k \int_A^B |\widetilde{g}(x)|^2 dx.
\end{aligned}$$

It follows that

$$\int_D u_y \widetilde{g}(x)h'(y) \le \frac{\varepsilon_{31}}{2}\int_D |\nabla u|^2 + \frac{\widetilde{c}k}{\varepsilon_{31}}\int_\Gamma |g(x)|^2. \tag{2.32}$$

Now we treat the second term on the right hand side of Equation (2.30). By the definition of η, $\int_D |\eta|^2 \le \int_D \frac{1}{k}|\widetilde{g}|^2 \le \frac{\widetilde{c}}{k}\int_\Gamma |g|^2$, where $\widetilde{c}$ depends on the length in the y direction, i.e.,

$y_2 - y_1$. It yields that

$$\begin{aligned} -\int_D k^2 u\overline{\eta} &\le \int_D \left(k^2|u|^2\right)^{\frac{1}{2}} \left(\int_D k^2|\eta|^2\right)^{\frac{1}{2}} \\ &\le \varepsilon_{31}\int_D k^2|u|^2 + \frac{1}{\varepsilon_{31}}\int_D k^2|\eta|^2 \\ &\le \varepsilon_{31}\int_D k^2|u|^2 + \frac{\widetilde{c}}{\varepsilon_{31}}\int_\Gamma k|g|^2. \end{aligned} \tag{2.33}$$

To treat the term $-\int_S \frac{\partial u}{\partial \mathbf{n}}\overline{\eta}$, we use an estimate for $\int_S |\eta|^2$, which is derived as follows. By the divergence theorem, we have

$$\int_D div(|\eta|^2 X) = \int_{\partial D} |\eta|^2 (X\cdot\mathbf{n}),$$

this is equivalent to

$$2\Re\int_D \overline{\eta}\nabla\eta\cdot X + \int_D |\eta|^2 div(X) = \int_\Gamma y_2|\eta|^2 + \int_S |\eta|^2(X\cdot\mathbf{n}).$$

It follows that

$$\begin{aligned} -\int_S |\eta|^2(X\cdot\mathbf{n}) &= -2\Re\int_D \overline{\eta}\nabla\eta\cdot X - \int_D |\eta|^2 div(X) + \int_\Gamma y_2|\eta^2| \\ &\le a_4\left[\int_D k|\eta|^2 + \int_D \frac{1}{k}|\nabla\eta|^2\right] + \int_\Gamma y_2|g|^2 \\ &\le a_5\int_\Gamma (|g|^2 + \frac{1}{k^2}|gx|^2). \end{aligned}$$

Then use the fact that $-(X\cdot\mathbf{n}) \ge p_1 > 0$, one obtains $\int_S |\eta|^2 \le \widetilde{c}\int_\Gamma (|g|^2 + \frac{1}{k^2}|gx|^2)$.

Hence

$$
\begin{aligned}
-\int_S \frac{\partial u}{\partial \mathbf{n}}\overline{\eta} \leq \left(\int_S |\frac{\partial u}{\partial \mathbf{n}}|^2\right)^{\frac{1}{2}} \left(\int_S |\eta|^2\right)^{\frac{1}{2}} &\leq \varepsilon_{32} \int_S |\frac{\partial u}{\partial \mathbf{n}}|^2 + \frac{1}{\varepsilon_{32}} \int_S |\eta|^2 \\
&\leq \varepsilon_{32} \int_S |\frac{\partial u}{\partial \mathbf{n}}|^2 + \frac{\widetilde{c}}{\varepsilon_{32}} \int_\Gamma (|g|^2 + \frac{1}{k^2}|g_x|^2).
\end{aligned}
\tag{2.34}
$$

Combining all the results in Equation (2.31), (2.32), (2.33) and (2.34), the estimate in the lemma is arrived. □

Now we are going to handle I_4, which yields the highest order of k in Theorem 2.1.1.

Lemma 2.4.5. *For I_4, we have the following estimate.*

$$
I_4 = 2\Re \int_{|\xi|\leq k} \left[-\xi\overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2-\xi^2}\widehat{u}(\xi)d\xi \leq \varepsilon_4 B_H + d_4 k^{\frac{7}{5}} G,
\tag{2.35}
$$

where ε_4 is a small positive number, and d_4 is chosen accordingly.

Remark: By direct usage of Schwarz inequality, and use the fact that $\widehat{u}'(\xi) = \widehat{(-ix)u}(\xi)$, we could arrive at an estimate of $I_4 \leq \varepsilon_4 B_H + Ck^3 G$. But we may lose some information since the relation between $\widehat{u}'(\xi)$ and u relies on the entire domain, while in fact only the portion $|\xi| \leq k$ is needed. Here our idea is to start directly from the frequency domain function defined in the lower frequency region, and introduce the corresponding spatial function by using inverse fourier transform, then through possible cancelation between positive and negative parts of the introduced function, it results in a lower power estimate in terms of k.

Proof. First I_4 is written as the sum of two integrals, i.e. $I_4 = I_{41} + I_{42}$ as shown below, where I_{41} represents the integral for the region $0 \leq \xi \leq k$ and I_{42} represents the integral for the region $-k \le \xi \le 0$. Note that

$$\begin{aligned} I_{41} &= 2\Re \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)\left[-i\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\overline{\widehat{u}'}(\xi)\right]d\xi \\ &= 2\Re \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)\overline{\left[i\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)\right]'}d\xi. \end{aligned}$$

Define $f_1(x) = \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)e^{i\xi x}d\xi$, then through integration by parts, it yields that

$$\int_0^k \left[i\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)\right]' e^{i\xi x}d\xi = -\int_0^k i\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)ixe^{i\xi x}d\xi = xf_1(x).$$

Thus by Parseval's identity,

$$I_{41} = 2\Re \int_R f_1(x)\overline{xf_1(x)}dx = 2\int_R x|f_1(x)|^2dx.$$

Furthermore, observe that

$$\begin{aligned} I_{42} &= -2\Re \int_{-k}^0 i\xi(k^2-\xi^2)^{\frac{1}{2}}\widehat{u}(\xi)\overline{\widehat{u}'}(\xi)d\xi \\ &= 2\Re \int_0^k i\xi(k^2-\xi^2)^{\frac{1}{2}}\widehat{u}(-\xi)\overline{\widehat{u}'}(-\xi)d\xi \\ &= 2\Re \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(-\xi)\overline{\left[i\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(-\xi)\right]'}d\xi. \end{aligned}$$

Define $f_2(x) = \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(-\xi)e^{i\xi x}d\xi$, then similar computation yields that

$$I_{42} = 2\int_R x|f_2(x)|^2dx.$$

Therefore $I_4 = 2\int_R x|f_1(x)|^2 dx + 2\int_R x|f_2(x)|^2 dx.$

Moreover, by the definitions of $f_1(x)$ and $f_2(x)$, we have

$$\int_R |f_1(x)|^2 = \int_0^k \xi(k^2-\xi^2)^{\frac{1}{2}}|\widehat{u}(\xi)|^2 d\xi \leq kA_L, \quad \int_R |f_2(x)|^2 \leq kA_L. \tag{2.36}$$

Note that by using Equation (2.36), for any given positive constant M, it follows that

$$\int_{|x|\leq M} \left[x|f_1(x)|^2 + x|f_2(x)|^2\right] \leq MkA_L \leq Mk(\frac{\delta}{k}B_H + \frac{1}{\delta}G). \tag{2.37}$$

Thus to estimate I_4, it remains to estimate $\int_{|x|>M} \left[x|f_1(x)|^2 + x|f_2(x)|^2\right]$ and choose an appropriate M.

By using the fact that $u(x, y_2)$ is supported on the interval $[a, b]$, it is easy to write $f_1(x)$ as follows,

$$f_1(x) = \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\widehat{u}(\xi)e^{i\xi x}d\xi = \int_0^k\int_a^b \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}e^{i\xi x}u(s)e^{-i\xi s}dsd\xi. \tag{2.38}$$

To simplify notations, define $H(z) = \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}e^{i\xi z}d\xi$, then

$$f_1(x) = \int_a^b u(s)H(x-s)ds.$$

Consequently $|f_1(x)|^2 = \int_a^b\int_a^b u(s)\overline{u(t)}H(x-s)\overline{H(x-t)}dsdt$. And $|f_1(-x)|^2$ can be represented by

$$|f_1(-x)|^2 = \int_a^b \int_a^b u(s)\overline{u(t)}H(-x-s)\overline{H(-x-t)}dsdt$$
$$= \int_a^b \int_a^b u(s)\overline{u(t)}\ \overline{H(x+s)}H(x+t)dsdt.$$

Then $\int_{|x|>M} x|f_1(x)|^2$ can be rewritten as a triple integral involving H term where cancelations take place.

$$\int_{|x|>M} x|f(x)|^2 = \int_{x>M} x\left[|f_1(x)|^2 - |f_1(-x)|^2\right]$$
$$= \int_a^b \int_a^b u(s)\overline{u(t)} \int_{x>M} \left[xH(x-s)\overline{H(x-t)} - xH(x+t)\overline{H(x+s)}\right] dxdsdt.$$

Define $J_1 = \displaystyle\int_{x>M} \left[xH(x-s)\overline{H(x-t)} - xH(x+t)\overline{H(x+s)}\right] dx$. By changing variable for possible cancelation, it follows that

$$J_1 = \int_{M-s}^{\infty} (x+s)H(x)\overline{H(x+s-t)}dx - \int_{M+t}^{\infty} (x-t)H(x)\overline{H(x-t+s)}dx$$
$$= \int_{M-s}^{\infty} sH(x)\overline{H(x+s-t)}dx + \int_{M+t}^{\infty} tH(x)\overline{H(x+s-t)}dx$$
$$+ \int_{M-s}^{M+t} xH(x)\overline{H(x+s-t)}dx.$$

Similar computations yield a representation of $\int_{|x|>M} x|f_2(x)|^2$, that is,

$$\int_{|x|>M} x|f_2(x)|^2 = \int_a^b \int_a^b u(s)\overline{u(t)}J_2 dsdt,$$

where

$$J_2 = -\int_{M+s}^{\infty} sH(x)\overline{H(x-s+t)}dx - \int_{M-t}^{\infty} tH(x)\overline{H(x-s+t)}dx$$
$$- \int_{M-t}^{M+s} xH(x)\overline{H(x-s+t)}dx.$$

Therefore the contribution to I_4 due to $|x| > M$, that is, $\int_{|x|>M} \left[x|f_2(x)|^2 + x|f_2(x)|^2\right]$, can be expressed using the following integral form.

$$\int_{|x|>M} \left[x|f_2(x)|^2 + x|f_2(x)|^2\right] = \int_a^b \int_a^b u(s)\overline{u(t)}(J_1 + J_2)dsdt$$
$$= \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right] J_1 dsdt = T_1 + T_2 + T_3,$$

where T_1, T_2, T_3 are denoted as

$$T_1 = \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right] \int_{M-s}^{\infty} sH(x)\overline{H(x+s-t)}dxdsdt,$$
$$T_2 = \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right] \int_{M+t}^{\infty} tH(x)\overline{H(x+s-t)}dxdsdt,$$
$$T_3 = \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right] \int_{M-s}^{M+t} xH(x)\overline{H(x+s-t)}dxdsdt.$$

Observe that the common term in T_1, T_2 and T_3 is $H(x)\overline{H(x+s-t)}$, so next we see how to express $H(x)\overline{H(x+s-t)}$ in more details using the expression of $H(x)$.

$$H(x) = \int_0^k \xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}} e^{i\xi x} d\xi = -\frac{1}{ix}\int_0^k \left[\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{1}{4}}\right]' e^{i\xi x} d\xi$$
$$= -\frac{1}{ix}\int_0^k \frac{k^2 - 2\xi^2}{\xi^{\frac{1}{2}}(k^2-\xi^2)^{\frac{3}{4}}} e^{i\xi x} d\xi = -\frac{k}{ix}\int_0^1 F(\eta) e^{ik\eta x} d\eta,$$

with $F(\eta) = \dfrac{1-2\eta^2}{\eta^{\frac{1}{2}}(1-\eta^2)^{\frac{3}{4}}}$.

Consider the integral $\psi(w) = \int_0^1 F(\eta)e^{iw\eta}d\eta$, and write $\psi(w) = \psi_1(w) + \psi_2(w)$, where $\psi_1(w) = \int_0^{\frac{1}{2}} F(\eta)e^{iw\eta}d\eta$ and $\psi_2(w) = \int_{\frac{1}{2}}^1 F(\eta)e^{iw\eta}d\eta$. To estimate $\psi_1(w)$, note that on the interval $[0, \frac{1}{2}]$, $\dfrac{1-2\eta^2}{(1-\eta^2)^{\frac{3}{4}}} = 1 + F_1(\eta^2)$, where $F_1(\eta^2)$ is continuous on $[0, \frac{1}{2}]$, thus

$$\begin{aligned}\psi_1(w) &= \int_0^{\frac{1}{2}} \eta^{-\frac{1}{2}}(1+F_1(\eta^2))e^{iw\eta}d\eta = \int_0^{\frac{1}{2}} \eta^{-\frac{1}{2}}e^{iw\eta}d\eta + \int_0^{\frac{1}{2}} \eta^{-\frac{1}{2}}F_1(\eta^2)e^{iw\eta}d\eta \\ &= \frac{1}{w^{\frac{1}{2}}}\int_0^{\frac{1}{2}w} \frac{e^{i\zeta}}{\sqrt{\zeta}}d\zeta + O\left(\frac{1}{w}\right) = \frac{1}{w^{\frac{1}{2}}}\int_0^{\infty} \frac{e^{i\zeta}}{\sqrt{\zeta}}d\zeta + O\left(\frac{1}{w}\right).\end{aligned}$$

It shows that

$$\psi_1(w) = \frac{C_1}{w^{\frac{1}{2}}} + O\left(\frac{1}{w}\right).$$

Furthermore, $\psi_2(w) = \int_{\frac{1}{2}}^1 F(\eta)e^{iw\eta}d\eta = \int_0^{\frac{1}{2}} F(1-\eta)e^{iw(1-\eta)}d\eta$, and on the interval $[0, \frac{1}{2}]$, write $F(1-\eta) = \dfrac{-1+4\eta-2\eta^2}{\eta^{\frac{3}{4}}(1-\eta)^{\frac{1}{2}}(2-\eta)^{\frac{3}{4}}} = \eta^{-\frac{3}{4}}\left(-2^{-\frac{3}{4}} + F_2(\eta)\right)$, where $F_2(\eta)$ is continuous on $[0, \frac{1}{2}]$, thus

$$\begin{aligned}&\psi_2(w) = e^{iw}\left[-2^{-\frac{3}{4}}\int_0^{\frac{1}{2}} \eta^{-\frac{3}{4}}e^{-iw\eta}d\eta + \int_0^{\frac{1}{2}} \eta^{-\frac{3}{4}}F_2(\eta)e^{-iw\eta}d\eta\right] \\ &= -\,2^{-\frac{3}{4}}e^{iw}\frac{1}{w^{\frac{1}{4}}}\int_0^{\frac{1}{2}w} \frac{1}{\zeta^{\frac{3}{4}}}e^{-i\zeta}d\zeta + O\left(\frac{1}{w}\right) = -2^{-\frac{3}{4}}e^{iw}\frac{1}{w^{\frac{1}{4}}}\int_0^{\infty} \frac{1}{\zeta^{\frac{3}{4}}}e^{-i\zeta}d\zeta + O\left(\frac{1}{w}\right)\end{aligned}$$

Therefore $\psi_2(w) = \dfrac{C_2e^{iw}}{w^{\frac{1}{4}}} + O\left(\dfrac{1}{w}\right)$. It follows that $\psi(w) = \dfrac{C_2e^{iw}}{w^{\frac{1}{4}}} + \dfrac{C_1}{w^{\frac{1}{2}}} + O\left(\dfrac{1}{w}\right)$, and the common term in T_1, T_2 and T_3 can be written as

$$H(x)\overline{H(x+s-t)} = \frac{k^2}{x(x+s-t)}\psi(kx)\overline{\psi[k(x+s-t)]}$$

$$= \frac{k^2}{x(x+s-t)}\left[\frac{C_2 e^{ikx}}{(kx)^{\frac{1}{4}}} + \frac{C_1}{(kx)^{\frac{1}{2}}} + O\left(\frac{1}{kx}\right)\right]\left[\frac{\overline{C_2} e^{-ik(x+s-t)}}{[k(x+s-t)]^{\frac{1}{4}}} + \frac{\overline{C_1}}{[k(x+s-t)]^{\frac{1}{2}}} + \right.$$

$$\left. O\left(\frac{1}{k(x+s-t)}\right)\right]$$

$$= \frac{k^2}{x^2}\left[|C_2|^2 \frac{e^{-ik(s-t)}}{(kx)^{\frac{1}{2}}} + O\left(\frac{1}{k^{\frac{1}{2}} x^{\frac{3}{2}}}\right) + O\left(\frac{1}{(kx)^{\frac{3}{4}}}\right)\right].$$

Accordingly, for $|x| > M$, we have

$$\Im x H(x)\overline{H(x+s-t)} = \frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 i \sin k(t-s) + O\left(\frac{k^{\frac{3}{2}}}{M^{\frac{5}{2}}}\right) + O\left(\frac{k^{\frac{5}{4}}}{M^{\frac{7}{4}}}\right).$$

Note that, the dominant term is $\frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 i \sin k(t-s)$, correspondly, the dominant term in T_3, denoted by T_{3d}, is given by

$$T_{3d} = \Re i \frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right](t+s)\sin k(t-s)\,dsdt$$

$$= \frac{2k^{\frac{3}{2}}}{M^{\frac{3}{2}}}\Re \int_a^b \int_a^b s\left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right] e^{ikt}e^{-iks}dsdt$$

$$= \frac{2k^{\frac{3}{2}}}{M^{\frac{3}{2}}}\Re \left[\int_a^b su(s)e^{-iks}ds\overline{\int_a^b u(t)e^{-ikt}dt} - \overline{\int_a^b su(s)e^{iks}ds}\int_a^b u(t)e^{ikt}dt\right]$$

$$= \frac{2k^{\frac{3}{2}}}{M^{\frac{3}{2}}}\Re\left\{\widehat{su}(k)\overline{\widehat{u}}(k) - \overline{\widehat{su}}(-k)\widehat{u}(-k)\right\}.$$

Note that in T_{3d}, $\widehat{su}(\xi) \le \|u\|_{L^2(\Gamma)}$, so it remains to consider $\widehat{u}(k)$. Since

$$
\begin{aligned}
|\widehat{u}(\xi) - \widehat{u}(k)| &= \left| \int_a^b u(x)[e^{-i\xi x} - e^{-ikx}]dx \right| \\
&\le \int_a^b \left| u(x)e^{-i\xi x} \right| \left| 1 - e^{i(\xi - k)x} \right| dx \quad \le \|u\|_{L^2(\Gamma)} |\xi - k|.
\end{aligned}
$$

Thus $\frac{1}{2}|\widehat{u}(k)|^2 - |\widehat{u}(\xi)|^2 \le |\widehat{u}(\xi) - \widehat{u}(k)|^2 \le \|u\|^2_{L^2(\Gamma)} |\xi - k|^2$, it follows that

$$
|\widehat{u}(\xi)|^2 \ge \frac{1}{2}|\widehat{u}(k)|^2 - \|u\|^2_{L^2(\Gamma)} |\xi - k|^2.
$$

Integrate the above inequality from $k - \delta$ to k, then we have

$$
\begin{aligned}
A_L &\ge \int_{k-\delta}^k \sqrt{k^2 - \xi^2} |\widehat{u}(\xi)|^2 d\xi \\
&\ge \frac{1}{2}|\widehat{u}(k)|^2 \int_{k-\delta}^k \sqrt{k^2 - \xi^2} d\xi - \|u\|^2_{L^2(\Gamma)} \int_{k-\delta}^k |\xi - k|^2 \sqrt{k^2 - \xi^2} d\xi \\
&\ge \frac{1}{2}|\widehat{u}(k)|^2 \sqrt{k} \delta^{\frac{3}{2}} - \|u\|^2_{L^2(\Gamma)} \sqrt{k} \delta^{\frac{7}{2}}.
\end{aligned}
$$

It provides an estimate for $|\widehat{u}(k)|^2$, which is bounded by $\|u\|^2_{L^2(\Gamma)}$ and A_L,

$$
|\widehat{u}(k)|^2 \le \delta^2 \|u\|^2_{L^2(\Gamma)} + \frac{1}{\sqrt{k}\delta^{\frac{3}{2}}} A_L.
$$

Using this inequality, T_3 can be bounded by

$$
|T_3| \le \frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}} \|u\|_{L^2(\Gamma)} \left[\delta \|u\|_{L^2(\Gamma)} + \frac{1}{k^{\frac{1}{4}} \delta^{\frac{3}{4}}} A_L^{\frac{1}{2}} \right] \le \frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}} \delta \|u\|^2_{L^2(\Gamma)} + \frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}} \frac{1}{k^{\frac{1}{2}} \delta^{\frac{5}{2}}} A_L.
$$

In particular, choose $M = k^{\frac{1}{5}}$ and $\delta = \epsilon k^{-\frac{1}{5}}$, then T_3 can be estimated by

$$|T_3| \leq \epsilon k \|u\|^2_{L^2(\Gamma)} + \frac{k^{\frac{6}{5}}}{\epsilon^{\frac{5}{2}}} A_L \leq \epsilon B_H + C k^{\frac{7}{5}} G.$$

Note that for T_1, for the integral $\int_{M-s}^{\infty} sH(x)\overline{H(x+s-t)}dx$, the dominant term is

$$s\frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 i \sin k(t-s);$$

and in T_2, for the integral $\int_{M+t}^{\infty} tH(x)\overline{H(x+s-t)}dx$, the dominant term is

$$t\frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 i \sin k(t-s).$$

Therefore, the dominant term for $T_1 + T_2$ is

$$\Re i\frac{k^{\frac{3}{2}}}{M^{\frac{3}{2}}}|C_2|^2 \int_a^b \int_a^b \left[u(s)\overline{u(t)} - u(t)\overline{u(s)}\right](t+s)\sin k(t-s)dsdt,$$

which is the same as T_{3d}. Therefore, T_1 and T_2 can also be controlled by $\epsilon B_H + Ck^{\frac{7}{5}}G$.

These estimates on T_1, T_2, and T_3 and the chosen M provides the final estimate for I_4,

$$\begin{aligned}
|I_4| &= \int_{|x|\leq M}\left[x|f_1(x)|^2 + x|f_2(x)|^2\right] + |T_1| + |T_2| + |T_3| \\
&\leq k^{\frac{1}{5}} k\left(\frac{\epsilon k^{-\frac{1}{5}}}{k} B_H + \frac{1}{\epsilon k^{-\frac{1}{5}}} G\right) + \epsilon B_H + Ck^{\frac{7}{5}} G \\
&\leq \epsilon_4 B_H + d_4 k^{\frac{7}{5}} G.
\end{aligned}$$

$\square$

Finally, we deal with I_5, which follows from the integration by parts and previous estimate on $\|u\|_{L^2(\Gamma)}$.

Lemma 2.4.6. *For any positive constant ε_5, we have*

$$I_5 = 2\Re \int_\Gamma x\overline{u}_x g dx \le \varepsilon_5 B_H + d_5 G + e_5 k^{-1} \|gx\|^2_{L^2(\Gamma)}, \tag{2.39}$$

where $d_5 = \dfrac{\varepsilon_5 C^2}{2} + \dfrac{2M_1}{\varepsilon_5 k}$, *and* $e_5 = \dfrac{2M_1}{\varepsilon_5}$.

Proof. Using Lemma 2.4.1, we have, for any $\varepsilon_5 > 0$,

$$\begin{aligned}
I_5 &= 2\Re \int_\Gamma x\overline{u}_x g dx = -2\Re \int_\Gamma \overline{u}(xg)_x dx \\
&\le 2\left(\int_\Gamma |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Gamma |(xg)_x|^2 dx\right)^{\frac{1}{2}} \le \frac{\varepsilon_5}{2} k \int_\Gamma |u|^2 dx + \frac{2}{\varepsilon_5} k^{-1} \int_\Gamma |(xg)_x|^2 dx \\
&\le \frac{\varepsilon_5}{2} k\left(\frac{2}{k} B_H + \frac{C^2}{k} G\right) + \frac{4}{\varepsilon_5} k^{-1} \int_\Gamma |g|^2 dx + \frac{4}{\varepsilon_5} k^{-1} \int_\Gamma |xg_x|^2 dx \\
&\le \varepsilon_5 B_H + \frac{C^2 \varepsilon_5}{2} G + \frac{4}{\varepsilon_5} k^{-1} \|g\|^2_{L^2(\Gamma)} + \frac{4M_1}{\varepsilon_5} k^{-1} \|gx\|^2_{L^2(\Gamma)} \\
&\le \varepsilon_5 B_H + d_5 G + e_5 k^{-1} \|gx\|^2_{L^2(\Gamma)}.
\end{aligned}$$

The lemma is proved. $\square$

By using the estimates obtained in Lemma 2.4.2 through 2.4.6, we can prove Theorem 2.1.1 as described below.

Proof of Theorem 2.1.1: Equation (2.23) can be estimated by

$$\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi - \int_S (X\cdot\mathbf{n})|\nabla u|^2$$

$$\leq \varepsilon B_H + m_1 G + m_2\|gx\|^2_{L^2(\Gamma)} + \varepsilon_{31}\int_D \left(|\nabla u|^2 + k^2|u|^2\right) + \varepsilon_{32}\int_S |\nabla u|^2, \qquad (2.40)$$

here $\varepsilon = \varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5$, $m_1 = d_1+d_2k+d_3k+d_4k^{\frac{7}{5}}+d_5+y_2$ and $m_2 = (e_3+e_5)k^{-1}$. By using the geometric assumption $X\cdot\mathbf{n} \leq -p_1 < 0$, we could choose ε_{32} such that $\varepsilon_{32} < p_1$, then by dropping the term associated with $\int_S |\nabla u|^2$, Equation (2.40) becomes:

$$(1-\varepsilon_{31})(\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)}) + \int_{|\xi|>k}(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{u}|^2 d\xi$$

$$\leq \varepsilon B_H + m_1 G + m_2\|gx\|^2_{L^2(\Gamma)}. \qquad (2.41)$$

Note here ε, ε_{31} can be chosen sufficiently small, choose C such that $m_1 \leq Ck^{\frac{7}{5}}$ and $m_2 \leq Ck^{-1}$, then Equation (2.2) is obtained.

2.5 Further Remarks

For the case where the cavity domain is of rectangular shape with $D = [0,\pi]\times[0,\pi]$, consider the solution $u = sin(mx)sin(\sqrt{k^2-m^2}y), m \in \mathbb{Z}$ and $m < k$. If in particular, $k^2 - m^2 = j^2, j \in \mathbb{Z}$, then $u = 0$ on the aperture of the cavity Γ where $y = \pi$, therefore $T(u) = 0$. In this case, we have an explicit formula for g, that is, $g = u_y = \sqrt{k^2-m^2}sin(mx)$. It follows that $\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} = \frac{\pi^2}{4}(m^2+(k^2-m^2)+k^2) = \frac{\pi^2}{2}k^2$, meanwhile $\|g\|^2_{L^2(\Gamma)} = \frac{\pi}{2}(k^2-m^2)$. It shows that when $j = 1$, i.e., $k^2 - m^2 = 1$, we would need a coefficient of k^2 such that $\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} \leq Ck^2\|g\|^2_{L^2(\Gamma)}$. This particular example shows that at least we would need k^2 in the estimate. Our estimate yields the order of $k^{\frac{7}{5}}$, which may not be the optimal because of techniques we used. It would be interesting to see whether the optimal order dependency on k is k for any g under the given geometric assumption.

Chapter 3

Stability Estimates under a Weak Geometric Assumption

3.1 Main Theorem and Outlines of the Approach

In this chapter, the cavity structure satisfies the following geometric assumptions:

(1) the angel θ between Γ and D satisfies $\theta \geq \theta_0 > 0$;

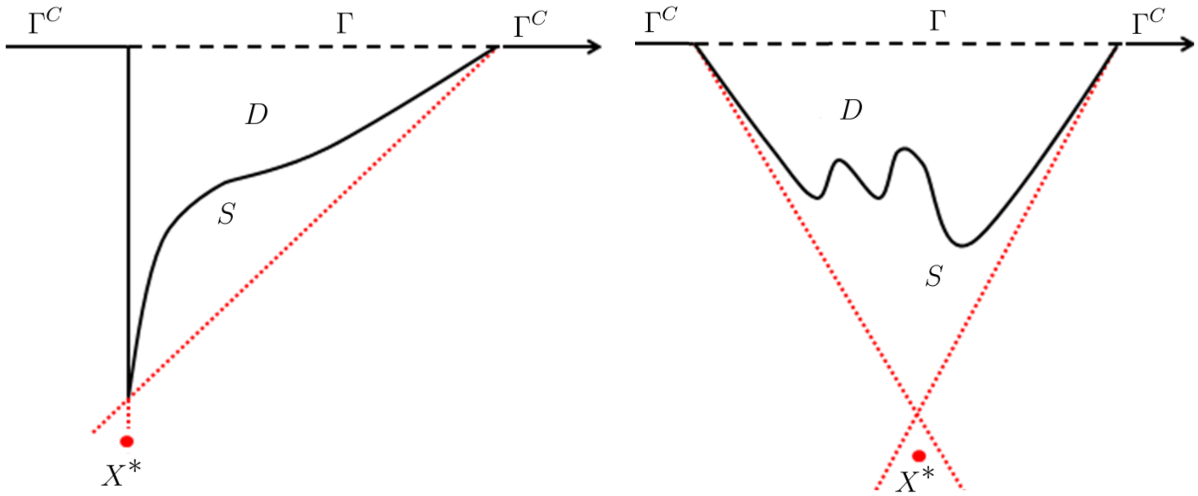
(2) there is a point $X^* = (x_0, y_0)$ on the plane and positive constants p such that

$$(X - X^*) \cdot \mathbf{n} \leq 0 \qquad \text{on} \;\; S. \tag{3.1}$$

(3) the domain D admits cusps of power sharpness $1 < \tau < 2$. One example is shown Figure 3.1. For domain with cusps satisfying the geometric assumption (3), the related trace theorem is established in [1], which states that for $u \in H^1(D)$, $\|u\|_{L^2(\partial D)} \leq C\|u\|_{H^1(D)}$, where the inclusion of $H^1(D) \subset L^r(D)$ for $2 \leq r \leq \frac{2(\alpha+1)}{\alpha-1}$ for domain with cusps [2] is used for the proof. The main result is Theorem 3.1.1 as stated below.

Theorem 3.1.1. *Under the geometric assumptions given in Equation (3.1), there exists a constant C such that for $k > k_0$,*

$$\|\nabla u\|_{L^2(D)} + |k|\|u\|_{L^2(D)} \leq C \left\{ k^{\frac{7}{10}} \|g\|_{L^2(\Gamma)} + \frac{1}{\sqrt{k}} \|g x\|_{L^2(\Gamma)} + k^{\frac{6}{5}} \left[|g(0)| + |g(\pi)| \right] \right\}.$$



(a) : $(X - X^*) \cdot \mathbf{n} \leq 0$ (b): $(X - X^*) \cdot \mathbf{n} \leq -p_1$

Figure 3.1: Domains of different shapes

See Figure (3.1)(b) for example, note that in Figure (3.1)(a), there is no such X^* which satisfies the geometric condition in Equation (2.1). It should be pointed out that even though Theorem 3.1.1 implies Theorem 2.1.1 in the case of $g(0) = g(\pi) = 0$. In general, $k^{\frac{7}{10}}\left[|g(0)| + |g(\pi)|\right]$ can not be controlled by $k^{\frac{6}{5}}\|g\|_{L^2(\Gamma)} + \frac{1}{\sqrt{k}}\|g_x\|_{L^2(\Gamma)}$. This can be seen when $g(x)$ is smooth and is supported in a small subinterval of $[0, \pi]$.

Remark 1: Note that in Chapter 2, the stronger geometric assumption is fully utilized in the final estimate. Specifically, the following energy inequality was obtained:

$$
\begin{aligned}
&\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} + \int_{|\xi|>k}\left(\sqrt{\xi^2 - k^2} + \frac{k^2}{\sqrt{\xi^2 - k^2}}\right)|\widehat{u}|^2 d\xi - \int_S (X \cdot \mathbf{n})|\nabla u|^2 \\
\leq &\Re \int_\Gamma g\overline{u}dx + y_2 \int_{|\xi|\leq k} 2(k^2 - \xi^2)|\widehat{u}|^2 d\xi + 2y_2 \Re \int_\Gamma T(u)\,\overline{g} \\
&+ 2\Re \int_{|\xi|\leq k}\left[-\xi\overline{\widehat{u}'}(\xi)\right] i\sqrt{k^2 - \xi^2}\,\widehat{u}(\xi)d\xi + 2\Re \int_\Gamma x\overline{u}_x g dx + y_2 \int_\Gamma |g|^2 dx.
\end{aligned} \tag{3.2}
$$

The integral $-\int_S (X \cdot \mathbf{n})|\nabla u|^2$ on the left hand side played an important role to control the term $2y_2 \Re \int_\Gamma T(u)\overline{g}$ is on the right hand side. Recall that in Lemma 2.4.4 , I is shown to be estimated as follows,

$$I \leq \varepsilon_{31} \int_D \left(|\nabla u|^2 + k^2 |u|^2 \right) + \varepsilon_{32} \int_S |\frac{\partial u}{\partial \mathbf{n}}|^2 + d_3 k^2 G + \frac{e_3}{k} \int_\Gamma |g_x|^2.$$

Under the strong geometric assumption, $-X \cdot \mathbf{n} \geq p_1 > 0$ on S, the term $\varepsilon_{32} \int_S |\frac{\partial u}{\partial \mathbf{n}}|^2$ can be absorbed as long as ε_{32} is chosen small enough. Now we have a weaker geometric assumption on S, thus the previous proof can not be extended directly.

Remark 2: Another natural way to estimate I is to rewrite the integral using Fourier transform. That is,

$$|I| \leq C \left| \int_\Gamma T(u)\, \overline{g} \right| = C \left| \int_{\mathbb{R}} T(u)\, \overline{\widetilde{g}} \right|$$

by extending g to be zero outside Γ. Hence

$$\left| \int_{\mathbb{R}} T(u)\, \overline{\widetilde{g}} \right| = \left| \int_{\mathbb{R}} \widehat{T(u)}\, \overline{\widehat{\widetilde{g}}} d\xi \right| = \left| \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2}\, \widehat{u}\, \overline{\widehat{\widetilde{g}}} \right|.$$

If Schwarz inequality is used, $|I| \leq C \left(\int_{\mathbb{R}} \sqrt{|k^2 - \xi^2}|\widehat{u}|^2 \right)^{\frac{1}{2}} \left(\sqrt{|k^2 - \xi^2|}|\overline{\widehat{\widetilde{g}}}| \right)^{\frac{1}{2}}$. Recall that we only have control on $\int_{\mathbb{R}} \sqrt{|k^2 - \xi^2}|\widehat{u}|^2$, we need the convergence of $\int_{\mathbb{R}} \sqrt{|k^2 - \xi^2|}|\overline{\widehat{\widetilde{g}}}|$. However this is not true if $|g(0)^2| + |g(\pi)^2| \neq 0$, since $\widehat{\widetilde{g}} \sim \frac{1}{|\xi|}$ as $|\xi| \to \infty$. If $g(0) = g(\pi) = 0$, we will have convergence of $\int_{\mathbb{R}} \sqrt{|k^2 - \xi^2|}|\overline{\widehat{\widetilde{g}}}|$ when $g_x(x) \in L^2(\Gamma)$. The new idea is to introduce a new auxiliary function u^* , and consider $w = u + u^*$, we have

$$\begin{cases} \Delta w + k^2 w = \Delta u^* + k^2 u^* \equiv f \quad \text{in } D, \\ w|_S = 0 \quad \text{if} \quad u^*|_S = 0, \\ w_y|_\Gamma = u_y + u^*_y = T(u) + g + u^*_y = T(w) - T(u^*) + g + u^*_y = T(w) + g_1 \text{ on } \Gamma, \end{cases} \tag{3.3}$$

where $g_1 = g - Tu^* + u^*_y$. Using this construction, we want $g_1(0) = g_1(\pi) = 0$.

To prove Theorem 3.1.1 under the weaker geometric assumption Equation (3.1), three main steps are involved.

Step 1. Introduce an auxiliary function u^* such that

1) $Tu^*(0, y_2) = g(0)$ and $Tu^*(\pi, y_2) = g(\pi)$,

2) $u^*(0, y_2) = u^*(\pi, y_2) = 0$, $u^*_y(0, y_2) = u^*_y(\pi, y_2) = 0$,

3) $u^* = 0$ on S. The trick is to construct such u^* so that the norms of f and u^* are as small as possible. But the construction is not trivial since Tu^* is not a local operator. Once u^* is chosen, then $w = u + u^*$ satisfies Equation (3.3).

Step 2. Note that this formulation has similar structure as that in Equation (1.2) with two major differences: 1) the governing equation here is nonhomogeneous with a right hand side source term f; 2) in the nonlocal boundary condition, g_1 vanishes on the boundary of Γ while g itself may not. It will be shown that

$$\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} \leq C_1 k^{\frac{7}{5}} \|g_1\|^2_{L^2(\Gamma)} + \frac{C_2}{k}\|g_{1x}\|^2_{L^2(\Gamma)} + C_3 k^{\frac{2}{5}} \int_D |f|^2. \quad (3.4)$$

The proof for the above inequality is presented in the last section since this part can be viewed as a modification of results in Chapter 2, where most of the ideas are similar; however, additional terms containing f are involved from the beginning of the proof, so each result need to be restated. Furthermore, since here g_1 vanishes on the boundary of Γ, an easier estimate could be used to treat the term $\int_\Gamma T(w)g_1$.

Step 3. To provide a final estimate for the wave energy on u, it remains to show that $\|g_1\|_{L^2(\Gamma)}$, $\|g_{1x}\|_{L^2(\Gamma)}$ and $\|f\|_{L^2(D)}$ terms are controlled by norms of g with suitable powers of k, which turned out to be

$$\begin{aligned}\int_D |f|^2 &\le C\left[|g(0)|^2 + |g(\pi)|^2\right],\\ \int_\Gamma |g_1|^2 &\le C\left[|g(0)|^2 + |g(\pi)|^2\right] + \int_\Gamma |g|^2,\\ \int_\Gamma |g_{1x}|^2 &\le Ck^2\left[|g(0)|^2 + |g(\pi)|^2\right] + \int_\Gamma |g_x|^2.\end{aligned}$$

Furthermore, it could be shown that

$$\|\nabla u^*\|^2_{L^2(D)} + k^2\|u^*\|^2_{L^2(D)} \le \frac{C}{k^2}\left[|g(0)|^2 + |g(\pi)|^2\right],$$

hence the final estimate could be established accordingly.

3.2 Some Basic Properties of the Auxiliary Function

In this section, we focus on the construction of the auxiliary function u^*, which essentially is a linear combination of two compact supported functions. The aim is to construct a function u^* such that it satisfies 1) $Tu^*(0, y_2) = g(0)$ and $Tu^*(\pi, y_2) = g(\pi)$, 2) $u^*(0, y_2) = u^*(\pi, y_2) = 0$, $u^*_y(0, y_2) = u^*_y(\pi, y_2) = 0$, and 3) $u^* = 0$ on S. In this section, we introduce two subdomains where u^* is supported, and analyze some basic properties of u^* .

3.2.1 The Construction of an Auxiliary Function

Define Ω_1 and Ω_2 as follows.

$$\Omega_1 = \{(x, y) : x \in [0, \frac{\pi}{j}], y \in [y_2 - \alpha \sin(jx), y_2]\},$$

and

$$\Omega_2 = \{(x, y) : x \in [\pi - \frac{\pi}{j}, \pi], y \in [y_2 - \alpha \sin(jx), y_2]\},$$

where α is a small constant that will be chosen later to optimize the order in k, which turns out to be $\alpha = \frac{\beta}{k}$, this is explained in Section 4.4, $j = \mu k$, μ is a small constant as indicated in the discussion in Section 3.2, and μ is chosen such that j is an integer. See Figure 3.2 for an illustration of Ω_1 and Ω_2. Once these two subdomains are defined, two

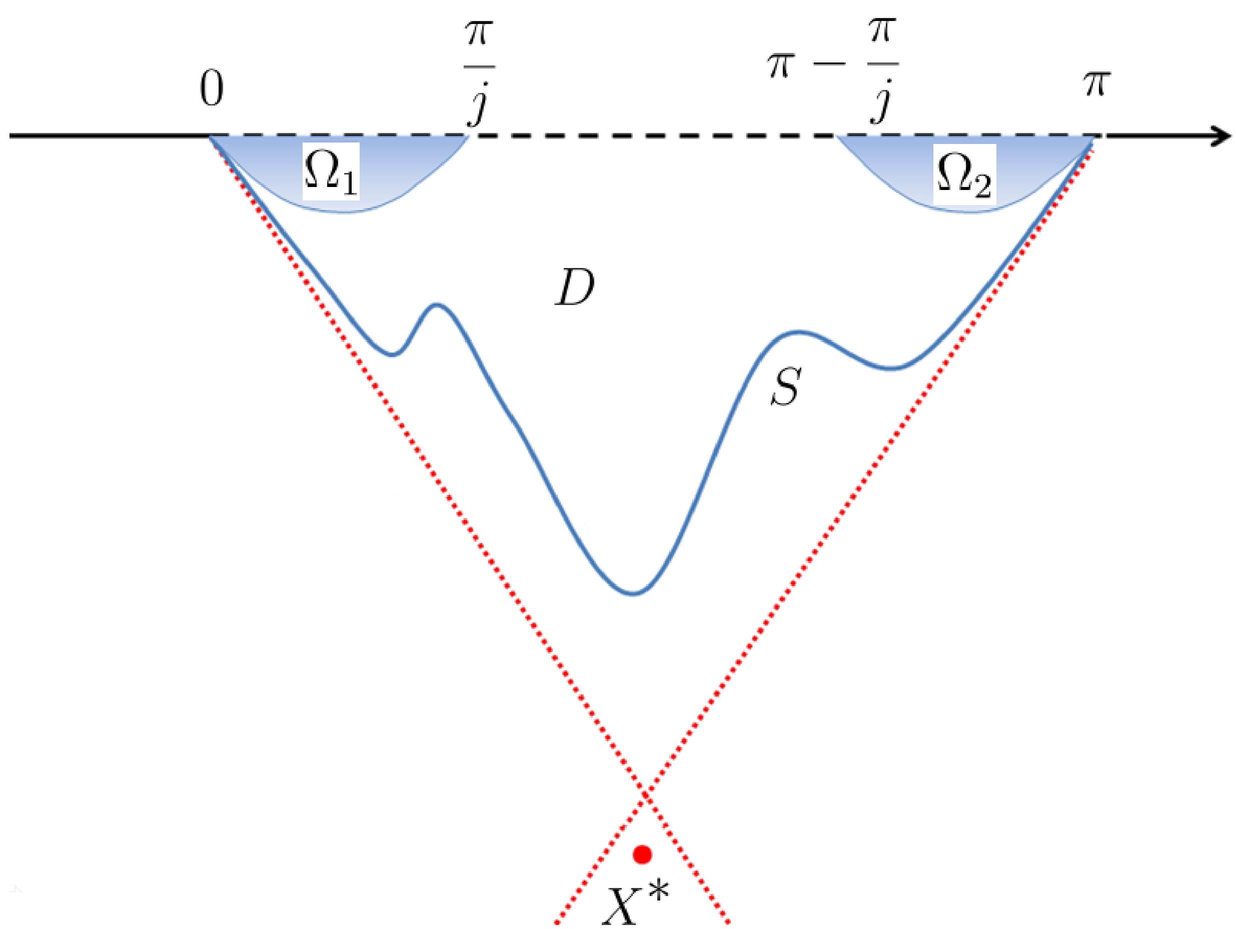


Figure 3.2: Cavity with subdomains where the auxiliary functions are supported

corresponding functions u_1^* and u_2^* are introduced accordingly, which are supported in Ω_1 and Ω_2 respectively. Thereafter u^* is defined as a linear combination of the two functions in a way that Tu^* cancels out the value of g on the boundary of Γ. Specifically, we write

$$u^* = a_1 u_1^* + a_2 u_2^*,$$

where

$$u_1^*(x,y) = \begin{cases} [y - y_2 + \alpha \sin(jx)]^3 & (x,y) \in \Omega_1, \\ 0, & \mathit{otherwise}. \end{cases}$$

and

$$u_2^*(x,y) = \begin{cases} [y - y_2 + \alpha \sin(jx)]^3 & (x,y) \in \Omega_2, \\ 0, & \mathit{otherwise}, \end{cases}$$

Furthermore, a_1, a_2 are constants chosen such that

$$\begin{cases} a_1 T u_1^*(0, y_2) + a_2 T u_2^*(0, y_2) = g(0) \\ a_1 T u_1^*(\pi, y_2) + a_2 T u_2^*(\pi, y_2) = g(\pi) \end{cases} \tag{3.5}$$

Note that in particular, $u_1^* = \alpha^3 \sin^3(jx)$ for $x \in [0, \frac{\pi}{j}]$ and $u_2^* = \alpha^3 \sin^3(jx)$ for $x \in [\pi - \frac{\pi}{j}, \pi]$ on Γ, this fact is used in the next subsection to analyze properties on a_1 and a_2.

3.2.2 The Existence and the Order of the Coefficients

To guarantee the existence of a_1 and a_2 in the auxiliary function, we show that the coefficient matrix

$$M = \begin{bmatrix} Tu_1^*(0, y_2) & Tu_2^*(0, y_2) \\ Tu_1^*(\pi, y_2) & Tu_2^*(\pi, y_2) \end{bmatrix} \tag{3.6}$$

is nonsingular. This could be proved by checking the leading order terms (in terms of order in k) for each component in the coefficient matrix in Equation (3.6). Notice that $T\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2}\widehat{\phi}(\xi)e^{i\xi x} d\xi$, thus the Fourier transform of u_1^* and u_2^* are needed for the further analysis on Tu_1^* and Tu_2^* . First, we derive the formula for $\widehat{u_1^*}$ and $\widehat{u_2^*}$.

Lemma 3.2.1. *The fourier transform of u_1^* and u_2^* can be represented by*

$$\begin{cases} \widehat{u_1^*}(\xi, y_2) = \dfrac{1}{\sqrt{2\pi}} \dfrac{6\alpha^3 j^3}{\xi^2 - 9j^2} \dfrac{e^{-i\xi\frac{\pi}{j}} + 1}{\xi^2 - j^2} \\ \widehat{u_2^*}(\xi, y_2) = \dfrac{1}{\sqrt{2\pi}} \dfrac{6\alpha^3 j^3}{\xi^2 - 9j^2} \dfrac{(-1)^{j+1} \left[e^{-i\xi\pi} + e^{-i\xi(\pi - \frac{\pi}{j})} \right]}{\xi^2 - j^2} \end{cases} \tag{3.7}$$

Proof. Since on Γ, $y = y_2$, therefore $u_1^*(x, y) = \alpha^3 \sin^3(jx)$ for $(x, y) \in [0, \frac{\pi}{j}] \times \{y = y_2\}$. We may assume $\alpha^3 = \sqrt{2\pi}$ from linearity of Fourier transform. $\widehat{u_1^*}(\xi, y_2)$ can be expressed as follows.

$$\begin{aligned}
\widehat{u_1^*}(\xi, y_2) &= \int_0^{\frac{\pi}{j}} \sin^3(jx)e^{-i\xi x}dx = \frac{1}{i\xi}\int_0^{\frac{\pi}{j}} 3\sin^2(jx)\cos(jx)je^{-i\xi x}dx \\
&= \frac{1}{(i\xi)^2}\int_0^{\frac{\pi}{j}} [6\sin(jx)\cos^2(jx)j^2 - 3\sin^3(jx)j^2]e^{-i\xi x}dx \\
&= \frac{1}{-\xi^2}\int_0^{\frac{\pi}{j}} [6\sin(jx)j^2 - 9\sin^3(jx)j^2]e^{-i\xi x}dx \\
&= \frac{1}{-\xi^2}[\int_0^{\frac{\pi}{j}} 6\sin(jx)j^2e^{-i\xi x}dx - 9j^2\widehat{u_1^*}(\xi, y_2)]
\end{aligned}$$

It shows that $(-\xi^2 + 9j^2)\widehat{u_1^*}(\xi, y_2) = j^2\int_0^{\frac{\pi}{j}} 6\sin(jx)e^{-i\xi x}dx$. Denote $C_j = \frac{3j^2}{-\xi^2+9j^2}$, then $\widehat{u_1^*}(\xi, y_2)$ can be written as

$$\begin{aligned}
&\widehat{u_1^*}(\xi, y_2) = C_j\int_0^{\frac{\pi}{j}} \frac{e^{ijx} - e^{-ijx}}{i}e^{-i\xi x}dx = C_j\frac{1}{i}\left[\frac{e^{i(j-\xi)x}}{(j-\xi)i} + \frac{e^{-i(j+\xi)x}}{(j+\xi)i}\right]\Bigg|_0^{\frac{\pi}{j}} \\
=&C_j\frac{1}{i}\frac{(e^{i(j-\xi)\frac{\pi}{j}} - 1)(j+\xi) + (e^{-i(j+\xi)\frac{\pi}{j}} - 1)(j-\xi)}{(j^2-\xi^2)i} = 2C_j\frac{e^{-i\xi\frac{\pi}{j}} + 1}{\xi^2 - j^2}.
\end{aligned}$$

$\widehat{u_2^*}(\xi, y_2)$ can be obtained by a suitable substitution.

$$\begin{aligned}
\widehat{u_2^*}(\xi, y_2) &= \int_{\pi-\frac{\pi}{j}}^{\pi} \sin^3(jx)e^{-i\xi x}dx = \int_0^{\frac{\pi}{j}} \sin^3[j(\pi - x_1)]e^{-i\xi(\pi - x_1)}dx_1 \\
&= \int_0^{\frac{\pi}{j}} (-1)^{j+1}\sin^3(jx_1)e^{i\xi x_1}dx_1e^{-i\xi\pi} = (-1)^{j+1}e^{-i\xi\pi}\widehat{u_1^*}(-\xi, y_2) \\
&= 2C_j\frac{(-1)^{j+1}\left[e^{-i\xi\pi} + e^{-i\xi(\pi - \frac{\pi}{j})}\right]}{\xi^2 - j^2}.
\end{aligned}$$

Lemma 3.2.1 is proved. $\square$

By using the explicit expressions for $\widehat{u_1^*}(\xi, y_2)$ and $\widehat{u_2^*}(\xi, y_2)$, the following facts can be obtained.

Corollary 3.2.2. *Using the connection between* $\widehat{u_1^*}(\xi, y_2)$ *and* $\widehat{u_2^*}(\xi, y_2)$ *as shown in Equation (3.8), we have the following relations.*

$$\begin{cases} Tu_2^*(0, y_2) = (-1)^{j+1} Tu_1^*(\pi, y_2), \\ Tu_2^*(\pi, y_2) = (-1)^{j+1} Tu_1^*(0, y_2). \end{cases} \tag{3.8}$$

Proof. Since the following relate holds for $\widehat{u_2^*}(\xi, y_2)$ and $\widehat{u_1^*}(-\xi, y_2)$,

$$\widehat{u_2^*}(\xi, y_2) = (-1)^{j+1} e^{-i\xi\pi} \widehat{u_1^*}(-\xi, y_2), \tag{3.9}$$

it follows that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2} \widehat{u_2^*}(\xi, y_2) d\xi &= (-1)^{j+1} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2} e^{-i\xi\pi} \widehat{u_1^*}(-\xi, y_2) d\xi \\ &= (-1)^{j+1} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2} e^{i\xi\pi} \widehat{u_1^*}(\xi, y_2) d\xi. \end{aligned}$$

Thus the first identity in Equation (3.8) is proved, the second identity could be easily derived after multiplying both sides of Equation (3.9) by $e^{i\xi\pi}$. □

From this corollary, in order to show that the coefficient matrix M in Equation (3.6) is nonsingular, note its determinant

$$det(M) = (-1)^{j+1} \left\{ \left[Tu_1^*(0, y_2)\right]^2 - \left[Tu_1^*(\pi, y_2)\right]^2 \right\},$$

thus the leading order for $Tu_1^*(0, y_2)$ and $Tu_1^*(\pi, y_2)$ in terms of power of k needs to be determined.

Lemma 3.2.3. *$Tu_1^*(0, y_2)$ and $Tu_1^*(\pi, y_2)$ can be expressed by the integrals containing only the higher frequency part.*

$$\begin{aligned} Tu_1^*(0, y_2) &= -\frac{1}{\pi}\int_{\xi>k}\sqrt{\xi^2-k^2}\frac{6\alpha^3 j^3}{\xi^2-9j^2}\frac{e^{i\xi\frac{\pi}{j}}+1}{\xi^2-j^2}d\xi, \\ Tu_1^*(\pi, y_2) &= -\frac{1}{\pi}\int_{\xi>k}\sqrt{\xi^2-k^2}\frac{6\alpha^3 j^3}{\xi^2-9j^2}\frac{e^{i\xi(\pi-\frac{\pi}{j})}+e^{i\xi\pi}}{\xi^2-j^2}d\xi. \end{aligned} \tag{3.10}$$

Proof. We first rewrite $Tu_1^*(0, y_2)$ using residue theorem. It should be pointed out that $i\sqrt{k^2-\xi^2}$ is not a restriction of an analytic function to the real line. However, we can express this as

$$i\sqrt{k^2-\xi^2} := \begin{cases} A(\xi) & \text{for } \xi < k, \\ A(\xi) - 2\sqrt{\xi^2-k^2} & \text{for } \xi > k, \end{cases}$$

where $A(z) = (z-k)^{\frac{1}{2}}(z+k)^{\frac{1}{2}}, \Im z \geq 0, z^{\frac{1}{2}} = |z|^{\frac{1}{2}}e^{\frac{i\arg z}{2}}$. $A(z), 0 < arg z < \pi$, is an analytic function. Then

$$\begin{aligned} Tu_1^*(0, y_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} i\sqrt{k^2-\xi^2}\widehat{u_1^*}(\xi, y_2)d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} i\sqrt{k^2-\xi^2}\widehat{u_1^*}(-\xi, y_2)d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A(\xi)\widehat{u_1^*}(-\xi, y_2)d\xi - \frac{2}{\sqrt{2\pi}}\int_{\xi>k}\sqrt{\xi^2-k^2}\widehat{u_1^*}(-\xi, y_2)d\xi \\ &= -\sqrt{\frac{2}{\pi}}\int_{\xi>k}\sqrt{\xi^2-k^2}\widehat{u_1^*}(-\xi, y_2)d\xi, \end{aligned}$$

since $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A(\xi)\widehat{u_1^*}(-\xi, y_2)d\xi = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{\partial B_R^+} A(\xi)\widehat{u_1^*}(-\xi, y_2)d\xi = 0$ by residue theorem. Similarly,

$$
\begin{aligned}
Tu_1^*(\pi, y_2) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\sqrt{k^2-\xi^2}\widehat{u_1^*}(\xi, y_2)e^{i\xi\pi}d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} A(\xi)\widehat{u_1^*}(\xi, y_2)e^{i\xi\pi}d\xi - \frac{2}{\sqrt{2\pi}} \int_{\xi>k} \sqrt{\xi^2-k^2}\widehat{u_1^*}(\xi, y_2)e^{i\xi\pi}d\xi \\
&= -\sqrt{\frac{2}{\pi}} \int_{\xi>k} \sqrt{\xi^2-k^2}\widehat{u_1^*}(\xi, y_2)e^{i\xi\pi}d\xi.
\end{aligned}
$$

Plug the expression of $\widehat{u_1^*}(\xi, y_2)$ into the two identities above, the two equations in (3.10) are obtained. $\square$

Using these expressions, we can compute the leading orders of $Tu_1^*(0, y_2)$ and $Tu_1^*(\pi, y_2)$ and therefore derive the bound for a_1 and a_2 in terms of k.

Lemma 3.2.4. *For large wave number k, the matrix M is nonsingular. Furthermore,*

$$
|a_1| \le \frac{C}{\alpha^3 k}(|g(0)| + |g(\pi)|), \quad |a_2| \le \frac{C}{\alpha^3 k}(|g(0)| + |g(\pi)|). \tag{3.11}
$$

Proof. To estimate the leading order of $Tu_1^*(0, y_2)$, rewrite the integral as the sum of two integrals, i.e.,

$$
Tu_1^*(0, y_2) = -\frac{6}{\pi}\alpha^3 j^3 \int_{\xi>k} \frac{\sqrt{\xi^2-k^2}}{(\xi^2-9j^2)(\xi^2-j^2)}(e^{i\xi\frac{\pi}{j}}+1)d\xi = -\frac{6}{\pi}\alpha^3 j^3 (I_1 + I_2),
$$

where

$$
I_1 = \int_{\xi>k} \frac{\sqrt{\xi^2-k^2}}{(\xi^2-9j^2)(\xi^2-j^2)} e^{i\xi\frac{\pi}{j}}\, d\xi \quad \text{and} \quad I_2 = \int_{\xi>k} \frac{\sqrt{\xi^2-k^2}}{(\xi^2-9j^2)(\xi^2-j^2)} d\xi,
$$

Assume that j is an integer such that $j = \mu k$ and $j < \frac{k}{3}$ in the following context. Consider

I_1 first, denote $f(\xi) = \frac{\sqrt{\xi^2-k^2}}{(\xi^2-9j^2)(\xi^2-j^2)}$, then

$$
\begin{aligned}
f'(\xi) &= \frac{\frac{\xi}{\sqrt{\xi^2-k^2}}(\xi^2-9j^2)(\xi^2-j^2) - \sqrt{\xi^2-k^2}\left[2\xi(\xi^2-9j^2+\xi^2-j^2)\right]}{(\xi^2-9j^2)^2(\xi^2-j^2)^2} \\
&= \frac{\xi\left[-3\xi^4+(10j^2+4k^2)\xi^2+j^2(9j^2-20k^2)\right]}{\sqrt{\xi^2-k^2}(\xi^2-9j^2)^2(\xi^2-j^2)^2}.
\end{aligned}
$$

Consider the term $q(x) = -3x^2+(10j^2+4k^2)x+j^2(9j^2-20k^2), x = \xi^2$ on the numerator. Use the fact that $j < k$, then $q(0) = j^2(9j^2-20k^2) < 0, q(k^2) = (k^2-j^2)(k^2-9j^2) > 0, q(4k^2) = -32k^4+20k^2j^2+9j^4 < 0$. It shows that the two zeros x_1, x_2 of $q(x)$ lie in the following intervals: $x_1 \in (0, k^2)$ and $x_2 \in (k^2, 4k^2)$. Consequently, there is a $\xi_0 \in (k, 2k), \xi_0 = \sqrt{x_2}$, such that $f'(\xi) > 0$ for $\xi \in (k, \xi_0)$ and $f'(\xi) < 0$ for $\xi \in (\xi_0, \infty)$. This indicates that $f(\xi)$ is a monotonic function in both the intervals (k, ξ_0) (increasing) and $((\xi_0, \infty)$ (decreasing), where $\xi_0 = \mu_0 k, \mu_0 \in (1, 2)$. Furthermore, note that $f(k) = 0$, $f(\xi_0) = \frac{\sqrt{\mu_0^2-1}}{(\mu_0^2-9\mu^2)(\mu_0^2-\mu^2)k^3}$, and $f(\xi) \to 0$ as $\xi \to \infty$, therefore, by the second mean value theorem, there exists $M_1 \in (k, \xi_0)$ and $M_2 \in (\xi_0, \infty)$, such that

$$
\begin{aligned}
|I_1| &= \left| f(\xi_0)\int_{M_1}^{\mu_0 k} e^{i\xi\frac{\pi}{j}}\,d\xi + f(\xi_0)\int_{\mu_0 k}^{M_2} e^{i\xi\frac{\pi}{j}}\,d\xi \right| \\
&\le \frac{\sqrt{\mu_0^2-1}}{(\mu_0^2-9\mu^2)(\mu_0^2-\mu^2)k^3}\frac{2j}{\pi} \le \frac{2\mu k}{\pi}\frac{\sqrt{\mu_0^2-1}}{(1-9\mu^2)(1-\mu^2)k^3} \\
&\le \frac{4\mu}{\pi(1-9\mu^2)(1-\mu^2)k^2}.
\end{aligned}
$$

Denote $\mu_1 = \frac{4\mu}{\pi(1-9\mu^2)(1-\mu^2)}$, then $|I_1| \le \frac{\mu_1}{k^2}$ and μ_1 is small for small μ. For I_2, rewrite it as $I_2 = I_{21} + I_{22}$, where

$$I_{21} = \int_{\xi>k} \frac{\sqrt{\xi^2 - 9j^2}}{(\xi^2 - 9j^2)(\xi^2 - j^2)} d\xi,$$

$$I_{22} = \int_{\xi>k} \frac{-k^2 + 9j^2}{(\sqrt{\xi^2 - k^2} + \sqrt{\xi^2 - 9j^2})} \frac{1}{(\xi^2 - 9j^2)(\xi^2 - j^2)} d\xi.$$

We have $|I_{21}| \geq \int_{\xi>k} \frac{1}{(\xi^2)^{\frac{1}{2}}\xi^2} = \int_{\xi>k} \frac{1}{\xi^3} = \frac{1}{2k^2}$. Furthermore, since $(\sqrt{\xi^2 - 9j^2})^2 \geq (\xi - 3j)^2$ for $\xi > k > 3j$, $\sqrt{\xi^2 - 9j^2} \geq \xi - 3j$, consequently,

$$|I_{22}| \leq (k^2 - 9j^2) \int_{\xi>k} \frac{1}{(\xi - 3j)^5} d\xi = \frac{1 - 9\mu^2}{4(1 - 3c)^4} \frac{1}{k^2}.$$

Denote $\mu_2 = \frac{1-9\mu^2}{4(1-3c)^4}$, then $|I_{22}| \leq \frac{\mu_2}{k^2}$ and $\mu_2$is close to $\frac{1}{4}$ when μ is small enough. From the estimates for $I_{11}, I_{12}, I_{21}, I_{22}$, it follows that

$$|\Re(I_1 + I_2)| \geq |I_{12}| - |I_{21}| - |I_{22}| \geq (\frac{1}{2} - 2\mu_1 - \mu_2)\frac{1}{k^2} \geq \frac{4\mu}{k^2},$$

and $|\Im(I_1)| \leq \frac{2\mu}{k^2}$, when μ is chosen small enough. It indicates that for small μ , the real part dominates the imaginary part in the leading term in terms of order in k for $Tu_1^*(0, y_2)$. On the other hand, note that

$$|I_1 + I_2| \leq \int_{\xi>k} \frac{2\sqrt{\xi^2 - k^2}}{(\xi^2 - 9j^2)(\xi^2 - j^2)} d\xi \leq \int_{\xi>k} \frac{2\sqrt{\xi^2}}{(\xi^2 - 9j^2)^2} d\xi = \frac{1}{\xi^2 - 9j^2} \leq \frac{C}{k^2}.$$

It shows that the real part of $|I_1 + I_2| \sim O(\frac{1}{k^2})$, which implies that

$$Tu_1^*(0, y_2) = -\frac{6}{\pi}\alpha^3 j^3 (I_1 + I_2) \sim O(\alpha^3 k),$$

where the real part dominates.

Next, we treat $Tu_1^*(\pi, y_2)$, where

$$Tu_1^*(\pi, y_2) = -\frac{6}{\pi}\alpha^3 j^3 \int_{\xi > k} \frac{\sqrt{\xi^2 - k^2}}{(\xi^2 - 9j^2)(\xi^2 - j^2)}(e^{i\xi(\pi - \frac{\pi}{j})} + e^{i\xi\pi})d\xi = -\frac{6}{\pi}\alpha^3 j^3 J,$$

Again by the second mean value theorem for integrals,

$$\begin{aligned}|J| &= \left| f(\xi_0)\int_{M_1}^{\mu_0 k}(e^{i\xi(\pi - \frac{\pi}{j})} + e^{i\xi\pi})d\xi + f(\xi_0)\int_{\mu_0 k}^{M_2}(e^{i\xi(\pi - \frac{\pi}{j})} + e^{i\xi\pi})d\xi \right| \\ &\leq \frac{\sqrt{\mu_0^2 - 1}}{(\mu_0^2 - 9\mu^2)(\mu_0^2 - \mu^2)k^3}(\frac{2}{\pi - \frac{\pi}{j}} + \frac{2}{\pi}) \leq \frac{8}{\pi(1 - 9\mu^2)(1 - \mu^2)k^3} \leq \frac{8}{k^3},\end{aligned}$$

when μ is small. This implies that $Tu_1^*(\pi, y_2) = -\frac{6}{\pi}\alpha^3 j^3 J \leq C\alpha^3$. The arguments above show that a_1, a_2 are the solutions to the following linear system

$$\begin{bmatrix} \alpha^3 k & o(\alpha^3 k) \\ (-1)^{j+1} o(\alpha^3 k) & (-1)^{j+1}\alpha^3 k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} g(0) \\ g(\pi) \end{bmatrix}. \tag{3.12}$$

Consequently, the matrix M is nonsingular and the result in the lemma as shown in Equation (3.11) holds. □

From the estimates of a_1 and a_2 in the lemma above, we can easily get the bound for $E(u^*)$. Moreover, the estimate of the energy terms involving w needs to be determined.

Remark: Note that w satisfies the nonhomogeneous equation in (3.3). To derive the dependency of the energy $E(w)$ on k, most of the ideas are similar to the previous work; however, additional terms containing f are involved from the beginning of the proof. Furthermore, since here g_1 vanishes on the boundary of Γ, a different estimate could be used for the estimate of the term $\int_\Gamma T(u)g_1$ as stated in Lemma, that's exactly the reason why we could use a weaker geometric constraint which is stated in Equation (3.1) compared to the previous work, which requires $(X - X^*)\cdot \mathbf{n} \le -p_1 < 0$ on S. So each result need to be restated and this part will be proved in the last section. It will be shown in the last section that

$$\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} \le C\left\{k^{\frac{7}{5}}\|g_1\|^2_{L^2(\Gamma)} + \frac{1}{k}\|g_{1x}\|^2_{L^2(\Gamma)} + k^{\frac{2}{5}}\int_D |f|^2\right\}. \quad (3.13)$$

3.3 Relations between New Source Terms and Original Source Term g

To obtain the main stability result for u associated with g instead of the new source contribution terms f, g_1, the dependence of f, g_1 and g_{1x} on g are needed, which are derived respectively in the following three lemmas.

Lemma 3.3.1. *For the right hand side source term f in Equation (3.3), we have*

$$\int_D |f|^2 \le C\max(\alpha k, \frac{1}{\alpha^3 k^3})\left[|g(0)|^2 + |g(\pi)|^2\right]. \quad (3.14)$$

If in particular choose $\alpha = \frac{\beta}{k}$, then $\int_D |f|^2 \le C\left[|g(0)|^2 + |g(\pi)|^2\right]$.

Proof. Note that f is defined as

$$f = \Delta u^* + k^2 u^* = a_1(u^*_{1xx} + u^*_{1yy} + k^2 u^*_1) + a_2(u^*_{2xx} + u^*_{2yy} + k^2 u^*_2).$$

From the definition of u^*_1, we can see that $u^*_1, Du^*_1, D^2 u^*_1$ vanish outside outside Ω_1. We will treat u^*_1 first, and u^*_2 can be handled in a similar way. In Ω_1,

$$\begin{cases} u^*_1 = [y - y_2 + \alpha \sin(jx)]^3, \\ u^*_{1x} = 3\alpha j[y - y_2 + \alpha \sin(jx)]^2 \cos(jx), \\ u^*_{1xx} = 6\alpha^2 j^2 [y - y_2 + \alpha \sin(jx)] \cos^2(jx) - 3\alpha j^2 [y - y_2 + \alpha \sin(jx)]^2 \sin(jx), \end{cases}$$

and

$$\begin{cases} u^*_{1y} = 3[y - y_2 + \alpha \sin(jx)]^2, \\ u^*_{1yy} = 6[y - y_2 + \alpha \sin(jx)]. \end{cases}$$

Note that in Ω_1, $f = a_1(u^*_{1xx} + u^*_{1yy} + k^2 u^*_1)$, thus it follows that

$$\int_{\Omega_1} |f|^2 = |a_1|^2 \int_0^{\frac{\pi}{j}} \int_{y_2 - \alpha \sin(jx)}^{y_2} \left| u^*_{1xx} + u^*_{1yy} + k^2 u^*_1 \right|^2 dydx. \tag{3.15}$$

Next, the expressions of u^*_{1xx},u^*_{1yy},and u^*_1 will be used to derive the estimate for $\int_{\Omega_1} |f|^2$.

For the first term in the expression of u^*_{1xx}, we have

$$
\begin{aligned}
&|a_1|^2 \int_0^{\frac{\pi}{j}} \int_{y_2-\alpha\sin(jx)}^{y_2} \left|6\alpha^2 j^2[y-y_2+\alpha\sin(jx)]\cos^2(jx)\right|^2 dydx \\
=&|a_1|^2 \int_0^{\frac{\pi}{j}} 12\alpha^4 j^4[\alpha\sin(jx)]^3\cos^2(jx)dx \\
\le&\left[\frac{C}{\alpha^3 k}\right]^2 (|g(0)|^2+|g(\pi)|^2)\frac{\pi}{j}\alpha^7 j^4 \le C\alpha k(|g(0)|^2+|g(\pi)|^2).
\end{aligned}
\tag{3.16}
$$

Similarly, for the second term in the expression of u^*_{1xx}, it follows that

$$
\begin{aligned}
&|a_1|^2 \int_0^{\frac{\pi}{j}} \int_{y_2-\alpha\sin(jx)}^{y_2} \left|3\alpha j^2[y-y_2+\alpha\sin(jx)]^2\sin(jx)\right|^2 dydx \\
\le& C\alpha k(|g(0)|^2+|g(\pi)|^2).
\end{aligned}
\tag{3.17}
$$

Moreover, for u^*_{1yy},

$$
\begin{aligned}
&|a_1|^2 \int_0^{\frac{\pi}{j}} \int_{y_2-\alpha\sin(jx)}^{y_2} \left|6[y-y_2+\alpha\sin(jx)]\right|^2 dydx = |a_1|^2 \int_0^{\frac{\pi}{j}} 12\,[\alpha\sin(jx)]^3\,dx \\
\le&\left[\frac{C}{\alpha^3 k}\right]^2 (|g(0)|^2+|g(\pi)|^2)\frac{\pi}{j}\alpha^3 \le \frac{C}{\alpha^3 k^3}(|g(0)|^2+|g(\pi)|^2).
\end{aligned}
\tag{3.18}
$$

And for the term $k^2 u^*_1$, the following estimate holds,

$$
\begin{aligned}
&|a_1|^2 \int_0^{\frac{\pi}{j}} \int_{y_2-\alpha\sin(jx)}^{y_2} \left|k^2[y-y_2+\alpha\sin(jx)]^3\right|^2 dydx = |a_1|^2 \int_0^{\frac{\pi}{j}} \frac{1}{7}\,[\alpha\sin(jx)]^7\,dx \\
\le&\left[\frac{C}{\alpha^3 k}\right]^2 (|g(0)|^2+|g(\pi)|^2)\frac{\pi}{j}k^4\alpha^7 \le C\alpha k(|g(0)|^2+|g(\pi)|^2).
\end{aligned}
\tag{3.19}
$$

□

Next, the integral involving g_1 is computed.

Lemma 3.3.2. *The relation between L_2 norm of g_1 and g can be expressed by*

$$\int_\Gamma |g_1|^2 \le C\left[|g(0)|^2 + |g(\pi)|^2\right] + \int_\Gamma |g|^2.$$

Proof. Recall that

$$g_1 = g - Tu^* + u_y^*,$$

where $Tu^* = a_1 Tu_1^* + a_2 Tu_2^*$. For $a_1 Tu_1^*(x, y_2)$, suitable contour integral and residue theorem are used like the computation of $Tu_1^*(0, y_2)$ and $Tu_1^*(\pi, y_2)$ in Section 2. The advantage of this is to convert the integral on $\mathbb{R}$ to $\xi > k$, which makes the estimate much easier. If$\frac{\pi}{j} \le x < \pi$,then

$$\begin{aligned}
|a_1 Tu_1^*(x, y_2)| &= \left| -\frac{1}{\pi} a_1 \int_{\xi > k} \sqrt{\xi^2 - k^2} \frac{6\alpha^3 j^3}{\xi^2 - 9j^2} \frac{e^{i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi \right| \\
&\le \frac{6\alpha^3 j^3 |a_1|}{\pi} \int_{\xi > k} \frac{\sqrt{\xi^2 - k^2}}{\xi^2 - 9j^2} \frac{2}{\xi^2 - j^2} d\xi \\
&\le \frac{6\alpha^3 j^3 |a_1|}{\pi} \int_{\xi > k} \frac{\xi}{(\xi^2 - 9j^2)^2} d\xi \\
&= \frac{6\alpha^3 j^3 |a_1|}{\pi} \frac{1}{k^2 - 9j^2} \le C(|g(0)| + |g(\pi)|).
\end{aligned}$$

If $0 < x < \frac{\pi}{j}$,then

$$\begin{aligned}
|a_1 Tu_1^*(x, y_2)| &= \left| \frac{a_1}{2\pi} \int_{\mathbb{R}} i\sqrt{k^2 - \xi^2} \frac{6\alpha^3 j^3}{\xi^2 - 9j^2} \frac{e^{-i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi \right| \\
&= \left| \frac{-2a_1}{2\pi} \int_{\xi > k} \sqrt{\xi^2 - k^2} \frac{6\alpha^3 j^3}{\xi^2 - 9j^2} \frac{e^{-i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi \right| \\
&\le C(|g(0)| + |g(\pi)|).
\end{aligned}$$

Similarly, $|a_2 T u_2^*(x, y_2)| \le C(|g(0)| + |g(\pi)|)$. Therefore,

$$\int_\Gamma |Tu^*|^2 = \int_0^\pi |a_1 T u_1^*|^2 + \int_0^\pi |a_2 T u_2^*|^2 \le C\left[|g(0)|^2 + |g(\pi)|^2\right].$$

Moreover, note that $u_y^* = a_1 u_{1y}^* + a_2 u_{2y}^*$, then

$$\int_\Gamma |u_y^*|^2 \le [\frac{C}{\alpha^3 k}]^2 \frac{\pi}{j}(|g(0)|^2 + |g(\pi)|^2)\alpha^5 \le \frac{C}{\alpha k^3}(|g(0)|^2 + |g(\pi)|^2).$$

Again if $\alpha = \frac{\beta}{k}$, then the result in this lemma is obtained. □

Furthermore, we have the following estimate for the integral in terms of g_{1x}.

Lemma 3.3.3. *The L_2 norm of g_{1x} can be estimated by*

$$\int_\Gamma |g_{1x}|^2 \le Ck^2\left[|g(0)|^2 + |g(\pi)|^2\right] + \int_\Gamma |g_x|^2.$$

Proof. Note that $g_{1x} = g_x - Tu_x^* + u_{yx}^*$, where $(Tu^*)_x = a_1(Tu_1^*)_x + a_2(Tu_2^*)_x$,

$$\begin{aligned} a_1 T u_1^*(x, y_2)_x &= \frac{1}{2\pi} a_1 \left[\int_{\mathbb{R}} i\sqrt{k^2 - \xi^2}\frac{6\alpha^3 j^3}{\xi^2 - 9j^2}\frac{e^{i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi\right]_x \\ &= \frac{1}{2\pi} a_1 \int_{\mathbb{R}} (i\xi) i\sqrt{k^2 - \xi^2}\frac{6\alpha^3 j^3}{\xi^2 - 9j^2}\frac{e^{i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi. \end{aligned}$$

Then if $\frac{\pi}{j} \leq x < \pi$, using the residue theorem as before, then

$$\begin{aligned}
|a_1 T u_1^*(x, y_2)_x| &= \left| -\frac{a_1}{\pi} \int_{\xi > k} (i\xi) \sqrt{\xi^2 - k^2} \frac{6\alpha^3 j^3}{\xi^2 - 9j^2} \frac{e^{i\xi(x - \frac{\pi}{j})} + e^{i\xi x}}{\xi^2 - j^2} d\xi \right| \\
&\leq \frac{24\alpha^3 j^3 |a_1|}{\pi} \int_{\xi > k} \frac{\xi^2}{\xi^2 (\xi - 3j)^2} d\xi \\
&= \frac{24\alpha^3 j^3 |a_1|}{\pi(k - 3j)} \leq Ck(|g(0)| + |g(\pi)|).
\end{aligned}$$

Follow the similar argument, if $0 < x < \frac{\pi}{j}$, then $|a_1 T u_1^*(x, y_2)_x| \leq Ck(|g(0)| + |g(\pi)|)$. Also,

$$|a_2 T u_2^*(x, y_2)_x| \leq Ck(|g(0)| + |g(\pi)|).$$

Thus $Tu^*(x)_x \leq Ck(|g(0)| + |g(\pi)|)$. It shows that

$$\int_\Gamma |(Tu^*)_x|^2 = \int_0^\pi |a_1 (Tu_1^*)_x|^2 + \int_0^\pi |a_2 (Tu_2^*)_x|^2 \leq Ck^2 \left[|g(0)|^2 + |g(\pi)|^2 \right].$$

Furthermore, we have $u_{yx}^* = a_1 u_{1yx}^* + a_2 u_{2yx}^*$, where

$$|a_1 u_{1yx}^*| = \left| 6a_1 \alpha \cos(jx) \sin(jx) j \right| \leq C\alpha k(|g(0)| + |g(\pi)|),$$

$$|a_2 u_{2yx}^*| = \left| 6a_2 \alpha \cos(jx) \sin(jx) j \right| \leq C\alpha k(|g(0)| + |g(\pi)|).$$

Based on the above expressions,

$$\int_\Gamma |u_{yx}^*|^2 \leq \frac{C}{j} \alpha^2 k^2 (|g(0)|^2 + |g(\pi)|^2) \leq C\alpha^2 k (|g(0)|^2 + |g(\pi)|^2).$$

Therefore, if α is chosen such that $\alpha = \frac{\beta}{k}$, then the lemma is proved. □

Remark: (1) the proof in Lemma (3.3.1) indicates that if $\alpha = \frac{\beta}{k}$, then the lowest power in k to control f is obtained. (2) the reason that u_1^* is chosen to be $(\alpha \sin(jx))^3$ on Γ instead of lower powers such as $\alpha \sin(jx)$ or $(\alpha \sin(jx))^2$ is that this guarantees the integrability of $(Tu^*)_x$.

Summarizing all the results proved in this section,

$$\int_D |f|^2 = \int_{\Omega_1} |f|^2 + \int_{\Omega_2} |f|^2 \leq C \left[|g(0)|^2 + |g(\pi)|^2 \right],$$

$$\int_\Gamma |g_1|^2 \leq C \left[|g(0)|^2 + |g(\pi)|^2 \right] + \int_\Gamma |g|^2,$$

$$\int_\Gamma |g_{1x}|^2 \leq Ck^2 \left[|g(0)|^2 + |g(\pi)|^2 \right] + \int_\Gamma |g_x|^2.$$

Combining these inequalities with Equation (3.13) yields that

$$\|\nabla w\|^2_{L^2(D)} + k^2 \|w\|^2_{L^2(D)} \leq C \left\{ k^{\frac{7}{5}} \int_\Gamma |g|^2 + \frac{1}{k} \int_\Gamma |g_x|^2 + k^{\frac{12}{5}} \left[|g(0)|^2 + |g(\pi)|^2 \right] \right\}.$$

Note that $w = u + u^*$, and similar as the computations in Lemma 3.3.1, we see that

$$\|\nabla u^*\|^2_{L^2(D)} + k^2 \|u^*\|^2_{L^2(D)} \leq \frac{C}{k^2} \left[|g(0)|^2 + |g(\pi)|^2 \right].$$

Hence, the final estimate could be established accordingly, that is,

$$\|\nabla u\|^2_{L^2(D)} + k^2 \|u\|^2_{L^2(D)} \leq C \left\{ k^{\frac{7}{5}} \int_\Gamma |g|^2 + \frac{1}{k} \int_\Gamma |g_x|^2 + k^{\frac{12}{5}} \left[|g(0)|^2 + |g(\pi)|^2 \right] \right\}.$$

3.4 The Estimate for the Nonhomogeneous System

In this last section, we establish the stability estimate for the formulation in Equation (3.3), or in general, we provide the stability estimate for the following problem

$$\begin{cases} \Delta w + k^2 w = f & \text{in } D, \\ w = 0 & \text{on } S, \\ \partial_{\mathbf{n}} w = T(w) + g_1 & \text{on } \Gamma. \end{cases} \tag{3.20}$$

where $g_1(a) = g_1(b) = 0$, $\Gamma = [a, b] \times \{y = y_2\}$, and $T(w) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - \xi^2}\, \widehat{w} e^{i\xi x} d\xi$, then the following theorem holds for the nonhomogeneous case.

Theorem 3.4.1. *Under the geometric assumptions given in Equation (3.1), there exists a constant C such that*

$$\|\nabla w\|^2_{L^2(D)} + k^2 \|w\|^2_{L^2(D)} \leq C \left\{ k^{\frac{7}{5}} \|g_1\|^2_{L^2(\Gamma)} + \frac{1}{k} \|g_{1x}\|^2_{L^2(\Gamma)} + k^{\frac{2}{5}} \|f\|^2_{L^2(D)} \right\} \tag{3.21}$$

Remark: The estimate for the case where $f = 0$ and g may not necessarily vanish on the boundary of Γ is established in the previous chapter where the stronger geometric assumption is imposed on S. Since most of the ideas are similar to the previous chapter, the arguments here emphasize on the portion when the nonhomogeneous terms are involved.

To simplify notation, we may again assume $X^* = (0, 0)$. Since in this chapter we use the fact that on Γ x coordinate starts from 0 to π in the construction of u^* which is related to g_1. So if we set $X^* = (0, 0)$, a translation of the coordinate system is applied here. That is why we assumed that $\Gamma = [a, b] \times \{y = y_2\}$.

The proof of Theorem 3.4.1 involves three major parts.

Part 1. The weak forms of the governing equation with two different test functions yield the following identity,

$$
\begin{aligned}
&\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{w}|^2 d\xi - \int_S (X\cdot \mathbf{n})|\nabla w|^2 \\
=&\Re\left(\int_\Gamma g_1\overline{w}\right) + \Re\left(\int_D f\overline{w}\right) + 2\Re\int_D f(X\cdot\nabla\overline{w}) \\
&- \int_\Gamma (X\cdot\mathbf{n})|\nabla w|^2 + k^2\int_\Gamma (X\cdot\mathbf{n})|w|^2 + 2\Re\int_\Gamma \frac{\partial w}{\partial \mathbf{n}}(X\cdot\nabla\overline{w}).
\end{aligned} \tag{3.22}
$$

Part 2. Appropriate usage of some basic facts related to Fourier transform, the nonlocal boundary condition in Equation (3.20) and the geometric assumption on S yield the following inequality

$$
\begin{aligned}
&\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} + \int_{|\xi|>k}\left(\sqrt{\xi^2-k^2} + \frac{k^2}{\sqrt{\xi^2-k^2}}\right)|\widehat{w}|^2 d\xi \\
\le&\Re\left(\int_\Gamma g_1\overline{w}\right) + \Re\left(\int_D f\overline{w}\right) + 2\Re\int_D f(X\cdot\nabla\overline{w}) + 2y_2\int_{|\xi|\le k}(k^2-\xi^2)|\widehat{w}|^2 d\xi \\
&+ 2y_2\Re\int_\Gamma T(w)\overline{g_1} - 2\Re\int_{|\xi|\le k}\left[\xi\overline{\widehat{w}}'(\xi)\right] i\sqrt{k^2-\xi^2}\,\widehat{w}(\xi)d\xi \\
&+ 2\Re\int_\Gamma x\overline{w}_x g_1 dx + y_2\int_\Gamma |g_1|^2 dx.
\end{aligned} \tag{3.23}
$$

Part 3. To fully use the higher frequency term on the left hand side in Equation (3.23), the relations between the higher frequency term on the left and the lower frequency component is needed, also the L_2 norm of w is expected to be controlled by the higher frequency component, these relations are derived in Lemma 3.4.5. Then by fully using the results in Lemma 3.4.5 and other tools, we estimate each term in the right hand side of Equation (3.23), which are named by I_1 through I_8.

3.4.1 Wave Energy Formulation

Next, we start from Part 1, where the goal is to formulate the wave energy identity.

Lemma 3.4.2. *For the lower frequency part of $\widehat{w}$, the following identity holds*

$$\int_{|\xi|\leq k} \sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi = -\Im\left(\int_\Gamma g_1\overline{w} + \int_D f\overline{w}\right). \tag{3.24}$$

And for the higher frequency part of $\widehat{w}$, it can be shown that

$$\|\nabla w\|^2_{L^2(D)} - k^2\|w\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi = \Re\left(\int_\Gamma g_1\overline{w} + \int_D f\overline{w}\right). \tag{3.25}$$

Proof. The weak form of the governing equation is given by the following equality

$$\int_D \nabla w\cdot\nabla\overline{v} - \int_D k^2 w\overline{v} - \int_{\partial D} \partial_{\mathbf{n}} w\overline{v} = \int_D f\overline{v} \tag{3.26}$$

for any $v \in H^1(D)$. Now choose $v = w$, then Equation (3.26) becomes

$$\int_D |\nabla w|^2 - \int_D k^2|w|^2 - \int_\Gamma (T(w)+g_1)\overline{w} = \int_D f\overline{w}, \tag{3.27}$$

where the boundary conditions $w = 0$ on S and $\partial_{\mathbf{n}} w = T(w) + g_1$ on Γ are used. Since w is supported on Γ, we can extend w such that $w = 0$ on Γ^C, then $\int_\Gamma T(w)\overline{w} = \int_R T(w)\overline{w} = \int_R \widehat{T(w)}\ \overline{\widehat{w}} = \int_{\mathbb{R}} i\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi$. Hence Equation (3.27) can be written as

$$\int_D |\nabla w|^2 - \int_D k^2|w|^2 - \int_{\mathbb{R}} i\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi = \int_\Gamma g_1\overline{w} + \int_D f\overline{w}. \tag{3.28}$$

Considering the fact that

$$-\int_{\mathbb{R}} i\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi = -\int_{|\xi|\leq k} i\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi + \int_{|\xi|>k} \sqrt{\xi^2-k^2}|\widehat{w}|^2 d\xi,$$

and taking the imaginary part and real part of Equation (3.28) respectively, the results in the lemma could be obtained consequently. $\square$

Notice that from Equation (3.25) in Lemma 3.4.2, the terms associated with the components in the wave energy, i.e. $\|\nabla w\|^2_{L^2(D)}$ and $-k^2\|w\|^2_{L^2(D)}$ are already there, unfortunately, the second term $-k^2\|w\|^2_{L^2(D)}$ carries a negative sign, while in the wave energy, we need the positive one, so another identity is needed such that both terms are positive. Therefore, another test function $v = X\cdot\nabla w$ is introduced here, where $X = (x, y)$, it yields the following lemma.

Lemma 3.4.3. *For $\epsilon > 0$, we have*

$$\begin{aligned} & -\int_S (X\cdot\mathbf{n})|\nabla w|^2 + 2k^2\int_D |w|^2 + \int_\Gamma (X\cdot\mathbf{n})|\nabla w|^2 - k^2\int_\Gamma (X\cdot\mathbf{n})|w|^2 \\ = \; & 2\Re\int_\Gamma \frac{\partial w}{\partial\mathbf{n}}(X\cdot\nabla\overline{w}) + 2\Re\int_D f(X\cdot\nabla\overline{w}). \end{aligned} \tag{3.29}$$

Proof. Plug this test function into Equation (3.26), it becomes

$$\int_D \nabla w\cdot\nabla(X\cdot\nabla\overline{w}) - \int_D k^2 w(X\cdot\nabla\overline{u}) - \int_{\partial D}\frac{\partial w}{\partial\mathbf{n}}(X\cdot\nabla\overline{w}) = \int_D f(X\cdot\nabla\overline{w}). \tag{3.30}$$

For the above equality, using divergence theorem, it is easy to check that the first two terms on the left hand side can be written as

$$2\Re \int_D \nabla w \cdot \nabla(X \cdot \nabla \overline{w}) = \int_{\partial D} (X \cdot \mathbf{n})|\nabla w|^2, \tag{3.31}$$

and

$$-2\Re \int_D k^2 w(X \cdot \nabla \overline{w}) = -k^2 \int_{\partial D} (X \cdot \mathbf{n})|w|^2 + 2k^2 \int_D |w|^2. \tag{3.32}$$

Then multiply Equation (3.30) by 2 and take its real part, and apply Equations.(3.31) and (3.32), we could arrive at the following equality

$$\begin{aligned} & \int_{\partial D} (X \cdot \mathbf{n})|\nabla w|^2 - k^2 \int_{\partial D} (X \cdot \mathbf{n})|w|^2 + 2k^2 \int_D |w|^2 - 2\Re \int_{\partial D} \frac{\partial w}{\partial \mathbf{n}} (X \cdot \nabla \overline{w}) \\ = \;& 2\Re \int_D f(X \cdot \nabla \overline{w}). \end{aligned} \tag{3.33}$$

Since ∂D contains two parts Γ and S with different boundary conditions, the two parts will be considered separately. For S part, note that $w = 0$ on S, $\nabla w = \dfrac{\partial w}{\partial \mathbf{n}} \mathbf{n}$, hence

$$\int_S (X \cdot \mathbf{n})|\nabla w|^2 - k^2 \int_S (X \cdot \mathbf{n})|w|^2 - 2\Re \int_S \frac{\partial w}{\partial \mathbf{n}} (X \cdot \nabla \overline{w}) = -\int_S (X \cdot \mathbf{n})|\nabla w|^2.$$

Therefore, Equation (3.33) becomes the equality shown in the lemma. $\square$

Obviously the term $2k^2 \int_D |w|^2$ in Equation (3.29) is helpful as this term provides positive contribution to the wave energy, so add Equation (3.25) and Equation (3.29) together, the identity involving the wave energy could be formulated as shown in Equation (3.22).

3.4.2 Analysis through Fourier Transform

On the left hand side of Equation (3.22), apart from the wave energy terms, the term $\int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{w}|^2 d\xi$ will be helpful to control the higher frequency part of $\widehat{w}$, the term $-\int_S (X \cdot \mathbf{n})|\nabla w|^2$ is nonnegative under the geometric assumption $X \cdot \mathbf{n} \leq 0$ on S, so it could be dropped from the estimate. For the right hand side, the term $\int_\Gamma \frac{\partial w}{\partial \mathbf{n}}(X \cdot \nabla \overline{w})$ is more involved, since ∇w on Γ can not be controlled by the wave energy terms directly, some further analysis is required. Since the operator T is simply a multiplication operator in terms of Fourier transform. Using the boundary condition on Γ and some basic identity from Fourier transform, the last 3 terms on the right hand side of Equation (3.22) can be easily converted into the frequency domain. For the details of the proof, see Chapter 2.

Lemma 3.4.4. *For the last three terms on the right hand side of Equation (3.22), it can be shown that*

$$\begin{aligned}
& -\int_\Gamma (X \cdot \mathbf{n})|\nabla w|^2 + k^2 \int_\Gamma (X \cdot \mathbf{n})|w|^2 + 2\Re \int_\Gamma \frac{\partial w}{\partial \mathbf{n}}(X \cdot \nabla \overline{w}) \\
\leq & -\int_{|\xi|>k} \frac{k^2}{\sqrt{\xi^2 - k^2}}|\widehat{w}|^2 d\xi 2y_2 \int_{|\xi|\leq k} (k^2 - \xi^2)|\widehat{w}|^2 d\xi + 2y_2 \Re \int_\Gamma T(w)\overline{g_1} \\
& - 2\Re \int_{|\xi|\leq k} \left[\xi \overline{\widehat{w}\,'}(\xi)\right] i\sqrt{k^2 - \xi^2}\widehat{w}(\xi) d\xi + 2\Re \int_\Gamma x\overline{w}_x g_1 dx + y_2 \int_\Gamma |g_1|^2 dx.
\end{aligned} \tag{3.34}$$

From the result in Lemma 3.4.4, we can see that the original term $2\Re \int_\Gamma \frac{\partial w}{\partial \mathbf{n}}(X \cdot \nabla \overline{w})$ actually contains lots of useful information. Especially, it produced a negative higher gre-quency component term, $-\int_{|\xi|>k} \frac{k^2}{\sqrt{\xi^2 - k^2}}|\widehat{w}|^2 d\xi$, this can be moved to the left hand side and helps with the final estimate. Therefore, by dropping the term $-\int_S (X \cdot \mathbf{n})|\nabla w|^2$ in Equation (3.22) using the geometric assumption and also use the result in Lemma 3.4.4, the result shown in Equation (3.23) is obtained.

Now we can see that the term ∇w has been canceled out by using the result shown in lemma 3.4.4; as a consequence, we obtain an energy inequality which contains lots of other terms on the right hand side.

3.4.3 Auxiliary Lemmas and Final Estimates

Next the estimate for each term on the right hand side of Equation (3.23) is derived. This is not straightforward as no direct connection is there for the higher frequency component on the left hand side and the terms on the right. Therefore, the connection between them is established in Lemma 3.4.5. To simplify the notations, we denote $A_L = \int_{|\xi|\leq k} \sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi$, and $B_H = \int_{|\xi|>k}(\sqrt{\xi^2-k^2}+\frac{k^2}{\sqrt{\xi^2-k^2}})|\widehat{w}|^2 d\xi$. As stated in the following lemma, A_L, the lower frequency part of w, could be controlled by B_H, which is connected to the higher frequency part of w.

Lemma 3.4.5. *There exists a positive constant C, such that*

$$A_L \leq \frac{\delta}{k}B_H + \frac{1}{\delta}||g_1||^2_{L^2(\Gamma)} + 2\,|\int_D f\overline{w}\,|, \tag{3.35}$$

where δ can be chosen such that $C\delta \leq k^{\frac{1}{2}}$. Furthermore,

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{2}{k}B_H + \frac{C^2}{k}\|g_1\|^2_{L^2(\Gamma)} + \frac{2C}{\sqrt{k}}\,|\int_D f\overline{w}\,|\,. \tag{3.36}$$

.

Proof. As shown in Lemma 2.4.1, there exists a constant C such that

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}} A_L + \frac{1}{k} B_H. \tag{3.37}$$

In addition, by using the identity Equation (3.24) and the above inequality, we have

$$\begin{aligned} A_L &\leq \|g_1\|_{L^2(\Gamma)} \|w\|_{L^2(\Gamma)} + \left| \int_D f\overline{w} \right| \leq \frac{\delta}{2}\|w\|^2_{L^2(\Gamma)} + \frac{1}{2\delta}\|g_1\|^2_{L^2(\Gamma)} + \left| \int_D f\overline{w} \right| \\ &\leq \frac{\delta}{2}\left(\frac{C}{\sqrt{k}} A_L + \frac{1}{k} B_H\right) + \frac{1}{2\delta}\|g_1\|^2_{L^2(\Gamma)} + \left| \int_D f\overline{w} \right| \end{aligned}$$

If δ is chosen such that $C\delta \leq \sqrt{k}$, then the following inequality holds,

$$\frac{1}{2} A_L \leq \left(1 - \frac{C\delta}{2\sqrt{k}}\right) A_L \leq \frac{\delta}{2k} B_H + \frac{1}{2\delta}\|g_1\|^2_{L^2(\Gamma)} + \left| \int_D f\overline{w} \right|.$$

Equation (3.35) is obtained accordingly. Plugging Equation (3.35) into Equation (3.37), the estimate for $\|w\|^2_{L^2(\Gamma)}$ could be arrived as shown below.

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}} \left(\frac{\delta}{k} B_H + \frac{1}{\delta}\|g\|^2_{L^2(\Gamma)} + 2\left| \int_D f\overline{w} \right| \right) + \frac{1}{k} B_H.$$

Choose $C\delta = \sqrt{k}$, then Equation (3.36) holds. □

Lemma 3.4.5 is crucial in the sense that it provides a way to relate the integrals of $|w|^2$ and $\sqrt{|\xi^2 - k^2|}|\widehat{w}|^2$ to the high frequency part B_H and also norms related to g_1; it is also important in determining the power of k in the estimate. Now we will estimate each term on the right hand side of Equation (3.23), which are named I_1 through I_8. Here similar techniques could be used as in the previous work for the terms I_1, I_3, I_5 and I_6, for other new terms, i.e., I_2 and I_8, they could be controlled by norms related to f and the energy terms on the left, and for I_4, considering the fact that $g_1 = 0$ on the boundary of Γ, a different way as shown in Lemma 3.4.9 can be used, which is relatively straightforward comparing to the previous work and here the geometric assumption is not needed for the estimate. The following lemmas, i.e., Lemma 3.4.6 through Lemma 3.4.12, provides the estimate for terms on the right hand side of the energy inequality.

Lemma 3.4.6. *The first term on the right hand side of Equation (3.23) can be estimated by*

$$I_1 = \Re \int_\Gamma g_1 \overline{w} \leq \frac{\varepsilon_1}{\sqrt{k}} B_H + \frac{d_1}{\sqrt{k}} \|g_1\|^2_{L^2(\Gamma)} + \varepsilon_1 \int_D k^2 |w|^2 + e_1 \frac{1}{k^2} \int_D |f|^2, \tag{3.38}$$

where ε_1 is any small positive number, and $d_1 = \dfrac{\varepsilon_1 C^2}{2} + \dfrac{1}{2\varepsilon_1}$ and $e_1 = \dfrac{\varepsilon_1 C^2}{4}$.

Proof. By applying the relation between $\|w\|^2_{L^2(\Gamma)}$ and B_H as shown in Lemma 3.4.5, it follows that

$$\begin{aligned}
I_1 &= \Re \int_\Gamma g_1 \overline{w} \leq \|g_1\|_{L^2(\Gamma)} \|w\|_{L^2(\Gamma)} \leq \frac{\varepsilon_1 \sqrt{k}}{2} \|w\|^2_{L^2(\Gamma)} + \frac{1}{2\varepsilon_1 \sqrt{k}} \|g_1\|^2_{L^2(\Gamma)} \\
&\leq \frac{\varepsilon_1 \sqrt{k}}{2} \left(\frac{2}{k} B_H + \frac{C^2}{k} \|g_1\|^2_{L^2(\Gamma)} + \frac{2C}{\sqrt{k}} \left| \int_D f \overline{w} \right| \right) + \frac{1}{2\varepsilon_1 \sqrt{k}} \|g_1\|^2_{L^2(\Gamma)} \\
&= \frac{\varepsilon_1}{\sqrt{k}} B_H + \frac{d_1}{\sqrt{k}} \|g_1\|^2_{L^2(\Gamma)} + \varepsilon_1 C \left| \int_D f \overline{w} \right| \\
&\leq \frac{\varepsilon_1}{\sqrt{k}} B_H + \frac{d_1}{\sqrt{k}} \|g_1\|^2_{L^2(\Gamma)} + \varepsilon_1 \int_D k^2 |w|^2 + e_1 \frac{1}{k^2} \int_D |f|^2,
\end{aligned} \tag{3.39}$$

where the constants are independent of k. □

It is easy to obtain the estimate for the second term on the right hand side of Equation (3.23) since a positive term $k^2|w|^2$ lies on the left hand side of the equation.

Lemma 3.4.7. *For I_2, we have*

$$I_2 = \Re \int_D f\overline{w} \le \varepsilon_2 \int_D k^2|w|^2 + e_2 \frac{1}{k^2} \int_D |f|^2, \tag{3.40}$$

where ε_2 is any small positive number, and $e_2 = \dfrac{1}{4\varepsilon_2}$.

Similarly, a direct usage of Schwarz inequality would lead to the estimate of I_3 as shown below.

Lemma 3.4.8.

$$I_3 = 2\Re \int_D f\,(X \cdot \nabla\overline{w}) \le \varepsilon_3 \int_D |\nabla w|^2 + e_3 \int_D |f|^2, \tag{3.41}$$

where ε_3 is any small positive number, and $e_3 = \dfrac{M_3}{4\varepsilon_8}$, $M_3 = max_{x\in\Gamma}\{x^2\}$.

The estimate on I_4 can be derived by using Lemma 3.4.5.

Lemma 3.4.9. *I_4 which contains only the lower frequency part can be estimated as follows.*

$$\begin{aligned} I_4 &= 2y_2 \int_{|\xi|\le k} (k^2 - \xi^2)|\widehat{w}|^2 d\xi \\ &\le \varepsilon_4 B_H + d_4 k \|g_1\|^2_{L^2(\Gamma)} + \varepsilon_4 \int_D k^2|w|^2 + e_4 \int_D |f|^2, \end{aligned} \tag{3.42}$$

where ε_4 is a small positive number, $d_4 = \dfrac{4y_2^2}{\varepsilon_4}$ and $e_4 = \dfrac{16y_2^2}{\varepsilon_4}$.

Proof. Using the fact that for $|\xi| \leq k$, $\sqrt{k^2 - \xi^2} < k$, then

$$\begin{aligned} I_4 &= 2y_2 \int_{|\xi| \leq k} (k^2 - \xi^2)|\widehat{w}|^2 d\xi \leq 2y_2 k A_L \\ &\leq 2y_2 k \left(\frac{\varepsilon_3}{2y_2 k} B_H + \frac{2y_2}{\varepsilon_4} \|g_1\|^2_{L^2(\Gamma)} + 2 \mid \int_D f\overline{w} \mid \right) \\ &= \varepsilon_4 B_H + d_4 k \|g_1\|^2_{L^2(\Gamma)} + 4y_2 k \int_D f\overline{w} \mid \\ &\leq \varepsilon_4 B_H + d_4 k \|g_1\|^2_{L^2(\Gamma)} + \varepsilon_4 \int_D k^2 |w|^2 + e_4 \int_D |f|^2. \end{aligned} \tag{3.43}$$

$\square$

Lemma 3.4.10. *It can be shown that I_5 satisfies the following inequality*

$$\begin{aligned} I_5 &= 2y_2 \Re \int_\Gamma T(w)\overline{g_1} \\ &\leq \varepsilon_5 B_H + d_5 \|g_1\|^2_{L^2(\Gamma)} + \widetilde{d_5} \frac{1}{k} \|g_1'\|^2_{L^2(\Gamma)} + \varepsilon_5 \int_D k^2 |w|^2 + \frac{e_4}{k} \int_D |f|^2, \end{aligned} \tag{3.44}$$

where ε_5 is a small positive number, $d_5 = \dfrac{\varepsilon_5 C^2}{2} + \dfrac{2y_2^2}{\varepsilon_5}, \widetilde{d_5} = \dfrac{2y_2^2}{\varepsilon_5}$ and $e_5 = \dfrac{C^2}{4\varepsilon_5} + \dfrac{2y_2^2}{\varepsilon_5}$.

Proof. Define $g_{1\Gamma}$ in the following way,

$$g_{1\Gamma} := \begin{cases} g_1 & \text{on } \Gamma, \\ 0 & \text{on } \Gamma^C, \end{cases}$$

then

$$
\begin{aligned}
I_5 &= 2y_2\Re\int_R T(w)\overline{g_{1\Gamma}} = 2y_2\Re\int_R \widehat{T(w)}\overline{\widehat{g_{1\Gamma}}} \\
&= 2y_2\Re\int_R i\sqrt{k^2-\xi^2}\widehat{w}\overline{\widehat{g_{1\Gamma}}} \le 2y_2\left(\int_{\mathbb{R}}|\widehat{w}|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|k^2-\xi^2||\widehat{g_{1\Gamma}}|^2 d\xi\right)^{\frac{1}{2}} \\
&\le \frac{\varepsilon_5 k}{2}\|w\|^2_{L^2(\Gamma)} + \frac{2y_2^2}{\varepsilon_5 k}\int_{\mathbb{R}}|k^2-\xi^2||\widehat{g_{1\Gamma}}|^2 d\xi \\
&\le \frac{\varepsilon_5 k}{2}\left(\frac{2}{k}B_H + \frac{C^2}{k}\|g_1\|^2_{L^2(\Gamma)} + \frac{2C}{\sqrt{k}}\,|\int_D f\overline{w}\,|\right) + \frac{2y_2^2}{\varepsilon_5 k}\int_{\mathbb{R}}\left(k^2|\widehat{g_{1\Gamma}}|^2 d\xi + \xi^2||\widehat{g_{1\Gamma}}|^2\right)d\xi \\
&= \varepsilon_5 B_H + \frac{\varepsilon_5 C^2}{2}\|g_1\|^2_{L^2(\Gamma)} + \varepsilon_5 C\sqrt{k}\,|\int_D f\overline{w}\,| + \frac{2y_2^2}{\varepsilon_5}k\|g_1\|^2_{L^2(\Gamma)} + \frac{2y_2^2}{\varepsilon_5 k}\|g_1'\|^2_{L^2(\Gamma)} \\
&\le \varepsilon_5 B_H + \frac{\varepsilon_5 C^2}{2}\|g_1\|^2_{L^2(\Gamma)} + \varepsilon_5\int_D k^2|w|^2 + \frac{e_5}{k}\int_D |f|^2 + \frac{2y_2^2}{\varepsilon_5}k\|g_1\|^2_{L^2(\Gamma)} + \frac{2y_2^2}{\varepsilon_5 k}\|g_1'\|^2_{L^2(\Gamma)} \\
&\le \varepsilon_5 B_H + d_5 k\|g_1\|^2_{L^2(\Gamma)} + \widetilde{d_5}\frac{1}{k}\|g_1'\|^2_{L^2(\Gamma)} + \varepsilon_5\int_D k^2|w|^2 + \frac{e_5}{k}\int_D |f|^2.
\end{aligned}
$$

□

Lemma 3.4.11. *By using Lemma 3.4.5 and further analysis in the frequency domain, we have*

$$
\begin{aligned}
I_6 &= -2\Re\int_{|\xi|\le k}\left[\xi\overline{\widehat{w}\,'}(\xi)\right] i\sqrt{k^2-\xi^2}\,\widehat{w}(\xi)d\xi \\
&\le \varepsilon_6 B_H + d_6 k^{\frac{7}{5}}\|g_1\|^2_{L^2(\Gamma)} + \varepsilon_6\int_D k^2|w|^2 + e_6 k^{\frac{2}{5}}\int_D |f|^2, \qquad (3.45)
\end{aligned}
$$

where ε_6 is any small positive number.

Proof. Following the conclusion in Lemma 2.4.5, the following inequality holds.

$$
\begin{aligned}
I_6 &= \varepsilon k\|w\|^2_{L^2(\Gamma)} + Ck^{\frac{6}{5}}A_L \\
&\leq \varepsilon k\left(\frac{2}{k}B_H + \frac{C^2}{k}\|g_1\|^2_{L^2(\Gamma)} + \frac{2C}{\sqrt{k}} \mid \int_D f\overline{w} \mid\right) \\
&+ Ck^{\frac{6}{5}}\left(\frac{\varepsilon}{k}B_H + \frac{k}{\varepsilon}\|g_1\|^2_{L^2(\Gamma)} + 2 \mid \int_D f\overline{w} \mid\right) \\
&\leq \varepsilon_6 B_H + d_6 k^{\frac{7}{5}}\|g_1\|^2_{L^2(\Gamma)} + \varepsilon_6\int_D k^2|w|^2 + e_6 k^{\frac{2}{5}}\int_D |f|^2.
\end{aligned}
$$

$\square$

Lemma 3.4.12. *It is shown that I_7 can be estimated by*

$$
\begin{aligned}
I_7 &= 2\Re\int_\Gamma x\overline{w}_x g dx \\
&\leq \varepsilon_7 B_H + d_7\|g_1\|^2_{L^2(\Gamma)} + \frac{\widetilde{d_7}}{k}\|g_{1x}\|^2_{L^2(\Gamma)} + \varepsilon_7\int_D k^2|w|^2 + e_7\frac{1}{k}\int_D |f|^2, \quad (3.46)
\end{aligned}
$$

ε_7 can be chosen as an arbitrary positive number, $d_7 = \frac{\varepsilon_7 C^2}{2} + \frac{4}{\varepsilon_7}$, $\widetilde{d_7} = \frac{2M_3}{\varepsilon_7}$, and $e_7 = \frac{C^2\varepsilon_7}{4}$

Proof. Using integration by parts and the connection between $\|w\|^2_{L^2(\Gamma)}$ and B_H in Lemma 3.4.5, it implies that

$$
\begin{aligned}
I_7 &= 2\Re\int_\Gamma x\overline{w}_x g_1 dx = -2\Re\int_\Gamma \overline{w}(xg_1)_x dx \\
&\leq \frac{\varepsilon_7 k}{2}\int_\Gamma |w|^2 dx + \frac{2}{\varepsilon_7 k}\int_\Gamma |(xg_1)_x|^2 dx \\
&\leq \frac{\varepsilon_7 k}{2}\left(\frac{2}{k}B_H + \frac{C^2}{k}\|g_1\|^2_{L^2(\Gamma)} + \frac{2C}{\sqrt{k}} \mid \int_D f\overline{w} \mid\right) + \frac{4}{\varepsilon_7 k}\|g_1\|^2_{L^2(\Gamma)} + \frac{2M_3}{\varepsilon_7 k}\|g_{1x}\|^2_{L^2(\Gamma)} \\
&\leq \varepsilon_7 B_H + d_7\|g_1\|^2_{L^2(\Gamma)} + \frac{\widetilde{d_7}}{k}\|g_{1x}\|^2_{L^2(\Gamma)} + \varepsilon_7\int_D k^2|w|^2 + e_7\frac{1}{k}\int_D |f|^2.
\end{aligned}
$$

$\square$

By using the estimates obtained in Lemma 3.4.6 through 3.4.12, we can see that Equation (3.23) can be estimated by:

$$
\begin{aligned}
&\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} + \int_{|\xi|>k} \Big(\sqrt{\xi^2 - k^2} + \frac{k^2}{\sqrt{\xi^2 - k^2}}\Big)|\widehat{w}|^2 d\xi \\
\leq & \varepsilon B_H + dk^{\frac{7}{5}}\|g_1\|^2_{L^2(\Gamma)} + \widetilde{d}\frac{1}{k}\|g_{1x}\|^2_{L^2(\Gamma)} + \varepsilon \left(\int_D k^2|w|^2 + \int_D |\nabla w|^2\right) + ek^{\frac{2}{5}} \int_D |f|^2,
\end{aligned}
\tag{3.47}
$$

here ε, d, $\widetilde{d}$ and e are chosen such that they are $\geq$ the sums of the corresponding components in the estimates. Since ε can be chosen as small positive number, so the B_H term on the right hand side of Equation (3.47) can be absorbed by the B_H term on the left, similarly, the energy term on the right hand side can be absorbed by the energy term on the left hand side. Therefore, it shows that there exists a constant C such that Theorem 3.4.1 holds.

Chapter 4

Stability Estimates for Rectangular-like Domains

4.1 Main Theorem and Sketches of Approach

There are other cavity domains which do not satisfies the aforementioned geometric assumptions in Chapter 2 and Chapter 3, such as rectangular-like domains (domains composed of rectangles of different sizes), one example is shown in Fig. 4.1. The main result is stated in the following theorem, where the outnormal vector $\mathbf{n} = (n_x, n_y)$.

Theorem 4.1.1. *For the cavity of rectangular-like shapes with $yn_y \leq 0$ on S, there exists a constant C such that*

$$\|\nabla u\|_{L^2(D)} + |k|\|u\|_{L^2(D)} \leq C[k^{\frac{7}{4}}\|g\|_{L^2(\Gamma)} + k^{\frac{3}{4}}\|g_x\|_{L^2(\Gamma)} + k^2(|g(0)| + |g(\pi)|)]. \tag{4.1}$$

To prove the theorem, the same u^* as in Chapter 3 is introduced, which satisfies: 1) $Tu^*(0, y_2) = g(0, y_2)$ and $Tu^*(\pi, y_2) = g(\pi, y_2)$, 2) $u^*(0, y_2) = u^*(\pi, y_2) = 0$, 3) $u^*_y(0, y_2) = u^*_y(\pi, y_2) = 0$, and 4) $u^* = 0$ on S. Then we set $w = u + u^*$, then w satisfies the following

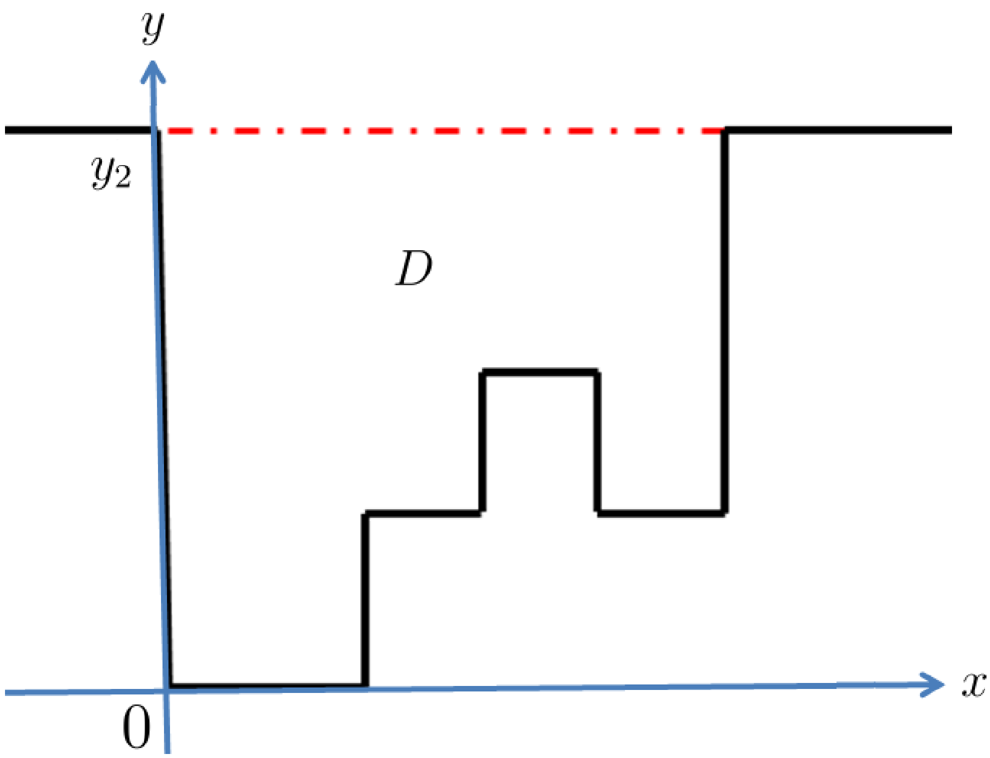


Figure 4.1: Rectangular-like domain

equations

$$
\begin{cases}
\Delta w + k^2 w = f & \text{in } D, \\
w = 0 & \text{on } S, \\
\partial_{\mathbf{n}} w = Tw + g_1 & \text{on } \Gamma,
\end{cases}
\tag{4.2}
$$

where

$$
\begin{cases}
f = \Delta u^* + k^2 u^*, \\
g_1 = g - Tu^* + u_y^*, \\
g_1(0) = g_2(\pi) = 0.
\end{cases}
\tag{4.3}
$$

The major difference lies in that we need to derive an energy estimate for w in the context of rectangular-like domains. Here two main procedures are involved to derive this energy estimate. First, we start from the weak formulation of w and also using appropriate test functions, the following inequality could be arrived, for any $\varepsilon > 0$,

$$\begin{aligned}&\int_D |w_y|^2 + \int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{w}|^2 d\xi \\ \leq & Ck^{\frac{3}{2}}\|g_1\|^2_{L^2(\Gamma)} + \frac{C}{\sqrt{k}}\|g_{1x}\|^2_{L^2(\Gamma)} + \varepsilon \int_D |w|^2 + \frac{C}{\varepsilon}k^2 \int_D |f|^2,\end{aligned} \tag{4.4}$$

Note that this inequality only contains the estimate for w_y, therefore, additional relations involving other terms in the energy needs to be explored, that is,

$$\int_D |w|^2 \leq C \int_D |w_y|^2. \tag{4.5}$$

Combining these two inequalities and also the result in Lemma 4.2.4, the following energy estimate for w is arrived:

$$k^2\|w\|^2_{L^2(D)} + \|\nabla w\|^2_{L^2(D)} \leq C \left[k^{\frac{7}{2}}\|g_1\|^2_{L^2(\Gamma)} + k^{\frac{3}{2}}\|g_{1x}\|^2_{L^2(\Gamma)} + k^4 \int_D |f|^2 \right]. \tag{4.6}$$

By using this energy estimate along with detailed analysis on u^*, the main result could be obtained.

4.2 Preliminary Lemmas

Since we use the same u^* as applied in Chapter 3, so here the details related to u^* are skipped for simplification and the resulting estimates related to u^* will be restated briefly when needed as for the clarity of our further proof. The main effort is to establish a stability estimate for the new function w by assuming the cavity region is of rectangular-like shape and furthermore we assume $yn_y \leq 0$. Here we start from the weak formulation of the problem for w, and apply two different test functions to obtain an important inequality. The weak form of the governing equation is given by the following equality

$$\int_D \nabla w \cdot \nabla \overline{v} - \int_D k^2 w\overline{v} - \int_{\partial D} \partial_{\mathbf{n}} w\overline{v} = \int_D f\overline{v}. \tag{4.7}$$

Similar as before, by choosing $v = w$ in Equation (4.7), and taking the imaginary part and real part respectively, one has

Lemma 4.2.1.

$$\int_{|\xi|\leq k} \sqrt{k^2 - \xi^2} |\widehat{w}|^2 d\xi = -Im \left(\int_\Gamma g_1 \overline{w} + \int_D f\overline{w} \right), \tag{4.8}$$

$$\|\nabla w\|^2_{L^2(D)} - k^2 \|w\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2 - k^2} |\widehat{u}|^2 d\xi = \Re \left(\int_\Gamma g_1 \overline{w} + \int_D f\overline{w} \right). \tag{4.9}$$

Notice that in Equation (4.9), the terms associated with the components in the wave energy, i.e. $\|\nabla w\|^2_{L^2(D)}$ and $-k^2\|w\|^2_{L^2(D)}$ are already there, unfortunately, the second term $-k^2\|w\|^2_{L^2(D)}$ carries a negative sign, while in the wave energy, we need the positive one, so another identity is needed to cancel out the negative term. Therefore, another test function $v = yw_y$ is introduced. Note that this is different from the test function used in Chapter 2 and Chapter 3 since a different geometric assumption is imposed here.

Pluging this test function into Equation (4.7) yields

$$\int_D \nabla w \cdot \nabla \overline{y w_y} - \int_D k^2 w \overline{y w_y} - \int_{\partial D} \frac{\partial w}{\partial \mathbf{n}} \overline{y w_y} = \int_D f \overline{y w_y}.$$

Multiply both sides of the above equation by 2 and take the real part, it becomes

$$2\Re \int_D \nabla w \cdot \nabla \overline{y w_y} - 2\Re \int_D k^2 w \overline{y w_y} - 2 \int_\Gamma y_2 |w_y|^2 = 2\Re \int_D f \overline{y w_y} + 2\Re \int_S \frac{\partial w}{\partial \mathbf{n}} \overline{y w_y} \tag{4.10}$$

By adding appropriate additional terms and using the divergence theorem, it follows that

$$\begin{aligned} & 2\Re \int_D \nabla w \cdot \nabla \overline{y w_y} = 2\Re \int_D [w_x y \overline{w_{xy}} + w_y y \overline{w_{yy}}] + 2|w_y|^2 \\ =& \int_D y(|\nabla w|^2)_y + 2|w_y|^2 = \int_D [y(|\nabla w|^2)]_y - |\nabla w|^2 + 2|w_y|^2 \\ =& \int_{\partial D} y n_y (|\nabla w|^2) - \int_D |\nabla w|^2 + 2 \int_D |w_y|^2 \\ =& \int_\Gamma y_2 (|\nabla w|^2) + \Re \int_S \frac{\partial w}{\partial \mathbf{n}} \overline{y w_y} - \int_D |\nabla w|^2 + 2 \int_D |w_y|^2. \end{aligned} \tag{4.11}$$

Furthermore, the other term $2\Re \int_D k^2 w \overline{y w_y}$ can be written as

$$\begin{aligned} & 2\Re \int_D k^2 w \overline{y w_y} = \int_D k^2 y (|w|^2)_y = \int_D k^2 (y|w|^2)_y - k^2 |w|^2 \\ =& k^2 \int_\Gamma y_2 |w|^2 - \int_D k^2 |w|^2. \end{aligned} \tag{4.12}$$

Plug the results in Equations (4.11) and (4.12) into Equation (4.10), it leads to

$$\begin{aligned} & 2 \int_D |w_y|^2 - \int_D |\nabla w|^2 + \int_D k^2 |w|^2 \\ =& - \int_\Gamma y_2 (|\nabla w|^2) + k^2 \int_\Gamma y_2 |w|^2 + 2 \int_\Gamma y_2 |w_y|^2 + 2\Re \int_D f \overline{y w_y} + \Re \int_S \frac{\partial w}{\partial \mathbf{n}} \overline{y w_y}. \end{aligned} \tag{4.13}$$

Note the geometric assumption $yn_y \leq 0$ on S guarantees that $\int_S \frac{\partial w}{\partial \mathbf{n}} \overline{y w_y} \leq 0$. Therefore the following lemma could be obtained.

Lemma 4.2.2. *For all $w \in H^1_S(D) \bigcap H^{\frac{3}{2}+\epsilon}(D), \epsilon > 0$,we have*

$$\begin{aligned} & 2\int_D |w_y|^2 - \int_D |\nabla w|^2 + k^2 \int_D |w|^2 \\ \leq \; & -\int_\Gamma y_2 |\nabla w|^2 + k^2 \int_\Gamma y_2 |w|^2 + 2\int_\Gamma y_2 |w_y|^2 + 2\Re \int_D f y \overline{w}_y. \end{aligned} \tag{4.14}$$

Consequently, the negative term on the left hand side of the above equation could be canceled out as shown in the following lemma.

Lemma 4.2.3. *By adding Equation (4.9) and Equation (4.2.2) together, it yields that*

$$\begin{aligned} & 2\int_D |w_y|^2 + \int_{|\xi|>k} \sqrt{\xi^2 - k^2} |\widehat{w}|^2 d\xi \\ \leq & \Re\left(\int_\Gamma g_1 \overline{w}\right) + \Re\left(\int_D f\overline{w}\right) - \int_\Gamma y_2 |\nabla w|^2 + k^2 \int_\Gamma y_2 |w|^2 \\ & + 2\int_\Gamma y_2 |w_y|^2 + 2\Re \int_D f y \overline{w}_y. \end{aligned} \tag{4.15}$$

Note that for the third to fifth term on the right hand side of the above identity, a direct computation yields the following result.

Lemma 4.2.4. *As detailed in Lemma 2.3.2 in Chapter 2, one has*

$$\begin{aligned} & -\int_\Gamma y_2 |\nabla w|^2 + k^2 \int_\Gamma y_2 |w|^2 + 2\int_\Gamma y_2 |w_y|^2 \\ \leq & 2y_2 \int_{|\xi|\leq k} (k^2 - \xi^2)|\widehat{w}|^2 d\xi + 2y_2 \Re \int_\Gamma T(w)\overline{g_1} + y_2 \int_\Gamma |g_1|^2 dx. \end{aligned} \tag{4.16}$$

Use the result in Lemma 4.2.4 and apply it in Lemma 4.2.3, the following lemma holds.

Lemma 4.2.5.

$$
\begin{aligned}
&2\int_D |w_y|^2 + \int_{|\xi|>k} \sqrt{\xi^2 - k^2}|\widehat{w}|^2 d\xi \\
\leq &\Re\left(\int_\Gamma g_1\overline{w}\right) + \Re\left(\int_D f\overline{w}\right) + 2y_2\int_{|\xi|\leq k}(k^2-\xi^2)|\widehat{w}|^2 d\xi \\
&+ 2y_2\Re\int_\Gamma T(w)\overline{g_1} + y_2\int_\Gamma |g_1|^2 dx + 2\Re\int_D fy\overline{w}_y.
\end{aligned} \tag{4.17}
$$

Observe that a higher frequency component lie on the left hand side of the above inequality, this could be used for the estimation on the right hand side, thus we need to build the connections between the higher frequency component and lower frequency component. By using the fact that w is supported on Γ and the result in Lemma 4.2.1 here, the connection could be eatablished. To simplify the notations, denote $A_L = \int_{|\xi|\leq k}\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi$, and $A_H = \int_{|\xi|>k}\sqrt{k^2-\xi^2}|\widehat{w}|^2 d\xi$. As stated in the following lemma, A_L, the lower frequency part of w, could be controlled by A_H, which is connected to the higher frequency part of w.

Lemma 4.2.6. *There exists positive constants C and d, such that*

$$
A_L \leq \frac{M}{\sqrt{k}}A_H + \frac{1}{M}||g_1||^2_{L^2(\Gamma)} + 2\left|\int_D f\overline{w}\right|, \tag{4.18}
$$

where M can be chosen such that $M \leq \frac{k^{\frac{1}{2}}}{d}$. Furthermore,

$$
\|w\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}}A_H + \frac{C}{k}\|g_1\|^2_{L^2(\Gamma)} + \frac{C}{\sqrt{k}}\left|\int_D f\overline{w}\right|. \tag{4.19}
$$

Remark. Note that the difference between this lemma and Lemma 3.4.5 lies in the power of k, here the power associated with A_H is $\frac{1}{\sqrt{k}}$ while in 3.4.5 the power associated with B_H is $\frac{1}{k}$, since in B_H additional higher order term exists. Here B_H is not involved due to the fact that a different test function is used. This power dependency difference essentially leads to the different power in the final estimate.

Proof. Here we could start from the idea of Lemma 3.5 in [8], that is, there exists a constant d, such that

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{d}{\sqrt{k}}(A_L + A_H). \tag{4.20}$$

In addition, by using the identity (4.8) and the above inequality, we have

$$\begin{aligned} A_L &\leq \|g_1\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)} + \left|\int_D f\overline{w}\right| \leq \frac{M}{2}\|w\|^2_{L^2(\Gamma)} + \frac{1}{2M}\|g_1\|^2_{L^2(\Gamma)} + \left|\int_D f\overline{w}\right| \\ &\leq \frac{M}{2}\frac{C}{\sqrt{k}}(A_L + A_H) + \frac{1}{2M}\|g_1\|^2_{L^2(\Gamma)} + \left|\int_D f\overline{w}\right| \end{aligned}$$

If M is chosen such that $dM \leq \sqrt{k}$, then the following inequality holds,

$$\begin{aligned} \frac{1}{2}A_L &\leq (1 - \frac{dM}{2\sqrt{k}})A_L \leq \frac{M}{2\sqrt{k}}A_H + \frac{1}{2M}\|g\|^2_{L^2(\Gamma)} + \left|\int_D f\overline{w}\right| \\ A_L &\leq \frac{M}{\sqrt{k}}A_H + \frac{1}{M}\|g\|^2_{L^2(\Gamma)} + 2\left|\int_D f\overline{w}\right| \end{aligned}$$

Considering the results obtained in Equation (4.20) and Equation (4.21) together, the estimate for $\|w\|^2_{L^2(\Gamma)}$ could be arrived as shown below.

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{d}{\sqrt{k}}\left(\frac{M}{\sqrt{k}}A_H + \frac{1}{M}\|g\|^2_{L^2(\Gamma)} + 2\left|\int_D f\overline{w}\right|\right) + \frac{d}{\sqrt{k}}A_H. \tag{4.21}$$

Choose $M = \frac{\sqrt{k}}{d}$, then

$$\|w\|^2_{L^2(\Gamma)} \leq \frac{C}{\sqrt{k}} A_H + \frac{C}{k}\|g\|^2_{L^2(\Gamma)} + \frac{C}{\sqrt{k}} \left| \int_D f\overline{w} \right| . \tag{4.22}$$

□

This is an important lemma as it relates the integrals of $|w|^2$ and $\sqrt{|\xi^2 - k^2|}|\widehat{w}|^2$ to the high frequency part A_H and also norms related to g_1. Now we will estimate each term on the right hand side of Equation (4.17), which are named I_1 through I_6. As most of the details are similar to the previous chapter, the only difference is the power of k as stated in the Remark, thus the details are skipped and only the estimates are given as below.

Lemma 4.2.7. *Similar to the computations as shown in Chapter 3, one can show that for $\varepsilon > 0$,*

$$\begin{aligned}
I_1 &= \Re \int_\Gamma g_1 \overline{w} \leq \frac{\varepsilon_1}{\sqrt{k}} A_H + \frac{C}{\varepsilon_1 k}\|g_1\|^2_{L^2(\Gamma)} + \frac{\varepsilon}{4}\int_D |w|^2 + \frac{C}{\varepsilon k}\int_D |f|^2, \\
I_2 &= \Re \int_D f\overline{w} \leq \frac{\varepsilon}{4}\int_D |w|^2 + \frac{C}{\varepsilon}\int_D |f|^2, && (4.23)\\
I_3 &= 2y_2 \int_{|\xi| \leq k} (k^2 - \xi^2)|\widehat{w}|^2 d\xi \\
&\leq \varepsilon_1 A_H + \frac{C}{\varepsilon_1} k^{\frac{3}{2}} \|g_1\|^2_{L^2(\Gamma)} + \frac{\varepsilon}{4}\int_D |w|^2 + \frac{C}{\varepsilon} k^2 \int_D |f|^2, && (4.24)\\
I_4 &= 2y_2 \Re \int_\Gamma T(w)\overline{g_1} \\
&\leq \varepsilon_1 A_H + \frac{C}{\varepsilon_1 \sqrt{k}}(\|g_1\|^2_{L^2(\Gamma)} + \|g_{1x}\|^2_{L^2(\Gamma)}) + \frac{\varepsilon}{4}\int_D |w|^2 + \frac{C}{\varepsilon}\int_D |f|^2, && (4.25)\\
I_6 &= 2\Re \int_D f y \overline{w}_y \leq \varepsilon_1 \int_D |w_y|^2 + \frac{C}{\varepsilon_1}\int_D |f|^2. && (4.26)
\end{aligned}$$

Combine all the estimates above, we have

$$\begin{aligned}
&(1-\varepsilon_1)\int_D |w_y|^2 + (1-3\varepsilon_1)\int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{w}|^2 d\xi \\
\leq &\frac{C}{\varepsilon}\left[k^{\frac{3}{2}}\|g_1\|^2_{L^2(\Gamma)} + \frac{1}{\sqrt{k}}\|g_{1x}\|^2_{L^2(\Gamma)}\right] + \varepsilon\int_D |w|^2 + \frac{C}{\varepsilon}k^2\int_D |f|^2,
\end{aligned} \tag{4.27}$$

As long as ε_1 is chosen as a positive number such that $1-3\varepsilon_1 \leq 0$, then we have

$$\begin{aligned}
&\int_D |w_y|^2 + \int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{w}|^2 d\xi \\
\leq &Ck^{\frac{3}{2}}\|g_1\|^2_{L^2(\Gamma)} + \frac{C}{\sqrt{k}}\|g_{1x}\|^2_{L^2(\Gamma)} + \varepsilon\int_D |w|^2 + \frac{C}{\varepsilon}k^2\int_D |f|^2,
\end{aligned} \tag{4.28}$$

So far, on the left hand side of the inequality, notice that only partial energy term is there, that is, $\int_D |w_y|^2$, and this inequality shows how $\int_D |w_y|^2$ is controlled by $\int_D |w|^2$ and other terms. To establish an energy estimate, the inequality of the reverse order, that is, how $\int_D |w|^2$ could be controlled by $\int_D |w_y|^2$ needs to be derived, which is the main goal of the next section.

4.3 Relation between $\int_D |w_y|^2$ and $\int_D |w|^2$

Using the divergence theorem, we see that

$$0 = \int_{\partial D}(y-y_2)|w|^2 n_y = \int_D [(y-y_2)|w|^2]_y = \int_D |w|^2 + 2\Re(y-y_2)w\overline{w_y}.$$

Then we can estimate $\int_D |w|^2$ by Schwarz inequality,

$$\int_D |w|^2 = 2\Re\int_D (y_2-y)w\overline{w_y} \leq \varepsilon\int_D |w|^2 + \frac{C}{\varepsilon}\int_D |w_y|^2.$$

Therefore, choose any $\varepsilon < 1$, it shows that

$$\int_D |w|^2 \leq C \int_D |w_y|^2. \tag{4.29}$$

4.4 Final Estimates

Note that earlier we have the following result,

$$\begin{aligned}
&\int_D |w_y|^2 + \int_{|\xi|>k} \sqrt{\xi^2 - k^2} |\widehat{w}|^2 d\xi \\
\leq& C \left[k^{\frac{3}{2}} \|g_1\|^2_{L^2(\Gamma)} + \frac{1}{\sqrt{k}} \|g_{1x}\|^2_{L^2(\Gamma)} \right] + \varepsilon \int_D |w|^2 + \frac{C}{\varepsilon} \int_D k^2 |f|^2.
\end{aligned} \tag{4.30}$$

Combine this with Equation (4.29) together, then

$$\begin{aligned}
&\int_D k^2 |w|^2 \leq Ck^2 \int_D |w_y|^2 \\
\leq& C \left[k^{\frac{7}{2}} \|g_1\|^2_{L^2(\Gamma)} + k^{\frac{3}{2}} \|g_{1x}\|^2_{L^2(\Gamma)} \right] + \frac{C}{\varepsilon} k^4 \int_D |f|^2 + \varepsilon \int_D k^2 |w|^2.
\end{aligned}$$

Note that again $\varepsilon \int_D k^2 |w|^2$ could be absorbed by the left hand side term, thus follows that the term $\int_D k^2 |w|^2$ could be estimated by

$$\int_D k^2 |w|^2 \leq C \left[k^{\frac{7}{2}} \|g_1\|^2_{L^2(\Gamma)} + k^{\frac{3}{2}} \|g_{1x}\|^2_{L^2(\Gamma)} + k^4 \int_D |f|^2 \right]. \tag{4.31}$$

Recall the second identity in Lemma 4.2.1,

$$\|\nabla w\|^2_{L^2(D)} - k^2 \|w\|^2_{L^2(D)} + \int_{|\xi|>k} \sqrt{\xi^2 - k^2} |\widehat{u}|^2 d\xi = \Re \left(\int_\Gamma g_1 \overline{w} + \int_D f \overline{w} \right). \tag{4.32}$$

Use the above identity and Equation (4.31), we have

$$
\begin{aligned}
&\|\nabla w\|^2_{L^2(D)} \leq k^2\|w\|^2_{L^2(D)} - \int_{|\xi|>k}\sqrt{\xi^2-k^2}|\widehat{u}|^2 d\xi + \Re\left(\int_\Gamma g_1\overline{w} + \int_D f\overline{w}\right) \\
\leq & k^2\|w\|^2_{L^2(D)} - A_H + \varepsilon\|w\|^2_{L^2(\Gamma)} + C\|g_1\|^2_{L^2(\Gamma)} + \varepsilon\|w\|^2_{L^2(D)} + C\|f\|^2_{L^2(D)} \\
\leq & k^2\|w\|^2_{L^2(D)} - A_H + \varepsilon\left(\frac{C}{\sqrt{k}}A_H + \frac{C}{k}\|g\|^2_{L^2(\Gamma)} + \frac{C}{\sqrt{k}}\left|\int_D f\overline{w}\right|\right) \qquad (4.33) \\
& + C\|g_1\|^2_{L^2(\Gamma)} + \varepsilon\|w\|^2_{L^2(D)} + C\|f\|^2_{L^2(D)} \\
\leq & C\left[k^{\frac{7}{2}}\|g_1\|^2_{L^2(\Gamma)} + k^{\frac{3}{2}}\|g_{1x}\|^2_{L^2(\Gamma)} + k^4\int_D |f|^2\right].
\end{aligned}
$$

Therefore, an estimate regarding to the wave energy of w is derived:

$$
k^2\|w\|^2_{L^2(D)} + \|\nabla w\|^2_{L^2(D)} \leq C\left[k^{\frac{7}{2}}\|g_1\|^2_{L^2(\Gamma)} + k^{\frac{3}{2}}\|g_{1x}\|^2_{L^2(\Gamma)} + k^4\int_D |f|^2\right]. \qquad (4.34)
$$

Next, we will use this relation together with the result in Section 3 for further analysis. As indicated in the previous subsection, an inequality related to the wave energy of w is derived, which could be controlled by norms of f and g_1. Recall that in the previous chapter, the following three inequalities are obtained, for which we can use to show the final estimate.

$$
\begin{gathered}
\int_D |f|^2 = \int_{\Omega_1} |f|^2 + \int_{\Omega_2} |f|^2 \leq C(|g(0)|^2 + |g(\pi)|^2), \\
\int_\Gamma |g_1|^2 \leq C(|g(0)|^2 + |g(\pi)|^2) + \int_\Gamma |g|^2, \\
\int_\Gamma |g_{1x}|^2 \leq Ck^2(|g(0)|^2 + |g(\pi)|^2) + \int_\Gamma |g_x|^2.
\end{gathered}
$$

By using these facts in Equation (4.34), it yields that

$$\|\nabla w\|^2_{L^2(D)} + k^2\|w\|^2_{L^2(D)} \leq C\left[k^{\frac{7}{2}}\int_\Gamma |g|^2 + k^{\frac{3}{2}}\int_\Gamma |g_x|^2 + k^4(|g(0)|^2 + |g(\pi)|^2)\right].$$

Note that $w = u + u^*$, and for u^*,

$$\|\nabla u^*\|^2_{L^2(D)} + k^2\|u^*\|^2_{L^2(D)} \leq \frac{C}{k^2}(|g(0)|^2 + |g(\pi)|^2),$$

thus, the final estimate could be established accordingly, that is,

$$\|\nabla u\|^2_{L^2(D)} + k^2\|u\|^2_{L^2(D)} \leq C\left[k^{\frac{7}{2}}\int_\Gamma |g|^2 + k^{\frac{3}{2}}\int_\Gamma |g_x|^2 + k^4(|g(0)|^2 + |g(\pi)|^2)\right].$$

Chapter 5

Conclusion and Some Open Questions

This work is a preliminary endeavor in studying the stability estimates of electromagnetic scattering from open cavity. We focus on the study of TM case and assume the media is homogeneous. Our main contribution is that explicit estimates are derived, on how the wave energy in the cavity region depends on the wave number k and incoming fields. We have provided the explicit relations in the context of different geometric features of the cavity domain using different techniques. The stability estimates can provide guidance for the numerical computations, as well as provide insights on the shape design of cavities.

On the other hand, the numerical results may give evidence on the dependency relations and help to find the optimal power dependency in terms of wave number k, which is an interesting topic to explore in the future. Also, the closely related problem would be the stability estimate for TE(Transverse Electric) polarization case in the two dimensional setting, where the bounded domain formulation is given by

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } D, \\ \partial_{\mathbf{n}} u = 0 \qquad\qquad \text{on } S, \\ u = T(\partial_{\mathbf{n}} u) + g \text{ on } \Gamma, \end{cases} \tag{5.1}$$

with $\widehat{T(\partial_{\mathbf{n}} u)} = \dfrac{i}{\sqrt{k^2 - \xi^2}} \widehat{\partial_{\mathbf{n}} u}$. The major difference lies in two parts: one is the boundary condition in particular the multiplication operator; another is that while in TM case u is supported on Γ, here u_y is supported on Γ while u and u_x may not be, a natural extension of the proof for TM case would fail in TE case; thus additional techniques are expected to derive the stability estimates. As long as the stability estimate for TE case could be obtained given the geometric assumption as in TM case, then the two dimensional problem would be solved under same geometric setting since the general case could always be decomposed into the sum of TM and TE case. For three dimensional problem, the bounded domain formulation is given by [4]

$$\begin{cases} \nabla \times E - i\omega\mu H = 0, \\ \nabla \times H + i\omega\epsilon E = 0, \\ \mathbf{n} \times E = 0 & \text{on } S, \\ -\mathbf{n} \times (\mathbf{n} \times H) = P(-\mathbf{n} \times (\mathbf{n} \times E)) + \mathbf{g} & \text{on } \Gamma. \end{cases} \tag{5.2}$$

where

$$-g^i = -\mathbf{n} \times (\mathbf{n} \times (E^i - pe^{ik_0 q\cdot x})) + P(-\mathbf{n} \times (\mathbf{n} \times (E^i - pe^{ik_0 q^*\cdot x})))$$

and

$$\begin{aligned} &P(-\mathbf{n} \times (\mathbf{n} \times E)) \\ =&\frac{1}{(2\pi)^2\omega\mu_0}\int_{\mathbb{R}^2}\{[-\frac{1}{\sqrt{k_0^2-\xi_1^2-\xi_2^2}}(\xi_1\widehat{g}_1+\xi_2\widehat{g}_2)\xi_2-\sqrt{k_0^2-\xi_1^2-\xi_2^2}\widehat{g}_2], \\ &\frac{1}{k_0^2-\xi_1^2-\xi_2^2}(\xi_1\widehat{g}_1+\xi_2\widehat{g}_2)\xi_1-\sqrt{k_0^2-\xi_1^2-\xi_2^2}\widehat{g}_1],0\}e^{i\xi\cdot x}d\xi, \end{aligned}$$

with $g_j(x_1,x_2) = E^j(x_1,x_2,0), j = 1,2,3$. Existence and uniqueness of the solutions for the model problem are established by a variational approach and the Hodge decomposition, the stability estimates are not derived yet. Furthermore the electromagnetic scattering in layered media, which is significantly important in many areas such as optics, geophysical probing, communication, remote sensing[12, 31], etc. Another interesting problem is concerned with the optimal design problem in inverse scattering, in aims to design the cavity shape and material to reduce or enhance the radar cross section [22]. For inverse scattering, there are some local stability result obtained for periodic structures and biperiodic structures [6, 9], and Li [22] proved the local stability for one particular case of the cavity problem, where the upper halfspace is filled with a lossless homogeneous medium above the flat ground surface; while the interior of the cavity is assumed to be filled with a lossy homogeneous medium accounting for the energy absorption; more stability results could be explored based on different consideration of the medium. In a word, there are still many open questions in terms of stability estimates in cavity problem, which are theoretically interesting and challenging.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] G. Acostaa, M. G. Armentanob, R. G. Durάb, and A. L. Lombardib. Nonhomogeneous neumann problem for the poisson equation in domains with an external cusp. *Journal of Mathematical Analysis and Applications*, 310(2):397–411, 2005.

[2] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[3] H. Ammari, G. Bao, and A. W. Wood. Analysis of the electromagnetic scattering from a cavity. *Japan J. Indust. Appl. Math.*, 19(2):301–310, 2002.

[4] H. Ammari, G. Bao, and A. W. Wood. A cavity problem for maxwell's equations. *Methods and applications of analysis*, 9(2):249–260, 2002.

[5] H. Anastassiu. A review of electromagnetic scattering analysis for inlets, cavities, and open ducts. *IEEE Antennas Propag. Mag.*, 45(6):27–40, 2003.

[6] G. Bao and A. Friedman. Inverse problems for scattering by periodic structures. *Arch. Ration. Mech. Anal.*, 132:49–72, 1995.

[7] G. Bao and W. Sun. A fast algorithm for the electromagnetic scattering from a large cavity. *SIAM J. Sci. Comput.*, 27:553–574, 2005.

[8] G. Bao, K. Yun, and Z. Zhou. Stability of the scattering from a large electromagnetic cavity in two dimensions. *SIAM J. Math. Anal.*, 44(1):383–404, 2012.

[9] G. Bao and Z. Zhou. An inverse problem for scattering by a doubly periodic structure. *Trans. Amer. Math. Soc.*, 350:4089–4103, 1998.

[10] R. J. Burkholder. A fast and rapidly convergent iterative physical optics algorithm for computing the rcs of open-ended cavities. *Applied Computational Electromagnetics Society Journal*, 16(1):53–60, 2001.

[11] R. J. Burkholder, R. C. Chou, and P. H. Pathak. Two ray shooting methods for computing the EM scattering by large open-ended cavities. *Computer Physics Communications*, 68(1-3):353–365, 1991.

[12] W. C. Chew. *Waves and Fields in Inhomogeneous Media.* IEEE Press, 1995.

[13] T. T. Chia, R. J. Burkholder, and R. Lee. The application of fdtd in hybrid methods for cavity scattering analysis. *IEEE Trans. Antennas Propagat.*, 43(10):1082–1090, 1995.

[14] F. Cummings and F. X. Sharp regularity coefficient estimates for complex-valued acoustic and elastic helmholtz equations. *Math. Models Methods. Appl. Sci.*, 16(1):139–160, 2006.

[15] S. Esterhazy and J. Melenk. On stability of discretizations of the helmholtz equation (extended version). *ASC Report*, No. 01:1 – 49, 2011.

[16] U. Hetmaniuk. Stability estimates for a class of helmholtz problems. *Commun. Math. Sci.*, 5(3):665 – 678, 2007.

[17] R. Hiptmair, A. Moiola, and I. Perugia. Stability results for the time-harmonic maxwell equations with impedance boundary conditions. *Mathematical Models and Methods in Applied Sciences (M3AS)*, 21(11):2263–2287, 2011.

[18] F. Ihlenburg and I. Babuska. Finite element solution of the Helmholtz equation with high wave number part I: The h-Version of the FEM. *Computers Math. Applic.*, 30(9):9 – 37, 1995.

[19] F. Ihlenburg and I. Babuska. Finite element solution of the helmholtz equation with high wave number part II: The h-p Version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[20] J. M. Jin and J. Volakis. A finite-element-boundary integral formulation for scattering by three-dimensional cavity-backed apertures. *IEEE Trans. Antennas Propagat.*, 39(1):97–104, 1991.

[21] C. S. Lee and S. W. Lee. RCS of a coated circular waveguide terminated by a perfect conductor. *IEEE Trans. Antennas Propagat.*, 35(4):391–398, 1987.

[22] P. J. Li, H. J. Wu, and W. Y. Zheng. An overfilled cavity problem for maxwell's equations. *Mathematical Methods in the Applied Sciences*, 35(16):1951–1979, 2012.

[23] H. Ling, R. C. Chou, and S. W. Lee. Shooting and bouncing rays: calculating the rcs of an arbitrarily shaped cavity. *IEEE Trans. Antennas Propagat.*, 37(2):194–205, 1989.

[24] H. Ling, S. W. Lee, and R. C. Chou. High-frequency RCS of open cavities with rectangular and circular cross sections. *IEEE Trans. Antennas Propagat.*, 37(5):648–654, 1989.

[25] J. Melenk. *On Generalized Finite Element Methods.* PhD thesis, The University of Maryland, 1995.

[26] J. W. Moll and R. G. Seecamp. Calculation of radar reflecting properties of jet engine intakes using a waveguide model. *IEEE Trans. Aerospace Electron. Syst.*, AES-6(5):675–683, 1970.

[27] F. Obelleiro, J. L. Rodriguez, and A. G. Pino. A progressive physical optics (PPO) method for computing the electromagnetic scattering of large open-ended cavities. *Microwave and optical technology letters*, 14(3):166–169, 1997.

[28] F. Obelleiro-Basteiro, J. L. Rodrı́uez, and R. J. Burkholder. An iterative physical optics approach for analyzing the electromagnetic scattering by large open-ended cavities. *IEEE Trans. Antennas Propagat.*, 43(4):356–361, 1995.

[29] P. H. Pathak and R. J. Burkholder. High-frequency electromagnetic scattering by open-ended waveguide cavities. *Radio Science*, 26(1):211–218, 1991.

[30] H. Peot. An electronic umbrella: the B1B defensive avionics systems. *Defense Syst. Rev.*, 2:15–18, 1984.

[31] F. Seydou, R. Duraiswami, N. A. Gumerov, and T. Seppänen. TM Electromagnetic Scattering from 2D Multilayered Dielectric Bodies - Numerical Solution. *Applied Computational Electromagnetics Society Journal*, 19(2):100–107, 2004.

[32] T. Van and A. W. Wood. Analysis of transient electromagnetic scattering from overfilled cavities. *SIAM J. Appl. Math.*, 64(2):688–708, 2003.

[33] Y. Wang, K. Du, and W. Sun. A second-order method for the electromagnetic scattering from a large cavity. *Numer. Math. Theor. Meth. Appl.*, 1(4):357–382, 2008.