ON SOME ASPECTS OF CLUSTER ALGEBRAS AND COMBINATORIAL HOPF ALGEBRAS

By

John Machacek

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ABSTRACT

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This dissertation deals with problems in cluster algebras and combinatorial Hopf algebras. Total positivity has been closely related to cluster algebras since their inception. Postnikov's totally nonnegative Grassmannian is a concrete example of total positivity with rich combinatorics. Our first problem is the computation of Plücker coordinates inside a generalization of the totally nonnegative Grassmannian. We provide a combinatorial formula in terms of edge weighted directed graphs embedded on a surface. The next problem we consider is the equality of a cluster algebra and its upper cluster algebra. Particular attention is paid to the coefficient ring of the cluster algebra. We give a sufficient condition for the cluster algebra and upper cluster algebra to coincide while allowing greater generality of coefficient ring than was previous known. The final problem we consider in cluster algebras is showing that log-canonical coordinates are as simple as possible (in a certain precise sense). Log-canonical coordinates are a fundamental part of the Poisson geometry approach to cluster algebras put forth by Gekhtman, Shapiro, and Vainshtein. In the theory of combinatorial Hopf algebras we compute a formula for the antipode in a Hopf algebra on simplicial complexes. This antipode formula generalizes Humpert and Martin's formula for graphs. We then use the character theory of Aguiar, Bergeron, and Sottile to realize a version of Stanley's chromatic symmetric function for simplicial complexes. We prove that the degree sequence of a uniform hypertree can be recovered from its chromatic symmetric function. We also show the chromatic symmetric function is not a complete invariant for uniform hypertrees.

To Melinda Pearl

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Chapter 1

Cluster algebra overview

This document consists of work in various areas of cluster algebras and combinatorial Hopf algebras. The overview in this chapter discusses the cluster algebra portion of this dissertation. The original research contributions in the area of cluster algebra theory are contained in Chapters 5, 6, and 7. Chapters 1–4 serve as background material putting the original results into context. Chapter 2 reviews Postnikov's totally nonnegative Grassmannian. Necessary notions in the theory of cluster algebras will be defined in Chapter 3. Poisson structures compatible with cluster algebra structures will be discussed in Chapter 4. We now explain some of the larger context of the particular results which will appear later. This chapter will be a general big picture overview. Any necessary definitions and precise statements will be given later.

Fomin and Zelevinsky's cluster algebras [FZ02] provide a framework to study a commutative algebra by dividing its generators into groups called clusters. Cluster algebras have an axiomatic definition through a process known as mutation which transforms one cluster to another. At first glance the axioms of a cluster algebra may seem artificial, but they are in fact natural. Cluster algebras were originally defined to study the canonical basis in Lusztig's theory of total positivity [Lus94, Lus98]. In addition to the original goal of studying the canonical basis in Lusztig's theory of total positivity and explaining relations between generators in certain coordinate rings, cluster algebra structures have since been

found to naturally appear many places in algebra, geometry, and physics. A (nonexhaustive) list of places where mutation and cluster algebras show up in other areas include: Seiberg duality in quiver gauge theories [Sei95], representation theory of quivers [BM06], Higher Teichmüller theory [FG06], Poisson geometry [GSV10], and computation of scattering amplitudes [GGS$^+$14]. We now further expand on the aspects of cluster algebras we will focus on.

1.1 Total positivity and scattering amplitudes

Postnikov's totally nonnegative Grassmannian [Pos06] is a particular example of Lusztig's more general theory of total nonnegativity. The totally nonnegative Grassmannian comes with a rich combinatorial structure. One aspect of this combinatorial structure is a parameterization of the totally nonnegative Grassmannian by edge weighted directed graphs embedded on a disk. These edge weighted directed graphs give rise to another combinatorial object, called an alternating strand diagram or Postnikov diagram, which can be used to define a cluster algebra structure on the homogenous coordinate ring of a Grassmannian [Sco06]. These graphs have also become useful in high energy physics for the computation of scattering amplitudes [AHBC$^+$16]. We will study a related construction which considers edge weighted directed graphs embedded on an orientable surface in Chapter 5.

The main content of Chapter 5 is a proof of a conjecture, appearing the physics literature, on a combinatorial formula for minors of a matrix associated to edge weight directed graphs embedded on an orientable surface [FGPW15]. Physicists have proposed a construction generalizing Postnikov's parameterization of the totally nonnegative Grassmannian by replacing the disk with any orientable surface. It is hoped this more general construction will

further aid in the computation of scattering amplitudes. For any orientable surface we show that the exists a combinatorial formula for Plücker coordinates which is similar to Talaska's formula in the case of the disk [Tal08]. Our proof makes use of Talaska's generalization of the Lindström-Gessel-Viennot lemma [Tal12]. We also remark that prior to the generalization to any orientable surface in the physics literature, Postnikov's parameterization of the totally nonnegative Grassmannian was first generalized to edge weighted directed graphs on the annuls in the context of Poisson geometry [GSV08].

1.2 Coordinate rings and upper cluster algebras

Cluster structures can be observed in coordinate rings of many natural algebraic varieties. Some examples include Grassmannians [Sco06] and double Bruhat cells of complex simple Lie groups [BFZ05]. A closely related object called the upper cluster algebra can, as it does for Double Bruhat cells [BFZ05, Theorem 2.10], coincide with the (homogeneous) coordinate ring of the variety. The cluster algebra is contained in the corresponding upper cluster algebra, and in general this can be a proper containment. When we have equality of the cluster algebra and upper cluster algebra, such coordinate rings can be approached by utilizing the explicit combinatorial description of the generators of the cluster algebra. So, determining when we have equality of the cluster algebra and upper cluster algebra is an important problem in the theory.

The question of equality of the cluster algebra and upper cluster algebra is considered in Chapter 6. An important choice in defining a cluster algebra is deciding which ground ring to generate the cluster algebra over. When considering cluster algebra structures on a coordinate ring, the choice of ground ring effects whether the cluster algebra structure is

actually on the coordinate ring of the variety or instead on the coordinate ring of an open subset. In their paper introducing upper cluster algebras Bernstein, Fomin, and Zelevinsky showed, with a coprimeness assumption, that there is equality of the cluster algebra and upper cluster algebra for acyclic cluster algebras [BFZ05, Corollary 1.19]. Muller has shown that this coprimeness assumption is not need and developed the theory of locally acyclic cluster algebras [Mul13, Mul14]. A locally acyclic cluster algebra is known to coincide with its upper cluster algebra for a certain choice of ground ring. Showing a cluster algebra is locally acyclic is a practical way to show it is equal to its upper cluster algebra. An example application is Muller and Speyer's result which states the cluster algebra structure on the homogenous coordinate ring of the Grassmannian is locally acyclic [MS16]. This can be used to conclude the cluster algebra coincides with the upper cluster algebra for the coordinate ring of an open subvariety known as a postiroid variety. In Chapter 6 we address the dependence of the choice of ground ring on deciding equality of the cluster algebra and upper cluster algebra. A condition for when there is equality of the cluster algebra and upper cluster algebra is given by using a variation of Muller's theory of cluster localization for more general ground rings. An explicit example exhibiting dependence on the ground ring is also provided.

1.3 Cluster algebras and Poisson geometry

Gekhtman, Shapiro, and Vainshtein have created an approach to the theory of cluster algebras considering certain compatible Poisson brackets [GSV10]. This approach makes use of Poisson brackets of a certain form called log-canonical. Given a cluster algebra, the log-canonical Poisson brackets give a "nice" coordinate system on an associated geometric

object. In Chapter 7 we provide an answer to how nice this coordinate system is. It turns out that, in a certain precise sense, it is the "nicest" possible coordinate system when one is only allowed a rational change of coordinates. In light of this we can consider log-canonical coordinates as an algebraic analog of Darboux coordinates from symplectic geometry.

Chapter 2

The Nonnegative Grassmannian

This chapter reviews the totally nonnegative Grassmannian and Postnikov's boundary measurement parameterization of the totally nonnegative Grassmannian [Pos06] via edge weighted directed graphs embedded on the disk.

We let $\mathrm{Gr}(k,n)$ denote the real Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Each subspace $V \in \mathrm{Gr}(k,n)$ can be represented by a $k \times n$ matrix A of rank k such that the rows of A span V. In this case we write $V = [A]$. The matrix A is unique up to left multiplication of elements of the general linear group. Thus,

$$\mathrm{Gr}(k,n) = \mathrm{GL_k}\backslash\{\mathrm{k} \times \mathrm{n} \text{ matrices of rank k}\}$$

and we obtain the *Plücker embedding*

$$\mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

$$[A] \mapsto (\Delta_I(A))_I$$

where $\Delta_I(A)$ denotes the maximal minor with columns indexed by a k-element subset I of $[n]$. A minor $\Delta_I(A)$ is called a *Plücker coordinate*. The *totally nonnegative Grassmannian*, denoted $\mathrm{Gr}^{\geq 0}(k,n)$, consists of all $V \in \mathrm{Gr}(k,n)$ such that there exists A where $V = [A]$ and $\Delta_I(A)$ is nonnegative for each I. Postnikov introduced the totally nonnegative Grassmannian

in the seminal preprint [Pos06]. There is a wealth of combinatorial objects which play on important roles in the study of $\mathrm{Gr}^{\geq 0}(k,n)$ including: positroids, decorated permutations, directed networks, and more. We will be mostly interested in the boundary measurement of directed networks. Boundary measurement will be introduced on the disk in Section 2.2 and Chapter 5 will focus on a generalization of boundary measurement. First we will describe positroids and certain stratifications of the Grassmannian in Section 2.1.

2.1 Positroids

Let $\binom{[n]}{k}$ denote the collection of k-element subsets of $[n] = \{1, 2, \ldots, n\}$. A *matroid* of rank k on the set $[n]$ is a subset $\mathcal{M} \subseteq \binom{[n]}{k}$ such that for all $I, J \in \mathcal{M}$ there exists $i \in I$ and $j \in J$ for which $(I \setminus \{i\}) \cup \{j\} \in \mathcal{M}$. Matroids generalize the concept of linear independence in vectors spaces. We will be particularly interested in matroids that come for linearly independence in real vector spaces. Given a full rank $k \times n$ real matrix A we obtain a matroid of rank k on $[n]$ known as an $\mathbb{R}$-*linear matroid* defined by

$$\mathcal{M}_A := \{I : \Delta_I(A) \neq 0\}.$$

Considering the columns $v_1, v_2, \ldots, v_n$ of A as vectors in $\mathbb{R}^k$, a subset $I \in \mathcal{M}_A$ if and only if $\{v_i : i \in I\}$ is a basis of $\mathbb{R}^k$. Given any $V \in \mathrm{Gr}(k,n)$ we define $\mathcal{M}_V := \mathcal{M}_A$ if $V = [A]$. Note this matroid is well-defined since A is unique up to multiplication by elements of the general linear group, and multiplication by an invertible matrix will not change the underlying

matroid. The Grassmannian then has the *matroid stratification*

$$\mathrm{Gr}(k,n)\bigcup_{\mathcal{M}} S_{\mathcal{M}}$$

where

$$S_{\mathcal{M}} = \{V \in \mathrm{Gr}(k,n) : \mathcal{M}_V = \mathcal{M}\}$$

are known as *matroid stratum*.

Example 2.1.1 (A real linear matroid)**.** If we consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

we then have

$$\mathcal{M}_A = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}.$$

A matroid $\mathcal{M}$ is called a *positroid* if $\mathcal{M} = \mathcal{M}_A$ for a matrix A which has $\Delta_I(A) \geq 0$ for all I. We then get a decomposition of the totally nonnegative Grassmannian

$$\mathrm{Gr}^{\geq 0}(k,n) = \bigcup_{\substack{\text{positroids} \\ \mathcal{M}}} S_{\mathcal{M}}^{\geq 0}$$

where $S_{\mathcal{M}}^{\geq 0} = S_{\mathcal{M}} \cap \mathrm{Gr}^{\geq 0}(k,n)$. The next example shows that not all matroids are positroids, or equivalently it shows that $S_{\mathcal{M}} \cap \mathrm{Gr}^{\geq 0}(k,n)$ may be empty.

Example 2.1.2 (Non-positroid)**.** We claim the $\mathbb{R}$-linear matroid from Example 2.1.1

$$\mathcal{M} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$$

of rank 2 on [4] is not a positroid. If $\mathcal{M}$ was a positroid, we would have to be $\mathcal{M} = \mathcal{M}_A$ for

$$A = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{bmatrix}$$

for $a, b \in \mathbb{R}$ with $a < 0$, $b > 0$, and $ab > 0$.

Each $S_{\mathcal{M}}^{\geq 0}$ is homeomorphic to an open ball, and the decomposition of $\mathrm{Gr}^{\geq 0}(k,n)$ into the cells $S_{\mathcal{M}}^{\geq 0}$ is a CW-complex [Pos06, Theorem 3.5]. Postnikov's approach of giving of CW-structure on $\mathrm{Gr}^{\geq 0}(k,n)$ agrees with a more general construction of cells in theory of total positivity given by Marsh and Rietsch [MR04].

The *positroid envelope* of a matroid $\mathcal{M}$ is denoted by $\mathcal{E}(\mathcal{M})$ and is the unique smallest positroid containing the matroid [KLS13, Section 3]. Given a positroid $\mathcal{M}$ the corresponding *open positroid variety* is

$$\Pi^{\circ}(\mathcal{M}) := \bigcup_{\substack{\mathcal{M}' \\ \mathcal{E}(\mathcal{M}')=\mathcal{M}}} S_{\mathcal{M}'}.$$

This decomposition groups matroid strata whose matroids have the same positroid envelope. We include this discussion of open positroid varieties because it will be relevant later when we consider cluster algebras and upper cluster algebras as coordinate rings in Section 3.4 and Chapter 6.

2.2 Boundary measurement

Postnikov [Pos06] has shown how to construct a matrix A such that $[A] = V$ for each $V \in \mathrm{Gr}^{\geq 0}(k,n)$. If $V \in \mathrm{Gr}^{\geq 0}(k,n)$, then the matrix A is constructed by considering an edge weighted directed graph embedded on the disk with n boundary vertices exactly k of which

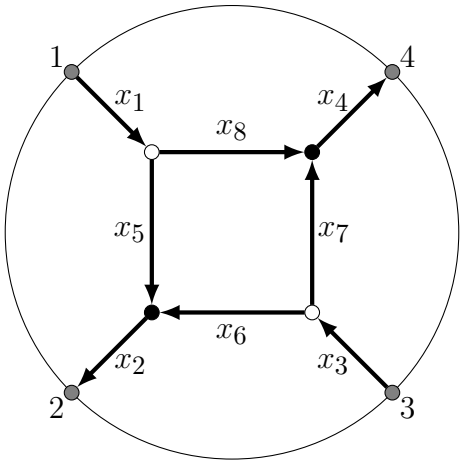

Figure 2.1: An edge weighted directed graph on the disk.

are sources. Such graphs on other surfaces will be considered in Chapter 5. We now briefly illustrate boundary measurement on the disk with an example.

We consider a directed graph G, modulo homotopy, embedded on a disk such that G has n vertices on the boundary of the disk which are labeled by $K = \{1, 2, \ldots, n\}$ in counterclockwise order. Each boundary vertex is either a source or sink. An example of such a graph can be found in Figure 2.1.

Remark 2.2.1. The directed graph in Figure 2.1 is an example of what is known as a *perfectly orientated network*. The definition of a perfectly orientated network will be given in Chapter 5. Also, the coloring of the vertices in the Figure 2.1 will be explained in Chapter 5.

Let $I \subseteq K$ denote the set of boundary sources. So, $I = \{1, 3\}$ and $K = \{1, 2, 3, 4\}$ for the graph in Figure 2.1. The weight of a directed path is taken to be the product of the weights of the edges in the path. Assume G has no directed cycles[1]. We then form the $I \times K$ *boundary measurement matrix* $B(G)$ with entry $B(G)_{ik}$ given by the sum of weights of all directed paths from $i \in I$ to $k \in K$ multiplied by $(-1)^{s_{ik}}$ where s_{ik} in the number of

[1]This assumption is not necessary. We make this assumption now for simplicity of presentation. Directed cycles will be allowed in a more general treatment in Chapter 5

element of I strictly between i and k. For the graph in Figure 2.1 we obtain the matrix

$$B = \begin{bmatrix} 1 & x_1x_5x_2 & 0 & -x_1x_8x_4 \\ 0 & x_3x_6x_2 & 1 & x_3x_7x_4 \end{bmatrix}.$$

The first row B records paths beginning at boundary vertex 1, while the second row of B records paths beginning at boundary vertex 3. We consider the empty path, which has weight equal to 1, to be the only path from a boundary source to itself. The only path from boundary vertex 1 to 2 has weight $x_1x_5x_2$ and the only path from boundary vertex 1 to 4 has weight $x_1x_8x_4$. We see $x_1x_5x_2$ has a positive sign in B since $s_{12} = 0$ while $x_1x_8x_4$ has a negative sign in B since $s_{14} = 1$.

The matrix B is a full rank 2×4 matrix, and hence represents an element of $\mathrm{Gr}(2, 4)$. The Plücker coordinates (listed in lexicographic order) in this case are

$$(x_3x_6x_2, 1, x_3x_7x_4, x_1x_5x_2, x_1x_2x_3x_4(x_5x_7 + x_6x_8), x_1x_8x_4)$$

which all evaluate to nonnegative values when each x_i is given a nonnegative real value. We can observe that the inclusion of the signs $(-1)^{s_{ij}}$ have, in this example, made all Plücker coordinates subtraction-free expressions. Boundary measurement matrices will be explored further in Chapter 5 where we study edge weighted directed graphs on surfaces other than the disk. Our main result in Chapter 5 will be a formula for the Plücker coordinates of these boundary measurement matrices which extends work of Talaska on the disk [Tal08].

Chapter 3

Cluster Algebras

In this chapter we give some background information on Fomin and Zelevinsky's cluster algebras [FZ02] . We will begin by defining the essential notion of mutation and then proceed to formally defining a cluster algebra. After defining a cluster algebra we will discuss the Laurent phenomenon, the upper cluster cluster algebra, and cluster algebras of geometric type. Once we have established these key objects in the theory of cluster algebras we will give an example of a cluster algebra and upper cluster algebra.

3.1 Defining mutation and cluster algebras

Let $\mathbb{P}$ be a semifield. This means that $\mathbb{P}$ is a torsion free abelian group whose operation is written multiplicatively. Additionally, $\mathbb{P}$ is equipped with an auxiliary addition $\oplus$, which is commutative, associative, and distributive over the multiplication of $\mathbb{P}$. We now want to consider a *ground ring*, $\mathbb{Z} \subseteq \mathbb{A} \subseteq \mathbb{ZP}$. This is sometimes called the coefficient ring of the cluster algebra. Let $\mathcal{F}$ be a field which contains $\mathbb{ZP}$. A *seed* of rank n in $\mathcal{F}$ is a triple $(\mathbf{x}, \mathbf{y}, B)$ consisting of three parts.

- The *cluster* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-tuple in $\mathcal{F}$ which freely generates $\mathcal{F}$ as a field over the fraction field of $\mathbb{ZP}$.
- The *coefficients* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are an n-tuple in $\mathbb{P}$.

- The *exchange matrix* B is an integral, skew-symmetrizable $n \times n$ matrix.

For a real number x we define $[x]_+ := \max\{0, x\}$. A seed $(\mathbf{x}, \mathbf{y}, B)$ may be mutated at an index $1 \leq k \leq n$, to produce a new seed $(\mu_k(\mathbf{x}), \mu_k(y\mathbf{y}), \mu_k(B))$. We say that the seed has been mutated in the direction k. Meaning that there are n different mutations that can be applied to a given seed. The mutation is given by the following rules:

- $\mu_k(\mathbf{x}) := (x_1, x_2, \ldots, x_{k-1}, x'_k x_{k+1}, \ldots, x_n)$, where

$$x'_k := \frac{y_k \prod x_j^{[B_{jk}]_+} + \prod x_j^{[-B_{jk}]_+}}{(y_k \oplus 1)x_k}$$

- $\mu_k(\mathbf{y}) := (y'_1, y'_2, \ldots y'_n)$, where

$$y'_i := \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{[B_{ki}]_+} (y_k \oplus 1)^{-B_{ki}} & \text{if } i \neq k \end{cases}$$

- $\mu_k(B)$ is defined by

$$\mu_k(B)_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k, \\ B_{ij} + \frac{1}{2}(|B_{ik}|B_{kj} + B_{ik}|B_{kj}|) & \text{otherwise} \end{cases}$$

Notice that μ_k^2 is the identity. We say that two seeds are *mutation equivalent* if one can be obtained by a sequence of mutations, up to permuting the indices of the seed.

Now that we have established the fundamental definitions of seeds and mutation we can define a *cluster algebra*. Given a seed $(\mathbf{x}, \mathbf{y}, B)$ we will call the union of all the seeds which are mutation equivalent to $(\mathbf{x}, \mathbf{y}, B)$ a set of *cluster variables* in $\mathcal{F}$. The *cluster algebra*,

$\mathcal{A}_{\mathbb{A}}(\mathbf{x}, \mathbf{y}, B)$, is the unital $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by the cluster variables. Notice that since we are allowed to freely mutate when generating the cluster variables, that two mutation equivalent seeds will generate the same cluster algebra. For this reason sometimes it is common to leave the seed out of the notation and simply refer to the algebra as $\mathcal{A}_{\mathbb{A}}$ or just $\mathcal{A}$ when the choice of ground ring is clear.

3.2 The Laurent phenomenon and the upper cluster algebra

The Laurent phenomenon [FZ02, Theorem 3.1] is a very importantly property of cluster algebras. It states that if our ground ring for $\mathcal{A}$ is $\mathbb{ZP}$, then $\mathcal{A}$ is a subalgebra of $\mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}] = \mathbb{ZP}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

In the initial work of Fomin and Zelevinsky they give a more generalized Laurent phenomenon, and outline conditions that allow for the Laurent phenomenon to hold even if we work over a potentially smaller ground ring $\mathbb{A} \subsetneq \mathbb{ZP}$ [FZ02, Theorem 3.2]. Throughout this dissertation, when we discuss the cluster algebra over a ground ring $\mathbb{A}$, we will assume that the Laurent phenomenon holds over $\mathbb{A}$ and that $\mathbb{A}$ contains all coefficients appearing in exchange relations. In this case we have:

$$\mathcal{A} \hookrightarrow \mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}] = \mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{3.1}$$

and

$$\frac{y_i}{1 \oplus y_i}, \frac{1}{1 \oplus y_i} \in \mathbb{A} \tag{3.2}$$

for any seed $(B, \mathbf{x}, \mathbf{y})$ of the cluster algebra.

Now we will consider an algebra closely related to the $\mathcal{A}$. This is the *upper cluster* algebra denoted by $\mathcal{U}$ or by $\mathcal{U}_{\mathbb{A}}((B, \mathbf{x}, \mathbf{y}))$. The upper cluster algebra is defined by

$$\mathcal{U} := \bigcap_{\mathbf{x} \in \mathcal{A}} \mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Since we have chosen a ground ring, $\mathbb{A}$, where the criteria for the Laurent phenomenon are met, we see that injections from each seed mean that $\mathcal{A} \subseteq \mathcal{U}$. In fact there are many occurrences where $\mathcal{A} = \mathcal{U}$ which gives an alternative way of understanding the cluster algebra $\mathcal{A}$. The choice of ground ring plays a substantial role in deciding whether or not we are in this nice situation. In Chapter 6 we will explore exactly how the choice of ground ring impacts whether $\mathcal{A} = \mathcal{U}$.

We now provide a proposition on normality generalizing [Mul13, Proposition 2.1] to our situation. Recall, an integral domain is called a *normal domain* if it is integrally closed in its field of fractions. A semifield $\mathbb{P}$ is always torsion-free, and thus any $\mathbb{Z} \subseteq \mathbb{A} \subseteq \mathbb{ZP}$ is an integral domain.

Proposition 3.2.1. *If $\mathbb{A}$ is a normal domain, then $\mathcal{U}_{\mathbb{A}}$ is a normal domain.*

Proof. Assume $\mathbb{A}$ is a normal domain. From the definition it follows that the intersection of normal domains is again a normal domain. So, it suffices to show that $\mathbb{A}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a normal domain for any seed $(B, \mathbf{x}, \mathbf{y})$. This is true since any polynomial ring over a normal domain is a normal domain [Sta18, Lemma 10.36.8] and any localization of a normal domain is a normal domain [Sta18, Lemma 10.36.5]. □

3.3 Cluster algebras of geometric type

The *tropical semifield* $\mathrm{Trop}(z_1, z_2, \ldots, z_m)$ is the free abelian group generated by $z_1, z_2, \ldots, z_m$ with auxiliary addition given by

$$\prod_{i=1}^{m} z_i^{a_i} \oplus \prod_{i=1}^{m} z_i^{b_i} = \prod_{i=1}^{m} z_i^{\min(a_i, b_i)}.$$

A cluster algebra is said to be of *geometric type* if it is defined over a tropical semifield. In this case $\mathbb{ZP} = \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_m^{\pm 1}]$ is the Laurent polynomial ring in $z_1, z_2, \ldots, z_m$. We are particularly interested in the polynomial ground ring $\mathbb{Z}[z_1, z_2, \ldots, z_m]$ which we will denote by $\mathbb{ZP}_+$. For cluster algebras of geometric type there is a sharpening of the Laurent phenomenon giving $\mathcal{A} \subseteq \mathbb{ZP}_+[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. This sharpening of the Laurent phenomenon can be found in [FWZ, Theorem 3.3.6].

A *quiver* is a directed graph without loops or directed 2-cycles, but parallel arrows are allowed. When $(B, \mathbf{x}, \mathbf{y})$ is a seed for a cluster algebra of geometric type and the matrix B is skew-symmetric, we will often consider a quiver which is equivalent data to the seed $(B, \mathbf{x}, \mathbf{y})$. The $n \times n$ skew-symmetric matrix B is considered as a signed adjacency matrix of a quiver. That is, the quiver Q has B_{ji} arrows $i \to j$ where negative arrows correspond to reversing the direction. The generators $z_1, z_2, \ldots, z_m$ of $\mathbb{P}$ correspond to additional vertices of Q called *frozen vertices*. Non-frozen vertices are known as *mutable vertices*. Mutable vertices of the quiver will be depicted as circles while frozen vertices are represented by squares. There are no arrows between frozen vertices. Arrows between mutable and frozen vertices are obtained by considering $\mathbf{y}$. If $y_i = z_1^{a_i} z_2^{a_2} \cdots z_m^{am}$, then the mutable vertex i has a_j arrows $i \to z_j$. As

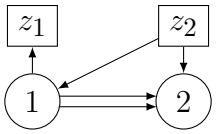

Figure 3.1: A quiver.

Figure 3.2: A quiver on the left and the resulting quiver after mutation at 1 on the right.

an example if $\mathbb{P} = \mathrm{Trop}(z_1, z_2)$ while our seed has $\mathbf{y} = (z_1 z_2^{-1}, z_2^{-1})$ and

$$B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

the corresponding quiver Q would be the quiver pictured in Figure 3.1. For a quiver Q we define *quiver mutation* at a vertex k to be the following process which produces a new quiver Q':

- For each oriented 2-path $i \to k \to j$ add an arrow $i \to j$ unless i and j are both frozen.
- Reverse each arrow incident to k. That is, any arrow $i \to k$ is removed and replaced by $k \to i$ while any arrow $k \to i$ is also removed and replaced by $i \to k$.
- Repeatably remove any pairs of arrows $i \to j$, $j \to i$ forming a 2-cycle until there are no longer any such pairs of arrows.

The process of quiver mutation exactly matches the process of seed mutation in the sense that if Q corresponds to a seed $(B, \mathbf{x}, \mathbf{y})$ then Q' corresponds to the seed $(B', \mathbf{x}', \mathbf{y}')$. An example of quiver mutation can be found in Figure 3.2.

Given any cluster $(x_1, x_2, \ldots, x_n)$ we call $(x_1, x_2, \ldots, x_{n+m})$ the *extended cluster* where $x_{n+i} = z_i$. There is then an $(n+m) \times n$ matrix $\tilde{B}$ called the *extended exchange matrix* such that the mutation relation takes the form

$$x_k x_k' := \prod x_j^{[\tilde{B}_{jk}]_+} + \prod x_j^{[-\tilde{B}_{jk}]_+}.$$

The extended cluster and extended exchange matrix will be important concepts in the compatible Poisson brackets which will be defined in Section 4.2.

3.4 An example cluster algebra

In this section we give an example of a cluster algebra to illustrate the definitions given in previous sections of this chapter. The example cluster algebra considered will turn out to be the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2,4)$. Consider the semifield $\mathbb{P} = \mathrm{Trop}(\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta 14)$ and ground ring $\mathbb{ZP}_+$. Here Δ_{ij} are a priori just formal tropical variables, but we will see (as the notation suggests) they will end up having meaning as Plücker coordinates for $\mathrm{Gr}(2,4)$ where $\Delta_{\{i,j\}}$ is abbreviated as Δ_{ij}. We take the initial seed $(B, \mathbf{x}, \mathbf{y})$ with $B = [0]$, $\mathbf{x} = (x)$, $\mathbf{y} = (\Delta_{12}\Delta_{34}\Delta_{14}^{-1}\Delta_{23}^{-1})$. The only other mutation equivalent seed is $(B', \mathbf{x}', \mathbf{y}')$ with $B' = [0]$, $\mathbf{x}' = (x')$, $\mathbf{y} = (\Delta_{12}^{-1}\Delta_{34}^{-1}\Delta_{14}\Delta_{23})$. The cluster variables x and x' satisfy the relation

$$xx' = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

which is exactly the Plücker relation in $\mathrm{Gr}(2,4)$ when we let $x = \Delta_{13}$ and $x' = \Delta_{24}$. Hence,

$$\mathcal{A}_{\mathbb{ZP}_+} = \mathbb{Z}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}]/(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$$

which (after tensoring with the field) is the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2,4)$. In this case our upper cluster algebra is

$$\mathcal{U}_{\mathbb{ZP}_+} = \mathbb{Z}[\Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34}, \Delta_{13}^{\pm 1}] \cap \mathbb{Z}\left[\Delta_{12}, \Delta_{14}, \Delta_{23}, \Delta_{34}, \left(\frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta 23}{\Delta_{13}}\right)^{\pm 1}\right]$$

which satisfies $\mathcal{A}_{\mathbb{ZP}_+} = \mathcal{U}_{\mathbb{ZP}_+}$. If we work on the ground ring $\mathbb{ZP}$ instead we find that $\mathcal{A}_{\mathbb{ZP}} = U_{\mathbb{ZP}}$ is the homogeneous coordinate ring not of $\mathrm{Gr}(2,4)$ but rather of the open positroid variety $\Pi^\circ(\binom{[4]}{2})$. Indeed we have

$$\mathcal{A}_{\mathbb{ZP}} = \mathbb{Z}[\Delta_{12}^{\pm 1}, \Delta_{13}, \Delta_{14}^{\pm 1}, \Delta_{23}^{\pm 1}, \Delta_{24}, \Delta_{34}^{\pm 1}]/(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$$

and so $\mathcal{A}_{\mathbb{ZP}}$ is of coordinate ring for

$$\bigcup_{\mathcal{M}} S_{\mathcal{M}}$$

where the union is taken over all matroids $\mathcal{M}$ with $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\} \subseteq \mathcal{M}$. We saw in Example 2.1.2 that $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ is not a positroid. It can be checked that the smallest positroid containing $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ is $\binom{[4]}{2}$ and hence $\mathcal{A}_{\mathbb{ZP}}$ is the coordinate ring of the open positroid variety $\Pi^\circ(\binom{[4]}{2})$.

The homogeneous coordinate ring of any Grassmannian is known to have a cluster algebra structure [Sco06]. It is also known that coordinate rings of open positroid varieties are cluster algebras which coincide with their upper cluster algebras [MS16]. This equality of cluster

algebra and upper cluster algebra can be shown using the theory of locally acyclic cluster algebras. In Section 6.1 we will further discuss locally acyclic cluster algebras.

We conclude this section by explaining how the cluster algebra structure on the homogeneous coordinate ring of $\mathrm{Gr}(2,4)$ generalizes to $\mathrm{Gr}(2,n)$. The homogeneous coordinate ring of $\mathrm{Gr}(2,n)$ in generated by $\{\Delta_{ij} : 1 \leq i < j \leq n\}$ subject to the 3-term Plücker relations

$$\Delta_{ik}\Delta_{j\ell} = \Delta_{ij}\Delta_{k\ell} + \Delta_{i\ell}\Delta_{jk}$$

for $1 \leq i < j < k < \ell \leq n$. To realize the cluster algebra structure we consider a regular n-gon with vertices labeled by $1, 2, \ldots, n$ in clockwise order. Clusters will correspond to triangulations of the polygon. We can draw a quiver associated to a triangulation so that mutable vertices are diagonals of the triangulation and frozen vertices are sides of the polygon. The cluster variable corresponding to the diagonal between i and j in the triangulation will be Δ_{ij}. A tropical variable $\Delta_{i,i+1}$ corresponds to each side of the polygon (with indices taken modulo n). In this way we will identify veritices are the quiver and edges in the triangulation. Given two vertices of our quiver in the same triangle we place are arrow between them so that each triangle is a clockwise oriented 3-cycle (with arrows omitted between frozen vertices). In this way quiver mutation will exactly correspond to removing a diagonal and replacing it with the (unique) another possible diagonal in the quadrilateral created. The two quivers for $\mathrm{Gr}(2,4)$ are shown in Figure 3.3 on top of the associated triangulations.

Figure 3.3: The two seeds for the cluster algebra structure on $\mathrm{Gr}(2,4)$.

Chapter 4

Poisson geometry

In this chapter we first define Poisson algebras, Poisson manifolds, and Poisson varieties. We then discuss an approach the studying cluster algebras using Poisson geometry where mutation appears as birational coordinate transformations respecting a Poisson structure.

4.1 Poisson algebras and varieties

Let P be an associative algebra. A *Poisson bracket* on P is a skew-symmetric bilinear map $\{\cdot,\cdot\} : P \times P \to P$ such that for any $a, b, c \in P$ both the *Leibnitz identity*

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

and the *Jacobi identity*

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

hold. A *Poisson algebra* is pair $(P, \{\cdot,\cdot\})$ where P is an associative algebra and $\{\cdot,\cdot\}$ is a Poisson bracket.

Notice that $\{\cdot,\cdot\}$ makes P a Lie algebra. So, we get the adjoint representation of P on itself sending $a \in P$ to $\mathrm{ad}_a \in \mathrm{End}(P)$, where $\mathrm{ad}_a(b) = \{a, b\}$. Note that the Jacobi identity

implies that ad_a is a Lie algebra derivation. Also observe that ad_a is a derivation of the associative algebra P by the Leibniz identity. If $a \in P^*$ is a unit, then the Leibniz identity implies that $\mathrm{ad}_{a^{-1}} = -a^{-2}\,\mathrm{ad}_a$. In particular, this implies that if $\{a, b\} = 0$ for some $a \in P^*$ and $b \in P$, then $\{a^{-1}, b\} = 0$.

Let M be a smooth manifold, and let $C^\infty(M)$ denote its algebra of smooth functions. A *Poisson structure* on M is a bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra. In this case we call $(M, \{\cdot,\cdot\})$ a *Poisson manifold*. For local coordinates $(x_1, \ldots, x_n)$ and $f, g \in C^\infty(M)$ the Poisson bracket is given by

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}. \tag{4.1}$$

and so the bracket is completely determined by the $\binom{n}{2}$ "structure functions" $\{x_i, x_j\}$, for $i < j$. Following [GSV10], a system of coordinates $(x_1, \ldots, x_n)$ is called *log-canonical* with respect to a Poisson bracket $\{\cdot,\cdot\}$ if there is a matrix of scalars $\Omega = (\omega_{ij})$ (necessarily skew-symmetric) such that the structure functions are given by $\{x_i, x_j\} = \omega_{ij} x_i x_j$. We note that this Poisson structure goes by many names in the literature. For example, it is called a *diagonal Poisson structure* in [LGPV13], *Poisson n-space* in [Oh06], and a *semi-classical limit of quantum affine space* in [GL09].

Example 4.1.1 (Standard Poisson-Lie)**.** Consider the special linear group SL_n, with coordinates (matrix entries) x_{ij}. The standard Poisson-Lie structure on SL_n is the quadratic bracket given by

$$\{x_{ij}, x_{k\ell}\} = c^{ij}_{k\ell} x_{i\ell} x_{kj}$$

where the coefficients are given by

$$c_{k\ell}^{ij} = \frac{1}{2}\left(\operatorname{sign}(k-i) + \operatorname{sign}(\ell - j)\right) = \begin{cases} 1 & \text{if } k > i,\ \ell > j \\ 0 & \text{if } k > i,\ \ell < j \\ \frac{1}{2} & \text{if } k > i, j = \ell \text{ or } k = i, \ell > j \end{cases}$$

For instance, when $n = 2$ we have

$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

with bracket relations:

$$\{a, b\} = \tfrac{1}{2}ab \quad \{c, d\} = \tfrac{1}{2}cd$$

$$\{a, c\} = \tfrac{1}{2}ac \quad \{b, d\} = \tfrac{1}{2}bd$$

$$\{a, d\} = bc \qquad \{b, c\} = 0$$

If we consider the Borel subgroup of upper triangular matrices in SL_2

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right\},$$

then the bracket is given by $\{\alpha, \beta\} = \frac{1}{2}\alpha\beta$. In particular, the standard Poisson-Lie structure gives a log-canonical bracket on the Borel subgroup.

In general, the local structure of Poisson manifolds is described by the following theorem of Weinstein.

Theorem ([Wei83]). *Let M be a Poisson manifold, and $p \in M$. Then there exists a neighborhood U containing p with coordinates $(x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_s)$, such that the bracket takes the form*

$$\{x_i, x_j\} = \{y_i, y_j\} = \{x_i, z_j\} = \{y_i, z_j\} = 0$$

$$\{x_i, y_j\} = \delta_{ij}$$

$$\{z_i, z_j\} = \varphi_{ij}$$

where $\varphi_{ij} \in C^\infty(U)$ depend only on $z_1, \ldots, z_s$, and $\varphi_{ij}(p) = 0$.

Example 4.1.2. If (M^{2n}, ω) is a symplectic manifold, then there is a standard Poisson structure induced by ω. In this special case, Weinstein's theorem is the classical Darboux theorem which says that locally ω has the form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

The local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ are commonly called canonical coordinates or Darboux coordinates.

Note that on a smooth Poisson manifold with a log-canonical system of coordinates $(x_1, \ldots, x_n)$ the system of coordinates $(y_1, \ldots, y_n) = (\log x_1, \ldots, \log x_n)$, defined on the open set where all x_i are positive, are similar to a system of canonical coordinates in the sense that the structure functions

$$\{y_i, y_j\} = \{\log x_i, \log x_j\} = \omega_{ij}$$

are all constants. This is indeed the intuition behind the terminology log-canonical. From

Theorem 7.1.8 it will follow that there does not exist any rational change of coordinates on any Zariski open subset such that the structure functions are constant in the new coordinates.

Similarly, let M be an algebraic variety and $\mathcal{O}(M)$ its algebra of regular functions. If there is a bracket making $\mathcal{O}(M)$ into a Poisson algebra, then we call $(M, \{\cdot,\cdot\})$ a *Poisson variety*. Suppose there is a system of coordinates $(x_1, \ldots, x_n)$ on some Zariski open subset of a Poisson variety M, then the bracket is given by Equation (4.1) just as in the smooth case (see for example [LGPV13] for details). We wish to investigate whether such a "simplification" of the structure functions is possible (analogous to the simplification in the Darboux/Weinstein Theorem, in the sense that all structure functions become lower degree polynomials), allowing only birational change-of-coordinates. It is suggested/conjectured in [Van01] that there are not canonical coordinates in general for an arbitrary Poisson variety, but that no specific counterexample has been demonstrated. In [GL11], it was shown that affine space with a log-canonical bracket is such a counterexample. We wish to demonstrate that this same example has the additional property that no rational change of coordinates can make the structure functions linear. The following example is given in [Van01] and demonstrates some of the nuances of the problem of finding canonical coordinates on an open set of a Poisson variety.

Example 4.1.3 ([Van01])**.** Consider affine space $\mathbb{C}^2$ with coordinates (x, y) and Poisson bracket given by $\{x, y\} = x$. Viewing $\mathbb{C}^2$ as a smooth manifold, there is a system of canonical local coordinates $(\log x, y)$ that is *not* algebraic. However, there is also $\left(\frac{1}{x}, -xy\right)$ which is a system of canonical local coordinates that *is* algebraic. That is, a system of canonical coordinates consisting of rational functions in x and y defined on the Zariski-open subset $\{(x, y) : x \neq 0\}$ of the variety $\mathbb{C}^2$. The example illustrates that there do exist Poisson varieties which admit a rational coordinate change on an open subset which make the structure

functions constant.

Example 4.1.4. More generally, consider $\mathbb{C}^2$ with coordinates (x, y) and Poisson bracket given by $\{x, y\} = x^a y^b$ for $(a, b) \in \mathbb{N} \times \mathbb{N}$. The case $(a, b) = (1, 1)$ gives a system of log-cononical coordinates. In all other instances, we can find a system of canonical coordinates as follows:

- If $a \neq 1$ and $b \neq 1$, then $\{x^{-(a-1)}, y^{-(b-1)}\} = (a-1)(b-1)$ is a nonzero constant.
- If $a = 1$ and $b \neq 1$, then $\{x^{-1}, xy^{-(b-1)}\} = (b-1)$ is a nonzero constant. The case $a \neq 1$ and $b = 1$ is similar using the fact that the bracket is antisymmetric.

Note that the previous example is the special case when $(a, b) = (1, 0)$. Although the specific example $(a, b) = (1, 0)$ does give a birational change of coordinates, this is not in general true for this family of examples. For instance, when either a or b is greater than 2, the inverse of the coordinate change is not a rational function.

Thus for $(a, b) \neq (1, 1)$ we can always find a pair of algebraically independent rational functions in two variables such that the bracket between these two functions is a nonzero constant. It is still unclear whether this example can be generalized to dimensions higher than 2. It will follow from Theorem 7.1.8 that $(a, b) = (1, 1)$ is the unique exception to the existence of two rational functions with nonzero constant bracket between them. This begs the following interesting, and more general, question.

Question 4.1.5. *Given a Poisson bracket whose structure functions are all (homogeneous) polynomials of a given degree, when is it possible to find a birational change of coordinates making the structure functions (homogeneous) polynomials of a smaller degree?*

We will demonstrate in Chapter 7 that the quadratic log-canonical Poisson structure has

the property that there is no rational change of coordinates making the Poisson bracket linear or constant.

4.2 Poisson brackets compatible with cluster algebras

In this section we will follow [GSV10] and discuss Poisson brackets compatible with cluster algebras of geometric type. Let $\mathcal{F}$ be the field of rational functions in $n+m$ independent variables with rational coefficients. Consider a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{F}$. In the context of cluster algebras and Poisson geometry we call functions $f_1, f_2, \ldots, f_{n+m}$ *log-canonical* if there exists $\omega_{ij} \in \mathbb{Z}$ such that

$$\{f_i, f_j\} = \omega_{ij} f_i f_j$$

for all $1 \leq i, j \leq n+m$. This is a special case of our previous definition of log-canonical where now we want the scalars ω_{ij} to be integers.

Let $\mathcal{A}$ be a rank n cluster algebra of geometric type over $\mathbb{P} = \mathrm{Trop}(z_1, z_2, \ldots, z_m)$. Take $\mathcal{F}$ to be the ambient field of the cluster algebra. A Poisson bracket on $\mathcal{F}$ is called *compatible* with $\mathcal{A}$ if every extended cluster is log-canonical. Clearly the trivial Poisson bracket with $\{f, g\} = 0$ for all $f, g \in \mathcal{A}$ is always compatible. If the extended exchange matrix of $\mathcal{A}$ is full rank, then there exist many nontrivial compatible Poisson brackets [GSV10, Theorem 4.3]. Moreover, the collection of compatible Poisson brackets forms of vector space which can be completely described.

A cluster structure in the field of rational functions of an algebraic variety is called *regular* if all cluster variables are regular functions. A main question is to construct explicitly a compatible regular cluster structure corresponding to a given variety equipped with an algebraic Poisson structure. For a simple complex Lie group, Belavin and Drinfled have given

a classification of Poisson-Lie structures coming from classical R-matrices [BD82]. The main conjecture of Gekhtman, Shapiro, and Vainshtein on cluster algebras and Poisson geometry states that for complex simple Lie groups the classification of regular cluster structures parallels the Belavin-Drinfled classification [GSV12, Conjecture 3.2].

A map $\varphi\colon M \to N$ between two Poisson manifolds (or Poisson varieties) is called a *Poisson map* if the pullback map φ^* is a homomorphism of Poisson algebras. A Lie group G is called a *Poisson-Lie group* if the multiplication map $G \times G \to G$ is a Poisson map. For further details, see [CP94]. The cluster structure on double Bruhat cells from [BFZ05] is known to correspond to a cluster structure compatible with the standard Poisson-Lie structure [GSV10, Chapter 4.3]. Other Poisson-Lie compatible cluster structures are called *exotic*. Thus an important problem in cluster algebras and Poisson geometry is to find exotic cluster structures. In Section 6.2 we will consider a particular exotic cluster structure which provides an example of sensitivity to the $\mathcal{A} = \mathcal{U}$ question on the choice of ground ring.

4.3 Poisson geometry for networks of the disk

In this section we explain the connection between the log-canonical Poisson structures which play a role in cluster algebras and the boundary measurement map that was defined in Chapter 2. We will focus our attention here to edge weighted directed graphs on the disk [GSV09]. Though we remark that Poisson structures exist for edge weighted directed graphs on the annulus which was the context for the first generalization on the boundary measurement map to a surface beyond the disk [GSV08]. In Chapter 5 we will consider a generalized version of the boundary measurement valid for any orientable surface.

As in [Pos06], we will use the term *directed network* $N = (V, E)$ to refer to the directed

graph $N = (V, E)$ along with edge weights. For any directed network N we defined its boundary measurement matrix $B(N)$ in Chapter 2. We will now describe a Poisson structures on the space on edge weights and on matrices so that the map associating a collection of edge weights to the boundary measurement matrix evaluated at those edge weights is a Poisson map. Here we will only consider directed networks such that each internal vertex is trivalent and neither a source nor sink, and each boundary vertex is univalent. Internal vertices are of two possible *types*. Either incident on one incoming edge and two outgoing edges, or incident on two incoming edges and one outgoing edge.

Remark 4.3.1. We can restrict to such directed networks with loss of generality. Postnikov [Pos06] has shown how to transform any directed network, without changing the boundary measurement, to one of the trivalent networks we consider in this section.

Given such a trivalent directed network N to each interval vertex v we assign a 3-dimensional space $(\mathbb{R} \setminus \{0\})^3_v$ along with a Poisson bracket $\{\cdot, \cdot\}_v$. We assign coordinates x^1_v, x^2_v, x^3_v to each $(\mathbb{R} \setminus \{0\})^3_v$ with a coordinate corresponding to each edge incident on v as depicted in Figure 4.1. To each boundary vertex v we assign a 1-dimensional space $(\mathbb{R}\setminus\{0\})_v$ with coordinate x^1_v. We then define $\mathcal{R}$ to be the direct sum the spaces associated to all vertices. The Poisson bracket $\{\cdot, \cdot\}_{\mathcal{R}}$ is the direct sum of brackets so that $\{x, y\}_{\mathcal{R}} = 0$ whenever x and y are defined are different spaces.

We next define edge weights $w_e = x^i_v x^j_u$ where $e = (u, v)$ with x^i_v and x^j_u corresponding to the edge e in the spaces $(\mathbb{R} \setminus \{0\})^3_v$ and $(\mathbb{R} \setminus \{0\})^3_u$. The space of edge weights $\mathcal{E}_N$ is then $(\mathbb{R} \setminus \{0\})^{|E|}$ and inherits a Poisson bracket $\{\cdot, \cdot\}_N$ from the pushfoward of the weight map $w : (\mathbb{R} \setminus \{0\})^{2|E|} \to (\mathbb{R} \setminus \{0\})^{|E|}$. Considering the case when the Poisson brackets $\{\cdot, \cdot\}_v$ only depends on the type of the vertex, there is a 2-parameter family of Poisson brackets on the space of matrices such that the boundary measurement $N \to B(N)$ is a Poisson

Figure 4.1: Coordinates on $(\mathbb{R} \setminus \{0\})_v^3$.

map [GSV10, Theorem 8.6]. Proceeding in this way Gekhtman, Shapiro, and Vainshtein are able to use directed networks to produce Poisson structures compatible with the standard cluster algebra structure on the Grassmannian [GSV10, Theorem 8.20]. This construction shows a connection between various topics (cluster algebras, log-canonical Poisson brackets, the Grassmannian, and boundary measurement) and provides a partial explanation for their inclusion together in this dissertation. We omit further details of this compatible Poisson structure via networks as details will not be needed in the remaining chapters.

Chapter 5

Boundary Measurement

This chapter is based on the article [Mac18].

5.1 Generalized boundary measurement definition

The totally nonnegative Grassmannian was defined by Postnikov [Pos06] and can be studied using edge weighted planar graphs embedded on a disk. These edge weighted planar graphs and the totally nonnegative Grassmannian are connected to the physics of scattering amplitudes and $\mathcal{N} = 4$ super Yang-Mills [AHBC$^+$16]. In the context of physics, the edge weighted planar graphs are usually called "on-shell diagrams." A key element of Postnikov's study of the totally nonnegative Grassmannian is the boundary measurement map which produces an element of the totally nonnegative Grassmannian for any edge weighted directed graph embedded in the disk. Under a mild hypothesis on the graph, Talaska [Tal08] gives a formula for the Plücker coordinates of the element of the totally nonnegative Grassmannian corresponding to a given graph. In [FGM14, FGPW15] a boundary measurement map for graphs on more general surfaces is proposed with the hopes of going beyond the "planar limit" of $\mathcal{N} = 4$ super Yang-Mills.

The definition of the boundary measurement map will be given later in this section, and in defining the boundary measurement we must make a choice of how to represent our directed graph in the plane. The boundary measurement map turns out to be independent

of this choice as we will see in Section 5.2. We will show in Section 5.3 how boundary measurement map can be obtained by signing the edges of a directed graph. This technique of signing edges will allow us to unify two formulas of Talaska [Tal08, Tal12]. A formula for the Plücker coordinates corresponding to the boundary measurement map is given in Section 5.4. In Section 5.5 we will show that the signs used in Section 5.3 are unique up to the gauge action.

5.1.1 Weighted Path Matrices

Let $N = (V, E)$ be a directed graph with finite vertex set V and finite edge set E. This means an edge $e \in E$ is an ordered pair $e = (i, j)$ for $i, j \in V$. If $e = (i, j)$ then the edge e is said to be directed from vertex i to vertex j. For each edge $e \in E$ of N we associate a formal variable x_e. We will work in $\mathbb{R}[[x_e : e \in E]]$ the ring of formal power series in the variables $\{x_e\}_{e\in E}$ with coefficients in $\mathbb{R}$. Recall, as in [Pos06], we will use the term *directed network* $N = (V, E)$ to refer to the directed graph $N = (V, E)$ along with edge weights $\{x_e\}_{e\in E}$.

A *path* is a finite sequence of edges $P = (e_1, e_2, \cdots, e_l)$ where $e_k = (i_{k-1}, i_k)$ for $1 \leq k \leq l$. If $P = (e_1, e_2, \cdots, e_l)$ where $e_1 = (i_0, i_1)$ and $e_l = (i_{l-1}, i_l)$, then P is said to be a path from i_0 to i_l. The path P is said to be *self avoiding* if $i_k \neq i_{k'}$ for $k \neq k'$. The path P is called a *cycle* if $i_0 = i_l$, and we say the cycle is a *simple cycle* when $i_k = i_{k'}$ if and only if $k = k'$ or $\{k, k'\} = \{0, l\}$. We use the notation $P : i \rightsquigarrow j$ to denote a path from i to j. When $P = (e_1, e_2, \cdots, e_l)$ we let

$$\mathrm{wt}(P) = \prod_{i=1}^{l} x_{e_i}$$

denote the *weight* of the path P.

We order our vertex set V and consider the $V \times V$ *weighted path matrix*

$$M = M(N, \{x_e\}_{e \in E})$$

with entries given by

$$M_{ij} = \sum_{P : i \rightsquigarrow j} \mathrm{wt}(P)$$

for all $(i, j) \in V \times V$.

We let $\mathcal{C}(N)$ denote the set of all collections $\mathbf{C}$ which consist of simple cycles that are pairwise vertex disjoint. For $\mathbf{C} \in \mathcal{C}(N)$ we define its weight as

$$\mathrm{wt}(\mathbf{C}) = \prod_{C \in \mathbf{C}} \mathrm{wt}(C)$$

and its sign as $\mathrm{sgn}(\mathbf{C}) = (-1)^{|\mathbf{C}|}$ where $|\mathbf{C}|$ denotes the number of cycles in the collection $\mathbf{C}$. The empty collection $\varnothing$ is in $\mathcal{C}(N)$ with $\mathrm{wt}(\varnothing) = 1$ and $\mathrm{sgn}(\varnothing) = 1$. We let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$ and consider elements $\pi \in S_n$ as bijections $\pi : [n] \to [n]$. For $\pi \in S_N$ and any $I, J \subseteq V$ with $I = \{i_1 < i_2 < \cdots < i_n\}$ and $J = \{j_1 < j_2 < \cdots < j_n\}$ we let $\mathcal{P}_{I,J,\pi}$ denote the set of collections $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ such that $P_k : i_k \rightsquigarrow j_{\pi(k)}$ is self avoiding for each $k \in [n]$, and P_k and $P_{k'}$ are vertex disjoint whenever $k \neq k'$. For $\mathbf{P} \in \mathcal{P}_{I,J,\pi}$ we define its weight as

$$\mathrm{wt}(\mathbf{P}) = \prod_{P \in \mathbf{P}} \mathrm{wt}(P)$$

and its sign as $\mathrm{sgn}(\mathbf{P}) = \mathrm{sgn}(\pi)$. Note if $\pi(k) = k$ we can have $P_k : i_k \rightsquigarrow i_k$ be the empty path P_k from i_k to i_k consisting of no edges, and in this case $\mathrm{wt}(P_k) = 1$. We then let

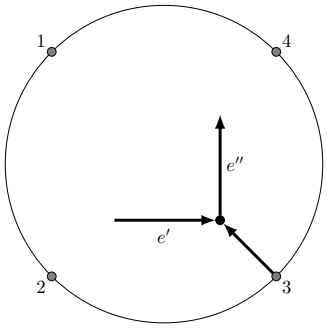

Figure 5.1: An example of the edge order induced by the boundary vertex order. Here we say $e' < e''$ given the boundary vertices are ordered as usual with $1 < 2 < 3 < 4$. The direction of the edges is irrelevant for the induced edge ordering.

$\mathcal{F}_{I,J}(N)$ denote the collection of *flows* from I to J. A flow from I to J is a pair $\mathbf{F} = (\mathbf{P}, \mathbf{C})$ such that $\mathbf{P} \in \mathcal{P}_{I,J,\pi}$ for some $\pi \in S_n$, $\mathbf{C} \in \mathcal{C}(N)$, and all paths in $\mathbf{P}$ and cycles in $\mathbf{C}$ are pairwise vertex disjoint. For $\mathbf{F} \in \mathcal{F}_{I,J}(N)$ with $\mathbf{F} = (\mathbf{P}, \mathbf{C})$ we define its weight as $\operatorname{wt}(\mathbf{F}) = \operatorname{wt}(\mathbf{P}) \operatorname{wt}(\mathbf{C})$ and its sign as $\operatorname{sgn}(\mathbf{F}) = \operatorname{sgn}(\mathbf{P}) \operatorname{sgn}(\mathbf{C})$.

Talaska's formula [Tal12] states

$$\Delta_{I,J}(M) = \frac{\sum_{\mathbf{F} \in \mathcal{F}_{I,J}(N)} \operatorname{sgn}(\mathbf{F}) \operatorname{wt}(\mathbf{F})}{\sum_{\mathbf{C} \in \mathcal{C}(N)} \operatorname{sgn}(\mathbf{C}) \operatorname{wt}(\mathbf{C})} \tag{5.1}$$

where $\Delta_{I,J}(M)$ denotes the minor of M with rows indexed by I and columns indexed by J. Equation (5.1) generalizes the Lindström-Gessel-Viennot lemma [Lin73, GV85] which only applies to directed networks without directed cycles. Fomin also provides of generalization of the Lindström-Gessel-Viennot lemma which allows for directed cycles [Fom01] where the sum is indexed by a minimal, but infinite, collection of paths.

5.1.2 Boundary Measurement Matrices

Now consider the directed network $N = (V, E)$ embedded in a closed orientable surface with boundary S. We call a vertex on the boundary of S a *boundary vertex* and an edge which is

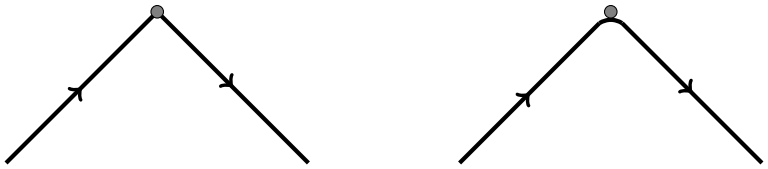

Figure 5.2: Smoothing a piecewise smooth curve at a vertex.

incident on a boundary vertex an *external edge*. We assume each boundary vertex is either a source or sink and that edges are embedded as smooth curves. Let $K \subseteq V$ be the collection of boundary vertices. Let b denote the number of boundary components of S and assume each boundary component is a smooth curve diffeomorphic to a circle. We make $b-1$ cuts between pairs of boundary components on the surface S to obtain a new surface T with a single boundary component. The cuts are made such that each cut is a smooth curve, no cut intersects any vertex of N, and cuts intersect edges of N transversally. The boundary ∂T is then a piecewise smooth curve homeomorphic to a single circle. We choose a piecewise smooth parameterization $\phi : [0,1] \to \partial T$ with $\phi(0) = \phi(1)$ (i.e. $\phi : S^1 \to \partial T$). Throughout we will assume all parameterizations are piecewise smooth with nowhere zero derivative. We order the boundary vertices so that they appear in order when traversing ∂T according to ϕ. Thus we have a linear ordering of the vertices in K which we denote by $<$ so that $K = \{i_1 < i_1 < \cdots < i_n\}$ with $i_j = \phi(t_j)$ for $0 \leq t_1 < t_2 < \cdots < t_n < 1$. The linear ordering of the boundary vertices induces an ordering on the set of edges incident on some external edge as demonstrated in Figure 5.1. We also have a cyclic ordering which we denote $\prec$. For $i, j, k \in K$ we write $i \prec j \prec k$ if $i < j < k$, $k < i < j$, or $j < k < i$.

When S is a closed orientable surface with boundary of genus $g = 0$ any network embedded on S can be drawn in the plane. In order to draw the directed network in the plane we must choose a boundary component of S called external and identify this external boundary component with a circle bounding a disk in the plane. We then draw the directed network

inside this disk. In Section 5.2 we will show that this choice of external boundary component does not have an impact on our results. Consider a network N on S embedded in the plane and overlay the cuts used to construct T. We will make use of both S and T. For any path $P : i \rightsquigarrow j$ where $(i, j) \in I \times K$ we form a closed curve $C(P)$ in the plane as follows:

1. Traverse the path P from i to j in S.

2. Follow the boundary of T in our specified direction from j to i.

We want $C(P)$ to be a smooth curve. Since we have assumed that all edges and boundary components are smooth curves the curve $C(P)$ will be piecewise smooth. In order to work with a smooth curve we will approximate $C(P)$ by a smooth curve at cut points and around each vertex as in Figure 5.2. We will make no distinction between $C(P)$ and the smooth curve we approximate it by, and in some cases we may draw a piecewise smooth curve in place of a smooth curve. Given any smooth closed curve in the plane define its *rotation number* to be the degree of the map $T \circ \psi : S^1 \to S^1$ where $\psi : S^1 \to C$ is a parameterization of C and $T : C \to S^1$ gives the unit tangent vector of each point. The choice of which smooth curve is used as an approximation will not effect the rotation number.

Now consider the case where S is a closed orientable surface with boundary S of genus $g > 0$. Similarly to the genus zero case, we want to construct a closed curve in the plane for each path $P : i \rightsquigarrow k$ where $(i, k) \in I \times K$. We choose generators of the first homology group for the underlying closed surfaced without boundary. The choice of homology generators does not affect our results as we will see in Section 5.2. The homology generators are chosen so that they do not intersect any vertices of N and so that all intersections with edges of N are transversal. Also, the homology generators are chosen so that they intersect transversally with the cuts used to form T. We then consider the *punctured fundamental polygon* of S

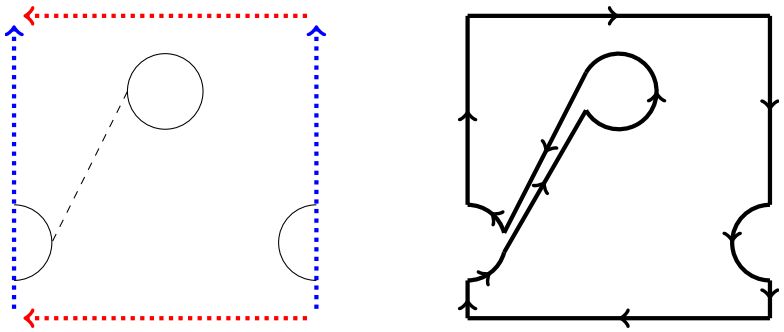

Figure 5.3: On the left we have a punctured fundamental polygon for S a torus with two boundary components with the cut between boundary components shown as a dashed line. On the right we have a closed curve around the boundary of T drawn inside the fundamental domain.

which is the usual fundamental polygon of the underlying closed surface without boundary where the sides of the polygon correspond to homology generators, but we must remove some number of disks to create the boundary of the surface. The punctured fundamental polygon has $4g$ sides with each corresponding to a homology generator, and when sides corresponding to the same homology generator are identified we obtain the surface S. Note no vertex appears on any side of the punctured fundamental polygon and no edge or cut ever runs parallel to any side of the punctured fundamental polygon. The punctured fundamental polygon represents a *fundamental domain* of our surface. See Figure 5.3 for an example of a punctured fundamental polygon.

When we have a network N on S with genus $g > 0$ we draw N in the plane inside a single fundamental domain of S and overlay the cuts used to construct the surface T. Given any path $P : i \rightsquigarrow j$ for $(i, j) \in I \times K$ we form a closed curve $C(P)$ in the plane, similarly to the genus zero case, by first traversing the path P from i to j and then following the boundary of T in our specified order from j to i. However, each time the closed curve leaves the chosen fundamental domain we connect the exit and entry points by following the sides of the punctured fundamental polygon clockwise from the exit point to the entry point.

Let $I \subseteq K$ be the collection of boundary vertices which are sources. We consider the $I \times K$ matrix $A = A(N, \{x_e\}_{e \in E})$ with entries given by $A_{ij} = M_{ij}$ for all $(i, j) \in I \times K$. So, A is obtained from M by restricting to rows I and columns K. We also consider the $I \times K$ *boundary measurement matrix* $B = B(N, \{x_e\}_{e \in E})$ with entries given by

$$B_{ij} = \sum_{P : i \rightsquigarrow j} (-1)^{s_{ij} + r_P + 1} \operatorname{wt}(P)$$

for all $(i, j) \in I \times K$. Here s_{ij} denotes the number of elements of I strictly between i and j with respect to $<$, and r_P denotes the rotation number of $C(P)$.

This definition of the boundary measurement matrix for any closed orientable surface with boundary S is due to Franco, Galloni, Penante, and Wen [FGPW15]. Postnikov [Pos06] gave the original definition on the boundary measurement matrix in the case where the surface is a disk. The boundary measurement matrix was considered for networks on the annulus by Gekhtman, Shapiro, and Vainshtein [GSV08] and for networks on any closed orientable genus zero surface with boundary by Franco, Galloni, and Mariotti [FGM14].

Consider specializing the formal variables x_e to real weights. Notice the boundary measurement matrix is then a real $|I| \times |K|$ matrix of rank $|I|$. Hence, for any directed network N and choice of real weights, the boundary measurement matrix $B(N)$ describes an element of the real Grassmannian $\operatorname{Gr}(|I|, |K|)$. This association of a directed network with real weights to an element of the Grassmannian is the known as the *boundary measurement map*. One feature of Postnikov's boundary measurement map applied to a directed network N embedded in the disk is that when the edge weights are positive real numbers, the boundary measurement matrix $B(N)$ represents an element of the totally nonnegative Grassmannian. The *totally nonnegative Grassmannian* is defined to be elements of the Grassmannian such

that all Plücker coordinates are nonnegative or nonpositive. That is, elements of the Grassmannian that can be represented by a matrix where each maximal minor is nonnegative.

When S is a disk it is shown in [Pos06] that the maximal minors of $B(N)$ are subtraction-free rational expressions in the edge weights. We call a directed network N *perfectly oriented* if each boundary vertex is a univalent source or sink and each interior vertex is trivalent and neither a source nor sink. When a network is perfectly oriented, the interior vertices are of one of two types. We distinguish the two types of interior vertices by coloring each interior vertex white or black. White vertices have one incoming edge and two outgoing edges, and black vertices have two incoming edges and one outgoing edge. For an example of a perfectly oriented network see Figure 5.6.

Remark 5.1.1. In [Pos06] it is shown how to transform any directed network N on the disk to a perfectly oriented network N' so that the boundary measurement matrix $B(N)$ is a specialization of the boundary measurement matrix of $B(N')$. All transformations needed take place locally around a vertex, and hence will work on more general surfaces.

If N is perfectly oriented Talaska [Tal08] gives the following formula

$$\Delta_{I,J}(B) = \frac{\sum_{\mathbf{F}\in\mathcal{F}_{I,J}(N)} \mathrm{wt}(\mathbf{F})}{\sum_{\mathbf{C}\in\mathcal{C}(N)} \mathrm{wt}(\mathbf{C})} \tag{5.2}$$

where $J \subseteq K$ with $|I| = |J|$. We notice Equation (5.2) is very similar to Equation (5.1) even though they describe minors of different matrices. In Theorem 5.3.3 we will show that the boundary measurement matrix B can be obtained from A by a simple change of variables which explains the similarity of the formulas. This theorem will also allow us to prove the following conjecture.

Conjecture 5.1.2 ([FGPW15])**.** *If $N = (V, E)$ is a perfectly oriented network embedded on*

a closed orientable surface with boundary, then for any $J \subseteq K$ *with* $|I| = |J|$

$$\Delta_{I,J}(B(N, \{x_e\}_{e \in E})) = \frac{\sum_{\mathbf{F} \in \mathcal{F}_{I,J}(N)} \sigma(\mathbf{F}) \operatorname{wt}(\mathbf{F})}{\sum_{\mathbf{C} \in \mathcal{C}(N)} \sigma(\mathbf{C}) \operatorname{wt}(\mathbf{C})}$$

for some $\sigma : \mathcal{F}_{I,J}(N) \cup \mathcal{C}(N) \to \{\pm 1\}$.

Our main result is that Conjecture 5.1.2 is true. It follows from Equation (5.1) and Theorem 5.3.3 which will be proven in the Section 5.3. Corollary 5.4.2 gives a formula for the maximal minors of the boundary measurement matrix where we explicitly describe the sign function σ in Conjecture 5.1.2. Recall, if we specialize are formal variables to take real values, the boundary measurement matrix represents an element of the real Grassmannian. In this context Conjecture 5.1.2 and Corollary 5.4.2 are formulas for the Plücker coordinates of this element of the Grassmannian.

5.2 Boundary Measurement Independence

Given a directed network N embedded on a closed orientable surface with boundary S, we must make some choices when computing the boundary measurement matrix $B(N)$. The first choice we must make is how to place the cuts on the surface S to obtain the surface T with a single boundary component. The boundary measurement does depend on this choice. For example, boundary measurement matrices are

$$B(N) = \begin{bmatrix} 1 & x \end{bmatrix} \qquad\qquad B(N') = \begin{bmatrix} 1 & -x \end{bmatrix}$$

for the directed networks in Figure 5.4.

Another choice we must make when computing the boundary measurement is how to

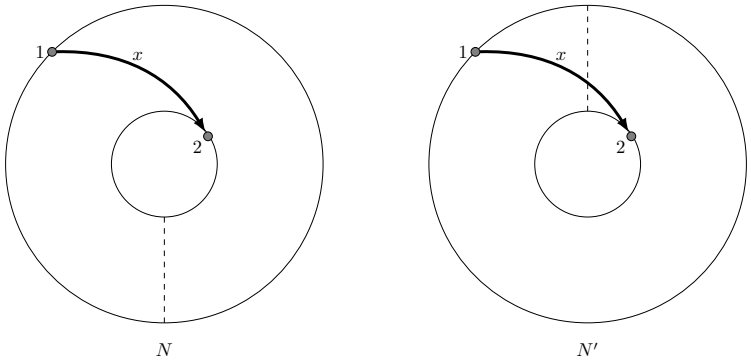

Figure 5.4: Two networks on the annulus each with a different choice of cut.

represent the closed orientable surface with boundary in the plane. For genus $g = 0$, we make a choice of which boundary component corresponds to the circle bounding the disk we draw our network inside. For genus $g > 0$, we choose a fundamental domain. In this section we will show that the boundary measurement does not depend on how we represent the surface in the plane.

Let $\psi : S^1 \to \mathbb{R}^2$ define a smooth closed curve C. When $\psi(t_1) = \psi(t_2)$ for $t_1 \neq t_2$ we call $\psi(t_1)$ a *self intersection point* of C. If $\psi(t_1)$ is a self intersetion point of C such that there exists a unique $t_2 \neq t_1$ with $\psi(t_1) = \psi(t_2)$ and $\{\psi'(t_1), \psi'(t_2)\}$ are linearly independent, we then call the self intersection point $\psi(t_1)$ *simple*. A smooth curve whose only self intersection points are simple is called *normal*. The rotation number of a normal curve differs in parity from the number of self intersections. This was proven by Whitney in [Whi37] where it is also proven that any smooth curve can be transformed into a normal curve by small deformations without changing the curves rotation number. When drawing closed curves inside a fundamental domain we sometimes may not connect exit and entry points along the sides of the punctured fundamental polygon, but rather draw some curve in the interior or exterior of the punctured fundamental polygon which has the same rotation number as the curve following the sides of the polygon. This will be done to simplify the drawing of the

curve and in some cases will be necessary to transform the curve into a normal curve.

Observe for any closed curve C on a closed orientable surface with boundary S, we can construct a closed curve in the plane in the same way we do for the closed curves which come from paths in our oriented network. Our next lemma will consider an arbitrary closed curve C on S. Given some representation of our surface in the plane, we will let $\hat{C}$ denote the corresponding closed curve in the plane. Also, in the proof of the lemma we will consider a lift C' of the closed curve C to the universal cover of S when the surface S has genus $g > 0$. Recall that each time a closed curve C leaves the fundamental domain, we connected the exit and entry points along the boundary of the punctured fundamental polygon when constructing the closed curve $\hat{C}$ which lives in a single fundamental domain. When doing this the tangent vector will make exactly one complete rotation. To account for this we construct another curve C'' on the universal cover of S. The curve C'' agrees with the curve C' except we add a loop each time it crosses a homology generator. See Figure 5.5 for an example of C, $\hat{C}$, C' and C''.

Lemma 5.2.1. *Let C be a closed curve on a closed orientable surface with boundary S and let $\hat{C}$ be the closed curve corresponding to C for some choice of representation of S in the plane. The parity of the rotation number of $\hat{C}$ does not depend on the choice of representation of S in the plane.*

Proof. Recall, the parity of the rotation number of a closed curve in the plane depends only on the number of self intersections of the curve. For the case genus $g = 0$, it is clear the number of self intersections of $\hat{C}$ does not depend of the representation of S in the plane.

Now consider the case genus $g > 0$. We denote the rotation number of $\hat{C}$ by r. We let C' be a lift of C to the universal cover of S. The universal cover is homeomorphic to $\mathbb{R}^2$.

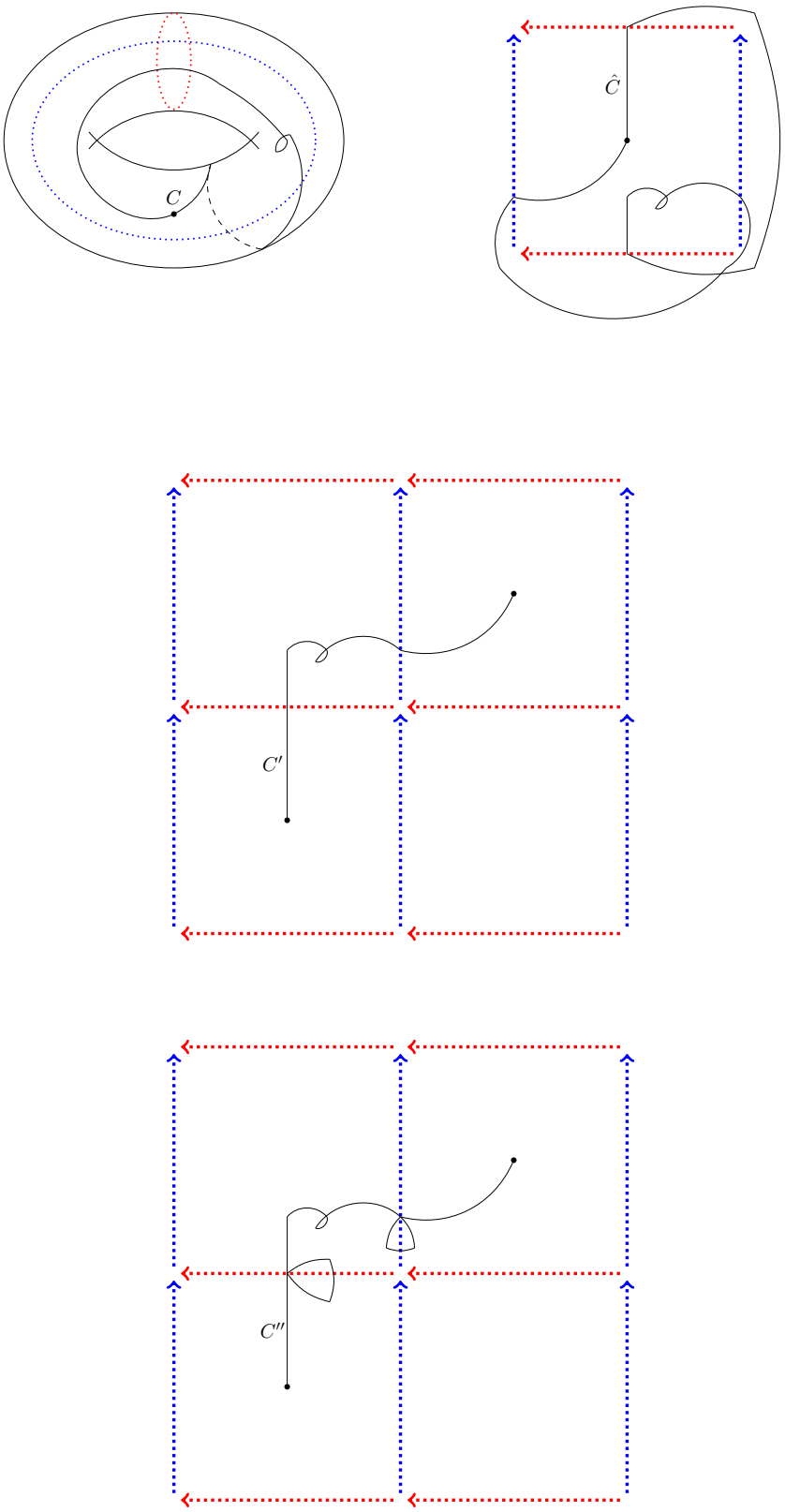

Figure 5.5: An example of a closed curve C on the torus and the corresponding curves $\hat{C}$, C' and C''.

Let h denote the number of times C intersects any homology generator. Notice the parity of h is determined by the homology class of C, and hence the parity of h is independent of the choice of fundamental domain used to represent S in the plane.

If C is null homologous, then C' is a closed curve in $\mathbb{R}^2$. If C is not null homologous, then C' is not a closed curve in $\mathbb{R}^2$. However, we can still define the rotation number of C' since the unit tangent vector at the starting point and ending point of C' will be the same. In any case, let r' denote the rotation number of C'. Notice each time C intersects any homology generator, the tangent vector to the curve $\hat{C}$ we make a complete rotation in the clockwise direction on the portion of the curve which connects the entry and exit points of the fundamental domain. We can modify the curve C' by adding a small loop in the clockwise direction each time C' intersects a lift of a homology generator. We let C'' denote this modified curve. We can construct C'' such that there is then a map $\phi : C'' \to \hat{C}$ such that $T = T \circ \phi$. Let r'' denote the rotation number of C'' It then follows that $r = r''$ and that $r'' = r' + h$. Therefore $r = r' + h$, and the parity of the rotation number of $\hat{C}$ is independent of how S is represented in the plane. □

Theorem 5.2.2. *The boundary measurement of a directed network N on a closed orientable surface with boundary S is independent of how we represent S in the plane.*

Proof. The only part of the boundary measurement matrix that depends on the representaion of S in the plane is the rotation numbers r_P of the closed curves $C(P)$ which correspond to paths P in N. In fact, the boundary measurement matrix only depends on the parity of r_P. Therefore, the theorem then follows immediately from Lemma 5.2.1. □

We have also made a choice to connect exit and entry points of a closed curve along the sides of the punctured fundamental polygon in the clockwise direction. Observe the

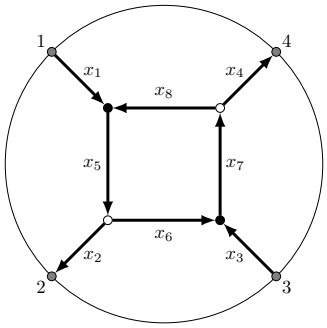

Figure 5.6: A perfectly oriented network on the disk.

proof of Lemma 5.2.1 can easily be modified if we chose to connect the exit points in the counterclockwise direction.

5.3 Signing Perfectly Oriented Networks

In this section we prove a theorem which shows a relationship between the weighted path matrix and the boundary measurement matrix. We show the boundary measurement matrix is the weighted path matrix with some edge weights thought of as negative. That is, by replacing x_e with $-x_e$ for some edges in $A(N, \{x_e\}_{e\in E})$ we obtain $B(N, \{x_e\}_{e\in E})$. We first look at an example of signing the edges of a network.

Let N be the network in Figure 5.6. Here the boundary vertices are labeled to respect the usual ordering of the natural numbers so that $1 < 2 < 3 < 4$. The weighted path matrix

and boundary measurement matrix for N are

$$A = \begin{bmatrix} 1 & \dfrac{x_1x_5x_2}{1-x_5x_6x_7x_8} & 0 & \dfrac{x_1x_5x_6x_7x_4}{1-x_5x_6x_7x_8} \\ 0 & \dfrac{x_3x_7x_8x_5x_2}{1-x_5x_6x_7x_8} & 1 & \dfrac{x_3x_7x_4}{1-x_5x_6x_7x_8} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & \dfrac{x_1x_5x_2}{1+x_5x_6x_7x_8} & 0 & \dfrac{-x_1x_5x_6x_7x_4}{1+x_5x_6x_7x_8} \\ 0 & \dfrac{x_3x_7x_8x_5x_2}{1+x_5x_6x_7x_8} & 1 & \dfrac{x_3x_7x_4}{1+x_5x_6x_7x_8} \end{bmatrix}$$

respectively. Notice that B can be obtained from A by replacing x_6 with $-x_6$. Theorem 5.3.3 shows that when N is perfectly oriented B can always be obtained from A by a change a variable which gives each edge of N a sign. However, there is not a unique way to obtained B from A. For example, replacing x_2 and x_5 with $-x_2$ and $-x_5$ respectively is another possibility. Theorem 5.5.1 characterizes all possible ways to sign the edges of N.

Before stating the main theorem of this section we prove two lemmas which will be needed.

Lemma 5.3.1. *Let N be a directed network embedded on a closed surface with boundary S and let T be the surface obtained after making cuts. If r_T is the rotation number of the closed curve which following the boundary of T in a chosen fundamental domain, then $r_T \equiv 1 \pmod 2$.*

Proof. For genus $g = 0$ it is clear that $r_T \equiv 1 \pmod 2$. For genus $g > 0$ we can choose homology generators so that they do not intersect the boundary of T. In this case it is again clear the $r_T \equiv 1 \pmod 2$. The general case for genus $g > 0$ then follows from Lemma 5.2.1. □

For any path $P : i \rightsquigarrow j$ we can form the closed curve $C'(P)$ by traversing the path P from

i to j and then following the boundary of T from j to i opposite to our choosen direction. We let r'_P denote the rotation number of $C'(P)$. Our next lemma shows that we can use r'_P in place of r_P and the boundary measurement matrix will not change.

Lemma 5.3.2. *Let N be a directed network embedded on a closed surface S with boundary and P is a path in N, then $r'_P \equiv r_P \pmod 2$.*

Proof. Let $P : i \rightsquigarrow j$ be a path from i to j in N. We claim $r_P - r'_P = r_T \pm 1$. To see this draw $C(P)$ and $C'(P)$ together in the same fundamental domain. We then reverse the direction of $C'(P)$ and observe that we traverse the boundary of T once and also traverse the path P once from i to j as well as once in reverse from j to i. Hence we can compute $r_P - r'_P$ by considering the rotation number of the closed curve obtained by first traversing P, then traversing the boundary of T, and finally traversing the path P in reverse. Thus $r_P - r'_P = r_T \pm 1$ and it follows by Lemma 5.3.1 that $r'_P \equiv r_P \pmod 2$. □

So, Lemma 5.3.2 shows that the direction in which we parameterize the boundary of S does not affect the boundary measurement. We now state and prove our theorem on signing edges.

Theorem 5.3.3. *If $N = (V, E)$ is a perfectly oriented network embedded on a closed orientable surface with boundary, then there exists a collection $\{\epsilon_e\}_{e\in E} \in \{\pm 1\}^E$ such that*

$$B(N, \{x_e\}_{e\in E}) = A(N, \{\epsilon_e x_e\}_{e\in E}).$$

Proof. Let N be a perfectly oriented network with vertex set V and edge set E. To show $B(N, \{x_e\}) = A(N, \{\epsilon_e x_e\})$ it suffices to show that the path $P : i \rightsquigarrow j$ for any $(i, j) \in I \times K$

has the following property:

$$\mathrm{wt}(P)|_{\{\epsilon_e x_e\}_{e\in E}} = (-1)^{s_{ij}+r_P+1}\,\mathrm{wt}(P) \qquad (\diamond)$$

When this is true for a choice of signs $\{\epsilon_e\}_{e\in E}$ we will say the path P has property $(\diamond)$. We assume the network has at least one boundary source, or else the in no boundary measurement matrix.

Recall that K denotes the set of boundary vertices of N and we have an ordering of the boundary vertices. We fix the following notation, if $j \in K$ is a boundary vertex we let e_j denote the unique external edge which is incident on j and write ϵ_j for ϵ_{e_j}. It can so happen that $e_{j_1} = e_{j_2}$ for $j_1 \neq j_2$, in this case we will consider distinct signs ϵ_{j_1} and ϵ_{j_2} on half edges with the sign on the edge being the product $\epsilon_{j_1}\epsilon_{j_2}$. We induct on the number of interior vertices. If there are no interior vertices, then the result is true since each path consists of a single edge.

For the inductive step we chose any boundary source i_0 and construct a network $\tilde{N}$ with one fewer interior vertex. The edge set of $\tilde{N}$ will be denoted $\tilde{E}$. We will inductively chose signs $\{\tilde{\epsilon}_e\}_{e\in\tilde{E}}$ so that each path in $\tilde{N}$ has property $(\diamond)$ and show how to modify these signs to give a collection $\{\epsilon_e\}_{e\in E}$ so that each path in the N has property $(\diamond)$. Recall that for two boundary vertices i and j of N we let s_{ij} denote the number of boundary sources strictly between them in N. For two boundary vertices i and j of $\tilde{N}$ we let $\tilde{s}_{ij}$ denote the number of boundary sources strictly between them in $\tilde{N}$. The inductive step falls into one of three cases depending on the boundary source i_0 and its unique neighboring vertex.

If i_0 is adjacent to a white vertex with outgoing edges e' and e'' we then remove the white vertex and split i_0 into two boundary sources $i_0' < i_0''$ as shown in Figure 5.7. Choose signs

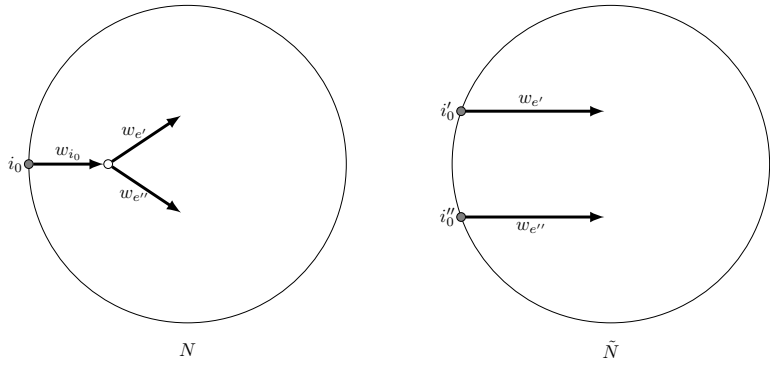

Figure 5.7: Splitting a white vertex

$\{\tilde{\epsilon}_e\}$ for the edges of $\tilde{N}$ by induction so that all paths in $\tilde{N}$ have property ($\diamond$). We define the signs $\{\epsilon_e\}$ as follows

$$\begin{aligned}
&\epsilon_j = \tilde{\epsilon}_j && \text{for } j \in K \text{ with } j < i_0 \\
&\epsilon_{e'} = \tilde{\epsilon}_{e'} \\
&\epsilon_{i_0} = +1 \\
&\epsilon_{e''} = -\tilde{\epsilon}_{e''} \\
&\epsilon_j = -\tilde{\epsilon}_j && \text{for } j \in K \text{ with } j > i_0 \\
&\epsilon_e = \tilde{\epsilon}_e && \text{otherwise}
\end{aligned}$$

and now verify the collection of signs $\{\epsilon_e\}$ are valid.

Consider a path $P : i \rightsquigarrow j$ in N with $i \neq i_0$. The path P corresponds to a path $\tilde{P} : i \rightsquigarrow j$ in $\tilde{N}$ with $r_P = r_{\tilde{P}}$. If $i, j < i_0$, then $s_{ij} = \tilde{s}_{ij}$ and P has property ($\diamond$) since the modification does not introduce any sign change to P. If $i < i_0 < j$ or $j < i_0 < i$, then $s_{ij} = \tilde{s}_{ij} - 1$ and P has property ($\diamond$) since the modification introduces one sign change to P. If $i, j > i_0$, then $s_{ij} = \tilde{s}_{ij}$ and P has property ($\diamond$) since the modification introduces two sign changes to P.

Next consider a path $P : i_0 \rightsquigarrow j$ in N. The path P corresponds either to a path $\tilde{P}' : i_0' \rightsquigarrow j$ or $\tilde{P}'' : i_0'' \rightsquigarrow j$. First consider the case P corresponds to $\tilde{P}'$. If $j < i_0$, then $s_{ij} = \tilde{s}_{ij}$ and P has property ($\diamond$) since the modification does not introduce any sign change to

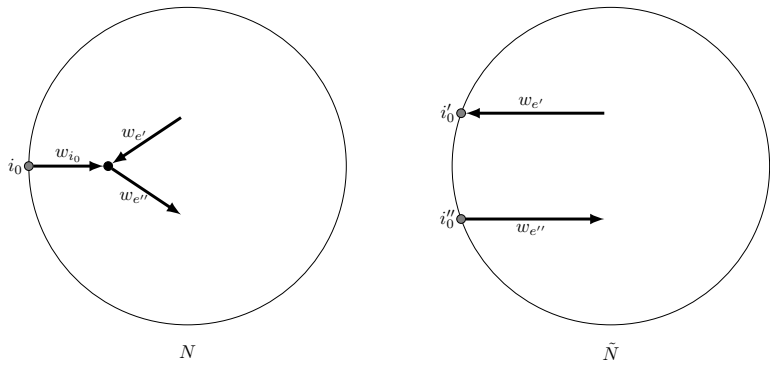

Figure 5.8: Splitting a black vertex in the case that i_0' is a sink and i_0'' is a source

P. If $i_0 < j$, then $s_{ij} = \tilde{s}_{ij} - 1$ and P has property ($\diamond$) since the modification introduces one sign change to P. Next consider the case P corresponds to $\tilde{P}''$. If $j < i_0$, then $s_{ij} = \tilde{s}_{ij} - 1$ and P has property ($\diamond$) since the modification introduces one sign change to P. If $i_0 < j$, then $s_{ij} = \tilde{s}_{ij}$ and P has property ($\diamond$) since the modification introduces two sign changes to P. Therefore the signs $\{\epsilon_e\}$ are valid in this case.

If i_0 is adjacent to a black vertex we then remove the black vertex and split i_0 into two boundary vertices $i_0' < i_0''$ one of which will be a sink and the other of which will be a source. We now consider the case where i_0' is a sink and i_0'' is a source as shown in Figure 5.8. Choose signs $\{\tilde{\epsilon}_e\}$ for the edges of $\tilde{N}$ by induction so that all paths in $\tilde{N}$ have property ($\diamond$). We define the signs $\{\epsilon_e\}$ as follows

$$\epsilon_{e'} = -\tilde{\epsilon}_{e'}$$

$$\epsilon_{i_0} = +1$$

$$\epsilon_e = \tilde{\epsilon}_e, \qquad \text{otherwise}$$

and now verify the collection of signs $\{\epsilon_e\}$ as defined satisfy our rule.

Consider a path $P : i \rightsquigarrow j$ in N with $i \neq i_0$. If P does not use the edge e', then P corresponds to a path $\tilde{P} : i \to j$ in $\tilde{N}$ with $r_P = r_{\tilde{P}}$ and $s_{ij} = \tilde{s}_{ij}$. If this is the case, then it is clear P has property ($\diamond$). Otherwise P traverses the edge e' some number of times.

Let $l+1$ be the number of times P traverses e' for $l \geq 0$. The path P corresponds to the concatenation of paths $\tilde{P}' : i \rightsquigarrow i'_0$, $\tilde{P}_k : i''_0 \rightsquigarrow i'_0$ for $1 \leq k \leq l$, and $\tilde{P}'' : i''_0 \rightsquigarrow j$. Now the sign of the product of the weights of these paths in $\tilde{N}$ is

$$(-1)^{\tilde{s}_{ii'_0}+r_{\tilde{P}'}+1}(-1)^{\sum_{k=1}^{l}(r_{\tilde{P}_k}+1)}(-1)^{\tilde{s}_{i''_0 j}+r_{\tilde{P}''}+1}$$

since $\tilde{s}_{i'_0 i''_0} = 0$. The sign of the path P in N will be

$$(-1)^{\tilde{s}_{ii'_0}+r_{\tilde{P}'}+1}(-1)^{\sum_{k=1}^{l}(r_{\tilde{P}_k}+1)}(-1)^{\tilde{s}_{i''_0 j}+r_{\tilde{P}''}+1}(-1)^{l+1}$$

since we pick up an addition factor of -1 each time we traverse e'. Simplifying the sign of P is

$$(-1)^{\tilde{s}_{ii'_0}+\tilde{s}_{i''_0 j}+1+r_{\tilde{P}'}+\sum_{k=1}^{l} r_{\tilde{P}_k}+r_{\tilde{P}''}}.$$

We observe that

$$\begin{aligned} s_{ij} &= \tilde{s}_{ii'_0} + \tilde{s}_{i''_0 j} + 1 \quad \text{if } i \prec i_0 \prec j \\ s_{ij} &= \tilde{s}_{ii'_0} + \tilde{s}_{i''_0 j} \qquad \text{if } j \prec i_0 \prec i, \end{aligned}$$

and so the sign of P is

$$\begin{aligned} &(-1)^{s_{ij}+r_{\tilde{P}'}+\sum_{k=1}^{l} r_{\tilde{P}_k}+r_{\tilde{P}''}} \qquad \text{if } i \prec i_0 \prec j \\ &(-1)^{s_{ij}+1+r_{\tilde{P}'}+\sum_{k=1}^{l} r_{\tilde{P}_k}+r_{\tilde{P}''}} \quad \text{if } j \prec i_0 \prec i. \end{aligned}$$

Finally we observe that

$$\begin{aligned} r_P + 1 &\equiv r_{\tilde{P}'} + \textstyle\sum_{k=1}^{l} r_{\tilde{P}_k} + r_{\tilde{P}''} \pmod 2 \quad \text{if } i \prec i_0 \prec j \\ r_P &\equiv r_{\tilde{P}'} + \textstyle\sum_{k=1}^{l} r_{\tilde{P}_k} + r_{\tilde{P}''} \pmod 2 \qquad \text{if } j \prec i_0 \prec i \end{aligned}$$

and it follows that P has property $(\diamond)$. See Figure 5.9 for the case of the disk. More generally when the surface is not the disk the boundary will still be a circle and Lemma 5.3.1 shows

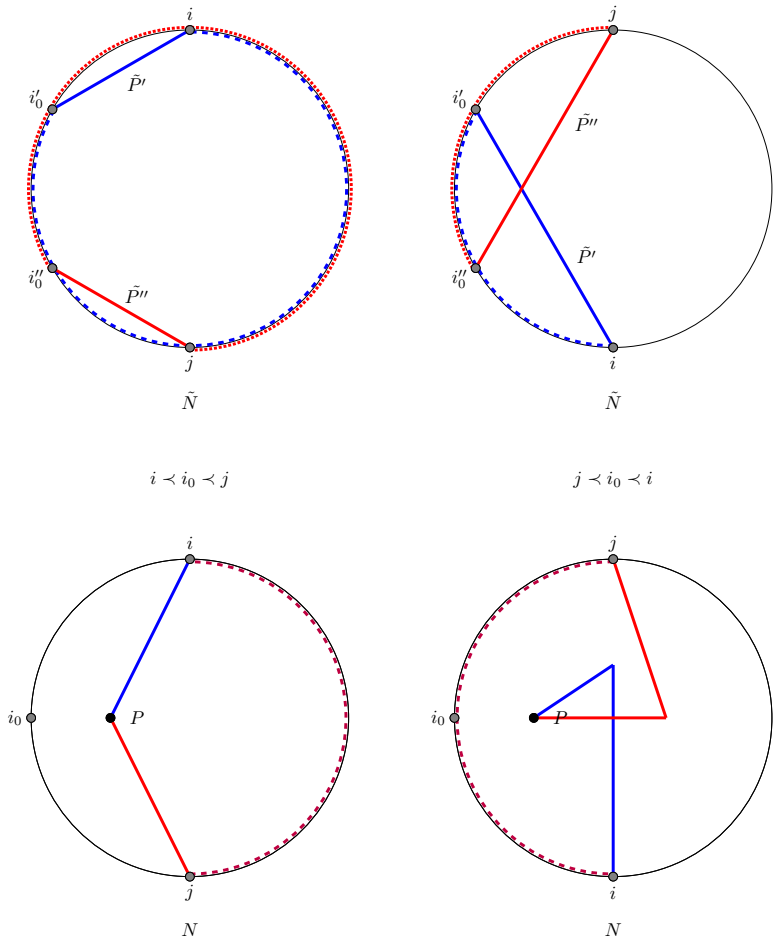

Figure 5.9: Closing paths in $\tilde{N}$. The case $i \prec i_0 \prec j$ is shown on the right while the case $j \prec i_0 \prec i$ is shown on the left.

that the rotation number of traversing the boundary will always be odd, and hence can be thought of as shown in Figure 5.9.

Now consider a path $P : i_0 \rightsquigarrow j$ in N. If P does not use the edge e', then P corresponds to a path $\tilde{P} : i''_0 \to j$ in $\tilde{N}$ with $r_P = r_{\tilde{P}}$ and $s_{ij} = \tilde{s}_{i''_0 j}$. If this is the case, then it is clear P has property ($\diamond$). Otherwise P traverses the edge e' some number of times. Let l be the number of times P traverses e' for $l > 0$. In this case P corresponds to the concatenation of paths $\tilde{P}_k : i''_0 \rightsquigarrow i'_0$ for $1 \leq k \leq l$, and $\tilde{P}'' : i''_0 \rightsquigarrow j$. Now the sign of the product of the

weights of these paths in $\tilde{N}$ is

$$(-1)^{\sum_{k=1}^{l}(r_{\tilde{P}_k}+1)}(-1)^{\tilde{s}_{i_0''j}+r_{\tilde{P}''}+1}$$

since $\tilde{s}_{i_0'i_0''} = 0$. The sign of the path P in N will be

$$(-1)^{\sum_{k=1}^{l}(r_{\tilde{P}_k}+1)}(-1)^{\tilde{s}_{i_0''j}+r_{\tilde{P}''}+1}(-1)^{l}$$

since we pick up an addition factor of -1 each time we traverse e'. Simplifying, the sign of P is

$$(-1)^{s_{ij}+\sum_{k=1}^{l} r_{\tilde{P}_k}+r_{\tilde{P}''}+1}$$

since $s_{ij} = \tilde{s}_{i_0''j}$. The equality

$$r_P = \sum_{k=1}^{l} r_{\tilde{P}_k} + r_{\tilde{P}''}$$

implies that P has property $(\diamond)$.

The final case is again i_0 is adjacent to a black vertex, and we remove the black vertex and split i_0 into two boundary vertices $i_0' < i_0''$. This time we consider the case where i_0' is a source and i_0'' is a sink as shown in Figure 5.10. This case will be identical to the previous case of splitting a black vertex, after applying Lemma 5.3.2 and forming closed curves in the opposite direction, with the subcases $i \prec i_0 \prec j$ and $j \prec i_0 \prec i$ reversed. □

Theorem 5.3.3 need not be true when N is not a perfectly oriented network. See Figure 5.11 for an example of a network for which Theorem 5.3.3 does not hold. The boundary

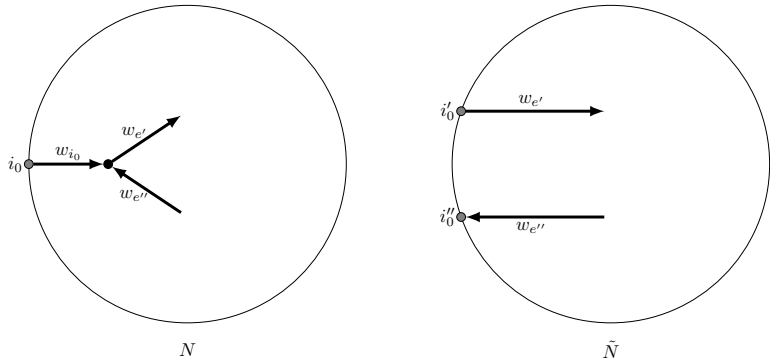

Figure 5.10: Splitting a black vertex in the case that i_0' is a source and i_0'' is a sink

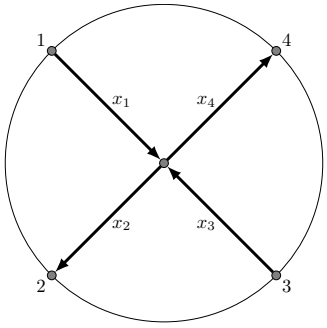

Figure 5.11: A network which is not perfectly oriented

measurement matrix for the network in Figure 5.11 is

$$B = \begin{bmatrix} 1 & x_1x_2 & 0 & -x_1x_4 \\ 0 & x_2x_3 & 1 & x_3x_4 \end{bmatrix}.$$

The matrix B cannot be obtained from the weighted path matrix for this example. Notice the second column of B would require x_1 and x_3 receive the same sign, while the fourth column of B would require x_1 and x_3 receive opposite signs. However, as mentioned in Remark 5.1.1 the network in Figure 5.11 can be transformed to a perfectly oriented network. In this case it turns out the perfectly oriented network we get after the transformation is the network in Figure 5.6 for which we have already seen how to sign the edges. We include Algorithm 1 for finding a signing of edges as in Theorem 5.3.3. Algorithm 1 makes use of

helper functions given in Algorithm 2 This recursive algorithm exactly corresponds to the induction used in the proof.

Algorithm 1 Signing edges of a perfectly oriented network

Require: A perfectly oriented network $N = (V, E)$.
  function FindSigns(N)
    if $\text{int}(V) = \varnothing$ **then**
      for $e = (i, j) \in E$ **do**
        $\epsilon_e = (-1)^{s_{ij}+r_e+1}$
      return $\{\epsilon_e\}_{e\in E}$
    else
      Choose boundary source $i_0 \in I$ adjacent to some interior vertex.
      Let $e_0 = (i_0, v_0)$ be the unique edge incident on i_0.
      Let e' and e'' be to two edges different from e_0 incident on v_0 with $e' < e''$.
      $\tilde{N} \leftarrow$ Split$(N, i_0, v_0, e_0, e', e'')$
      $\{\tilde{\epsilon}_e\}_{e\in E(\tilde{N})} \leftarrow$ FindSigns$(\tilde{N})$
      return ModifySigns$(N, i_0, v_0, e_0, e', e'', \{\tilde{\epsilon}_e\}_{e\in E(\tilde{N})})$

5.4 A Formula for Plücker Coordinates

Notice that from Theorem 5.3.3 it follows that Conjecture 5.1.2 is true. We now want to give an explicit formula for the minors of the boundary measurement matrix. In order to do this we must first review some concepts and results that can be found in [Pos06] and [Tal08]. Take $I, J \subseteq [n]$ with $|I| = |J|$. Let $\pi : I \to J$ be a bijection with $\pi(i) = i$ for all $i \in I \cap J$. A pair $(i_1, i_2) \in I \times I$ where $i_1 < i_2$ is called a *crossing* of π if the following condition holds

$$(i_1 - \pi(i_2))(\pi(i_2) - \pi(i_1))(\pi(i_1) - i_2))(i_2 - i_1) < 0$$

This condition is equivalent to the chord for i_1 to $\pi(i_1)$ crossing the chord from i_2 to $\pi(i_2)$ when the elements of $[n]$ are placed in cyclic order on the boundary of a disk. Thinking of I as a collection of boundary sources and J as a collection of boundary vertices, the condition

Algorithm 2 Helper functions for signing edges of a perfectly oriented network

function SPLIT(N,i_0,v_0,e_0,e',e'')
  $\tilde{V} \leftarrow (V \setminus \{i_0, v_0\}) \cup \{i_0', i_0''\}$
  $\tilde{K} = (K \setminus \{i_0\}) \cup \{i_0', i_0''\}$
  Order $\tilde{K}$ by $i_1 < i_0' < i_0'' < i_2$ for all $i_1, i_2 \in K$ such that $i_1 < i_0 < i_2$
  if $e' = (v_0, x)$ **then** $\tilde{e}' \leftarrow (i_0', x)$
  if $e' = (x, v_0)$ **then** $\tilde{e}' \leftarrow (x, i_0')$
  if $e'' = (v_0, x)$ **then** $\tilde{e}'' \leftarrow (i_0'', x)$
  if $e'' = (x, v_0)$ **then** $\tilde{e}'' \leftarrow (x, i_0'')$
  $\tilde{E} \leftarrow (E \setminus \{e_0, e', e''\}) \cup \{\tilde{e}', \tilde{e}''\}$
  $\tilde{N} \leftarrow (\tilde{V}, \tilde{E})$
  return $\tilde{N}$
function MODIFYSIGNS($N, i_0, v_0, e_0, e', e'', \{\tilde{\epsilon}_e\}_{e \in E(\tilde{N})}$)
  if $\tilde{e}' = (v_0, x)$ and $\tilde{e}'' = (v_0, y)$ **then**
    $\epsilon_{i_0} \leftarrow +1$
    $\epsilon_{e'} \leftarrow \tilde{\epsilon}_{\tilde{e}'}$
    $\epsilon_{e''} \leftarrow \tilde{\epsilon}_{\tilde{e}''}$
    for $j \in K$ with $j < i_0$ **do**
      $\epsilon_j \leftarrow \tilde{\epsilon}_j$
    for $j \in K$ with $j > i_0$ **do**
      $\epsilon_j \leftarrow -\tilde{\epsilon}_j$
    for All other edges $e \in E$ **do**
      $\epsilon_e \leftarrow \tilde{\epsilon}_e$
  if $\tilde{e}' = (x, v_0)$ and $\tilde{e}'' = (v_0, y)$ **then**
    $\epsilon_{i_0} \leftarrow +1$
    $\epsilon_{e'} \leftarrow -\tilde{\epsilon}_{\tilde{e}'}$
    $\epsilon_{e''} \leftarrow \tilde{\epsilon}_{\tilde{e}''}$
    for All other edges $e \in E$ **do**
      $\epsilon_e \leftarrow \tilde{\epsilon}_e$
  if $\tilde{e}' = (v_0, x)$ and $\tilde{e}'' = (y, v_0)$ **then**
    $\epsilon_{i_0} \leftarrow +1$
    $\epsilon_{e'} \leftarrow \tilde{\epsilon}_{\tilde{e}'}$
    $\epsilon_{e''} \leftarrow -\tilde{\epsilon}_{\tilde{e}''}$
    for All other edges $e \in E$ **do**
      $\epsilon_e \leftarrow \tilde{\epsilon}_e$

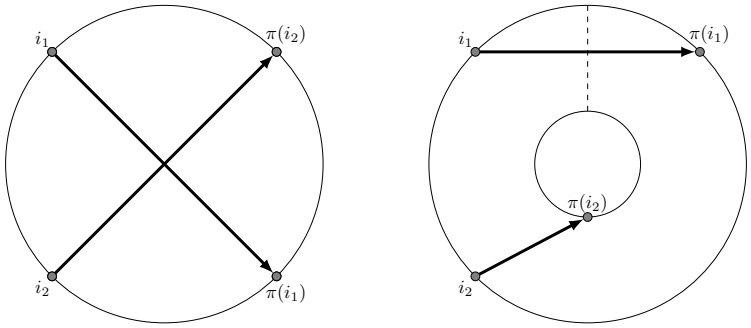

Figure 5.12: Crossings on the disk and the annulus. Here boundary vertices are ordered $i_1 < i_2 < \pi(i_1) < \pi(i_2)$.

of being a crossing means that a path $P_1 : i \rightsquigarrow \pi(i_1)$ must intersect any path $P_2 : i_2 \rightsquigarrow \pi(i_2)$ in any network embedded on a disk. However, when our surface is not a disk it can happen that (i_1, i_2) is a crossing of π but paths $P_1 : i \rightsquigarrow \pi(i_1)$ and $P_2 : i_2 \rightsquigarrow \pi(i_2)$ do not intersect. See Figure 5.12 for a pictorial representation of a crossing on the disk and an example of paths of the annulus which come from a crossing but do not intersect. We let $\mathrm{xing}(\pi)$ denote the number of crossings of π.

If $|I| = |J| = k$ then a bijection $\pi : I \to J$ determines a unique permutation $\pi \in S_k$ by standardizing I and J. We let $\mathrm{inv}(\pi)$ denote the number of inversion of π when view as an element of S_k. We let $s_{i,j}$ denote the number of elements of I strictly between i and j.

Lemma 5.4.1 ([Tal08])**.** *If $I, J \subset [n]$ with $|I| = |J|$ and $\pi : I \to J$ is a bijection such that $\pi(i) = i$ for all $i \in I \cap J$, then*

$$(-1)^{\mathrm{xing}(\pi)} = (-1)^{\mathrm{inv}(\pi)} \prod_{i \in I} (-1)^{s_{i,\pi(i)}}.$$

Proof. This is shown during the proof of [Tal08, Proposition 2.12]. □

We now give our formula for the Plücker coordinates of the boundary measurement map.

Corollary 5.4.2. *If $N = (V, E)$ is a perfectly oriented network embedded on a closed orientable surface with boundary, then for any $J \subseteq K$ with $|I| = |J|$*

$$\Delta_{I,J}(B(N, \{x_e\}_{e\in E})) = \frac{\sum_{\mathbf{F}\in\mathcal{F}_{I,J}(N)}(-1)^{c(\mathbf{F})}\operatorname{wt}(\mathbf{F})}{\sum_{\mathbf{C}\in\mathcal{C}(N)}\operatorname{wt}(\mathbf{C})}$$

where $c(\mathbf{F}) = \operatorname{xing}(\pi) + \sum_{P\in\mathbf{P}}(r_P + 1)$ if $\mathbf{F} = (\mathbf{P}, \mathbf{C})$.

Proof. We first take $\{\epsilon_e\}_{e\in E}$ such that $B(N, \{x_e\}_{e\in E}) = A(N, \{\epsilon_e x_e\}_{e\in E})$ which necessarily exists by Theorem 5.3.3. Using Equation (5.1) we obtain

$$\Delta_{I,J}(B(N, \{x_e\}_{e\in E})) = \frac{\sum_{\mathbf{F}\in\mathcal{F}_{I,J}(N)}\operatorname{sgn}(\mathbf{F})\left(\prod_{e\in\mathbf{F}}\epsilon_e\right)\operatorname{wt}(\mathbf{F})}{\sum_{\mathbf{C}\in\mathcal{C}(N)}\operatorname{sgn}(\mathbf{C})\left(\prod_{e\in\mathbf{C}}\epsilon_e\right)\operatorname{wt}(\mathbf{C})}.$$

Since traversing a cycle in a perfectly oriented network will always change the rotation number by exactly one, it follows that $\prod_{e\in\mathbf{C}}\epsilon_e = (-1)^{|\mathbf{C}|}$ for any $\mathbf{C} \in \mathcal{C}(N)$. Also, $\operatorname{sgn}(\mathbf{C}) = (-1)^{|\mathbf{C}|}$ for any $\mathbf{C} \in \mathcal{C}(N)$ and thus

$$\operatorname{sgn}(\mathbf{C})\left(\prod_{e\in\mathbf{C}}\epsilon_e\right) = 1$$

and the denominator in the corollary is correct.

It remains to show the numerator in the corollary is correct. That is we must show $\operatorname{sgn}(\mathbf{F})\left(\prod_{e\in\mathbf{F}}\epsilon_e\right) = (-1)^{\operatorname{xing}(\pi)+\sum_{P\in\mathbf{P}}(r_P+1)}$ for any $\mathbf{F} \in \mathcal{F}_{I,J}(N)$. Take $\mathbf{F} = (\mathbf{P}, \mathbf{C}) \in$

$\mathcal{F}_{I,J}(N)$ and let $\pi : I \to J$ be the bijection determined by $\mathbf{P}$, then

$$\begin{aligned}
\operatorname{sgn}(\mathbf{F}) \left(\prod_{e \in \mathbf{F}} \epsilon_e \right) &= \operatorname{sgn}(\mathbf{P}) \operatorname{sgn}(\mathbf{C}) \left(\prod_{e \in \mathbf{P}} \epsilon_e \right) \left(\prod_{e \in \mathbf{C}} \epsilon_e \right) \\
&= \operatorname{sgn}(\pi) \left(\prod_{e \in \mathbf{P}} \epsilon_e \right) \\
&= (-1)^{\operatorname{inv}(\pi)} \left(\prod_{i \in I} (-1)^{s_{i,\pi(i)}} \right) \left(\prod_{P \in \mathbf{P}} (-1)^{r_P + 1} \right) \\
&= (-1)^{\operatorname{xing}(\pi) + \sum_{P \in \mathbf{P}} (r_P + 1)}
\end{aligned}$$

where we have made use of Lemma 5.4.1. □

In the case our surface S is a disk it is easy to see that the formula in Corollary 5.4.2 contains no negative terms. On the disk for any flow $\mathbf{F} = (\mathbf{P}, \mathbf{C})$ we must have $\operatorname{xing}(\pi) = 0$ and $r_P = \pm 1$ for all $P \in \mathbf{P}$. Thus, $c(\mathbf{F})$ is even for any flow $\mathbf{F}$ in the disk. Hence we recover Equation (5.2). For more general surfaces we no longer have positivity, for example see Figure 5.4.

5.5 The Gauge Action and Uniqueness of Signs

Given a directed network $N = (V, E)$ embedded on a surface the *gauge group* $\mathcal{G} = \mathcal{G}(N) := (\mathbb{R}^*)^{\operatorname{int}(V)}$ where $\operatorname{int}(V) = V \setminus K$ denotes the set of interior vertices of N and $\mathbb{R}^*$ denotes the nonzero real numbers. We also define the *weight space* $\mathcal{X} = \mathcal{X}(N)$ to be the set of all collections $\{a_e x_e\}_{e \in E}$ where $a_e \in \mathbb{R}^*$. Notice here to each edge $e \in E$ we associate a nonzero real number a_e and a formal variable x_e. An element of the gauge group $g = (g_v)_{v \in \operatorname{int}(V)} \in \mathcal{G}$

acts on an element of the weight space $X = \{a_e x_e\}_{e \in E} \in \mathcal{X}$ as follows

$$g \cdot X = \{(g \cdot a_e) x_e\}_{e \in E}$$

where if $e = (i, j)$ then $g \cdot a_e = g_j^{-1} a_e g_i$ (with the convention that $g_i = 1$ if $i \in K$ is a boundary vertex). It follows that

$$A(N, X) = A(N, g \cdot X)$$

for all $g \in \mathcal{G}$ and $X \in \mathcal{X}$. When $X, Y \in \mathcal{X}(N)$ are such that $Y = g \cdot X$ for some $g \in \mathcal{G}(N)$ we call X and Y *gauge equivalent*.

Algorithm 3 Finding Gauge Transformation

Require: A directed network $N = (V, E)$ embedded on a closed orientable surface with boundary such that each vertex is contained in some path between boundary vertices and $X, Y \in \mathcal{X}(N)$ are such that $A(N, X) = A(N, Y)$.

function FindGauge(X,Y)

Let $X = \{a_e x_e\}_{e \in E}$ and $Y = \{b_e x_e\}_{e \in E}$

$g \leftarrow (1)_{v \in \text{int}(V)}$

$\mathcal{O} \leftarrow \text{int}(V)$

$\mathcal{C} \leftarrow V \setminus \text{int}(V)$

while $\mathcal{O} \neq \varnothing$ **do**

Choose $(u, v) \in E$ such that $(u, v) \in \mathcal{C} \times \mathcal{O}$

$g_v \leftarrow \frac{g \cdot a_{(u,v)}}{b_{(u,v)}}$

$\mathcal{C} \leftarrow \mathcal{C} \cup \{v\}$

$\mathcal{O} \leftarrow \mathcal{O} \setminus \{v\}$

return g ▷ g returned will be such that $g \cdot X = Y$

Theorem 5.5.1. *Let $N = (V, E)$ be a directed network embedded on a closed orientable surface with boundary such that every vertex in contained in some path between boundary vertices, then $A(N, X) = A(N, Y)$ for $X, Y \in \mathcal{X}(N)$ if and only if X and Y are gauge equivalent.*

Proof. Our proof will show that Algorithm 3 returns $g \in \mathcal{G}(N)$ such that $g \cdot X = Y$. Let $X = \{a_e x_e\}_{e \in E}$ and $Y = \{b_e x_e\}_{e \in E}$. First note that Algorithm 3 will always terminate since each vertex of N is contained in some path between boundary vertices and $\mathcal{C}$ initially consists of all the boundary vertices. Furthermore, when the algorithm terminates $\mathcal{C} = V$. Also, observe that if $v \in \mathcal{C} \cap \text{int}(V)$ at some stage of the algorithm there is a directed path from some boundary vertex to the vertex v passing through only vertices in $\mathcal{C}$. Lastly, we note that at a given stage of the algorithm $g_v = 1$ whenever $v \notin \mathcal{C}$. It suffices to show that at each step of Algorithm 3 we have the following property:

$$g \cdot a_e = b_e \text{ for all } e \in \mathcal{C} \times \mathcal{C} \tag{$\star$}$$

Initially $\mathcal{C}$ consists of only the boundary vertices and $g = (1)_{v \in \text{int}(V)}$. At this stage we have $g \cdot a_e = a_e$ for all $e \in E$, and $a_e = b_e$ whenever $e \in \mathcal{C} \times \mathcal{C}$ by the assumption that $A(N, X) = A(N, Y)$. So, initially we have property $(\star)$.

We now consider extending the set of vertices $\mathcal{C}$. Suppose we are at some stage of the algorithm where $g \cdot a_e = b_e$ for all $e \in \mathcal{C} \times \mathcal{C}$. Consider $(u, v) \in \mathcal{C} \times \mathcal{O}$ and let $\mathcal{C}' = \mathcal{C} \cup \{v\}$ and g' be such that $g'_v = g \cdot a_{(u,v)} / b_{(u,v)}$ and $g'_x = g_x$ for $x \neq v$. Now we must show for all $e \in \mathcal{C}' \times \mathcal{C}'$ that $g' \cdot a_e = b_e$. We need only consider edges $e \in \mathcal{C}' \times \mathcal{C}'$ incident on v as

$g' \cdot a_e = g \cdot a_e = b_e$ for $e \in \mathcal{C}' \times \mathcal{C}'$ with e not incident on v. First we compute

$$\begin{aligned} g' \cdot a_{(u,v)} &= (g'_v)^{-1} a_{(u,v)} g'_u \\ &= \frac{b_{(u,v)} a_{(u,v)} g'_u}{g \cdot a_{(u,v)}} \\ &= \frac{b_{(u,v)} (g \cdot a_{(u,v)})}{g \cdot a_{(u,v)}} \\ &= b_{(u,v)} \end{aligned}$$

and conclude $g' \cdot a_{(u,v)} = b_{(u,v)}$.

Consider $(w, v) \in E$ such that $w \in \mathcal{C}$. We can find paths $P_u : i_u \rightsquigarrow u$ and $P_w : i_w \rightsquigarrow w$ passing through only vertices of $\mathcal{C}$ for $i_u, i_w \in I$. Choose some path $P : v \rightsquigarrow j$ for $j \in K$ so we get paths $P_1 = P_u(u, v)P : i_u \rightsquigarrow j$ and $P_2 = P_w(w, v)P : i_w \rightsquigarrow j$. It then follows that

$$\prod_{e \in P_1} a_e = \prod_{e \in P_1} b_e \qquad\qquad \prod_{e \in P_2} a_e = \prod_{e \in P_2} b_e$$

and so also

$$\prod_{e \in P_1} g' \cdot a_e = \prod_{e \in P_1} b_e \qquad\qquad \prod_{e \in P_2} g' \cdot a_e = \prod_{e \in P_2} b_e.$$

Considering ratios we see

$$\frac{\left(\prod_{e \in P_u} g' \cdot a_e\right) (g' \cdot a_{(u,v)}) \left(\prod_{e \in P} g' \cdot a_e\right)}{\left(\prod_{e \in P_w} g' \cdot a_e\right) (g' \cdot a_{(w,v)}) \left(\prod_{e \in P} g' \cdot a_e\right)} = \frac{\left(\prod_{e \in P_u} b_e\right) (b_{(u,v)}) \left(\prod_{e \in P} b_e\right)}{\left(\prod_{e \in P_w} b_e\right) (b_{(w,v)}) \left(\prod_{e \in P} b_e\right)}$$

and recalling $g' \cdot a_e = b_e$ for $e \in P_u \cup P_w \subseteq \mathcal{C}$ we can conclude $g' \cdot a_{(w,v)} = b_{(w,v)}$ as desired.

Consider $(v, w) \in E$ such that $w \in \mathcal{C}$. We can find paths $P_u : i_u \rightsquigarrow u$ and $P_w : i_w \rightsquigarrow w$ passing through only vertices of $\mathcal{C}$ for $i_u, i_w \in I$. Choose some path $P : w \rightsquigarrow j$ for $j \in K$ so we get paths $P_1 = P_u(u, v)(v, w)P : i_u \rightsquigarrow j$ and $P_2 = P_w P : i_w \rightsquigarrow j$. It then follows that

$$\prod_{e \in P_1} a_e = \prod_{e \in P_1} b_e \qquad\qquad \prod_{e \in P_2} a_e = \prod_{e \in P_2} b_e$$

and so also

$$\prod_{e \in P_1} g' \cdot a_e = \prod_{e \in P_1} b_e \qquad\qquad \prod_{e \in P_2} g' \cdot a_e = \prod_{e \in P_2} b_e.$$

Considering ratios we see

$$\begin{aligned} &\frac{\left(\prod_{e \in P_u} g' \cdot a_e\right)(g' \cdot a_{(u,v)})(g' \cdot a_{(v,w)})\left(\prod_{e \in P} g' \cdot a_e\right)}{\left(\prod_{e \in P_w} g' \cdot a_e\right)\left(\prod_{e \in P} g' \cdot a_e\right)} \\ &\qquad = \frac{\left(\prod_{e \in P_u} b_e\right)(b_{(u,v)})(b_{(v,w)})\left(\prod_{e \in P} b_e\right)}{\left(\prod_{e \in P_w} b_e\right)\left(\prod_{e \in P} b_e\right)} \end{aligned}$$

and recalling $g' \cdot a_e = b_e$ for $e \in P_u \cup P_w \subseteq \mathcal{C}$ and we can conclude $g' \cdot a_{(w,v)} = b_{(w,v)}$ as desired. Therefore property $(\star)$ extends at each step of Algorithm 3 and the theorem is proven. □

Theorem 5.5.1 has the following corollary which says that the choice of signs guaranteed by Theorem 5.3.3 is unique up to gauge transformation provided each vertex is contained in some path between boundary vertices.

Corollary 5.5.2. *If $N = (V, E)$ is a directed network embedded on a closed orientable surface with boundary such that every vertex in contained in some path between boundary vertices and there exists a collections $\{\epsilon_e\}_{e\in E}, \{\epsilon'_e\}_{e\in E} \in \{\pm 1\}^E$ such that*

$$B(N, \{x_e\}_{e\in E}) = A(N, \{\epsilon_e x_e\}_{e\in E}) \quad \text{and} \quad B(N, \{x_e\}_{e\in E}) = A(N, \{\epsilon'_e x_e\}_{e\in E})$$

then $\{\epsilon_e\}_{e\in E}$ and $\{\epsilon'_e\}_{e\in E}$ are gauge equivalent.

Chapter 6

Upper cluster algebras and ground rings

This chapter is based on the preprint [BMS18] which is joint work with Eric Bucher and Michael Shapiro.

6.1 Locally isolated and Locally acyclic cluster algebras

We first review Muller's notion of locally acyclic cluster algebras [Mul13]. This section closely follows [Mul14] while describing some changes needed to adapt the theory of locally acyclic cluster algebras for ground rings other than $\mathbb{ZP}$. Let $(B, \mathbf{x}, \mathbf{y})$ be a seed of rank n and $\mathbb{A}$ be a ground ring satisfying (3.1) and (3.2). Denote the corresponding cluster algebra by $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(B, \mathbf{x}, \mathbf{y})$ and upper cluster algebra by $\mathcal{U} = \mathcal{U}_{\mathbb{A}}(B, \mathbf{x}, \mathbf{y})$.

The *freezing* of $\mathcal{A}$ at $x_n \in \mathbf{x}$ is the cluster algebra $\mathcal{A}^\dagger = \mathcal{A}_{\mathbb{A}^\dagger}(B^\dagger, \mathbf{x}^\dagger, \mathbf{y}^\dagger)$ defined as follows

- The new semifield is $\mathbb{P}^\dagger = \mathbb{P} \times \mathbb{Z}$ with x_n as the generator of the free abelian group $\mathbb{Z}$. The auxiliary addition is extended as

$$(p_1 x_n^a) \oplus (p_2 x_n^b) = (p_1 \oplus p_2) x_n^{\min(a,b)}.$$

- The new ground ring $\mathbb{A}^\dagger \subseteq \mathbb{ZP}^\dagger$ is $\mathbb{A}^\dagger = \mathbb{A}[x_n^{\pm 1}]$.
- The new ambient field is $\mathcal{F}^\dagger = \mathbb{Q}(\mathbb{P}^\dagger, x_1, x_2, \ldots, x_{n-1})$ and $\mathbf{x}^\dagger = (x_1, x_2, \ldots, x_{n-1})$.
- The new coeffients are $\mathbf{y}^\dagger = (y_1^\dagger, y_2^\dagger, \ldots, y_{n-1}^\dagger)$ where $y_i^\dagger = y_i x_n^{B_{ni}}$.
- The new exchange matrix $B^\dagger$ is obtained from B by deleting the nth row and column.

In this setting the upper cluster algebra corresponding to $\mathcal{A}^\dagger$ will be denoted $\mathcal{U}^\dagger$. Notice in a cluster algebra arising from a quiver that a freezing exactly corresponds to replacing the mutable vertex labeled by x_n with a frozen vertex. By considering a permutation of indices we can freeze at any $x_i \in \mathbf{x}$. We can freeze at some subset of cluster variables by iteratively freezing at individual cluster variables. This process is independent of the order of freezing. A freezing $\mathcal{A}^\dagger$ of $\mathcal{A}$ at $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \in \mathbf{x}$ is called a *cluster localization* if $\mathcal{A}^\dagger = \mathcal{A}[(x_{i_1} x_{i_2} \cdots x_{i_m})^{-1}]$. A *cover* of $\mathcal{A}$ is a collection $\{A_i\}_{i \in I}$ of cluster localizations such that for any prime ideal $P \subseteq \mathcal{A}$ there exists $i \in I$ where $\mathcal{A}_i P \subsetneq \mathcal{A}_i$. Lemma 6.1.1 below is not exactly [Mul14, Lemma 1], but follows immediately and will be all that is used for our purposes.

Lemma 6.1.1 ([Mul14, Lemma 1])**.** *If $\mathcal{A}^\dagger$ is a freezing of a cluster algebra $\mathcal{A}$ at cluster variables $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ and $\mathcal{A}^\dagger = \mathcal{U}^\dagger$, then $\mathcal{A}^\dagger = \mathcal{A}[(x_{i_1}, x_{i_2}, \ldots, x_{i_m})^{-1}]$ is a cluster localization.*

The seed $(B, \mathbf{x}, \mathbf{y})$ is called *isolated* if B is the zero matrix. A cluster algebra defined by an isolated seed is also referred to as *isolated*. In terms of quivers, isolated means that there are no arrows between mutable vertices. The seed $(B, \mathbf{x}, \mathbf{y})$ is said to be *acyclic* if there are not $i_1, i_2, \ldots, i_\ell \in \{1, 2, \ldots, n\}$ with $B_{i_{j+1} i_j} > 0$ for $1 \leq j < \ell$ and $i_1 = i_\ell$. A cluster algebra defined by an acyclic seed is called an acyclic cluster algebra. A *locally isolated*, respectively

locally acyclic, cluster algebra is a cluster algebra for which there exists a cover by isolated, respectively acyclic, cluster algebras.

In [Mul14] the only ground ring considered is $\mathbb{ZP}$; however, proofs of the following results go through without change for more general ground rings.

Lemma 6.1.2 ([Mul14, Lemma 2]). *Let $\{\mathcal{A}_i\}_{i\in I}$ be a cover of $\mathcal{A}$. If $\mathcal{A}_i = \mathcal{U}_i$ for all $i \in I$, then $\mathcal{A} = \mathcal{U}$.*

Proposition 6.1.3 ([Mul14, Proposition 3]). *Let $\mathcal{A}$ be an isolated cluster algebra. Then $\mathcal{A} = \mathcal{U}$.*

This immediately gives the following result.

Theorem 6.1.1. *If $\mathcal{A}$ is a locally isolated cluster algebra, then $\mathcal{A} = \mathcal{U}$.*

The definition of a locally isolated cluster algebra and Theorem 6.1.1 are not explicitly stated in [Mul14]. This is because over the ground ring $\mathbb{ZP}$ being locally isolated is equivalent to being locally acyclic as every acyclic cluster algebra over $\mathbb{ZP}$ admits a cover by isolated cluster algebras [Mul14, Proposition 4]. The equivalence is not true over other ground rings.

In Section 6.2 we give an example of a cluster algebra of geometric type which is locally acyclic over $\mathbb{ZP}$, but for which the cluster algebra and upper cluster algebra do not coincide over a different natural choice of ground ring. We show this example cluster algebra is locally acyclic by using Muller's Banff algorithm [Mul13, Theorem 5.5]. A pair of vertices (i_1, i_2) in a quiver Q is called a *covering pair* if (i_1, i_2) is an arrow of Q that is not contained in any bi-infinite path of mutable vertices. The notion of covering pair is needed to run the Banff algorithm. The (reduced) Banff algorithm can be found as Algorithm 4. The reduced version of the algorithm deletes vertices rather then freezes them. This makes for a simpler check

that a cluster algebra is locally acyclic, but has the down side that it is does not compute the actual cover.

Algorithm 4 The (reduced) Banff algorithm to determine if a quiver is locally acyclic.

Require: A (mutable part of a) quiver Q_0
  function BANFF(Q_0)
    $\mathbf{A} \leftarrow \emptyset$
    $\mathbf{B} \leftarrow \{Q_0\}$
    while $\mathbf{B} \neq \emptyset$ **do**
      Remove a quiver Q from $\mathbf{B}$
      if Q is mutation equivalent to an acyclic quiver **then**
        $\mathbf{A} \leftarrow \mathbf{A} \cup \{Q\}$
      else if Q is mutation equivalent to a quiver Q' with a covering pair (i_1, i_2) **then**
        Let Q_j be the quiver obtained from Q' by freezing (deleting) i_j for $j = 1, 2$
        $\mathbf{B} \leftarrow \mathbf{B} \cup \{Q_1, Q_2\}$
      else
        The algorithm **fails**
    return A ▷ **A** returned will be a finite set of acyclic quivers

In the remainder of this section we analyze the proof of [Mul14, Proposition 4] consider conditions which imply a cluster algebra is locally isolated. This allows the application of Theorem 6.1.1 to conclude the cluster algebra is equal to its upper cluster algebra. An index $i \in \{1, 2, \ldots, n\}$ is a *source* in the seed $(B, \mathbf{x}, \mathbf{y})$ if $B_{ki} \geq 0$ for all $1 \leq k \leq n$. In this case mutation in the direction i gives

$$x_i x_i' = \frac{y_i}{y_i \oplus 1} \prod_{B_{ki}>0} x_k^{B_{ki}} + \frac{1}{y_i \oplus 1}. \tag{6.1}$$

A key step in showing an acyclic cluster algebra over $\mathbb{ZP}$ is covered by isolated cluster algebras is as follows[1]. Let i be a non-isolated source choose j with $B_{ji} > 0$, then after multiplying

[1]The argument that follows is the "source version" and a corresponding "sink version" holds with corresponding modification. The "sink version" is used in [Mul14].

by $y_i \oplus 1$ and rearranging (6.1) we obtain

$$((y_i \oplus 1)x_i')x_i - \left(y_i \prod_{\substack{B_{ki}>0 \\ k \neq j}} x_k^{B_{ki}} \right) x_j^{B_{ji}} = 1. \tag{6.2}$$

It is implied by (6.2) that $1 \in \mathcal{A}x_i + \mathcal{A}x_j$ and it follows that $\{\mathcal{A}[x_i^{-1}], \mathcal{A}[x_j^{-1}]\}$ cover $\mathcal{A}$ provided these are cluster localizations. Here we see that do not necessarily need to be working over $\mathbb{ZP}$. The essential fact is that $1 \in \mathcal{A}x_i + \mathcal{A}x_j$. This leads to the following definition. Call the seed $(B, \mathbf{x}, \mathbf{y})$ a *source freezing seed* with respect to a ground ring $\mathbb{A}$ if $y_i \oplus 1 \in \mathbb{A}$ for all $1 \leq i \leq n$.

Lemma 6.1.4. *Let $(B, \mathbf{x}, \mathbf{y})$ be a source freezing seed with respect to $\mathbb{A}$. If i is a source and $B_{ji} > 0$, then $\{\mathcal{A}[x_i^{-1}], \mathcal{A}[x_j^{-1}]\}$ cover $\mathcal{A}$ provided they are cluster localizations.*

Proof. In the case $(B, \mathbf{x}, \mathbf{y})$ a *source freezing seed* with respect to $\mathbb{A}$, i is a source, and $B_{ji} > 0$ we have that $1 \in \mathcal{A}x_i + \mathcal{A}x_j$ by (6.2). Given any prime $\mathcal{A}$-ideal P we must have either $x_i \notin P$ or $x_j \notin P$ since $P \neq \mathcal{A}$ is a proper ideal. If $x_i \notin P$, then $\mathcal{A}[x_i^{-1}]P \subsetneq \mathcal{A}[x_i^{-1}]$. If $x_j \notin P$, then $\mathcal{A}[x_j^{-1}]P \subsetneq \mathcal{A}[x_j^{-1}]$. Thus if $\mathcal{A}[x_i^{-1}]$ and $\mathcal{A}[x_j^{-1}]$ are cluster localization they form a cover. □

Lemma 6.1.5. *Let $(B, \mathbf{x}, \mathbf{y})$ be a source freezing seed, and let $(B^\dagger, \mathbf{x}^\dagger, \mathbf{y}^\dagger)$ the freezing at x_i for any $1 \leq i \leq n$. The seed $(B^\dagger, \mathbf{x}^\dagger, \mathbf{y}^\dagger)$ is a source freezing seed with respect to $\mathcal{A}^\dagger = \mathcal{A}[x_i^{\pm 1}]$.*

Proof. Since $(B, \mathbf{x}, \mathbf{y})$ is a source freezing seed, $1 \oplus y_j \in \mathbb{A}$ for each $1 \leq j \leq n$. We must

show that $1 \oplus y_j^\dagger \in \mathbb{A}^\dagger$ for each $1 \leq j \leq n$ where $y_j^\dagger = y_j x_i^{B_{ij}}$. We compute

$$\begin{aligned} 1 \oplus y_j^\dagger &= 1 \oplus y_j x_i^{B_{ij}} \\ &= (1 \oplus y_j) x_i^{\min(0, B_{ij})} \in \mathbb{A}^\dagger = \mathbb{A}[x_i^{\pm 1}], \end{aligned}$$

and the lemma is proven. □

Theorem 6.1.2. *If $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(B, \mathbf{x}, \mathbf{y})$ where $(B, \mathbf{x}, \mathbf{y})$ is an acyclic source freezing seed with respect to $\mathbb{A}$, then $\mathcal{A} = \mathcal{U}$.*

Proof. Assume $(B, \mathbf{x}, \mathbf{y})$ is an acyclic source freezing on rank n. We will induct on the rank. If $n \leq 1$, we are done by Proposition 6.1.3 since B must be isolated. Now we may assume the seed in non-isolated, otherwise we are done by Proposition 6.1.3. Since the seed in non-isolated and acyclic there must be a non-isolated source. We choose a non-isolated source i and then pick j with $B_{ji} > 0$. Freezing at x_i, the seed $(B^\dagger, \mathbf{x}^\dagger, \mathbf{y}^\dagger)$ is an acyclic source freezing seed with respect to $\mathbb{A}^\dagger$ by Lemma 6.1.5, and hence $\mathcal{A}^\dagger = \mathcal{U}^\dagger$ by induction since $\mathcal{A}^\dagger$ is of rank $n-1$. Similarly freezing at x_j, the seed $(B^{\dagger\dagger}, \mathbf{x}^{\dagger\dagger}, \mathbf{y}^{\dagger\dagger})$ is an acyclic source freezing seed with respect to $\mathbb{A}^{\dagger\dagger}$ by Lemma 6.1.5 and $\mathcal{A}^{\dagger\dagger} = \mathcal{U}^{\dagger\dagger}$ also. Now $\mathcal{A}^\dagger$ and $\mathcal{A}^{\dagger\dagger}$ both must be cluster localizations by Lemma 6.1.1. Thus Lemma 6.1.4 says that $\{\mathcal{A}^\dagger, \mathcal{A}^{\dagger\dagger}\}$ is a cover of $\mathcal{A}$. We conclude $\mathcal{A} = \mathcal{U}$ using Lemma 6.1.2. □

Theorem 6.1.2 can be used to produce example of geometric type with $\mathcal{A} = \mathcal{U}$ over the polynomial ground ring $\mathbb{ZP}_+$. Call a quiver a *source freezing quiver* if all arrows involving frozen vertices are directed from a mutable vertex to a frozen vertex. An acyclic source freezing quiver is pictured in Figure 6.1. Combining Theorem 6.1.2 with the sharpening of the Laurent phenomenon [FWZ, Theorem 3.3.6] for cluster algebras of geometric type we

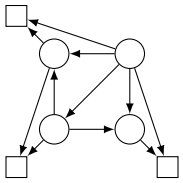

Figure 6.1: An acyclic source freezing quiver

obtain the following corollary since any source freezing quiver defines a source freezing seed with respect to $\mathbb{ZP}_+$.

Corollary 6.1.6. *If Q is an acyclic source freezing quiver defining a cluster algebra $\mathcal{A}$, then $\mathcal{A} = \mathcal{U}$ over the ground ring $\mathbb{ZP}_+$.*

6.2 The Cremmer-Gervais example

In this section we consider an exotic cluster structure constructed by Gekhtman, Shapiro, and Vainshtein known as the *Cremmer-Gervais* cluster structure on SL_n [GSV14, GSV17]. This cluster structure extends to a cluster structure on the set of $n \times n$ matrices denoted Mat_n. We let $\mathcal{A}_{\mathbb{A}}(CG, n)$ and $\mathcal{U}_{\mathbb{A}}(CG, n)$ denote the cluster algebra and upper cluster algebra over the ground ring $\mathbb{A}$ coming from the Cremmer-Gervais cluster structure on Mat_n. The quiver $Q(CG, n)$ defines this cluster algebra. Figure 6.2 shows the quiver $Q(CG, 3)$. It is know that for any n the upper cluster algebra $\mathcal{U}_{\mathbb{ZP}_+}(CG, n)$ is naturally isomorphic to the ring of regular functions on Mat_n [GSV17, Theorem 3.1]. The following result shows the sensitivity of the $\mathcal{A} = \mathcal{U}$ question on the ground ring

Proposition 6.2.1. *We have equality $\mathcal{A}_{\mathbb{ZP}}(CG, 3) = \mathcal{U}_{\mathbb{ZP}}(CG, 3)$ over $\mathbb{ZP}$ but strict containment $\mathcal{A}_{\mathbb{ZP}}(CG, 3) \subsetneq \mathcal{U}_{\mathbb{ZP}}(CG, 3)$ over $\mathbb{ZP}_+$.*

Proof. The equality $\mathcal{A}_{\mathbb{ZP}}(CG, 3) = \mathcal{U}_{\mathbb{ZP}}(CG, 3)$ follows from the fact that $\mathcal{A}_{\mathbb{ZP}}(CG, 3)$ is a locally acyclic cluster algebra over $\mathbb{ZP}$. This can be checked by applying the Banff algorithm.

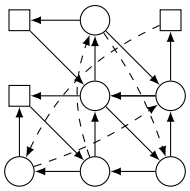

Figure 6.2: The quiver $Q(CG, 3)$.

A visual representation of the reduced Banff algorithm is applied to $Q(CG, 3)$ is shown in Figure 6.3. In Figure 6.3 mutation equivalence is denoted by $\Leftrightarrow$ and covering pairs used are displayed in thick red. The strict containment $\mathcal{A}_{\mathbb{ZP}}(CG, 3) \subsetneq \mathcal{U}_{\mathbb{ZP}}(CG, 3)$ is [GSV14, Theorem 4.1]. □

6.3 Relationship with reddening

A maximal green sequence or reddening sequence is a special sequence of mutations whose existence gives rise to additional properties of the underlying cluster algebra. These sequences of mutations were introduced by Keller to study quantum dilogarithm identities and Donaldson-Thomas invariants [Kel11]. It has been observed in the literature that the existence of a maximal green sequence or reddening sequence seems to coincide with equality of the cluster algebra and upper cluster algebra [CLS15]. We will exhibit a maximal green sequence for $Q(CG, 3)$ in this section.

The existence of such a sequence depends only on the mutable part of a quiver. Given a mutable quiver Q, the *framed quiver* $\hat{Q}$ is obtain by adding a new frozen vertex i' for each (mutable) vertex i along with an arrow $i \to i'$. A example of a framed quiver is given in Figure 6.4. A mutable vertex i is called *green* if all arrows involving i and a frozen vertex j' are directed $i \to j$. Otherwise the vertex i is called *red*. All mutable vertices of a framed quiver start as green. A *maximal green sequence* is a sequence of mutations starting with

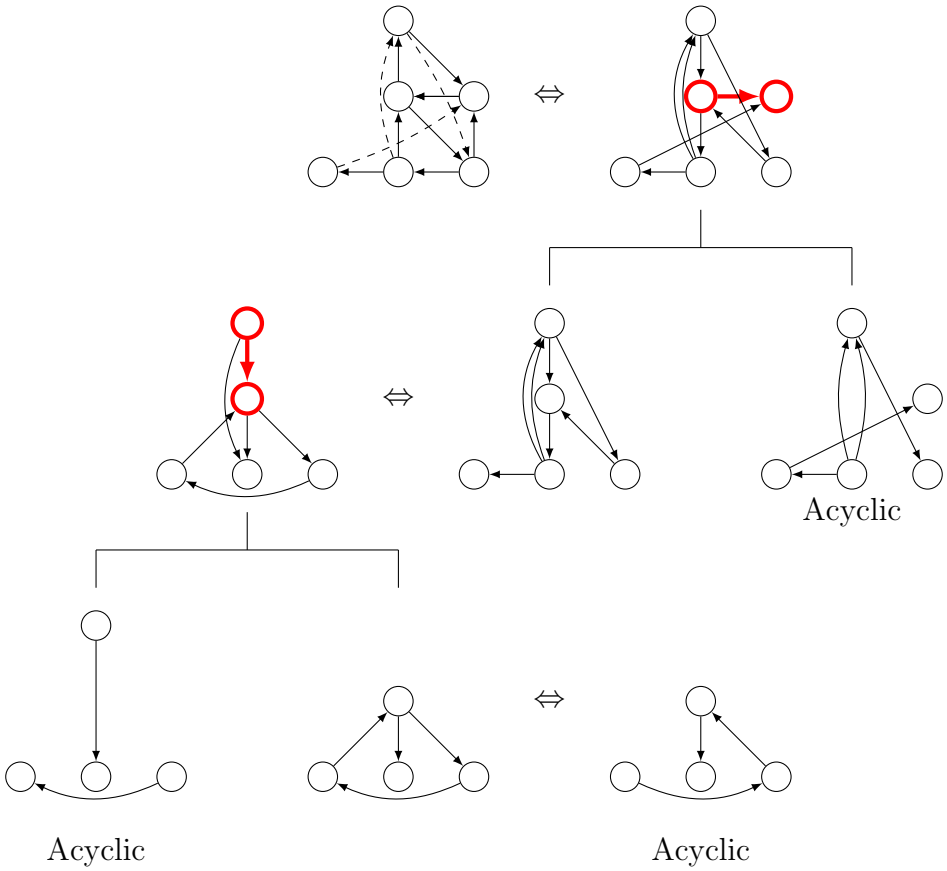

Figure 6.3: The reduced Banff algorithm applied to the quiver $Q(CG, 3)$.

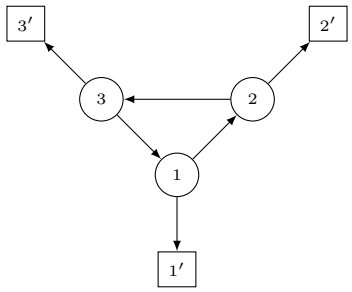

Figure 6.4: A framed quiver.

Figure 6.5: A maximal green sequence.

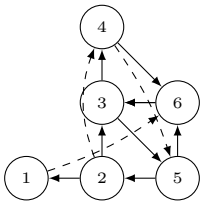

Figure 6.6: The mutable part of the quiver $Q(CG, 3)$.

$\hat{Q}$ such that each mutation occurs at a green vertex and all mutable vertices are red after applying the sequence of mutations.

The quiver with vertices $\{1, 2\}$ and arrow set $\{1 \to 2\}$ has exactly 2 maximal green sequences which are mutations at $(1, 2)$ and $(2, 1, 2)$. The maximal green sequence $(2, 1, 2)$ is illustrated in Figure 6.5.

The quiver $Q(CG, 3)$ provides an interesting case regarding the connection between maximal green sequences and the upper cluster algebra. The mutable part of the quiver $Q(CG, 3)$ with vertices labeled is shown in Figure 6.6. With this labeling of vertices it can be checked that $(2, 3, 4, 1, 5, 1, 2, 6, 3)$ is a maximal green sequence. Hence, we see the relationship of reddening sequences and equality of the cluster algebra and upper cluster algebra is again sensitive to the choice of ground ring since $\mathcal{A}_{\mathbb{ZP}_+}(CG, 3) \neq \mathcal{U}_{\mathbb{ZP}_+}(CG, 3)$ but $\mathcal{A}_{\mathbb{ZP}}(CG, 3) = \mathcal{U}_{\mathbb{ZP}}(CG, 3)$. The maximal green sequence exhibited for $Q(CG, 3)$ can be found using a new technique called *component preserving mutations*. The idea of component preserving mutations along with further examples will be given in [BMR$^+$].

Chapter 7

Log-Canonical Coordinates

This chapter is based on the article [MO17] which is joint work with Nicholas Ovenhouse. We will be interested in Poisson algebras of rational functions. Let $\mathbb{K}$ be a field and $\Omega = (\omega_{ij})$ a skew-symmetric matrix. Consider the algebra $\mathcal{R}_\Omega = \mathbb{K}(x_1, \ldots, x_n)$ of rational functions in n variables with a Poisson bracket in which the functions $x_1, \ldots, x_n$ form a system of log-canonical coordinates:

$$\{x_i, x_j\} = \omega_{ij} x_i x_j$$

Here we wish to show that the bracket $\{\cdot, \cdot\}$ has the simplest expression in the coordinates $x_1, \cdots, x_n$. In particular, we want to show that no rational change of coordinates can make the structure functions constant or linear (homogeneous or non-homogeneously linear). We wish to investigate the following conjecture of Michael Shapiro.

Conjecture 7.0.1. *If $f_1, \cdots, f_n \in \mathcal{R}_\Omega$ are rational functions such that*

$$\{f_i, f_j\} = \sum_{k=1}^{n} c_{ij}^k f_k + d_{ij}$$

with $c_{ij}^k, d_{ij} \in \mathbb{K}$ for $1 \leq i, j, k \leq n$, then $\{f_i, f_j\} = 0$ for $1 \leq i, j \leq n$.

We prove this conjecture in Theorem 7.2.4. Note that the conjecture implies that for any log-canonical Poisson structure on affine space, the answer to Question 4.1.5 is "no." That is, there is no system of coordinates whose structure functions are polynomials of degree less

than two.

7.1 Nonexistence of Constant Bracket

We have discovered that some of the results of this section have already appeared in [GL11]. However, we include this section for completeness. The results in this section will be built upon to prove our main theorem. Given some $n \times n$ skew-symmetric matrix $\Omega = (\omega_{ij})$, recall that $\mathcal{R}_\Omega = \mathbb{K}(x_1, \cdots, x_n)$ is the algebra of rational functions in n variables with the Poisson bracket given by

$$\{x_i, x_j\} = \omega_{ij} x_i x_j$$

for $1 \leq i, j \leq n$. For $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, the corresponding Laurent monomial is written $\mathbf{x}^I = x_1^{i_1} \cdots x_n^{i_n}$. For $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ in $\mathbb{Z}^n$, let A_J^I be the 2-by-n matrix whose rows are I and J. Let $\Delta_{ij}(A_J^I)$ be the 2-by-2 minor of A_J^I with columns indexed by i and j. Also define M_J^I to be the weighted sum of the $\Delta_{ij}(A_J^I)$ given by the following formula

$$M_J^I := \sum_{k<\ell} \omega_{k\ell} \Delta_{k\ell}(A_J^I) = \sum_{k<\ell} \omega_{k\ell} \begin{vmatrix} i_k & i_\ell \\ j_k & j_\ell \end{vmatrix} = \sum_{k<\ell} \omega_{k\ell}(i_k j_\ell - i_\ell j_k).$$

Note that if $e_1, \ldots, e_n$ is a basis for $\mathbb{Z}^n$, with $e^1, \ldots, e^n$ the dual basis, we can define the two-form

$$\omega = \sum_{k<\ell} \omega_{k\ell}\, e^k \wedge e^\ell,$$

and then $M_J^I = \omega(I, J)$. In particular, the expression M_J^I is $\mathbb{Z}$-bilinear and skew-symmetric with respect to I and J. We now compute an explicit formula for the bracket of two Laurent polynomials.

Lemma 7.1.1. *If $I, J \in \mathbb{Z}^n$, then*

$$\{\mathbf{x}^I, \mathbf{x}^J\} = M_J^I \, \mathbf{x}^{I+J}.$$

Proof. Let $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$. For $1 \leq k \leq n$ let

$$I_k = (i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n)$$

and

$$J_k = (j_1, \ldots, j_{k-1}, 0, j_{k+1}, \ldots, j_n).$$

By using Equation 4.1, we find

$$\begin{aligned}
\{\mathbf{x}^I, \mathbf{x}^J\} &= \sum_{1 \leq k,\ell \leq n} \frac{\partial \mathbf{x}^I}{\partial x_k} \frac{\partial \mathbf{x}^J}{\partial x_\ell} \{x_k, x_\ell\} \\
&= \sum_{1 \leq k,\ell \leq n} i_k \, j_\ell \, \mathbf{x}^{I_k + J_\ell} \{x_k, x_\ell\} \\
&= \sum_{1 \leq k,\ell \leq n} i_k \, j_\ell \, \mathbf{x}^{I_k + J_\ell} \omega_{k\ell} x_k \, x_\ell \\
&= \sum_{1 \leq k,\ell \leq n} \omega_{k\ell} i_k \, j_\ell \, \mathbf{x}^{I+J} \\
&= \sum_{1 \leq k < \ell \leq n} \omega_{k\ell} (i_k j_\ell - i_\ell j_k) \mathbf{x}^{I+J} \\
&= M_J^I \mathbf{x}^{I+J}
\end{aligned}$$

□

In order to prove our first theorem we want to work with iterated Laurent series. We will give a brief overview of the theory of iterated Laurent series which will be needed for our purpose. For a more in depth treatment of iterated Laurent series we refer the reader to [Xin04]. For us a *formal Laurent series* in variables $x_1, \ldots, x_n$ over $\mathbb{K}$ will mean any formal sum

$$f = \sum_{I \in \mathbb{Z}^n} \alpha_I \mathbf{x}^I$$

with $\alpha_I \in \mathbb{K}$ for all $I \in \mathbb{Z}^n$. For any $I \in \mathbb{Z}^n$ let $[\mathbf{x}^I]f$ denote the coefficient of $\mathbf{x}^I$ in f. In particular, $[\mathbf{1}]f$ denotes the constant term of f. Also, we let $\mathrm{supp}(f)$ denote the set $I \in \mathbb{Z}^n$ such that $[\mathbf{x}^I]f \neq 0$. The set of formal Laurent series is a $\mathbb{K}$-vector space, but it is not a $\mathbb{K}$-algebra as we cannot multiply any two formal Laurent series in general.

However, certain subsets of the set of formal Laurent series form a $\mathbb{K}$-algebra. Define $\mathbb{K}\langle\langle x \rangle\rangle := \mathbb{K}((x))$ to be the field of Laurent series in a single variable. That is, $\mathbb{K}((x))$ consists of formal Laurent series $\sum_{i \in \mathbb{Z}} a_i x^i$ containing only finitely many negative exponents. Now define

$$\mathbb{K}\langle\langle x_1, \cdots, x_{i+1} \rangle\rangle := \mathbb{K}\langle\langle x_1, \cdots, x_i \rangle\rangle((x_{i+1}))$$

iteratively. We then let $\mathcal{L} = \mathbb{K}\langle\langle x_1, \cdots, x_n \rangle\rangle$ be the field of *iterated Laurent series* in n variables. We have the following immediate corollary of Lemma 7.1.1 which holds for Laurent polynomials. In the remainder of this section we will show that this corollary can be extended to hold for any iterated Laurent series.

Corollary 7.1.2. *Let $f, g \in \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be Laurent polynomials, with $\mathcal{I} = \mathrm{supp}(f)$ and*

$\mathcal{J} = \operatorname{supp}(g)$. *Then their bracket is given by*

$$\{f, g\} = \sum_{(I,J)\in\mathcal{I}\times\mathcal{J}} \alpha_I \beta_J M_J^I \mathbf{x}^{I+J}.$$

Remark 7.1.3. Note that we have the inclusion $\mathbb{K}[x_1, \ldots, x_n] \hookrightarrow \mathcal{L}$. Since $\mathcal{L}$ is a field and $\mathcal{R}_\Omega$ is the field of fractions of $\mathbb{K}[x_1, \ldots, x_n]$, we also have the inclusion $\mathcal{R}_\Omega \hookrightarrow \mathcal{L}$. Hence, $\mathcal{R}_\Omega$ is a $\mathbb{K}$-subalgebra of $\mathcal{L}$.

Remark 7.1.4. Notice the order in which we adjoin our variables is relevant. For instance, consider the rational function $\frac{1}{x+y}$. As an element of $\mathbb{K}\langle\langle x, y\rangle\rangle$, it can be written as

$$\frac{1}{x+y} = \sum_{n\geq 0} (-1)^n x^{-(n+1)} y^n$$

However, since there is no lower bound on the powers of x, this does not give an element of $\mathbb{K}\langle\langle y, x\rangle\rangle$. Instead, to represent it as an element in the latter field, we must write

$$\frac{1}{x+y} = \sum_{n\geq 0} (-1)^n x^n y^{-(n+1)}.$$

Remark 7.1.5. Any iterated Laurent series $f \in \mathcal{L}$ can be expressed as a formal Laurent series. That is, we can write

$$f = \sum_{I\in\operatorname{supp}(f)} \alpha_I \mathbf{x}^I.$$

Given $f, g \in \mathcal{L}$ where

$$f = \sum_{I \in \text{supp}(f)} \alpha_I \mathbf{x}^I \qquad\qquad g = \sum_{J \in \text{supp}(g)} \beta_J \mathbf{x}^J$$

their product is

$$fg = \sum_{(I,J) \in \text{supp}(f) \times \text{supp}(g)} \alpha_I \beta_J \mathbf{x}^{I+J}.$$

This product fg is also an iterated Laurent series, since $\mathcal{L}$ is a field. In particular this means fg is a formal Laurent series with the property that for any $K \in \mathbb{Z}^n$, the set

$$\{(I, J) \in \text{supp}(f) \times \text{supp}(g) : I + J = K\}$$

is finite. In fact, we have the following result, which will be useful later[1]:

Proposition 7.1.6 ([Xin04, Proposition 2-2.1])**.** *Let f be a formal Laurent series. Then $f \in \mathcal{L}$ if and only if* $\text{supp}(f)$ *is well-ordered with respect to the reverse lexicographic ordering.*

Lemma 7.1.7. *The Poisson bracket on $\mathcal{R}_\Omega$ extends uniquely to a Poisson bracket on $\mathcal{L}$.*

Proof. Note that by bilinearity, any Poisson bracket on $\mathcal{L}$ is determined by the brackets of all Laurent monomials. Thus by Lemma 7.1.1, any bracket extending the one on $\mathcal{R}_\Omega$ must be given by the same formula on monomials. We claim that the same formula in Corollary 7.1.2 gives the bracket on $\mathcal{L}$. It suffices to show that for $f, g \in \mathcal{L}$ that $\{f, g\} \in \mathcal{L}$. That is we must show that given $f, g \in \mathcal{L}$, the formula from Corollary 7.1.2 yields an element of $\mathcal{L}$.

[1] We have chosen to use the iterated Laurent construction, and hence must show the well-ordered support property in Proposition 7.1.6. Alternatively, we could have started from the well-ordered support property and shown that we obtain a ring structure. A formal series with well-order support are sometimes called a *Hahn series* or a *Mal'cev-Neumann series* and exponents can be taken from any ordered abelian group.

Let $f, g \in \mathcal{L}$, and again use the notation $\mathcal{I} = \text{supp}(f)$ and $\mathcal{J} = \text{supp}(g)$. Note that since $fg \in \mathcal{L}$, then $\text{supp}(fg)$ is well-ordered by Proposition 7.1.6. The formula from Corollary 7.1.2 also implies that $\text{supp}(\{f, g\}) \subseteq \text{supp}(f) + \text{supp}(g)$, where "+" is used to denote Minkowski addition:

$$\text{supp}(f) + \text{supp}(g) = \{I + J \mid I \in \text{supp}(f),\ J \in \text{supp}(g)\}.$$

Being a subset of a well-ordered set, we see that $\text{supp}(\{f, g\})$ is itself well-ordered. Hence, $\{f, g\} \in \mathcal{L}$ by Proposition 7.1.6.

□

The remaining results in this section are restatements of the indicated results from [GL11]. The next theorem is a simple but powerful observation which is an essential ingredient to our proof of Conjecture 7.0.1.

Theorem 7.1.8 ([GL11, Proposition 5.2 (a)])**.** *If $f, g \in \mathcal{L}$, then $[\mathbf{1}]\{f, g\} = 0$.*

Proof. As usual, let $\mathcal{I}$ and $\mathcal{J}$ be the supports of f and g, and let

$$f = \sum_{I \in \mathcal{I}} \alpha_I \mathbf{x}^I \qquad\qquad g = \sum_{J \in \mathcal{J}} \beta_J \mathbf{x}^J$$

be expressions for f and g as formal Laurent series. Computing using Corollary 7.1.2 we see that

$$\{f, g\} = \sum_{(I,J) \in \mathcal{I} \times \mathcal{J}} \alpha_I \beta_J M_J^I \mathbf{x}^{I+J}$$

and so

$$[\mathbf{1}]\{f, g\} = \sum_{\substack{(I,J) \in \mathcal{I} \times \mathcal{I} \\ I+J=0}} \alpha_I \beta_J M_J^I.$$

However, if $I + J = 0$, then $J = -I$ and $M_J^I = 0$. Here we have used that M_J^I is skew-symmetric with respect to I and J. □

Corollary 7.1.9 ([GL11, Corollary 5.3])**.** *If $f_1, \cdots, f_n \in \mathcal{R}_\Omega$ are rational functions such that $\{f_i, f_j\} = c_{ij}$ with $c_{ij} \in \mathbb{K}$ for $1 \leq i, j \leq n$, then $c_{ij} = 0$ for $1 \leq i, j \leq n$.*

Proof. By Lemma 7.1.7, $\mathcal{R}_\Omega$ is a Poisson subalgebra of $\mathcal{L}$. The corollary then follows immediately from Theorem 7.1.8. □

7.2 Nonexistence of Linear Bracket

As in the previous section, we consider the Poisson algebra of rational functions $\mathcal{R}_\Omega$, in n variables, with bracket given by

$$\{x_i, x_j\} = \omega_{ij} x_i x_j$$

for some skew-symmetric matrix $\Omega = (\omega_{ij})$ with coefficients in $\mathbb{K}$. It is the goal of this section to prove the aforementioned Conjecture 7.0.1, which states that there is no rational change of coordinates making the bracket linear. That is, if there are rational functions $f_1, \ldots, f_n$ such that $\{f_i, f_j\} = \sum_{k=1}^n c_{ij}^k f_k + d_{ij}$ for constants $c_{ij}^k, d_{ij} \in \mathbb{K}$, then in fact all the coefficients c_{ij}^k and d_{ij} must be zero.

We now prove a lemma which will be used later.

Lemma 7.2.1. *There do not exist linearly independent $f, g \in \mathcal{R}_\Omega$ such that $\{f, g\} = af + bg$ for $a, b \in \mathbb{K}$ with a and b not both zero.*

Proof. Assume there do exist linearly independent rational functions f and g so that $\{f, g\} = af + bg$ for some $a, b \in \mathbb{K}$. Then the linear span of f and g is a two-dimensional Lie subalgebra of $\mathcal{R}_\Omega$. Up to isomorphism, there is a unique two-dimensional non-abelian Lie algebra, with

the bracket given by $\{f, g\} = f$. Explicitly, the isomorphism is given by $f \mapsto f + \frac{b}{a}g$, $g \mapsto \frac{1}{a}g$ (assuming $a \neq 0$). So, we may assume without loss of generality that $a = 1$ and $b = 0$, thus $\{f, g\} = f$. But then we have that $\frac{1}{f}\{f, g\} = 1$. Note that since $\mathrm{ad}_f = \{f, \cdot\}$ is a derivation, we have $\frac{1}{f}\{f, g\} = \left\{f, \frac{g}{f}\right\}$. This in turn implies that $\{f, \frac{g}{f}\} = 1$. But this directly contradicts Corollary 7.1.9, which says that the bracket of any two rational functions cannot be a nonzero constant. □

A useful consequence of this lemma is that the adjoint maps ad_f can have no non-zero eigenvalues.

Corollary 7.2.2. *If $f, g \in \mathcal{R}_\Omega$ with $g \neq 0$ and $\{f, g\} = \lambda g$ for some $\lambda \in \mathbb{K}$, then $\lambda = 0$.*

The next result says that the adjoint maps ad_f cannot be nonzero and nilpotent.

Lemma 7.2.3. *If $f, g \in \mathcal{R}_\Omega$ and $\{f, g\} \neq 0$, then $\{f, \{f, g\}\} \neq 0$.*

Proof. Take $f, g \in \mathcal{R}_\Omega$ and assume that $\{f, g\} \neq 0$ but $\{f, \{f, g\}\} = 0$. Then we know that $\left\{f, \frac{1}{\{f,g\}}\right\} = 0$. Computing, we see that

$$\left\{f, \frac{g}{\{f, g\}}\right\} = g\left\{f, \frac{1}{\{f, g\}}\right\} + \frac{1}{\{f, g\}}\{f, g\} = 1$$

which is a contradiction to Corollary 7.1.9. □

We are now ready to prove the main result.

Theorem 7.2.4. *If $f_1, \cdots, f_n \in \mathcal{R}_\Omega$ are rational functions such that*

$$\{f_i, f_j\} = \sum_{k=1}^{n} c_{ij}^k f_k + d_{ij}$$

with $c_{ij}^k, d_{ij} \in \mathbb{K}$ for $1 \leq i, j, k \leq n$, then $\{f_i, f_j\} = 0$ for $1 \leq i, j \leq n$.

Proof. Assume first that $\mathbb{K} = \overline{\mathbb{K}}$ is algebraically closed. Let $f_1, \ldots, f_n \in \mathcal{R}_\Omega$ be rational functions such that $\{f_i, f_j\} = \sum_{k=1}^n c_{ij}^k f_k + d_{ij}$ for some $c_{ij}^k, d_{ij} \in \mathbb{K}$. This means that $1, f_1, \ldots, f_n$ generate a finite dimensional Lie algebra inside $\mathcal{R}_\Omega$. Let $F \leq \mathcal{R}_\Omega$ denote this finite dimensional Lie algebra generated by $1, f_1, \ldots, f_n$. For any $f \in F$ we have the linear map $\mathrm{ad}_f : F \to F$, and by Corollary 7.2.2 all eigenvalues of ad_f are zero. It follows, since $\mathbb{K}$ is algebraically closed, that ad_f is nilpotent. However, Lemma 7.2.3 implies that if ad_f is nilpotent we must have $\mathrm{ad}_f = 0$. The theorem then follows.

In the case that $\mathbb{K}$ is *not* algebraically closed, consider $\overline{\mathcal{R}_\Omega} := \overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{R}_\Omega = \overline{\mathbb{K}}(x_1, \ldots, x_n)$. The relations $\{f_i, f_j\} = \sum_{k=1}^n c_{ij}^k f_k + d_{ij}$ still hold. Thus $1, f_1, \ldots, f_n$ will generate some finite dimensional Lie algebra inside $\overline{\mathcal{R}_\Omega}$, and we can complete the argument just as in the algebraically closed case. □

Remark 7.2.5. Given a Poisson algebra P, the *quadratic Poisson Gel'fand-Kirillov problem* is to determine if the field of fractions of P is isomorphic to $\mathcal{R}_\Omega$ for some Ω. This problem was first defined in [GL11] and further studied in [LL16]. In this section, we have shown a number of properties of the Poisson algebra $\mathcal{R}_\Omega$. Hence, any Poisson algebra isomorphic to $\mathcal{R}_\Omega$ must also have these properties, and the results in this section can be viewed as necessary conditions for a Poisson algebra to be a solution to the quadratic Poisson Gelfand-Kirillov problem.

7.3 Generalizations

The results of the previous section are not specific to only the Poisson algebra $\mathcal{R}_\Omega$. Let P be a Poisson algebra P with the following two properties:

- P is a field.

- For any $a, b \in P$ we have $\{a, b\} = 0$ whenever $\{a, b\} \in \mathbb{K}$.

Call such an algebra a *nonconstant Poisson field* (since there are no elements with $\{f, g\} = 1$). Then versions of the results in the previous section hold for P since the proofs only use the conditions above. In particular we will have a version of Theorem 7.2.4 which says that P can have no finite dimensional non-abelian Lie subalgebra. Before proving this theorem, let us collect some of the essential parts of the proofs from the previous section into a useful general lemma:

Lemma 7.3.1. *Let P be a Poisson $\mathbb{K}$-algebra which is a field. Then the following are*

(a) *There exist $f, g \in P$ such that $\{f, g\} = 1$.*

equivalent:

(b) *There exist $f, g \in P$ such that $\{f, g\} = g$.*

(c) *There exist $f, g \in P$ with $\{f, g\} \neq 0$ but $\{f, \{f, g\}\} = 0$.*

Proof. $(b) \Rightarrow (a)$: Follows from proof which is identical to the proof of Lemma 7.2.1.

$(c) \Rightarrow (a)$: Follows from proof which is identical to the proof of Lemma 7.2.3

$(a) \Rightarrow (c)$: If $\{f, g\} = 1$, then $\{f, \{f, g\}\} = \{f, 1\} = 0$.

$(a) \Rightarrow (b)$: Suppose that $\{f, g\} = 1$, and define $x = fg$ and $y = g$. Then

$$\{x, y\} = \{fg, g\} = \{f, g\}g = g = y.$$

□

Analogous to the above definition, define a *nonlinear Poisson field* as a Poisson field P which has no finite-dimensional nonabelian Lie subalgebras. This means there are no finite

collections $f_1, \ldots, f_k \in P$ and constants c_{ij}^{ℓ} such that $\{f_i, f_j\} = \sum_{\ell} c_{ij}^{\ell} f_{\ell}$. The next result says that for a Poisson field, being nonconstant is a sufficient condition to being nonlinear.

Theorem 7.3.2. *Any nonconstant Poisson field is also nonlinear.*

Proof. We assume that we are working over an algebraically closed field, if not we can modify just as in the proof of Theorem 7.2.4. Suppose that there exist some $f_1, \ldots, f_k \in P$ for some $k > 1$ and constants c_{ij}^{ℓ} so that

$$\{f_i, f_j\} = \sum_{\ell} c_{ij}^{\ell} f_{\ell}$$

Then $f_1, \ldots, f_k$ generate a finite-dimensional Lie subalgebra $F \leq P$. Each map ad_{f_i} is an endomorphism of F. Note that ad_{f_i} cannot have any nonzero eigenvalues. If it did, there would be some $g \in F$ and $\lambda \neq 0$ so that $\{f_i, g\} = \lambda g$. Then for $\tilde{f}_i = \frac{1}{\lambda} f_i$, we have $\{\tilde{f}_i, g\} = g$. By the previous theorem, there must also exist some $u, v \in P$ so that $\{u, v\} = 1$. But this contradicts the assumption on P. So in fact ad_{f_i} can have only zero eigenvalues, and hence must be nilpotent. Again, by the previous theorem, if ad_{f_i} is nonzero and nilpotent, then there would exist $u, v \in P$ with $\{u, v\} = 1$. So it must be that $\mathrm{ad}_{f_i} = 0$, and thus F is an abelian Lie algebra. □

In the spirit of Question 4.1.5, let us call a system of coordinates *(homogeneous) algebraically reduced* if all structure functions are (homogeneous) polynomials of a given degree, and there does not exist any rational change of coordinates making the structure functions (homogeneous) polynomials of a smaller degree. In Theorem 7.2.4 we provided an answer to Question 4.1.5 for any log-canonical system of coordinates and showed that they are algebraically reduced. It is natural to look for other (homogeneous) algebraically reduced coordinate systems.

Let us now consider systems of coordinates for which all structure functions are monomials. In dimension 2 with coordinates (x, y) so that $\{x, y\} = x^a y^b$ we have seen that such a monomial system of coordinates is algebraically reduced if and only if $(a, b) = (0, 0)$ or $(a, b) = (1, 1)$. In dimension 3 with coordinates (x, y, z) and bracket relations

$$\{x, y\} = Ax^{a_1} y^{a_2} z^{a_3}$$
$$\{x, z\} = Bx^{b_1} y^{b_2} z^{b_3}$$
$$\{y, z\} = Cx^{c_1} y^{c_2} z^{c_3}$$

we can extend by skew-symmetry, but must also ensure the Jacobi identity holds. Computing we obtain

$$\begin{aligned}\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = (b_1 - a_1)ABx^{a_1+b_1-1} y^{a_2+b_2} z^{a_3+b_3} \\ + (c_2 - a_2)ACx^{a_1+c_1} y^{a_2+c_2-1} z^{a_3+c_3} \\ + (c_3 - b_3)BCx^{b_1+c_1} y^{b_2+c_2} z^{b_3+c_3-1}.\end{aligned}$$

If $a_1 = b_1, a_2 = c_2$, and $b_3 = c_3$ the Jacobi identity will hold. In that case the bracket relations are

$$\{x, y\} = A(x^{a_1} y^{a_2} z^{b_3}) z^{a_3-b_3}$$
$$\{x, z\} = B(x^{a_1} y^{a_2} z^{b_3}) y^{b_2-a_2}$$
$$\{y, z\} = C(x^{a_1} y^{a_2} z^{b_3}) x^{c_1-a_1}$$

Consider the simplest example of the case above, where $(a_1, a_2, b_3) = (0, 0, 0)$. One such

example is the bracket on $\mathbb{K}(x, y, z)$ given by

$$\{x, y\} = az^2 \qquad \{x, z\} = by^2 \qquad \{y, z\} = cx^2$$

for some $a, b, c \in \mathbb{K}^*$. This bracket gives a candidate for another homogeneous quadratic algebraically reduced system of coordinates which differs from the log-canonical case. By the above discussion, it would suffice to show that this bracket makes $\mathbb{K}(x, y, z)$ a nonconstant Poisson field. However, unlike the log-canonical case, this bracket can produce non-zero constant terms, as exhibited by the following examples:

$$\left\{x, \frac{y}{z^2}\right\} = a - 2b\left(\frac{y}{z}\right)^3$$

$$\left\{\frac{x}{z}, \frac{y}{z}\right\} = a - b\left(\frac{y}{z}\right)^3 + c\left(\frac{x}{z}\right)^3$$

As such, the arguments used previously do not apply, since everything followed from Theorem 7.1.8, which said that the constant term of $\{f, g\}$ (viewed as a Laurent series) is always zero. However, it is possible that this bracket makes $\mathbb{K}(x, y, z)$ a nonconstant Poisson field, despite the fact that Theorem 7.1.8 does not hold. It seems to be an interesting problem to find other algebraically reduced brackets on $\mathbb{K}(x_1, \ldots, x_n)$.

Chapter 8

Combinatorial Hopf Algebras

This chapter considers *combinatorial Hopf algebras* as defined by Aguiar, Bergeron, and Sottile [ABS06]. A particular Hopf algebra of simplicial complexes will be studied in Chapter 9. Hopf algebras in combinatorics predate the 2006 Aguiar, Bergeron, and Sottile definition. In 1979 Joni and Rota [JR79] showed that Hopf algebra structures can be a valuable tool in studying the assembly and disassembly of many combinatorial objects. A key addition to the study of Hopf algebras in combinatorics made by Aguiar, Bergeron, and Sottile was a map called a *character*. The character allows one to associate a (quasi)symmetric function to each combinatorial object underlying the Hopf algebra. We will study (a generalization of) the symmetric function the arises from the Hopf algebra in Chapter 9.

8.1 Hopf algebras

We now define (combinatorial) Hopf algebras. Let $\mathcal{H}$ be a vector space over a field $\mathbb{K}$. We will assume that $char(\mathbb{K}) = 0$. Let Id be the identity map on $\mathcal{H}$. We call $\mathcal{H}$ an *associative* $\mathbb{K}$-*algebra with unit* 1 when $\mathcal{H}$ is equipped with a $\mathbb{K}$-linear map $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and an

element $1 \in \mathcal{H}$ satisfying

$$m \circ (m \otimes \mathrm{Id}) = m \circ (\mathrm{Id} \otimes m);$$

$$m \circ (\mathrm{Id} \otimes u) = m \circ (u \otimes \mathrm{Id}) = \mathrm{Id}.$$

Here, u stands for the $\mathbb{K}$-linear map $\mathbb{K} \to \mathcal{H}$ defined by $t \mapsto t \cdot 1$.

A *coassociative* $\mathbb{K}$*-coalgebra with counit* ϵ is a $\mathbb{K}$-vector space $\mathcal{D}$ over $\mathbb{K}$ equipped with a coproduct $\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ and a counit $\epsilon : \mathcal{D} \to \mathbb{K}$. Both Δ and ϵ must be $\mathbb{K}$-linear maps. The coproduct is coassociative so that $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$ and must be compatible with ϵ so that

$$(\epsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id}.$$

If an algebra $(\mathcal{H}, m, u)$ is also equipped with a coalgebra structure given by Δ and ϵ, then we say that $\mathcal{H}$ is a *bialgebra* provided Δ and ϵ are algebra homomorphisms.

The maps m and Δ can be applied iteratively as follows. Letting $\mathcal{H}^{\otimes k} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ denote the k-fold tensor, define the iterated product map $m^{(k-1)} : \mathcal{H}^{\otimes k} \to \mathcal{H}$ inductively by setting $m^{(-1)} = u$, $m^{(0)} = \mathrm{Id}$ and for $k \geq 1$ let $m^{(k)} = m \circ (\mathrm{Id} \otimes m^{(k-1)})$. Similarly, the iterated coproduct map $\Delta^{(k-1)} : \mathcal{H} \to \mathcal{H}^{\otimes k}$ is given inductively by $\Delta^{(k)} = (\mathrm{Id} \otimes \Delta^{(k-1)}) \circ \Delta$, where $\Delta^{(-1)} = \epsilon$ and $\Delta^{(0)} = \mathrm{Id}$.

Definition 8.1.1. A *Hopf algebra* $\mathcal{H}$ is a $\mathbb{K}$-bialgebra together with a $\mathbb{K}$-linear map $S : \mathcal{H} \to \mathcal{H}$ called the antipode. This map must satisfy the following

$$m \circ (S \otimes \mathrm{Id}) \circ \Delta = m \circ (\mathrm{Id} \otimes S) \circ \Delta = u \circ \epsilon.$$

Example 8.1.2 (Group Hopf algebra)**.** If G is a group, then the group algebra $\mathbb{K}G$ is Hopf

algebra with

$$\Delta(g) = g \otimes g$$

$$\epsilon(g) = 1$$

$$S(g) = g^{-1}$$

for all $g \in G$.

Example 8.1.3 (Polynomial Hopf algebra)**.** The polynomial ring $\mathbb{K}[x]$ is a Hopf algebra with

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

$$\epsilon(x) = 0$$

$$S(x) = -x.$$

8.2 Combinatorial Hopf algebras

We say that a bialgebra $\mathcal{H}$ is *graded* if it is decomposed into a direct sum

$$\mathcal{H} = \bigoplus_{n \geq 0} H_n$$

where $m(H_i \otimes H_j) \subseteq H_{i+j}$, $u(\mathbb{K}) \subseteq H_0$, $\Delta(H_n) \subseteq \bigoplus_{i=0}^{n} H_i \otimes H_{n-i}$, and $\epsilon(H_n) = 0$ for $n \geq 1$. We call $\mathcal{H}$ *connected* if $H_0 \cong \mathbb{K}$. For each $n \geq 0$ we refer to elements in H_n as *homogeneous elements of degree* n.

Any graded and connected $\mathbb{K}$-bialgebra is a Hopf algebra since the antipode can be

defined recursively. In many instances computing the antipode of a given Hopf algebra is a very difficult problem. However, we will provide an explicit cancellation-free formula for the antipode in the Hopf algebra of finite simplicial complexes in Chapter 9. We will compute the antipode using Takeuchi's formula for the antipode in a graded connected Hopf algebra [Tak71]. This formula states

$$S = \sum_{k \geq 0} (-1)^k m^{k-1} \pi^{\otimes k} \Delta^{k-1} \tag{8.1}$$

where $\pi : \mathcal{H} \to \mathcal{H}$ is the projection given by linearly extending

$$\pi|_{H_n} = \begin{cases} 0 & n = 0 \\ \\ \mathrm{Id} & n > 0. \end{cases}.$$

A *combinatorial Hopf algebra* is a pair $(\mathcal{H}, \zeta)$ where $\mathcal{H}$ is a graded connected Hopf algebra and $\zeta : \mathcal{H} \to \mathbb{K}$ is a algebra morphism called a *character*. Given combinatorial Hopf algebras $(\mathcal{H}, \zeta)$ and $(\mathcal{H}', \zeta')$ a map $\phi : \mathcal{H} \to \mathcal{H}'$ is a combinatorial Hopf algebra morphism if ϕ is a Hopf algebra morphism and $\zeta = \zeta' \circ \phi$. In the category of combinatorial Hopf algebras the Hopf algebra of quasisymmetric functions with a canonical character is a terminal object [ABS06, Theorem 4.1]. The Hopf algebra of symmetric functions with a canonical character is a terminal object in the catergory of cocommutative combinatorial Hopf algebas [ABS06, Theorem 4.3]. This gives one explanation for the ubiquity of symmetric functions and quasisymmetric functions.

The Hopf algebra of quasisymmetric functions $\mathcal{QS}ym$ is graded as

$$\mathcal{QS}ym = \bigoplus_{n \geq 0} \mathcal{QS}ym_n$$

where $\mathcal{QS}ym_n$ is spanned linearly over $\mathbb{K}$ by $\{M_\alpha\}_{\alpha \vDash n}$. Here M_α is the formal power series is countable many variables $x_1, x_2, \ldots$ defined by

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$$

where $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a composition of n. The basis given by $\{M_\alpha\}$ is known as the *monomial basis of* $\mathcal{QS}ym$. We have $M_{()} = 1$, which spans $\mathcal{QS}ym_0$, where $()$ is the composition of 0 with no parts. The Hopf algebra of symmetric functions $Sym = \bigoplus_{n \geq 0} Sym_n$ is the Hopf subalgebra of $\mathcal{QS}ym$ where Sym_n is spanned by the monomial symmetric functions

$$m_\lambda = \sum_\alpha M_\alpha$$

where λ is a integer partition and the sum ranges are all integer compositions which are rearrangements of λ. The algebra structure on $\mathcal{QS}ym$ and Sym is given by the usual multiplication of formal power series. Further discussion the Hopf algebra structure is omitted because will not be needed for our purposes. We will only be concerned with the canonical morphism which exists from any combinatorial Hopf algebra to $\mathcal{QS}ym$.

Example 8.2.1. As an example we consider to two monomial quasisymmetric functions

$$M_{(1,2)} = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \cdots$$
$$M_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots$$

and the monomial symmetric function

$$m_{(2,1)} = x_1x_2^2 + x_1^2x_2 + x_1x_3^2 + x_1^2x_3 + x_2x_3^2 + x_2^2x_3 + \cdots$$

which satisfy $m_{(2,1)} = M_{(1,2)} + M_{(2,1)}$.

Let the map $\zeta_{\mathcal{Q}} : \mathcal{QS}ym \to \mathbb{K}$ be defined as $\zeta_{\mathcal{Q}}(f) = f(1, 0, 0, \dots)$ for a quasisymmetric function $f(x_1, x_2, x_3, \dots)$. Given that $\zeta_{\mathcal{Q}}$ is an evaluation map, it is also an algebra map and hence a character of $\mathcal{QS}ym$. This endows $\mathcal{QS}ym$ with a combinatorial Hopf algebra structure. Moreover, Theorem 4.1 of [ABS06] states that given a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ there is a unique combinatorial Hopf algebra homomorphism

$$\Psi_\zeta : \mathcal{H} \to \mathcal{QS}ym$$

given by

$$\Psi_\zeta(h) = \sum_{\alpha=(\alpha_1,\dots,\alpha_\ell)\vDash n} \zeta_\alpha(h) M_\alpha \tag{8.2}$$

for h homogeneous of degree n, where ζ_α is the composition of functions

$$\mathcal{H} \xrightarrow{\Delta^{(\ell-1)}} \mathcal{H}^{\otimes\ell} \longrightarrow H_{\alpha_1} \otimes H_{\alpha_2} \otimes \cdots \otimes H_{\alpha_\ell} \xrightarrow{\zeta^{\otimes\ell}} \mathbb{K}.$$

Here the unlabeled map is the canonical projection and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. It is straight forward to see that if $\mathcal{H}$ is cocommutative, then Ψ_ζ is a map $\Psi_\zeta : \mathcal{H} \to Sym$.

Chapter 9

Simplicial Complexes

In this chapter we define a cocommutative Hopf algebra structure on simplicial complexes. By this we mean that we define a Hopf algebra whose basis is indexed by isomorphism classes of simplicial complexes. We then see are certain choices of character the morphism induced to *Sym* gives rise the symmetric functions related to coloring. Results in this chapter are also present in the article [BHM16] with is joint work with Carolina Benedetti and Joshua Hallam.

9.1 The Hopf algebra structure

A finite *(abstract) simplicial complex*, Γ, is a nonempty collection of subsets of some finite set V such that $\{v\} \in \Gamma$ for all $v \in V$, and $X \in \Gamma$ implies $Y \in \Gamma$ for all $Y \subseteq X$. By convention all our simplicial complexes contain the empty set. We denote by $\varnothing$ the simplicial complex with empty vertex set. So $\varnothing$ is the unique simplicial complex whose vertex set is empty and whose only face is the empty set. The elements of Γ are called *faces* and the maximal (with respect to inclusion) faces are called *facets*. Notice that the facets completely determine the simplicial complex. If X is a face of Γ then the *dimension* of X is $\dim X = |X| - 1$. A face of dimension s is called an *s-simplex*. The faces of dimension 0 are called *vertices* of Γ and the set of vertices will be denoted $V(\Gamma)$ where we identify $\{v\}$ with v. For instance, if Γ has facets $\{1, 2, 3\}$ and $\{3, 4\}$ then $V(\Gamma) = \{1, 2, 3, 4\}$. The dimension of Γ, written as $\dim \Gamma$, is

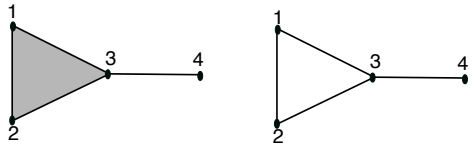

Figure 9.1: A simplicial complex Γ and its 1-skeleton $\Gamma^{(1)}$.

the maximum of the dimensions of its facets.

If Γ and Θ are simplicial complexes with vertex sets V_1 and V_2, the *disjoint union* of Γ and Θ is the simplicial complex $\Gamma \uplus \Theta$ with vertex set $V_1 \uplus V_2$ and faces X such that $X \in \Gamma$ or $X \in \Theta$. If k is a nonnegative integer, the *k-skeleton* of Γ is the collection of faces of Γ with dimension no greater than k. We will denote the k-skeleton of Γ by $\Gamma^{(k)}$. For example, if Γ has facets $\{1, 2, 3\}$ and $\{3, 4\}$, then $\Gamma^{(1)}$ is the simplicial complex with facets $\{1, 2\}, \{1, 3\}, \{2, 3\}$ and $\{3, 4\}$. Figure 9.1 provides a pictorial representation of this example. Notice that a simple graph gives rise to a simplicial complex of dimension 1 or less. Conversely, a simplicial complex of dimension 1 or less can be thought of as a simple graph.

Let Γ and $V(\Gamma)$ be defined as above. Given $T \subseteq V(\Gamma)$, define the *induced simplicial complex* of Γ on T, denoted by Γ_T, to be the simplicial complex with faces $\{X \cap T \mid X \in \Gamma\}$. So if we return to our example with Γ having facets $\{1, 2, 3\}, \{3, 4\}$ and if $T = \{1, 3, 4\}$, then Γ_T has facets $\{1, 3\}$ and $\{3, 4\}$.

Now we define a Hopf algebra structure on simplicial complexes. Let $\mathcal{A} = \bigoplus_{n \geq 0} A_n$ where A_n is the free $\mathbb{K}$-vector space on the set of isomorphism classes of simplicial complexes on n vertices. Given a simplicial complex, Γ, we will denote its isomorphism class by $[\Gamma]$.

Define the product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$m([\Gamma] \otimes [\Theta]) = [\Gamma \uplus \Theta].$$

Notice that with this multiplication, the unit $u : \mathbb{K} \to \mathcal{A}$ is given by

$$u(1) = [\varnothing].$$

The coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, is given by

$$\Delta([\Gamma]) = \sum_{T \subseteq V(\Gamma)} [\Gamma_T] \otimes [\Gamma_{V(\Gamma)\setminus T}].$$

Additionally, define the counit of $\mathcal{A}$ by

$$\epsilon([\Gamma]) = \delta_{[\Gamma],[\varnothing]}$$

where $\delta_{[\Gamma],[\varnothing]}$ is the Kronecker delta.

It follows that $\mathcal{A}$ is a graded, connected $\mathbb{K}$-bialgebra and hence a Hopf algebra. Also, it is not hard to see that $\mathcal{A}$ is commutative and cocommutative. From now on, we will drop the brackets from the notation $[\Gamma]$, keeping in mind that we are considering isomorphism classes of simplicial complexes.

9.2 A cancellation-free formula for the antipode

Before stating the main result in this section, we review some basic concepts from graph theory. Suppose $G = (V, E)$ is a graph. A subset U of V is called *stable* if there is no edge between any pair of vertices in U. A *flat*, F, of G is a collection of edges such that in the graph with vertex set V and edge set F, each connected component is an induced subgraph of G. If F is a flat then we will denote the subgraph of G with vertex set V and edge set

F by $G_{V,F}$ and its number of connected components by $c(F)$. The set of flats of a graph G will be denoted by $\mathcal{F}(G)$. We denote by G/F the graph obtained from G by contracting the edges in F. Recall that an orientation of a graph is called *acyclic* if it does not contain any directed cycles. The number of acyclic orientations of a graph G will be denoted by $a(G)$. Given an orientation O of G, a vertex $v \in V$ is called a *source of* O if for every edge $\{v, u\} \in E$, $\{v, u\}$ is oriented away from v.

Let Γ be a simplicial complex. Any face X of Γ gives rise to a simplicial complex, namely, the simplicial complex formed by all the subsets of X. Given a flat F in $\Gamma^{(1)}$ define $\Gamma_{V,F}$ to be the subcomplex of Γ, with vertex set $V = V(\Gamma)$, whose faces are given by

$$\{X \in \Gamma \ : \ X^{(1)} \subseteq (\Gamma^{(1)})_{V,F}\}.$$

For example, if we again take Γ to have facets $\{1, 2, 3\}, \{3, 4\}$ and let $F = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ then $\Gamma_{V,F}$ is the simplicial complex with facets $\{1, 2, 3\}, \{4\}$.

Theorem 9.2.1. *Let $\Gamma \in A_n$ be a simplicial complex where $n \geq 1$. Then*

$$S(\Gamma) = \sum_{F \in \mathcal{F}(\Gamma^{(1)})} (-1)^{c(F)} a(\Gamma^{(1)}/F) \Gamma_{V,F}$$

where the sum runs over all flats of the 1-skeleton of Γ.

Proof. The strategy to show this result is to define, for every acyclic orientation of each of the graphs $\Gamma^{(1)}/F$, a sign-reversing involution with a unique fixed point. We will only illustrate the proof when F is the empty flat. Namely, we will show in this case that the coefficient of $\Gamma_{V,F}$ equals $a(\Gamma^{(1)})$. At the end of the proof, we will explain how to extend this proof when $F \neq \emptyset$.

Denote by $\mathcal{O}(\Gamma^{(1)})$ the set of acyclic orientations of the graph $\Gamma^{(1)}$ and identify the vertex set V of Γ with $[n] := \{1, 2, \ldots, n\}$. Using Takeuchi's formula for the antipode in a Hopf algebra (see [Tak71]), given $\Gamma \in A_n$ we obtain

$$S(\Gamma) = \sum_{(V_1,\ldots,V_\ell)\models[n]} (-1)^\ell \, \Gamma_{V_1} \uplus \cdots \uplus \Gamma_{V_\ell} \tag{9.1}$$

summing over all ordered set partitions $(V_1, \ldots, V_\ell)$ of $[n]$ where all of the V_i are nonempty. Notice that Takeuchi's formula does not provide a cancellation-free expression for the antipode in general.

A term in (9.1) can be thought as the union of simplicial subcomplexes of Γ such that $\Gamma^{(1)}_{V_i}$ is an induced subgraph of $\Gamma^{(1)}$. In particular, notice that when $(V_1, \ldots, V_\ell)$ is such that $\dim(\Gamma_{V_i}) = 0$ for each i, then $\Gamma_{V_1} \uplus \cdots \uplus \Gamma_{V_\ell}$ is (isomorphic to) the zero-dimensional subcomplex of Γ on the set $[n]$ denoted by $\Gamma_{[n],\emptyset}$. Note that different ordered set partitions $(V_1, \ldots, V_\ell) \models [n]$ may contribute to the coefficient of $\Gamma_{[n],\emptyset}$ in (9.1).

Let $A_\emptyset = \{(V_1, \ldots, V_\ell) \models [n] \mid \Gamma_{V_1} \uplus \cdots \uplus \Gamma_{V_\ell} = \Gamma_{[n],\emptyset}\}$ and define the function

$$\rho : A_\emptyset \to \mathcal{O}(\Gamma^{(1)})$$

that assigns to $(V_1, \ldots, V_\ell) \in A_\emptyset$ an orientation in $\mathcal{O}(\Gamma^{(1)})$ to each edge $\{i, j\}$ in $\Gamma^{(1)}$ as follows:

$$i \to j \quad \text{if} \quad i \in V_r, j \in V_s \quad \text{and} \quad r < s.$$

Now, given $\sigma = (V_1, \ldots, V_\ell)$ define the *sign* of σ to be $sign(\sigma) = (-1)^\ell$. Let $O \in \mathcal{O}(\Gamma^{(1)})$. We will think of O not just as an acyclic orientation but also as the directed graph it induces on the vertex set $[n]$. Such O gives rise to a canonical ordered set partition of $[n]$ in the

following manner. Let v_1 be the biggest source of $O_1 = O$, then let v_2 be the biggest source of $O_2 = O_1 - \{v_1\}$, and in general let v_{k+1} be the biggest source of $O_{k+1} = O_k - \{v_k\}$ for $k = 0, \ldots, n-1$. Then we obtain the ordered set partition $\pi_O = (\{v_1\}, \ldots, \{v_n\}) \models [n]$ such that v_i is the largest source in O_i. Since $\rho(\pi_O) = O$, ρ is a surjection.

For fixed $O \in \mathcal{O}(\Gamma^{(1)})$ define a sign reversing involution ι_O on the set $\rho^{-1}(O)$ in the following way. Set $\iota_O(\pi_O) = \pi_O$. For $\sigma = (V_1, \ldots, V_\ell) \in \rho^{-1}(O)$ such that $\sigma \neq \pi_O$ let i be the smallest index such that $V_i \neq \{v_i\}$, where $(\{v_1\}, \ldots, \{v_n\}) = \pi_O$ as above. The choice of i implies that $v_i \in V_i \cup \cdots \cup V_\ell$. Let V_j be the block in σ containing v_i. If $|V_j| > 1$ define

$$\iota_O(\sigma) = (V_1, \ldots, V_{j-1}, V_j - \{v_i\}, \{v_i\}, V_{j+1}, \ldots, V_\ell)$$

Otherwise, if $|V_j| = 1$ define

$$\iota_O(\sigma) = (V_1, \ldots, V_{j-2}, V_{j-1} \cup V_j, V_{j+1}, \ldots, V_\ell).$$

In the latter case, since v_i is the largest source in O_i, the vertices in V_{j-1} are vertices in O_i as well and hence, $V_{j-1} \cup V_j$ is a stable set of vertices. Notice that in both cases, $sign(\iota_O(\sigma)) = -sign(\sigma)$. Moreover, $\iota_O(\iota_O(\sigma)) = \sigma$ and π_O is the unique fixed point of ι_O. We conclude that for each acyclic orientation O, the involution ι_O has a unique fixed point. Hence the coefficient of $\Gamma_{[n],\emptyset}$ in (9.1) is $(-1)^n a(\Gamma^{(1)})$.

The proof for the coefficient of $\Gamma_{V,F}$ when $F \neq \emptyset$ can be done using the same argument as above with slight modifications. Namely, each connected component of $\Gamma_{V,F}$ can be identified with a single vertex and a similar sign reversing involution can be defined for the graph $\Gamma^{(1)}/F$ whose vertex set has cardinality $c(F)$. $\square$

Remark 9.2.2. Obtaining a cancellation-free a formula is, in general, a difficult problem when studying Hopf algebras arising in combinatorics. Theorem 9.2.1 fits into a family of results on antipode formulas for graphs and various generalizations of graphs. In [HM12, Theorem 3.1] Humpert and Martin, the authors provide a cancellation-free formula for the antipode of graphs using induction. On the other hand, in [BS17, Theorem 7.1] Benedetti and Sagan make use of sign-reversing involutions on combinatorial objects to obtain cancellation-free formulas for antipodes of several Hopf algebras including the graph Hopf algebra. Our proof makes use of a sign-reversing involution for graphs generalized to simplicial complexes.

Aguiar and Ardila [AA17] have also recovered the antipode formula for both graphs and simplicial complexes using a Hopf monoids of generalized permutahedra [AA17]. In the context the similarity of the formula and proofs is explained by the fact that the generalized permutahedra associated to a simplicial complex and its 1-skeleton have the same normal fan. Benedetti and Bergeron [BB16] have given a simplified, but not cancellation-free, formula for the antipode of hypergraphs. The fact the formula is not cancellation-free has since been explained by Benedetti, Bergeron, and Machacek in [BBM17] where it is shown that the coefficients in the antipode formula of a hypergraph are Euler characteristics of certain polyhedral complexes.

Let us return to our previous example with Γ generated by the facets $\{1, 2, 3\}$ and $\{3, 4\}$. Using the information in Table 9.1 we obtain the expression in Figure 9.2. Looking at the expression for the antipode in this example, we see that if we add all the coefficients together we obtain 1. It turns out that the sum of the coefficients of the antipode of a simplicial complex is always $(-1)^n$ where n is the number of vertices of the simplicial complex. We will derive this fact using characters and quasisymmetric functions in the next section (see Corollary 10.2.2).

$F \in \mathcal{F}(\Gamma^{(1)})$	$(-1)^{c(F)}$	$a(\Gamma^{(1)}/F)$
$\emptyset$	$(-1)^4$	12
$\{1,2\}$	$(-1)^3$	4
$\{1,3\}$	$(-1)^3$	4
$\{2,3\}$	$(-1)^3$	4
$\{3,4\}$	$(-1)^3$	6
$\{1,2\},\{3,4\}$	$(-1)^2$	2
$\{1,3\},\{3,4\}$	$(-1)^2$	2
$\{2,3\},\{3,4\}$	$(-1)^2$	2
$\{1,2\},\{1,3\},\{2,3\}$	$(-1)^2$	2
$\{1,2\},\{1,3\},\{2,3\},\{3,4\}$	$(-1)^1$	1

Table 9.1: Information to compute the antipode of Γ.

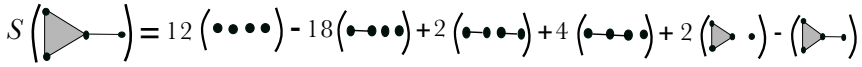

Figure 9.2: Antipode of an element in A_4.

Now, note that once we have computed $S(\Gamma)$, we can easily find the antipode of the simplicial complex $\Gamma^{(1)}$ by just taking the 1-skeleton of each of the terms in the sum for the antipode. So we immediately get that

$$S(\Gamma^{(1)}) = 12\overline{K_4} - 18(K_2 \uplus \overline{K_2}) + 2(K_2 \uplus K_2) + 4(P_3 \uplus K_1) + 2(K_3 \uplus K_1) - \Gamma^{(1)}.$$

where K_n is the complete graph on n vertices, $\overline{K_n}$ is the complement of the complete graph on n vertices, and P_n is the path on n vertices.

More generally, let $\mathcal{A}^{(k)}$ be the $\mathbb{K}$-linear span of isomorphism classes of simplicial complexes of dimension at most k. That is, complexes $\Gamma \in \mathcal{A}$ such that $\Gamma^{(k)} = \Gamma$. For each

$k \geq 0$, we define the map

$$\phi_k : \mathcal{A} \to \mathcal{A}^{(k)}$$
$$\Gamma \mapsto \Gamma^{(k)}$$

which takes the k-skeleton of a simplicial complex. We extend this map linearly to all of $\mathcal{A}$.

Proposition 9.2.3. *For any nonnegative integer k, $\mathcal{A}^{(k)}$ is a Hopf subalgebra of $\mathcal{A}$ and the map $\phi_k : \mathcal{A} \to \mathcal{A}^{(k)}$ is a Hopf algebra homomorphism.*

Proof. Let Γ and Θ be simplicial complexes. Since $\dim \Gamma \uplus \Theta = \max\{\dim \Gamma, \dim \Theta\}$ and $\dim \Gamma_T \leq \dim \Gamma$ for any $T \subseteq V(\Gamma)$ it follows that $\mathcal{A}^{(k)}$ is a Hopf subalgebra. Observe that

$$(\Gamma \uplus \Theta)^{(k)} = \{X : X \in \Gamma \uplus \Theta, |X| \leq k+1\} = \{X \in \Gamma : |X| \leq k+1\} \cup \{X \in \Theta : |X| \leq k+1\}.$$

Therefore $(\Gamma \uplus \Theta)^{(k)} = \Gamma^{(k)} \uplus \Theta^{(k)}$ and ϕ_k is an algebra homomorphism. Next, since

$$(\Gamma_T)^{(k)} = \{X \in \Gamma : X \subseteq T, |X| \leq k+1\} = (\Gamma^{(k)})_T$$

we have

$$\sum_{T \subseteq V(\Gamma)} (\Gamma_T)^{(k)} \otimes (\Gamma_{V(\Gamma)-T})^{(k)} = \sum_{T \subseteq V(\Gamma)} (\Gamma^{(k)})_T \otimes (\Gamma^{(k)})_{V(\Gamma)-T}$$

and so ϕ_k is also a coalgebra homomorphism. We conclude that ϕ_k is a Hopf algebra homomorphism. □

Using Proposition 9.2.3 along with the fact that for any Hopf algebra homomorphism $\beta : \mathcal{H}_1 \to \mathcal{H}_2$ one has $\beta(S_{\mathcal{H}_1}(h)) = S_{\mathcal{H}_2}(\beta(h))$ for all $h \in \mathcal{H}_1$ (see [GR, Proposition 1.46])

we can conclude that $S \circ \phi_k = \phi_k \circ S$. This means that once we know $S(\Gamma)$ for some simplicial complex Γ we can find $S(\Gamma^{(k)})$ for any k.

Chapter 10

The chromatic symmetric function

Now that we have endowed $\mathcal{A}$ with a Hopf algebra structure, we will proceed to define a family of characters on $\mathcal{A}$. This will give rise to a family of combinatorial Hopf algebras. The morphism to the Hopf algebra of symmetric functions will map each simplicial complex to a generalized of Stanley's chromatic symmetric function [Sta95].

For each $s > 0$, define the map $\zeta_s : \mathcal{A} \to \mathbb{K}$ by

$$\zeta_s(\Gamma) = \begin{cases} 1 & \dim \Gamma < s, \\ 0 & \dim \Gamma \geq s, \end{cases}$$

and extend linearly to $\mathcal{A}$. Each map ζ_s is multiplicative, i.e. $\zeta_s(\Gamma \uplus \Theta) = \zeta_s(\Gamma)\zeta_s(\Theta)$. Thus, for each s the pair $(\mathcal{A}, \zeta_s)$ is a combinatorial Hopf algebra. Moreover, since $\mathcal{A}$ is cocommutative, equation (8.2) implies that Ψ_ζ is actually a symmetric function.

10.1 Coloring in simplicial complexes

Let $\mathbb{P}$ denote the set of positive integers and let G be a graph with vertex set V. A *coloring* of G is a map $f : V \to \mathcal{P}$. We refer to $f(u)$ as the color of u. A *proper coloring* of V is a coloring such that $f(u) \neq f(v)$ whenever uv is an edge of G. Given a simplicial complex Γ

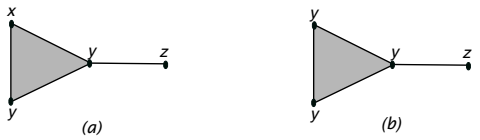

Figure 10.1: A 2-coloring of Γ in *(a)* and a 3-coloring of Γ in (b).

and $s \in \mathbb{N}$, define an *s-simplicial coloring*[1] to be a coloring of $V(\Gamma)$ such that there is no monochromatic face of dimension s. Notice that any 1-simplicial coloring of Γ is simply a proper coloring of its 1-skeleton $\Gamma^{(1)}$. In Figure 10.1 we use our earlier example and depict two colorings of Γ using the colors $\{x, y, z\} \subseteq \mathcal{P}$.

Given a graph G, the number of proper colorings $f : V(G) \to \{1, 2, \ldots, t\}$ is the well-known chromatic polynomial, $\chi(G; t)$. For a simplicial complex Γ the number of s-simplicial colorings $f : V(\Gamma) \to \{1, 2, \ldots, t\}$ is called the *s-chromatic polynomial*, $\chi_s(\Gamma; t)$, and defined in [Nor12, MN16]. Although it is not obvious that $\chi_s(\Gamma; t)$ is a polynomial, we will see that this is the case once we realize it as an evaluation of a certain symmetric function.

Stanley provided a generalization (see [Sta95]) of the chromatic polynomial of a graph G by defining

$$\psi(G; x_1, x_2, \ldots) = \sum_f \prod_{i \geq 1} x_i^{|f^{-1}(i)|}$$

where the sum is over proper colorings $f : V \to \mathbb{P}$. This formal power series is known as *Stanley's chromatic symmetric function*. For a simplicial complex Γ we define the *s-chromatic symmetric function* as

$$\psi_s(\Gamma; x_1, x_2, \ldots) = \sum_f \prod_{i \geq 1} x_i^{|f^{-1}(i)|}$$

where now the sum is over s-simplicial colorings $f : V \to \mathbb{P}$ and $V = V(\Gamma)$. Notice that when

[1] In [DMN] the authors use the term (P, s)-coloring for an *s-simplicial coloring* which uses some palette of colors $P \subseteq \mathcal{P}$. To avoid confusion with terminology in graphs, we have adopted the term *s-simplicial coloring.*

$s = 1$ we obtain Stanley's chromatic symmetric function. Given $f(x_1, x_2, \dots) \in \mathcal{QS}ym$ its *principal specialization at* t is defined by $ps^1(f)(t) = f(1, \dots, 1, 0, 0, \dots)$ where $t \in \mathbb{P}$ and only the first t variables are specialized to 1. It turns out that $ps^1(f)(t)$ gives rise to a *unique* polynomial in t (see [GR, Proposition 7.7]). The s-chromatic polynomial $\chi_s(\Gamma; t)$ is the polynomial determined $ps^1(\psi_s(\Gamma))(t)$. We now show how the s-chromatic symmetric function arises from the CHA $(\mathcal{A}, \zeta_s)$.

Theorem 10.1.1. *Fix s and consider the combinatorial Hopf algebra $(\mathcal{A}, \zeta_s)$. If Γ is a simplicial complex, then $\Psi_{\zeta_s}(\Gamma) = \psi_s(\Gamma; x_1, x_2, \dots)$.*

Proof. Consider the formula in equation (8.2). Given a simplicial complex $\Gamma \in A_n$ and a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \vDash n$, we get that the coefficient of M_α is the number of ordered set partitions $V_1 \uplus V_2 \uplus \cdots \uplus V_\ell$ of $V(\Gamma)$ such that $|V_i| = \alpha_i$ and $\dim \Gamma_{V_i} < s$ for each i. In an s-simplicial coloring, every element of a subset T of $V(\Gamma)$ can be assigned the same color if and only if $\dim \Gamma_T < s$. Thus the coefficient of M_α counts s-simplicial colorings using only colors $\{j_1 < j_2 < \cdots < j_\ell\} \subseteq \mathcal{P}$ where $|f^{-1}(j_i)| = \alpha_i$ for each i. The result follows. □

We discuss now the expansion of the symmetric function $\Psi_{\zeta_s}(\Gamma; x_1, x_2, \dots)$ in terms of the *power sum basis*. The *power sum symmetric function* of degree n, denoted by p_n, in the variables $x_1, x_2, \dots$ is given by $p_n = \sum_{i \geq 1} x_i^n$ and for $\lambda = (\lambda_1, \dots, \lambda_\ell)$ an integer partition define

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

Take a simplicial complex Γ and let $V = V(\Gamma)$. For any $s > 0$, we denote the collection of s-simplices of Γ by $F_s(\Gamma)$. Given any $A \subseteq F_s(\Gamma)$ we let $\Gamma_{V,A}$ be the simplicial complex on the vertex set V generated by A. That is, the faces of $\Gamma_{V,A}$ of dimension greater than

0 are subsets $X \subseteq V$ such that $X \subseteq Y$ for some $Y \in A$. For $A \subseteq F_s(\Gamma)$ we define an integer partition $\lambda(A)$ which has length the number of connected components of $(\Gamma_{V,A})^{(1)}$ and whose parts are given by the number of vertices in each connected component. The s-chromatic symmetric function has the following expansion in the power sum basis

$$\Psi_{\zeta_s}(\Gamma) = \sum_{A \subseteq F_s(\Gamma)} (-1)^{|A|} p_{\lambda(A)} \tag{10.1}$$

which can be proven analogously to [Sta95, Theorem 2.5].

Let Γ denote the $(n-1)$-simplex, i.e., the simplicial complex on $[n]$ whose only facet is the set $[n]$ itself. We now look at the monomial and Schur expansions of $\Psi_{\zeta_s}(\Gamma)$. We have

$$\Psi_{\zeta_s}(\Gamma) = \sum_{\substack{\mu \vdash n \\ \mu_1 \leq s}} \binom{n}{\mu_1, \cdots, \mu_\ell} m_\mu$$

where, for every $s \geq 1$, the sum is over partitions $\mu = (\mu_1, \cdots, \mu_\ell)$ of n such that $\mu_1 \leq s$.

In the above case when $s = n$ we get

$$\Psi_{\zeta_n}(\Gamma) = \sum_{\lambda \vdash n} f^\lambda s_\lambda$$

where f^λ is the number of standard Young tableaux of shape λ. This follows since

$$\sum_{\mu \vdash n} \binom{n}{\mu_1, \cdots, \mu_\ell} m_\mu = (m_{(1)})^n = (s_{(1)})^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

Moreover, since $m_{(n)} = p_n = \sum_{i=0}^{n}(-1)^i s_{(n-i,1^i)}$ we conclude that the symmetric function

$$\Psi_{\zeta_{n-1}}(\Gamma) = \sum_{\lambda \vdash n} f^\lambda s_\lambda - m_{(n)}$$

is Schur positive as well. Unfortunately, the functions $\Psi_{\zeta_s}(\Gamma)$ are not always Schur positive. An instance of this is when $n = 4$ and $s = 2$. In this case,

$$\begin{aligned}\Psi_{\zeta_s}(\Gamma) &= 6m_{(2,2)} + 12m_{(2,1,1)} + 24m_{(1,1,1,1)} \\ &= 6s_{(2,2)} + 6s_{(2,1,1)} - 6s_{(1,1,1,1)}.\end{aligned}$$

It would be interesting to determine other families of simplicial complexes that give Schur positivity of the functions Ψ_{ζ_s} for different values of s.

10.2 Acyclic orientations and evaluations

In this section, we use our antipode formula along with the characters defined above to interpret certain evaluations of the s-chromatic polynomial. Given any character $\zeta : \mathcal{A} \to \mathbb{K}$, the following identity holds (see [ABS06, Section 1])

$$\zeta^{-1} = \zeta \circ S$$

where S is the antipode in $\mathcal{A}$ and ζ^{-1} is the inverse of ζ under convolution. In other words, $\zeta^{-1}\zeta = u \circ \epsilon$ where $\zeta^{-1}\zeta = m \circ (\zeta^{-1} \otimes \zeta) \circ \Delta$.

Now, since $ps^1(\psi_s(\Gamma))(t) = \chi_s(\Gamma; t)$, using [GR, Proposition 7.7 (iii)] yields

$$\zeta_s \circ S(\Gamma) = \zeta_s^{-1}(\Gamma) = ps^1(\psi_s(\Gamma))(-1) = \chi_s(\Gamma; -1). \tag{10.2}$$

This allows us to prove the following theorem.

Theorem 10.2.1. *Let $\Gamma \in A_n$ be a simplicial complex and let s be a positive integer. Then*

$$\chi_s(\Gamma; -1) = \sum_{\substack{F \in \mathcal{F}(\Gamma^{(1)}) \\ \dim \Gamma_{V,F} < s}} (-1)^{c(F)} a(\Gamma^{(1)}/F).$$

Proof. Using equation (10.2), the fact that $\zeta_s^{-1} = \zeta_s \circ S$, and our antipode formula in Theorem 9.2.1 yields

$$\begin{aligned}
\chi_s(\Gamma; -1) &= \zeta_s(S(\Gamma)) \\
&= \sum_{F \in \mathcal{F}(\Gamma^{(1)})} (-1)^{c(F)} a(\Gamma^{(1)}/F) \zeta_s(\Gamma_{V,F}) \\
&= \sum_{\substack{F \in \mathcal{F}(\Gamma^{(1)}) \\ \dim \Gamma_{V,F} < s}} (-1)^{c(F)} a(\Gamma^{(1)}/F)
\end{aligned}$$

and so the result is proven. □

This result shows that like the chromatic polynomial for graphs, the evaluation at $t = -1$ of the s-chromatic polynomial for simplicial complexes has a combinatorial interpretation in terms of counting acyclic orientations. If we $s = 1$ in Theorem 10.2.1, we recover Stanley's classical result [Sta73] that $\chi_1(\Gamma; -1) = (-1)^n a(\Gamma^{(1)})$. In [HM12, Example 3.3] the authors

preform a similar calculation with characters for the Hopf algebra of graphs. In addition to this result, we also get the following corollary.

Corollary 10.2.2. *Let Γ be a simplicial complex on n vertices, then we have the following*

$$(-1)^n = \sum_{F \in \mathcal{F}(\Gamma^{(1)})} (-1)^{c(F)} a(\Gamma^{(1)}/F). \tag{10.3}$$

Proof. If we take $s > \dim \Gamma$, then $\chi_s(\Gamma; t) = t^n$ since there is no restriction on coloring. So, $\chi_s(\Gamma; -1) = (-1)^n$. Meanwhile, the sum in Theorem 10.2.1 runs over all $F \in \mathcal{F}(\Gamma^{(1)})$ because the condition $\dim \Gamma_{V,F} < s$ is always satisfied. □

10.3 The f-vector

Given a simplicial complex Γ, the *f-vector* of Γ is defined to be $(f_0, f_1, \dots)$ where f_s is the number of s-simplices in Γ. For example, if Γ is the simplicial complex generated by the facets $\{1, 2, 3\}$ and $\{3, 4\}$, then Γ has f-vector $(4, 4, 1, 0, 0, \dots)$. In this section we show how to obtain the f-vector of a simplicial complex from the symmetric functions $\{\Psi_{\zeta_s}\}_{s>0}$.

Let $[p_\lambda]\Psi_{\zeta_s}(\Gamma)$ denote the coefficient of p_λ in the power sum expansion of $\Psi_{\zeta_s}(\Gamma)$. If Γ is a simplicial complex on n vertices and $A \subseteq F_s(\Gamma)$, then $\lambda(A) = (s+1, 1^{n-s-1})$ if only if A consists of a single s-simplex. By considering equation (10.1) we obtain the following proposition.

Proposition 10.3.1. *If Γ is a simplicial complex with $|V(\Gamma)| = n$ and $s > 0$, then*

$$f_s = -[p_{(s+1,1^{n-s-1})}]\Psi_{\zeta_s}(\Gamma).$$

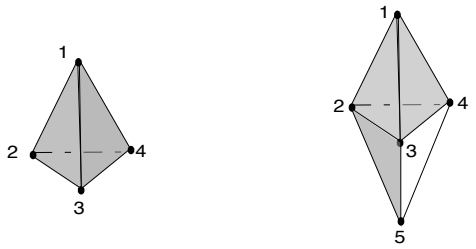

Figure 10.2: Z (left) has 2-simplicies $123, 124, 134, 234$ and Y (right) has 2-simplices $123, 124, 134, 235$.

Given a simplicial complex Γ, denote its s^{th} homology group by $H_s(\Gamma)$ for each $s \geq 0$. The s^{th} Betti number is denoted $\beta_s(\Gamma)$ and defined to be the rank of $H_s(\Gamma)$. One useful fact about homology groups is that if $\dim \Gamma = k$, then $H_s(\Gamma) = 0$ for $s > k$. In particular, this means $\beta_s(\Gamma) = 0$ for $s > k$. Hence, Proposition 10.3.1 allows us to recover the Euler characteristic χ_Γ of Γ, since $\chi_\Gamma = \sum_{s\geq 0}(-1)^s f_s = \sum_{s\geq 0}(-1)^s \beta_s$ where $\beta_s = \beta_s(\Gamma)$.

Since we can determine the f-vector from the s-chromatic symmetric functions, it is natural to wonder if we can also determine the Betti numbers. If Γ is a graph, i.e. if $\dim(\Gamma) = 1$, then β_0 equals the number of its connected components. This number can also be recovered by means of the chromatic polynomial of Γ. Thus, in this case we recover the sequence of Betti numbers $(\beta_0, \beta_1, 0, 0, ...)$. However, for higher dimensional simplicial complexes this is not always the case as we see in the next example.

Example 10.3.2. We now consider two simplicial complexes Γ and Θ such that $\Psi_{\zeta_s}(\Gamma) = \Psi_{\zeta_s}(\Theta)$ for all $s > 0$, but Γ and Θ have different Betti numbers. We set $\Gamma = X \uplus Y$ and $\Theta = Z \uplus W$ where X, Y, Z, and W are given in Table 10.1. The simplicial complexes Y and Z are shown Figure 10.2.

	Vertices	Facets
X	$1, 2, 3, 4$	$12, 13, 14, 23, 24, 34$
Y	$1, 2, 3, 4, 5$	$123, 124, 134, 235, 45$
Z	$1, 2, 3, 4$	$123, 124, 134, 234$
W	$1, 2, 3, 4, 5$	$12, 13, 14, 23, 24, 25, 34, 35, 45$

Table 10.1: Vertices and facets of the complexes X, Y, Z, and W.

Since $\Gamma^{(1)} = \Theta^{(1)}$, it follows $\Psi_{\zeta_1}(\Gamma) = \Psi_{\zeta_1}(\Theta)$. Also,

$$\Psi_{\zeta_2}(\Gamma) = (p_{(1^4)})(p_{(1^5)} - 4p_{(3,1^2)} + 3p_{(4,1)}) = p_{(1^9)} - 4p_{(3,1^6)} + 3p_{(4,1^5)}$$

$$\Psi_{\zeta_2}(\Theta) = (p_{(1^4)} - 4p_{(3,1)} + 3p_{(4)})(p_{(1^5)}) = p_{(1^9)} - 4p_{(3,1^6)} + 3p_{(4,1^5)}$$

and Γ and Θ have the same 2-chromatic symmetric function.

Since Γ and Θ are both 2-dimensional simplicial complexes, we conclude $\Psi_{\zeta_s}(\Gamma) = \Psi_{\zeta_s}(\Theta)$ for all $s > 0$. However, the Betti numbers of Γ and Θ are not the same since $\beta_2(\Theta) = 1$ while $\beta_2(\Gamma) = 0$.

10.4 Hypertrees

In this final section of this chapter we turn our attention to the chromatic symmetric function of hypertres. Results in this section can also be found in the article [Mac17]. A *hypergraph* H is a pair $H = (V, E)$ where E a collection of nonempty subsets of V. We call the elements of V vertices and elements of of E hyperedges. If for each $e \in E$ we have that $|e| = s$, then we call H an s-uniform hypergraph. A map $f : V \to \mathcal{P}$ is a proper coloring of H if it produces no monochromatic hyperedge.

We observe that coloring in simplicial complexes can be thought of as coloring in uniform

hypergraphs and conversely. Given a simplicial complex Γ and a nonnegative integer s we get an $(s+1)$-uniform hypergraph $H^{(s)}(\Gamma) = (V(\Gamma), E^{(s)}(\Gamma))$ with edge set defined by

$$E^{(s)}(\Gamma) := \{A \in \Gamma : |A| = s+1\}.$$

Lemma 10.4.1. *A map f is an s-simplicial coloring of a simplicial complex Γ if and only if f is a proper coloring of $H^{(s)}(\Gamma)$.*

Next we show that coloring in a uniform hypergraph can be thought of as an instance of coloring in a simplicial complex. Given any $H = (V, E)$ we get a simplicial complex $\Gamma(H)$ on the vertex set V defined by

$$\Gamma(H) = \{A : A \subseteq e \in E\} \cup \{\{v\} : v \in V\}.$$

Lemma 10.4.2. *A map f is a proper coloring of an $(s+1)$-uniform hypergraph H if and only if f is an s-simplicial coloring of $\Gamma(H)$.*

It is an open problem, first considered in [Sta95], to determine if the chromatic symmetric function distinguishes trees up to isomorphism. For some partial results on this problem see [MMW08, APZ14]. Russel has verified that the chromatic symmetric function distinguishes trees on 25 or fewer vertices up to isomorphism [Rus12]. We will investigate the analogous question for uniform hypertrees.

Throughout the remainder of this section section let $H = (V, E)$ be a hypergraph on n vertices. Let E be a set of subsets of V and assume that $|e| > 1$ for all $e \in E$. A *walk* of

length $\ell > 0$ between $v_1 \in V$ and $v_\ell \in V$ is a sequence

$$(v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell, v_{\ell+1})$$

such that $e_i \in E$ with $v_i, v_{i+1} \in e_i$ for all i. If all the vertices and hyperedges are distinct, then the walk is called a *path*. In the case all vertices and hyperedges are distinct with the exception that $v_1 = v_{\ell+1}$ we call the walk a *cycle*. The hypergraph H is *connected* if for any $v, v' \in V$ there exists a path between v and v'. A *hypertree* is a connected hypergraph with no cycles. We call H a *linear hypergraph* if $|e_1 \cap e_2| \leq 1$ for all $e_1, e_1 \in E$ such that $e_1 \neq e_2$. Notice a hypertree is necessarily linear, otherwise for distinct hyperedges $e_1, e_2 \in E$ and distinct vertices $v_1, v_2 \in e_1 \cap e_2$ there is a cycle a length 2

$$(v_1, e_1, v_2, e_2, v_1).$$

For H let $(a_i)_{i=2}^n$ be the sequence defined by

$$a_i := |\{e \in E : |e| = i\}|$$

which records the number of hyperedges of each size in the hypergraph. The *hyperedge magnitude* of H is defined to be the sum

$$\sum_{i=2}^{n} (i-1)a_i.$$

In [GK05] is it shown that a connected hypergraph on n vertices is a hypertree if and only if the hyperedge magnitude is $n-1$. We give the following lemma which extends this result

and is a generalization of the corresponding well known fact for trees.

Lemma 10.4.3. *Let $H = (V, E)$ be a hypergraph on n vertices, and consider the following conditions:*

(i) H is connected.

(ii) H is acyclic.

(iii) H has hyperedge magnitude equal $n - 1$.

Any two of the above conditions together imply the third. Hence, to show that a hypergraph H is a hypertree is suffices to prove that any two of the above conditions hold for H.

Proof. From [GK05] we already know that (i) and (ii) together imply (iii), and also that (i) and (iii) together imply (ii). It remains to show that (ii) and (iii) together imply (i). Assume that $H = (V, E)$ is a acylic hypergraph on n vertices with hyperedge magintude equal $n - 1$. We order the hyperedges $E = \{e_1, e_2, \ldots, e_m\}$ and let $H_i = (V, E_i)$ for where $E_i = \{e_1, e_2, \ldots, e_i\}$ for $1 \leq i \leq m$. Also let $H_0 = (V, \emptyset)$ Notice H_i will be an acyclic hypergraph for $1 \leq i \leq m$. Since each hypergraph is acyclic it follows that if H_i has c connected components, then H_{i+1} has $c - |e_{i+1}| + 1$ connected components. Now H_0 has n connected components and so it follows that $H = H_m$ has c connected components where

$$c = n - \sum_{i=1}^{m} (|e_i| - 1) = n - (n - 1) = 1.$$

Here we have used the assumption that H has edge magintude $n - 1$. Therefore we have shown H is connected and completed the proof. □

If G is a graph on n vertices, then G is a tree if and only if $\chi_G(t) = t(t-1)^{n-1}$. There is a similar result for s-uniform hypertrees when we restrict to linear hypergraphs. It is proven

in [BŁ07, Theorem 5] that if H is a linear hypergraph on n vertices, then H is an s-uniform hypertree with m hyperedges if and only if $\chi_H(t) = t(t^{s-1}-1)^m$. Here we observe some similar behavior between trees and uniform hypertrees when we restrict to linear hypergraphs. In what follows we will show some of the results on the chromatic symmetric which can be proven from trees can also be proven for uniform hypertrees. However, we will also exhibit two 3-uniform hypertrees which are not isomorphic yet have the same chromatic symmetric function.

Given any set partition $\pi = B_1/B_2/\cdots/B_\ell$ of $[n]$ we let $\operatorname{type}\pi$ be the integer partition of n given by the sizes of the blocks in π. Given $A \subseteq E$ we let $\lambda(A) = \operatorname{type}\pi(A)$. The chromatic symmetric function X_H of a hypergraph H has the expansion

$$X_H := \sum_{A \subseteq E} (-1)^{|A|} p_{\lambda(A)}$$

which we will take as a definition. Of course the chromatic symmetric function can also be defined in the monomial basis as a sum of proper coloring, but we will only need the chromatic symmetric of a hypertree in the powersum basis. Let $c_{\lambda(H)}$ denote the coefficient of p_λ is the powersum expansion of X_H so that

$$X_H = \sum_{\lambda} c_{\lambda(H)} p_\lambda,$$

and let $c_i(H) = c_{(i,1,1,\ldots,1)}(H)$. Notice that X_H is homogeneous of degree $|V|$ and when H is s-uniform $-c_s(H) = |E|$. Thus, we can always recover the number of vertices from X_H, and we can recover the number of hyperedges in the case of uniform hypergraphs.

Now assume that H is s-uniform and acyclic. For every $A \subseteq E$ the hypergraph (V, A)

has $n-(s-1)|A|$ connected components. For any integer partition λ we let $\operatorname{len}\lambda$ denote the length of the partition. Thus for s-uniform acyclic hypergraphs

$$\operatorname{len}\lambda(A) = n-(s-1)|A|$$

for any $A \subseteq E$. It then follows

$$c_\lambda = (-1)^{\frac{n-k}{s-1}} |\{A \subseteq E : \lambda(A) = \lambda\}|$$

for $\lambda \vdash n$ with $\operatorname{len}\lambda = k$. This implies the relation

$$(-1)^{\frac{n-k}{s-1}} \sum_{\substack{\lambda \vdash n \\ \operatorname{len}\lambda = k}} c_\lambda(H) = \binom{m}{\frac{n-k}{s-1}}$$

where $m = |E|$.

For a vertex $v \in V$, the *degree* of v in H is $\deg v := |\{e \in E : v \in e\}|$. The *degree sequence* of H is the collection of the degrees of all vertices of H arranged in weakly decreasing order. Our next result shows that the chromatic symmetric function of a uniform hypertree determines its degree sequence. In [MMW08, Corollary 5] it was shown that the chromatic symmetric function determines the degree sequence of a tree.

Proposition 10.4.4. *If H is a uniform hypertree, then the degree sequence of H can be determined from X_H.*

Proof. Let $H = (V, E)$ be an s-uniform hypertree on n vertices. Thus H must have $m = \frac{n-1}{s-1}$ hyperedges. Let $X_H = \sum c_\lambda p_\lambda$ and let D_i denote the number of vertices of a degree i in H. It suffices to show that we can determine the numbers D_i for $1 \leq i \leq m$. Since H is a

hypertree and hence connected, we must have $D_0 = 0$. For any $\lambda \vdash n$ let $1(\lambda)$ denote the number of parts of size 1 in λ. Recall that if $A \subseteq E$, then $\operatorname{len} \lambda(A) = n - (s-1)|A|$. Any 1 in the partition $\lambda(A)$ must come from a vertex of degree at most $m - |A|$. Now for any integer $0 \leq i \leq m$ let us consider partitions λ with $\operatorname{len} \lambda = k_i$ where $k_i = n - (s-1)(m-i)$. Exactly the vertices of H of degree at most i will contribute to the sum

$$(-1)^{m-i} \sum_{\substack{\lambda \vdash n \\ \operatorname{len} \lambda = k_i}} c_\lambda \cdot 1(\lambda).$$

Note that a vertex of degree j will contribute to the sum exactly $\binom{m-j}{i-j}$ times. It follows that

$$(-1)^{m-i} \sum_{\substack{\lambda \vdash n \\ \operatorname{len} \lambda = k_i}} c_\lambda \cdot 1(\lambda) = \sum_{j=1}^{i} \binom{m-j}{i-j} D_j.$$

This gives a triangular system that we can solve for each D_i. Therefore X_H determines the degree sequence of a hypertree H. □

We conclude this section by showing that the chromatic symmetric function is not a complete invariant among uniform hypertrees. We give two pairs on 3-uniform hypertrees on 21 vertices which are not isomorphic, but have the same chromatic symmetric function. These hypertrees were found by using nauty [MP14] to enumerate all 3-uniform hypertrees up to isomorphism and then using SageMath [Dev16] to compute the chromatic symmetric functions. The computation indicates that the examples are minimal. That is, there does not exist a pair of hypertrees on fewer than 21 vertices which are not isomorphic but have the same chromatic symmetric function. Let $H_1 = (V, E_1)$, $H_2 = (V, E_2)$, $H_3 = (V, E_3)$,

and $H_4 = (V, E_4)$ where $V = \{0, 1, \ldots, 20\}$ and

$$E_1 = \{\{0,1,2\},\{0,3,4\},\{1,5,6\},\{0,7,8\},\{2,9,10\},\{1,11,12\},\{9,13,14\},$$
$$\{16,3,15\},\{17,18,7\},\{19,20,13\}\}$$

$$E_2 = \{\{0,1,2\},\{0,3,4\},\{1,5,6\},\{0,7,8\},\{2,9,10\},\{1,11,12\},\{9,13,14\},$$
$$\{16,3,15\},\{17,18,5\},\{19,20,15\}\}$$

$$E_3 = \{\{0,1,2\},\{0,3,4\},\{1,5,6\},\{0,7,8\},\{5,9,10\},\{5,11,12\},\{0,13,14\},$$
$$\{16,2,15\},\{1,17,18\},\{19,20,15\}\}$$

$$E_4 = \{\{0,1,2\},\{0,3,4\},\{1,5,6\},\{0,7,8\},\{2,9,10\},\{1,11,12\},\{0,13,14\},$$
$$\{16,9,15\},\{17,18,9\},\{3,19,20\}\}.$$

One can check that H_1, H_2, H_3, and H_4 are all 3-uniform hypertrees on 21 vertices and that $X_{H_1} = X_{H_2}$ and $X_{H_3} = X_{H_4}$ However, H_1 is not isomorphic to H_2 and H_3 is not isomorphic to H_4. The hypertrees H_1 and H_2 are shown in Figure 10.3.

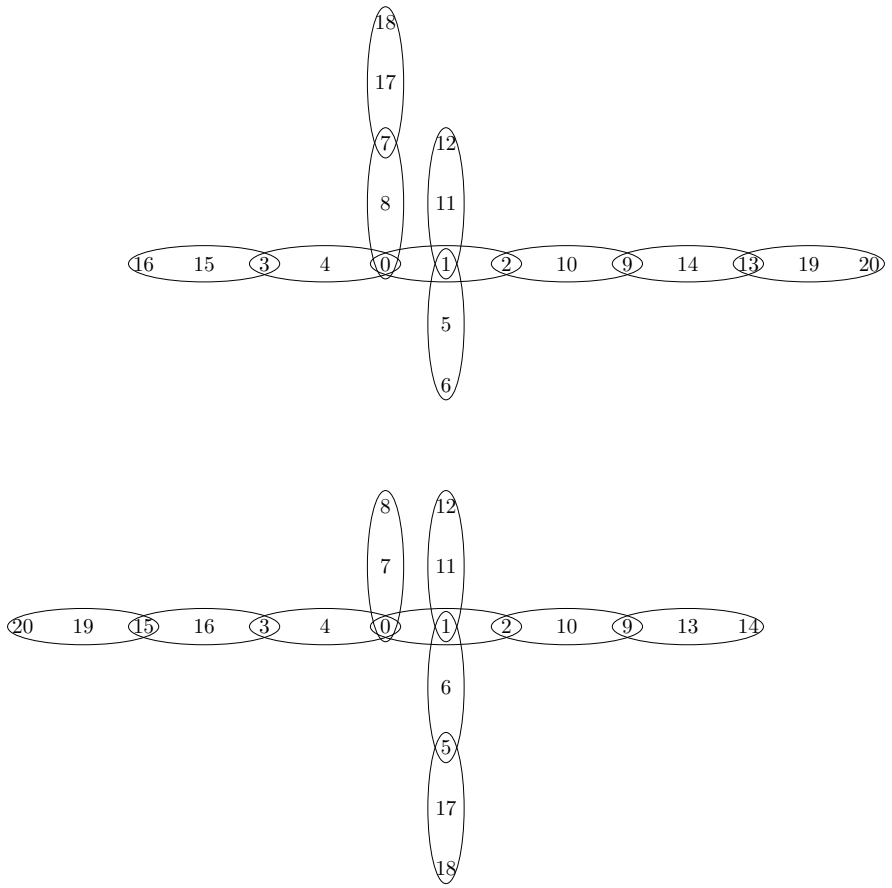

Figure 10.3: The hypertree H_1 above and the hypertree H_2 below which are not isomorhpic but have the same chromatic symmetric function.

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