101
994
THS

THE n-SEGMENT PROPERTY FOR CONTINUOUS FUNCTIONS DEFINED IN THE UPPER HALF PLANE

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY

Charles Leonard Belna

1969

THESIS

LIBRARY
Michigan State
University

This is to certify that the

thesis entitled

The n-Segment Property for
Continuous Functions Defined
in the Upper Half Plane

presented by

Charles Leonard Belna

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Peter A. Zoppan
Major professor

Date May 13, 1969

O-169

ABSTRACT

THE n-SEGMENT PROPERTY FOR CONTINUOUS FUNCTIONS
DEFINED IN THE UPPER HALF PLANE

By

Charles Leonard Belna

Let f be a function from the upper half plane H into the Riemann sphere W and let p be a point on the real line R. Then f is said to possess the n-segment property at p if there exist n-segments $S_1(p),\ldots,S_n(p)$ at p such that $\bigcap_{j=1}^{n} CS_j(p) = \phi$, where $CS_j(p)$ is the cluster set of f at p along $S_j(p)$. It is shown that the set of such points is countable if f is a homeomorphism of H onto a Jordan domain U.

Various techniques which hinge on the Lebesgue Density Theorem are used to establish the following main results. The set of points at which a continuous function has the n-segment property relative to either n or $n-1$ fixed directions is an F_σ set of first category and measure zero. The set of points at which a continuous function has the n-segment property relative to n fixed points in H is an F_σ set of first category and measure zero. Finally, the set of points at which a continuous function has the essential-n-segment property relative to n fixed directions is of measure zero.

Concerning local behavior, the following are proved. A point at which a meromorphic function possesses the 3-segment property is

in some sense "very close" to being an ambiguous point. There exists a holomorphic function having the strong 3-segment property at $p = 0$ which does not have the strong n-segment property at 0 for any other $n \geq 2$. Also, if the principal cluster set is empty at a point p, then either p is an ambiguous point or the infimum of the diameters of all arc cluster sets at p is positive.

THE n-SEGMENT PROPERTY FOR CONTINUOUS FUNCTIONS DEFINED IN THE UPPER HALF PLANE

By

Charles Leonard Belna

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1969

G57236
9-3-69

ACKNOWLEDGMENT

I wish to thank Professor Peter Lappan for suggesting the investigation of the arc properties and for his patient guidance throughout the preparation of this thesis.

TABLE OF CONTENTS

I. INTRODUCTION 1

II. THE n-SEGMENT PROPERTY RELATIVE TO k DIRECTIONS 9

III. THE n-SEGMENT PROPERTY RELATIVE TO n FIXED POINTS 20

IV. THE ESSENTIAL-n-SEGMENT PROPERTY RELATIVE TO n DIRECTIONS 24

V. BEHAVIOR AT POINTS WHERE THE ARC PROPERTIES OCCUR 30

BIBLIOGRAPHY 37

I. INTRODUCTION

The following notations will be used throughout this paper:

With Z denoting the finite complex plane, we set

$$C = \{z \in Z : |z| = 1\} ,$$

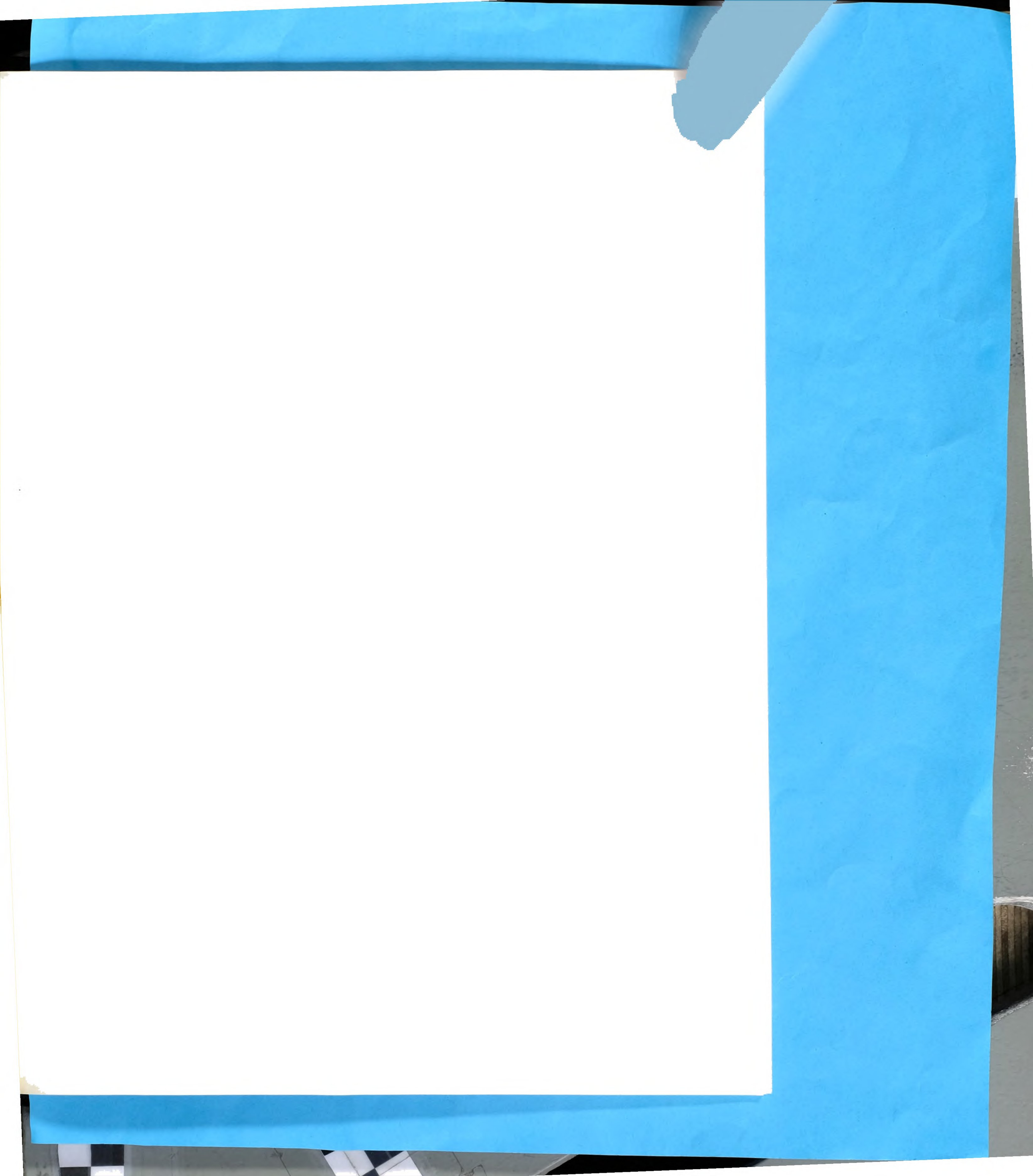
$$D = \{z \in Z : |z| < 1\} ,$$

$$H = \{z \in Z : \mathrm{Im}(z) > 0\} ,$$

$$R = \{z \in Z : \mathrm{Im}(z) = 0\} , \text{ and}$$

$$R^{+} = \{p \in R : p > 0\} .$$

Let f be a function from the upper half plane H into the Riemann sphere W and let p be a point on the real line R. The cluster set, $C(f,p)$, of f at p is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_k\}$ of points in H with $z_k \to p$ and $f(z_k) \to w$. If σ is an arc in H such that $\sigma \cup \{p\}$ is a Jordan arc with one endpoint at p, then σ is said to be an arc at p and the cluster set, $C(f,p,\sigma)$, of f at p along σ is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_k\}$ of points in σ with $z_k \to p$ and $f(z_k) \to w$. We say that f possesses the n-arc property at p, for some integer $n \geq 2$, if there exist n arcs $\sigma_1, \sigma_2, \ldots, \sigma_n$ at p such that the intersection of all n of the sets $C(f,p,\sigma_1), C(f,p,\sigma_2), \ldots, C(f,p,\sigma_n)$ is empty. If, in addition, the intersection of any $n-1$ of these n sets is non-empty we say that f possesses the strong n-arc property at p. Similarly, if the n



arcs can be chosen to be rectilinear segments, we say that f possesses the n-segment property or, correspondingly, the strong n-segment property at p. A point $p \in R$ at which f has the 2-arc property is called an ambiguous point of f. Analogous definitions are made for functions from the open unit disk D into W and for points p on the unit circle C.

In 1955, Bagemihl [1] proved the following theorem which either sharpened or generalized various results of Blumberg [4], Schmeiser [21], and Jarník [11].

BAGEMIHL'S AMBIGUOUS POINT THEOREM. The set of ambiguous points of an arbitrary function from D into W is at most countable.

A few years ago, Mathews [16] noted that Bagemihl's theorem remains true when "ambiguous points" is replaced by "extended ambiguous points". A point $p \in C$ is an extended ambiguous point of a function f if there exist two arcs σ_1 and σ_2 lying in $\overline{D} - \{p\}$ except for one endpoint at p such that

$$EC(f,p,\sigma_1) \cap EC(f,p,\sigma_2) = \phi ,$$

where $EC(f,p,\sigma_j)$ is the extended arc cluster set of f at p along σ_j which is defined analogous to the left and right boundary cluster sets.

In a lecture presented at the Thirteenth Congress of Scandinavian Mathematicians in 1957, Lohwater [13] proposed the investigation of the n-arc property for $n > 2$. Up to that time, only a few results pertaining to this more general property had been

obtained: A result of Lindelöf [12, p. 28] in 1915 implied that for each $n \geq 2$, a schlicht function possesses the n-arc property nowhere. In 1919, Gross [10, p. 57, Section 20] gave a partial extension of Lindelöf's result to meromorphic functions which furnished the corollary that a meromorphic function from D into W does not possess the n-arc property for any $n \geq 2$ at a point $p \in C$ where the cluster set $C(f,p)$ is nowhere dense. Finally, in 1936, Jarník [11] constructed a function having the 3-segment property at an uncountable set of points. Thus Bagemihl's theorem is not true for the 3-segment (hence 3-arc) property.

As an initial response to Lohwater's proposal, Piranian [19], in 1959, proved that there exists a continuous function from D into W which possesses the 3-arc property at each point of C.

Later that year Bagemihl, Piranian and Young [3] gave the following results on the 3-segment and 3-arc properties: There exists a function from D into W which has the 3-segment property at each point of C; there exists a continuous function from D into W which has the 3-segment property at each point of a perfect set; the elliptic modular function has the 3-arc property at every boundary point; and, there exists a Blaschke product which has the 3-arc property at each point of a perfect set on the unit circle.

In the following year, the first of these four results was improved by Erdős and Piranian [9] to read: There exists a function f in H for which each point p in R is the common endpoint of a family $\{L\}_p$ of rectilinear segments in H such that (1) $\{L\}_p$ contains $2^{\aleph_0}$ elements, and the set of their directions is a set

of second category and (ii) the intersection of the cluster sets of f along any three segments in $\{L\}_p$ is empty.

A different type of result concerning the strong n-arc property was obtained in 1961. Young [22] proved that if f is a bounded analytic function from D into W which possesses the 2-arc property at a point $p \in C$, then f possesses the strong n-arc property at p for all $n > 2$. Five years later, Mathews [17] established this result for arbitrary meromorphic functions and conjectured that if f is a meromorphic function from D into W which has the strong n-arc property at a point $p \in C$, then f has the strong k-arc property at p for all $k > n$.

In 1963, Rung [20, Theorem 5, p. 50] and Bagemihl [2, consequence of lemma, p. 4] independently established: If f is a normal meromorphic function from D into W, then for any integer $n \geq 2$, the set of points at which f has the n-segment property is a set of first category and measure zero on C.

It is still unknown as to whether the above theorem can or can not be stated for arbitrary meromorphic functions or at least for arbitrary holomorphic functions in D. To the author's knowledge, the only progress toward this determination is attributed to McMillan [14, Theorem 4, p. 10 and 15, Theorem 8, p. 195]. In his 1965 paper [14], he proved: If f is a holomorphic function from D into W and if f has no ambiguous points in the open arc α of C, then the set of points at which f possesses the n-arc property for any $n \geq 2$ is of first category on α. The following year in [15] he proved: If f is a holomorphic function

from D into W and if α is an open arc of C such that the set of points at which f has a point asymptotic value is dense on α while f has the point asymptotic value ∞ at no point of α, then the complement of the set of points at which f has the n-arc property for some $n \geq 2$ is both metrically dense on α and residual on α.

Finally, in their 1966 paper, Bruckner and Goffman [5, Theorem 2, p. 512] proved: If f is a continuous function from H into R and if s_1 and s_2 are distinct directions in $(0,\pi)$, then for every point $p \in R$, except for a set of first category, the supremum of the essential cluster set of f at p in the direction s_1 is not exceeded by the infimum of the essential cluster set of f at p in the direction s_2.

The main purpose of this paper is to investigate the character of the sets of points at which continuous functions possess the n-segment property with various restrictions placed on the directions of a certain number of the segments involved. In section II we show among other things that the set of points $p \in R$ at which a continuous function in H has the n-segment property relative to either n or $n-1$ fixed directions is an F_σ set of first category and measure zero. Then in section III we prove that the set of points $p \in R$ at which a continuous function has the n-segment property relative to n fixed points in H is again an F_σ set of first category and measure zero.

It is shown in section IV that the set of points $p \in R$ at which a continuous function has the essential-n-segment property

relative to n fixed directions is of measure zero.

In the last section we investigate the local behavior of a function f at points where f has one of the various arc properties. To mention one result, it is shown that even though a continuous function f in H may have the 3-segment property at a point $p \in R$ without p being an ambiguous point of f, each such point is in some sense "very close" to being an ambiguous point if f is meromorphic.

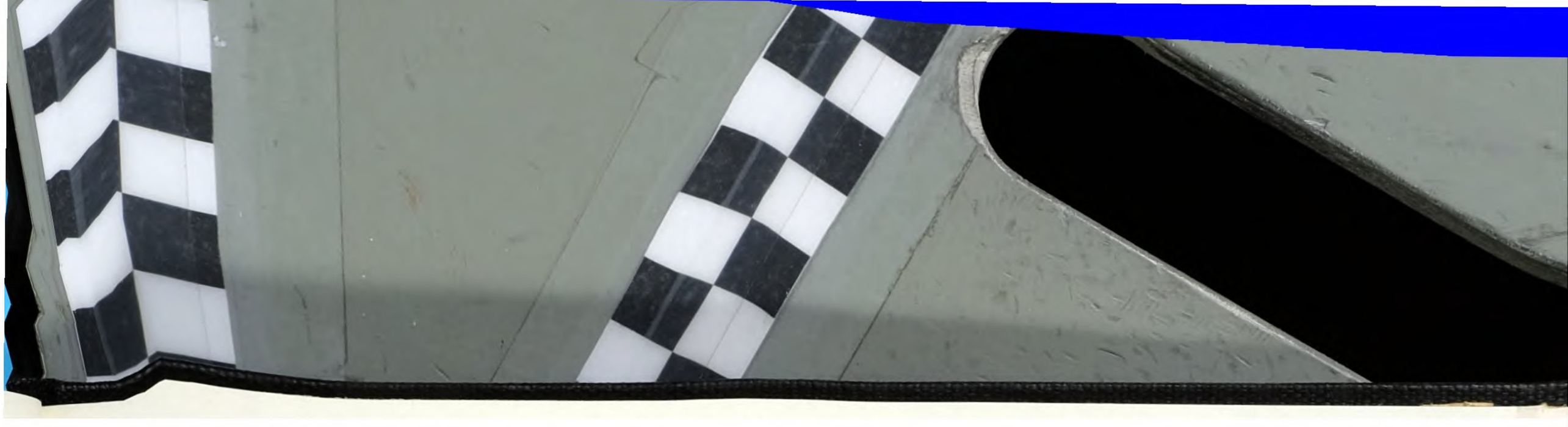
The remainder of this introduction will be used to indicate other notations and definitions along with an important theorem that will be used throughout this paper.

OPEN SETS IN W. Let $\rho(w_1,w_2)$ denote the (three-dimensional) Euclidean distance between the points w_1 and w_2 of W, let $\mathcal{B}$ be a countable basis for the topology induced by ρ on W, and let $\mathcal{G}$ be the collection of all open sets expressible as a finite union of sets $B \in \mathcal{B}$. Then for each integer $n \geq 2$, let $\mathcal{G}^*(n)$ be the set of subsets $\{G_1,\ldots,G_n\}$ of $\mathcal{G}$ for which the intersection of all n of the sets $\overline{G}_1,\ldots,\overline{G}_n$ is empty, where $\overline{G}_j$ is the closure of G_j in W.

SEGMENTS IN H. If p is a point on R and if s is a real number in $(0,\pi)$, the set

$$\{z \in Z : \arg(z - p) = s,\ 0 < \mathrm{Im}(z) \leq 1\}$$

is called a <u>segment at</u> p and s is called the <u>direction</u> of this segment. The symbols $S(p)$ and $S_j(p)$ (where j is a positive integer) will be used to denote various segments at p and the



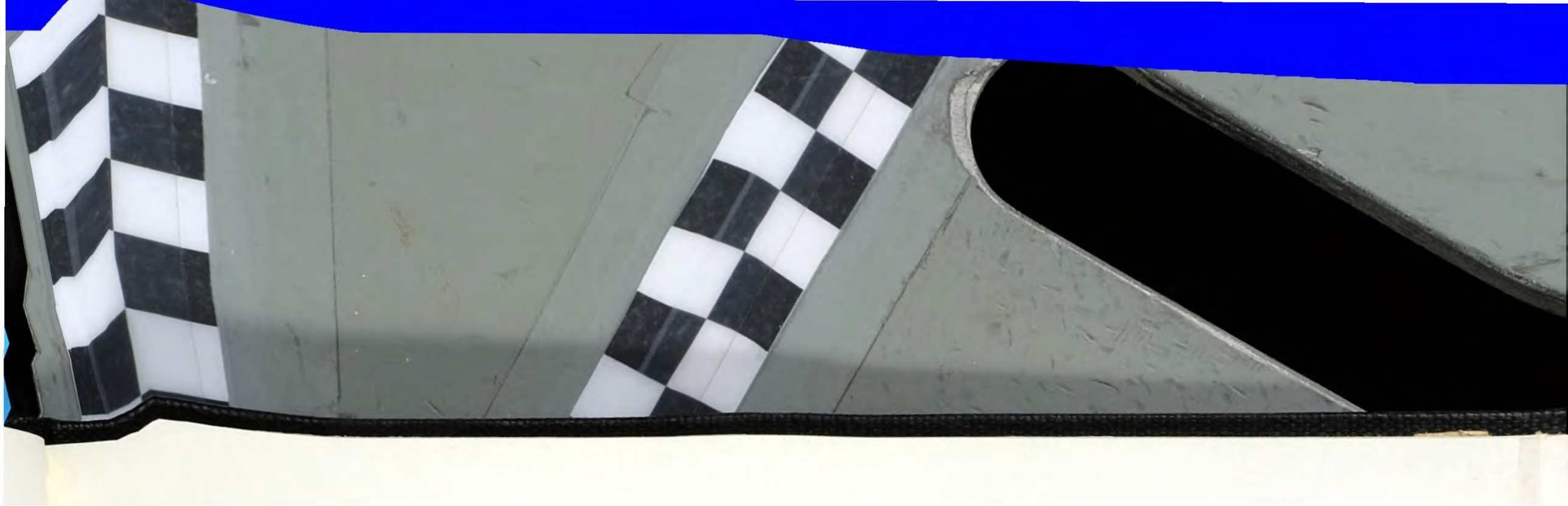
symbols dir S(p) and dir $S_j(p)$, respectively, will denote their directions. The following comments pertain to the symbol S(p) but equally apply to the symbol $S_j(p)$. The particular choice of the segment at p which S(p) is to represent will depend on the context in which the choice is made. In order to specify the segment at p which S(p) is to represent, we need only specify its direction dir S(p). For example, if S(p) were chosen to represent the vertical segment at p, we would say that S(p) is the segment at p for which dir $S(p) = \pi/2$.

For each real number $r \in (0,1]$, S(p,r) will denote the set $S(p) \cap \{z : 0 < \mathrm{Im}(z) \leq r\}$.

Since every segment S(p) is an arc at p, the cluster set C(f,p,S(p)) of f at p along S(p) has already been defined. For convenience we shall use $C_f S(p)$, or simply CS(p) if f is clearly understood, to denote this cluster set.

METRIC DENSITY AT A POINT. Let E be a Lebesgue measurable subset of R. If A is an interval of R containing the point p (which may not belong to E), the inferior and superior limits of $m(E \cap A)/mA$ as mA approaches 0 are called the lower and upper metric densities of E at p respectively. If these are equal, their common value will be called the metric density of E at p. If the metric density of E at p is 1, then we say p is a point of density of E. In this paper, we will make considerable use of the following well-known theorem on density.

LEBESGUE DENSITY THEOREM. If E is measurable, then the metric density of E exists and is equal to 1 at every point of



E except for a set of measure zero.

If the interval A is chosen so that p is the left (right) endpoint of A relative to an observer at z = -i, we can analogously define the terms: lower and upper right (left) metric densities of E at p; right (left) metric density of E at p; and point of right (left) density of E.

II. THE n-SEGMENT PROPERTY RELATIVE TO k DIRECTIONS

Bagemihl, Piranian and Young [3] have listed a number of open questions pertaining to intersections of cluster sets, one of which is the combination of the following two questions.

QUESTION A. *Does there exist a continuous function from* D *into* W *having the 3-segment property at each point of a set of positive measure on* C?

QUESTION B. *Does there exist a continuous function from* D *into* W *having the 3-segment property at each point of a set of second category on* C?

The attempt to answer these questions led to the results of parts 2 and 3 of this section. In part 2 we show that the answer to both questions is in the negative if two of the three directions of the segments are fixed. Then in part 3 we show that the answer to Question B is still in the negative if just one of the three directions is fixed. To make more explicit what is meant by fixing a certain number of the directions of the segments, we give the following definition.

DEFINITION 1. *Let* f *be a function from* H *into* W, *let* n *be an integer* ≥ 2 *and let* k *be an integer satisfying* $1 \leq k \leq n$. *Then* f *is said to possess the* n*-segment property at the point* p *in* R *relative to the* k *directions* $s_1,\ldots,s_k \in (0,\pi)$ *if there exist* n *segments* $S_1(p),\ldots,S_n(p)$ *at* p *with* $\dir S_j(p) = s_j$ *for* $1 \leq j \leq k$ *such that the intersection of all* n *of the sets*

$CS_1(p),\ldots,CS_n(p)$ *is empty. If, in addition, the intersection of any* n-1 *of these sets is non-empty we say that* f *possesses the strong* n-*segment property at* p *relative to the* k *given directions.*

NOTATION. For any set $\{s_1,\ldots,s_k\}$ of $k \geq 1$ directions in $(0,\pi)$, the set of points p in R at which f has the n-segment property $(n \geq k)$ relative to these k fixed directions is denoted by $E[n; s_1,\ldots,s_k]$. Furthermore, $E[n]$ denotes the set of points at which f possesses the n-segment property. We remark that $E[n; s_1,\ldots,s_k]$ is a subset of $E[n]$ and that $E[n; s_1,\ldots,s_k]$ is a subset of $E[n; s_1^*,\ldots,s_m^*]$ for any subset $\{s_1^*,\ldots,s_m^*\}$ of $\{s_1,\ldots,s_k\}$.

In part 1 it is shown that the sets $E[n; s_1,\ldots,s_k]$ and $E[n]$ are measurable if f is a continuous function.

In part 4, the set $E[n]$ is shown to be countable for homeomorphisms of H onto a Jordan domain U.

1. Measurability of the sets $E[n; s_1,\ldots,s_k]$ and $E[n]$

LEMMA 1. *Let* f *be an arbitrary function from* H *into* W *and let* p *be a point of* R. *If* $S_1(p),\ldots,S_n(p)$ *are segments at* p *for which the intersection of all* n *of the sets* $CS_1(p),\ldots,CS_n(p)$ *is empty, then there exists a set* $\{G_1,\ldots,G_n\} \in \mathcal{G}^*(n)$ *and a positive rational number* $r \leq 1$ *such that* $f[S_j(p,r)] \subset \overline{G}_j$ *for* $1 \leq j \leq n$.

Proof. Since each set $CS_j(p)$ is a closed subset of W, there exists an $\epsilon > 0$ such that the intersection of all n of the sets $[CS_1(p)]_\epsilon,\ldots,[CS_n(p)]_\epsilon$ is empty, where

$$[CS_j(p)]_\epsilon = \{w \in W : \rho(w,[CS_j(p)]) < \epsilon\} .$$

Due to the normality of W, there exist open sets U_j $(1 \le j \le n)$ such that

$$CS_j(p) \subset U_j \subset \overline{U}_j \subset [CS_j(p)]_\epsilon .$$

Then, since each set $CS_j(p)$ is compact, there exist open sets $G_j \in \mathscr{G}$ $(1 \le j \le n)$ satisfying $CS_j(p) \subset G_j \subset U_j$.

It is clear that $CS_j(p) = \cap \overline{f[S_j(p,1/k)]}$, where the intersection is taken over all positive integers k. Each set $\overline{f[S_j(p,1/k+1)]}$ is compact and a subset of $\overline{f[S_j(p,1/k)]}$. Thus, for each j, there exists a positive integer $k(j)$ such that $\overline{f[S_j(p,1/k)]} \subset G_j$ for all $k > k(j)$.

Choosing $(1/r) = \max\{k(j) : 1 \le j \le n\}$, we have $f[S_j(p,r)] \subset \overline{G}_j$ for $1 \le j \le n$ with $\bigcap_{j=1}^{n} \overline{G}_j = \phi$ and hence the lemma is proved.

In order to simplify to some degree the wording of the next result, we define sets $\Sigma^*(n)$ as follows: Let Σ denote the set of closed intervals in $(0,\pi)$ which have rational endpoints. Then, for each positive integer n, let $\Sigma^*(n)$ be the collection of all sets consisting of n mutually disjoint intervals in Σ, i.e. $\{\Delta_1,\ldots,\Delta_n\} \in \Sigma^*(n)$ if $\Delta_i \in \Sigma$ $(i = 1,2,\ldots,n)$ and $\Delta_i \cap \Delta_j = \phi$ for $i \ne j$.

With the use of Lemma 1, the validity of the following decomposition is readily verified.

DECOMPOSITION A. Let f be an arbitrary function from H into W, let n be an integer ≥ 2 and let $s_1,\ldots,s_k$ be any k $(1 \leq k \leq n)$ fixed directions in $(0,\pi)$. For each set $\{\Delta_1,\ldots,\Delta_{n-k}\} \in \Sigma^*(n-k)$, each set $\{G_1,\ldots,G_n\} \in \mathscr{G}^*(n)$ and each positive rational number $r \leq 1$, define the set

$$E[n;\ s_1,\ldots,s_k;\ \Delta_1,\ldots,\Delta_{n-k};\ G_1,\ldots,G_n;\ r]$$

to be the set of points p in R at which there exist n segments $S_1(p),\ldots,S_n(p)$ for which the following three conditions are satisfied:

(1) $\operatorname{dir} S_j(p) = s_j$ for $1 \leq j \leq k$,

(2) $\operatorname{dir} S_j(p) \in \Delta_{j-k}$ for $k+1 \leq j \leq n$, and

(3) $f[S_j(p,r)] \subset \overline{G}_j$ for $1 \leq j \leq n$.

Then $E[n;\ s_1,\ldots,s_k]$ is the countable union of these sets. Likewise, $E[n]$ is the countable union of sets

$$E[n;\ \Delta_1,\ldots,\Delta_n;\ G_1,\ldots,G_n;\ r]$$

which are defined in an obvious way.

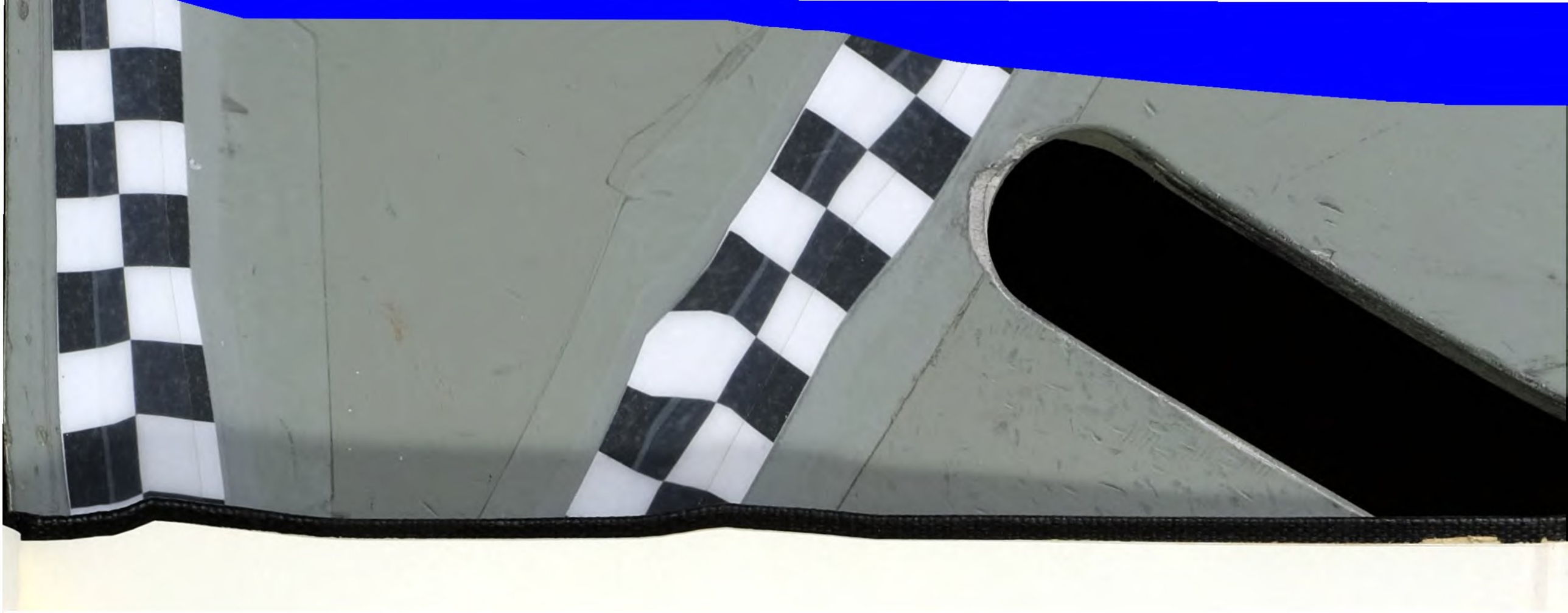
THEOREM 1. Let f be a continuous function from H into W, let n be an integer ≥ 2, and let $s_1,\ldots,s_k$ be k $(1 \leq k \leq n)$ fixed directions between 0 and π. Then $E[n;\ s_1,\ldots,s_k]$ and $E[n]$ are F_σ sets. In particular, the sets

$$E[n;\ s_1,\ldots,s_k;\ \Delta_1,\ldots,\Delta_{n-k};\ G_1,\ldots,G_n;\ r]$$

and

$$E[n;\ \Delta_1,\ldots,\Delta_n;\ G_1,\ldots,G_n;\ r]$$

are closed sets.



Proof. Let $E[n; s_1,\ldots,s_k; \Delta_1,\ldots,\Delta_{n-k}; G_1,\ldots,G_n; r]$ be an arbitrary set from Decomposition A and denote it simply by E. Suppose $\{p_t\}$ is a sequence of points in E which converges to a point p. We need only show that $p \in E$.

For each positive integer t, we have $\operatorname{dir} S_j(p_t) = s_j$ for $1 \le j \le k$ and $\operatorname{dir} S_j(p_t) \in \Delta_{j-k}$ for $k+1 \le j \le n$, where $S_j(p_t)$ $(1 \le j \le n)$ are the segments at p_t guaranteed by $p_t \in E$. Choosing a subsequence if necessary, we may assume $\lim_{t\to\infty} \operatorname{dir} S_j(p_t) = s_j^*$ for $1 \le j \le n$. Then $s_j^* = s_j$ for $1 \le j \le k$ and $s_j^* \in \Delta_{j-k}$ for $k+1 \le j \le n$.

Let $S_1(p),\ldots,S_n(p)$ be the segments at p for which $\operatorname{dir} S_j(p) = s_j^*$ for $1 \le j \le n$. By the continuity of f, we have $f[S_j(p,r)] \subset \overline{G}_j$ for $1 \le j \le n$. Hence $p \in E$ and E is closed. If E^* is an arbitrary set in the decomposition of $E[n]$, then a slight modification of the above proof shows that E^* is closed and the theorem is proved.

If Q is any countable set of directions between 0 and π, then, by taking the union of the sets $E[n; s_1,\ldots,s_k]$ over all integers $n \ge 2$ and all subsets $\{s_1,\ldots,s_k\}$ $(1 \le k \le n)$ of Q, we obtain the following result.

COROLLARY. *Let f be a continuous function from H into W and let Q be a countable set of directions between 0 and π. Then the set of points at which f possesses the n-segment property for any $n \ge 2$ relative to some k $(1 \le k \le n)$ directions in Q is an F_σ set.*

2. The cases $k = n$ and $k = n-1$

We will utilize the following lemma which is a consequence of a result of Doob [8, Lemma 2.1, p. 158].

LEMMA 2. *Let E be a closed subset of the real line R and let $p = 0$ be a point of right density of E. If h is a continuously differentiable homeomorphism from $p \geq 0$ into $p \geq 0$ such that $h(0) = 0$ and $h'(0+) > 0$, then $p = 0$ is a point of right density of the set $h(E \cap R^{+})$.*

THEOREM 2. *Let f be a continuous function from H into W and let n be an integer ≥ 2. Then the set of points at which f possesses the n-segment property relative to either n or $n-1$ fixed directions is of first category and measure zero on R.*

Proof. The result for $n = 2$ follows directly from Bagemihl's ambiguous point theorem. Thus we consider $n \geq 3$ in what follows. Furthermore, since the result for $n-1$ directions implies the result for n directions, we prove the former.

Choose any $n-1$ directions $s_1,\ldots,s_{n-1} \in (0,\pi)$ and consider the set $E[n; s_1,\ldots,s_{n-1}]$. Pick an arbitrary set

$$E[n; s_1,\ldots,s_{n-1}; \Delta_1; G_1,\ldots,G_n; r]$$

from Decomposition A and denote it by E. We now show that no point $p \in E$ is a point of density of E.

Suppose $q \in E$ is a point of density of E. We may assume $q = 0$ and that $\operatorname{dir} S_n(0) < s_j$ for $1 \leq j \leq n-1$, where $S_1(0),\ldots,S_n(0)$ are the segments at 0 guaranteed by $0 \in E$.

For each $j = 1,\ldots,n-1$ define the function μ_j from $I_j = \{p: p > 0, S_n(0) \cap S_j(p) \neq \emptyset\}$ into $S_n(0)$ by

$$\{\mu_j(p)\} = S_n(0) \cap S_j(p)$$

where $\operatorname{dir} S_j(p) = s_j$. Then define the function h_j from $I_j \cup \{0\}$ into $p \geq 0$ by $h_j(p) = |\mu_j(p)|$ for $p > 0$ and $h_j(0) = 0$. It is easily verified that

$$h_j(p) = (\sin s_j / \sin[s_j - \operatorname{dir} S_n(0)])(p).$$

By Lemma 2, the point 0 is a point of right density of $h_j(E \cap I_j)$ for $1 \leq j \leq n-1$. This implies that 0 is a point of right density of $\bigcap_{j=1}^{n-1} h_j(E \cap I_j)$. Therefore, there exists a positive number b in this intersection sufficiently close to 0 such that if $p_j = \mu_j^{-1}(z)$, where z is the point on $S_n(0,r)$ for which $|z| = b$, then $z \in S_j(p_j,r)$ for $1 \leq j \leq n-1$. Consequently

$$z \in S_n(0,r) \cap \left[\bigcap_{j=1}^{n-1} S_j(p_j,r)\right]$$

and, since $0 \in E$ and $p_j \in E$ for $1 \leq j \leq n-1$, we have $f(z) \in \overline{G}_1 \cap \overline{G}_2 \cap \ldots \cap \overline{G}_n$ which contradicts the fact that this intersection is empty. Thus E contains none of its points of density.

The Lebesgue Density Theorem gives $mE = 0$ and the arbitratiness of E gives $mE[n; s_1,\ldots,s_{n-1}] = 0$ in view of Decomposition A. Furthermore, since E is closed (Theorem 1) and can not contain an interval (each interior point of which would be a point of density of E), $E[n; s_1,\ldots,s_{n-1}]$ is a first category set and the proof of the theorem is complete.

Remark. In [3] Bagemihl, Piranian and Young construct a continuous function in H for which $E[3; \pi/4, \pi/2, 3\pi/4]$ contains a perfect set. Hence, Theorem 2 for $n = 3$ is the best possible result in the sense that first category and measure zero can not be replaced by countable.

The following corollary is established in the same manner as was the corollary to Theorem 1.

COROLLARY. *Let f be a continuous function from H into W and let Q be any countable set of directions in $(0,\pi)$. Then the set of points at which f possesses the n-segment property for any $n \geq 2$ relative to some set of n or $n-1$ directions from Q is of first category and measure zero on R.*

3. The case $k = 1$

THEOREM 3. *If f is a continuous function from H into W, then the set of points at which f possesses the strong n-segment property relative to one fixed direction is of first category on R.*

Proof. Choose any direction $s \in (0,\pi)$ and let E be the set of points at which f possesses the strong n-segment property relative to s. If $p \in E$, then there exist segments $S_1(p),\ldots,S_n(p)$ at p with $\operatorname{dir} S_1(p) = s$ such that the intersection of all n of the sets $CS_1(p),\ldots,CS_n(p)$ is empty while the intersection of the sets $CS_2(p),\ldots,CS_n(p)$ is non-empty. Therefore $CS_1(p) \neq C(f,p)$. Due to a result of Collingwood [6, Theorem 1, p. 5], the set of points p at which $CS(p) \neq C(f,p)$ for $\operatorname{dir} S(p) = s$ is a set of first category. As a subset of this set, E is also a set of first

category and the theorem is proved.

The following corollary is established by taking a countable union of sets considered in Theorem 3.

COROLLARY. Let f be a continuous function from H into W and let Q be a countable set of directions in $(0,\pi)$. Then the set of points at which f possesses the strong n-segment property for any $n \geq 2$ relative to some direction in Q is of first category on R.

If a function f has the 3-segment property at a point p relative to a given direction s, then either f has the strong 3-segment property at p relative to s or p is an ambiguous point of f. In view of this fact, the following theorem is an immediate consequence of Theorem 3.

THEOREM 4. If f is a continuous function from H into W, then the set of points at which f possesses the 3-segment property relative to one fixed direction is of first category on R.

4. The n-segment property

THEOREM 5. If f is a homeomorphism of the upper half plane H onto a Jordan domain U, then the set of points at which f possesses the n-segment property for some $n \geq 2$ is countable.

Proof. We first consider $U = D$ and show that $E[n] - E[n-1]$ is a subset of the set of ambiguous points of f.

Suppose $S_1(p),\ldots,S_n(p)$ are n distinct segments at some point $p \in R$ for which the intersection of all n of the sets $CS_1(p),\ldots,CS_n(p)$ is empty while the intersection of any n-1 of these sets is non-empty, and suppose that

$$\text{dir } S_1(p) < \text{dir } S_2(p) < \ldots < \text{dir } S_n(p).$$

Then $CS_1(p) \cap CS_n(p) = \alpha_1 \cup \alpha_2$, where α_1 and α_2 are closed connected subsets of C with $\alpha_1 \cap \alpha_2 = \phi$ [note that one of the α_j might be empty and one or both of them might be a single point].

Set $\Delta = \{z: \text{dir } S_1(p) < \arg(z-p) < \text{dir } S_n(p),\ 0 < \text{Im}(z) < 1\}$. If no point $q \in CS_1(p) \cap CS_n(p)$ is accessible (see e.g. [7, p. 168]) from $f(\Delta)$, then for at least one of the sets α_j, say α_1, we have $\phi \neq \alpha_1 \subset CS(p)$ for all segments $S(p)$ at p for which $\text{dir } S_1(p) \leq \text{dir } S(p) \leq \text{dir } S_n(p)$. In particular, $\alpha_1 \subset CS_j(p)$ for $j = 1,\ldots,n$ which violates the supposition that $\bigcap_{j=1}^{n} CS_j(p) = \phi$. Therefore, there exists a point $q \in CS_1(p) \cap CS_n(p)$ which is accessible through $f(\Delta)$. But this implies the existence of an arc σ in Δ with $\overline{\sigma} \cap C = \{p\}$ such that the cluster set $C(f,p,\sigma) = \{q\}$. Consequently $C(f,p,\sigma) \cap CS_j(p) = \phi$ for some j $(1 \leq j \leq n)$ and p is an ambiguous point of f. Due to the arbitrariness of p, $E[n] - E[n-1]$ is a subset of the set of ambiguous points of f.

Letting E denote the set of points $p \in R$ at which f possesses the n-segment property for some $n \geq 2$, we have

$$E = E[2] \cup \left(\bigcup_{n=3}^{\infty} (E[n] - E[n-1]) \right)$$

and the desired result follows from Bagemihl's ambiguous point theorem.

Now, let f be a homeomorphism of H onto an arbitrary Jordan domain U whose boundary is the Jordan curve J. Let g be a one-one conformal mapping of U onto D and set $h = gof$.

Suppose there exist n distinct segments $S_1(p),\ldots,S_n(p)$ at a point $p \in R$ for which the intersection of all n of the sets $CS_1(p),\ldots,CS_n(p)$ is empty. Let g^* be the extension of g

that maps $U \cup J$ homeomorphically onto $D \cup C$ and for each subset A of J, set $g^*[A] = \{g^*(a): a \in A\}$. Then

$$\cap\, C_h S_j(p) = \cap\, g^*[C_f S_j(p)] = g^*[\cap\, C_f S_j(p)] = g^*[\phi] = \phi$$

where each intersection is taken over $j = 1,\ldots,n$.

It follows from the first part of this proof that p is an ambiguous point of h. That is, there exist two arcs σ_1 and σ_2 at p such that

$$C(h,p,\sigma_1) \cap C(h,p,\sigma_2) = \phi.$$

Then

$$\begin{aligned} C(f,p,\sigma_1) \cap C(f,p,\sigma_2) &= g^{*-1}[C(h,p,\sigma_1)] \cap g^{*-1}[C(h,p,\sigma_2)] \\ &= g^{*-1}[C(h,p,\sigma_1) \cap C(h,p,\sigma_2)] \\ &= g^{*-1}[\phi] = \phi \end{aligned}$$

and so p is an ambiguous point of f and the proof is complete.

It is noted that, in the above proof, the n segments $S_1(p),\ldots,S_n(p)$ could have been replaced by n mutually disjoint arcs at p. Thus we can state the following theorem.

THEOREM 6. *Let f be a homeomorphism of H onto a Jordan domain U. Then the set of points $p \in R$ for which there exist n mutually disjoint arcs $\sigma_1,\ldots,\sigma_n$ at p such that the intersection of all n of the sets $C(f,p,\sigma_1),\ldots,C(f,p,\sigma_n)$ is empty is a countable set.*

III. THE n-SEGMENT PROPERTY RELATIVE TO n FIXED POINTS

Throughout this section, when considering points $P \in H$, we will assume without loss of generality that $Im(P) \leq 1$.

DEFINITION 2. *Let* f *be a function from* H *into* W. *We say that* f *possesses the* n-*segment property* ($n \geq 2$) *at a point* p *in* R *relative to the* n *points* $P_1,\ldots,P_n \in H$ *if the intersection of all* n *of the sets* $CS_1(p),\ldots,CS_n(p)$ *is empty where* $S_j(p)$ *is the segment at* p *which passes through the point* P_j *for* $j = 1,\ldots,n$.

The purpose of this section is to prove the following result.

THEOREM 7. *If* f *is a continuous function from* H *into* W, *then the set of points at which* f *possesses the* n-*segment property relative to* n *fixed points in* H *is an* F_σ *set of first category and measure zero on* R.

Before stating a lemma which we will utilize in the proof of the above theorem, we give some notations that will simplify its statement.

For any subset M of the real line R and any positive real number a, we set $M(a) = M \cap (0,a)$. Then for each point $P \in H$ and each positive real number $h \leq Im(P)$, we set

$$Q[M(a),h,P] = \bigcup S(p,h)$$

where the union is taken over all $p \in M(a)$ and where $S(p)$ is the segment at p which passes through P.

LEMMA 3. Let E be a closed subset of R for which $m[E \cap (0,a)]/a \to 1$ as $a \to 0$ and let P be a point in H. Then, given an integer $n \geq 2$, there exists a real number $b > 0$ such that for any $h \in (0,Im(P)]$ and any $a \in (0,b)$ we have

$$m[Q[E(a),h,P] \cap \Delta(a,h)] > [(n-1)/n][\text{area } \Delta(a,h)] ,$$

where $\Delta(a,h)$ is any triangular region in H of height h whose base is the interval $(0,a)$ and which satisfies $\Delta(a,h) \subset Q[R(a),h,P]$.

Proof. If E' denotes the complement of E in R and if a is a positive real number, then $E'(a) = \bigcup_{j=1}^{\infty} I_j$ where the I_j are mutually disjoint open intervals of length I_j^*. Thus, for each $h \in (0,Im(P)]$, we have (setting $q = Im(P)$)

$$\begin{aligned} m[Q[E'(a),h,P] \cap \Delta(a,h)] &\leq mQ[E'(a),h,P] \\ &= \sum_{j=1}^{\infty} [hI_j^*/2q][2q-h] \\ &= [h(2q-h)/2q][mE'(a)] \\ &< [h][mE'(a)] \\ &= [ah/2n][2n(mE'(a)/a)]. \end{aligned}$$

Choose a real number $b > 0$ so that $[mE'(a)/a] < (1/2n)$ for each $a \in (0,b)$ to get

$$m[Q[E'(a),h,P] \cap \Delta(a,h)] < (ah/2n) = (1/n)[\text{area } \Delta(a,h)]$$

which establishes the lemma.

Proof of Theorem 7. Choose any n points $P_1,\ldots,P_n \in H$ and set $d = \min\{Im(P_j): 1 \leq j \leq n\}$. For each positive real number

$r < d$ and each set $\{G_1,\ldots,G_n\}$ in $\mathscr{S}^*(n)$, define $E[G_1,\ldots,G_n; r]$ to be the set of points $p \in R$ for which $f[S_j(p,r)] \subset \overline{G}_j$ for $1 \leq j \leq n$, where $S_j(p)$ is the segment at p which passes through the point P_j for $j = 1,\ldots,n$.

If E is the set of points at which f possesses the n-segment property relative to the n fixed points $P_1,\ldots,P_n$, then

$$E = \cup\, E[G_1,\ldots,G_n; r]$$

where the union is taken over all positive rational numbers $r < d$ and all sets $\{G_1,\ldots,G_n\}$ in $\mathscr{S}^*(n)$. Since each of the sets $E[G_1,\ldots,G_n; r]$ is clearly a closed set, E is an F_σ set.

Pick an arbitrary one of the sets $E[G_1,\ldots,G_n; r]$ and denote it by E^*. We now show that no point of E^* is a point of density of E^*. Suppose $q \in E^*$ is a point of density of E^* and for convenience let $q = 0$.

For each real number $a > 0$, the intersection

$$\bigcap_{j=1}^{n} Q[R(a),\mathrm{Im}(P_j),P_j]$$

is an open triangular region $\Delta(a,h_a)$ of height h_a $(0 < h_a \leq d)$ whose base is the interval $(0,a)$.

It follows from Lemma 3 that for each $j = 1,\ldots,n$ there exists a real number $b_j > 0$ such that for each real number $a \in (0,b_j)$, we have

$$m[Q[E^*(a),h_a,P_j] \cap \Delta(a,h_a)] > [(n-1)/n][\text{area } \Delta(a,h_a)].$$

If we set $b = \min\{b_1,\ldots,b_n\}$, then for each $j = 1,\ldots,n$ and for each real number $a \in (0,b)$ we have

$$m[Q[E^*(a),h_a,P_j] \cap \Delta(a,h_a)] > [(n-1)/n][\text{area } \Delta(a,h_a)].$$

Consequently, for each $a \in (0,b)$, there exists a point z_a which is in each set $Q[E^*(a),h_a,P_j] \cap \Delta(a,h_a)$ for $j = 1,\ldots,n$. Hence, if we choose a point $a \in (0,b)$ for which $h_a < r$, $f(z_a) \in \overline{G}_j$ for each $j = 1,\ldots,n$. This contradicts $\{G_1,\ldots,G_n\} \in \mathscr{G}^*(n)$ and thus no point of E^* is a point of density of E^*.

The Lebesgue Density Theorem gives $mE^* = 0$ and thus $mE = 0$ by the arbitrariness of E^*. Furthermore, since E^* is closed and can not contain an interval (each interior point of which would be a point of density of E^*), E is of first category and the proof is complete.

Taking a countable union of the sets considered in the previous theorem, we obtain the following result.

COROLLARY. *Let f be a continuous function from H into W and let $\mathscr{P}$ be a countable set of points in H. Then the set of points at which f possesses the n-segment property for some $n \geq 2$ relative to some n points from $\mathscr{P}$ is an F_σ set of first category and measure zero on R.*

IV. THE ESSENTIAL-n-SEGMENT PROPERTY RELATIVE TO n DIRECTIONS

Let f be a continuous function from H into W, let $S(p)$ be a segment at a point $p \in R$ and let mA denote the linear measure of A. A subset V of H is said to have <u>positive upper density at p relative to $S(p)$</u> if

$$\limsup_{t \to 0} [m(V \cap S(p,t))/m\, S(p,t)] > 0 .$$

The set V is said to have <u>density d at p relative to $S(p)$</u> if

$$\lim_{t \to 0} [m(V \cap S(p,t))/m\, S(p,t)] = d ;$$

and, if $d = 1$, p is said to be a <u>point of density of V relative to $S(p)$</u>. With $B[w,\epsilon] = \{w' \in W : \rho(w',w) < \epsilon\}$, the <u>essential cluster set</u>, $C_e S(p)$, <u>of f at p along $S(p)$</u> is defined as follows: the point $w \in W$ is in $C_e S(p)$ if for every $\epsilon > 0$, the set $f^{-1}(B[w,\epsilon])$ has positive upper density at p relative to $S(p)$.

DEFINITION 3. <u>If f is a continuous function from H into W, then f is said to possess the essential-n-segment property ($n \geq 2$) at the point $p \in R$ relative to n given directions $s_1,\ldots,s_n \in (0,\pi)$ if the intersection of all n of the sets $C_e S_1(p),\ldots,C_e S_n(p)$ is empty, where $S_j(p)$ $(1 \leq j \leq n)$ is the segment at p for which $\operatorname{dir} S_j(p) = s_j$.</u>

The purpose of this section is to prove the following theorem.

THEOREM 8. <u>If f is a continuous function from H into W, then the set of points $p \in R$ at which f possesses the essential-</u>

n-segment property ($n \geq 2$) relative to n fixed directions is of measure zero.

We now prove three lemmas which will be used in the proof of Theorem 8.

LEMMA 4. If f is a continuous function from H into W, then $C_e S(p)$ is a closed set for each $p \in R$ and each segment $S(p)$ at p.

Proof. Suppose $\{w_n\}$ is a sequence of points in $C_e S(p)$ with $w_n \to w$. Let $\epsilon > 0$ be given. For some positive integer $N = N(\epsilon)$, we have $B[w_N, \epsilon/2] \subset B[w, \epsilon]$. Consequently

$$f^{-1}(B[w_N, \epsilon/2]) \subset f^{-1}(B[w, \epsilon])$$

and hence the set $f^{-1}(B[w, \epsilon])$ has positive upper density at p relative to $S(p)$. Since ϵ was arbitrary, $w \in C_e S(p)$ and the proof is complete.

LEMMA 5. Let f be a continuous function from H into W and let $S(p)$ be a segment at a point $p \in R$. If $C_e S(p) \subset G$ for some open set $G \in \mathcal{G}$, then p is a point of density of $f^{-1}(G)$ relative to $S(p)$.

Proof. If not, $f^{-1}(G')$ has positive upper density at p relative to $S(p)$, where G' is the complement of G in W. Since $G' \cap C_e S(p) = \emptyset$, to each $w \in G'$ there corresponds a real number $\epsilon(w) > 0$ for which $f^{-1}(B[w, \epsilon(w)])$ has density 0 at p relative to $S(p)$. Then G' is contained in the union of the sets $B[w, \epsilon(w)]$ taken over all $w \in G'$. Since G' is compact,

$$G' \subset \bigcup_{j=1}^{k} B[w_j, \epsilon(w_j)]$$

for some finite subset $\{w_1, \ldots, w_k\}$ of G'. Thus

$$f^{-1}(G') \subset \bigcup_{j=1}^{k} f^{-1}(B[w_j, \epsilon(w_j)])$$

which contradicts the assumption that $f^{-1}(G')$ has positive upper density at p relative to $S(p)$ and the lemma is proved.

Before stating Lemma 6, we introduce some notations that will be used throughout the remainder of this section.

For any subset M of the real line R and any positive real number a, set $M(a) = M \cap (-a,a)$. Then for any direction $s \in (0,\pi)$ and any positive real number $h \leq 1$, set

$$T[M(a),h,s] = \bigcup S(p,h)$$

where the union is taken over all $p \in M(a)$ and where $S(p)$ is the segment at p for which dir $S(p) = s$.

LEMMA 6. *Let E be a Borel subset of R and let s be a direction in $(0,\pi)$. For positive numbers a and h $(h \leq 1)$, let $\Delta(a,h)$ be any open triangular region in H of height h whose base is the interval $(-a,a)$ and which satisfies $\Delta(a,h) \subset T[R(a),h,s]$. If U is an open subset of H for which there exists a real number $q \in (0,1)$ such that $m[U \cap S(p,h)] > q[m\, S(p,h)]$ for each $p \in E(a)$ where $S(p)$ is the segment at p for which dir $S(p) = s$, then*

$$m[U \cap \Delta(a,h)] > [q(mE(a)/a) - 1][\text{area } \Delta(a,h)].$$

Proof. If μ is the characteristic function of $E(a)$, then

$$m(U \cap T[E(a),h,s]) > (\sin s) \int_{-a}^{a} q[mS(p,h)]\, \mu(p)\, dp$$

$$= \int_{-a}^{a} (qh)\, \mu(p)\, dp$$

$$= (qh)\, mE(a)$$

$$= q[mE(a)/a][\text{area } \Delta(a,h)].$$

Then, since $[\text{area } \Delta(a,h)] = (1/2)(\text{area } T[R(a),h,s])$, we have

$$m[U \cap \Delta(a,h)] \geq m(U \cap T[R(a),h,s]) - [\text{area } \Delta(a,h)]$$

$$> q[mE(a)/a][\text{area } \Delta(a,h)] - [\text{area } \Delta(a,h)]$$

as asserted in the lemma.

Proof of Theorem 8. Choose any $n \geq 2$ directions $s_1,\ldots,s_n \in (0,\pi)$ and let E be the set of points $p \in R$ at which f has the essential-n-segment property relative to these n directions. If p is a point in E, then the intersection of the sets $C_e S_1(p),\ldots,C_e S_n(p)$ is empty, where $S_j(p)$ $(1 \leq j \leq n)$ is the segment at p for which $\operatorname{dir} S_j(p) = s_j$. Since the sets $C_e S_j(p)$ are closed (Lemma 4), there exists a set $\{G_1,\ldots,G_n\} \in \mathcal{G}^*(n)$ such that $C_e S_j(p) \subset G_j$ for each $j = 1,\ldots,n$. Furthermore, by Lemma 5, p is a point of density of $f^{-1}(G_j)$ relative to $S_j(p)$ $(1 \leq j \leq n)$. Hence, given an ϵ $(0 < \epsilon < 1/2n)$, there exists a rational number $r > 0$ such that

$$m[f^{-1}(G_j) \cap S_j(p,t)] > [(2n-1)/2n(1-\epsilon)]\, mS_j(p,t)$$

for each $j = 1,\ldots,n$ and each $t \in (0,r)$.

Let $E[G_1,\ldots,G_n;\ r;\ \epsilon]$ be the set of points $p \in R$ such that

$$m[f^{-1}(G_j) \cap S_j(p,t)] > [(2n-1)/2n(1-\epsilon)]mS_j(p,t)$$

for each rational $t \in (0,r)$ and each $j = 1,\ldots,n$ where $S_j(p)$ is the segment at p for which dir $S_j(p) = s_j$. Then

$$E \subset \cup\, E[G_1,\ldots,G_n;\ r;\ \epsilon]$$

with the union being taken over all rationals $r > 0$, all rationals $\epsilon \in (0,1/2n)$ and all sets $\{G_1,\ldots,G_n\}$ in $\mathscr{G}^*(n)$. Each of the sets $E[G_1,\ldots,G_n;\ r;\ \epsilon]$ is a G_δ set since its complement can be expressed as the union of the closed sets

$$\{p : m[f^{-1}(G_j) \cap S_j(p,t)] \leq [(2n-1)/2n(1-\epsilon)]mS_j(p,t)\}$$

taken over all rational numbers $t \in (0,r)$ and each $j = 1,\ldots,n$.

Pick any one of the sets $E[G_1,\ldots,G_n;\ r;\ \epsilon]$ and denote it by E^*. To establish the theorem we need only show that $mE^* = 0$. We now show that no point of E^* is a point of density of E^*. Suppose $p^* \in E^*$ is a point of density of E^*, and without loss of generality set $p^* = 0$. Choose a real number $a > 0$ for which $mE^*(a) > 2a(1-\epsilon)$ and for which the intersection of the sets $T[R(a),r,s_1],\ldots,T[R(a),r,s_n]$ is a triangular region Δ of height $h < r$. Then, by Lemma 6, we have

$$m[f^{-1}(G_j) \cap \Delta] > ([(2n-1)/2n(1-\epsilon)][2(1-\epsilon)] - 1)[\text{area } \Delta]$$

$$= [(n-1)/n][\text{area } \Delta]$$

for each $j = 1,2,\ldots,n$. Hence $\bigcap_{j=1}^{n} [f^{-1}(G_j) \cap \Delta] \neq \phi$ in violation

of the fact that $\{G_1,\ldots,G_n\}$ is an element in $\mathscr{S}^*(n)$. Hence, no point of E^* is a point of density of E^*. Then, in view of the Lebesgue Density Theorem, we have $mE^* = 0$ and the proof is complete.

V. BEHAVIOR AT POINTS WHERE THE ARC PROPERTIES OCCUR

The purpose of this section is to investigate the behavior of a function f at a point where f possesses either the n-arc or the n-segment property.

We previously remarked (see p. 16) that continuous functions having the 3-segment property at an uncountable number of points do exist. Due to Bagemihl's ambiguous point theorem, we conclude that a point at which a continuous function f has the 3-segment property need not be an ambiguous point of f. However, if f is a meromorphic function, it is shown in Theorem 9 that such a point is either an ambiguous point of f or in some sense "very close" to being an ambiguous point of f.

Combining Theorem 10 with the remark following it, we show that a holomorphic function can have the strong n-segment property at a point p for some integer $n \geq 2$ and fail to have the strong k-segment property at p for any other integer $k \geq 2$.

Finally, in Theorem 11 we show that if f is an arbitrary complex-valued function having the n-arc property at a point p for some integer $n \geq 2$, then either p is an ambiguous point of f or the infimum of the diameters of all the arc cluster sets at p is positive.

THEOREM 9. *Let f be a meromorphic function from H into W. If f possesses the 3-segment property at the point p in R, then there exist two disjoint Stoltz arcs σ_1 and σ_2 at p for*

<u>which</u> $C(f,p,\sigma_1) \cap C(f,p,\sigma_2)$ <u>is at most a finite set of points in</u> W.

<u>Remark</u>. An arc σ at $p \in R$ is said to be a <u>Stoltz</u> arc at p if there exist two distinct real numbers a and b in $(0,\pi)$ such that

$$\sigma \subset \{z \in H : a \leq \arg(z-p) \leq b\}.$$

<u>Proof of Theorem</u> 9. For any point $w' \in W$ and any positive real number $d \leq 1$, let $B[w',d]$ denote the set of all points $w \in W$ for which $\rho(w',w) < d$ and let $C[w',d]$ denote the set of all points $w \in W$ for which $\rho(w',w) = d$. Then let $\mathcal{B}(f)$ denote the collection of all the sets $B[w',d]$ whose boundary $C[w',d]$ contains none of the images of the points $z \in H$ for which $f'(z) = 0$, and let $\mathcal{G}(f)$ denote the collection of all open sets G in W that are expressible as a finite union of sets $B \in \mathcal{B}(f)$.

There exist three segments $S_1(p)$, $S_2(p)$ and $S_3(p)$ at p with $\operatorname{dir} S_1(p) < \operatorname{dir} S_2(p) < \operatorname{dir} S_3(p)$, three open sets G_1, G_2 and G_3 in $\mathcal{G}(f)$ with $\bigcap_{j=1}^{3} G_j = \phi$, and a positive rational number $r < 1$ such that $\overline{f(S_j(p,r)} \subset G_j$ for $j = 1,2,3$. Furthermore, due to the fact that the boundary ∂G of each G in $\mathcal{G}(f)$ is the union of a finite number of circular arcs, we may also stipulate that $\partial G_1 \cap \partial G_3$ is at most a finite set of points in W.

For $1 \leq i < j \leq 3$ and $0 < t \leq r$, set

$$\Delta(i,j;r) = \{z : \operatorname{dir} S_i(p) < \arg(z-p) < \operatorname{dir} S_j(p),\ 0 < \operatorname{Im}(z) < r\}$$

and set

$$\ell(i,j;t) = \{z : \operatorname{dir} S_i(p) < \arg(z-p) < \operatorname{dir} S_j(p),\ \operatorname{Im}(z) = t\}.$$

Let Λ be the countable set of components λ of the set $\{z : z \in \Delta(1,2;r),\ f(z) \in \partial G_1\}$ for which $\overline{\lambda} \cap S_2(p,r) \neq \phi$ (where $\overline{\lambda}$ is the closure of λ in H). Each component $\lambda \in \Lambda$ is a homeomorphic image of the open unit interval $0 < x < 1$.

If $\Lambda = \phi$, then either $\overline{f(S_2(p,r))} \subset G_1$ or $\overline{f(S_2(p,r))} \subset W - \overline{G}_1$. In the former case, we have

$$\overline{f(S_3(p,r))} \cap \overline{f(S_2(p,r))} \subset G_3 \cap (G_1 \cap G_2) = \phi$$

and hence the desired Stoltz arcs at p are $\sigma_1 = S_2(p)$ and $\sigma_2 = S_3(p)$. In the latter case, the desired Stoltz arcs at p are $\sigma_1 = S_1(p)$ and $\sigma_2 = S_2(p)$.

Suppose $\Lambda \neq \phi$. If a component $\lambda \in \Lambda$ has p as a limit point, then there exists an arc α at p which is contained in λ and $C(f,p,\alpha) \subset \partial G_1$. Therefore $C(f,p,\alpha) \cap CS_1(p) = \phi$ and, in this case, $\sigma_1 = \alpha$ and $\sigma_2 = S_1(p)$ are the desired Stoltz arcs at p.

Suppose none of the $\lambda \in \Lambda$ have p as a limit point. Since f is a local homeomorphism on $\{z : f(z) \in \partial G_1\}$, at most finitely many of the $\lambda \in \Lambda$ intersect the line segment $\ell(1,2;t)$ for any positive real number $t \leq r$. Furthermore, none of them intersect $S_1(p,r)$. Consequently, there exists an arc σ_1 at p contained in

$$(S_2(p,r) \cap \{z : f(z) \in G_1\}) \cup (\cup\lambda)$$

where the second union is taken over all $\lambda \in \Lambda$. It follows that

$$C(f,p,\sigma_1) \subset (G_1 \cap G_2) \cup \partial G_1.$$

In a similar manner we obtain an arc σ_2 at p that is contained in

$$(S_2^{\cdot}(p,r) \cap \{z : f(z) \in G_3\}) \cup (\cup\lambda_*)$$

where the second union is taken over all components λ_* of the set $\{z : z \in \Delta(2,3;r), f(z) \in \partial G_3\}$ with $\overline{\lambda}_* \cap S_2(p,r) \neq \phi$. Hence

$$C(f,p,\sigma_2) \subset (G_2 \cap G_3) \cup \partial G_3.$$

Thus, in view of $\bigcap_{j=1}^{3} G_j = \phi$, we have

$$C(f,p,\sigma_1) \cap C(f,p,\sigma_2) \subset \partial G_1 \cap \partial G_3$$

and the proof of the theorem is complete.

THEOREM 10. *There exists a holomorphic function* f *in* H *having the strong 3-segment property at* 0 *which does not have the strong n-segment property at* 0 *for any other integer* $n \geq 2$.

Proof. Let $S_1(0)$, $S_2(0)$ and $S_3(0)$ be the segments at 0 with $\operatorname{dir} S_j(0) = j\pi/4$ for $j = 1,2,3$ and set $U = \bigcup_{j=1}^{3} S_j(0)$.

For positive integers n, define the sets

$$K_n = \{z : \operatorname{Im}(z) = 1/n, \ |\arg(z) - j\pi/4| \geq 1/n \text{ for } j = 1,2,3\}$$

and

$$R_n = \{z : -n \leq \operatorname{Re}(z) \leq n, \ 1/n \leq \operatorname{Im}(z) \leq n\}.$$

Also set $R_0 = \phi$.

Let g_0 be a continuous function from H into H which has the strong 3-segment property at 0 relative to the directions $\pi/4$, $\pi/2$ and $3\pi/4$. For each $j = 1,2$ and 3, choose an element

$a_j \in C_{g_0} S_j(0) \cap C_{g_0} S_{j+1}(0)$ where we consider $S_4(0) = S_1(0)$.

By a well-known theorem of Mergelyan [18], there exists a polynomial $P_1(z)$ such that

$$|P_1(z) - g_0(z)| < 1/2 \quad \text{on} \quad R_3 \cap U$$

and

$$|P_1(z) - a_j| < 1/2 \quad \text{on} \quad R_3 \cap K_j \quad \text{for} \quad j = 1,2,3.$$

Define $g_1(z)$ to be a continuous function in H such that

$$g_1(z) = P_1(z) \quad \text{on} \quad R_3,$$

$$g_1(z) = g_0(z) \quad \text{for} \quad Im(z) = 1/6 \quad \text{and} \quad z \in U,$$

and

$$|g_1(z) - g_0(z)| < 1/2 \quad \text{on} \quad U \cap [R_6 - R_3].$$

Suppose that we have defined continuous functions $g_0(z)$, $g_1(z),\ldots,g_{n-2}(z)$ in H and polynomials $P_1(z), P_2(z),\ldots,P_{n-1}(z)$ such that for each $k = 1,2,\ldots,n-2$

1) $g_k(z) = P_k(z)$ on R_{3k},

2) $g_k(z) = g_0(z)$ for $Im(z) = 1/(3k+3)$ and $z \in U$,

3) $|g_k(z) - g_0(z)| < 1/2^k$ on $U \cap [R_{3k+3} - R_{3k}]$

and for each $k = 1,2,\ldots,n-1$

4) $|P_k(z) - g_{k-1}(z)| < 1/2^k$ on $R_{3k-3} \cup (U \cap R_{3k})$, and

5) $|P_k(z) - a_j| < 1/2^k$ on $R_{3k} \cap K_{3q+j}$ for $q = 0,1,\ldots,k-1$ and $j = 1,2,3$.

Again by Mergelyan's theorem, if we define a continuous function on $g_{n-1}(z)$ in H such that conditions 1), 2) and 3) are

satisfied for $k = n-1$, there exists a polynomial $P_n(z)$ such that conditions 4) and 5) are satisfied for $k = n$. Hence we can obtain a sequence $\{P_k(z)\}$ of polynomials and a sequence $\{g_{k-1}\}$ of continuous functions such that all five of the above conditions are satisfied for each $k = 1,2,3,\ldots$.

It is readily verified that the sequence $\{P_k(z)\}$ converges uniformly on each R_{3k} and hence $f(z) = \lim_{k\to\infty} P_k(z)$ is a holomorphic function in H. Furthermore, $|f(z) - g_0(z)| \to 0$ as $z \to 0$ with $z \in U$ and for each $j = 1,2,3$, $|f(z) - a_j| \to 0$ as $z \to 0$ with $z \in \cup K_{3q+j}$ where the union is taken over all integers $q = 0,1,2,3,\ldots$ Then, since for each segment $S(0)$ at 0 with $\operatorname{dir} S(0) \neq j\pi/4$ $(j = 1,2,3)$ there exists a positive integer N such that $S(0) \cap K_n \neq \phi$ for each $n > N$, f is the desired function.

Remark. A construction similar to the one above enables us to show that given any integer $n \geq 2$ there exists a holomorphic function in H having the strong n-segment property at 0 which does not have the strong k-segment property at 0 for any other integer $k \geq 2$.

If f is a function from H into W, the principal cluster set of f at $p \in R$ is defined to be the set

$$\Pi(f,p) = \cap\, C(f,p,\sigma)$$

where the intersection is taken over all arcs σ at p.

THEOREM 11. Let f be an arbitrary function from H into W. If $\Pi(f,p) = \phi$ for some point $p \in R$, then either p is an ambiguous point of f or there exists a positive number h such that the diameter of $C(f,p,\sigma)$ is greater than h for all arcs σ at p.

Proof. Suppose there exists a sequence $\{\sigma_k\}$ of arcs at p for which diameter $[C(f,p,\sigma_k)] \to 0$. Without loss of generality, we may assume $C(f,p,\sigma_k) \to w$ for some $w \in W$.

Assume p is not an ambiguous point of f and let σ be an arc at p. Then, for each positive integer k, there exists a point

$$w_k \in C(f,p,\sigma_k) \cap C(f,p,\sigma).$$

Hence $w \in C(f,p,\sigma)$ since $C(f,p,\sigma)$ is closed and $w_k \to w$. Since σ was an arbitrary arc at p, we have $w \in \Pi(f,p)$ in violation of our hypothesis. Thus p is an ambiguous point of f and the theorem is proved.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. F. Bagemihl, Curvilinear cluster sets of arbitrary functions, Proc. Nat. Acad. Sci. U.S.A. 41(1955), 379-382.

2. F. Bagemihl, Some approximation theorems for normal functions, Ann. Acad. Sci. Fenn. Ser. A I 335(1963), 1-5.

3. F. Bagemihl, G. Piranian, and G.S. Young, Intersections of cluster sets, Bul. Inst. Politehn. Iaşi (N.S.) 5 (9), (1959), no. 3-4, 29-34.

4. H. Blumberg, A theorem on arbitrary functions of two variables with applications, Fund. Math. 16(1930), 17-24.

5. A.M. Bruckner and C. Goffman, The boundary behavior of real functions in the upper half plane, Rev. Roumaine Math. Pures Appl. 11(1966), 507-518.

6. E.F. Collingwood, Cluster sets and prime ends, Ann. Acad. Sci. Fenn. Ser. A I 250/6(1958), 1-12.

7. E.F. Collingwood and A.J. Lohwater, The theory of cluster sets, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press, 1966.

8. J.L. Doob, The boundary values of analytic functions, Trans. Amer. Math. Soc. 34(1932), 153-170.

9. P. Erdös and G. Piranian, Restricted cluster sets, Math. Nachr. 22(1960), 155-158.

10. W. Gross, Zum Verhalten der konformen abbildung am rande, Math. Z. 3(1919), 43-64.

11. V. Jarník, Sur les fonctions de deux variables réelles, Fund. Math. 27(1936), 147-150.

12. E. Lindelöf, Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme, Acta Soc. Sci. Fenn. 46(1915) (4), 1-35.

13. A.J. Lohwater, The cluster sets of meromorphic functions, Treizième Congrès des Mathématiciens Scandinaves, tenu à Helsinki 18 - 23 août 1957, pp. 171-177. Mercators Tryckeri, Helsinki, 1958.

14. J.E. McMillan, On local asymptotic properties, the asymptotic value set, and ambiguous properties of functions meromorphic in the unit disk, Ann. Acad. Sci. Fenn. Ser. A I 384(1965), 1-12.

15. J.E. McMillan, Principal cluster values of continuous functions, Math. Z. 91(1966), 186-197.

16. H.T. Mathews, A note on Bagemihl's ambiguous point theorem, Math. Z. 90(1965), 138-139.

17. H.T. Mathews, The n-arc property for functions meromorphic in the disk, Math. Z. 93(1966), 164-170.

18. S.N. Mergelyan, On the representation of functions by series of polynomials on closed sets (in Russian), Doklady Akad. Nauk SSSR, N.S. 78(1951), 405-408. Amer. Math. Soc. Transl. No. 85, Providence (1953).

19. G. Piranian, On a problem of Lohwater, Proc. Amer. Math. Soc. 10(1959), 415-416.

20. D.C. Rung, Boundary behavior of normal functions defined in the unit disk, Michigan Math. J. 10(1963), 43-51.

21. M. Schmeiser, Some properties of arbitrary functions, Fund. Math. 22(1934), 70-76.

22. G.S. Young, Types of ambiguous behavior of analytic functions, Michigan Math. J. 8(1961), 193-200.

MICHIGAN STATE UNIVERSITY LIBRARIES
3 1293 03057 7047