APPLICATIONS OF GEOMETRIC MEASURE THEORY TO COMPLEX AND QUASICONFORMAL ANALYSIS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics — Doctor of Philosophy

2018

ABSTRACT

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There are many intersections between complex analysis, geometric measure theory, and harmonic analysis; the interactions between these fields yield many important results and applications. In this work, we focus on two aspects of these connections: the regularity theory of quasiconformal maps and the quantitative study of rectifiable sets.

Quasiconformal maps are orientation-preserving homeomorphisms that satisfy certain distortion inequalities; infinitesimally, they map circles to ellipses of uniformly bounded eccentricity. Such maps have many useful geometric distortion properties, and yield a flexible and powerful generalization of conformal mappings. These maps arise naturally in the study of elasticity, in complex dynamics, and in the analysis of partial differential equations. We study the singularities of these maps; in particular, we consider the size and structure of the sets where a quasiconformal map can exhibit given stretching and rotation behavior. We improve the previously known results to give examples of stretching and rotation sets with non-sigma-finite measure at the critical Hausdorff dimension. We further improve this to give examples with positive Riesz capacity at the critical homogeneity, as well as positivity of measure for a broad class of gauged Hausdorff measures at the critical dimension.

The local distortion properties of quasiconformal maps also give rise to a certain degree of global regularity and Hölder continuity. We give new lower bounds for the Hölder continuity of these maps, relating both the structure of the underlying partial differential equation for the maps and the geometric distortion they can exhibit; the analysis is based on combining the isoperimetric inequality with a study of the length of quasicircles. Furthermore, the extremizers for Hölder continuity are characterized, and we give a natural application to solutions of elliptic partial differential equations.

Finally, given a set in the plane, the average length of its projections in all directions is called the Favard length of a set; it is closely related to the Buffon needle probability of the set. This quantity measures the size and structure of a set, and is closely related to metric and geometric properties of the set such as rectifiability, Hausdorff dimension, and analytic capacity. We develop new geometrically motivated techniques for estimating Favard length. We will give a new proof relating Hausdorff dimension to the decay rate of the Favard length of neighborhoods of a set. We will also show that, for a large class of self-similar one-dimensional sets, the sequence of Favard lengths of the generations of the set is convex; this leads directly to lower bounds on Favard length for various fractal sets.

ACKNOWLEDGMENTS

Love. You can learn all the math in the 'Verse, but you take a boat in the air that you don't love, she'll shake you off just as sure as the turning of the worlds. Love keeps her in the air when she oughta fall down, tells you she's hurtin' 'fore she keens. Makes her a home.

- *Malcolm Reynolds, Serenity*

These years of study have been a wonderful and amazing experience, especially because of the kindness and support of the people I have encountered. So many people have contributed to my work, and I owe them all a great debt of gratitude; none of this would have been possible without them. First and foremost, I am thankful for my advisor, Ignacio Uriarte-Tuero. Beginning with the little bit of geometric measure theory you showed us as qual students, you have continually motivated, aided, and taught me. Your advice and counsel over the years have been absolutely indispensable, and your decency and humility are inspiring.

Embarking on this journey as a graduate student would not have been possible without the incredible education and opportunities I found at Colorado State University - Pueblo. I am especially thankful to have met Darren Funk-Neubauer; taking your calculus class and talking to you at math club was my original inspiration for changing my major to math. Bruce Lundberg taught my first real and complex analysis classes and supervised my undergraduate research, setting me on the path I've followed since; I learned so much from you, and you were always happy to answer my questions. Your kindness, advice, and hospitality whenever I am back in Pueblo have been wonderful. I had my first reading course with James Louisell; seeing the beauty of advanced math for the first time had a huge impact on me, and your advice to apply to Michigan State was invaluable. Janet Barnett's historical perspective and incredible passion for teaching inspired me as an educator. Jonathan Poritz's reading course in number theory showed me some of the great beauty of complex analysis. Learning chemistry from Rick Farrer was incredible; your energy and passion made for a great experience. I am also very thankful for the courses and discussions I was able to have with Paul Chacon, Rick Kreminski, and Marta Wallin. My REU with Jorge Garcia at CSUCI was an amazing experience, and I am happy I had the chance to work with Walter. Finally, I owe a great deal to my time studying with Brena, Travis, Dan, and Gary; the discussions we had and the help you gave me were incredibly useful, and your friendship meant so much to me.

The faculty and staff of Michigan State have been a constant source of help for me over the years. I am deeply grateful to Tsveta Sendova, Andy Krause, Alec Drachman, and Jane Zimmerman; without the support, advice, and opportunities you gave me, I would not have developed into the teacher I am today. I am so happy that I had opportunities to take courses with and learn from so many people, and I am especially thankful for the work of my committee members Alexander Volberg, Dapeng Zhan, and Gabor Francsics. Many other people have contributed to my growth as a graduate student, whether through discussing math or careers or teaching: Jun Kitagawa, Russell Schwab, Sheldon Newhouse, Jeanne Wald, James Dudziak, Vladimir Peller, Max Gilula, Jonathan Hall, Zhengfang Zhou, and Shiv Karunakaran. I am also especially thankful for the help of the departmental staff, especially Debie Lecato and Leslie Aitcheson.

This work was generously financially supported by several funding agencies. I am grateful for the support of the Michigan State University College of Natural Science via fellowships, as well as the National Science Foundation under grant DMS-1056965.

My time in graduate school would not have been possible without the support of my friends here. Especially to Ben and Lauren - those times of Beggars, haiku, Halo, and climbing will stay with me forever, and your friendship changed my life. My path into analysis would not have been the same without Guillermo, José, and Paata; your passion for math and all the tools and ideas you showed me over the years were really wonderful. I owe a great debt to all those people that I studied with throughout those first years; especially Chris and Tom. I am glad I had the chance to share an office with Emily; your skill and advice in teaching has been so helpful to me. My time here has also been brightened by so many other wonderful people, especially Hitesh, Sami, Charlotte, Rami, Rani, Christos, Michael, and Jingyun. I am so deeply thankful for Reshma; your companionship and loving support mean more to me than you can know and you've brought so much joy into my life. *Meri zindagi mein aaye ho aur aise aaye ho tum, jo ghul gayaa hai saanson mein vo geet laaye ho tum.*

Finally, and most importantly, I am grateful for my family. Mom and Dad, you made me who I am today. You didn't just teach me in school, you taught me how to live; it was only with your support that I was able to pursue my dreams. Gram, you've always been there for me and that's meant the world. Grandpa, even from those times making paper airplanes over pancakes, you inspired me to pursue science and math. Grandma Betty and Grandpa Ken and Uncle Frank, your loving support has been deeply important to me.

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East Lansing, Michigan

27 March 2018

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KEY TO SYMBOLS

The following is a list of some of the notation used throughout this paper.

- $\hat{\mathbb{C}}$, the Riemann sphere
- $\mathbb{D}$, the unit disk in the complex plane
- $B(z, r)$, the closed disk of radius r centered at z
- ∂_α, the directional derivative in direction α
- Df, the differential matrix of $f = u + iv$,

$$Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

- $|Df|$, the operator norm of the matrix Df
- ∂_z, the complex derivative $\frac{1}{2}(\partial_x - i\partial_y)$
- $\partial_{\overline{z}}$, the $\overline{z}-$derivative $\frac{1}{2}(\partial_x + i\partial_y)$
- $|E|$, Lebesgue measure of E; could be one- or two-dimensional according to context
- $\mathcal{H}^d$, the $d-$dimensional Hausdorff measure
- $C^\alpha(\Omega)$, the set of locally α-Hölder continuous functions on a domain Ω; we will also refer to these functions as belonging to a Lipschitz class
- J_f, the Jacobian of a function f
- $L^p(\Omega)$, the space of functions f such that $|f|^p$ is integrable on Ω.
- $L^p_{\text{loc}}(\Omega)$, the space of functions f such that f^p is integrable on each compact subset of Ω
- $A \lesssim B$ if there exists a constant C (possibly depending on parameters, but not A and B) such that $A \leq CB$
- $A \sim B$ if $A \lesssim B$ and $B \lesssim A$; that is, there exists a universal C such that $\frac{1}{C}B \leq A \leq CB$
- $W^{1,p}(\Omega)$, the space of functions in $L^p(\Omega)$ with one weak derivative also in $L^p(\Omega)$
- $W^{1,p}_{\text{loc}}(\Omega)$, the space of functions in $L^p_{\text{loc}}(\Omega)$ with one weak derivative also in $L^p_{\text{loc}}(\Omega)$

Chapter 1

Introduction

Quasiconformal maps were first introduced by Grötzsch [32] in order to study the following problem:

Problem 1.0.1. *Given rectangles R and S in the plane, find a homeomorphism $f : R \to S$ which is as close to a conformal map as possible.*

This problem is closely related to modules of ring domains and extremal length [3]. The Riemann mapping theorem shows a conformal map between a connected domain Ω with smooth boundary is uniquely specified by its values at three boundary points. Therefore, once the images under a conformal map of three vertices of a rectangle are given, the fourth is constrained. This suggests that the geometry of the rectangle is important (that is, the ratio of the side lengths), and that a different class of functions is needed. These functions ought to allow more distortion than conformal maps can, and so we must have a more flexible definition than conformality allows.

This leads to a natural definition of quasiconformality. In order to discuss the infinitesimal distortion, it is useful to impose some degree of regularity on the maps; as such, as will assume that they lie in the Sobolev space $W^{1,2}_{\text{loc}}$ of functions with one (locally) square integrable weak derivative.

Definition 1.0.2. *Let $f : \Omega \to \Omega'$ be an orientation-preserving homeomorphism of two domains in the complex plane, and assume that $f \in W^{1,2}_{loc}(\Omega)$. We say that f is K-*

quasiconformal if at almost every $z \in \Omega$, we have

$$\max_\alpha |\partial_\alpha f| \leq K \min_\alpha |\partial_\alpha f| \tag{1.0.1}$$

where ∂_α is a directional derivative in direction α. We say that f is quasiconformal if there exists a $K \in [1, \infty)$ such that f is K-quasiconformal.

This inequality can be interpreted geometrically: a K-quasiconformal map sends infinitesimal circles to infinitesimal ellipses with uniformly bounded eccentricity; it is precisely this change between circles and ellipses that provides the flexibility to study Grötzsch's problem.

Since 1-quasiconformal maps are conformal, we now have a useful generalization of the idea of conformality. The freedom gained by relaxing the definition of conformality makes quasiconformal maps appropriate for many applications. They arise naturally in complex dynamics (due to their connections with holomorphic motions), in elasticity problems (from their geometric distortion properties), in fluid dynamics and partial differential equations (by generalizing harmonicity), and many other areas. For a general overview of quasiconformal maps, see the books by Astala, Iwaniec and Martin [8] or Ahlfors [4].

Many of the analytic properties of quasiconformal maps follow from a differential inequality equivalent to (1.0.1), as discussed throughout [8]. It is possible to rewrite (1.0.1) as $|Df(z)|^2 \leq KJ_f(z)$. This may be rewritten further using the Wirtinger derivatives f_z and $f_{\overline{z}}$; combining the facts that $|Df| = |f_z| + |f_{\overline{z}}|$ and $J_f = |f_z|^2 - |f_{\overline{z}}|^2$, we find that

$$|f_{\overline{z}}| \leq \frac{K-1}{K+1}|f_z|. \tag{1.0.2}$$

This inequality leads to a partial differential equation underlying any quasiconformal map.

We may define $\mu(z) = f_{\overline{z}}/f_z$ whenever $f_z \neq 0$ (and 0 otherwise, which can only occur on a set of zero measure); therefore any K-quasiconformal map satisfies the *Beltrami equation*

$$\partial_{\overline{z}} f = \mu(z) \partial_z f \quad \text{with} \quad \|\mu\|_{L^\infty(\mathbb{C})} \leq \frac{K-1}{K+1}. \tag{1.0.3}$$

The function μ is called the *Beltrami coefficient*, and represents the complex dilatation of the map f. This representation allows the usage of the techniques of elliptic PDEs to study quasiconformal maps.

Additionally, this equation gives a tool for constructing quasiconformal maps with given dilatation. Given a compactly supported coefficient μ satisfying $\|\mu\|_\infty \leq \frac{K-1}{K+1}$, there is a (unique, once normalized appropriately) K-quasiconformal map satisfying $\partial_{\overline{z}} f = \mu \partial_z f$; this is the measurable Riemann mapping theorem of Ahlfors and Bers [2]. The proof of this unites harmonic analysis and complex analysis: the solution can be written explicitly with the Beurling transform, which is a singular integral operator that intertwines ∂_z and $\partial_{\overline{z}}$. The regularity properties of quasiconformal maps (such as integrability of the derivatives) are intimately related with the operator-theoretic properties of the Beurling transform and are connected to important open problems about the L^p spaces where the Beurling transform is bounded (see, e.g. [35]).

Many modern results about quasiconformal maps, including Astala's area distortion theorem and the consequential distortion of Hausdorff dimension [7] (as well as the sharp integrability and applications to elasticity of [13]) are proven by combining harmonic analysis techniques with complex dynamics. Of fundamental importance is the fact that quasiconformal maps naturally embed within a global holomorphic motion (see, e.g. [40, 58] and Chapter 12 of [8] for some of the development of this theory). By using such an embedding, we may introduce the techniques of classical complex analysis (such as growth estimates for bounded analytic functions) to analyze the regularity of a map and the integrability of its derivatives. This leads to many of the self-improvement properties of quasiconformal maps; for example, a quasiconformal map lies within a Sobolev space with more integrability than $W^{1,2}_{\text{loc}}$ suggests.

In Chapters 2 and 3 of this work, we will consider some aspects of the regularity theory of quasiconformal maps. In Chapter 2, we will consider the notion of stretching and rotation for a quasiconformal map (giving a more precise notion of Hölder continuity and adapting it to the argument of a map) from [10] and give examples of very irregular quasiconformal maps. These maps will be irregular on a set of positive Riesz capacity and non-σ-finite Hausdorff measure at the appropriate dimension, showing a fundamental limitation in the techniques used to prove regularity. The constructions here are motivated by techniques of geometric measure theory, and will involve highly non-self-similar Cantor sets. Chapter 3 will go on to demonstrate new bounds on the Lipschitz class that a quasiconformal map lies in, based on its complex dilatation and the local geometric distortion of a map. We will also give a classification of the quasiconformal extremizers for regularity.

Another aspect of the connection between complex analysis, harmonic analysis, and geometric measure theory is in the study of rectifiability. A set $E \subseteq \mathbb{C}$ is called *rectifiable* if there is a countable collection of Lipschitz maps $f_i : \mathbb{R} \to \mathbb{C}$ such that $E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R})$ has zero length. A set is called purely unrectifiable if it has no subset of positive length which is rectifiable. Rectifiability is closely related to the geometry of a set: smooth curves are rectifiable, while highly irregular sets such as Cantor sets are not. Morally, rectifiable sets and Lipschitz maps play a role analogous to manifolds and smooth charts.

Unrectifiable sets can be detected using Favard length:

Definition 1.0.3. *The Favard length of a set* $E \subseteq \mathbb{C}$ *is*

$$\mathrm{Fav}(E) = \int_0^{2\pi} |\pi_\theta E| \, d\theta \tag{1.0.4}$$

where π_θ *is projection onto a line through the origin at angle* θ *above the positive* $x-$*axis and* $|\cdot|$ *is the length measure within this line.*

Favard length gives a geometrically motivated measure of the size of a set, and carries important metric and geometric information about the set. It is comparable to the *Buffon needle probability*; that is, the likelihood that a long, thin needle dropped near a set will interstect a given set. A theorem of Besicovitch [17] shows that a set with positive and finite length is unrectifiable if and only if it has Favard length zero. Pure unrectifiability further implies that a set cannot support a measure with positive and finite density with respect to length [16, 54].

Favard length and unrectifiability are closely related to removability problems in complex analysis. A classical question asks for which sets E in the complex plane there are no non-constant, bounded, holomorphic functions on $\mathbb{C} \setminus E$ (in which case E is called removable). It is a deep result that a compact subset of a rectifiable curve is removable if and only if it has length zero; this follows from the L^2-boundedness of the Cauchy transform on Lipschitz graphs (which was first proved by Calderón in [24] for graphs with small Lipschitz constant, and later improved by many authors; see, for example, [26]).

As a consequence of Besicovitch's theorem, the Favard lengths of the r-neighborhoods of bounded unrectifiable sets of finite length must tend to zero as the scale r does. A more quantitative understanding of this fact motivates the following problem:

Problem 1.0.4. *How is the decay rate of the Favard length of the neighborhoods of a set* *related to the metric and geometric properties of the set?*

In Chapter 4, we consider several aspects of this problem. First, we give a new proof that sufficiently fast decay leads to upper bounds on Hausdorff dimension. We also develop a new geometrically motivated technique for estimating the Favard length decay for a large class of self-similar unrectifiable sets in the plane; in particular, this work applies to the four-corner Cantor set.

1.1 Regularity of Quasiconformal Maps

Before coming to the main problems considered in later chapters, we will outline the current state of the art of regularity theory for quasiconformal maps. It is well known due to Mori [46] that a K-quasiconformal map f is Hölder continuous with exponent $\alpha \geq 1/K$; that is, for any compact set E in the domain Ω of f,

$$\sup_{z,w\in E} \frac{|f(z)-f(w)|}{|z-w|^{\alpha}} < \infty.$$

Alternatively, this can be stated as the result that f lies in the Lipschitz class $C^{\alpha}(\Omega)$.

The degree of smoothness of a quasiconformal map is closely related to the integrability of f_z and J_f. If the Jacobian J_f lies in an L^p_{loc} space, then it is quite direct to show that f is Hölder continuous with exponent at least $1-1/p$. To see this, note that applications of quasisymmetry and Jensen's inequality allow us to estimate that

$$\begin{aligned}|f(z+r)-f(z)|^2 \sim_K \left|f\big(B(z,r)\big)\right| &= \pi r^2 \int_{B(z,r)} J_f \frac{dA}{\pi r^2} \\ &\leq \pi r^2 \left(\int_{B(z,r)} J_f^p \frac{dA}{\pi r^2}\right)^{1/p}\end{aligned}$$

$$\lesssim r^{2-2/p} \|J_f\|_{L^p(B(z,1))}$$
$$\lesssim r^{2-2/p}$$

for $r < 1$. Taking a square root gives the desired result.

Following this idea, it is important to understand the optimal Sobolev class within which quasiconformal maps lie. Improved integrability beyond $W^{1,2}_{\text{loc}}$ was proven by Boyarskiĭ[23], and the sharp Sobolev exponents were conjectured by Gehring and Reich [31]. Integrability up to the sharp exponent $2K/(K-1)$ was proved by Astala in [7] and the question of integrability at the borderline was studied in [13]. In the converse direction, it was conjectured (e.g. in [35]) that Hölder continuity above the exponent $1/K$ implies improved integrability above exponent $2K/(K-1)$; however, this was disproved by Koskela in [38] and further studied by Clop in [25].

Furthermore, recalling that the Beltrami equation (1.0.3) may be solved rather explicitly with the Beurling transform, many authors have studied the L^p bounds for the Beurling transform

$$(\mathcal{S}\varphi)(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\tau)}{(z-\tau)^2}\, d\tau.$$

The integral here may be understood in the principal value sense, as the Beurling transform is a Calderón-Zygmund convolution operator with a nonintegrable kernel. The L^p properties of quasiconformal maps are closely tied to the sharp L^p bounds of $\mathcal{S}$; it is conjectured that $\|\mathcal{S}\|_{L^p \to L^p} = \max\{p-1, \frac{1}{p-1}\}$. The correct asymptotic behavior (that the bound is $O(p)$ as $p \to \infty$) is known, but the sharp bounds are still open; see, for example, [34, 35, 36, 12, 50, 29, 14] for some of the progression of this conjecture. Furthermore, the sharp bounds are closely connected to questions about rank-one convexity and quasiconvexity [11].

Many recent results about regularity hinge upon the fact that quasiconformal maps can be embedded within a holomorphic motion. Given a set A in the Riemann sphere, we say that a map $\Phi : \mathbb{D} \times A \to \hat{\mathbb{C}}$ is a holomorphic motion if Φ is holomorphic in the first variable for any fixed $a \in A$, injective in the second variable for any fixed λ, and $\Phi(0, a) = a$ for all $a \in A$. A fundamental result is the following application of Słodkowski's extended λ-lemma [58] by Astala [7]:

Theorem 1.1.1. *If $\Phi : \mathbb{D} \times A \to \hat{\mathbb{C}}$ is a holomorphic motion of $A \subset \mathbb{C}$, then Φ has an extension to $\tilde{\Phi} : \mathbb{D} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that each $\tilde{\Phi}_\lambda$ is a K-quasiconformal self-homeomorphism of $\hat{\mathbb{C}}$ with*

$$K \leq \frac{1 + |\lambda|}{1 - |\lambda|}.$$

Conversely, if $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, then there is a holomorphic motion $\Phi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ such that $f(z) = F\left(\frac{K-1}{K+1}, z\right)$ for all $z \in \mathbb{C}$.

The utility of this theorem is that it enables the usage of classical techniques from complex and harmonic functions to give estimates for quasiconformal maps. The key idea of the proof of the second statement is to embed a quasiconformal map f with Beltrami coefficient μ into a flow via solving

$$\partial_{\overline{z}} f^\lambda = \lambda \frac{\mu}{\|\mu\|_\infty} \partial_z f^\lambda. \tag{1.1.1}$$

Varying λ allows us to transfer information about the functions f^λ from the origin ($\lambda = 0$, in which case f^0 is the identity) out to other points on the disk, using classical growth estimates of bounded analytic functions on the disk. As a particular usage, consider the following self-improvement property from [7], which has deep consequences:

Theorem 1.1.2. *If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, then $f \in W^{1,p}_{loc}(\mathbb{C})$ for all $p < \frac{2K}{K-1}$.*

The key estimate leading to this result is based on embedding f within a holomorphic flow $f^\lambda(z)$ as indicated in (1.1.1), noting that $\lambda \mapsto |(f^\lambda)'(z)|^2$ is harmonic for each fixed z, and applying Harnack's inequality in λ. In a similar vein, sharpening Harnack's inequality leads to results on quasicircles and quasilines [59, 53].

Complementary to these Hölder regularity results, it is also possible to study more refined notions of smoothness. To this end, we define the stretching and rotation sets of a function, first introduced in [10]:

Definition 1.1.3. *Given $z \in \mathbb{C}$, we say that f stretches with exponent α and rotates with exponent γ at z if there exist scales $r_n \to 0$ with*

$$\lim_{n\to\infty} \frac{\log|f(z+r_n)-f(z)|}{\log r_n} = \alpha \text{ and } \lim_{n\to\infty} \frac{\arg(f(z+r_n)-f(z))}{\log|f(z+r_n)-f(z)|} = \gamma.$$

The set of points z where f stretches with exponent α and rotates with exponent γ simultaneously is denoted $E_f(\alpha,\gamma)$.

Here, the argument is understood in terms of the winding number: it is the number of times that the image of the ray $[z+r_n \to \infty)$ wraps around $f(z)$ (up to a small, but fixed, term depending on the choice of branch of the argument). Note that by applying the classical Hölder regularity results to f and f^{-1}, we are only interested in $\alpha \in [K^{-1}, K]$.

Given Definition 1.1.3, we can now state the main question considered in Chapter 2 of this work:

Problem 1.1.4. *How large can the stretching and rotation set $E_f(\alpha,\gamma)$ of a quasiconformal map be?*

The question is necessarily vague, as there are many (not necessarily compatible) notions of the size of a set. Motivated by [5], we will address several different interpretations of size in Chapter 2.

This question was introduced and studied by Astala, Iwaniec, Prause, and Saksman in [10], in which they gave an upper bound on the size in terms of Hausdorff dimension (as a function of α, γ and K); they also showed this bound to be optimal at the level of dimension. The main idea in giving the upper bound was to use a holomorphic motion first studied in [9] to give efficient bounds on the integrability of $|f_z^\beta|$ for complex exponents β, while the lower bound was proven by a direct construction. Hitruhin sharpened this lower bound in [33] to the level of Hausdorff measure by constructing quasiconformal maps whose stretching and rotation sets have positive and finite Hausdorff measure at the critical dimension.

We improve these results by constructing far larger Cantor sets that are stretching and rotation sets; this is done not only in the sense of Hausdorff measure (where we extend the previous results to both the non-σ-finite setting but also gauged Hausdorff measures), but also in terms of Riesz capacity. By analogy with many results in area distortion (e.g. the studies of distortion of Hausdorff measure and Riesz capacity of [39, 5]), it was expected that the stretching and rotation set could not be larger than σ-finite with respect to the Hausdorff measure at the critical dimension. Thus our result is rather surprising and shows a limitation of the techniques used to prove regularity. The details of this work are the content of Chapter 2.

The constructions in Chapter 2 are heavily based on summing or modifying maps like $z \mapsto z|z|^{\alpha(1+i\gamma)}$. This is the prototypical map for studying stretches and rotations, since it can be easily computed to have $E_f(\alpha, \gamma) = \{0\}$ and $E_f(1, 0) = \mathbb{C} \setminus \{0\}$; it also has a simple Beltrami coefficient. As a complementary question to the work in Chapter 2, we are interested in determining which maps can exhibit the worst-case stretching behavior:

Problem 1.1.5. *Which K-quasiconformal maps are no better than $1/K$-Hölder continuous, and what can we say about the structure of such maps? In an appropriate sense, do they need to be "close" to the pure radial stretch $z|z|^{1/K-1}$?*

We address this question in Chapter 3. It turns out that the appropriate notion of "closeness" is in terms of the Beltrami coefficient. Using the isoperimetric inequality together with computation of the length of quasicircles, we are able to give a new formula for the Hölder exponent of the solutions to $\partial_{\overline{z}} f = \mu \partial_z f$. The new idea here is to vary a technique of Morrey [47]: we may write

$$\int_{\mathbb{D}_t} J_f d\mathcal{L}^2 = |f(\mathbb{D}_t)| = \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1(S_t)^2} \mathcal{H}^1(S_t)^2. \tag{1.1.2}$$

The first term is bounded above by the isoperimetric inequality, while the second term can be explicitly computed in terms of the Jacobian J_f and the Beltrami coefficient μ. Having replaced the integral over $\mathbb{D}_t$ with an integral over its boundary $\mathbb{S}_t$, we can find an inequality for $\int_{\mathbb{D}_t} J_f$ in terms of its t-derivative, leading eventually to pointwise estimates in t. The details of this are provided in Section 2 of Chapter 3.

As a further application of these ideas, we gain substantial new information about the Hölder continuity extremizers. The key realization is that if a K-quasiconformal map is no more regular than $C^{1/K}$, all the inequalities arising from (1.1.2) must be sharp. We may use this to not only constrain the structure of the Beltrami coefficient, but also understand how much distortion such a map can introduce to small circles. This gives an answer to Problem 1.1.5: the Beltrami coefficient of such a map must be a small perturbation of the coefficient of $z|z|^{1/K-1}$. This is explored in detail in Section 3 of Chapter 3.

The techniques introduced in Chapter 3 are not limited to quasiconformal maps, but have consequences for solutions to elliptic partial differential equations as well. Following the classical correspondence

$$\partial_{\bar{z}} f = 0 \iff \Delta u = 0$$

between the Cauchy-Riemann equations (encoding analyticity of a function f) and Laplace's equation (encoding harmonicity of the real part $u = \operatorname{Re} f$), there is a generalization to

$$\partial_{\bar{z}} f = \mu \partial_z f \iff \operatorname{div}(A \nabla u) = 0 \tag{1.1.3}$$

between quasiconformality and solving a divergence-form elliptic PDE where the matrix A is positive definite, symmetric, measurable, has determinant 1, and satisfies an ellipticity bound.

Regularity theory of elliptic PDEs has a rich history. Explicit bounds for the Hölder exponents were given by Morrey [47], improved by Piccinini and Spagnolo [52], and improved further by Ricciardi [56, 57]. Following these results, the correspondence (1.1.3) is explored in detail in Section 4 of Chapter 3. The main result is to give estimates of the Lipschitz class that solutions lie in. Furthermore, the Stoilow factorization theorem allows us to give a classification of the Hölder continuity extremizers. Finally, we state a generalization to a large class of non-linear elliptic operators.

1.2 Favard Length

Before discussing the new techniques and results introduced in Chapter 4, we will give a brief background and motivation of Favard length. We are interested in an old problem of Painlevé: give a metric or geometric characterization of those sets in the complex plane which are removable for analytic functions. It is well known that sets of length zero are removable, and that sets of Hausdorff dimension greater than 1 are not; as such, dimension 1 is the critical point where the problem is most interesting. Vitushkin conjectured that a compact set with positive and finite length is removable if and only if it is purely unrectifiable; this was eventually proved by David in [27].

Just as Favard length gives a quantitative measurement associated to rectifiability, it is useful to introduce the *analytic capacity* as a quantitative object associated to removability. Given a compact $E \subseteq \mathbb{C}$, its analytic capacity is

$$\gamma(E) = \sup \left\{ |f'(\infty)| : f \text{ is analytic on } \mathbb{C} \setminus E \text{ and } \|f\|_\infty \leq 1 \right\} \tag{1.2.1}$$

where $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. It is immediate that a set is removable if and only if it has analytic capacity zero, and the capacity gives another measure of the size and distribution of sets. Connecting analytic capacity zero and Favard length zero would give a great deal of information about Painlevé's problem. In particular, we are interested in whether $\gamma(E) \sim \text{Fav}(E)$ (or similar inequalities) can hold for a large class of sets E.

In [30], Garnett used the four-corner Cantor set to give an example of a removable set with positive length. The generations of this set are denoted $\mathcal{K}_n$, and each consist of 4^n squares of side-length 4^{-n}; see Chapter 4, Section 3 for the full definition. This showed that the geometry of the underlying set is an important factor in removability, and not just its metric properties. Later, Calderón's proof [24] of the weak $(1, 1)$ and strong (p, p) bounds of the Cauchy operator on Lipschitz graphs of small Lipschitz constant (and the extensions of this result to arbitrary Lipschitz constant, such as in [26]) was used to prove that sets of positive length contained in a rectifiable curve are non-removable; this is proved by a direct construction of non-constant bounded holomorphic functions via the Cauchy transform. It is worth mentioning that there are now numerous proofs with a variety of different techniques for the boundedness of the Cauchy transform (e.g. [26, 6] and many others).

Mattila [43] showed that for any $f : \mathbb{R}^2 \to \mathbb{R}^2$ with two continuous derivatives, there is a set E such that $\mathrm{Fav}(E) = 0 < \mathrm{Fav}(fE)$. Thus zero Favard length is not a conformal invariant, while zero analytic capacity clearly is; this shows that $\mathrm{Fav}(E) \sim \gamma(E)$ must be false. Jones and Murai [37] gave an example of a set with positive capacity and zero Favard length, so $\gamma(E) \lesssim \mathrm{Fav}(E)$ cannot hold. The other inequality is still open, and we state it as a problem:

Problem 1.2.1. *Given a compact set E, is it true that* $\mathrm{Fav}(E) \lesssim \gamma(E)$*?*

Cantor sets are frequently used to test this problem, since they have many geometric properties that simplify estimates for analytic capacity. The sharp estimate of analytic capacity for a broad class of Cantor sets (including the four-corner Cantor set) was given by Tolsa [61] in terms of a quantity called positive capacity (later shown to be comparable to analytic capacity in [62]). Mateu, Tolsa, and Verdera [42] also classified the Cantor sets of analytic capacity zero and gave upper bounds on analytic capacity in general. These papers generally rely on finding estimates of the curvature of measures supported on an unrectifiable set. Combining the results, it is known that the n-th generation $\mathcal{K}_n$ of the four-corner Cantor set has analytic capacity $\gamma(\mathcal{K}_n) \sim n^{-1/2}$.

We are also interested in estimating the Favard lengths of the generations of Cantor sets. It was shown by Mattila [44] that the Favard length of the r-neighborhood of any 1-dimensional set decays no faster than $(\log 1/r)^{-1}$, regardless of the geometric structure; in particular, this implies that $\mathrm{Fav}(\mathcal{K}_n) \gtrsim n^{-1}$. Upper bounds on $\mathrm{Fav}(\mathcal{K}_n)$ were found by

Murai [48] and Peres and Solomyak [51]; this latter paper also showed that the expected Favard length of the n-th generation of certain random cantor sets is of order n^{-1}. Tao [60] gave a quantitative version of the Besicovitch projection theorem that gave upper bounds on the Favard lengths of a broad class of fractal sets.

More recently, there have been a number of improved bounds for the four-corner Cantor set, as well as other fractal sets with similar arithmetic and product structure. Bateman and Volberg [15] used a square-counting technique to improve the lower bound to $\mathrm{Fav}(\mathcal{K}_n) \gtrsim (\ln n)/n$. Nazarov, Peres, and Volberg [49] used Fourier analytic techniques based on the structure of $\mathcal{K}_n$ to improve the upper bound to $\mathrm{Fav}(\mathcal{K}_n) \lesssim n^{\delta-1/6}$ for any $\delta > 0$. Bond, Łaba and Volberg [19] generalized this technique to a large class of self-similar product sets. These are currently the best-known bounds; as such, it is not possible to say whether $\mathrm{Fav}(\mathcal{K}_n) \lesssim \gamma(\mathcal{K}_n)$.

In Chapter 4, we introduce new techniques for estimating Favard length. In Section 2, we use a new geometrically motivated argument to study sets with Hausdorff dimension less than 1. This new, simpler technique recovers a result of [44] to compute the power law for decay of Favard length in terms of the Hausdorff dimension. In Section 3, we develop a new method for studying self-similar sets. In particular, we will show that the sequence of Favard lengths of the generations of a 1-dimensional set generated by similitudes is convex; this has powerful consequences in terms of the asymptotic behavior of the Favard length.

Chapter 2

Stretching and Rotation Sets of Quasiconformal Maps

The work in this chapter first appeared in [21].

2.1 Introduction

We say that a map $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ is K-quasiconformal if it is an orientation preserving homeomorphism and satisfies the distortion inequality $\max_\alpha |\partial_\alpha f| \leq K \min_\alpha |\partial_\alpha f|$ almost everywhere, where ∂_α is a directional derivative. Geometrically, f maps infinitesimal circles to infinitesimal ellipses; these can be viewed as perturbations of conformal maps, which are 1-quasiconformal. Such maps are also realized as solutions to the Beltrami equation

$$\partial_{\overline{z}} f = \mu(z) \partial_z f$$

where the coefficient μ satisfies $\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1$.

We are interested in geometric distortion properties of these maps. Given $z \in \mathbb{C}$, we say that f stretches with exponent α and rotates with exponent γ at z if there exist scales

$r_n \to 0$ with

$$\lim_{n\to\infty} \frac{\log|f(z+r_n)-f(z)|}{\log r_n} = \alpha \text{ and } \lim_{n\to\infty} \frac{\arg(f(z+r_n)-f(z))}{\log|f(z+r_n)-f(z)|} = \gamma.$$

Here, the argument is interpreted as the total angular change with respect to $f(z)$ along the image of the ray $[z+r_n,\infty)$; see section 2 or [10] for the full definitions.

A classical theorem of Mori (see [46]) states that every K-quasiconformal map is locally $1/K$-Hölder continuous, which implies that $1/K \le \alpha \le K$. In the more recent [10], Astala, Iwaniec, Prause and Saksman improved this substantially to give the exact range of both stretching and rotation exponents which can be realized by a K-quasiconformal map f: if we let $B_K \subset \mathbb{C}$ be the open disk centered at $\frac{1}{2}(K+\frac{1}{K})$ with radius $\frac{1}{2}(K-\frac{1}{K})$, then f can stretch like α and rotate like γ if and only if $\alpha(1+i\gamma) \in \overline{B_K}$. As a particular application, this gives the precise rotation behavior that a bilipschitz map can exhibit. Moreover, this work gave the precise multifractal spectrum $F_K(\alpha,\gamma)$ - that is, the maximal possible Hausdorff dimension of the simultaneous stretching and rotation set of such maps; the sharp result was the following theorem.

Theorem 2.1.1. *If $f:\mathbb{C}\to\mathbb{C}$ is a K-quasiconformal mapping with $K>1$, and $\alpha(1+i\gamma) \in B_K$, then the Hausdorff dimension of the stretching and rotation set E_f of f is bounded by*

$$\dim_{\mathcal{H}} E_f \le F_K(\alpha,\gamma) := 1+\alpha - \frac{K+1}{K-1}\sqrt{(1-\alpha)^2 + \frac{4K\alpha^2\gamma^2}{(K+1)^2}}$$

and this result is sharp at the level of dimension.

The techniques used to prove this theorem mainly involved improved integrability estimates for complex powers of the derivatives of f. There is very substantial overlap with the techniques used in studying area distortion, and as such it is a natural conjecture that the Hausdorff measure at the critical dimension should be finite, in analogy with Theorem 1.2 in [5]. However, we will show that this is not the case.

In the direction of lower bounds, that paper gives constructions to attain all dimensions below the bound $F_K(\alpha, \gamma)$. Hitruhin improved this in [33] to give examples of quasiconformal maps whose stretching and rotation sets have positive and finite Hausdorff measure at the critical dimension. That paper used a Cantor set construction from [63] to prove this; the work gives a construction of a quasiconformal map whose distortion of a family of disks is completely understood.

In this work, we improve the above results beyond finite measure, showing that the stretching and rotation set can actually have positive measure with respect to many gauged Hausdorff measures which are much smaller than the typical $\mathcal{H}^d$. Our main theorem is

Theorem 2.1.2. *Let Λ be a gauge function of the form $\Lambda(r) = r^d h(r)$ where h is a non-negative, nondecreasing function satisfying the growth condition $h(r)/h(s) \geq C_\epsilon (r/s)^\epsilon$ for all $\epsilon > 0$ and $0 < r \leq s$ sufficiently small. Select parameters $\alpha < 1$ and γ such that $d > 0$ is the maximum allowed Hausdorff dimension of the corresponding stretching and rotation set; that is, $d = F_K(\alpha, \gamma)$. Then there is a K-quasiconformal mapping f and a set E with $\mathcal{H}^\Lambda(E) > 0$ such that E is the stretching and rotation set for f.*

We have a generalization to stretching exponents $\alpha > 1$ under an additional constraint on the gauge function Λ. Furthermore, as a corollary, there is an application to an interesting class of gauge functions:

Corollary 2.1.3. *There are positive measure stretching and rotation sets associated to the gauges $\Lambda(r) = r^d \left(\log \frac{1}{r}\right)^{-\beta}$ for every $\beta > 0$.*

As an interesting second corollary, we can extend this to positive Riesz capacity $\dot{C}_{\beta,p}$ for all parameter choices (β, p) with homogeneity matching the dimension d. Again, this is a surprising result: by analogy with the work of [5], it is reasonable to expect that at the critical homogeneity, the $\dot{C}_{\beta,p}$ Riesz capacity would be zero for some range of indices. However, this conjecture is also false, and we have the following corollary:

Corollary 2.1.4. *Fix any parameter $\tau = \alpha(1 + i\gamma) \in B_K$ with $\alpha \in (1/K, 1)$, and a pair (β, p) with $1 < p < \infty$ and $2 - \beta p = F_K(\alpha, \gamma)$. There is a K-quasiconformal map f and a set E such that f stretches with exponent α and rotates with exponent γ at every point in E, and E has positive $(\beta, p)-$Riesz capacity.*

The paper is organized as follows. In Section 2, we give a brief recollection of some notions involving quasiconformal mappings, and a more precise definition of the rotation. In Section 3, we analyze the Hausdorff dimension zero case; our main results here will be a construction of a quasiconformal mapping that stretches on any given countable set, as well as a first construction of a map with $\mathcal{H}^d$ non-σ-finite stretching and rotation set, where $d = F_K(\alpha, \gamma)$. In Section 4, we will prove the main theorem and indicate applications to particular gauges and Riesz capacities.

2.2 Prerequisites

Following [10], given a quasiconformal map f, we will say that it *stretches like* α at a point z_0 if there exists a sequence of scales r_n decreasing to zero for which

$$\lim_{n\to\infty} \frac{\log|f(z_0 + r_n) - f(z_0)|}{\log r_n} = \alpha.$$

Rotation is similar, but a little more subtle. For a principal quasiconformal map f, that is a map whose domain and codomain are both $\mathbb{C}$ and $f(z) = z + O\left(\frac{1}{z}\right)$ as $|z| \to \infty$, we can select a branch of $\log f$. We can find a corresponding choice of argument, and using this we can understand $\arg(f(z_0 + r) - f(z_0))$ as the total rotation around the point $f(z_0)$ of the image of the ray $[z_0 + r, \infty)$ under f. Using this interpretation, we will say that f *rotates like* γ at a point z_0 if

$$\lim_{n\to\infty} \frac{\arg(f(z_0 + r_n) - f(z_0))}{\log |f(z_0 + r_n) - f(z_0)|} = \gamma$$

for a sequence $r_n \to 0$. It is worth noting that the stretch and rotation at a point are not uniquely defined; it is possible that a quasiconformal map stretches like α and α' at a point with $\alpha \neq \alpha'$ (or rotates with two different behaviors); this is due to the dependence on the particular choice of sequence r_n.

Given a quasiconformal mapping f, we set $E_f(\alpha, \gamma)$ to be its simultaneous rotation-like-γ and stretching-like-α set; when it is clear from context, this will be abbreviated as E_f. Finally, we have the multifractal spectrum

$$F_K(\alpha, \gamma) = \sup \left\{ \dim_{\mathcal{H}}(E_f(\alpha, \gamma)) : f \text{ is } K\text{-quasiconformal} \right\}$$

where this $F_K(\alpha, \gamma)$ is that of Theorem 2.1.1, as proved in [10].

2.3 Dimension Zero

There are two complementary senses in which we will improve upon results with positive measure. The first is to give particular examples of stretching and rotation sets with very large measure, perhaps uncountable or having positive measure with respect to some gauged

Hausdorff measure. The second is to give a broader class of examples of sets, in particular including that every countable set can appear as a stretching set. Before the constructions, we will start with a useful lemma that will allow us to simplify some of the subsequent computations involving stretching. Although it was not stated as a separate result, the computation here is more or less contained in [33].

Lemma 2.3.1. *Suppose that z is a point with the following property: there is a sequence of balls $B_n = B(z_n, r_n)$ such that $z \in B_n$ for each n, $r_n \to 0$, and*

$$\frac{\log |f(B_n)|}{\log |B_n|} = \alpha + \epsilon_n$$

with error $\epsilon_n \to 0$. Then f stretches like α at z.

The utility of this lemma is that we can transfer stretching information at a central point not only to points at difference r away, but to *all* nearby points. As an idea of an application, it is frequently possible to get stretching at exponent α on an entire Cantor set just by taking a quasiconformal map that stretches like α at each of the points used at successive scales to generate the Cantor set.

Proof. Fix n. We can rotate using quasisymmetry. Fix a point $w \in \partial\mathbb{D}(z_n, r_n)$ that is equidistant with z and z_n (e.g. an intersection point of the perpendicular bisector of $\overline{zz_n}$ with the boundary of the circle). Then

$$\begin{aligned}
\log |f(z + r_n) - f(z)| &= \log |f(z + r_n e^{i\theta}) - f(z)| + C_K \\
&= \log |f(w) - f(z)| + C_K \\
&= \log |f(w) - f(w + |z - w| e^{i\nu})| + C_K
\end{aligned}$$

$$= \log|f(w) - f(z_n)| + C'_K$$

$$= \log|f(z_n + |w - z_n|) - f(z_n)| + C''_K$$

$$= \log|f(z_n + r_n) - f(z_n)| + C''_K$$

given appropriate choices of ν and θ; the constants C_K, C'_K and C''_K are unimportant except in that they are bounded in terms of K only. Dividing by $\log r_n$ and letting $n \to \infty$, we find that

$$\begin{aligned}\frac{\log|f(z + r_n) - f(z)|}{\log r_n} &= \frac{\log|f(z_n + r_n)| + C''_K}{\log r_n} \\ &= \frac{\frac{1}{2}\log|f(B_n)| + C'''_K}{\frac{1}{2}\log|B_n| - \frac{1}{2}\log\pi} \\ &= \alpha + \epsilon_n + \frac{2C'''_K}{\log|B_n|} + o(1)\end{aligned}$$

following a final application of quasisymmetry. The result follows. □

Note that we can replace the measures of the balls with their radii. We can actually extract a little more information: if C is a fixed constant, and z is a point for which $|z - z_n| \leq Cr_n$, the same result holds. To see this, notice that there is a polygonal path connecting z to z_n where each segment has length r_n, and the number of segments is uniformly bounded by a constant only involving C. Repeating the double-rotation idea of the proof, we now lose a constant several times (but a uniformly bounded number), which does not impact the result.

Moreover, the same result holds for rotations:

Lemma 2.3.2. *Suppose that z is a point with the following property: there is a sequence of* *balls* $B_n = B(z_n, r_n)$ *such that* $z \in B_n$ *for each* n, $r_n \to 0$, *and*

$$\frac{\arg(f(z_n + r_n) - f(z))}{\log |f(z_n + r_n) - f(z_n)|} = \gamma + \epsilon_n$$

with error $\epsilon_n \to 0$. *Then* f *rotates like* γ *at* z.

Proof. We will only give a brief description of the technique of the proof, as it is rather similar to the previous one. Fix n, and consider the rays $[z_n, \infty)$ and $[z, \infty)$ parallel to the positive x-axis. By a rotation, which changes the cumulate argument by an $O(1)$ factor, we may assume that z lies on the ray $[z_n, \infty)$. Now reusing the double rotation argument of the previous lemma, the denominators of the rotation are the same up to an $O(1)$ error, which is enough. □

Our first result will be a large dimension zero set which has the most extreme stretching and rotation allowed by the multifractal spectrum bounds of [10]. The construction will be a sort of Cantor set built from disks, within which we can explicitly keep track of the stretching and rotation.

Theorem 2.3.3. *For any pair* (α, γ) *for which* $z|z|^{\alpha(1+i\gamma)-1}$ *is* K*-quasiconformal, there is a* K*-quasiconformal map* f *and an uncountable set* E_f *for which* f *stretches like* α *and rotates like* γ *at every point in* E_f.

Proof. Start with $B_{0,1} = \mathbb{D}$ and $f(z) = z$ on all of $\mathbb{C}$. Now assume that $B_{n,i}$ has been defined and has radius r_n, and that there are complex numbers $\beta_{n,i}, w_{n,i}$ for which $f(z) = \beta_{n,i} z + w_{n,i}$ in a neighborhood of $B_{n,i}$. Choose a number $\tilde{r}_n$ (which will be substantially smaller than r_n); take a concentric ball $A_{n,i}$ within $B_{n,i}$ of radius $\tilde{r}_n$, and place two disjoint balls $B_{n+1,j}$ within $A_{n,i}$ each with radius $\frac{1}{4}\tilde{r}_n$. We now modify the construction of f; without loss of generality, we may assume that $w_{n,i} = 0$ and $f(w_{n_i}) = 0$ - otherwise, pre- and post-compose with an appropriate translation (this only simplifies the notation). Now modify the definition of f to become

$$f(z) = \begin{cases} \beta_{n,i} z & \text{near } B_{n,i} \text{ but in } B^c_{n,i} \\ \beta_{n,i} z \left| \frac{z}{r_n} \right|^{\alpha(1+i\gamma)-1} & z \in B_{n,i} \setminus A_{n,i} \\ \beta_{n,i} \left(\frac{\tilde{r}_n}{r_n} \right)^{\alpha - 1} e^{i\theta} z & z \in A_{n,i} \end{cases}$$

where $e^{i\theta}$ is chosen so that f is continuous across $\partial A_{n,i}$, and

$$\beta_{n+1,j} = \beta_{n,i} \left(\frac{\tilde{r}_n}{r_n} \right)^{\alpha - 1} e^{i\theta}.$$

Note that the original function f is injective; on the other hand, the construction only carries out a local modification by stretching and rotating the ball $A_{n,i}$, and remains injective. Moreover, the limiting function of the construction is K-quasiconformal as long as the parameters (α, γ) are chosen to allow this. In particular, following [33], we can choose α, γ to be any pair for which $F_K(\alpha, \gamma) = 0$.

We just need to compute the change in argument induced by crossing the annulus between $B_{n,i}$ and $A_{n,i}$, find the corresponding stretching on scale $\tilde{r}_n$ with respect to the center point, and choose the sequence of radii carefully. Since

$$\left| \frac{z}{r} \right|^{\alpha(1+i\gamma)} = \left| \frac{z}{r} \right|^{\alpha} e^{i\alpha\gamma \log |z/r|}$$

it is immediate that the change in argument across the annulus is $\alpha\gamma \log \frac{\tilde{r}_n}{r_n} + O(1)$. The numerator of the stretching with respect to the center point of $B_{n,i}$ on scale $\tilde{r}_n$ is

$$\log \left| \beta_{n,i} \left(\frac{\tilde{r}_n}{r_n} \right)^{\alpha-1} e^{i\theta} \tilde{r}_n \right| = \alpha \log \tilde{r}_n + \log |\beta_{n,i}| - (\alpha - 1) \log r_n.$$

As a consequence, we see that the overall stretching of f with respect to the center point is

$$\frac{\log |f(\tilde{r}_n) - f(0)|}{\log \tilde{r}_n} = \alpha + \frac{\log |\beta_{n,i}|}{\log \tilde{r}_n} - (\alpha - 1) \frac{\log r_n}{\log \tilde{r}_n} \tag{2.3.1}$$

while the overall rotation is

$$\frac{\arg\left(f(\tilde{r}_n) - f(0)\right)}{\log |f(\tilde{r}_n) - f(0)|} = \frac{\alpha\gamma \log \tilde{r}_n - \alpha\gamma \log r_n + O(1)}{\alpha \log \tilde{r}_n + \log |\beta_{n,i}| - (\alpha - 1) \log r_n}. \tag{2.3.2}$$

Each $\beta_{n,i}$ has the same modulus β_n; the only potential difference is the exact rotation. We can easily compute this number from its definition, finding that

$$\beta_n = \left[\prod_{k=0}^{n-1} \frac{\tilde{r}_k}{r_k} \right]^{\alpha-1}$$

As a consequence, we have that

$$\frac{\log |\beta_{n,i}|}{\log \tilde{r}_n} = (\alpha - 1) \sum_{k=0}^{n-1} \frac{\log \tilde{r}_k - \log r_k}{\log \tilde{r}_n}$$

Because $\tilde{r}_k < r_k < 1$, we can estimate all the terms roughly by the final term (provided that $\tilde{r}_k / r_k$ is decreasing, which it will be), finding

$$\left| \frac{\log |\beta_{n,i}|}{\log \tilde{r}_n} \right| \leq 2(1 - \alpha) n \frac{\log \tilde{r}_{n-1}}{\log \tilde{r}_n} \tag{2.3.3}$$

We have already defined $r_{k+1} = \frac{1}{4}\tilde{r}_k$, and now we make the selection that

$$\tilde{r}_k = r_k^{k^2}$$

and the above error estimate (2.3.3) tends to zero. As an immediate consequence of this selection, we have that the stretching tends towards α, while the rotation tends towards γ. This completes the proof. $\square$

Now we will go in the other direction, finding that any countable set is a stretching set with the worst possible exponent. As a nice application, this shows that an interesting multifractal spectrum bound in the style of [10] is not possible for Minkowski dimension; see, e.g. Chapter 5 of [45] for constructions of countable sets with large Minkowski dimension. There are countable sets whose lower Minkowski dimension is arbitrarily close to 2, and these can exhibit stretching of exponent $1/K$ at every point. The key idea here will be that sums of radial stretches are quasiconformal maps; in general, it is quite rare for a sum of quasiconformal maps to be quasiconformal (let alone injective). This idea will not work for rotations.

Note, however, that this contrasts starkly with the possibilities in other dimensions. For example, a one dimensional set containing a smooth curve or a segment can never be a stretching set for an exponent other than 1. To see this, consider the fact that if f stretches with exponent $\alpha > 1$ at every point within a line segment, f is flat at every point within that line. Explicitly, if f is viewed as a single-variable function on this line, it is (classically) differentiable with derivative zero at every point, hence non-injective. Considering f^{-1} shows why f cannot stretch with exponent $\alpha < 1$.

Theorem 2.3.4. *Given a countable set $\Lambda \subseteq \mathbb{D}$, there is a K-quasiconformal mapping f such* *that for each $\lambda \in \Lambda$ there is a sequence of scales r_m decreasing to zero for which*

$$\lim_{m \to \infty} \frac{\log |f(\lambda + r_m) - f(\lambda)|}{\log r_m} = \frac{1}{K}.$$

Recall that $1/K$ is the most extreme possible exponent due to [10].

Proof. Let us begin with the radial stretches

$$f_\lambda(z) = (z - \lambda)|z - \lambda|^{\frac{1}{K} - 1} + \lambda$$

when $|z - \lambda| \leq 1$, and the identity otherwise. These are K-quasiconformal mappings that satisfy a Beltrami equation with coefficient μ_{λ_n}. Moreover, their derivatives $\partial_z f_\lambda$ have constant sign where they are defined. To wit,

$$\partial_z f_\lambda = \left(\frac{1}{2K} + \frac{1}{2} \right) |z - \lambda|^{\frac{1}{K} - 1}$$

within the disk $\lambda + \mathbb{D}$, and 1 outside. It follows that if we sum such solutions, we can still have a solution to a Beltrami equation; in particular, assuming that derivatives and sums commute in this context, we have

$$\begin{aligned} \left| \partial_{\overline{z}} \sum_{n=1}^{\infty} \frac{1}{2^n} f_{\lambda_n}(z) \right| &= \left| \sum_{n=1}^{\infty} \frac{1}{2^n} \partial_{\overline{z}} f_{\lambda_n}(z) \right| \\ &= \left| \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_{\lambda_n}(z) \partial_z f_{\lambda_n}(z) \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|\mu_{\lambda_n}\|_\infty |\partial_z f_{\lambda_n}(z)| \\ &= \frac{K-1}{K+1} \sum_{n=1}^{\infty} \frac{1}{2^n} \partial_z f_{\lambda_n}(z) \end{aligned}$$

$$= \frac{K-1}{K+1} \partial_z \sum_{n=1}^{\infty} \frac{1}{2^n} f_{\lambda_n}(z)$$

Now given a countable set, we can therefore define a function

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{\lambda_n}(z). \tag{2.3.4}$$

Modulo swapping the derivatives and the sum, we have shown that f satisfies a Beltrami equation with coefficient bounded by $(K-1)/(K+1)$. This condition will follow very quickly from the dominated convergence theorem. Fix a test function $\varphi \in C_0^\infty(\mathbb{C})$ and integrate by parts:

$$\int f \partial_x \varphi = \int \lim_{n\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} f_{\lambda_n} \partial_x \varphi$$
$$= \lim_{n\to\infty} \sum_{n=1}^{N} \int \frac{1}{2^n} f_{\lambda n} \partial x \varphi$$

where we have used the fact that $|f(z)| \leq \sum_n \frac{1}{2^n} |f_{\lambda_n}(z)| \leq \sum_n \frac{1}{2^n}(|\lambda| + 1 + |z|) \leq 2 + |z|$ from the estimate $|f_\lambda(z)| \leq |z-\lambda|^{1/K} + |\lambda| \leq 2$ within the disk $\lambda + \mathbb{D}$, and $|z|$ otherwise. Thus f is bounded on the support of φ, and the above follows. Now integrate by parts in each summand to get

$$\int f \partial_x \varphi = -\lim_{n\to\infty} \int \sum_{n=1}^{N} \frac{1}{2^n} \partial_x f_{\lambda_n} \varphi$$

Now φ is bounded on its support, and $|\partial_x f_{\lambda_n}| \lesssim_K |z-\lambda_n|^{1/K-1}$ is locally integrable (as $1/K - 1 > -1$), and summing in n does not change this. Taking $\sum_{n=1}^{\infty} \frac{1}{2^n} |z-\lambda_n|^{1/K-1} |\varphi|$

as our dominating function, we again interchange the limits and find that

$$\int f \partial_x \varphi = -\int \left(\sum_{n=1}^{\infty} \partial_x f_{\lambda_n} \right) \varphi$$

as desired. Now we have that f has a weak derivative, which is a convergent sum of locally L^2 integrable functions. The same holds for ∂_y, and hence both ∂_z and $\partial_{\overline{z}}$. Now it follows immediately that $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ and satisfies a Beltrami equation; thus, the measurable Riemann mapping theorem (see, for example, Theorem 5.3.2 of [8]) gives us the following lemma:

Lemma 2.3.5. *Given a countable set* $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$*, the function* f *defined in (2.3.4) is* K*-quasiconformal.*

We now claim that this function f has the correct stretching behavior at each point in Λ. Fix $\lambda_n \in \Lambda$; we can assume that $\lambda_n = 0$. Morally, we proceed as follows: there are contributions to the stretching from terms on two scales, the nearby and the far away. We can arrange it so that nearby points λ_m only have very large indices, so that the exponentially decaying weights will render this negligible; on the other hand, far away points have the advantage of the smoothness of the radial stretches.

Let us make this precise. We will show that

$$|f(r) - f(0)| = cr^{1/K} + o(r^{1/K}) \tag{2.3.5}$$

with a non-zero constant c, from which the theorem will follow. First of all, it is clear that the term $n = m$ contributes exactly $\frac{1}{2^m} r^{1/K}$; we will estimate away the remaining terms.

To this end, we have for terms with $m \neq n$ that the difference is

$$\sum_{m \neq n} \frac{1}{2^m}(r-\lambda_m)|r-\lambda_m|^{\frac{1}{K}-1} - \frac{1}{2^m}(-\lambda_m)|-\lambda_m|^{\frac{1}{K}-1}$$

After factoring a term $-\lambda_m|-\lambda_m|^{\frac{1}{K}-1}$ from each summand and applying the triangle inequality, we need to estimate

$$\sum_{m \neq n} \frac{1}{2^m}|\lambda_m|^{1/K} \left| \left(1-\frac{r}{\lambda_m}\right)\left|1-\frac{r}{\lambda_m}\right|^{1/K-1} - 1 \right|.$$

To deal with the term within the absolute value, we need a simple estimate of a particular function:

Lemma 2.3.6. *If* $K > 1$,

$$\left|(1+z)|1+z|^{1/K-1} - 1\right| \leq C_0 \min\left\{|z|, |z|^{1/K}\right\}.$$

for a constant C_0 *depending only on* K.

Proof. For large values of $|z|$, the triangle inequality implies that this is controlled by a constant multiple of $|z|^{1/K}$, which is smaller (up to a constant) than $|z|$. So let us assume that $|z|$ is small, e.g. $|z| \leq \frac{1}{2}$. Write $|1+z| = 1+y$ with y real and $|y| \leq |z|$.

If $y = 0$, $|1+z| = 1$ and

$$(1+z)|1+z|^{1/K-1} - 1 = z.$$

Otherwise, select λ so that $\lambda y = z$; then Taylor expansion gives

$$\begin{aligned}(1+z)|1+z|^{1/K-1} - 1 &= (1+\lambda y)(1+y)^{1/K-1} - 1 \\ &= 1 + \left(\lambda + \frac{1}{K} - 1\right) y + O(y^2) - 1 \\ &= \left(\lambda + \frac{1}{K} - 1\right) y + O(y^2) \\ &= z + \left(\frac{1}{K} - 1\right) y + O(y^2) \\ &= z + O(|z|) + O(|z|^2)\end{aligned}$$

from which the lemma follows. □

Now we are ready to make the division into two scales. The cutoff point is to separate in the following way: Since the sequence is fixed, we can choose r small enough that

$$\frac{r}{|\lambda_m|} \geq \left(\frac{1}{2^{n+1}C_0}\right)^{\frac{1}{1-1/K}} \implies m \geq n + a + 10$$

where a is chosen so that $2^a > C_0$; C_0 here is the constant of Lemma 2.3.6. That is, when $|\lambda_m|$ is smaller than a very large constant multiple of r, the index must be very large.

The far scale is for terms when $(r/|\lambda_m|)^{1-1/K} < 1/2^{n+1}C_0$. In this case we have the lemma's linear estimate available, and the sum over these indices m is at most

$$C \sum_{m \text{ far}} \frac{1}{2^m} |\lambda_m|^{1/K} \frac{r}{|\lambda_m|} = C_0 r^{1/K} \sum_{m \text{ far}} \frac{1}{2^m} \left(\frac{r}{|\lambda_m|}\right)^{1-1/K} < \frac{r^{1/K}}{2^{n+1}}$$

which is enough. Note that we have no control over the index m here.

Next is the nearby scale where we have the opposite inequality; now m must be large but we have worse control on the summands. Using the non-linear estimate from the lemma, we find that the contribution is at most

$$C_0 \sum_{m \text{ near}} \frac{1}{2^m} |\lambda_m|^{1/K} \left(\frac{r}{|\lambda_m|} \right)^{1/K} = C_0 \sum_{m \text{ near}} \frac{1}{2^m} r^{1/K} \leq \frac{C_0}{2^{n+a+9}} r^{1/K} < \frac{r^{1/K}}{2^{n+9}}$$

having used the fact that $\sum_{m \geq N} \frac{1}{2^m} = \frac{1}{2^{N+1}}$.

Combining these two estimates, the contribution from all indices $m \neq n$ is of the order $r^{1/K}$ with constant significantly less than 2^{-n}. This proves (2.3.5) and is the desired result.

□

2.4 Dimension Greater than Zero

To prepare for the main result, we will define a particular class of gauge functions. These will be gauges which lead to minor perturbations of the pure Hausdorff meaures, without changing the dimension. The perturbations should be chosen to tend to zero slowly enough to guarantee this, and will contain some sort of embedded convexity condition.

Definition 2.4.1. *We will say that a gauge function $\Lambda(r) = r^d h(r)$ is* ***admissible*** *if $h(r)$ is continuous, nonnegative, non-decreasing on $[0, \infty)$, and satisfies the following decay condition at the origin: For every $\epsilon > 0$, there exists a constant C_ϵ such that for any $0 < r \leq s \leq 1$,*

$$\frac{h(r)}{h(s)} \geq C_\epsilon \left(\frac{r}{s} \right)^\epsilon.$$

It will be proven later that functions of the form $(\log(1/r))^{-\beta}$ for $\beta > 0$ are admissible, giving a rich class of examples. We now come to the first theorem of the section.

Theorem 2.4.2. *Let Λ be an admissible gauge function. Fix K and $\alpha \in (1/K, 1)$, setting*

$d = F_K(\alpha, 0)$. *Then there is a set E with positive gauged Hausdorff measure $\mathcal{H}^\Lambda(E)$ and a K-quasiconformal map f so that f stretches like α at every point in E.*

Proof. The main construction of the proof is taken from [63], although our choice of parameters will be different. We retain the notation from that paper, and for the sake of self-containment give a brief description of the construction. At each stage of the construction, we will pack a disk completely with disjoint disks, and then shrink these disks appropriately to build a set of the desired Hausdorff dimension. The quasiconformal map will stretch these shrunken disks appropriately.

Step 1. Select $m_{1,1}$ disjoint disks $D^i_{1,1}$ of radius $R_{1,1}$ within the unit disk, followed by $m_{1,2}$ disjoint disks (and disjoint from the previously constructed disks as well) $D^i_{1,2}$ of radius $R_{1,2}$, and so on. In this manner we pack the unit disk completely in area, leading to

$$\sum_{j=1}^{\infty} m_{1,j} R_{1,j}^2 = 1.$$

It is important to note that we can assume that every $R_{1,j}$ is smaller than some fixed $\delta_1 > 0$, which is as small as we desire. Also for each radius associate a parameter $\sigma_{1,j} > 0$; these will be chosen later, but are all quite small.

Next, we construct a first approximation of our quasiconformal map. Denote the center of the disk $D^i_{1,j}$ as $z^i_{1,j}$. Let $\psi^i_{1,j}(z) = z^i_{1,j} + (\sigma_{1,j})^K R_{1,j} z$, and define disks

$$D^i_j = D(z^i_{1,j}, R_{1,j}) = \frac{1}{(\sigma_{1,j})^K} \psi^i_{1,j}(\mathbb{D})$$

$$(D^i_j)' = D(z^i_{1,j}, (\sigma_{1,j})^K R_{1,j}) = \psi^i_{1,j}(\mathbb{D})$$

Then our first approximation is

$$\varphi_1(z) = \begin{cases} (\sigma_{1,j})^{1-K}(z - z^i_{1,j}) + z^i_{1,j}, & z \in (D^i_j)' \\ \left|\frac{z - z^i_{1,j}}{R_{1,j}}\right|^{\frac{1}{K}-1} (z - z^i_{1,j}) + z^i_{1,j}, & z \in D^i_j \setminus (D^i_j)' \\ z, & z \notin \bigcup D^i_j. \end{cases}$$

This is K-quasiconformal, being a modification of a radial stretch, and is conformal except for the annular regions between small disks $(D^i_j)'$ and their dilates D^i_j. In particular, it is important to note that φ_1 maps the disks of radius $(\sigma_{1,j})^K R_{1,j}$ onto other disks of radius $\sigma_{1,j} R_{1,j}$.

Step 2. We repeat the idea of the construction from the previous step. Choose $m_{2,1}$ disjoint disks $D^i_{2,1}$ with centers $z^i_{2,1}$ of radius $R_{2,1}$, and so on; again these will be subject to the constraint

$$\sum_{j=1}^{\infty} m_{2,j} R^2_{2,j} = 1.$$

Again, we can choose $R_{2,j}$ to be bounded by some $\delta_2 > 0$, but as small as needed; this is the difference from step 1, as we may wish to have $\delta_2 < \delta_1$. Next, we choose $\sigma_{2,j} > 0$.

As before, we follow this with an approximation of the quasiconformal map. Set $\psi^n_{2,k}(z) = z^n_{2,k} + (\sigma_{2,k})^K R_{2,k} z$, a radius $r_{\{2,k\},\{1,j\}} = R_{2,k}\sigma_{1,j}R_{1,j}$ and define disks

$$D^{i,n}_{j,k} = D(z^{i,n}_{j,k}, r_{\{2,k\},\{1,j\}}) = \varphi_1\left(\frac{1}{(\sigma_{2,k})^K}\psi^i_{1,j} \circ \psi^n_{2,k}(\mathbb{D})\right)$$

$$(D^{i,n}_{j,k})' = D(z^{i,n}_{j,k}, (\sigma_{2,k})^K r_{\{2,k\},\{1,j\}}) = \varphi_1\left(\psi^i_{1,j} \circ \psi^n_{2,k}(\mathbb{D})\right)$$

Now we define

$$g_2(z) = \begin{cases} (\sigma_{2,k})^{1-K}(z - z^{i,n}_{j,k}) + z^{i,n}_{j,k}, & z \in (D^{i,n}_{j,k})' \\ \left|\frac{z - z^{i,n}_{j,k}}{r_{\{2,k\},\{1,j\}}}\right|^{\frac{1}{K}-1} (z - z^{i,n}_{j,k}) + z^{i,n}_{j,k}, & z \in D^{i,n}_{j,k} \setminus (D^{i,n}_{j,k})' \\ z, & \text{otherwise.} \end{cases}$$

Finally, our second approximation is $\varphi_2 = g_2 \circ \varphi_1$. As before, this is a K-quasiconformal map equal to the identity outside the unit disk; the most important thing to note is that this map behaves essentially as a radial stretch, sending certain disks of radius $(\sigma_{1,j}\sigma_{2,k})^K R_{1,j} R_{2,k}$ to certain other disks of radius $(\sigma_{1,j}\sigma_{2,k}) R_{1,j} R_{2,k}$.

Induction step. Assuming that $N-1$ steps of the construction have been fulfilled, we repeat the process, getting disks $D^i_{N,j}$ with centers $z^q_{N,p}$, radii $R_{N,p}$ and satisfying

$$\sum_{j=1}^{\infty} m_{N,j} R^2_{N,j} = 1.$$

As before, we have a constraint $R_{N,j} < \delta_N$ and parameters $\sigma_{N,j} > 0$.

We proceed with the next approximation of the quasiconformal map. Define radii

$$r_{\{N,p\},\{N-1,h\},\ldots,\{1,j\}} = R_{N,p}\sigma_{N-1,h} r_{\{N-1,h\},\ldots,\{1,j\}}$$

and maps $\psi^q_{N,p}(z) = z^q_{N,p} + (\sigma_{N,p})^K R_{N,p} z$. For multiindices $I = (i_1, \ldots, i_N)$ and $J = (j_1, \ldots, j_N)$, we define disks

$$D^I_J = D(z^I_J, r_{\{N,p\},\ldots,\{1,j\}}) = \varphi_{N-1}\left(\frac{1}{(\sigma_{N,p})^K}\psi^{i_1}_{1,j_1} \circ \cdots \circ \psi^{i_N}_{N,j_N}(\mathbb{D})\right)$$

$$(D^I_J)' = D(z^I_J, (\sigma_{N,p})^K r_{\{N,p\},\dots,\{1,j\}}) = \varphi_{N-1}\left(\psi^{i_1}_{1,j_1} \circ \cdots \circ \psi^{i_N}_{N,j_N}(\mathbb{D})\right)$$

As usual, we set

$$g_N(z) = \begin{cases} (\sigma_{N,p})^{1-K}(z - z^I_J) + z^I_J, & z \in (D^I_J)' \\ \left|\frac{z - z^I_J}{r_{\{N,p\},\dots,\{1,j\}}}\right|^{\frac{1}{K}-1}(z - z^I_J) + z^I_J, & z \in D^I_J \setminus (D^I_J)' \\ z, & \text{otherwise.} \end{cases}$$

This map is K-quasiconformal, conformal outside of the union of all the annuli and preserves the disks D^I_J. We finally set $\varphi_N = g_N \circ \varphi_{N-1}$, noting that this is the identity outside the unit disk and maps disks of radius $(\sigma_{1,j_1} \cdots \sigma_{N,j_N})^K R_{1,j_1} \cdots R_{N,j_N}$ to disks of radius $(\sigma_{1,j_1} \cdots \sigma_{N,j_N}) R_{1,j_1} \cdots R_{N,j_N}$.

We now take the limits resulting from this construction. As φ_N is a K-quasiconformal map which is the identity outside of $\mathbb{D}$, compactness of quasiconformal maps allows us to select a K-quasiconformal limit

$$f = \lim_{n\to\infty} \varphi_N$$

with convergence in the Sobolev space $W^{1,2}_{\text{loc}}$.

To recap, the result of the above construction is a Cantor type set E whose building blocks at generation N are disks with radius

$$s_{j_1 \dots j_N} = \left((\sigma_{1,j_1})^K R_{1,j_1}\right) \dots \left((\sigma_{N,j_N})^K R_{N,j_N}\right)$$

which are mapped to disks of radius

$$t_{j_1 \dots j_N} = \left(\sigma_{1,j_1} R_{1,j_1}\right) \dots \left(\sigma_{N,j_N} R_{N,j_N}\right)$$

where we can choose σ_{i,j_i} more or less freely, subject to the constraint that they are all small.

Now we will select our parameters σ_{k,j_k}. We will choose them subject to the governing equation

$$\begin{aligned} R_{1,j_1}^2 \cdots R_{N,j_N}^2 = (R_{1,j_1} \cdots R_{N,j_N})^d (\sigma_{1,j_1} \cdots \sigma_{N,j_N})^{Kd} \\ \cdot h(R_{1,j_1} \cdots R_{N,j_N} \sigma_{1,j_1}^K \cdots \sigma_{N,j_N}^K). \end{aligned} \tag{2.4.1}$$

If we write $\sigma_{k,j_k} = R_{k,j_k}^{\frac{2-d}{Kd}} \eta_{k,j_k}$, the condition is equivalent to

$$1 = \eta_{1,j_1}^{Kd} \cdots \eta_{N,j_N}^{Kd} h\left(R_{1,j_1}^{2/d} \cdots R_{N,j_N}^{2/d} \eta_{1,j_1}^K \cdots \eta_{N,j_N}^K\right). \tag{2.4.2}$$

To see the relevance of the governing equation, note that if we sum over all the building blocks of our construction at level N, our choice of parameters gives us

$$\sum_{j_1,\dots,j_N} m_{1,j_1} \cdots m_{n,j_n} s_{j_1,\dots,j_n}^d h(s_{j_1,\dots,j_n}) = \sum_{j_1,\dots,j_N} (R_{1,j_1} \cdots R_{N,j_N})^2 = 1$$

This is suggestive of the desired result, namely that the constructed set has positive measure in the gauge $r^d h(r)$.

We now have three questions left to address: whether we can actually select our parameters σ in this manner, whether the Cantor set will exhibit the correct stretching, and whether the set has positive measure with respect to $\mathcal{H}^\Lambda$.

First, we consider the satisfiability of the governing equation for σ_{k,j_k}; the selection is made inductively. Looking at the second form of our governing equation, and recalling that h is continuous, it is immediately clear that we can select η_{N,j_N} to satisfy the equation - the right hand side tends to zero as η_{N,j_N} does, and to infinity as η_{N,j_N} does. The only concern is that η_{N,j_N} might be so large as to defeat our requirement that σ_{N,j_N} is small. First, notice that $R_{N,j_N}\sigma_{N,j_N}^K < 1$; if it were not, then we would have

$$\begin{aligned}
R_{1,j_1}^2 \cdots R_{N,j_N}^2 &= (R_{1,j_1} \cdots R_{N-1,j_{N-1}})^d (\sigma_{1,j_1} \cdots \sigma_{N-1,j_{N-1}})^{Kd} \\
&\quad \cdot h\left(R_{1,j_1} \cdots R_{N,j_N} \sigma_{1,j_1}^K \cdots \sigma_{N,j_N}^K\right) (R_{N,j_N}\sigma_{N,j_N}^K)^d \\
&\geq (R_{1,j_1} \cdots R_{N-1,j_{N-1}})^d (\sigma_{1,j_1} \cdots \sigma_{N-1,j_{N-1}})^{Kd} \\
&\quad \cdot h\left(R_{1,j_1} \cdots R_{N-1,j_{N-1}} \sigma_{1,j_1}^K \cdots \sigma_{N-1,j_{N-1}}^K\right) \\
&= R_{1,j_1}^2 \cdots R_{N-1,j_{N-1}}^2
\end{aligned}$$

contradicting the fact that each R_{k,j_k} is much smaller than 1.

The above computation also suggests how to bound each σ_{N,j_N}, by playing the governing equation off itself at different generations. In this manner, essentially just rearranging the above, we find that

$$R_{N,j_N}^2 = R_{N,j_N}^d \sigma_{N,j_N}^{Kd} \frac{h\left(R_{1,j_1} \cdots R_{N,j_N} \sigma_{1,j_1}^K \cdots \sigma_{N,j_N}^K\right)}{h\left(R_{1,j_1} \cdots R_{N-1,j_{N-1}} \sigma_{1,j_1}^K \cdots \sigma_{N-1,j_{N-1}}^K\right)}$$

Rearranging for σ_{N,j_N} and applying our growth condition with exponent ϵ, we find that

$$\sigma_{N,j_N}^{Kd} \leq R_{N,j_N}^{2-d} \left(\frac{1}{R_{N,j_N}\sigma_{N,j_N}}\right)^{\epsilon} \frac{1}{C_\epsilon}.$$

Consequently,

$$\sigma_{N,j_N} \leq \frac{1}{C_\epsilon^{1/K(d+\epsilon)}} R_{N,j_N}^{\frac{2-d-\epsilon}{Kd}}.$$

As long as ϵ is chosen small enough that $2 - d - \epsilon > 0$, we may choose all δ_N small enough to result in $\sigma_{N,j_N} < 1/100$ as desired.

Next, we proceed to the stretching. Following the general approximation lemma 2.3.1, it is sufficient to show that

$$\frac{\log t_{j_1,\ldots,j_N}}{\log s_{j_1,\ldots,j_N}} \to \alpha$$

as $N \to \infty$. In this direction, observe that

$$\begin{aligned} \frac{\log t_{j_1,\ldots,j_N}}{\log s_{j_1,\ldots,j_N}} &= \frac{\sum_{i=1}^N \log R_{i,j_i} + \sum_{i=1}^N \log \sigma_{i,j_i}}{\sum_{i=1}^N \log R_{i,j_i} + K \sum_{i=1}^N \log \sigma_{i,j_i}} \\ &= \frac{\left(1 + \frac{2-d}{Kd}\right) \sum_{i=1}^N \log R_{i,j_i} + \sum_{i=1}^N \log \eta_{i,j_i}}{\left(1 + K\frac{2-d}{Kd}\right) \sum_{i=1}^N \log R_{i,j_i} + \sum_{i=1}^N \log \eta_{i,j_i}}. \end{aligned}$$

Now provided that the perturbation terms are negligible with comparison to the radii terms, the stretching result follows. Indeed, in that case the quotient tends towards

$$\frac{1 + \frac{2-d}{Kd}}{1 + K\frac{2-d}{Kd}} = \frac{2 + (K-1)d}{2K} = \alpha.$$

Thus, we need to prove that

$$S_N := \frac{\sum_{i=1}^N \log \eta_{i,j_i}}{\sum_{i=1}^N \log R_{i,j_i}}$$

tends to zero as N grows.

To get this result, first notice that S_N is negative: the product of all η_{i,j_i} is greater than

1 (as h is small), while each R_{i,j_i} is less than 1; see (2.4.2). From this, it follows that

$$\begin{aligned} 0 \geq KdS_N &= \frac{Kd\sum_{i=1}^N \log \eta_{i,j_i}}{\sum_{i=1}^N \log R_{i,j_i}} \\ &= \frac{-\log h\left(R_{1,j_1}^{2/d} \cdots R_{N,j_N}^{2/d} \eta_{1,j_1}^K \cdots \eta_{N,j_n}^K\right)}{\sum_{i=1}^N \log R_{i,j_i}} \\ &\geq \frac{-\log\left(C_\epsilon R_{1,j_1}^{2\epsilon/d} \cdots R_{N,j_N}^{2\epsilon/d} \eta_{1,j_1}^{K\epsilon} \cdots \eta_{N,j_N}^{K\epsilon}\right)}{\sum_{k=1}^N \log R_{i,j_i}} \\ &= \frac{-\log C_\epsilon}{\sum_{i=1}^N \log R_{i,j_i}} - \frac{2\epsilon}{d} - K\epsilon S_N \end{aligned}$$

where in the inequality we have used that $h(r) \geq C_\epsilon r^\epsilon$ provided that r is sufficiently small, e.g. that N is sufficiently large; this is the admisibility condition (2.4.1) applied with $s = 1$. To be precise, we require that N is large enough that $R_{1,j_1}^{2\epsilon/d} \cdots \eta_{N,j_n}^{K\epsilon} < 1$. Now rearranging the result, we get

$$0 \geq S_N \geq \left(\frac{1}{K(d+\epsilon)}\right)\left(-\frac{\log C_\epsilon}{\sum_{i=1}^N \log R_{i,j_i}} - \frac{2\epsilon}{d}\right)$$

It follows that we have

$$|S_N| \leq \frac{\log C_\epsilon}{N \log 2} + O(\epsilon) = O(\epsilon)$$

provided that N is chosen large enough given ϵ. Taking ϵ to zero gives the required stretching.

Now all that remains is to show positivity of the measure of the Cantor set. Our starting point is an estimate analogous to equation (3.17) in [63]; if D is a building block at generation $N-1$,

$$\sum_{B_n \text{ children of } D} r(B_n)^d h(r(B_n)) = \sum_{j_N} m_{N,j_N} \Lambda\left(R_{1,j_1} \cdots R_{N,j_N} \sigma_{1,j_1}^K \cdots \sigma_{N,j_N}^K\right)$$

$$
\begin{aligned}
&= \left[R_{1,j_1} \cdots R_{N-1,j_{N-1}} \sigma_{1,j_1}^K \cdots \sigma_{N-1,j_{N-1}}^K\right]^d \\
&\quad \cdot \sum_{j_N} m_{N,j_N} (R_{N,j_N} \sigma_{N,j_N}^K)^d h\left(R_{1,j_1} \cdots R_{N,j_N} \sigma_{1,j_1}^K \cdots \sigma_{N,j_N}^K\right) \\
&= \sum_{j_N} m_{N,j_N} R_{1,j_1}^2 \cdots R_{N,j_N}^2 \\
&= R_{1,j_1}^2 \cdots R_{N-1,j_{N-1}}^2 \\
&= (R_{1,j_1} \cdots R_{N-1,j_{N-1}})^d (\sigma_{1,j_1} \cdots \sigma_{N-1,j_{N-1}})^{Kd} \\
&\quad \cdot h\left(R_{1,j_1} \cdots R_{N-1,j_{N-1}} \sigma_{1,j_1}^K \cdots \sigma_{N-1,j_{N-1}}^K\right) \\
&= \Lambda(r(D)) \qquad (2.4.3)
\end{aligned}
$$

where we have used the governing equation (2.4.1) at generations N and $N-1$. As a consequence, we can iterate this result to find that if $\{B_n\}$ is a finite collection of building blocks all contained in D (not necessarily of the same generation), and $B_{N,k}$ are the generation N descendents of B_n,

$$\sum_{B_n} \Lambda(r(B_n)) = \sum_{B_{N,k}} \Lambda(r(B_{N,k})).$$

We now wish to prove a Carleson style packing condition, from which positivity of measure will follow. We will state this as a separate lemma, similar to Lemma 3.2 of [63].

Lemma 2.4.3. *Let B be an arbitrary disk and B_n disjoint building blocks of E. There is an absolute constant C_1 independent of the family $\mathcal{C} = \{B_n\}$ such that*

$$\sum_{\substack{B_n \in \mathcal{C} \\ B_n \subset B}} \Lambda(r(B_n)) \leq C_1 \Lambda(r(B)).$$

Once the lemma has been proven, the positivity of the gauged Hausdorff measure follows immediately. So let us fix such a family $\mathcal{C}$; we may assume that $r(B) \leq 1$, since the above computation (2.4.3) shows that the lemma holds when $B = \mathbb{D}$. Choose the integer H such that all the B_n are contained in some building block at generation $H-1$, but not at generation H; then let $\{B^H_{k_p}\}_{p=0}^m$ be the complete list of ancestors at generation H of our family. Note that the lemma holds with $B = B^{H-1}_{i_0}$ (by the same reasoning that it holds for $B = \mathbb{D}$) and so we will assume that

$$r(B) \leq r(B^{H-1}_{i_0}) = s_{j_1,\dots,j_{H-1}}.$$

For each of these generation H disks, let $\widetilde{B^H_{k_p}}$ be the concentric dilate with radius

$$r(\widetilde{B^H_{k_p}}) = \frac{s_{j_1,\dots,j_H}}{\sigma^K_{H,j_H}}.$$

Provided that the multiindices $I = (i_1, ..., i_H)$ and $J = (j_1, ..., j_H)$ are chosen appropriately, these disks are the disks $(D^I_J)'$ from the construction of the Cantor set; now as each $\sigma_{N,p}$ is small (e.g. less than 1/100) and since B meets each $B^H_{k_p}$, we find that

$$2r(B) \geq \frac{99}{100} r(\widetilde{B^H_{k_p}}).$$

Moreover, we have the containment $\widetilde{B^H_{k_p}} \subseteq 4B$. We now can compute:

$$\begin{aligned}\sum_{B_n \in \mathcal{C}} \Lambda(r(B_n)) &\leq \sum_{p=0}^m \Lambda(r(B^H_{k_p})) \\ &= \left[\sigma^K_{1,j_1} R_{1,j_1} \cdots \sigma^K_{H-1,j_{H-1}} R_{H-1,j_{H-1}}\right]^d \\ &\quad \cdot \sum_{p=0}^m \left(\sigma^K_{H,j_{H_{k_p}}} R_{H,j_{H_{k_p}}}\right)^d h\left(R_{1,j_1} \cdots \sigma^K_{H,j_{H_{k_p}}}\right)\end{aligned}$$

$$= s^d_{j_1,\dots j_{H-1}} h(s_{j_1,\dots j_{H-1}}) \sum_{p=0}^{m} R^2_{H,j_{H_{k_p}}}$$
$$= s^d_{j_1,\dots j_{H-1}} h(s_{j_1,\dots j_{H-1}}) \frac{1}{\pi} \sum_{p=0}^{m} \text{Area}(D_p)$$

where we have defined $D_p = D(z^{k_p}_{H,j_{H_{k_p}}}, R_{H,j_{H_{k_p}}})$, recalling that these are disks chosen during the induction step of the Cantor set's construction, called $D^i_{N,j}$. The second to last equality follows from applications of the governing equation at generations H and $H-1$.

Now since

$$r(\widetilde{B^H_{k_p}}) = \frac{s_{j_1,\dots,j_{H_{k_p}}}}{\sigma^K_{H,j_{H_{k_p}}}}$$

and

$$r(D_p) = R_{H,j_{H_{k_p}}} = \frac{r(\widetilde{B^H_{k_p}})}{s_{j_1,\dots,j_{H-1}}}$$

it follows that

$$\sum_{B_n \in \mathcal{C}} \Lambda(r(B_n)) \leq s^d_{j_1,\dots j_{H-1}} h(s_{j_1,\dots j_{H-1}}) \left[\frac{r(4B)}{s_{j_1,\dots,j_{H-1}}} \right]^2$$
$$\lesssim r(B)^d h(s_{j_1,\dots,j_{H-1}}) \left[\frac{r(B)}{s_{j_1,\dots,j_{H-1}}} \right]^{2-d}$$

Finally, recall the condition (2.4.1) that for any $0 < x < y \leq 1$, we have

$$\frac{h(x)}{h(y)} \geq C \left(\frac{x}{y} \right)^{2-d}$$

Applying this to the above, it follows that

$$\sum_{B_n \in \mathcal{C}} \Lambda(r(B_n)) \lesssim r(B)^d h(r(B)) = \Lambda(r(B))$$

as desired. ☐

We now move to the rotation results.

Theorem 2.4.4. *Let Λ be an admissible gauge function. Fix K and parameters α, γ so that $\alpha(1+i\gamma) \in B_K$ and $\alpha < 1$, setting $d = F_K(\alpha, \gamma)$. Then there is a set E with positive gauged Hausdorff measure $\mathcal{H}^{\Lambda}(E)$ and a K-quasiconformal map f so that f stretches like α and rotates like γ at every point in E.*

Proof. This proof will very closely follow Hitruhin's modifications in [33] to add rotation to the previous theorem. We select $\overline{K} < 1/\alpha$ and let $\overline{f}$ be the $\overline{K}$-quasiconformal map previously constructed; the corresponding Cantor set has positive $\mathcal{H}^{r^{\overline{d}} h(r)}$ measure, where

$$\overline{d} = 1 + \alpha - \frac{\overline{K}+1}{\overline{K}-1}(1-\alpha).$$

Now all we need to do is modify the construction of $\overline{\varphi}_n$ for each n by replacing the old $\overline{g}_n$ by

$$g_n(z) = \begin{cases} (\sigma_{n,j_n})^{1-\overline{K}}(z - z_J^I)e^{i\theta_J^I} + z_J^I, & z \in (D_J^I)' \\ \left|\frac{z - z_J^I}{r(D_J^I)}\right|^{\frac{1}{\overline{K}} - 1 + i\alpha\gamma\frac{\overline{K}-1}{\overline{K}(1-\alpha)}}(z - z_J^I) + z_J^I, & z \in D_J^I \setminus (D_J^I)' \\ z, & \text{otherwise.} \end{cases}$$

where the change in argument over the annulus $D_J^I \setminus (D_J^I)'$ is θ_J^I, and makes the map continuous across the boundary crossings. Let f denote the resulting map using φ_n and g_n, rather than the old versions $\overline{\varphi}_n$ and $\overline{g}_n$.

Since the paper [33] has already shown that $\overline{d} = F_K(\alpha, \gamma)$ is the desired dimension, and the previous theorem improves this to the perturbed Hausdorff gauge function, all that remains is to check that the rotational behavior is correct. That is, we need to show that

$$\lim_{n\to\infty} \frac{\arg(f(z_0 + r_n) - f(z_0))}{\log|f(z_0 + r_n) - f(z_0)|} = \gamma$$

for a suitable choice of scales $r_n \to 0$, and z_0 in a large subset of the Cantor set. Following the argument in [33], we end up with the result that the total rotation as we move from ∞ to a disk at scale n is

$$\arg\left(f(z_0 + r_n) - f(z_0)\right) = \alpha\gamma \frac{\overline{K} - 1}{(1 - \alpha)} \sum_{k=1}^{n-1} \log \sigma_{k,j_k} + O(n)$$

Now we select our parameters σ_{k,j_k} as before, but with $\overline{d}$ and $\overline{K}$ replacing d and K respectively. With our usual notation

$$\sigma_{k,j_k} = R_{k,j_k}^{\frac{1-\alpha}{\alpha\overline{K}-1}} \eta_{k,j_k}$$

we can compute that

$$\begin{aligned}
&\frac{\arg(f(z_0 + r_n) - f(z_0))}{\log|f(z_0 + r_n) - f(z_0)|} \\
&= \frac{\alpha\gamma\frac{\overline{K}-1}{1-\alpha}\left[\frac{1-\alpha}{\alpha\overline{K}-1}\sum_{k=1}^{n-1}\log R_{k,j_k} + \sum_{k=1}^{n-1}\eta_{k,j_k}\right]}{\alpha\left(\overline{K}\frac{1-\alpha}{\alpha\overline{K}-1}\sum_{k=1}^{n-1}\log R_{k,j_k} + \sum_{k=1}^{n-1}\log R_{k,j_k} + \sum_{k=1}^{n-1}\eta_{k,j_k}\right)} \\
&\approx \frac{\alpha\gamma\frac{\overline{K}-1}{\alpha\overline{K}-1}}{\alpha(\overline{K}\frac{1-\alpha}{\alpha\overline{K}-1} + 1)}
\end{aligned}$$

$$= \frac{\alpha\gamma(\overline{K}-1)}{\alpha\overline{K}(1-\alpha)+\alpha\overline{K}-1}$$

$$= \gamma$$

as desired, where we have used the previous result that $\sum_{k=1}^{N} \eta_{k,j_k}$ is negligible in comaprison to $\sum_{k=1}^{N} R_{k,j_k}$. In particular, letting $n \to \infty$, the infinitesimal rotation is exactly γ. □

Now we would like to generalize this theorem to include stretching exponents greater than 1; this can be done by considering the inverse function f^{-1}, which inverts the stretching exponent and changes the sign of the rotation exponent. However, without assuming additional constraints on the perturbation h, it does not seem (to the best of the author's knowledge) possible to identify a gauge function h' for which

$$\mathcal{H}^{r^{d'}h'(r)}(f(E)) > 0.$$

It turns out that the key obstacle is a lack of decay in h; taking Section 4 of [63] as inspiration, we will impose the additional condition that for all $t > 0$, $h(t) \lesssim h(t^K)$. Powers of logarithms such as $(\log 1/r)^{-\beta}$ clearly satisfy this condition, so we still have a useful class of examples.

Theorem 2.4.5. *Let $\Lambda(r) = r^d h(r)$ be an admisible gauge function, with the additional constraint that $h(r^K) \lesssim h(r)$ for all $r > 0$. Let E and f be the stretching and rotation set and quasiconformal map constructed in Theorem 2.4.4, with exponents α and γ. Then f^{-1} stretches with exponent $1/\alpha$ and rotates with exponent $-\gamma$ at every point in $f(E)$ and $f(E)$ has positive measure with respect to the gauge function*

$$\Lambda'(r) = r^{d'} h^{d'/Kd}(r).$$

Since we have the additional decay constraint on h, the proof of this is a minor modification of that of Theorem 4.2 of [63], again proceeding through a Carleson type estimate. Rather than repeat a sketch of the argument, we will compare our conditions on h to those of Uriarte-Tuero. First of all, for technical reasons, it is important to use only a finite family of disks (as in [63]) at each generation of the Cantor set (rather, what is important is that there is a minimum choice of $R_{n,j}$ at each generation n, so that the construction of the next scale takes place on stricly smaller scales). In particular, choosing a sequence $\epsilon_n \to 0$ very quickly and packing an $(1-\epsilon_n)$ portion of the unit disk at each generation will only change the measure by a factor $\prod_{n=1}^{\infty}(1-\epsilon_n) \approx 1$, so the finiteness condition is not an obstacle.

Secondly, it is required in Uriarte-Tuero's construction that $h(t)$ is a (strictly) increasing function for which $t^\alpha/h(t) \to 0$ as $t \to 0$ for each $\alpha > 0$, that $h^{1/(2-d)}(t)/t$ is decreasing in t, and the logarithmic-type condition $h(t) \lesssim h(t^K)$. The first and fourth conditions hold here, as does the second by the definition of admissibility. Furthermore, admissibility applied with exponent $\epsilon = 2-d$ gives us that if $r < s$,

$$\frac{h^{\frac{1}{2-d}}(r)/r}{h^{\frac{1}{2-d}}(s)/s} \geq \frac{s}{r} C_\epsilon \left(\frac{r}{s}\right)^{\frac{\epsilon}{2-d}} = C_\epsilon$$

Although this does not show that $h^{1/(2-d)}(t)/t$ is decreasing, it is almost decreasing (and in fact, if s is small enough we may assume that $C_\epsilon > 1$, in which case we do have a decreasing function); it turns out that this is enough for the proof of the theorem to go through with minor modifications of the constants. It is worth pointing out, however, that the logarithmic-type decay condition is independent of the admissibility condition in the sense that neither is strong enough to imply the other.

Now to show the usefulness of the theorems, it would be nice to give an explicit and interesting gauge function. Fortunately, logarithmic perturbations of r^d are admissible, so we have a variety of gauges for which the theorem holds.

Corollary 2.4.6. *There are positive measure stretching and rotation sets associated to the gauges* $\Lambda(r) = r^d \left(\log \frac{1}{r}\right)^{-\beta}$ *for every* $\beta > 0$.

To be precise, this is not a well-defined gauge function for $r \geq 1$; we ought to cut it off at some point between 0 and 1 so it does not blow up. However, as we only really care about the behavior as r tends to zero, this is a point we will ignore; we will assume that $s \leq \frac{1}{100}$.

Proof. We will show that the gauge functions $\Lambda(r) = r^d \left(\log \frac{1}{r}\right)^{-\beta}$ are admissible for all $\beta > 0$; all that we need to prove is the growth condition. Fix $0 < r \leq s$ with s small, and $\epsilon > 0$. We need to show that there exists a constant $C_{\epsilon,\beta}$ for which

$$\frac{h(r)}{h(s)} \geq C_{\epsilon,\beta} \left(\frac{r}{s}\right)^{\epsilon}$$

or alternatively, that

$$\frac{s^{\epsilon}(\log(1/s))^{\beta}}{r^{\epsilon}(\log(1/r))^{\beta}}$$

is bounded below independent of r and s. We may just as well consider the functions

$$g(r,s) = \frac{s^{\epsilon/\beta} \log s}{r^{\epsilon/\beta} \log r}.$$

on the triangular domain $\{(r,s) : 0 < r \leq s \leq \frac{1}{100}\}$.

First, let us fix s; we minimize the function over r. The r-derivative is

$$\frac{\partial g}{\partial r} = \frac{s^{\epsilon/\beta} r^{\epsilon/\beta - 1}}{(r^{\epsilon/\beta} \log r)^2} (-\log s) \left[\frac{\epsilon}{\beta} \log r + 1\right],$$

which changes sign from negative to positive at $r = e^{-\beta/\epsilon}$. We now split into two cases, depending on the size of s.

The first case is that $s \geq e^{-\beta/\epsilon}$, so that $g(r, s)$ is in fact minimized at $r = e^{-\beta/\epsilon}$. In this case, we have

$$g(r, s) \geq g(e^{-\beta/\epsilon}, s) = -\frac{\epsilon e}{\beta} s^{\epsilon/\beta} \log s.$$

If we again differentiate, but in s, we get

$$-\frac{\epsilon e}{\beta} s^{\epsilon/\beta - 1} \left[\frac{\epsilon}{\beta} \log s + 1\right]$$

which is negative due to the fact that $s \geq e^{-\beta/\epsilon}$. Hence, this quantity is minimized when $s = \frac{1}{100}$; this gives a lower bound of

$$g(r, s) \geq g\left(e^{-\beta/\epsilon}, \frac{1}{100}\right) \qquad \forall\, 0 < r \leq s, s \geq e^{-\beta/\epsilon}.$$

This of course only depends on β and ϵ, which is good enough.

The second case is that $s < e^{-\beta/\epsilon}$. Here, we may compute the s-derivative, finding that

$$\frac{\partial g}{\partial s} = \frac{1}{r^{\epsilon/\beta} \log r} \left[\frac{\epsilon}{\beta} s^{\epsilon/\beta - 1} \log s + s^{\epsilon/\beta} +\right] = \frac{s^{\epsilon/\beta - 1}}{r^{\epsilon/\beta}} \frac{1}{\log r} \left[\frac{\epsilon}{\beta} \log s + 1\right].$$

Since $r < 1$, this is positive; therefore, g increases from its minimum value of 1 at the bottom of the domain where $r = s$, and is again bounded below. □

We can also apply this technique to get positive results for Riesz capacities. Recall that for a set E, the $(\beta,p)-$Riesz capacity $\dot{C}_{\beta,p}$ is defined by

$$\dot{C}_{\beta,p}(E) = \inf\left\{\|g\|_p : g * I_\beta \geq \chi_E\right\}$$

where up to a normalization, $I_\beta(z) = |z|^{-(2-\beta)}$ is the Riesz kernel; see, e.g. [1] for more details. There is also a dual characterization by Wolff's theorem that

$$\dot{C}_{\beta,p}(E) \simeq \sup\left\{\mu(E) : \mathrm{supp}(\mu) \subseteq E, \dot{W}^{\mu}_{\beta,p}(z) \leq 1\, \forall z \in \mathbb{C}\right\}$$

where the homogeneous Wolff potential $\dot{W}^{\mu}_{\beta,p}$ is

$$\dot{W}^{\mu}_{\beta,p}(z) = \int_0^\infty \left(\frac{\mu\big(B(z,r)\big)}{r^{2-\beta p}}\right)^{p'-1} \frac{dr}{r}.$$

Furthermore, it is important to note that the Riesz capacity is homogeneous of degree $2-\beta p$, which will correspond with the Hausdorff dimension of the set under consideration. Our main result here is the following theorem:

Theorem 2.4.7. *Fix any parameter $\tau = \alpha(1+i\gamma) \in B_K$ with $\alpha \in (1/K, 1)$, and a pair (β,p) with $1 < p < \infty$ and $2-\beta p = F_K(\alpha,\gamma)$. There is a K-quasiconformal map f and a set E such that f stretches with exponent α and rotates with exponent γ at every point in E, and E has positive $(\beta,p)-$Riesz capacity.*

In particular, this shows that there cannot be a theorem improving the results of [10] to the level of Riesz capacity zero for any choice of parameters with the correct homogeneity. This stands in sharp contrast with the results of [5], in which Riesz capacities were used to give sharper results than gauge functions alone can give. In [5], at the critical homogeneity, there was a range of parameters (β, p) in which extremal examples could exist, beyond which there was a negative result showing the sharpness of Riesz capacities. However, in our case, all possible indices have associated examples; thus an analogue of their theorem is not possible.

Proof. This theorem is actually much easier to prove than the last one, as the Riesz capacities of these Cantor type sets have already been estimated in [5]. We will first make the construction for a fixed (β, p), and then extend it in such a way that the set will have positive Riesz capacities for all parameter choices simultaneously. Let E be a Cantor type set as constructed in Theorem 2.4.2; our choice of parameters will be

$$\sigma_{k,j_k} = R_{k,j_k}^{\frac{2-d}{dK}} \left(\frac{k+1}{k}\right)^{\delta}$$

with δ to be chosen soon. Following the techniques of the previous proofs, we can compute that the stretching exponent is α at every point of E, while the rotation exponent is γ on a large subset of E.

Now it remains to understand the Riesz capacity of this set. Per Lemma 8.1 of [5], if ν is the naturally distributed measure on E, its Wolff potential is

$$\dot{W}_{\beta,p}^{\nu} \simeq \sum_{n=2}^{\infty} \frac{1}{n^{dK(p'-1)\delta}}$$

at each $x \in E$. If we select, e.g.

$$\delta = 1 + \frac{1}{dK(p'-1)}$$

then this series is convergent, the Wolff potential is uniformly bounded, and therefore the set has positive $(\beta, p)-$Riesz capacity.

Now we need to extend this from a particular parameter choice to all simultaneously. Carry out the above construction, but localized to a disk of radius 1/2. Fix a new choice (β_1, p_1) with $p_1 > p$, and carry out the construction with this parameter choice (meaning, with the updated value of δ) in a disjoint disk of radius 1/4. Continue in this manner with (β_2, p_2) with $p_2 > p_1$, and so on; this gives a set with positive Riesz capacity for a sequence (β_n, p_n) with $2 - \beta_n p_n = d$ for every n. If $p_n \to \infty$, a comparison theorem (e.g. Theorem 5.5.1(b) of [1]) shows that E has positive capacity for all parameter choices. □

It is worth remarking that this theorem actually follows from the previous one, with the correct choice of h (at least for these Cantor type sets). If we choose the gauge to be $h(r) = (\log 1/r)^{-1}$ for small enough r, then the resulting Cantor set must be larger, in a sense, than one only with positive Riesz capacity. This follows from an estimate of the η_{k,j_k}. Recall the generating relationship (2.4.2):

$$1 = \left(\eta_{1,j_1} \cdots \eta_{N,j_N}\right)^{Kd} h\left(R_{1,j_1}^{2/d} \cdots R_{N,j_N}^{2/d} \eta_{1,j_1}^K \cdots \eta_{N,j_N}^K\right).$$

We will show that the choice of η_{k,j_k} to satisfy this equation with this choice of gauge is typically larger than $(1+1/k)^\delta$; in particular, that means that the Cantor set naturally associated to this gauge is significantly larger than that constructed for positive Riesz capacity. To this end, suppose that $\eta_{k,j_k} \leq (1+1/k)^\delta$ for all k. Then we have

$$\begin{aligned} 1 &= \left(\eta_{1,j_1} \cdots \eta_{N,j_N}\right)^{Kd} h\left(R_{1,j_1}^{2/d} \cdots R_{N,j_N}^{2/d} \eta_{1,j_1}^K \cdots \eta_{N,j_N}^K\right) \\ &= (N+1)^{Kd\delta} h\left(R_{1,j_1}^{2/d} \cdots R_{N,j_N}^{2/d} (N+1)^{K\delta}\right) \end{aligned}$$

$$= \frac{(N+1)^{Kd\delta}}{\frac{2}{d}\sum_{k=1}^{N}\log\frac{1}{R_{k,j_k}} - K\delta\log(N+1)}.$$

However, we can choose the radii R_{k,j_k} as small as we desire, making the right hand side of this equation arbitrarily small. This leads to a contradiction, showing that this uniformly bounded selection of η_{k,j_k} was in fact too small. Hence at least some of the selections η_{k,j_k} must have been larger than $(1+1/k)^{\delta}$, a contradiction. Moreover, asymptotically, the choices of η_{k,j_k} must be much larger than $(1+1/k)^{\delta}$, and larger choices of η_{k,j_k} lead to a larger Cantor set.

Chapter 3

Improved Hölder Continuity of Quasiconformal Maps

This work first appeared in [22].

3.1 Introduction

A K-quasiconformal map f is an orientation-preserving homeomorphism between two domains in the complex plane, lying in the Sobolev space $W^{1,2}_{\text{loc}}$ and satisfying the distortion inequality

$$\max_{\beta} |\partial_\beta f| \le K \min_{\beta} |\partial_\beta f|$$

for almost all z, where ∂_β is the directional derivative in the direction β. These maps can be realized as solutions to the Beltrami equation

$$\partial_{\overline{z}} f = \mu(z) \partial_z f \tag{3.1.1}$$

where the Beltrami coefficient $\mu(z)$ has the bound $\|\mu\|_\infty \le \frac{K-1}{K+1} < 1$ and represents the complex distortion of the function f. Such maps have useful geometric and regularity properties, and provide a natural framework for generalizing conformal maps. They arise naturally in a number of applications, and are closely related with the solutions to elliptic PDEs in the plane.

In this paper, we will be concerned with the precise degree of regularity and smoothness properties of quasiconformal maps; we will be most interested in determining what Hölder continuity such maps have (that is, which Lipschitz class the functions lie in). A function f defined on an open set Ω is said to be locally α-Hölder continuous if for each compact set $E \subseteq \Omega$ there exists a constant $C = C(f, E)$ with

$$|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in E$; equivalently, f lies in the Lipschitz class $C^\alpha(\Omega)$. It is well known that K-quasiconformal maps are Hölder continuous with exponent $1/K$, due to an old theorem of Mori [46]. More recently, quantitative upper and lower bounds on the size (in the sense of Hausdorff measure and dimension) of the set where f can attain the worst-case Hölder regularity were given by Astala, Iwaniec, Prause, and Saksman [10] and the author [21].

However, the exponent $1/K$ is not always optimal for a K-quasiconformal map. For example, there are bilipschitz K-quasiconformal maps defined on $\mathbb{C}$ which are not $(K - \epsilon)$-quasiconformal for any $\epsilon > 0$. A particular case of this is a map exhibiting rotation, such as $z|z|^{i\gamma}$ for an appropriately chosen exponent $\gamma \in \mathbb{R}$. Therefore, it is apparent that the exact Hölder regularity of a quasiconformal map depends on more than just the magnitude of the complex distortion and should instead encode something about the structure of the distortion. A result of Ricciardi [56] gave a great deal of information; in that paper, it was shown that if f is a solution to the Beltrami equation $\partial_{\overline{z}} f = \mu \partial_z f$ then f is α-Hölder continuous with exponent

$$\alpha \geq \left(\sup_{S_{\rho,x} \subset \Omega} \frac{1}{|S_{\rho,x}|} \int_{S_{\rho,x}} \frac{|1 - \overline{\eta}^2 \mu|^2}{1 - |\mu|^2} \, d\sigma \right)^{-1} \tag{3.1.2}$$

where $S_{\rho,x}$ is a circle with radius ρ centered at $x \in \Omega$, η is an outward unit normal, and $d\sigma$ is the arclength measure. This was proven through a sharp Wirtinger inequality. The integrand here also appeared in [55] in the context of ring modules; for more information, as well as some related estimates and theorems about extremizers, see the book [18] of Bojarski, Gutlyanskii, Martio, and Ryazanov.

An important application of the regularity results for quasiconformal maps is their connection with solutions to elliptic partial differential equations of the form

$$\operatorname{div}(A \nabla u) = 0 \tag{3.1.3}$$

where $z \mapsto A(z)$ is an essentially bounded, symmetric, measurable, matrix-valued function satisfying $\lambda\langle \xi, \xi \rangle \leq \langle \xi, A(z)\xi \rangle \leq \Lambda\langle \xi, \xi \rangle$ for some $0 < \lambda \leq \Lambda < \infty$ at almost every z. Just as there is a correspondence between the Cauchy-Riemann equation $\partial_{\overline{z}} f = 0$ and the Laplacian $\Delta u = 0$ (where $f = u + iv$ and v is the harmonic conjugate of u), there is a correspondence between the $\mathbb{C}$-linear Beltrami equation (3.1.1) and the divergence form elliptic equation (3.1.3), where $f = u + iv$ and v is the A-harmonic conjugate of u. The exact details of this correspondence can be found in, e.g. Chapter 16 of [8].

Solutions to these equations are known to be Hölder continuous, and the study of their regularity has a long history. Hölder continuity of solutions to (3.1.3) was shown by De Giorgi [28]; later, Piccinini and Spagnolo [52] gave a quantitative estimate that the Hölder continuity exponent of u is at least $\sqrt{\lambda/\Lambda}$ (as well as further improved bounds for the case of an isotropic matrix A). More recently, Ricciardi [57] showed that the Hölder exponent is at least

$$\alpha \geq \left(\sup_{S_{\rho,x} \subset \Omega} \frac{1}{|S_{\rho,x}|} \int_{S_{\rho,x}} \langle \eta, A\eta \rangle d\sigma \right)^{-1}.$$

The main result of this paper will be an improvement of the Hölder continuity exponent given in (3.1.2) to incorporate an extra term involving the geometry of the underlying map. In particular, we will show that

Theorem 3.1.1. *Let $f : \Omega \to \Omega'$ be a continuous and $W^{1,2}_{loc}$ solution to the Beltrami equation $\partial_{\overline{z}} f = \mu(z) \partial_z f$ with $|\mu(z)| \leq \frac{K-1}{K+1}$. Then f is α-Hölder continuous for some exponent α satisfying*

$$\alpha \geq \left[4\pi \sup_{S_{\rho,x} \subset \Omega} \frac{|f(\mathbb{D}_{\rho,x})|}{\mathcal{H}^1\big(f(S_{\rho,x})\big)^2} \sup_{S_{\rho,x} \subset \Omega} \frac{1}{|S_{\rho,x}|} \int_{S_{\rho,x}} \frac{|1 - \bar{\eta}^2 \mu|^2}{1 - |\mu|^2} d\sigma \right]^{-1}$$

where $S_{\rho,x}$ is the circle centered at x with radius ρ, η is the outward unit normal, and σ is arclength measure.

Here, it is important to note that the isoperimetric inequality guarantees that

$$4\pi \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1(f(\mathbb{S}_t))^2} \leq 1$$

for all t (and is frequently strictly less than 1); here, $\mathbb{D}_t$ is the disk centered at the origin with radius t. Our result therefore gives an improvement over the previously known regularity whenever we can impose an upper bound on $4\pi \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1(f(\mathbb{S}_t))^2}$; for example, any affine map which stretches differently in two orthogonal directions will exhibit this. Furthermore, we can use this information to determine the structure of the extremizers for Hölder continuity. We have the following definition:

Definition 3.1.2. *Let f be a K-quasiconformal map, a K-quasiregular map, or a solution to* (3.1.3) *with ellipticity constants satisfying $\sqrt{\Lambda/\lambda} = K$. We say that f is an extremizer for Hölder continuity at the origin if f is not more than $1/K$-Hölder continuous there. In particular, for each $\epsilon > 0$, there is a sequence $r_n \to 0$ such that $|f(r_n) - f(0)| \geq r_n^{1/K+\epsilon}$.*

We can now give the form of the complex distortion of a K-quasiconformal map that exhibits the worst-case regularity. Motivated by the fact that the Beltrami coefficient of the radial stretch $z|z|^{1/K-1}$ is $-kz/\overline{z}$ with $k = \frac{K+1}{K-1}$, we have the following result:

Theorem 3.1.3. *Suppose that f is K-quasiconformal and an extremizer for Hölder continuity at the origin. Write the Beltrami coefficient in the form $\mu(z) = \frac{z}{\overline{z}}(-k + \epsilon(z))$. Then there is a sequence of scales $t_n \to 0$ for which*

$$\int_{t_n}^{1} \frac{1}{r} \left(\frac{1}{|S_r|} \int_{S_r} \operatorname{Re} \epsilon(z)\, d\sigma \right) dr = o\left(\log \frac{1}{t_n} \right).$$

There is an analogous theorem for the geometric distortion properties of such extremizers, and how far they can be from a map which preserves circularity:

Theorem 3.1.4. *Suppose that f is K-quasiconformal and an extremizer for Hölder continuity at the origin. Define a function δ by $\frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1(f(S_t))^2} = \frac{1}{4\pi(1+\delta(t))}$. Then there is a sequence $t_n \to 0^+$ such that*

$$\int_{t_n}^{1} \frac{\delta(r)}{r}\, dr = o\left(\log \frac{1}{t_n} \right).$$

Our last main results are on the regularity and Hölder continuity extremizers of solutions to the elliptic equation (3.1.3). Suppose $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a continuous solution to $\operatorname{div}(A\nabla u) = 0$ on a simply connected domain Ω, that A is symmetric, measurable, and satisfies the ellipticity bound $\frac{1}{K}|\xi|^2 \leq \langle \xi, A(z)\xi \rangle \leq K|\xi|^2$ almost everywhere. Let v be the A-harmonic conjugate of u and $f = u + iv$. We will show that

Theorem 3.1.5. *If u is an extremizer for Hölder continuity at the origin, u must be of the form*

$$u = \Phi \circ g$$

where Φ is harmonic with non-vanishing gradient at $g(0)$, g is K-quasiconformal, and g exhibits the worst-case regularity for a K-quasiconformal map. In particular, the bounds of the previous two theorems apply to the Beltrami coefficient and circular distortion of g.

The outline of this paper is as follows. In Section 2, we prove the main theorem on Hölder continuity of quasiconformal maps. In Section 3, we use this information to classify the extremizers and study geometric distortion, and extend the results to quasiregular maps. In Section 4, we apply the results of the previous sections to solutions to elliptic PDEs.

3.2 Estimate of Hölder Exponent

The main result of this section is estimate the exact degree of Hölder continuity of a K-quasiconformal map through the behavior of the associated Beltrami coefficient.

Theorem 3.2.1. *Let $f : \Omega \to \Omega'$ be a continuous and $W^{1,2}_{loc}(\Omega)$ solution to the Beltrami equation $\partial_{\overline{z}} f = \mu(z)\partial_z f$ with $|\mu(z)| \leq \frac{K-1}{K+1}$. Then f is α-Hölder continuous for some exponent α satisfying*

$$\alpha \geq \left[4\pi \sup_{S_{\rho,x} \subset \Omega} \frac{|f(\mathbb{D}_{\rho,x})|}{\mathcal{H}^1\big(f(S_{\rho,x})\big)^2} \sup_{S_{\rho,x} \subset \Omega} \frac{1}{|S_{\rho,x}|} \int_{S_{\rho,x}} \frac{|1 - \overline{\eta}^2 \mu|^2}{1 - |\mu|^2} d\sigma \right]^{-1}$$

where $S_{\rho,x}$ is the circle centered at x with radius ρ, η is the outward unit normal, and σ is arclength measure.

Here, the suprema can be regarded as the essential supremum over the radii ρ for each point x. Note that the isoperimetric inequality guarantees that

$$4\pi \sup_{S_{\rho,x}\subset\Omega} \frac{|f(\mathbb{D}_{\rho,x})|}{\mathcal{H}^1\big(f(S_{\rho,x})\big)^2} \leq 1$$

and so we recover the result (3.1.2) of Ricciardi for the case of the homogeneous Beltrami equation; moreover,

$$\int_{S_{\rho,x}} \frac{|1-\bar{\eta}^2\mu|^2}{1-|\mu|^2} d\sigma \leq \int_{S_{\rho,x}} \frac{(1+|\mu|)^2}{1-|\mu|^2} d\sigma = \int_{S_{\rho,x}} \frac{1+|\mu|}{1-|\mu|} d\sigma \leq K|S_{\rho,x}|$$

for almost every circle (that is, for almost every positive radius). This recovers the classic exponent of $1/K$.

The theorem also shows that Hölder continuity of a quasiconformal map is locally determined by the structure of the Beltrami coefficient. For example, if there is an open set E where $\|\mu\chi_E\|_\infty < \|\mu\|_\infty$, then the quasiconformal map displays better-than-expected Hölder continuity on the entirety of the open set. As will be proved later on, this idea also gives powerful constraints on the complex distortion of a map which has the worst-case Hölder continuity (even at a single point). It is worth mentioning that these results also follow from Stoilow factorization. We now turn to the proof of Theorem 3.2.1.

Proof. Without loss of generality, we will look at circles centered at the origin. Our starting point will be an adaptation of a classical argument of Morrey [47]. To this end, define $\varphi(t) = \int_{\mathbb{D}_t} J_f = |f(\mathbb{D}_t)|$. If we can show that $\varphi(t) \leq \varphi(1)t^{2c}$, then quasisymmetry shows that the worst case length distortion is controlled by

$$|f(te^{i\theta}) - f(0)|^2 \sim_K |f(\mathbb{D}_t)| \le \varphi(1)t^{2c}$$

which implies a Hölder exponent no worse than c. Our task is therefore to estimate φ, which we will do by controlling φ by its derivative.

In order to do this, we will compute the circumference of the quasicircle $f(S_t)$ explicitly, where S_t has radius t and is centered at the origin. Parameterize the quasicircle by $\gamma(\theta) = f(te^{i\theta})$ for $\theta \in [0, 2\pi]$; note that for almost every t, the quasicircle has positive and finite length. Indeed, since $f \in W^{1,2}_{\text{loc}}$, f is absolutely continuous on the circle S_t for almost every $t \in [0, \infty)$. Likewise, since the Beltrami coefficient μ is defined almost everywhere with respect to the area measure, Fubini's theorem guarantees that μ is defined at almost every point (with respect to arclength) on almost every circle. We also have that $f_{\overline{z}} = \mu f_z$ almost everywhere in the plane, so almost everywhere on almost every circle. Thus, we can compute the length by

$$\text{Length} = \int_0^{2\pi} \left| \frac{d}{d\theta}\gamma(\theta) \right| \, d\theta.$$

Writing $f = u + iv$, we have that

$$\begin{aligned}
\frac{1}{t^2}\left|\frac{d}{d\theta}\gamma\right|^2 &= \frac{1}{t^2}\left|\frac{d}{d\theta}\left[u(t\cos\theta, t\sin\theta) + iv(t\cos\theta, t\sin\theta)\right]\right|^2 \\
&= \frac{1}{t^2}\left|-tu_x \sin\theta + tu_y\cos\theta + i\big[-tv_x\sin\theta + tv_y\cos\theta\big]\right|^2 \\
&= u_x^2\sin^2\theta + u_y^2\cos^2\theta - 2u_xu_y\sin\theta\cos\theta \qquad (3.2.1) \\
&\qquad + v_x^2\sin^2\theta + v_y^2\cos^2\theta - 2v_xv_y\sin\theta\cos\theta \\
&= |f_x|^2\sin^2\theta + |f_y|^2\cos^2\theta - 2\sin\theta\cos\theta\big(u_xu_y + v_xv_y\big) \qquad (3.2.2)
\end{aligned}$$

If we write the Beltrami equation in terms of $x-$ and $y-$derivatives rather than the Wirtinger derivatives, we find that

$$f_x + if_y = \mu(f_x - if_y) \implies f_x(1-\mu) = -if_y(1+\mu) \implies f_y = i\frac{1-\mu}{1+\mu}f_x. \tag{3.2.3}$$

Furthermore,

$$\operatorname{Re}(\overline{f_x}f_y) = \operatorname{Re}\big((u_x - iv_x)(u_y + iv_y)\big) = u_xu_y + v_xv_y.$$

Thus, the final term in (3.2.2) can be replaced with

$$\begin{aligned}
u_xu_y + v_xv_y &= \operatorname{Re}(\overline{f_x}f_y) \\
&= \operatorname{Re}\left(\overline{f_x}i\frac{1-\mu}{1+\mu}f_x\right) \\
&= |f_x|^2\left(-\operatorname{Im}\frac{1-\mu}{1+\mu}\right) \\
&= -\frac{|f_x|^2}{|1+\mu|^2}\operatorname{Im}\big((1-\mu)(1+\overline{\mu})\big) \\
&= -\frac{|f_x|^2}{|1+\mu|^2}\operatorname{Im}(\overline{\mu}-\mu) \\
&= \frac{2|f_x|^2\operatorname{Im}\mu}{|1+\mu|^2} \qquad (3.2.4)
\end{aligned}$$

We also wish to rewrite f_x in terms of the Jacobian, so as to relate φ to φ'. We have

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = (1-|\mu|^2)|f_z|^2.$$

On the other hand, $f_x = f_z + f_{\overline{z}} = (1+\mu)f_z$, so that

$$J_f = \frac{1-|\mu|^2}{|1+\mu|^2}|f_x|^2 \tag{3.2.5}$$

Combining the equations (3.2.2)-(3.2.5), we find that the length $\mathcal{H}^1\big(f(S_t)\big)$ of the quasicircle $f(S_t)$ can be written as

$$\int_0^{2\pi} \frac{|1+\mu|}{\sqrt{1-|\mu|^2}} J_f^{1/2} \sqrt{\sin^2\theta + \left|\frac{1-\mu}{1+\mu}\right|^2 \cos^2\theta - 4\sin\theta\cos\theta \frac{\operatorname{Im}\mu}{|1+\mu|^2}}\, t d\theta$$

$$= \int_0^{2\pi} \frac{J_f^{1/2}}{\sqrt{1-|\mu|^2}} \sqrt{|1+\mu|^2\sin^2\theta + |1-\mu|^2\cos^2\theta - 4\sin\theta\cos\theta\operatorname{Im}\mu}\, t d\theta. \tag{3.2.6}$$

It remains to simplify the term within the square root. We can expand it to find that

$$\begin{aligned}
|1+\mu|^2\sin^2\theta + &|1-\mu|^2\cos^2\theta - 4\sin\theta\cos\theta\operatorname{Im}\mu \\
&= 1+|\mu|^2 + 2\operatorname{Re}\mu\sin^2\theta - 2\operatorname{Re}\mu\cos^2\theta - 4\sin\theta\cos\theta\operatorname{Im}\mu \\
&= 1+|\mu|^2 - 2\operatorname{Re}\mu\cos 2\theta - 2\operatorname{Im}\mu\sin 2\theta \\
&= 1+|\mu|^2 - 2\operatorname{Re}\big((\cos 2\theta - i\sin 2\theta)(\operatorname{Re}\mu + i\operatorname{Im}\mu)\big) \\
&= 1+|\mu|^2 - 2\operatorname{Re}(e^{-2i\theta}\mu) \\
&= |1-e^{-2i\theta}\mu|^2
\end{aligned} \tag{3.2.7}$$

Noting that $e^{i\theta} = \eta$ is the outer normal from the circle, combining this with (3.2.6) we arrive at

$$\mathcal{H}^1\big(f(S_t)\big) = \int_0^{2\pi} \left(\frac{|1-\overline{\eta}^2\mu|^2}{1-|\mu|^2}\right)^{1/2} J_f^{1/2}\, t\, d\theta. \tag{3.2.8}$$

We are now ready to make the estimate of φ. Denote

$$A = 4\pi \sup_t \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1\big(f(S_t)\big)^2} \qquad \text{and} \qquad C = \sup_t \frac{1}{|S_t|}\int_{S_t} \frac{|1-\overline{\eta}^2\mu|^2}{1-|\mu|^2}\, d\sigma.$$

recalling that $d\sigma = td\theta$ is arclength. We then have for almost every t that

$$\begin{aligned}
\varphi(t) &= \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1\big(f(S_t)\big)^2}\mathcal{H}^1\big(f(S_t)\big)^2 \\
&\leq \frac{1}{4\pi}A\left(\int_0^{2\pi}\left(\frac{|1-\overline{\eta}^2\mu|^2}{1-|\mu|^2}\right)^{1/2} J_f^{1/2}\, t\, d\theta\right)^2 \\
&\leq \frac{1}{4\pi}A\int_0^{2\pi}\frac{|1-\overline{\eta}^2\mu|^2}{1-|\mu|^2}\, t\, d\theta \int_0^{2\pi} J_f\, t\, d\theta \\
&= \frac{1}{2}At\left(\frac{1}{2\pi t}\int_{S_t}\frac{|1-\overline{\eta}^2\mu|^2}{1-|\mu|^2}\, d\sigma\right)\left(\int_0^{2\pi} J_f\, t\, d\theta\right) \\
&\leq \frac{1}{2}ACt\varphi'(t) \qquad (3.2.9)
\end{aligned}$$

for almost every t. We then find that

$$\frac{d}{dt}\left[t^{-2/AC}\varphi(t)\right] = t^{-2/AC-1}\left[-\frac{2}{AC}\varphi(t) + t\varphi'(t)\right] \geq 0.$$

almost everywhere. Integrating this inequality over $[t, 1]$ leads to $\varphi(t) \leq \varphi(1)t^{2/AC}$, which is the desired result. □

3.3 Extremizers for Hölder Continuity

Here we will study the structure of the Beltrami equation for the extremizers of Hölder continuity. Recall that by Mori's Theorem, a K-quasiconformal map is at least $\frac{1}{K}$-Hölder continuous. We will show that, in some sense, the extremizers must have Beltrami coefficients which are very close to the coefficient for a pure radial stretch. Denote $k = \frac{K-1}{K+1}$; since the

Beltrami coefficient has $|\mu| \le k$, we can always write

$$\mu(z) = e^{2i\theta}\left(-k + \epsilon(z)\right) \tag{3.3.1}$$

with θ being the argument of z, and ϵ some function such that has nonnegative real part. Note that $-ke^{2i\theta}$ is precisely the Beltrami coefficient of the radial stretch $z|z|^{1/K-1}$. We now have our result:

Theorem 3.3.1. *Suppose that f is K-quasiconformal and an extremizer for Hölder continuity at the origin. Write the Beltrami coefficient in the form* (3.3.1)*. Then there is a sequence of scales $t_n \to 0$ for which*

$$\int_{t_n}^{1} \frac{1}{r} \int_{S_r} \operatorname{Re} \epsilon(z)\, d\tau\, dr = o\left(\log \frac{1}{t_n}\right)$$

where $d\tau = \frac{d\sigma}{2\pi r}$ is normalized arclength.

Morally, this says that the circular averages of $\operatorname{Re} \epsilon(z)$ are tending to zero in some quantitative sense - otherwise, the integral $\int_{t_n}^{1} \frac{dr}{r}$ would give some non-zero fraction of $\log 1/t_n$.

Proof. We proceed through two steps; the first is to estimate the impact that our perturbation by ϵ has on

$$g(t) := \int_{S_r} \frac{|1 - e^{-2i\theta}\mu|^2}{1 - |\mu|^2}\, d\tau$$

and the second is to sharpen the estimate of $|f(\mathbb{D}_t)|$ accordingly. Recall that Theorem 3.2.1 (and in particular equation (3.2.9) with the estimate $A \le 1$) tells us that

$$\varphi(r) \le \frac{rg(r)}{2}\varphi'(r)$$

for almost every $r \in [0,1]$. Integrating this differential inequality, we find that

$$\ln \frac{\varphi(1)}{\varphi(t)} = \int_t^1 \frac{\varphi'(r)}{\varphi(r)}\, dr \geq \int_t^1 \frac{2}{rg(r)}\, dr \tag{3.3.2}$$

Later on, we will prove that there is a constant $c_1 > 0$ such that

$$g(r) \leq K - c_1 \int_{S_r} \operatorname{Re} \epsilon \, d\tau \tag{3.3.3}$$

so that

$$\frac{1}{g(r)} \geq \frac{1}{K} \cdot \frac{1}{1 - (c_1/K) \int_{S_r} \operatorname{Re} \epsilon \, d\tau} \geq \frac{1}{K} \left[1 + c \int_{S_r} \operatorname{Re} \epsilon \, d\tau \right]$$

with $c = c_1/K > 0$. With this estimate in mind, we can continue (3.3.2) to find that

$$\begin{aligned} \ln \frac{\varphi(1)}{\varphi(t)} &\geq \int_t^1 \frac{2}{r} \cdot \frac{1}{K} \left[1 + c \int_{S_r} \operatorname{Re} \epsilon \, d\tau \right] dr \\ &= \ln t^{-2/K} + \frac{2c}{K} \int_t^1 \frac{1}{r} \int_{S_r} \operatorname{Re} \epsilon \, d\tau dr \end{aligned} \tag{3.3.4}$$

Rearranging this gives us

$$\varphi(t) \leq \varphi(1) t^{2/K} \exp \left(-2 \frac{c}{K} \int_t^1 \frac{1}{r} \int_{S_r} \operatorname{Re} \epsilon \, d\tau dr \right) \tag{3.3.5}$$

Now if f is no more regular than $1/K$-Hölder continuous and $\gamma > 0$, there is a sequence of scales $r_n \to 0$ (depending on γ) for which $\varphi(r_n) \geq r_n^{2/K+\gamma} \varphi(1)$ for all n. We therefore have

$$r_n^{2/K+\gamma} \leq r_n^{2/K} \exp \left(-2 \frac{c}{K} \int_{r_n}^1 \frac{1}{r} \int_{S_r} \operatorname{Re} \epsilon \, d\tau dr \right) \tag{3.3.6}$$

which is the key estimate behind our constraint on the structure of $\operatorname{Re}\epsilon$. Equivalently,

$$\begin{aligned} \gamma \log r_n &\leq -2\frac{c}{K}\int_{r_n}^{1}\frac{1}{r}\int_{S_r}\operatorname{Re}\epsilon\, d\tau dr \\ \implies \frac{\gamma K}{2c}\log\frac{1}{r_n} &\geq \int_{r_n}^{1}\frac{1}{r}\int_{S_r}\operatorname{Re}\epsilon\, d\tau dr \end{aligned} \tag{3.3.7}$$

Taking a sequence of $\gamma_m \to 0$ and choosing scales t_m appropriately gives the desired estimate.

All that remains now is to show (3.3.3). Writing $\mu = e^{2i\theta}(-k + \operatorname{Re}\epsilon)$ and choosing $w = -k + \operatorname{Re}\epsilon$ we have that

$$\frac{|1 - e^{-2i\theta}\mu|^2}{1 - |\mu|^2} = \frac{|1 - w|^2}{1 - |w|^2}.$$

We will show that

$$\frac{|1 - w|^2}{1 - |w|^2} - \frac{1 + k}{1 - k} \lesssim -\operatorname{Re}\epsilon \tag{3.3.8}$$

where the implied constant only depends on k; the estimate (3.3.3) follows immediately from integrating this over S_r, recalling that $\frac{1+k}{1-k} = K$ and that $d\tau$ is a probability measure. It is worth mentioning that the estimate $|1 - w|^2/(1 - |w|^2) - (1 + k)/(1 - k) \leq 0$ is immediate from the triangle inequality, but that we need to sharpen it a little bit. To carry out this estimate, observe that since w is real we have

$$\begin{aligned} \frac{|1 - w|^2}{1 - |w|^2} - \frac{1 + k}{1 - k} &= \frac{1 - w}{1 + w} - \frac{1 + k}{1 - k} \\ &= \frac{(1 - w)(1 - k) - (1 + k)(1 + w)}{(1 + w)(1 - k)} \\ &= \frac{-2(w + k)}{(1 + w)(1 - k)} \end{aligned}$$

$$= \frac{-2\operatorname{Re}\epsilon}{(1+w)(1-k)} \tag{3.3.9}$$

Now $w \le k$ since $\epsilon(z)$ lies within the disk centered at k with radius k, and so we have

$$\frac{|1-w|^2}{1-|w|^2} - \frac{1+k}{1-k} \le \frac{-2\operatorname{Re}\epsilon}{(1+k)(1-k)}$$

which proves (3.3.3). □

In a similar manner, we can study the local distortion properties of an extremizer. Infinitesimally, a quasiconformal map must take circles to ellipses with bounded eccentricity; we will show that (in an appropriate sense), the extremizers for Hölder continuity must almost take circles to circles. Our approach to this is similar to the previous theorem: we will write $|f(\mathbb{D}_t)|/\mathcal{H}^1(f(S_t))^2$ as a perturbation of $1/4\pi$ and control the perturbation.

Theorem 3.3.2. *Suppose that f is K-quasiconformal and an extremizer for Hölder continuity at the origin. Define functions h and δ by*

$$h(t) := \frac{|f(\mathbb{D}_t)|}{\mathcal{H}^1(f(S_t))^2} = \frac{1}{4\pi(1+\delta(t))}.$$

Then there is a sequence $t_n \to 0^+$ such that

$$\int_{t_n}^1 \frac{\delta(r)}{r}\, dr = o\left(\log \frac{1}{t_n}\right).$$

Proof. As a first remark, the isoperimetric inequality shows that $h(t) \le \frac{1}{4\pi}$ for all t, so $\delta(t) \ge 0$ everywhere; also, $\delta(t) < \infty$ almost everywhere. In analogy with the previous theorem, we can conclude from (3.2.9) that

$$\varphi(t) \leq 2\pi K t h(t) \varphi'(t).$$

Rearranging and integrating leads to

$$\varphi(t) \leq \varphi(1) \exp\left(-\int_t^1 \frac{dr}{2\pi K r h(r)}\right) \tag{3.3.10}$$

Now since f is an extremizer for Hölder continuity, for any sequence $\gamma_n \to 0^+$ there is a sequence of scales $t_n \to 0^+$ for which $\varphi(t_n) \geq \varphi(1) t_n^{2/K(1+\gamma_n)}$; combining this with (3.3.10) leads to

$$t_n^{\frac{2}{K}(1+\gamma_n)} \leq \exp\left(-\int_{t_n}^1 \frac{dr}{2\pi K r h(r)}\right).$$

Taking a logarithm and rearranging, we find that

$$\begin{aligned}
\frac{2}{K}(1+\gamma_n)\log t_n &\leq -\int_{t_n}^1 \frac{dr}{2\pi K r h(r)} \\
&= -\int_{t_n}^1 \frac{dr}{2\pi K r \frac{1}{4\pi(1+\delta(r))}} \\
&= -\frac{2}{K}\int_{t_n}^1 \frac{1+\delta(r)}{r}\, dr \\
&= \frac{2}{K}\log t_n - \frac{2}{K}\int_{t_n}^1 \frac{\delta(r)}{r}\, dr.
\end{aligned} \tag{3.3.11}$$

Rearranging this leads to

$$\int_{t_n}^1 \frac{\delta(r)}{r}\, dr \leq \gamma_n \log \frac{1}{t_n}$$

as desired. □

Corollary 3.3.3. *Suppose f is an extremizer for Hölder continuity at the origin and δ is* *defined as in Theorem 3.3.2. Then for any $\delta_0 > 0$, the set $\{r : \delta(r) > \delta_0\}$ has zero lower density at* 0.

Proof. Fix $\delta_0 > 0$ and suppose, intending a contradiction, that the lower density of $\{r : \delta(r) > \delta_0\}$ had lower density at least $\eta > 0$ at zero. Then there exists a scale $\epsilon > 0$ such that for all $\gamma < \epsilon$,

$$\frac{|\{r : \delta(r) > \delta_0\} \cap [0, \gamma]\}|}{\gamma} > \frac{\eta}{2}.$$

Consequently, there exists an M depending only on η such that for all $\gamma < \epsilon$,

$$\frac{|\{r : \delta(r) > \delta_0\} \cap [\gamma/M, \gamma]\}|}{\gamma} > \frac{\eta}{100}.$$

Therefore, we can estimate that

$$\begin{aligned} \int_{\gamma/M}^{\gamma} \frac{\delta(r)}{r}\, dr &\geq \frac{\delta_0}{\gamma} \left|[\gamma/M, \gamma] \cap \{r : \delta(r) > \delta_0\}\right| \\ &> \frac{\delta_0}{\gamma} \cdot \frac{\gamma\eta}{100} \\ &= \frac{\delta_0 \eta}{100} \end{aligned} \tag{3.3.12}$$

Summing (3.3.12) over intervals $[\gamma/M, \gamma], [\gamma/M^2, \gamma/M]$ and so on, it is immediate that

$$\int_t^1 \frac{\delta(r)}{r}\, dr \gtrsim \log \frac{1}{t}$$

for all sufficiently small t. This contradicts the result of Theorem 3.3.2 and the corollary follows. □

Finally, as usual, there is a natural generalization of the quasiconformal result to the quasiregular result.

Theorem 3.3.4. *A K-quasiregular map g is an extremizer for Hölder continuity at the origin if and only if it is of the form $g = \Phi \circ f$, where f is K-quasiconformal and an extremizer for Hölder continuity at the origin, and Φ is conformal in a neighborhood of $f(0)$.*

Proof. Suppose g is a K-quasiregular extremizer. By the Stoilow factorization theorem, there exists a holomorphic Φ and K-quasiconformal f such that

$$g = \Phi \circ f.$$

Since Φ is a smooth function, $\Phi \circ f$ is at least as regular as f is (in particular, if f is α-Hölder continuous, then so is g); but since g is not more than $1/K$-Hölder continuous, neither is f. Furthermore, we must have $\Phi'(f(0)) \neq 0$ (which implies conformality); otherwise, $\Phi(f(z)) - \Phi(f(0))$ vanishes to at least second order at 0, and $\Phi \circ f$ is Hölder continuous at 0 with exponent at least $\min\{1, 2/K\} > 1/K$.

On the other hand, if $g = \Phi \circ f$ with f an extremizer and Φ conformal at $f(0)$, we can locally invert Φ as a smooth function, so that $f = \Phi^{-1} \circ g$. Hence f is at least as regular as g is; but since f has the worst-case regularity, so must g. □

3.4 Applications to Elliptic PDEs

Next, we will use the relationship between quasiconformal maps and solutions to elliptic partial differential equations in order to deduce regularity results and classify the extremizers for Hölder continuity. The starting point for this work is the correspondence laid down in [8], Chapter 16. Throughout, we will assume that $A(z)$ is a matrix valued function which is measurable, symmetric, and satisfies the ellipticity bound

$$\frac{1}{K}|\xi|^2 \leq \langle A(z)\xi, \xi \rangle \leq K|\xi|^2$$

at almost every $z \in \Omega$. Note that this implies that $A(z)$ is positive definite, and an equivalent formulation is the unified inequality

$$|\xi|^2 + |A(z)\xi|^2 \leq \left(K + \frac{1}{K}\right) \langle A(z)\xi, \xi \rangle.$$

Consider the divergence form equation

$$\operatorname{div} A(z)\nabla u = 0. \tag{3.4.1}$$

If Ω is a simply connected domain and $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a solution to (3.4.1), the Poincaré lemma guarantees that there exists an A-harmonic conjugate v; that is, $v \in W^{1,2}_{\text{loc}}(\Omega)$ solves

$$\nabla v = *A(z)\nabla u$$

where $*$ is the Hodge star operator, viewed as the matrix

$$* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Define $f = u + iv$; we now claim that the ellipticity condition implies that f is K-quasiregular. Following Theorem 3.3.4, we will be able to use this to determine the regularity and extremizers for Hölder continuity. To see that f is actually quasiregular, we may compute that

$$\|Df\|^2 = |\nabla u|^2 + |\nabla v|^2 = |\nabla u|^2 + |*A(z)\nabla u|^2 \leq \left(K + \frac{1}{K}\right)\langle A(z)\nabla u, \nabla u\rangle.$$

It is immediate to check that $\langle A(z)\nabla u, \nabla u\rangle = J(z, f)$ is the Jacobian of f, and we therefore have $\|Df\|^2 \leq (K + \frac{1}{K})J(z, f)$. Rewriting the Hilbert-Schmidt norm in terms of Wirtinger derivatives, this implies that $|\partial_{\bar{z}} f| \leq \frac{K-1}{K+1}|\partial_z f|$ as desired. This brings us to the first theorem on regularity, which is a direct application of Theorem 3.2.1:

Theorem 3.4.1. *Let $u \in W^{1,2}_{loc}(\Omega)$ be a continuous $W^{1,2}_{loc}(\Omega)$ solution to* (3.4.1) *on a simply connected domain Ω, where v is its A-harmonic conjugate, and $f = u + iv$. Let $\mu(z)$ denote the complex distortion of f. Then u is α-Hölder continuous with some exponent α, where*

$$\alpha \geq \left[4\pi \sup_{S_{\rho,x} \subset \Omega} \frac{|f(\mathbb{D}_{\rho,x})|}{\mathcal{H}^1\big(f(S_{\rho,x})\big)^2} \sup_{S_{\rho,x} \subset \Omega} \frac{1}{|S_{\rho,x}|} \int \frac{|1 - \bar{\eta}^2 \mu|^2}{1 - |\mu|^2} d\sigma\right]^{-1}.$$

In particular, $u = \operatorname{Re} f$ is Hölder continuous with exponent at least $1/K$.

In a similar manner, we can find the extremizers for Hölder continuity.

Theorem 3.4.2. *With the assumptions and notation of Theorem 3.4.1, suppose that u is an extremizer for Hölder continuity at the origin. Then there is a harmonic map Φ and a K-quasiconformal map g such that g is an extremizer for Hölder continuity at the origin, Φ has non-vanishing gradient in a neighborhood of $g(0)$, and $f = \Phi \circ g$. In particular, the Beltrami coefficient and circular distortion of g satisfy the bounds of Theorems 3.3.1 and 3.3.2 respectively.*

Proof. This is essentially the same idea as the proof of Theorem 3.3.4. Since u is the real part of a K-quasiregular map f, we can use Stoilow factorization to write $u = (\operatorname{Re}\Psi) \circ g$

with Ψ holomorphic and g being K-quasiconformal. As before, Ψ must have non-vanishing gradient at $g(0)$ (so as not to improve the regularity), and g must be an extremizer for Hölder continuity at the origin; the result follows. □

Finally, we also have a generalization of the result on extremizers to nonlinear elliptic PDEs.

Theorem 3.4.3. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and suppose $\mathcal{A} : \Omega \times \mathbb{C} \to \mathbb{C}$ is measurable in $z \in \Omega$ and continuous in $\xi \in \mathbb{C}$ and satisfies the ellipticity condition*

$$|\xi|^2 + |\mathcal{A}(z,\xi)|^2 \leq \left(K + \frac{1}{K}\right) \langle \xi, \mathcal{A}(z,\xi)\rangle.$$

Then if $u \in W^{1,2}_{loc}(\Omega)$ is a solution to

$$\operatorname{div} \mathcal{A}(z, \nabla u) = 0 \tag{3.4.2}$$

Then if u is an extremizer for Hölder continuity at the origin, u is the form of Theorem 3.4.2.

Proof. By Theorem 16.1.8 of [8], every solution $u \in W^{1,2}_{\text{loc}}(\Omega)$ of (3.4.2) solves a linear elliptic equation

$$\operatorname{div} \mathbb{A}(z)\nabla u = 0$$

with $\mathbb{A}(z)$ a positive definite, symmetric measurable matrix field of determinant 1. Furthermore, we have the ellipticity bound $\frac{1}{K}|\xi|^2 \leq \langle \mathbb{A}(z)\xi, \xi\rangle \leq K|\xi|^2$ and the result follows from Theorem 3.4.2. □

Chapter 4

Geometric Bounds for Favard Length

This work first appeared in [20].

4.1 Introduction

Given a set E in the plane, its Favard length is the average

$$\mathrm{Fav}(E) = \int_0^{2\pi} |\pi_\theta E| \, d\theta$$

where π_θ is orthogonal projection onto a line L_θ through the origin at angle θ to the positive x-axis, and $|.|$ is the length measure within the line L_θ. This quantity is comparable to the Buffon needle probability of the set E; this is the probability that a needle dropped near the set E passes through it.

The Favard length of a set carries a great deal of metric and geometric information about the set. It is deeply related to rectifiability; Bescovitch proved in [17] that a set with positive and finite length is purely unrectifiable if and only if it has Favard length zero. In such a case, the dominated convergence theorem implies that the Favard lengths of the r-neighborhoods of a bounded set E (that is, the set $E(r)$ of points of distance less than r from E) must tend to zero as r does. The exact rate of decay is another measure of the size of a set, and is related to Minkowski dimension. It is also conjectured that Favard length is controlled by analytic capacity in many circumstances.

In this paper, we will give geometrically motivated proofs for various properties of Favard length. First, we will reprove a result of Mattila from [44] that connects the decay rate of the Favard lengths of the neighborhoods of a set with the Hausdorff dimension of the underlying set:

Theorem 4.1.1. *Fix $s \in (0,1)$ and suppose that $E \subseteq \mathbb{R}^2$ is measurable, and $A \subseteq S^1$ is measurable with positive (arc-length) measure. Suppose there exists a sequence of scales $r_n \to 0$ such that*

$$\int_A |\pi_\theta (E(r_n))| \, d\theta \leq C r_n^s$$

for some $C < \infty$. Then $\dim_{\mathcal{H}} E \leq 1 - s$.

The original proof of this theorem relies on potentials and estimates of the energy of a measure; we will prove this result here with a direct geometric argument.

In the next section, we will show how self-similarity leads to new and useful properties of the sequence of Favard lengths. In particular, we will show that:

Theorem 4.1.2. *Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets such that $A_{n+1} \subseteq A_n$ for all n, that each generation can be written as a union*

$$A_{n+1} = \bigcup_{i=1}^{N} r_i A_n + \beta_i$$

for some fixed set of contraction ratios $r_i > 0$, and that $\sum_i r_i = 1$. Then for each θ, the sequence $\{|\pi_\theta A_n|\}_{n\in\mathbb{N}}$ is convex.

Note that since the sum of contraction ratios is 1, the sequence of sets converges to an attractor which is a self-similar set of Hausdorff dimension at most 1 (and if the similitudes satisfy the open set condition, it has Hausdorff dimension equal to 1). See, e.g., Chapter 4 of [45] for more details.

Convexity gives a powerful constraint on the decay rate of Favard lengths: the decay within the first few generations controls the decay until much later stages. In particular, it is very easy to recover the result that:

Corollary 4.1.3. *If $\mathcal{K}_n$ is the n-th generation of the four-corner Cantor set, then* $\mathrm{Fav}(\mathcal{K}_n) \gtrsim 1/n$.

Before we begin the proofs, we first define some notation. Given a set A, we will denote its Lebesgue measure by $|A|$; depending on the context, this could mean the Lebesgue measure within a line, or area measure in the plane, or arc-length measure in the circle. If it is clear from context which one of these is meant, we will not specify.

4.2 Dimension and Favard Length

In this section, we will prove that sufficiently quick decay of Favard length of neighborhoods of a set controls the Hausdorff dimension of the set. For a set E in some Euclidean space and $r > 0$, we denote the r-neighborhood of E by

$$E(r) = \{x : \mathrm{dist}(x, E) < r\}.$$

The decay rate of the Lebesgue measure of $E(r)$ as $r \to 0$ is connected with the Minkowski dimension of the underlying set, as well as other notions of size. Our main result is a new proof of the following theorem:

Theorem 4.2.1. *Fix $s \in (0, 1)$ and suppose that $E \subseteq \mathbb{R}^2$ is measurable, and $A \subseteq S^1$*

is measurable with positive (arc-length) measure. Suppose there exists a sequence of scales $r_n \to 0$ *such that*

$$\int_A |\pi_\theta\left(E(r_n)\right)|\, d\theta \leq C r_n^s$$

for some $C < \infty$*. Then* $\dim_{\mathcal{H}} E \leq 1 - s$*.*

The contrapositive of this theorem appeared in [44] with sharper bounds involving both the measure of E with respect to an appropriate measure and the measure of the angle set A; here, we only have a result at the level of dimension. Mattila's argument relies on studying the energy of a measure; here, we use a direct geometric argument. The previous proof relies on being able to find a measure supported on the set that satisfies certain decay conditions, which is guaranteed for compact (or Suslin) sets by Frostman's lemma. Our technique has the advantage of avoiding questions of the existence of such a measure, so we do not need any additional topological assumptions about the set.

Proof. We proceed in three steps. First, we need to find a particular direction where the projection $\pi_\theta E$ has full dimension while simultaneously having almost sufficiently quick decay of $|\pi_\theta E(r_n)|$. Secondly, we will use a Hölder inequality to control the sum of lengths over a natural cover on the projection side; this gives control on the Hausdorff measure of the projection. Finally, we tighten the bounds by adjusting exactly how quickly $|\pi_\theta E(r_n)|$ decays. Note that we do not need to differentiate between the sets $\pi_\theta(E(r))$ and $(\pi_\theta E)(r)$ (that is, the neighborhood of a projection within a line); they are equal.

First, note that E has Hausdorff dimension at most 1; otherwise, a result of Marstrand in [41], Chapter II, would imply that E has strictly positive projection length in almost every direction, which would contradict that $\int_A |\pi_\theta(E)|\, d\theta \leq \int_A |\pi_\theta(E(r_n))|\, d\theta \leq C r_n^s \to 0$. (Of course, this follows from Mattila's work in [44] or Chapter 9 of [45]; however, we are trying to avoid the use of potentials). For each n, we can consider a set of angles

$$A_n = \left\{\theta \in A : \dim_{\mathcal{H}}(\pi_\theta E) = \dim_{\mathcal{H}} E \text{ and } |\pi_\theta E(r_n)| \leq r_n^{s-\epsilon}\right\}.$$

The first condition holds for almost all θ; this also follows from Chapter II of [41]. Secondly, since $\int_A |\pi_\theta E(r_n)| \leq C r_n^s$, we can estimate the size of the exceptional set A_n^c by

$$|A \cap A_n^c| \leq C r_n^{\epsilon}.$$

Passing to a subsequence of scales (which we also denote as r_n) if necessary, we can assume that $\sum_n |A \setminus A_n| < |A|$; thus, there exists an angle $\varphi \in \bigcap_n A_n$. In particular, $\pi_\varphi E$ has Hausdorff dimension equal to that of E itself.

Next, we will control the Hausdorff measure of $\pi_\varphi E$ at dimensions a little above s. Note that $\pi_\varphi E(r_n)$ consists of a union of disjoint intervals $I_{n,k}$, each having length at least $2r_n$. This forms a natural cover of $\pi_\varphi(E)$. We can estimate the number of intervals in the cover via

$$r_n^{s-\epsilon} \geq |\pi_\varphi E(r_n)| = \sum_k |I_{n,k}|$$

Using that $|I_{n,k}| \gtrsim r_n$, we can rearrange this to find that there are at most $r_n^{s-\epsilon-1}$ such intervals. We are now in a position to estimate sums of the form $\sum_k |I_{n,k}|^p$ for $p \in (0,1)$. A direct application of Hölder's inequality shows that if $p \in (0,1)$, q satisfies $1/p - 1/q = 1$, and μ is any measure,

$$\int fg\, d\mu \geq \left(\int f^p\, d\mu\right)^{1/p} \left(\int g^{-q}\, d\mu\right)^{-1/q}$$

holds for measurable functions. Taking this with counting measure, we find that

$$\begin{aligned} r_n^{s-\epsilon} &\geq \sum_k |I_{n,k}| \\ &\geq \left(\sum_k |I_{n,k}|^p\right)^{1/p} \left(\sum_k 1^{-q}\right)^{-1/q} \\ &\gtrsim \left(\sum_k |I_{n,k}|^p\right)^{1/p} \left(r_n^{s-\epsilon-1}\right)^{-1/q} \end{aligned} \tag{4.2.1}$$

Rearranging this leads to

$$\left(\sum_k |I_{n,k}|^p\right)^{1/p} \lesssim r_n^{s-\epsilon+\frac{1}{q}(s-\epsilon-1)}.$$

The exponent can be simplified as

$$\left(1+\frac{1}{q}\right)(s-\epsilon) - \frac{1}{q} = \frac{1}{p}\left(s-\epsilon-(1-p)\right).$$

As long as $s-\epsilon-(1-p) \geq 0$, we can give a uniform upper bound on $\sum_k |I_{n,k}|^p$; this works provided that $p \geq 1-s+\epsilon$. Furthermore, one can see from (4.2.1) that each $I_{n,k}$ has radius no larger than $r_n^{s-\epsilon}$, which tends to zero as n grows. Combining these observations leads to

$$\mathcal{H}^{1-s+\epsilon}\left(\pi_\varphi E\right) < \infty.$$

so that $\dim_{\mathcal{H}} \pi_\varphi(E) \leq 1-s+\epsilon$.

Finally, take a smaller ϵ and rerun the argument with a (potentially) new choice of φ.

Taking a sequence $\epsilon_m \to 0$, we then get a sequence φ_m of angles and

$$\dim_{\mathcal{H}} E = \dim_{\mathcal{H}} \pi_\varphi(E) \le 1 - s + \epsilon_m \to 1 - s.$$

This is the desired bound on dimension. □

4.3 Self-similar Sets

In this section, we will show how self-similarity can be used to give lower bounds on the Favard length, as well as control the behavior of the sequence of projection lengths. Our result is

Theorem 4.3.1. *Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets such that $A_{n+1} \subseteq A_n$ for all n, that each generation can be written as a union*

$$A_{n+1} = \bigcup_{i=1}^{N} r_i A_n + \beta_i$$

for some fixed set of contraction ratios $r_i > 0$, and that $\sum_i r_i = 1$. Then for each θ, the sequence $\{|\pi_\theta A_n|\}_{n\in\mathbb{N}}$ is convex.

Proof. Let us define $E_{n,\theta} = \pi_\theta A_n$ and $\alpha_n(\theta) = |E_{n,\theta}|$; note that on the projection side, $E_{n,\theta}$ is also self-similar and is generated by similitudes of the form $T_i : x \mapsto r_i x + \pi_\theta \beta_i$, where x is measured within the line L_θ. We then have

$$\begin{aligned}\alpha_n(\theta) - \alpha_{n+1}(\theta) &= |E_{n,\theta}| - |E_{n+1,\theta}| \\ &= |E_{n,\theta} \setminus E_{n+1,\theta}|\end{aligned}$$

$$
\begin{aligned}
&= \left| \bigcup_{i=1}^{N} T_i(E_{n-1,\theta}) \setminus \bigcup_{j=1}^{N} T_j(E_{n,\theta}) \right| \\
&\leq \left| \bigcup_{i=1}^{N} \big(T_i(E_{n-1,\theta}) \setminus T_i(E_{n,\theta}) \big) \right| \\
&= \left| \bigcup_{i=1}^{N} T_i(E_{n-1,\theta} \setminus E_{n,\theta}) \right| \\
&\leq \sum_{i=1}^{N} r_i |E_{n-1,\theta} \setminus E_{n,\theta}| \\
&= \alpha_{n-1}(\theta) - \alpha_n(\theta).
\end{aligned}
$$

where we have used that $\{E_{n,\theta}\}_{n\in\mathbb{N}}$ is a decreasing sequence of sets for each θ, that each T_i is a contraction by r_i along with a translation, and that $\sum_i r_i = 1$. Rearranging this, we find that

$$\alpha_n(\theta) \leq \frac{\alpha_{n-1}(\theta) + \alpha_{n+1}(\theta)}{2}$$

which is the desired result. □

As a corollary, we can easily deduce lower bounds on the Favard length of the four-corner Cantor set. Recall that the generations of this set are constructed by taking $\mathcal{K}_0 = [0,1]^2$; then $\mathcal{K}_n$ is constructed by taking each square in $\mathcal{K}_{n-1}$, dividing it into four sections of equal width horizontally and vertically, and taking the four subsquares at the corners. The result is a family of 4^n squares of sidelength 4^{-n} each. Alternatively, the set is generated by the similitudes $f_i(z) = \frac{1}{4}z + \beta_i$, with $\{\beta_i : 1 \leq i \leq 4\} = \{(0,0), (0,3/4), (3/4,0), (3/4,3/4)\}$. We have the following result:

Corollary 4.3.2. *The Favard lengths of the generations of the four-corner Cantor set satisfy* $\mathrm{Fav}(\mathcal{K}_n) \gtrsim \frac{1}{n}$.

Proof. Fix $n \in \mathbb{N}$. Note that if we take $\theta^* = \arctan 1/2$, the projection π_{θ^*} maps the four components of $\mathcal{K}_1$ to four intervals that only overlap on the boundaries (and therefore, $\pi_\theta \mathcal{K}_n$ is the same interval for all n, as an application of self-similarity). Therefore, $\alpha_0(\theta^*) - \alpha_1(\theta^*) = 0$. Furthermore, $\theta \mapsto \alpha_0(\theta) - \alpha_1(\theta)$ is piecewise C^1 and the derivative is bounded by 10 (which follows from a direct computation of the function $\alpha_0 - \alpha_1$); moreover, $\alpha_0(\theta) \geq 1$ for all θ. Hence, there is an interval I_n of length $\frac{1}{20n}$ centered at θ^* such that

$$0 \leq \alpha_0(\theta) - \alpha_1(\theta) \leq \frac{1}{2n}$$

for all θ in the interval. Applying convexity iteratively leads to $\alpha_n(\theta) \geq \frac{1}{2}$ for all $\theta \in I_n$, and so

$$\mathrm{Fav}(\mathcal{K}_n) \geq \int_{I_n} \alpha_n(\theta)\, d\theta \geq \frac{1}{40n}$$

as desired. $\square$

Note that the key idea here is that there is a special angle at which the projection acts (more or less) bijectively on components. It follows that this technique is applicable to a broad class of self-similar sets with such an angle - the Sierpinski gasket is another important example. One hopes that tightening the losses of this technique would be sufficient to improve the estimate past $1/n$; it was proved in [15] that the Favard length of $\mathcal{K}_n$ is actually at least $c \ln n / n$.

It is worth mentioning that the function $n \mapsto |E(4^{-n})|$ is not generally convex without the self-similarity assumption. For example, the set

$$\{0, 1/4, 1/2, 3/4, ..., 100\}$$

(or a small neighborhood of it) serves as a counterexample. A modification of this example (using ever finer lattices around carefully selected points in the set), one can find examples where $n \mapsto |E(4^{-n})|$ is neither eventually convex nor eventually concave. Rather, the sequence exhibits "see-saw" behavior as it decays to zero.

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