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COMPARISON OF NATIVE-ENGLISH AND NATIVE–KOREAN SPEAKING UNIVERSITY STUDENTS' DISCOURES ON INFINITY AND LIMIT

By

Dong-Joong Kim

A DISSERTATION

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ABSTRACT

COMPARISON OF NATIVE-ENGLISH AND NATIVE–KOREAN SPEAKING UNIVERSITY STUDENTS' DISCOURES ON INFINITY AND LIMIT

By

Dong-Joong Kim

This study investigated and compared how native-English and native-Korean speaking university students, who received their education respectively in the U.S. and in Korea, thought about the concepts of *infinity* and *limit*. The primary motivation for this study was the discontinuity in Korean and the continuity in English between the non-mathematical and mathematical discourses on *infinity* and *limit*: non-mathematical uses of the mathematical words infinity and limit appear only in English. Based on the communicational approach to cognition, according to which mathematics is a kind of discourse, the characteristics of students' discourse on the topics were identified. The participants' discourse was scrutinized with an eye to the common characteristics as well as culture- and education-related differences.

The setting for the study was a calculus class for university students in the U.S. and Korea. Methodology involved surveys and interviews. A total of 132 English speakers and 126 Korean speakers participated in the survey. Within each linguistically distinct group, twenty paired representatives were selected from the survey participants for follow-up interviews. The detailed discourse analyses of the interview transcripts generated preliminary hypotheses in the interviewees' discourse on *infinity* and *limit*. Overall analyses of response patterns involved searching for frequencies and percentages of the survey participants' responses and comparing the proportions of the two groups through chi-square analysis to confirm the emerging hypotheses. Data from both sources

were used to elaborate and verify hypotheses. In order to attribute the specific source of differences in mathematical discourses of the two groups, students' backgrounds which were likely to influence their mathematical discourses on *infinity* and *limit* were also investigated in this study.

It is concluded from this study that the mathematical discourse of U.S. English speakers seemed to be growing continuously from their non-mathematical discourse, but there was no such continuity in the case of Korean speakers. The use of the mathematical nouns *infinity* and *limit* by English speakers was processual, just as it was in their non-mathematical, colloquial uses, whereas Korean speakers' language was more formally mathematical and structural. This shows in developing their discourses on *infinity* and *limit*, English speakers seemed to build more on their previous everyday experience with discourses on *infinity* and *limit*, whereas Korean speakers had no choice but to rely on the structural discourse of mathematical textbooks. These differences in the use of the words *infinity* and *limit* between U.S. and Korean groups seemed to result in differences in *endorsed narratives* accepted by the students as true, *routines* they used in problem solving, and *visual mediators* they applied. In general, none of the mathematical discourses, either of English or Korean speakers, was fully compatible with the canonical mathematical discourse.

The results of this study provide strong support for the conjecture that non-mathematical discourse shapes mathematical discourse of students – a fact that should be kept in mind in both the research on advanced mathematical concepts and teaching advanced mathematics.

DEDICATION

For my parents, my wife, Noah, and Ian

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CHAPTER I

INTRODUCTION AND RATIONALE

The purpose of this study is to investigate and compare how native-English and native-Korean speaking university students who received their education in the U.S. and in Korea, think about the concepts of *infinity* and *limit*. Based on the *communicational approach to cognition*, according to which mathematics is a kind of discourse (Sfard, 1998), Sfard's frame for analyzing discourse is used to investigate students' thinking about infinity and limit. Infinity is the conceptual basis for mathematical topics such as the number line and infinite decimals. Since the 19th century the concept of limit has been essential to mathematical analysis. As known to teachers and as confirmed by researchers, most students have considerable difficulty with the notions of infinity and limit.

Since the mid 1960s large-scale international comparative studies have been carried out to investigate student learning. These studies show that East Asian students, such as those in China, Japan, and Korea, outperform their U.S. counterparts in mathematics. However, these large-scale studies do not examine the reasons for this disparity in the results. Nor do they provide a satisfactory method of helping students overcome their learning difficulties (Wang and Lin, 2005). In order to have a better understanding of why and how East Asian students outperform U.S. students, more grounded qualitative studies are needed. This study is an attempt to investigate students' thinking about infinity and limit using a different conceptual framework of learning — not from a cognitivist, *acquisitionist* perspective, but rather using a *participationist* view with attention to socio-cultural contexts.

The first basic assumption in this study is derived from the *communicational approach to cognition*. The communicational framework conceives of *thinking* as *communication with oneself*, of mathematics as a form of discourse, and of learning mathematics as a change in ways of communicating (Sfard, 2000). The second assumption is that when students are thinking, *meta-discursive rules* guide them to do it in a certain patterned way. Throughout discursive activity, routines as repetitive patterns are implemented with other features of mathematical discourses (uses of words, visual mediators, and endorsed narratives). The third important assumption is that when students come to the classroom to learn mathematical concepts they already have a certain amount of knowledge about these concepts that come from daily experience. This is based on the view that mathematical learning involves thinking with and through informal everyday language, generalizing everyday concepts associated with mathematical words, and abstracting those concepts with schooling.

This study attempts to answer the questions raised by these assumptions. One specific question about learning mathematics as changing mathematical discourse is: What are the leading characteristics of non-mathematical and mathematical uses of *infinity* and *limit* both between and within each of two linguistically distinct groups of university students? Because mathematical discourse may have culture-specific characters, two critical questions derived from the third assumption include: "What are the similarities and differences between native-English and native-Korean speakers in their former mathematics education and curricular experience with the words *infinity* and *limit*?" and "What are the similarities and differences between native-English and native-

Korean speakers in their non-mathematical and mathematical discourses on *infinity* and *limit*?"

This study may be of significance to the larger field of mathematics education for several reasons. First, discourse analysis may lead to methods for helping students overcome their difficulties with the central concepts of infinity and limit, with important implications for teacher education and K-16 curriculum. It may provide insight into the role of language in mathematical thinking in general and into the basis of the common phenomenon known as misconceptions in particular. Second, such investigations will address some previously unanswered questions such as "What is the role of the knowledge that students bring to the classroom from out-of-school settings?" and "Why do people have different uses for what they learned in different contexts?" Third, the communicational approach may prove fruitful for investigating advanced mathematical thinking in other areas. Findings of this study regarding infinity and limit may enlighten us about the ways that students and teachers deal with other notions such as continuity, differentiability, and integration in calculus. Finally, applying this approach to culturally diverse groups of students may shed additional light on an issue of great importance, vigorously pursued currently by researchers, namely how culture affects student learning in mathematics.

CHAPTER II

THEORETICAL BACKGROUND

This chapter sets forth epistemology and history of the infinity and limit concepts to describe the genesis and evolution of both of the notions, as well as the interdependence between them. This chapter also includes a review of previous research on learning infinity and limit. In addition, literature reviews on two topics, colloquial use of language in learning mathematics and comparative studies of language in learning mathematics, are presented. Next, the communication approach to mathematical learning that guides this study and contributes to the research design is discussed. Finally, my pre-dissertation research as a pilot study is summarized.

Epistemology and History of the Notions of Infinity and Limit

The histories of the mathematical concepts of *infinity* and *limit* have been interwoven since their beginning. The story of infinity begins with the ancient Greeks. For the Greeks, infinity did not exist in actuality, but rather as a potential construct. Although there was the notion of bounded processes, there was no concept of limit as a concrete bounding entity.

In the Middle Ages, Christianity came to value infinity as a divine property. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find methods for calculating the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). Throughout the 18th century, calculus lacked firm conceptual foundations. At the end of the 18th century, mathematicians

became acutely aware of inconsistencies which plagued the theory of infinitesimal magnitudes.

Today's notion of limit emerged gradually in the 19th century as a result of attempts to remedy the uncertainties existing within mathematical analysis at that time. Cauchy and Weierstrass were pioneers of the movement toward a rigorous calculus. At this time, limit turned into an arithmetical rather than geometrical concept, as it was before, in the context of infinitesimals. Infinity was now actual rather than potential. In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of the infinite set.

Previous Research on Student Learning on Infinity and Limit

Various aspects of the learning about infinity and limit have been investigated over the last few decades. Anchoring their research in the analysis of the *mathematical structure* of the notions, Cottrill et al. (1996) report that there are two reasons for student difficulties with limits. One reason is the need to mentally coordinate two processes: $x \to a$, and $f(x) \to L$. The other is the need for a good understanding of quantification related to ε and δ. Borasi (1985) suggests several alternative rules about how to compare infinities based on students' intuitive notions (within this tradition, see also Cornu, 1992; Tall, 1992).

Other research has focused on *misconceptions* and *cognitive obstacles* related to infinity and limit. Fischbein, Tirosh, and Hess (1979) and Tall (1992) emphasized the role of intuition. One source of difficulty with the notion of infinity is the belief that a part must be smaller than the whole. Other researchers (Cornu, 1992; Davis and Vinner, 1986) stress the influence of language. Students may have had many life experiences with

boundaries, speed limits, minimum wages, etc. that involved the word "limit". These everyday linguistic uses interfere with students' mathematical understandings (Davis and Vinner, 1986). Przenioslo (2004) focuses on the key elements of students' *concept images* of the limits of functions. Still others have focused on *informal models* that act as cognitive obstacles (Fischbein, 2001; Williams, 2001). According to Williams, informal models based on the notion of actual infinity are a primary cognitive obstacle to students' learning.

Finally, some researchers address students' difficulties through the lens of the theory of actions, processes, objects, and schemas (APOS; see Weller, Brown, Dubinsky, McDonald, and Stenger, 2004). Weller et al. discuss the cognitive mechanisms of *interiorization*, *encapsulation*, and *thematization* that are used to build and connect actions, processes, objects, and schemas.

Previous Research on Use of Colloquial Language in Learning Mathematics

To investigate the use of language in learning mathematics, many researchers have examined several mathematical topics on the basis of three different approaches to everyday language. One of the approaches focuses on the *meanings* of everyday language and their effects on learning mathematics. Some research has shown that students' misconceptions are influenced by the meanings of everyday language in learning mathematics (Cornu, 1992; Davis and Vinner, 1986; Pimm, 1984; Raiker, 2002; Schwarzenberger and Tall, 1978). Students continue to rely on their misconceptions stemming from everyday language because even though the everyday and mathematical meanings of words are similar, their different uses in mathematics classrooms are significant. In the same vein, other research addresses the use of language in bilingual or

multilingual contexts (Clarkson, 1992; Gutierrez, 2002; Moschkovich, 1996; Setati, 2002). Bilingual students employ not only different uses of concepts, but also different perspectives on mathematical tasks.

Another approach stresses the effects of the *structures* of everyday language on mathematical learning (Bintz and Moore, 2002; Cotter, 2000; Leung, 2005). For instance, the difficulty of the patterns for counting in the English language (e.g., "1-ten 3" in the Asian number pattern instead of thirteen in English) interferes with American students' mathematical understandings of counting.

Most previous research on the role of colloquial language in learning mathematics has been grounded in the meanings and structures of everyday language. Comparatively, the approach used in this dissertation views mathematical learning as a *discursive* activity that employs mathematical objects including language as well as meta-level factors. Previous research based on the discursive approach to mathematical learning is not extensive (Anderson et al., 2004, Barwell, 2003; Ben-Yehuda et al., 2005). Based on the communicational framework, Ben-Yehuda and colleagues (2005) characterize the arithmetic discourses of two 18-year old girls with learning difficulties in terms of four features of mathematical discourse including routines as a meta-level factor. They claim that the use of colloquial arithmetic discourse in learning literate mathematical discourse is important.

Previous Research on Comparative Studies of Language in Learning Mathematics

There is a limited number of comparative studies on language in learning mathematics. Song and Ginsburg (1988) are among the earliest to conduct a comparative study between two languages in learning mathematics. They investigate how far

American and Korean preschool children can count. There are two different systems of number words in the Korean language. One is the everyday system, which is irregular like the English language number system, and the other is the academic system, which is regular like the Chinese number system. Song and Ginsburg report that the four year-old Korean had an initial difficulty with both counting systems (everyday and academic systems) and their ability to count was less developed than that of the American counterparts. However, after learning the academic system of counting, the six year-old Koreans attain a higher level of performance in counting than their U.S. counterparts. It is concluded that the everyday number system provides the Korean a cognitive advantage for understanding a base-ten structure at school (within the comparison of counting systems, see also Fuson and Kwon, 1992; Miura, Okamoto, Kim, Chang, Steere, and Fayol, 1994).

Few comparative studies focusing on the Chinese language have in mathematics been conducted (Han and Ginsburg, 2001; Li and Nuttall, 2001). One representative of experimental studies is a work by Han and Ginsburg (2001). They ask 48 judges to rate the clarity of 71 mathematical words in Chinese, English, and "Chinglish" which is an English equivalent to the Chinese words. Chinese mathematical words are rated higher in clarity than English mathematical words. Chinglish mathematical words also tend to be rated higher in clarity than English words. After that, a mathematics test including 71 mathematical words is given to three groups of Chinese-American eighth graders: 33 Chinese-only speakers, 29 bilingual speakers of Chinese and English, and 20 English-only speakers. The result of the test reveals that the Chinese-only speakers and the Chinese-English bilinguals outperform the English-only speakers. Therefore, Han and

Ginsburg hypothesize that the clarity of Chinese mathematics language may contribute to the better performance of the groups of the Chinese-only and bilingual speakers.

Communicational Approach to the Study of Mathematical Learning

The Need for an Additional Approach to Study the Learning of Infinity and limit

In spite of the mutual interdependence of the concepts of infinity and limit in the history of mathematics, there has been little research to examine students' understandings of and difficulties with both concepts simultaneously. For a better understanding of the notions of infinity and limit and of their relationships, more research about student learning on the concepts together is needed.

As influenced by the work of Piaget, mathematical learning has been considered as a process of active cognitive acquisition. Most previous research on the learning of infinity and limit has been grounded in the acquisitionist framework. The acquisitionist tenets have brought important insights (e.g., cognitive structures of infinity and limit, students' misconceptions, and the hierarchical relationships between categories in these concepts) into mathematical learning and teaching on the basis of the three different cognitive approaches to the learning of infinity and limit. The acquisitionist research, however, has not led to satisfactory solutions to account for inter-personal and cross-situational differences and the source of human development (Sfard, 2006). They also underestimate not only the inherently social nature of student thinking, but also the role of discourse and communication in learning and in other intellectual activities.

Most previous research on the use of colloquial language in learning mathematics has not moved beyond attempts to characterize the meanings and structures of everyday language and their impact on mathematical learning. Some researchers in the discursive

view on mathematics learning reported mathematical object-level and meta-level factors (such as patterns of attention, and routines) in mathematical discourse. However, previous discursive research focused only on primary and secondary school mathematics. Previous comparative studies emphasizing the importance of language in learning mathematics have also been conducted mainly with primary and secondary mathematics. Thus, there has been little comparative research on the role of colloquial discourse in learning mathematics at the post-secondary level. More comparative studies on advanced mathematical topics using the discursive approach to mathematical learning are warranted for a better understanding of how students think in general.

Communicational Approach

While Piaget focuses on the individual development, Vygotsky concentrates on the social activity of the mind as a starting point. From Vygotsky's point of view, speech for communicating with others comes before internal speech in children's internalization of higher mental processes. The claim is vividly made in the following excerpt:

> Every function in the child's cultural development appears twice: first on the social level, and later, on the individual level; first between people (inter-psychological), and then inside the child (intra-psychological). (Vygotsky, 1978, p.57)

Thinking arises from an individualized version of interpersonal communication because of the inherently social nature of human activities. Higher mental processes are developed through the procedures of internalization from public speech to inner private speech, and are tightly related to tool-mediated activity (Vygotsky, 1986). For instance, symbols and mathematical language are important tools in mathematics. The properties of such tools are inseparable from the cognitive processes of the uses of the tools (Rogoff, 1990). Thus, thought and language are inseparable; so are thought and symbols. Thought must be transferred through meanings and only then through words as a tool.

According to Vygotsky (1986, p.256), "the word is a direct expression of the historical nature of human consciousness". In other words, consciousness can be investigated as it is expressed in words.

In the communication approach, thinking is a special form of communication which is the process of internalization from public speech to inner private speech. Because thinking arises from an individualized version of interpersonal communication, it is simply communication with oneself and thus learning is a change in ways of communicating (Ben-Yehuda et al., 2005). In particular, learning mathematics is seen as becoming more skillful and articulate in mathematical discourse (Sfard, 2001a).

The word *discourse* signifies any type of communicative activities, whether with others or with oneself, whether verbal or not, whether synchronic (like in a face-to face conversation) or asynchronous (like in an exchange of emails or reading a book) (Sfard and Lavie, 2005). A discourse is regarded as *mathematical* if it deals with mathematical objects, such as infinity and limits (Ben-Yehuda et al., 2005). When we use mathematical words such as limit in an everyday context, we call it a colloquial use in colloquial discourse. Thus, to discover the mechanisms of mathematical thinking, the discourses that students use in both the contexts of everyday life and mathematics are important because they can be used not only to express thinking processes, but to reveal what students already know.

Whenever mathematical discourses are being analyzed, compared, and watched for changes over time, four distinctive features are often considered: *words and their use*, *mediators and their use*, *endorsed narratives*, and *discursive routines* (Ben-Yehuda et al.,

2005; Sfard, 2001b; Sfard and Lavie, 2005). In this study, all four features are considered to identify characteristics of mathematical discourses.

Uses of words are the ways the participants use keywords of colloquial and mathematical discourse regarding *infinity* and *limit*. *Word uses* are important as a feature of colloquial and mathematical discourses because they can reveal how students understand those words. For instance, how students use their mathematical words can be assessed by the degree of objectification. The degree of objectification will be assessed according to the frequency with which students speak about infinity and limit, as if these words referred to self-sustained, discourse-independent objects. The degree of objectification can indicate, among others, how students make the use of the words *infinity* and *limit* similar to the discourse on material objects such as apples or atoms.

Discursive routines are the patterns of repetitive actions in students' discourses. For example, we can observe such patterns in the mathematical discourses of comparing, counting, and defining. When students mention mathematical use of keywords, certain discursive routines can also be detected. Therefore, discursive routines are the central concept in discourse analysis. Routines are a set of *meta-rules*. When we consider meta-rules, two important aspects need to be emphasized. One of them is *when* the routine procedure is used and the other is *how* it is implemented. *When* refers to the cues for beginning discursive routines. *How* refers to the kinds of patterns which exist in discursive routines.

Endorsed narratives are propositions that students accept as true. A narrative endorsed by an interviewee does not have to be a quotation from the interviewee. It may be an interpretation of assumptions that tacitly guides him or her. For instance, the

expression "Limit is something ..." can be inferred from statements actually made by students. Thus, through discursive activity, students can produce *endorsed narratives.*

Visual mediators are means that the participants use for their communication activity. While colloquial discourses are communicated with the help of concrete objects or individual images, mathematical discourses are mediated by informal and formal symbols. For example, to identify infinity in mathematical discourse, students can use and refer to such mediators as an infinity symbol (∞), three dots (...), and an "eight sideways" (∞) in multiple ways.

The Question to be Answered in This Research

American and Korean students differ considerably in their mathematics achievements, as shown in large-scale international comparative studies. Difference in language is one of many cultural differences between these two groups that are relevant to understanding the relationship between mathematics learning and culture. Therefore, the difference in relations between colloquial and mathematical discourses on infinity and limit in Korean-speaking and English-speaking students are explored in this study.

The Pilot Study

A pilot study was conducted in preparation for this study (see Kim, 2006). I interviewed four American and four Korean students in fall 2004. Each ethnically distinct group included one elementary school student, one middle school student, one high school student, and one undergraduate. The participants within the same grade level (elementary, middle and high schools, and college) were selected based on the criteria of the same age, grade, and educational institution. Their discourse on *infinity* and *limit* was scrutinized with an eye to common characteristics as well as culture-, age-, and

education-related differences. Because the interviews were conducted in English, the four Korean students who were selected had been living in the United States and attending US schools for more than three years. The interview questionnaire consisted of 29 questions in eight categories. The first two categories aimed at scrutinizing students' *colloquial* discourses on infinity and limit, whereas the remaining six categories were targeted at investigating students' *mathematical* discourses on the topics (see Appendix A for the full questionnaire). The interviews were audio- and video-taped and then transcribed in their entirety for further analysis. Data were analyzed so as to identify and describe the four distinctive features of the respondents' discourses: *word use*, *discursive routines*, *mediators*, and *endorsed narratives*.

One significant finding was that the Korean and American students used *infinite* and *infinity* in different ways, and used *limited* and *limit* in similar ways in both colloquial and mathematical discourses. There were few noticeable differences between the American and Korean groups in the colloquial and mathematical discourses on infinite and infinity (e.g., the *element-based* approach to the word *infinity* in the Korean group versus the *set-based* approach in the American group). However, there was no systematic difference between the American and Korean students in their discourses on limited and limit. Another important finding was that there were no significant systematic differences with age in the colloquial and mathematical discourses on *infinity*. In contrast, some age-related characteristics of the students' *word use* and *endorsed narratives* on *limit* were found in the study (e.g., *objectified word use* as a particular number and the endorsement of limit as an unreachable object for the older students).

Although the sample size of the pilot study was too small to allow for generalization, the findings served as a basis for the hypotheses that were tested in this study. According to the results of the pilot study, colloquial discourse seems to have an impact on mathematical discourse as evidenced by certain clear differences between the mathematical discourse on *infinity* of the American and the Korean students. These differences can be ascribed to the fact that only in English do the mathematical words infinity and infinite (as well as *set* and *element*) appear in colloquial discourse. The absence of systematic differences between the two ethnic groups in their discourses on *limited* and *limit* might be ascribed to the fact that unlike the word infinity, the words limit and limited are quite commonly used in English colloquial discourses. Due to the fact that all the Korean students had been living in the United States and attending US school for more than three years, they may have been influenced by the colloquial and mathematical English discourses on infinity and limit. Therefore, in the present study, native Korean students, who have not experienced English colloquial and literate discourses on infinity and limit, will be chosen and interviewed about their discourses on infinity and limit *in Korean.*

Summary

The interconnectedness between infinity and limit in the history of mathematics justifies this study's focus on both *infinity* and *limit* simultaneously. Because of the inherently social nature of student thinking, previous cognitive approaches to the learning difficulties of infinity and limit provide justification for the need for discourse analysis as a research methodology, with the promise of answering some previously unanswered questions within the traditional cognitive framework. Literature reviews on the use of

colloquial language and comparative studies of language in learning mathematics justify the need for comparative studies of language on learning advanced mathematics at the post-secondary level to better understand how students think in general. As noted earlier, communication is of principal importance in learning mathematics as a tool-mediated activity. These different perspectives as well as the pilot study laid the groundwork for the design of this study.

CHAPTER III

DESIGN OF STUDY

This study investigated how native-English and native-Korean speaking university students thought about the concepts of *infinity* and *limit* and compared their colloquial and mathematical discourses. The method chosen to fulfill this purpose was the *communicational approach to cognition*, according to which mathematics is a kind of discourse. This chapter describes the research questions and method. In addition, connections are drawn between the pilot study and certain methodological choices for this study.

Research Questions

The purpose of this study, to characterize how students think about infinity and limit, led to the following research questions:

- What are the relationships between colloquial and mathematical uses of the English words infinity and limit? What are the relationships between colloquial and mathematical uses of the corresponding Korean words?
- What are the similarities and differences between native-English and native-Korean speakers in their previous mathematics education experiences, their learning habits and their curricular experience with the words *infinity* and *limit*?
- What are the similarities and differences between non-mathematical and mathematical discourses on *infinity* and *limit* of native-English and native-Korean speakers?

Overview of the Method

Two groups of university students (native-English and native-Korean speakers) participated in the study. The comparisons of these two groups were based on analyses of students' responses to a survey questionnaire and discourse analyses of paired interviews. After collecting the survey data, I first analyzed and categorized 132 English-speaking and 126 Korean-speaking university students' written responses to the survey within each

group, and chose their representatives from each group for the interview study. Twenty representatives in each group were interviewed in pairs, and discourse analyses on those interview data was conducted on the basis of the communicational framework. Finally, the written responses to mathematics problems on the survey were reanalyzed to make conjectures about the generality of characteristics observed in interviews.

Method

Context

Sites

The two sites, the United States and Korea, were chosen to investigate the three research questions (see Appendix B for an overview of the educational systems at the two sites). The specific reason for the selection of native-English and native-Korean speakers is the discontinuity in Korean and the continuity in English between the colloquial and mathematical discourses on infinity and limit. For the sake of the clarity in the following discussion, let us denote the colloquial and mathematical words for infinity and limit as specified in Table 3.1.

Table 3.1 Colloquial and mathematical uses of infinity and limit in English and Korean

	English		Korean	
	Infinity	Limit	Infinity	Limit
Colloquial	$\text{Infinity}_{\text{EColl}}$	$\text{Limit}_{\text{EColl}}$	$\text{Infinity}_{\text{KColl}}$ 한없음 (han-up-um), 무수(*moo-soo*),...	$\text{Limit}_{\text{KColl}}$ 제한 (*jae-han*), 경계 (*kyung-gae*), ...
Mathematical	$\text{Infinity}_{\text{EMath}}$	$\text{Limit}_{\text{EMath}}$	$\text{Infinity}_{\text{KMath}}$ 무한 (*mu-han*) and 무한대 (*mu-han-dae*)	$\text{Limit}_{\text{KMath}}$ 극한 (*gŭk-han*) and 극한값 (*gŭk-han-gab*)

As can be seen from the table, the relation between the colloquial and mathematical words may now be presented as follows:

$$\text{Infinity}_{\text{EColl}} = \text{Infinity}_{\text{EMath}}, \text{ whereas } \text{Infinity}_{\text{KColl}} \neq \text{Infinity}_{\text{KMath}}$$

$$Limit_{EColl} = Limit_{EMath}, \text{ whereas } Limit_{KColl} \neq Limit_{KMath}$$

Here, the equality symbol '=' means that the words *sound* the same, *not* that their use is the same. $Infinity_{KColl}$ and $Limit_{KColl}$ signify the colloquial Korean words that can count as rough translations of $Infinity_{EColl}$ and $Limit_{EColl}$, respectively.

In English, the words *infinity* and *limit* appear in colloquial language and students are likely to be familiar with the terms before they encounter them in a formal mathematical context. Even young children are likely to be acquainted with expressions such as "infinity of problems" or "speed limit" that feature the same words which they will latter learn in its mathematical version. This is not the case for Korean children. The formal mathematical words for infinity and limit learned in school or university do not originate in colloquial discourse and the odds are that when the Korean children hear the Korean mathematical words for infinity and limit ($Infinity_{KMath}$ and $Limit_{KMath}$) for the first time, these words (the sounds) are completely new for them. Later, these words may make their way into their colloquial discourse, but this is irrelevant to this study.

This difference between the Korean and American children's linguistic experiences may be consequential. One can hypothesize that the early colloquial uses of the English words *infinity* and *limit* would influence, even possibly interfere with, the way students learn the formal mathematical use of these words. The pilot study brought an initial corroboration of this thesis when it showed, for example, that English speakers tend to use $limit_{EMath}$ the way mathematicians use the term "upper boundary" – the use that corresponds to the $limit_{EColl}$, as appearing in the colloquial expressions such as "speed limit" or "There is a limit to my patience." Since there is no colloquial counterpart to the

$limit_{KMath}$, it is reasonable to assume that there will be no such bias in the Korean students' use of $limit_{KMath}$ (the use of the Korean mathematical counterpart of $limit_{EMath}$). This linguistic hypothesis, if it is empirically corroborated, may explain at least some of the phenomena known as "misconceptions about infinity and limits" and, more generally, may shed light on the linguistic mechanisms that produce such phenomena. Thus, I tried to examine the possible impact of the discontinuity in Korean and the continuity in English on students' mathematical thinking about infinity and limit.

Research Sites

A major university in the capital city of each site, Michigan and Korea, was selected for collection of data for this study. The Korean participants were selected from Korea University (KU). KU is located in Seoul, the capital city of Korea. KU is a private university with an enrollment of approximately 20,000 undergraduate and graduate students. Almost all the students in KU are Korean and international students comprise less than 3% of the total student population. At KU the Department of Mathematics offers only one calculus course. All freshmen whose major was science took the calculus course as a requirement. If students took calculus as one of the four elective courses in high school, they were expected to have some background in calculus. All calculus classes were taught by faculty and instructors in the Department of Mathematics. Each class size was under 60 students. *Thomas' Calculus* (Thomas, 2005) was used as the textbook.

The U.S. participants were selected from Michigan State University (MSU). MSU is located in East Lansing, three miles east of Michigan's capitol in Lansing. MSU is a public university with approximately 45,000 undergraduate and graduate students. More

than 7% of the total student body is international students. There were more than 200 programs of study. MSU students came from all 83 counties in Michigan, all 50 states in the United States, and about 125 other countries[1]. All students whose major was natural science or engineering at MSU were required to take the calculus course MTH 132 as a requirement. If students had taken Pre-calculus as an elective course in high school, they might come to the university class with a background in infinity or limit. Faculty in the Department of Mathematics taught all MTH 132 classes. Each class size was under 35 students. In all MTH 132 calculus classes, the same Thomas' calculus book was used. The characteristics of the two universities, MSU and KU, are summarized in Table 3.2.

Table 3.2 Summary of the characteristics of MSU and KU

	MSU	KU
Location	East Lansing, MI	Seoul, Korea
Number of students	About 45,000	About 20,000
International students	More than 7% of the total	Less than 3% of the total
Calculus as a requirement	Students whose major is natural science or engineering	Students whose major is science
Each calculus class size	Under 35 students	Under 60 students

Opportunities for Learning on Mathematical Infinity and Limit

International comparative studies are meaningful and useful when they include a good alignment of achievement and opportunity to learn (see Wolfe, 1999 for positive correlations between achievement and opportunity profiles in the Second International Mathematics Study (SIMS) data and the International Assessment of Educational Progress (IAEP) data). In order to analyze students' opportunities to learn infinity and limit, I used student responses about the textbooks they had used for learning infinity and limit as well as policy documents such as state and national standards.

[1] From the website, http://newsroom.msu.edu/snav/184/page.htm

1) Opportunities of native-English speakers in the United States

Characterizing students' opportunity to learn about infinity and limit is complex in the United States because different curricula are used across schools and universities and this implies different experiences. However, Michigan's Grade Level Content Expectations (GLCE) for grades K-8 and High School Content Expectations (HSCE) in mathematics can be used to address students' expected opportunity in general, in Michigan[2]. In order to describe what students should know and be able to do in mathematics, the Michigan's GLCE addresses content standards categorized into the five strands at each grade level. The five strands consist of the following: 1) number and operations; 2) algebra; 3) measurement; 4) geometry; and 5) data and probability. The Michigan's HSCE describes content standards in each course (Algebra I, Algebra II, Geometry, Pre-calculus, and Statistics and Probability) on the basis of the four strands: 1) quantitative literacy and logic; 2) algebra and functions; 3) geometry and trigonometry; and 4) statistics and probability.

Michigan's GLCE prescribe that non-terminating and non-repeating decimals are assessed at grade 8.[3] In algebra I, students are expected to learn about recursively defined functions, the behavior of a function as x approaches infinity, and asymptotic behaviour at infinity[4]. In algebra II, students are taught infinite geometric sequences and asymptotes

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[2] From the website, http://www.michigan.gov/mde/0,1607,7-140-28753---,00.html

[3] N. ME.08.03: Understand that in decimal form, rational numbers either terminate or eventually repeat, and that calculators truncate or round repeating decimals; locate rational numbers on the number line; know fraction forms of common repeating decimals, e.g., $0.\bar{1} = \frac{1}{9}$; $0.\bar{3} = \frac{1}{3}$.

N. ME.0804: Understand that irrational numbers are those that cannot be expressed as the quotient of two integers, and cannot be represented by terminating or repeating decimals; approximate the position of familiar irrational numbers, e.g., $\sqrt{2}$, $\sqrt{3}$, π, on the number line.

[4] **A2.1.5** Recognize that functions may be defined recursively. Compute values of and graph simple recursively defined functions (e.g., $f(0) = 5$, and $f(n) = f(n\text{-}1) + 2$).

of rational functions[5]. If students take pre-calculus, they would be exposed to an opportunity to learn about the limit of a function as x approaches a number or infinity as well as infinite geometric series[6].

2) Opportunities of native-Korean speakers in Korea

Unlike in the United States, similar curricula are used across schools in Korea due to a centralized educational system. The mathematical word 무한 (*mu-han*) for *infinity* without using the Chinese characters is first used in Korean schools to introduce the concept of infinite sets in the 7th grade school curriculum (see Appendix B for the general development of the Korean language). As explained in the appendix, one thing to note is that Chinese characters show meanings and etymologies of words. In a 7th grade textbook, a set is defined as a collection whose objects are clearly delineated. In the same textbook, an element is then described as a member of a set. After the introduction, 무수히 (*mu-soo-hee*) as a Korean colloquial word without using Chinese characters is used to define an infinite set. In the word *mu-soo-hee*, the meaning of the first character

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A2.1.6 Identify the zeros of a function and the intervals where the values of a function are positive or negative. Describe the behavior of a function as x approaches positive or negative infinity, given the symbolic and graphical representations.
A2.3.1 Identify a function as a member of a family of functions based on its symbolic or graphical representation. Recognize that different families of functions have different asymptotic behavior at infinity and describe these behaviors.

[5] ***L2.2.4** Compute sums of infinite geometric sequences.
A2.9.2 Analyze graphs of simple rational functions (e.g., $f(x) = \frac{2x+1}{x-1}$; $g(x) = \frac{x}{x^2-4}$) and understand the relationship between the zeros of the numerator and denominator and the function's intercepts, asymptotes, and domain.

[6] **P1.7** Understand the concept of limit of a function as x approaches a number or infinity. Use the idea of limit to analyze a graph as it approaches an asymptote. Compute limits of simple functions (e.g., find the limit as x approaches 0 of $f(x) = 1/x$) informally.
P8.4 Compute the sums of infinite geometric series. Understand and apply the convergence criterion for geometric series.

(*mu*) is none or nothing and the second character (*soo*) means number or count and the last character is a Korean suffix to make the word *mu-soo* as an adverb. Thus, an infinite set is introduced as a set which includes countlessly many elements using a Korean colloquial word *mu-soo-hee*. In the 10th grade mathematics curriculum, the other mathematical term 무한대 (*mu-han-dae*) for *infinity* and its symbol (∞) are introduced along with the concept of infinite sequence, as well as the new symbol ($n \to \infty$: as '*n* is getting bigger endlessly').

As in the case of infinity, there are also two Korean mathematical words for limit only written with Korean letters: 극한 (*gŭk-han*) and 극한값 (*gŭk-han-gab*). The meaning of the first character (*gŭk*) is an extreme and the second character (*han*) means bound. The word 값(*gab*) as a pure Korean word means value or amount. Thus the first mathematical word *gŭk-han* for limit means "the utmost boundary." According to a Korean dictionary, the Korean word *gŭk-han* is equivalent to the word *gŭk-han-gab*. In the dictionary, the second mathematical word *gŭk-han-gab*, used only in mathematics, is defined as "a value to which a function approaches, as independent variables approach to a number endlessly." The word *gŭk-han* for *limit* without using Chinese characters is first introduced to signify the limit of a sequence in the 10th grade mathematics curriculum. By the end of grade 12, students whose major will be science are expected to learn about the limit of a function, the limit of an infinite sequence, and sums of infinite series, whereas students who will study liberal arts at a university are not taught those concepts.

Opportunities of native-English and native-Korean speakers in the United States and Korea respectively for learning about infinity and limit are different. For native-English speakers in the U.S., the colloquial use of the English words infinity and limit precedes the mathematical use because native-English speakers first experience colloquial meanings of those words in everyday language and later they experience the mathematical meanings in high school. However, for native-Korean speakers in Korea, the mathematical words infinity and limit are first introduced at grades 7 and 10 respectively.

Participants

Students from several calculus classes at MSU and KU were recruited, yielding a sample of 132 English-speaking and 126 Korean-speaking university students for the study. Only MTH 132 calculus classes at MSU were included in this study because the content in MTH 132 was equivalent to that of the calculus course at KU in terms of using the same textbook (Thomas, 2005). All freshmen who majored in science were required to take the calculus course at KU, whereas most students who studied natural science or engineering took the MTH 132 calculus class at MSU. In Spring 2007, when the data were gathered, about 1740 students distributed in 29 sections were taking the calculus class at KU, whereas 533 students in 16 sections were registered for the MTH 132 class at MSU. Of those students, almost all the students at KU and about 70 percent of the total students at MSU were freshmen.

Out of the undergraduate students who volunteered to participate, 20 students in each site were selected for paired interviews on the basis of their responses to the survey. The selection process was designed to maximize difference in degrees of exposure to and

experience with the concepts of infinity and limit throughout their formal education as well as to include students whose responses to the questions investigating colloquial and mathematical discourses on these concepts fell into various categories (see the section of procedures for collecting data about how to select 20 interviewees in detail). As the results of the pilot study (Kim, 2006) suggested, native Korean students who *have not studied mathematics in English or any other language* were chosen and interviewed *in Korean* regarding both their colloquial and literate discourses on infinity and limit. I selected interview subjects who had no experience in any language other than their native language in learning mathematics. Data from these native-Korean speakers might help explain the characteristics of their discourses described in the first research question. In a similar way, native-English speakers, who had not studied mathematics in Korean or any other language, were chosen and interviewed in English.

Data

Questionnaire

The primary instrument for data collection was a questionnaire (see Appendix C for the entire questionnaire, with both the English and Korean versions). The questionnaire consisted of two sections: *Section I, Background*; and *Section II, Discourse*. Section I aimed at examining students' linguistic background, formal mathematics education, and curricular experience with the concepts of infinity and limit. Section II was composed of questions regarding creating sentences with infinity and limit and solving mathematical problems about these concepts. The discourse questionnaire (Section II) was piloted and subsequently revised. The questions from the discourse questionnaire were used for paired interviews to scrutinize students' non-mathematical and mathematical discourses

on infinity and limit. Because the questionnaire was used in English and Korean, three mathematics educators and one linguist who were all Korean-English bilinguals were selected to evaluate the Korean translations of the questions into English.

To address the first research question (about the leading characteristics of non-mathematical and mathematical discourses on infinity and limit), the problems for investigating non-mathematical and mathematical discourses in the discourse questionnaire were open-ended, providing the participants with an opportunity to freely express their thinking. Responses to open-ended questions permitted an investigation of students' thought processes. Because of the nature of the questions focusing on specific features, only a subset of the four features (word use, endorsed narratives, visual mediators, and routines) in mathematical and non-mathematical discourses was dealt with within each question. Then, common characteristics (in terms of the four features) in discourse analysis on the interviews of ten pairs of native-Korean speakers were compared and contrasted to those of their native-English counterparts.

To answer the second and third research questions (about the differences between native-English and native-Korean speakers in terms of their colloquial and mathematical discourses, formal mathematics education, and curricular experience), I hypothesized about relationships between characteristics of mathematical and colloquial discourses, formal mathematics education, and curricular experience with infinity and limit. Finally, I compared and contrasted the characteristics in these relationships between native-English and native-Korean speakers.

The background questionnaire (*Section I*) included 21 questions mainly asking about students' linguistic background, formal education, and the textbooks that they used when

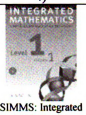
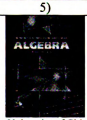
learning infinity and limit. Questions 1 and 2 ask about students' year and native language. Question 5 is aimed at examining whether students have studied mathematics in any language other than their native language. Questions 6 through 9 ask in which grade and in which mathematics course students had first heard the mathematical words of infinity and limit. Questions 1 through 9 are shown in Figure 1.

1. Year: □ Freshman □ Sophomore □ Junior □ Senior
2. Native Language: □ Korean □ English □ Other (Specify):
3. How often do you speak your native language at home?
   □ Always □ Almost always □ Sometimes □ Never
4. About how many books are there in your home? (Do not count magazines, newspapers, or your school books.)
   □ None or very few (0-10 books) □ Enough to fill one shelf (11-25 books)
   □ Enough to fill one bookcase (26-100 books) □ Enough to fill two bookcases (101-200 books)
   □ Enough to fill three or more bookcases (more than 200 books)
5. Have you studied mathematics in any language other than your native language?
   □ Yes □ No
   If yes, specify the languages in which you learned mathematics: _____
6. Can you remember the first mathematics course in which you heard the word (mathematical) *infinity*?
   □ Yes □ No
7. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *infinity*? Grade(s):_____ Course:_____.
8. Can you remember the first mathematics course in which you heard the word (mathematical) *limit*?
   □ Yes □ No
9. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *limit*?
   Grade(s):_____ Course:_____.

Figure 1: Questions 1 through 7 from the background questionnaire

Questions 10 through 14 were intended to investigate students' different levels of exposure to and experience with infinity and limit in their formal education. Question 10 regarding the textbooks which students used in high school is included. A set of widely used textbooks was provided to the students. In question 11, the titles of sequences within each curriculum were used to investigate different levels of courses that students took in high school. Because students may take advanced mathematics courses other than the examples in question 11, question 12 was targeted to examine the most advanced mathematics course they took in high school. Questions 11 and 12 are included to investigate the topics in which students may have learned about infinity and limit in their

10. Please specify from which high school textbook(s) series you studied among the examples below. If you used textbooks other than those in the examples, please check the box for *Others*.

1)	2)	3)	4)
			
Core-Plus Mathematics Project (CPMP)	Mathematics Connections	Interactive Mathematics Program (IMP)	SIMMS: Integrated Mathematics
5)	6)	7)	*Others*
			
The University of Chicago School Mathematics Project (UCSMP)	Mathematics: Modeling Our World	Prentice Hall	

11. In high school, which course(s) in the series below did you take? Please check all the boxes that apply to you (under the textbook series you used).

CPMP	Math Connections	IMP	Integrated Math
☐ Course 1	☐ Algebra	☐ Year 1	☐ Level 1
☐ Course 2	☐ Geometry	☐ Year 2	☐ Level 2
☐ Course 3	☐ Advanced Algebra	☐ Year 3	☐ Level 3
☐ Course 4		☐ Year 4	☐ Level 4

UCSMP	Mathematics: Modeling Our World	Prentice Hall	*Others*
☐ Transition Math	☐ Course 1	☐ Pre-Algebra	
☐ Algebra	☐ Course 2	☐ Algebra 1	
☐ Geometry	☐ Course 3	☐ Geometry	
☐ Advanced Algebra		☐ Algebra 2	
☐ Functions, Statistics and Trigonometry			
☐ Precalculus and Discrete Math			

12. In high school, what is the most advanced mathematics course you took other than those in #9?

13. Please check the box(es) next to topics which you have learned. Where and when did this learning occur?

- ☐ Infinite processes
- ☐ Infinite sequence or series
- ☐ Comparison of infinite sets
- ☐ Infinitely small or infinitesimal
- ☐ Infinite decimal fractions
- ☐ The sum of infinite geometric series
- ☐ Countable and uncountable sets
- ☐ Other topic (Describe):

14. Please check the box(es) next to topics which you have learned. Where and when did this learning occur?

- ☐ Limits to infinity
- ☐ Limit of a sequence or series
- ☐ Limit of a function geometrically
- ☐ Limit and the definition of the derivative
- ☐ The ε-δ (epsilon-delta) definition of limit
- ☐ Limit of a function numerically
- ☐ Limit and the definition of continuity
- ☐ Other topic (Describe):

Figure 2: Questions 10 and 14 from the background questionnaire

former education. Seven possible responses to each question are organized in a hierarchy and specified at several different levels with regard to the degrees of difficulties of infinity and limit. Questions 10 and 14 from the survey are shown in Figure 2.

In addition to examining students' former mathematics education and curricular experience with infinity and limit, the background questionnaire was used to investigate their current mathematics learning and attitudes toward mathematics. Questions 15 and 16 were intended to examine students' view of themselves as mathematics learners and the role of mathematics in their lives respectively. Questions 17 through 21 were aimed at investigating students' self-report on their learning of mathematics, out-of-school activities, extra lessons and homework (see Appendix C for details).

The discourse questionnaire (*Section II*) consisted of 16 questions organized into eight items (see the section on data analysis for how the questions connected the four features in non-mathematical and mathematical discourses). Open-ended questions were employed so that the participants had an opportunity to freely express their thinking. Students' responses to open-ended questions and the method of paired problem solving reduced interviewer's effects and biases on interviewee's responses. The first item aimed at scrutinizing students' colloquial discourses on infinity and limit. The next seven items were designed to investigate students' mathematical discourse on the topic. The distinction between non-mathematical and mathematical discourses in this study is important in this investigation of students' thinking about infinity and limit. Students' discourse on infinity and limit in response to items II through VIII was regarded as mathematical, not only because the questions framed the discourse as mathematical (and thus the student volunteers agreed to participate in this discourse), but also because the

participants did it by adjusting their colloquial discourse to the rules of school mathematical discourse.

The descriptions of each questionnaire item are as follows. In the first item, students were asked to create a sentence for a given word to investigate their non-mathematical discourse on infinity and limit. In the Korean survey questionnaire, some representative words of $\text{infinity}_{\text{KColl}}$, $\text{infinite}_{\text{KColl}}$, $\text{limit}_{\text{KColl}}$, and $\text{limited}_{\text{KColl}}$ were used. The questions in the first item are shown in Figure 3.

I. (First Item) Create a sentence with the following word (term).	
1. (Colloquial) Limited	2. (Colloquial) Limit
3. (Colloquial) Infinite	4. (Colloquial) Infinity

Figure 3: The questions in the first item

The remaining seven items were designed to investigate students' mathematical discourses on infinity and limit. In the second item, students were asked to compare two sets (i.e., fingers and toes, odd numbers and even numbers, and odd numbers and integers). The aim of questions in the second item was to examine whether the routines for comparing finite sets were the same as the routines for comparing infinite sets. The infinite cardinalities in the second item are one of the most important concepts in infinity. Different degrees of infinity have been controversial in the history of mathematics. Cantor himself could not believe the different cardinalities between rational and real numbers when he first created them. Thus, questions involving comparisons of infinite sets have been one of the main methods for investigating student thinking about mathematical infinity (Borasi, 1985; Fischbein, Tirosh, and Hass, 1979; Tsamir and Dreyfus, 2002).

In the third item, students also were asked to compare integers with even numbers. Unlike the second item, integers and even numbers were presented horizontally in order

to investigate students' approaches to a contradiction. If students used the part-whole comparison in the second item, they may have reached a contradiction as they saw no consistency between part-whole and one-to-one correspondence comparisons in different representations of infinite sets. The third item was intended to raise a contradiction and promote student's awareness of their contradictory conclusions when comparing infinite sets (Tsamir and Dreyfus, 2002). Thus, the third item could reveal meta-discursive rules in the ways of viewing a contradiction on the process of mathematical problem solving.

The fourth item was targeted to examine students' conceptions of the word *larger* in comparing infinite sets. Students may use different criteria in order to claim that one infinite set A is larger than the other infinite set B. As shown in the pilot study, one criterion is based on a competition (or race) between two running lists of elements within each infinite set. Another criterion is to compare two amounts of entire sets of elements. Thus, the fourth item could reveal criteria that students use for comparing two infinite sets. The questions in the second, third and fourth categories are shown in Figure 4.

II. (Second Item) Of which are there more? Please check one of the boxes. How do you know?

5. ☐ Of your fingers or ☐ Of your toes – because
6. ☐ Of odd numbers or ☐ Of even numbers – because
7. ☐ Of odd numbers or ☐ Of integers – because

III. (Third Item) Which of the two sets A and B is bigger? How do you know?

A = { 1, 2, 3, 4, 5, 6, 7, ... }

B = { 2, 4, 6, 8, 10, 12, 14, ... }

IV. (Fourth Item) A student claims that each of the two sets A and B is infinite, but A is larger than B. What can this mean?

Figure 4: The questions in the second, third, and fourth items

In the history of mathematics, one aspect of students' learning difficulties with the notion of limit is the failure to link geometry with numbers (Cornu, 1991). In other words, the geometric interpretation in the concept of limit differs from the understanding of the notion of a numerical limit. Thus, both arithmetic and geometric contexts often have been used to investigate students' difficulties regarding limit (Davis and Vinner; Monaghan,

1991, Szydlik, 2000). Specifically, the limit of a sequence in an arithmetic context and the limit of a function in a geometric context have been focused on in previous studies on limit. Thus, the intention of the fifth item was to investigate how students calculated the limit of a given sequence in arithmetic context. Graphical representations in the sixth item were utilized to examine students' mathematical discourse on limit by using spatial principles in geometric context. Students might use different ways of mathematical discourse to approach different mathematical contexts. The questions in the fifth and sixth categories are shown in Figure 5.

V. (Fifth Item) What do you think will happen later in the columns of this table?

x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
0.01	0.099990
0.005	0.099995
0.001	0.099999
...	...

VI. (Sixth Item) What will happen to the curve in #8 as x approaches 0 from left or from right? What will happen to the curve in #9 and #10 when x goes to positive infinity?

8. $\frac{1}{x}$ 9. $\frac{x^2}{1+x}$ 10. $\frac{x^2}{(1+x)^2}$

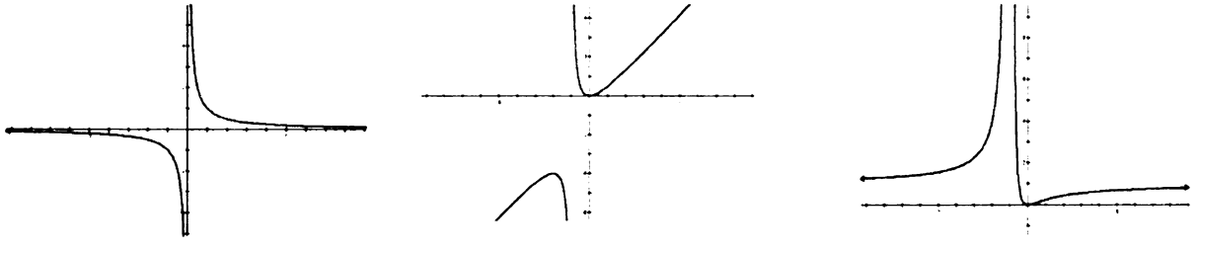
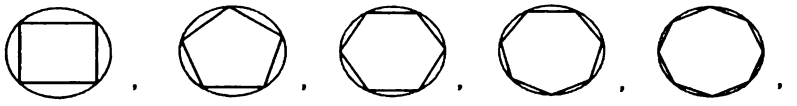


Figure 5: The questions in the fifth and sixth items

In the seventh item, concrete geometric representations in limiting processes were used in the questions to examine different characteristics of students' mathematical discourse on limit from the fifth and sixth categories. The questions in the seventh item could draw a result which explicitly explains how students think and their approach to the

limit in geometric progressing processes of regular polygons inscribed in a circle. The eighth item was intended to assess how students identity the limit of a sequence as either infinity or a number. Another purpose of the questions in the eighth item was to evaluate how students utilize the definition of limit in their minds in answering the questions. The questions in the seventh and eighth categories are shown below in Figure 6.

VII. (Seventh Item) Examine the sequence of the square, the regular pentagon, hexagon (6 sides), heptagon (7 sides), ..., which is inscribed in the circle.



What happens if you continue increasing the number of sides of a regular polygon inscribed in a circle? Explain your reasoning.

VIII. (Eighth Item)

- **A student claims that a sequence $a_1, a_2, a_3, \ldots$ goes to infinity. What does this mean?**
- **A student claims that the number P is the limit of a sequence $a_1, a_2, a_3, \ldots$ What does this mean?**

Figure 6: The questions in the seventh and eighth items

Unlike the English questionnaire, a ninth item was considered in the Korean questionnaire. With the Korean mathematical words of infinity and limit, the same question asked in the first item was used in the ninth item of the Korean questionnaire to thoroughly investigate Korean students' discourse. In summary, the discourse questionnaire included a total of 16 questions in eight categories to investigate students' colloquial discourse on infinity and limit and their mathematical discourse about these concepts in the different mathematical contexts.

Procedures for Collecting Data

During data collection, two main types of data were collected: students' responses to the questionnaire and interviews with pairs of students solving the same tasks in the discourse questionnaire. First, I contacted instructors (e.g., teaching assistants or professors) who taught calculus classes at MSU and KU to ask about their willingness to

allow me to collect data in their classrooms. After obtaining approvals from the instructors, all the students in each identified classroom were asked during their classes to participate in this study during their classes. The native-English participants at MSU were provided a consent form for soliciting their participation which was a part of the MSU Institutional Review Board (IRB) requirements. The consent form described the purpose of this study, its importance, and the procedures involved. The benefits and risks were also explained (see Appendix C for details). From several calculus classes, 132 English speakers and 126 Korean speakers were recruited to take the survey on the basis of instructors' agreement to data collection in their classroom and students' willingness to participate. The volunteers were informed that they might be selected for the further paired interviews after completing the questionnaire.

Of those volunteers in each university, twenty students were selected for paired interviews based on their responses to the questionnaire. The following procedures were used in selecting interview participants. First, I looked at whether the participants had studied mathematics before in any language other than their native language; any participant who had done so was not considered as a possible interviewee. Second, different kinds of students were selected in terms of their responses to the questionnaire based on their previous curricular experience with infinity and limit. The selection of those students in terms of their curricular experience improved the validity of the data for the proposed study (Patton, 2002), because the selection of students from various levels can be more representative of the population.

To examine students' intended curricular experience with *infinity* and *limit*, the textbooks students used in high school were analyzed. Table 3.3 summarizes percentages

of the English-speaking participants' and Korean-speaking participants' written responses to the question about high school textbooks.

Table 3.3 Percentages of students' responses to the question about high school textbooks

Korean Textbooks	K-participants	K-interviewees	U.S. Textbooks	E-participants	E-interviewees
K1	9 (7.1%)	2 (10.0%)	US1	1 (0.8%)	0 (0.0%)
K2	12 (9.5%)	2 (10.0%)	US2	1 (0.8%)	0 (0.0%)
K3	10 (7.9%)	2 (10.0%)	US3	0 (0.0%)	0 (0.0%)
K4	1 (0.8%)	0 (0.0%)	US4	14 (10.6%)	4 (20.0%)
K5	9 (7.1%)	2 (10.0%)	US5	20 (15.2%)	4 (20.0%)
K6	26 (20.6%)	4 (20.0%)	US6	4 (3.0%)	2 (10.0%)
K7	13 (10.3%)	2 (10.0%)	US7	24 (18.2%)	4 (20.0%)
Mixed	31 (24.6%)	6 (30.0%)	Mixed	7 (5.3%)	0 (0.0%)
Others	9 (7.1%)	0 (0.0%)	Others	53 (40.2%)	6 (30.0%)
No response	6 (4.8%)	0 (0.0%)	No response	8 (6.1%)	0 (0.0%)
Total	126 (100%)	20 (100%)		132 (100%)	20 (100%)

In order to ensure that the selected interviewees were representative of each population in terms of curricular experience, I selected the number of interviewees within each category on the basis of the ratio of the number of students in each category to the total number of students. For instance, because the ratio in the K6 category is 21% (the number of students (n = 26) / the total number of the Korean students), four interviewees were selected within the K6 category among the total of twenty selected interviewees. After deciding the number of interviewees within each category based on their curricular experience, a random selection procedure was used to choose interviewees within the category. For instance, in the US7 category, 4 students were randomly selected out of the total of 24 students. If a student declined to participate in paired interview, another random selection was conducted with all of the students in each category, excluding the student who declined to participate in the interview.

The following paragraph describes the interview method for this study. The method of interviewing pairs of students, who solved problems together with minimal participation of the interviewer, was used because this type of interview may further

reveal mechanisms of student thinking (especially in terms of routines in the first research question) rather than interviewing a single student by a researcher. That is, student thinking processes were monitored through paired problem solving. A pair of students was able to watch each other's errors, probe the other student's understanding, and help each other to reach a deeper level of comprehension. To ensure the occurrence of the listed three advantages of the paired problem solving, I helped pairs of students to continuously verbalize their thoughts aloud and express the major steps they used to solve each mathematical problem on the questionnaire. I also assisted each pair of students to clarify his or her thoughts for details whenever the listener was not sure of how the problem solver was thinking (Whimbey and Lochhead, 1984). The following possible prompts were used: "Can you explain how to come to your answer step by step?", "What do you mean by the word you just used?", and "Can you explain more about what you just said?" This was a way to monitor student thinking and reduce interviewer's effects and biases on students' responses.

The interviews, which were conducted in the participants' first language, took as much time as the participants needed to respond to the problems in the discourse questionnaire. For pairs of students to concentrate on a given problem, one small card listing each term or each problem was prepared and shown to interviewees during the interviews. For instance, the problems in the first category were given to interviewees on a card with only one word written on it, such as *infinity* and *limit*. The interviews took place in a university office, or another convenient location. All interviews were audio-taped, video-taped, and transcribed in their entirety for further analysis. As compensation, each interviewee received 25 dollars. The Korean-English bilinguals (three mathematics

educators and one linguist who helped the Korean translations of the survey questionnaire into English) assisted in transcribing the Korean interviews into English.

Analysis

Analysis of Former Education

To analyze and categorize students' written responses to the questions regarding former education in the background questionnaire, the columns of course and content were used to search for variations in different levels of previous education. For instance, the following categories were used for responses to the questions about limit (see Table 3.4). Compared to those university students who did not learn about the concept of limit in their secondary education, university students who learned about limits when taking Pre-calculus or Calculus in high school may respond differently to the notion of limit because of their different degrees of exposure to limit. Similarly, the content column was used to select different levels of students in terms of different experiences and degrees of difficulties with the concept of limit.

Table 3.4 The course and content categories for limit

Course	Content
Algebra	Limits to infinity
Geometry	The ε-δ (epsilon-delta) definition of limit
Advanced Algebra	Limit of a sequence or series
Pre-calculus	Limit of a function numerically
Calculus	Limit of a function geometrically
	L Limit and the definition of continuity
	Limit and the definition of the derivative

Analysis of Intended Curricula

Intended curriculum in this study means planned learning experiences in terms of standards or frameworks at the national or state level, as well as instructional guides and

textbooks. Thus, instructional materials and policy documents were used for intended curricular emphases. As noted, information about instructional materials was collected on the basis of students' written responses to the questions regarding the textbooks that they used. However, due to the complexity of curricular experience, interviewees were selected as representative of each population in terms of their curricular experience rather than analysis of their textbooks.

Discourse Analysis

The transcribed data from paired interviews were used as a primary source to analyze students' non-mathematical and mathematical discourses. Through paired problem solving, students' non-mathematical and mathematical discourses on infinity and limit were recorded, transcribed and coded according to the four features of discourse. Then, interview data were analyzed in order to identify and describe those four features. Because of the nature of the questions asked, only a subset of the four features was depicted in each category. At the next stage, the analysis of the data was guided by the three research questions of this study.

In each of the four features, the representative analysis samples were elicited from the pilot study (see Appendix B for details). Based on the four features, I deduced hypotheses to relate characteristics in mathematical discourse to those in non-mathematical discourse and curricular experience. Finally, I examined similarities and differences of these hypothesized relationships between native-English and native-Korean speakers. To decide what to include and focus on from the entirety of the conversations, my analytic decisions were based on representatives of general themes through the interview rather than on outliers. Repeated parts were also dropped.

Therefore, only a sample of representative utterances was presented in the results. The theoretical components (the four distinctive features of colloquial and mathematical discourses) which each item assesses are summarized in Table 3.5.

Table 3.5 Theoretical components assessed by each item

Category	Theoretical components			
	Word use	Visual mediators	Routines	Endorsed narratives
I	√			√
II	√		√	√
III	√		√	√
IV	√		√	
V	√	√	√	
VI	√		√	
VII	√			√
VIII	√			√

CHAPTER IV

ANALYSIS AND FINDINGS

This chapter is divided into two main sections. The first section describes the overall characteristics of the survey participants' backgrounds relative to mathematical learning, using descriptive and inferential statistical results of all the participants' responses to the survey questionnaire. The second section provides a characterization of the participants' colloquial and mathematical discourses, drawing on analyses of all the participants' written responses to the survey, as well as analyses of the interview data of twenty representatives in pairs within each linguistically distinct group. The analyses in the second section are organized to identify the primary features of the participants' colloquial and mathematical discourses on *infinity* and *limit*.

Background Data about the Students and their Past and Present Mathematical Learning

In this section, I will discuss students' past and present mathematical learning, and will present an analysis of the 126 Korean and 132 English speaking students' written responses to the background questions in the survey. For the second research question, this analysis of the background data will generate differences between the Korean and English speaking groups in their former mathematics education and curricular experiences with *infinity* and *limit*. To investigate whether or not there is a statistically significant background difference between the two groups on categorical variables, Pearson's chi-square statistics (with alpha set at .01) in Statistical Package for the Social Sciences 16.0 for Windows (SPSS Inc., 2007) were used because most of the items on the background questions measure categorical variables. One of the important assumptions of chi-square is that the minimum expected cell frequency should be 5 or greater (Pallant,

2005). If the minimum expected count violated this assumption, two or three categories were collapsed into one category to use chi-square analysis. For background items measuring a continuous variable, an independent samples *t* test was conducted to compare the mean score for the two groups. Chi-square analysis compares the proportions of the two groups rather than their means as in the case of the *t* test.

The background survey consists of 21 questions. Most of the questions were either adapted or borrowed from the student questionnaire of the Third International Mathematics and Science Study (TIMSS, 2003). These 21 questions are aimed at investigating demographic information about the participants, their attitudes toward mathematics learning, and characteristics of their formal education related to *infinity* and *limit*. This questionnaire provides not only the information about students' background, but also their past and present mathematics learning about *infinity* and *limit*.

Students' Background

Personal background

Table 4.1 presents a summary of the demographic characteristics of the Korean and English speaking participants (K-participants and E-participants) in the survey study. In total, the 126 Korean and 132 English speaking university students participated in this study from three and six classrooms respectively. The 126 Korean speakers (K-speakers) consisted of 31 (24.6%) females and 95 (75.4%) males. Of the 132 English speakers (E-speakers), 37 (28.0%) students were female and 95 (72.0%) were male. The grade levels of the K-speakers were as follows: 123 (97.6%) freshmen, 1 (0.8%) sophomore, and 2 (1.6%) juniors. The E-speakers were composed of 109 (82.6%) freshmen, 15 (11.4%) sophomores, 6 (4.5%) juniors, and 2 (1.5%) seniors.

One hundred percent of the K-participants said that they spoke Korean as their native language. The majority (84.1%) of the E-participants stated that they had English as their native language and 15.9% (n=21) participants, including five Koreans, reported "other" as their native language. The majority (83.3%) of the K-speakers responded that they were studying calculus in English due to the use of the English calculus textbook in their classroom. However, the language of instruction in class was not English, but Korean. The majority (86.4%) of the E-speakers replied that they had not been exposed to other languages in learning mathematics. Both their instruction and their textbook was in English.

Table 4.1 Demographic characteristics of the survey participants

		K-speakers (n = 126)	E-speakers (n = 132)
0. Gender	Male	31 (24.6%)	37 (28.0%)
	Female	95 (75.4%)	95 (72.0%)
1. Year	Freshman	123 (97.6%)	109 (82.6%)
	Sophomore	1 (0.8%)	15 (11.4%)
	Junior	2 (1.6%)	6 (4.5%)
	Senior	0 (0.0%)	2 (1.5%)
2. Native Language	Korean	126 (100.0%)	5 (3.8%)
	English	0 (0.0%)	111 (84.1%)
	Other	0 (0.0%)	16 (12.1%)
3. How often do you speak your native language at home?	Always	120 (95.2%)	114 (86.4%)
	Almost always	4 (3.2%)	16 (12.1%)
	Sometimes	2 (1.6%)	1 (0.8%)
	Never	0 (0.0%)	0 (0.0%)
	No response	0 (0.0%)	1 (0.8%)
4. About how many books are there in your home? (Do not count magazines, newspapers, or your school books)	0-10 books	7 (5.6%)	4 (3.0%)
	11-25 books	13 (10.3%)	10 (7.6%)
	26-100 books	33 (26.2%)	39 (29.5%)
	101-200 books	28 (22.2%)	33 (25.0%)
	More than 200 books	45 (35.7%)	46 (34.8%)

On the question regarding how often the participants speak their native language at home, 95.2% of the Korean speaking group (K-group) and 86.4% of the English speaking group (E-group) answered "always." With respect to the question on the number of books which the participants estimate having at home, this survey indicated

that there was no significant difference between two linguistically different groups in terms of the chi-square statistic ($\chi^2 = 1.99$, $p > .01$).

Mathematics background

To investigate and categorize variations in different levels of former education, students' written responses to the background questions regarding textbooks used and courses taken in high school were analyzed.

Table 4.2 Summary of textbooks used in high school

Korean Textbooks	K-speakers (n = 126)	U.S. Textbooks	E-speakers (n = 132)
K 1	9 (7.1%)	US 1	1 (0.8%)
K 2	12 (9.5%)	US 2	1 (0.8%)
K 3	10 (7.9%)	US 3	0 (0.0%)
K 4	1 (0.8%)	US 4	14 (10.6%)
K 5	9 (7.1%)	US 5	20 (15.2%)
K 6	26 (20.6%)	US 6	4 (3.0%)
K 7	13 (10.3%)	US 7	24 (18.2%)
Mixed	31 (24.6%)	Mixed	7 (5.3%)
Others	9 (7.1%)	Others	53 (40.2%)
No response	6 (4.8%)	No response	8 (6.1%)

Table 4.2 indicates that about 90% of the K-speakers said that they studied from one or more of the seven representative textbook examples in the survey, whereas only 53.7% of the E-speakers answered that they used one of the seven examples in high school (see Appendix C for the examples). Of the K-speakers, 24.6 % reported that they used multiple textbooks in the examples rather than one textbook series, and 5.3 % of the E-speakers reported the same thing.

Table 4.3 Summary of course series taken in high school

Korean courses	K-speakers (n = 126)	U.S. courses	E-speakers (n = 132)
Math Application	4 (3.2%)	Pre-Algebra	13 (9.9%)
Mathematics 10	110 (87.3%)	Algebra or Algebra I	62 (47.0%)
Mathematics I	115 (91.3%)	Geometry	83 (62.9%)
Mathematics II	114 (90.5%)	Advanced Algebra/Algebra 2	91 (68.9%)
Calculus	112 (88.9%)	Pre-Calculus/Calculus	114 (86.4%)
Others	0 (0.0%)	Others	4 (3.0%)
No response	10 (7.9%)	No response	29 (22.0%)

Table 4.3 demonstrates that about 90% of the K-speakers who responded to the question about courses taken in high school said that they took courses consistently in the series from Mathematics 10 to Calculus. In contrast, only 47.0%, 62.9%, and 68.9% of the E-speakers reported that they took Algebra 1, Geometry, and Algebra 2 respectively, in high school.

Another question was asked in order to inquire about the most advanced mathematics course that students took in high school. Of the K-speakers, 90.5 % answered that they did not take any course other than calculus. However, 88.9% of the K-speakers reported that they took calculus in high school as part of a course series taken in high school. Comparatively, 86.4% of the E-speakers claimed that they took either Calculus or Pre-Calculus in high school as the most advanced mathematics course taken in high school.

Experience with infinity and limit

To examine the participants' experience with the mathematical words *infinity* and *limit*, they were asked to recall the first mathematics course in which they heard the terms. Table 4.4 shows a summary of frequencies and percentages of the participants' responses to the question about their experiences with the words *infinity* and *limit*.

Seventy seven percent of the K-speakers responded that they remembered the first mathematics course in which they heard the word *infinity*, compared to about 50 % of the E-speakers. Of the K-speakers, 88.9 % reported that they recalled the first use of the word *limit* in a mathematics course, whereas 84.1 % of the E-speakers said that they remembered hearing the word in a mathematics class. Chi-square analysis of this distribution indicated that the K-speakers were significantly more likely to report

remembering the first mathematics course in which they heard *infinity* ($\chi^2 = 19.15, p < .001$), but there was no significant difference between the K-group and E-group in remembering the first use of *limit* in their mathematics course ($\chi^2 = 1.27, p > .01$).

Table 4.4 Summary of the participants' recollection of experiences with *infinity* and *limit*

		K-speakers (n = 126)	E-speakers (n = 132)
6. Can you remember the first math course in which you heard the word (mathematical) *infinity*?	Yes	97 (77.0%)	67 (50.8%)
	No	29 (23.0%)	65 (49.2%)
7. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *infinity*.	2	0 (0.0%)	1 (0.8%)
	4	4 (3.2%)	2 (1.5%)
	5	2 (1.6%)	3 (2.3%)
	6	5 (4.0%)	3 (2.3%)
	7	16 (12.7%)	9 (6.8%)
	8	18 (14.3%)	8 (6.1%)
	9	7 (5.6%)	7 (5.3%)
	10	17 (13.5%)	13 (9.8%)
	11	26 (20.6%)	13 (9.8%)
	12	0 (0.0%)	7 (5.3%)
	14	0 (0.0%)	1 (0.8%)
	No response	31 (24.6%)	65 (49.2%)
8. Can you remember the first math course in which you heard the word (mathematical) *limit*?	Yes	112 (88.9%)	111 (84.1%)
	No	14 (11.1%)	21 (15.9%)
9. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *limit*.	6	0 (0.0%)	1 (0.8%)
	8	0 (0.0%)	2 (1.5%)
	9	6 (4.8%)	6 (4.5%)
	10	29 (23.0%)	16 (12.1%)
	11	77 (61.1%)	42 (31.8%)
	12	0 (0.0%)	33 (25.0%)
	13	0 (0.0%)	10 (7.6%)
	14	0 (0.0%)	1 (0.8%)
	No response	14 (11.1%)	21 (15.9%)

On the question regarding the grades in which they heard the words *infinity* and *limit*, overall response rates are the same as the percentage of the participants who answered "yes" for the previous question. For instance, about 75% of the K-speakers and 50% of the E-speakers did specify grades in which they recalled having first heard the word *infinity*. Among the students who specified a grade, the range of grades was from 4th to 11th grade in the K-speakers' responses. The range of grades of the E-speakers' responses was more varied, from 2nd to 14th grade. This survey result also indicated that the majority (84.1%) of the E-speakers' responses had a variety of grades relative to first

hearing the word *limit*, ranged from 6th to 14th grade as compared to the majority (88.9%) of the K-speakers who said that they first heard the word *limit* between 9th and 11th grade.

Table 4.5 Grades in which students recalled first hearing the words *infinity* and *limit*

Word	K-speakers (n=126)	E-speakers (n=132)		
	M (SD)	M (SD)	*df*	*t*-score
Infinity	8.75 (1.99)	9.01 (2.37)	160	- .78
Limit	10.63 (0.59)	11.15 (1.22)	158.09	- 4.06*

The grade items were analyzed using a *t* test because grade is a continuous outcome variable. In the case of the question about the participants' recollection of the grade in which they were first introduced to the word *infinity*, equal variance was assumed because the significant value of Levene's test for equality of variances was larger than 0.05. The *t* test [t (160) = -.78, $p > .01$] confirmed no significant difference between the K-group and E-group in the range of grades in which they first heard *infinity*. In the case of the same question about the word *limit*, equal variance is not assumed because the result of Levene's test was significant. Interestingly, the *t* test result [t (158.09) = -4.06, $p < .01$] revealed that the mean of the E-speakers' responses to grade was significantly higher than that of the K-speakers, as shown in Table 4.5.

In order to indentify the topics related to the words *infinity* and *limit* which students studied in high school, seven concept categories within each term were considered. Tables 4.6 and 4.7 show percentages of students reporting that they had studied particular topics related to *infinity* and *limit* in high school as well as the results of chi-square comparing the percentages of each concept category in the two groups. Pearson's chi-square statistics were employed to compare the percentages of the K-group and the E-group in each concept category.

Table 4.6 Percentages of students reporting having studied specific topics related to *infinity* in high school and the results of chi-square

13. Please check the box(es) to topic which you have studied.	K-speakers (n =126)	E-speakers (n = 132)	Chi-square	
			Value	Significance Level
a. Infinite processes	42.9%	47.0%	3.69	.06
b. Infinite decimal fractions	84.9%	45.5%	34.95	.000*
c. Infinite sequence or series	91.3%	61.4%	23.94	.000*
d. The sum of infinite geometric series	75.4%	50.8%	8.48	.004*
e. Comparison of infinite sets	46.8%	32.6%	2.00	.16
f. Countable and uncountable sets	69.8%	18.9%	58.51	.000*
g. Infinitely small or infinitesimal	46.0%	36.4%	.33	.57

Note. * The percentage difference is significant at the 0.01 level.

In the case of *infinity*, the chi-square analysis confirmed a significant difference between the K-group and E-group in categories b, c, d, and f. The results revealed that the percentage of K-speakers who reported having learned each of the four concepts was significantly different from the percentage of E-speakers who reported having learned the same concept.

Table 4.7 Percentages of students reporting having studied specific topics related to *limit* in high school and the results of chi-square

14. Please check the box(es) to topic which you have studied.	K-speakers (n = 126)	E-speakers (n =132)	Chi-square	
			Value	Sig.
a. Limits to infinity	86.5%	77.3%	5.21	.022
b. The ε-δ definition of limit	14.3%	31.8%	11.22	.001*
c. Limit of a sequence or series	88.1%	53.0%	44.23	.000*
d. Limit of a function numerically	61.1%	65.9%	.66	.42
e. Limit of a function geometrically	60.3%	52.3%	1.90	.17
f. Limit and the definition of continuity	92.9%	65.2%	38.00	.000*
g. Limit and the definition of the derivative	81.0%	65.2%	9.86	.002*

Note. Sig. = significance level. * The percentage difference is significant at the 0.01 level.

In the case of *limit*, the differences in categories b, c, f, and g were large enough to be statistically significant. It is notable that the percentage (31.8%) of the E-speakers who reported having studied the ε-δ definition of *limit* in high school was significantly higher than that (14.3%) of the K-speakers, whereas the percentages of the E-speakers who stated that they learned the other topics were less than those of the K-speakers at statistically significant levels.

Summary

In summary, most E-speakers and K-speakers in the survey were freshmen and the ratio of males to females was similar between the E-group and K-group. The majority of the E-speakers said that they were exposed only to English in learning mathematics, and the majority of the K-speakers answered that they studied mathematics in Korean with the use of an English textbook (Thomas, 2005) in their calculus classrooms. However, in most cases the K-speakers were exposed to the English textbook only recently. This English textbook was used at the university, whereas high schools in Korea use Korean mathematics textbooks.

The E-speakers reported that they used a variety of textbooks in high school other than the representative examples in the survey, whereas the majority of the K-speakers said that they studied from the seven textbook examples. E-speakers also claimed that they took courses less consistently in the course sequences than the K-speakers. It is noteworthy that even though the majority of the E-speakers and K-speakers reported taking either Pre-calculus or Calculus in high school, K-speakers indicate that the calculus course is required, whereas E-speakers imply that U.S. pre-calculus or calculus is optional. Note that the non-response rate of the E-speakers is more than double that of the K-speakers on the item about course series taken in high school. This is probably not only because some of the E-speakers took different course sequences from the Algebra 1 – Geometry – Algebra 2 sequence, perhaps through the use of integrated curricula, but also because they studied from textbooks other than the survey examples.

As for experience with *infinity* and *limit*, it seems to be easier for students to remember when they first heard the word *limit* in a mathematics course than to remember

when they first heard the word *infinity*. More K-speakers responded that they remembered the first mathematics course experience about *infinity* compared to E-speakers. However, there was no significant difference between the two groups in the range of grades in which they first heard *infinity*. In the case of *limit*, there was no significant difference between the two groups in remembering the first mathematics course experience about *limit*. However, the E-speakers varied significantly in their responses to the question regarding the grade in which they heard the word *limit*. As for the topics related to the words infinity and limit which students studied in high school, K-speakers reported that they had studied most of the seven categories related to the words *infinity* and *limit* at higher rates than the E-speakers. This is probably because the K-speakers had studied more topics related to *infinity* and *limit* in their high school curricula.

Students' Current Mathematics Learning

Description of the calculus classes in the U.S. and Korea

The calculus courses in the U.S. institution which is the site of this study are 3-credit classes taught by faculty and post-doctoral instructors. The U.S. class meets on Monday, Wednesday, and Friday for 50 minutes, and has a class size that is usually under 35 students. Chapters 2 through 5 of Thomas' calculus book (Thomas, 2005) are covered during the semester. The U.S. calculus course is planned so that students take four one-hour tests and a two-hour final examination. Each test covers approximately one chapter in Thomas' calculus book. The final exam is a comprehensive examination which covers all the content from chapters 2 through 5. A lab course is designed for the U.S. calculus course. This lab course as a 2-credit class is offered on a voluntary basis for students who want extra work and more credits. This lab class meets Monday, Wednesday, and Friday

and students are required to attend class. In the lab class, students complete worksheets and solve problems to review the content of the calculus course. In the fall of 2007, about 5 % (30 out of 983 students) of the total number of students in the calculus course took the lab course.

The calculus class in Korea University is a 4-credit class taught by faculty and post-doctoral instructors. The Korean class meets on either Monday and Wednesday or Tuesday and Thursday for one hour and fifteen minutes. The size of each class is generally under 60 students. Chapters 1 through 12 of Thomas' calculus book (Thomas, 2005) are covered during the semester. Some sections of the textbook are not covered during the course. In the calculus class, students are required to take two two-hour tests and a two-hour final examination. Each test covers about four chapters in Thomas' calculus book. However, the final is not a comprehensive examination and covers the content learned after the second test. Part of the class also includes a 50- to 80-minute drill period every week. During the period, students solve problems in the calculus textbook.

Student's self-report on their learning of mathematics

As context for interpreting the particular findings of this research relative to E-speakers' and K-speakers' understanding of *infinity* and *limit,* data were gathered to determine the speakers' perception of the calculus instructional setting. In order to understand how students perceive their learning of mathematics in the classroom, each classroom environment item was rated with the four categories of frequency in mathematics lessons ('Never,' 'Some lessons,' 'About half the lessons,' and 'Every or almost every lesson'). Table 4.8 shows percentages of the participants' responses to the

questions about their classroom environments and the chi-square results of the comparison of the E-speakers' and K-speakers' responses about the mathematics lesson items. The lowest expected frequency in any cell should be 5 (about four percent) or more to use chi-square statistics. If the expected frequency was less than 5, a method of collapsing two or three categories into one category was used. For the classroom environment questions with the minimum expected counts less than 5 (e.g., 17-a, 17-b, and 17-f), for instance, the 'Never' category and the 'Some lessons' category were combined together into one category and the 'Almost every lesson' category and the 'About half lessons' category were collapsed into another category.

Table 4.8 Percentages of students' responses to the classroom environment items and the chi-square results

17. How often do you do these things in your mathematics lessons?		Almost every lesson	About half the lessons	Some lessons	Never	No response	Chi-square		
							df	Value	Sig.
17-a. We work together in small groups.	K	3.9	6.4	20.6	68.3	0.8	1	2.32	.13
	E	2.3	3.0	25.8	68.9	0.0			
17-b. We relate what we learning in math to daily lives.	K	2.4	7.1	45.2	45.2	0.0	1	.12	.91
	E	4.6	5.3	33.3	56.1	0.8			
17-c. We explain our answers.	K	2.4	12.7	50.0	34.1	0.8	3	59.52	.000*
	E	33.3	25.0	29.5	12.1	0.0			
17-d. We decide on our own procedures for solving problems	K	8.7	34.9	45.2	9.5	1.6	3	25.84	.000*
	E	9.9	16.7	39.4	33.3	0.8			
17-e. We review our homework	K	15.9	27.8	38.9	16.7	0.8	3	20.85	.000*
	E	19.7	14.4	26.5	39.4	0.0			
17-f. We listen to the teacher's a lecture-style presentation	K	70.6	19.0	7.9	1.6	0.8	1	2.52	.11
	E	90.2	5.3	2.3	2.3	0.0			
17-g. We work problems on our own.	K	18.3	37.3	38.1	5.6	0.8	3	36.59	.000*
	E	38.6	18.2	21.2	22.0	0.0			
17-h. We use calculators	K	0.8	1.6	27.8	69.0	0.8	3	19.40	.000*
	E	9.9	9.1	18.2	62.9	0.0			

Note. K = K-speakers; E = E-speakers; Sig. = significance level. *The distribution differences are significant ($p \leq .01$).

There were statistically significant differences between the E-group and K-group in items 17-c, 17-d, 17-e, 17-g, and 17-h. In classrooms, the K-speakers said that they had more opportunities to use their own procedures for solving complex problems and spent more class time to review their homework, as shown in items 17-d and 17-e respectively.

In contrast, items 17-c, 17-g, and 17-h show that the E-speakers reported that they had more opportunities to explain their answers, work on problems on their own, and use calculators in their classrooms. Note also that there were no significant differences in E-speakers' and K-speakers' reports about use of small group work in the calculus classroom (neither group reports this occurring very much); that both groups report few opportunities to relate calculus to their daily lives, and that both groups report a similar prevalence of lecture-based instruction.

Table 4.9 Percentage distributions of extra lessons and homework and the chi-square results

Question	Categories	K-speakers (n = 126)	E-speakers (n = 132)	Chi-square		
				df	Value	Sig.
19. During this school year, how often have you had extra lessons or tutoring in math that is not part of your regular class?	Every or almost every day	0.8	2.3	1	10.05	.002*
	Once or twice a week	1.6	10.6			
	Sometimes	0.0	24.2			
	Never or almost never	97.6	62.1			
	No response	0.0	0.8			
20. How often does your instructor give you homework in math?	Every day	1.6	60.6	1	195.59	.000*
	3 or 4 times a week	0.0	27.3			
	1 or 2 times a week	97.6	6.1			
	Less than once a week	0.8	0.8			
	Never	0.0	4.6			
	No response	0.0	0.8			
21. When your instructor gives you math homework, about how many minutes are you usually given?	Fewer than 15 minutes	2.4	18.9	4	47.78	.000*
	15-30 minutes	2.4	13.6			
	31-60 minutes	25.4	34.1			
	61-90 minutes	37.4	14.4			
	More than 90 minutes	32.5	16.7			
	No response	0.0	2.3			

Note. Sig. = significance level. *The distribution differences are significant ($p \leq .01$).

Questions on extra lessons in mathematics that were not a part of regular class and mathematics homework were also asked. Table 4.9 shows the percentage distributions of extra lessons and homework in mathematics and the chi-square results of the comparison of the E-speakers' and K-speakers' responses about the extra lessons and mathematics homework items. Because the expected frequency is less than 5 in the question on extra lessons in mathematics, the first two categories ('Every or almost every day' and 'Once or twice a week') were combined together into one category and the other two categories

('Sometimes' and 'Never or almost never') were collapsed into another category to use chi-square statistics. In the next question on frequency of mathematics homework, because the minimum expected count was .98, a method of collapsing categories was to combine the first two categories ('Every day' and '3 or 4 times a week') into one category and the other three categories ('1 or 2 times a week,' 'Less than once a week,' and 'Never') into another category. Finally, in the question on length of mathematics homework, no collapsing method was considered because the minimum expected count is bigger than 10.

There was a statistically significant association between the E-speakers and extra lessons. In other words, the E-speakers reported that they had more extra lessons or tutoring than the K-speakers ($\chi^2 = 10.05, p < .01$). Seventeen of 132 E-speakers had extra lessons at least once a week, compared with three of 126 K-speakers. The results of the chi-square analysis revealed that there were statistically significant differences between the K-group and E-group for frequency and length of mathematics homework. The statistically significant difference ($\chi^2 = 195.59$, $p < .001$) was observed in frequency of mathematics homework. The statistical result shows that the E-speakers were assigned more homework than the K-speakers. Chi-square analysis of this distribution indicated a significant difference ($\chi^2 = 47.78$, $p < .001$). The K-speakers were significantly more likely to report spending time doing mathematics homework than the E-speakers.

Out-of-school activities

Nine items were asked to explore possible differences between the two linguistically distinct groups in the ways that out-of-school time was spent on a typical school day. Table 4.10 displays a summary of percentages of students' responses to the questions

regarding activities before or after school and the results of chi-square to the extent of time spent out-of-school activities.

Table 4.10 Percentages of students' responses to the out-of-school activity items and the chi-square results

18. On a normal school day, how much time do you spend before/after school doing each of these things?		No time	Less than 1 hour	1-2 hours	More than 2 hours	No response	Chi-square		
							df	Value	Sig.
18-a. I watch television and videos	K	30.2	43.7	17.5	8.7	0.0	3	24.28	.000*
	E	10.6	37.1	31.8	20.5	0.0			
18-b. I play computer games	K	36.5	32.5	19.8	10.3	0.8	3	10.08	.02
	E	54.5	25.8	9.9	9.1	0.8			
18-c. I play or talk with friends	K	1.6	31.0	38.1	27.8	1.6	1	16.20	.000*
	E	0.0	12.1	39.4	48.5	0.0			
18-d. I do jobs at home	K	44.4	44.4	6.4	1.6	3.2	1	7.66	.006*
	E	27.3	52.3	14.4	6.1	0.0			
18-e. I work at a paid job	K	73.8	2.4	11.1	12.7	0.0	1	1.70	.19
	E	64.4	4.6	6.1	25.0	0.0			
18-f. I play sports	K	32.5	48.4	15.1	3.2	0.8	3	26.61	.000*
	E	22.7	29.5	31.8	15.9	0.0			
18-g. I read a book for enjoyment	K	37.3	48.4	10.3	3.2	0.8	1	1.01	.32
	E	42.4	39.4	15.2	3.0	0.0			
18-h. I use the internet	K	0.8	20.6	47.6	30.2	0.8	1	2.82	.09
	E	0.0	13.6	38.6	47.7	0.0			
18-i. I do homework	K	3.9	22.2	46.8	27.0	0.0	1	10.51	.001*
	E	0.0	10.6	43.9	45.5	0.0			

Note. K = K-speakers; E = E-speakers; Sig. = significance level. *The distribution differences are significant ($p \leq .01$).

Each item on activities before or after school was encoded on the basis of the following four categories ('No time,' 'Less than 1 hour,' '1-2 hours,' and 'More than 2 hours'). Because the minimum expected frequencies of items 18-c, 18-d, 18-e, 18-g, 18-h, and 18-i were less than 5 (about four percent), the first two categories ('No time' and 'Less than 1 hour') and the other two categories ('1-2 hours' and 'More than 2 hours') were collapsed into one and another categories respectively to use chi-square. The chi-square results indicate that the E-speakers answered that they used more time than the K-speakers on accomplishing most activities.

In order to investigate the extent of time spent out-of-school activities on average, each item on activities before or after school was encoded based on the following scores

(0 = 'no time,' 1 = 'less than 1 hour,' 2 = '1-2 hours,' and 3 = 'more than 2 hours'). The mean (M) and the standard deviation (SD) results for each item were summarized in Table 4.11.

Table 4.11 Mean (M) and SD results to the extent of time spent out-of-school

Item	K-speakers		E-speakers	
	M	SD	M	SD
18-a. I watch television and videos.	1.05	0.91	1.62	0.93
18-b. I play computer games.	1.03	1.00	0.73	0.97
18-c. I play or talk with friends.	1.90	0.84	2.36	0.69
18-d. I do jobs at home.	0.62	0.68	0.99	0.82
18-e. I work at a paid job.	0.63	1.11	0.92	1.31
18-f. I play sports.	0.88	0.78	1.41	1.01
18-g. I read a book for enjoyment.	0.79	0.76	0.79	0.81
18-h. I use the internet.	2.06	0.76	2.34	0.71
18-i. I do homework.	1.97	0.81	2.35	0.67

The descriptive statistic of item 18-i, for instance, shows that the E-group had a mean score of 2.35 with a SD of .67 in time spent on doing homework on a usual school day, which indicates that the E-speakers reported spending about one hour and 20 minutes on average doing homework on a typical school day. In contrast, the K-group had a mean score of 1.97 with a SD of .81, which reveals that the K-speakers spent about an hour on average.

Students' way of life

The results about students' engagement in mathematics outside of the course (e.g., extra lessons, homework, etc.) reported in Table 4.9 raised additional questions about whether there were indeed differences in what was happening outside of class with students in these courses, and what the nature of those differences might be. In order to gain additional insights about how students were studying mathematics, all English speaking and Korean speaking interviewees (E-interviewees and K-interviewees) were asked about the length of time they were spending studying and their study methods, to

provide descriptions of their typical weekday in their typical week via email. In the emails, the following two questions were asked: 1) Could you please describe how many hours and in what kinds of format (e.g., group study, tutoring, study alone) you studied calculus in a typical week last semester? 2) Could you please describe your typical weekday? Your weekends? And your holidays?

Table 4.12 Summary of students' descriptions of how to study calculus

Students	Students' descriptions of how to study calculus
K-speakers	I mainly reviewed contents at night on Monday and Wednesday when I had class. If there was a test coming up, I studied at night on Monday, Tuesday, and Friday. I studied by looking at the textbook and solving examples...I studied alone for about 2 hours at night on Monday and Wednesday. Because 2 o'clock on Saturday would be the examination time, I studied alone from about 5 pm when class was over to 12 o'clock and Friday night and Saturday morning...I followed my class schedule during the week.
	In class, I roughly wrote the professor's notes from the blackboard in my workbook ...And then I reviewed contents that were covered in class by transcribing them from the workbook into my notebook after class was over...So I worked in this way. I studied contents in the textbook alone and solved problems related to them. I solved homework problems first alone, and then gathered with friends later and compared answers. If there were problems which I did not solve alone, I solved them with the help of friends...I tried to be faithful to my school life during the week. (In the given time table, she wrote "reviewing notebook" on Tuesday and Thursday and "group study" on Friday)
E-speakers	I often make the unwise choice of waiting until the night before a test or quiz to start studying for it...The only way I can learn is if I sit down in a quiet place and read the text book while doing example problems, or having a tutor sit down next to me and personally explain the material...I have good intentions to study, but I usually end up wasting a lot of time sitting around talking to my friends. (In the given time table, he wrote "alone" several times on Monday through Sunday)
	On weekdays, I go to class and do whatever homework is due the following day around 9 or 10 at night. On weekends, I finish all the homework I possibly can for the upcoming week, and any reading that I missed over the preceding week. (In the given time table, she wrote "1 (hour)" on Monday, Wednesday, and Friday and "alone or Math Learning Center for 4 (hours)" on Sunday)
	My typical weekday involves going to my classes and then relaxing for a little while once I get home. I usually do homework before and after dinner... My weekends involve studying and spending time with friends. (In the given time table, she wrote "study alone" several times on Monday through Sunday and "study alone or Math Learning Center for "a half hour" on Thursday)
	(In the given time table, he just wrote "do problems out of the book" and "2 (hours)" on Tuesday and Thursday and "1 (hour)" on Sunday)

Four E-interviewees and two K-interviewees responded to those questions in March and April 2008 (see Appendix D for their entire responses; notice that K-interviewees were interviewed in May 2007, compared to E-interviewees in September and October 2007 due to different systems of academic year). Table 4.12 is a summary of the

students' descriptions of how they studied calculus. The students' responses provide a more refined view on how the two groups of students are studying. Two K-interviewees reported that they had everything systematically planned including study methods, periods of rest and they spent time exactly according to that plan. In contrast, the four E-interviewees appeared to have planned less and admitted to not being able to follow it too tightly even if they had a plan. These responses, although from a very small sample of students, could suggest that the K-speakers are more serious about their studying, more carefully organized, and devote more thought and more time to it than the E-speakers. For the K-speakers interviewed here, studying is more central in their lives, compared to the E-speakers.

Summary

Students in this study reported that they listened to teachers' lecture-style presentation in both U.S. and Korean classrooms for more than half of the lessons. In classrooms, the K-speakers replied that they used their own procedures more flexibly for problem solving. However, the E-speakers responded that they had more opportunities to explain their answers and use calculators. According to student responses, K-speakers may be spending more class time reviewing their homework than the E-speakers, probably because of the 50- to 80-minute drill period every week.

The E-speakers reported that they spent more time on homework at home on a normal school day than the K-speakers. However, when talking about mathematics homework specifically, the K-speakers replied that they spent about 60 minutes twice a week on average, while the E-speakers reported that they spent less than 30 minutes three times a week. The E-speakers seem to be more engaged in homework for several subjects,

in addition to mathematics. The frequency of mathematics homework (three times per week in E-speakers' responses and twice per week in K-speakers' responses) seems to be related to the number of times class meets per week.

As for extra lessons in mathematics that were not a part of regular class, the K-speakers said that they have almost never had extra lessons or tutoring (see question 19 in Table 4.9). In contrast, 37.1% of the E-speakers reported that they had extra lessons or tutoring more than "sometimes." In fact, some E-speakers said that they received extra lessons from the Mathematics Learning Center (MLC) offered by the Department of Mathematics at MSU[1]. The MLC is staffed primarily by teaching assistants and is open approximately 40 hours per week for students in lower level mathematics courses. In terms of students' descriptions of how they study calculus in their typical week, K-speakers' and E-speakers' responses could be interpreted as an indication that for K-speakers, learning was more central in their lives because they reported systematically devoting more time to it, compared with E-speakers.

Regarding students' activities before or after school, the E-speakers reported that they were more involved in the out-of-school activities given in the survey than the K-speakers. The K-speakers may have spent time on other out-of-school activities which were not listed in the survey examples, such as departmental activities and club activities, as shown in two K-interviewees' explanations[2].

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[1] Some interviewees, who said that they had extra lessons or tutoring "sometimes," were asked how they studied Calculus in a typical week via email. Two E-interviewees responded to the email. According to their responses, they used the MLC for extra lessons and tutoring.

[2] All Korean and E-interviewees were asked to describe their typical weekday via email. Two K-interviewees answered the question. One Korean interviewee responded that she participated in many activities of her department. The other Korean interviewee reported that he did club activities after classes.

Attitudes toward Mathematics

Students' views of themselves as mathematics learners

Another potential important component of background is in the area of students' views of themselves as mathematics learners and their attitudes about mathematics learning. Six items on the written survey addressed these issues. To investigate whether or not there was a statistically significant difference between the K-speakers and E-speakers in their attitudes toward mathematics learning, an independent samples *t*-test was employed. Each item was scored 4 = 'agree a lot,' 3 = 'agree a little,' 2 = 'disagree a little,' and 1 = 'disagree a lot.' The higher the mean scores are, the more strongly each group agrees with statements in the attitude items. Table 4.13 presents the descriptive statistics and *t*-test results of the survey participants' response to each item.

K-speakers had a significantly lower score (that is, bigger than 2.5) on item 15-a than the E-speakers, which indicates that the E-speakers expressed a stronger agreement with the statement that they usually do well in mathematics because a 2.5 indicates a neutral response. Item 15-e reveals that the K-speakers had a significantly higher score (that is, less than 2.5) than the E-speakers, indicating that the E-speakers had a stronger disagreement on the statement that "mathematics is not one of my strengths" than the K-speakers. In other words, the E-speakers were more likely that the K-speakers to claim that mathematics is one of their strengths. The K-speakers had a significantly lower score (less than 2.5) than the E-speakers in item 15-d. In other words, the K-speakers had a stronger disagreement than E-speakers with the statement that when they do not understand a new topic initially then they will never understand it, suggesting that they believe that they can eventually understand a topic that is difficult initially.

Table 4.13 Students' views of themselves as mathematics learners and the *t*-test results of the comparison of percentage of the Korean and E-speakers' responses

15. How much do you agree with these statements about learning mathematics?		Agree a lot	Agree a little	Disagree a little	Disagree a lot	M	SD	Sig.
15-a. I usually do well in mathematics.	K	10.3	61.1	23.8	4.8	2.77	.70	.00*
	E	53.0	40.2	6.1	0.8	3.45	.65	
15-b. I would like to take more mathematics in school.	K	20.6	44.4	27.8	7.1	2.79	.85	.91
	E	26.5	40.2	17.4	15.9	2.77	1.01	
15-c. Mathematics is more difficult for me than for many of my classmates.	K	4.8	23.0	54.8	17.5	2.15	.76	.29
	E	5.3	21.2	46.2	27.3	2.05	.84	
15-d. When I do not initially understand a new topic in mathematics, I know that I will never really understand it.	K	1.6	6.4	24.6	67.5	1.42	.69	.00*
	E	2.3	16.7	36.4	44.7	1.77	.81	
15-e. Mathematics is not one of my strengths.	K	8.7	29.4	44.4	17.5	2.29	.86	.00*
	E	6.8	13.6	38.6	40.9	1.86	.90	
15-f. I learn things quickly in mathematics.	K	7.9	60.3	28.6	3.2	2.73	.65	.15
	E	18.2	53.8	23.5	4.6	2.86	.76	

Note. K = K-speakers; E = E-speakers; M = mean; SD = standard deviation; Sig. = significance level.
*Differences are significant ($p \leq .01$).

The role of mathematics in students' lives

Five items were included to examine students' sense of the role of mathematics in their lives. These were scored on the scale 1, 2, 3, and 4 (1 = 'disagree a lot,' 2 = 'disagree a little,' 3 = 'agree a little,' and 4 = 'agree a lot'). In order to check for group differences between the K-speakers and E-speakers in their view of the role of mathematics in their lives, the statistical significance of the mean difference was determined by an independent samples *t*-test. The descriptive statistics and *t*-test results are summarized in Table 4.14.

Overall mean scores of both the K-speakers and E-speakers in the five attitude items were higher than a neutral response (that is a 2.5). More than 60% of both the E-speakers and K-speakers agreed with the statements in the five items regarding the need of mathematics in their lives. These descriptive statistics imply that students agreed on the need for mathematics. However, as shown in items 16-a and 16-b, the E-speakers had a significantly stronger agreement on the need for mathematics in their daily lives as well as for other school subjects than the K-speakers. There was also a significant difference

between the two groups in the mean scores of item 16-c, which indicates that the K-speakers were more convinced about the role of mathematics to get into the university of their choice.

Table 4.14 The role of mathematics in students' lives and the *t*-test results of the comparison of percentage of the Korean and E-speakers' responses

16. How much do you agree with these statements about mathematics?		Agree a lot	Agree a little	Disagree a little	Disagree a lot	M	SD	Sig.
16-a. I think learning mathematics will help me in my daily life.	K	18.3	47.6	25.4	8.7	2.75	.86	.002*
	E	34.1	47.0	12.9	6.1	3.09	.84	
16-b. I need mathematics to learn other school subjects.	K	37.3	33.3	21.4	7.9	3.00	.96	.01*
	E	43.9	43.2	9.8	3.0	3.28	.77	
16-c. I need to do well in mathematics to get into the university of my choice.	K	77.0	20.6	0.8	1.6	3.73	.56	.000*
	E	46.2	41.7	9.8	1.5	3.34	.72	
16-d. I would like a job that involved using mathematics.	K	23.0	36.5	33.3	7.1	2.75	.89	.92
	E	25.8	38.6	22.0	13.6	2.77	.99	
16-e. I need to do well in mathematics to get the job I want.	K	43.7	36.5	15.9	4.0	3.20	.85	.36
	E	40.9	33.3	20.5	5.3	3.10	.91	

Note. K = K-speakers; E = E-speakers; M = mean; SD = standard deviation; Sig. = significance level.
*Differences are significant ($p \leq .01$).

Summary

The E-speakers in this study showed more confidence in their capability of learning mathematics than the K-speakers. As for the need of mathematics, there was a significant difference between the E-group and K-group. The E-speakers agreed more with the practical need for mathematics in their daily lives, whereas the K-speakers showed a stronger agreement on the need for mathematics to get into the university of their choice.

Comparability of the E-group and K-group

Similarities and differences between the E-group and K-group

Background analyses showed similarities and differences between the K-group and E-group in terms of their past experience with *infinity* and *limit*, current mathematics learning, and attitudes toward learning mathematics. In students' past mathematics learning regarding *infinity* and *limit*, the K-speakers seemed to have had more opportunities to learn concepts related to *infinity* and *limit* in high school than the E-

speakers. In addition, the grades in which students first heard the word *limit* were less varied in the K-group. However, there was no significant difference between the E-group and K-group in the range of grades in relation to first hearing *infinity*.

In students' present mathematics learning, the E-speakers reported that they were more involved in out-of-school activities and they devoted less time to learning, compared with the K-speakers. Data from the follow-up questionnaire suggested that for K-speakers learning was more important, and thus they systematically devoted more time to it. In students' attitudes toward learning mathematics, the E-speakers showed more confidence in learning mathematics than the K-speakers. As for the role of mathematics in students' lives, the E-speakers had more focused on its application to their daily lives and other school subjects than the K-speakers. However, there was no statistically significant difference between the E-group and K-group in their perceived need of mathematics for their jobs.

Summary

The primary purpose of this study is to examine potential differences in mathematical discourse between E-speakers and K-speakers. However, any differences that might emerge would need to be understood and interpreted with understanding of the differences in students' background that may contribute to those differences in mathematical discourse. While analysing students' discourses on *infinity* and *limit*, all these differences in the background analyses should be kept in mind as possible attributions for observed differences in mathematical discourses on *infinity* and *limit*.

Students' Discourse about Infinity and Limit

In this section, I shall identify primary features of the participants' colloquial and mathematical discourses on *infinity* and *limit*. These features were drawn from two analyses. The first analysis is based on the paired interview data of 20 representatives within each linguistically distinct group. The other is an analysis of all the participants' written responses (n=258) to the discourse questions in each item in the survey. Differences in characteristics of the Korean speaking group (K-group) and the English speaking group (E-group) were identified on the basis of the paired interview results, and then with the help of chi-square analysis applied to the responses to the written questionnaire, it was checked whether the finding could be generalized, that is, whether the differences were statistically significant.

In reporting the data from paired interviews, the following symbolization was introduced. To refer to interviewee members within the E-group and K-group, the symbols such as K6a and K6b were used for the first pair of the K-interviewees who used the Korean 6th textbook in the survey. The symbols of Eoc and Eod represent the second pair of the E-interviewees who used other textbooks. The first characters *K* and *E* represent the Korean and English speakers respectively. The second characters *6* and *o* mean the Korean 6th textbook and other English textbooks respectively. The last characters *a* and *b* in K6*a* and K6*b* represent the first pair of the K-speakers who used the Korean 6th textbook, whereas the characters *c* and *d* in Eo*c* and Eo*d* refer to the second pair of the E-speakers who used other English textbooks.

This section is divided into two main subsections: 2.1 Discourse on *infinity* and 2.2 Discourse on *limit*. Each subsection presents the properties of the given discourse as

reflected in the responses to the relevant items in the questionnaire. Each subsection closes with a summary of characteristics of colloquial and mathematical discourses on infinity and limit respectively.

Discourse on *Infinity*

Items I and IX

Table 4.15 Summary of interview responses using the word *infinite* in items I and IX

I and IX. Create a sentence with the following word								
	E-interviewees			K-interviewees				
		$Infinite_E$ Response	context		$Infinite_{KColl}$ response	context	$Infinite_{KMath}$ response	context
Infinite (adjective), Infinitely (adverb)	E4a	[1] There's an infinite amount of numbers	Math	K1a	[15] Numbers seem to be infinite	Math	[27] The set of A has infinite numbers of elements	Math
	E4b	[2] There's an infinite number of ways to call something different	Non	K1b				
	E4c	[3] The number of y values is infinite	Math	K2a	[16] In an infinite sea, [I] take a walk	Non	[28] The sum of infinite sequences is an infinite series	Math
	E4d	[4] It just never ends	Non	K2b			[29] Infinite spiritual circulation is a key truth of Buddhism	Non
	E5a	[5] There are infinite numbers	Math	K3a	[17] Across the ocean, there is an infinite horizon	Non	[30] I am spinning around an infinite space	Non
	E5b	[6] The function is infinite	Math	K3b				
	E5c E5d	[7] There is an infinite amount of grains of sand	Non	K5a K5b	[18] In the infinite universe, the earth is no more than a dot	Non	[31] My teacher has infinite (*Moo-han-euh*) patience	Non
	E6a E6b	[8] Polynomials have an infinite amount of solutions	Math	K6a	[19] An infinite decimal is a number in which numbers fewer than its decimal point is infinite	Math	[32] I want to be flooded with infinite (*Moo-han-euh*) money	Non
				K6b	[20] I feel deeply thankful for parents' infinite love	Non		
	E7a E7b	[9] The number of careers dealing with mathematics is infinite	Non	K6c K6d	[21] We live through infinite obstacles	Non	[33] An infinite number is sideways 8	Non
	E7c E7d	[10] There's an infinite amount of numbers	Math	K7a	[22] I was locked in infinite sorrow	Non	[34] Do something by ambition of infinite youthfulness	Non
				K7b	[23] Infinite seashore…	Non	[35] Infinite possibility	Non
	Eoa Eob	[11] There are infinite answers to the question	Math	Kma Kmb	[24] He is a child whose complaint is infinite	Non	[36] I had infinite possibility	Non
	Eoc Eod	[12] There are an infinite number of ways that you can do something	Non	Kmc Kmd	[25] Infinite seashore and breathe in clear air	Non	[37] I will take on infinite challenges	Non
							[38] Time has infinite repeatability	Non
	Eoe	[13] There's an infinite amount of stars	Non					
	Eof	[14] Infinite number of possibilities	Non	Kme Kmf	[26] Your arrogance is infinite	Non	[39] The thing putting the side of 8 down which represents infinite state…that's not a number	Non
	Eof	[14] Infinite number of possibilities	Non					

Table 4.16 Summary of interview responses using the word *infinity* in items I and IX

		E-interviewees			K-interviewees				
		I and IX. Create a sentence with the following word							
		Infinity_{E} Response	context		Infinity_{KColl} response	Context		Infinity_{KMath} response	context
Infinity (noun)	E4a	[40] This could go on for infinity	Non	K1a	[56] There are *infinitely* (*Moo-soo-hee*) many answers	Non	[69] The cardinal number of elements *is infinite* (*Moo-han-ha-da*)		Math
	E4b	[41] Nothing goes to infinity. It's not really a number	Non	K1b					
	E4c	[42] *x* approaches infinity…It just keeps going	Math	K2a	[57] There are *infinitely* many people	Non	[70] The set of all numbers is an *infinite* (*Moo-han-han*) set		Math
	E4d	[43] Unending time or space goes forever	Non	K2b					
	E5a	[44] The function goes to infinity	Math	K3a	[58] Among *infinitely* many people, I	Non	[71] Our possibilities *are infinite*		Non
	E5b	because it has no limit		K3b	am one				
	E5c	[45] Nothing can pass infinity	Non	K5a	[59] Stars of the universe are *infinitely* many	Non	[72] Your potentialities *are infinite*		Non
	E5d	[46] Infinity is a definition of something that cannot be reached	Non	K5b					
	E6a	[47] The limit as x approaches infinity of one over x is zero	Math	K6a	[60] I pray for an *infinite* (*Moo-goong-han*) progress	Non	[73] Sequence diverging to infinity		Math
	E6b	[48] It seems to be an infinite amount of something	Non	K6b	[61] Even numbers are the half of *infinite* (*Moo-soo-han*) numbers	Math			
	E7a	[49] It's just such a never-ending process	Non	K6c	[62] Our potentials *are infinite* (*Moo-goong-ha-da*)	Non	[74] People's greed *is infinite*		Non
	E7b			K6d	[63] Integers and rational numbers *are so infinite* (*Moo-soo-ha-da*)	Math	[75] People have *infinite* possibilities		Non
	E7c	[50] Numbers range from negative infinity to positive infinity…go on forever	Math	K7a	[64] Possibility *is infinite*	Non	[76] Human's greed *is infinite*		Non
	E7d			K7b	[65] For *infinite* (*Moo-goong-han*) glory of motherland and nation	Non			
	Eoa	[51] As *x* approaches infinity, *y* continuously decreases	Math	Kma	[66] Idea *is infinite* (*Moo-goong-moo-jin-ha-da*)	Non	[77] At the restaurant there are infinite side dish refills (infinity refills in Korean)		Non
	Eob			Kmb					
	Eoc	[52] It's the biggest number	Non	Kmc	[67] In seashore, there are *infinitely* (*Moo-soo-hee*) many grains of sand	Non	[78] Infinite Challenge [TV program] is funny (Infinity Challenge in Korean)		Non
	Eod	[53] Find a limit as x goes to infinity…it just keeps going	Math	Kmd					
	Eoe	[54] You can never reach infinity	Non	Kme	[68] Among *infinitely* many majors, the department of mathematics education is of course the best	Non	[79] My physical strength *is infinite*		Non
	Eof	[55] It's ongoing like never-ending	Non	Kmf					

Table 4.17 Distribution of the contexts of use of the word *infinite* and the chi-square results of the comparison of difference between the E-group and K-group

Students	Words	Non-mathematical context	Mathematical context	None	Chi-square		
					df	Value	Sig.
E-interviewees (n = 20)	$Infinite_E$	45 %	55 %	0 %			
K-interviewees (n = 20)	$Infinite_{KColl}$	85 %	15 %	0 %			
	$Infinite_{KMath}$	65 %	15 %	20 %			
E-participants (n = 132)	$Infinite_E$	34.1 %	52.3 %	13.6 %	1	55.69	.000*
K-participants (n =126)	$Infinite_{KColl}$	81.7 %	12.7 %	5.6 %			
	$Infinite_{KMath}$	36.5 %	38.9 %	24.6 %	1	1.69	.19

*Differences are significant ($p \leq .01$).

A. *Infinite* and *infinitely*: the type of discourse. E-speakers used the adjective, but *not the adverb*. The use was mainly in *non-mathematical discourse*. K-speakers used both the adjective and the adverb. They used the adverb when they were asked to create a sentence with the noun *infinity* (either mathematical or colloquial). Their uses of *infinitely* and *infinite* was mainly in non-mathematical context (even more so than in the E-group).

The E-interviewees used the adjective predominantly in non-mathematical contexts ([2], [4], [7], [9], [12], [13], and [14]; 7 responses out of 14 in Table 4.15). Among the K-interviewees, the use of mathematical contexts was rare even for the mathematical version of the word *infinite* (2 responses out of 12 for $infinite_{KColl}$ and 2 out of 13 for $infinite_{KMath}$ in Table 4.15). The chi-square results in Table 4.17 show that the distribution of the word use of the E-participants and K-participants in the survey between mathematical and non-mathematical contexts is significantly different when the word *infinite* is colloquial ($\chi^2 = 55.69, p = .000$), but not significantly different when the word is mathematical ($\chi^2 = 1.69, p > .01$).

Table 4.18 Distribution of types of entities described with the word *infinite*

Students	amount or number of something described as *infinite*	self-sustained entities described by *infinite*	Chi-square		
			df	Value	Sig.
E-interviewees (n = 20)	60 %	10 %			
K-interviewees (n = 20)	5 %	60 %			
E-participants (n = 132)	30.3 %	9.9 %	1	67.94	.000*
K-participants (n = 126)	0.8 %	48.4 %			

Table 4.19 Distribution of the use of the word *infinitely* as a descriptor of many

Students	*Infinitely* as a descriptor of many	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	0 %			
K-interviewees (n = 20)	60 %			
E-participants (n = 132)	0 %	1	60.41	.000*
K-participants (n =126)	37.3 %			

*Differences are significant ($p \leq .01$).

B. *Infinite* and *infinitely*: the type of use (meaning). E-speakers' use of the adjective was in the context of *large sets*. In their sentences, the adjective *infinite* appeared in the expressions "infinite number of" or "infinite amount of," that is, as a descriptor of a size of a set. K-speakers used the adjective *infinite* as descriptors of concrete things known for their large dimensions or of spiritual matters. They used the adverb *infinitely* as a descriptor of *many*, in the context of sets.

The majority of E-interviewees used *infinite* as descriptors of *amount of* or *number of* elements in a set such as grains of sand and stars (see [1], [2], [7-8], [10], and [12-14] in Table 4.15). The set was thus treated as one entity. The majority of K-interviewees, with just one exception ([27]), applied the adjective *infinite*, either in its colloquial or mathematical version, directly to self-sustained entities (concrete objects: sea ocean, seashore, earth, and universe – 5 out of 12 responses; spiritual matters: sorrow, love, complains, possibilities, and potentialities in Table 4.15). Only K-interviewees used the adverb, and they did it when asked to use the noun $infinity_{KColl}$ (they added the suffix of *hee* to $infinity_{KColl}$ which turned the noun *infinity* into the adverb *infinitely*). They used it as a descriptor of "many" (see [56-59] and [67-68]). These phenomena were clearly seen also in the written questionnaire. According to the chi-square results in Table 4.18, there is a significant difference between the E-group and K-group in the percentage of sentences in which the word *infinite* is used as describing entities. Chi-square analysis in

Table 4.19 indicates a significant difference between these two groups ($\chi^2 = 60.41, p = .000$) in the use of *infinitely* to describe many.

Table 4.20 Distribution of sentences not using the noun *infinity*

Students	Sentences not using the noun *infinity*	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	0 %			
K-interviewees (n = 20)	95 %			
E-participants (n = 132)	0 %	1	179.91	.000*
K-participants (n =126)	77.8 %			

*Differences are significant ($p \leq .01$).

Table 4.21 Distribution of the use of *infinity* as process

Students	*Infinity* is unreachable and unpassable end-point of never-ending process	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	45 %			
K-interviewees (n = 20)	0 %			
E-participants (n = 132)	15.2 %	1	17.86	.000*
K-participants (n =126)	0.8 %			

*Differences are significant ($p \leq .01$).

C. *Infinity.* E-speakers' use of the word *infinity* was in association with processes, which were said to be *never ending.* The infinity itself was said to be *unreachable* or *impassable.* There was no association of *infinity* with largeness. K-speakers did not use the noun *infinity*, either in its colloquial or mathematical version. When asked to construct a sentence with the $infinity_{KMath}$, they converted the noun into adjective (*infinite*) and when asked to construct a sentence with the $infinity_{KColl}$, they converted the noun into adverb (*infinitely*).

Only one E-interviewee gave a fully structural response that described *infinity* as a *number* ([52]). Others were talking about it in the context of something in an unending process (see the use of verbs "go on/to," "pass," "reach," and "approach" - 14 out of 16 responses in Table 4.16). None of the K-interviewees' responses contained any of these verbs. The single response of a K-interviewee that did contain the noun *infinity* featured the verb *diverge* (in the expression "sequence diverges to infinity" [73]). For some E-

interviewees, *infinity* was tantamount to a never-ending process ([49], [55]). For others, it was an object that was an inextricable part of a process. This object was applied predominantly in the sense of an unreachable endpoint of processes that do not end. The word *infinity* appeared in some other object (x, function, and numbers) that goes to infinity (in an expression such as "x approaches infinity" [42]). Table 4.20 shows that about 78 % of the K-participants used other grammatical forms to create a sentence rather than using the noun. In contrast, the only grammatical forms employed by the E-participants were nouns. A chi-square test in Table 4.21 indicated that the E-participants were significantly more likely to report having used the word *infinity* as an unreachable and unpassable end-point.

Item II

Table 4.22 Distribution of endorsed narratives for comparing pairs of two sets in item II

Comparisons	Students	Endorsed narratives				Chi-square		
		A = B	A > B	A < B	Can't	*df*	Value	Sig.
II-a: Fingers (A) and toes (B)	E-interviewees	25 %	10 %	60 %	5 %			
	K-interviewees	60 %	40 %	0 %	0 %			
	E-participants	39.4 %	14.4 %	38.6 %	0 %	2	47.94	.000*
	K-participants	56.3 %	28.6 %	3.2 %	0 %			
II-b: Odd (A) and even numbers (B)	E-interviewees	45 %	25 %	25 %	0 %			
	K-interviewees	30 %	15 %	30 %	20 %			
	E-participants	41.7 %	17.4 %	22.0 %	9.1 %	3	13.53	.004*
	K-participants	21.4 %	20.6 %	19.8 %	19.8 %			
II-c: Odd numbers (A) and integers (B)	E-interviewees	10 %	10 %	80 %	0 %			
	K-interviewees	5 %	5 %	70 %	20 %			
	E-participants	15.2 %	8.3 %	61.4 %	3.8 %	3	15.61	.001*
	K-participants	6.4 %	1.6 %	73.0 %	10.3 %			

Note. "A=B" = A is the same as B; "A>B" = A has more than B; "A<B" = B has more than A; "Can't" = A and B are incomparable. *Differences are significant ($p \leq .01$). Missing data were not included in the table.

D. II-a: A = fingers and B = toes. Almost all E-speakers compared the two sets according to the number of elements, and if they did not choose the option A = B, it was only because of the fact that they interpreted the word *finger* as not referring to the thumb. The 40% of K-speakers who chose an option other than A = B did so because their

comparison was non-numerical. In other words, they responded fingers mainly because they thought fingers were used more often than toes.

E. II-b: A = odds and B = evens. Only two fifths of E-speakers chose the canonical response A = B. Other responses were distributed evenly between A < B and A > B. Only half as many K-speakers chose the canonical option A = B. Other responses were distributed evenly between A < B, A > B, and "can't say". Some E-speakers (17.4 %) and K-speakers (20.6 %) determined that there were more odd numbers because counting started at one. Other E-speakers (22.0 %) and K-speakers (19.8 %) concluded that there were more even numbers because zero as a starting number was an even number. However, only K-speakers (19.8 %) replied that even numbers could not be compared with odd numbers because they were both infinite and countless.

F. II-c: A = odds and B = integers. Only 10%-15% of E-speakers chose the canonical response A = B. An overwhelming majority (80% in the interview and 61.4% in the survey) chose the option A < B. The percentage of K-speakers who chose the canonical response A = B was even smaller than that of E-speakers (5% in the interview and 6.4% in the survey). Approximately 70% chose the option A < B.

Table 4.23 Comparisons between fingers and toes in item II-a

II. Of which are there more? (a) A: Your fingers, B: Your toes			
E4a E4b	[1] Toes. I don't count thumbs as a finger [2] Neither. I don't know if you can do that	K1a K1b	[17] They are the same because fingers are 10 and toes are 10 and the cardinal numbers are decided exactly
E4c	[3] Toes because thumbs aren't part technically fingers	K2a	[18] Same because of ten and ten
E4d	[4] Fingers because they're longer and there's more mass to your finger	K2b	[19] Fingers because fingers are more sensitive
E5a	[5] Fingers because they're longer	K3a	[20] Fingers because fingers are used more
E5b	[6] Numerically they're the same	K3b	[21] Fingers have senses and can move on their own
E5c E5d	[7] There are more toes because technically thumbs aren't fingers	K5a K5b	[22] Same...they are natural numbers [23] From what is being used, fingers are more
E6a E6b	[8] Toes...you only have eight fingers and ten toes	K6a K6b	[24] They are the same...the numbers to be seen are the same
E7a E7b	[9] Your toes because thumb doesn't count [10] There are exactly as many fingers or toes so there's neither more	K6c K6d	[25] Fingers are used more than toes [26] They are both 10 and 10, so they are the same
E7c E7d	[11] Toes...Because thumbs don't count	K7a K7b	[27] I counted them so that they were the same
Eoa Eob	[12] Neither one of them are more [13] I checked both boxes...because you have an equal number	Kma Kmb	[28] Same...each of them is 10...because they are finite
Eoc	[14] Toes, because thumbs aren't fingers	Kmc	[29] We use fingers much more...there are more fingers
Eod	[15] Neither...ten fingers and ten toes	Kmd	[30] Fingers are ten and toes are ten
Eoe Eof	[16] Toes...because thumbs don't count	Kme Kmf	[31] Their cardinal numbers are the same...there are 10 and 10 [32] Fingers are more highly valued in utilization

Table 4.24 Comparisons between odd and even numbers in item II-b

II. Of which are there more? (b) A: Odd numbers, B: Even numbers			
E4a	[33] Even numbers only if you count zero…I think it's the same…you go in either direction 'cause like either way there's no set limit	K1a	[52] They are same…because if we count from 1 to 10
E4b	[34] If you keep going with the numbers…because there's not like a set value of how many there are…It's still kind of the same thing	K1b	[53] If I send 2k and 2k+1 to infinity, the cardinal numbers are infinite…they are the same because it's a half of natural numbers
E4c	[35] It'd be even [same] because the numbers go on for infinity	K2a	[54] Even numbers because it starts from 0 with a similar thought
E4d	[36] You start with zero…zero is before all the numbers and so…more even numbers, but that's not right. So, I think there's the same too	K2b	[55] Odd numbers are more because numbers start from 1…because one who wins first round finally gains a victory
E5a	[37] There are 1 and -1 and 2 and -2…They're even…unless zero's an even number…there would be more evens	K3a	[56] Because there are both even and odd numbers within odd numbers, odd numbers seem to be more
E5b	[38] They're the same because I was thinking of the whole number line like it goes to infinity	K3b	[57] I checked even…odd numbers were incomplete numbers. If there is an odd number, you can add one and make it an even
E5c	[39] I said even 'cause I thought zero was an even number…I realized that he's more right on than I was	K5a	[58] They are incomparable because they are countless
E5d	[40] Odd numbers 'cause you start with one…same amount or more odd	K5b	[59] As n goes to infinity, the ratio, 2n over 2n-1 equal 1…same
E6a	[41] Odd numbers and even numbers go to infinity…for every even	K6a	[60] Even numbers…0 seems to be included in even numbers
E6b	number there is an odd number that comes after it…for every odd number there is an even…if you're including zero….one more even	K6b	[61] We cannot count infinite things. We cannot say that one is more than the other
E7a	[42] Amount infinite…odds...the number line starts with an odd number	K6c	[62] 2, 1st even, is a greater number than 1, 1st odd…more even
E7b	[43] Neither, because…an infinite number in a set…can't put a cap on…it [infinity] is really the same thing	K6d	[63] Odd, even, odd, even, in this pattern…it is the same. If there is another 0, [the answer is] even number
E7c	[44] Because numbers go to infinity…you can't really have more [same]	K7a	[64] More even numbers…0 is even…either equal or more evens
E7d	[45] Odd numbers because you start at one and one is an odd number	K7b	[65] They're infinite…comparison of cardinal numbers impossible
Eoa	[46] I wrote odd numbers because counting always starts at 1…but I agree with what she said more…because numbers go on forever	Kma	[66] If it's infinite, we cannot count the cardinal number…[infinities are] same in that state
Eob	[47] There's the same number of both because they're infinite, like you could keep counting…	Kmb	[67] Because we correspond by one-to-one, the cardinal numbers are same
Eoc	[48] There's more even numbers because 0 is even…keep going out…if you don't think of zero, then there are equal amounts	Kmc	[68] Even numbers, if we multiply odd numbers by 2, they become even numbers…they are more than or as same as odd numbers
Eod	[49] Neither [same] because you have an infinite amount of numbers…go to infinity. I forgot about 0 so that makes more sense…more even	Kmd	[69] They seem to be the same…it's not an inclusive concept…because starting points are different and comparison is impossible
Eoe	[50] There's more or less equal amount, because infinity goes on and on	Kme	[70] It's not possible to decide their cardinal numbers…infinite
Eof	[51] Two different thoughts…infinite amount of numbers so it goes on and on, but, some people consider zero to be an even number	Kmf	[71] Odd numbers because even numbers can be made from odd numbers…once by additions

Table 4.25 Comparisons between odd numbers and integers in item II-c

II. Of which are there more? (c) A: Odd numbers, B: Integers			
E4a	[72] Integers because I thought that you'll go on for infinity and it'd be like twice as more	K1a K1b	[88] Because within integers, there are natural numbers…within natural numbers, there are odd numbers...because it's a bigger set…a half of the cardinal number this set [A] is [that of B]
E4b	[73] Keep going up to infinity…same amount…I had to say the same		
E4c	[74] Integers because there are infinite numbers…they go on…for every odd number you have, you get an integer.	K2a	[89] Integers because we don't know where the end is…odd numbers are included in integers
E4d	[75] Every whole number is an integer…that's twice as many as odds	K2b	[90] Between integers 1 and 2 there are a lot of odd numbers…1.1
E5a	[76] There's more odd numbers because of that there's like one point whatever…it's like infinity squared versus infinity	K3a	[91] Because there are even and odd numbers within integers
E5b	[77] Because they both go to infinity…I feel like they have to be the same	K3b	[92] Odd and even numbers are natural numbers. In addition, 0 and negative numbers are also parts of integers
E5c E5d	[78] Integers because it's both odd and even numbers	K5a	[93] Because both are infinitely many…cardinal numbers are incomparable
		K5b	[94] Odd numbers are included in integers
E6a E6b	[79] There are more integers… odd numbers plus even numbers equals integers	K6a K6b	[95] Integers are divided into odd and even numbers. Because integers are a bigger range, the cardinal number of integers is more
E7a	[80] Its rate of reaching infinity…So it not only has the set of odd numbers in it but also it has even numbers.	K6c K6d	[96] Integers are more…odd numbers and integers are infinite … countless numbers…integers include odd numbers
E7b	[81] Integers would always be twice the amount of odd numbers.		
E7c E7d	[82] Integers…because every number is an integer	K7a	[97] More integers, because integers include negative integers
		K7b	[98] It not possible to compare two infinities…comparison of cardinal numbers is impossible
Eoa	[83] Integers…because it [an integer] is just a whole number so it can be even or odd.	Kma Kmb	[99] Because infinity or multiplication of infinities or infinity multiplied by 3…they all are infinity, they are same
Eob	[84] More odd numbers because that would include decimals and fractions…I changed my answer…because I didn't know…integer		[100] Integers…continues to go to there [negative direction]
Eoc	[85] It was tied because you have an infinite number of odd numbers and integers… that [integers are more] makes more sense	Kmc Kmd	[101] Integers…because integers include odd numbers
Eod	[86] Integers because integers just include all numbers		
Eoe Eof	[87] Integers…odd numbers are included inside the set of integers. But then says numbers up to end or still infinite, you really can't say that for sure	Kme	[102] Because they become one-to-one correspondence…because they both are infinite, it's not possible to compare them
		Kmf	[103] It's not possible to compare infinity with infinity. When within the produced sets [through an operation], the opposite set is included, the set is larger…integers

Table 4.26 Distribution of phrase-driven and structural uses of infinity

Comparisons	Students	Phrase-driven use	Structural use	Chi-square		
				df	Value	Sig.
II-b. Even and odd numbers	E-interviewees	55 %	0 %			
	K-interviewees	5 %	40 %			
	E-participants	30.3 %	0.8 %	2	73.84	.000*
	K-participants	1.6 %	33.3 %			
II-c. Odd numbers and integers	E-interviewees	30 %	5 %			
	K-interviewees	0 %	35 %			
	E-participants	29.5 %	2.3 %	2	73.98	.000*
	K-participants	0 %	34.1 %			

Note. *Differences are significant ($p \leq .01$).

G. K-speakers' discourse on infinite sets appeared more *structural* than that of E-speakers, notwithstanding the fact that only E-speakers used the noun *infinity*. K-speakers explicitly referred to sets and to cardinal numbers associated with sets, whereas such reference was practically non-existent in the E-speakers' responses. The E-speakers' use of the noun *infinity* was phrase-driven rather than structural. In other words, the noun *infinity* appeared in phrases such as "goes [on] to infinity".

Many E-interviewees spoke about infinite sets in items II-b and II-c in terms of processes ("goes to infinity", "goes on and on", etc.; see [34-35], [38], [44], [46-51], [72-74], and [77] in Tables 4.24 and 4.25). Such reference was rare among K-interviewees and when it did appear, it was often formulated in a formal mathematical way (e.g., "I send 2k and 2k+1 to infinity" [53] or "As n goes to infinity..." [59] in Table 4.24). The majority of K-interviewees made reference to cardinal numbers associated with sets (see [53], [65-67], [70], [88], [93], [95], [98], and [103] in Tables 4.24 and 4.25). The chi-square results in Table 4.26 show that the difference between E-respondents and K-respondents found in the interview also appeared in written survey responses.

H. Eleven routines for comparing sets were identified in the interviews and the survey:

1. Comparing by counting: Some interviewees counted elements in A and elements in B, compared the counting of elements in A with those in B, and then endorsed that A or B had more elements because the counting of elements for the set A or B ended up with a larger number (see [52] in Table 4.24).

2. Comparing that involves treating size of infinite sets as subject to the same rules as that of finite sets (doubling increases, halving makes smaller): The students calculated the number of elements in the given set from some data. For example, the students would state that two sets are equal because they are "halves" of the same infinity. The amount of elements for given infinite sets consists of half parts of the same infinity (see [53] in Table 4.24 and [72], [81], and [88] in Table 4.25).

3. Comparing the numbers of elements in two sets without specifying how the numbers were found.

4. Comparing the size of corresponding elements of the two sets: There were interviewees who created a one-to-one mapping between two sets A and B and checked whether the number of elements of one set was consistently bigger than the counterparts from the other set. If the elements of the set A were bigger than those of B, for instance, they endorsed that A had more, A was bigger, or there were more elements in A (see [62] and [71] in Table 4.24).

5. Comparing the size of corresponding partial sums of the two sets: Other interviewees focused on comparing the magnitude of corresponding partial sums. They interpreted that if the partial sums of one set could be shown to be consistently

greater than those of the other set, they endorsed that the set that was greater in partial sums, had more, or was larger (see [8] in Table 4.32).

6. Checking two sets for inclusion of one of them in the other: Some other interviewees checked whether one set was included in the other (e.g. by checking the relative density of the two sets on the number line). If one set included the other, they concluded that the including set had more or was larger (see [40] in Table 4.24 and [94] in Table 4.25).

7. Comparing the numerical range covered by the elements of the sets: The students would deem a set of numbers as 'bigger' if it spanned a larger segment of the number line (see [45] in Table 4.24).

8. Constructing one-to-one mapping between the sets: There were interviewees who built a one-to-one mapping between A and B in any way. If there was a mapping from A to B, then they endorsed either that A and B were equal if the mapping was a surjection on B, or that B had more (or B was greater than A) if the mapping was an injection on B (see [67] in Table 4.24).

9. The use of the assumption that any two infinite sets are equal: Some interviewees endorsed that A and B were equal if they saw that both sets were infinite (see [43] in Table 4.24).

10. The use of the assumption that infinite sets are incomparable: Other interviewees endorsed that A and B could not be compared if they saw that both sets were infinite (see [61] in Table 4.24).

11. Non-numerical comparisons (see [19] in Table 4.23).

In addition, in several cases we opted for the following categories

12. Unidentified routine: This is the case, for example, when the student says that one set has 'more elements' than the other.

13. No answer

Table 4.27 Distribution of *routines* in comparison tasks in item II

Routine	E-speakers				K-speakers			
	interview		Survey		interview		survey	
	II-b	II-c	II-b	II-c	II-b	II-c	II-b	II-c
1) Comparing by counting	0 %	0 %	0.8 %	2.3 %	5 %	0 %	2.4 %	2.4 %
2) Comparing that involves treating size of infinite sets as subject to the same rules as that of finite sets	0 %	15 %	0.8 %	7.6 %	5 %	0 %	1.6 %	0.8 %
3) Comparing the number of elements in two sets without specifying how the numbers were found	0 %	0 %	0 %	0 %	0 %	0 %	0 %	0 %
4) Comparing the size of corresponding elements of the two sets	0 %	0 %	2.3 %	0 %	20 %	0 %	7.9 %	0 %
5) Comparing the size of corresponding partial sums of two sets	0 %	0 %	0 %	0 %	0 %	0 %	0 %	0 %
6) Checking two sets for inclusion of one of them in the other	0 %	75 %	0 %	50.0 %	0 %	80 %	0 %	67.5 %
7) Comparing the numerical range covered by the elements of the sets	40 %	0 %	32.6 %	0 %	25 %	0 %	31.0 %	0 %
8) Constructing one-to-one mapping between the sets	0 %	0 %	0.8 %	0 %	10 %	0 %	9.5 %	4.0 %
9) The use of the assumption that any two infinite sets are equal	60 %	10 %	44.7 %	16.7 %	5 %	5 %	1.6 %	2.4 %
10) The use of the assumption that infinite sets are incomparable	0 %	0 %	6.1 %	2.3 %	20 %	15 %	22.2 %	10.3 %
11) Non-numerical comparisons	0 %	0 %	1.5 %	0 %	0 %	0 %	3.2 %	0.8 %
12) Unidentified routine	0 %	0 %	3.8 %	9.9 %	0 %	0 %	6.4 %	3.2 %
13) No answer	0 %	0 %	9.1 %	11.4 %	0 %	0 %	14.3 %	8.7 %

I. **In comparing A = odds and B = evens (II-b),** E-speakers tended to claim that two infinite sets are always equal (routine 9, 44%). Only very few K-speakers used routine 9 (5% and 1.6% in the interview and in the survey, respectively). They relied more strongly than E-speakers on the endorsed narratives that *infinite sets cannot be compared* (routine 10; 22.2% for K-speakers versus 6.1% for E-speakers). This is due to the fact that the Korean word for infinity was *countless*, that is, "which cannot be counted," and the Korean participants simply derived that what cannot be counted cannot be compared.

J. In comparing A = odds and B = integers (II-c), both groups relied strongly on the endorsed narrative "proper subset is smaller/has less elements than the whole set" (or "infinite set that includes 'half as much elements' as another infinite set is smaller) (routine 6; E-speakers: 50% and K-speakers: 67%)

Table 4.28 Distribution of contradictory answers in items II-b and II-c

Item				E-speakers		K-speakers	
II-b: odds (A) and even (B)		II-c: odds (A) and integers (B)					
Answer	Routine	Answer	Routine	Interview	survey	Interview	survey
Equal	2	A is smaller	6	0 %	0 %	5 %	1.6 %
Equal	8	A is smaller	6	0 %	0 %	5 %	4.0 %
Equal	9	A is smaller	6	40 %	26.5 %	10 %	7.9 %
Incomparable	10	A is smaller	6	0 %	1.5 %	10 %	11.1 %

K. In items II-b and II-c, there were respondents who produced two contradictory answers. The contradiction was usually the result of the fact that the respondent endorsed two narratives: 1) If A is included in B then A is smaller and 2) If A and B are infinite then they are equal (or cannot be compared). This contradiction appeared as the result of applying two different routines of comparison. The respondents' routines are presented in Table 4.28.

The following Episodes 1 and 2 are excerpts from the conversations with an English speaking pair (E-pair) and a Korean speaking pair (K-pair) respectively in comparing even numbers with odd numbers and odd numbers with integers (see Appendix E for the Korean original conversation). These two episodes exemplify two contradictory answers, and interview procedures between the two interviewees and the interviewer indicated by "I."

Episode 1. E-pair in comparing odds with evens and odds with integers

Speaker	What was said	What was done
38. I	**[show the card "A: Odd numbers B: Even numbers"]** What about, you know, odd number and even number?	Show the card
39. E4c	We do technically...I'd believe it'd be even because the numbers go on for, you know, infinity, and there are...they're infinite numbers...so every time you go up one, go down one, etcetera. It's either an odd	

	number or even number and it's never ending. So, I'd say that it would be even because we know that there won't be one more odd number than one more even number.	
40. E4d	Yeah, that's pretty much it, that's what I put too, I think.	
41. I	You first responded, you know...	
42. E4d	[take a look at his questionnaire] Oh, I put even numbers I think. Yeah.	
43. DJ	Right	
44. E4d	I don't know what I think...I was just trying to figure out the answer so I put even numbers because, uh, this doesn't even make sense now that I look at it again. So, it was just, kind of, putting something down, but, I mean, I couldn't think of a way that you could have more even numbers or more odd numbers. So, I think there's the same too, I just thought we had to check a box, so I checked a box.	
45. I	But then first, you know, when you checked even numbers, did you think of anything?	
46. E4d	Uh, just kind of, um, zero is an even number	
47. I	Uh-huh	
48. E4d	And so, you start with zero, and then, for some reason I was thinking, the numbers, it just seems like zero is before all the numbers and so there would be more even numbers, but that's not right...I don't think.	
49. I	**[show the card "A: Odd numbers B: Integers"]** What about odd number and integers?	Show the card
50. E4c	Uh, I said integers because an odd number can be an integer. So there are, uh, and then even numbers are also integers	
51. E4d	Yes	
52. E4c	So for every odd number you have, you get an integer anyway, so even though that they go on, for, um, you know, there...there are infinite numbers, or infinite integers, it's still odd numbers are integers. So if there was a breaking point, there'd be more integers.	
53. E4d	Yeah, it's...I have integers too just because every whole number is an integer, and any, I think, isn't it? OK. And, uh, I just thought I'd make sure before I said that, um, and, I mean, that's twice as many as there are odd numbers if you have the odd and even.	

Episode 2. K-pair in comparing odds with evens and odds with integers

45. I	Now then **[show the card "A: Odd numbers B: Even numbers"]** of which are there more? Please check one of the boxes. How do you know?	Show the card
46. K6b	I didn't check this one either	
47. I	Um...	
48. K6b	Because odd numbers and even numbers are not finite	
49. K6a	I checked on even numbers.	
50. I	Um...	
51. K6a	Just 0 is...in my feeling, 0 seems to be included in even numbers. So, I feel such a feeling though, but it seems not to be.	
52. K6b	So, looking back, it seems not to be...	
53. I	Please talk with each other about the problem because you have different thoughts.	
54. K6b	Key point is whether or not 0 is an even number.	
55. K6a	But, if it is seen from a certain point, we seem not to be able to decide that 0 is only either an even or an odd number. In my feeling, 0 looks like so...	
56. K6b	But, when they are expressed with n, even numbers are expressed as 2n and odd numbers as 2n-1 like these....but if we substitute 0 for	

	n...then....Uh! It seems to be right. [...] But, my thought is ....and....after listening to that, it seems to be right....what I thought...so, we can count finite numbers, can't we? Because I thought that we cannot count infinite things, I answered so.	
57. I	So, because they both are infinite...	
58. K6b	Yes, they are infinite.	
59. I	Um...	
60. K6b	As a result, because there is no end in a single word...I thought that we cannot say that one is more than the other.	
61. I	Um...then **[show the card "A: Odd numbers B: Integers"]** What do you think about the next problem? About odd numbers and integers	Show the card
62. K6b	Oh, integers	
63. K6a	Me too. Integers	
64. I	The reason is?	
65. K6a	I thought that integers are....integers are divided into odd and even numbers. So, because integers are a bigger range...I thought that the cardinal number of integers is more.	
66. K6b	Because I thought that the set of odd numbers is included in integers.	
67. I	Do you have anything else to add?	
68. K6b	No	

Item III

Item III in the questionnaire was very similar to the third question in item II except for three differences. First, in item III, odd numbers were replaced with even numbers to be compared with integers. Second, the sets A (integers) and B (even numbers) in item III were not described in words, but represented horizontally and in parallel by using the set builder notation. Finally, the wording of the question in item III was different from that in item II. In item III, the question, "Which of the two sets A and B is bigger?" was used, whereas the question, "Of which are there more?" was asked in item II. The intention of item III was to check the impact of these technical differences, especially of the different wording by the use of the adjective "bigger," which implicates reification of the infinite set, instead of the adverb "more" that is less dependent on reification.

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Table 4.29 Distribution of *endorsed narratives* in item II-c and item III

Comparisons	Students	Endorsed narratives				Chi-square		
		A = B	A > B	A < B	Can't	*df*	Value	Sig.
III. Integers (A) and evens (B)	E-interviewees	15 %	30 %	55 %	0 %			
	K-interviewees	15 %	60 %	0 %	25 %			
	E-participants	23.5 %	19.7 %	47.7 %	4.5 %	3	93.68	.000*
	K-participants	18.3 %	56.3 %	0.8 %	19.8 %			
II-c. Odds (A) and integers (B)	E-interviewees	10 %	10 %	80 %	0 %			
	K-interviewees	5 %	5 %	70 %	20 %			
	E-participants	15.2 %	8.3 %	61.4 %	3.8 %	3	15.61	.001*
	K-participants	6.4 %	1.6 %	73.0 %	10.3 %			

Note. *Differences are significant ($p \leq .01$). Missing data in item III were not included in the table (4.5 % of the E-participants and 4.8 % of the K-participants).

L. The changes in the question III in comparison to II-c did have an impact on the responses of E-participants but not on those of K-participants. K-participants' responses to item III showed similar distribution of the four endorsed narratives (Notice integers (B) in item II-c and integers (A) in item III; 73 % and 56.3% in items II-c and III respectively). In other words, a dominant endorsed narrative in the K-group was that integers were bigger (or more) than evens (or odds). However, a major endorsement in the E-group had been changed from item II-c to III (Integers; 61.4% in item II-c versus evens; 47.7% in item III). Many E-participants changed their endorsed narratives between items II-c and III because of different prompts (*more* in item II-c and *bigger* in item III).

Table 4.30 Distribution of the word use of *bigger*

Students	More numbers	Higher values	Chi-square		
			df	Value	Sig.
E-interviewees	45 %	50 %			
K-interviewees	100 %	0 %			
E-participants	38.6 %	46.2 %	2	82.12	.000*
K-participants	86.5 %	0 %			

*Differences are significant ($p \leq .01$).

M. The change in wording influenced strongly the way E-participants implemented the comparison of sets, but it did not change K-participants' choices. The word

bigger seemed to guide the E-speakers to two different understandings: more numbers and higher values. About 50% of the E-participants indicated that the meaning they ascribed to *bigger* meant *increasing faster*. Some of the E-interviewees explicitly hesitated between the two possible interpretations of *bigger* in this context. In contrast, all of the K-interviewees accepted this question as being the same as the previous one in item II ("Of which are there more?").

Table 4.31 Distribution of *routines* in comparison tasks in items II-c and III

Routine	E-speakers				K-speakers			
	Interview		Survey		interview		survey	
	II-c	III	II-c	III	II-c	III	II-c	III
1) Comparing by counting	0 %	10 %	2.3 %	0.8 %	0 %	0 %	2.4 %	10.3 %
2) Comparing that involves treating size of infinite sets as subject to the same rules as that of finite sets	15 %	0 %	7.6 %	1.5 %	0 %	5 %	0.8 %	2.4 %
3) Comparing the number of elements in two sets without specifying how the numbers were found	0 %	0 %	0 %	0 %	0 %	0 %	0 %	0 %
4) Comparing the size of corresponding elements of the two sets	0 %	50 %	0 %	43.2 %	0 %	0 %	0 %	0.8 %
5) Comparing the size of corresponding partial sums of two sets	0 %	5 %	0 %	2.3 %	0 %	0 %	0 %	0 %
6) Checking two sets for inclusion of one of them in the other	75 %	20 %	50.0 %	15.9 %	80 %	55 %	67.5 %	42.1 %
7) Comparing the numerical range covered by the elements of the sets	0 %	0 %	0 %	2.3 %	0 %	0 %	0 %	0 %
8) Constructing one-to-one mapping between the sets	0 %	0 %	0 %	0.8 %	0 %	10 %	4.0 %	12.7 %
9) The use of the assumption that any two infinite sets are equal	10 %	15 %	16.7 %	23.5 %	5 %	5 %	2.4 %	3.2 %
10) The use of the assumption that infinite sets are incomparable	0 %	0 %	2.3 %	3.8 %	15 %	25 %	10.3 %	21.4 %
11) Non-numerical comparisons	0 %	0 %	0 %	0 %	0 %	0 %	0.8 %	0 %
12) Unidentified routine	0 %	0 %	9.9 %	1.5 %	0 %	0 %	3.2 %	2.4 %
13) No answer	0 %	0 %	11.4 %	4.6 %	0 %	0 %	8.7 %	4.8 %

N. A dominant *routine* in the E-group was changed from routine 6 in item II-c to routine 4 in item III but not in the K-group. In both items II-c and III, a dominant comparison routine in the K-group was routine 6 (67% and 42% respectively). In contrast, 43 % of the E-participants used routine 4 in item III, compared to routine 6 (50%) in item

II-c. In other words, they employed the routines of "which one gets to infinity faster?" with the conception of increasing faster in the use of the word *bigger*.

The following Episodes 3 and 4 exemplify the differences in *word use* and *routine* between the E-group and K-group.

Episode 3. E-pair in comparing integers (A) with evens (B)

Speaker	What was said	What was done
50. I	**[show the card about item III]** What about this problem...which of the two sets A and B is bigger? How do you know?	Show the card
51. E7a	Um I said A since um even though the uh set continues um it includes more integers within uh... a set amount of uh like say if it was like 1 to 100 um... A would include more integers than B would.	
52. E7b	Um this reminded me of the set before it except that I took the opposing stance this time. I said because B is increasing at a greater rate than A um... it would just be increasing to infinity just at a faster rate so that's why I decided that B uh I thought was larger...Mm...It depends 'cause bigger is such a vague term...like you can...	
53. E7a	Yeah it's...I didn't really know whether to say like contains more values or has a greater amount inside of it or...	
54. E7b	Mm-hmm. Because they're the same number of values in each set...it's just the range on...on B is greater except A contains more...I guess...precise values or...	
55. E7a	Yeah	
56. E7b	Or uh smaller interval between values so...I guess that just depends on your uh idea of bigger and I just went in terms of its rate of increase.	
57. I	So do you want uh say something more-	
58. E7a	Um...	
59. I	About the other's opinion?	
60. E7b	It's...it's really hard to determine like you know differ between opinions since everything is-in infinity its all subjective to what your views on infinity are...it's like um...I mean...I...I mean I can see from his standpoint how-where he comes from his conclusions so it...it's logical.	

Episode 4. K-pair in comparing integers (A) with evens (B)

111. I	Then **[show the card about item III]**	Show the card
112. K3b	I said it was A.	
113. K3a	Me too.	
114. I	Why did you say it was A?	
115. K3b	When explaining, I interpreted it this way. Of course there are preconditions and even though you don't know what happens later, if [numbers are] listed according to this rule, A will be a set of natural numbers. The set B will be a multiple of 2, which is also a set of even numbers. Similar to the reasoning above, natural numbers can be divided into even and odd numbers, so B is included in A. Because of this reasoning, I thought A was bigger.	
116. K3a	Because they are sets whose cardinal numbers cannot be counted, I thought we could show what is big and small by the method of taking away if you can't count them. If you are figuring out who is tall and who is small, you can find out who is tall by a difference when they are	

standing. Just like this, you can see who is tall without measuring. Based on this concept, you can take away B from A and you will still have something left, so I answered that A was 100% bigger.

117. I Do you have any other thoughts about this question?

118. K3a No

Table 4.32 Comparisons between even numbers and integers in item III

III. Which of the two sets A and B is bigger? How do you know? A = {1, 2, 3, 4, 5, 6, 7, 8, ...} and B = {2, 4, 6, 8, 10, 12, 14, 16, …}			
E4a	[1] More numbers…that'd be the same…B because it goes up by two	K1a	[21] A…natural numbers are a bigger set than even numbers
E4b	[2] If you're going to infinity, they would be the same	K1b	[22] Half of the cardinal number of A is the cardinal number of B
E4c	[3] They are equal because they keep going on	K2a	[23] Numbers in B are to A, but those in A do not exist to B
E4d	[4] B because I thought they meant bigger numbers, and so if you add them all up, it's going to be a bigger, even though it's an infinite series	K2b	[24] A goes out more slowly than B. So, because [A] increases by 1 and B increases by 2
E5a	[5] A…because it includes every single number to infinity and B would only have half as many numbers	K3a	[25] You can take away B from A and you will still have something left, so I answered that A was 100% bigger
E5b	[6] B cause B is always ahead of A…for any value of A, B will be twice that. It depends on how you define bigger like more numbers or like higher values	K3b	[26] A…a set of natural numbers. B…a multiple of 2, a set of even numbers. B is included in A. A was bigger
E5c	[7] A would have more numbers because it is just going up by integers	K5a	[27] Incomparable because both are infinite…not cardinal number
E5d	[8] Since these numbers are larger in size, if you added them up, it'd be a bigger total [B]. Now, I would agree with Andrew…the set bigger	K5b	[28] It means set A includes set B… incomparable because both of them are infinite (corresponded one-to-one and nothing left by K5a)
E6a	[9] B…this increases by one every time and this…by two every time…this [B] gets to infinity faster	K6a	[29] A because the cardinal number of A is more…even though they both are infinite
E6b	[10] Since B is increasing faster…it will reach higher numbers faster	K6b	[30] Because they are infinite, we cannot decide whether or not one is more than the other
E7a	[11] The set continues…a set amount…A includes more integers than B. I didn't really know…more values or a greater amount inside of it	K6c	[31] A was bigger because A includes B
E7b	[12] B is increasing at a greater rate than A…bigger is a vague term	K6d	[32] They are the same…if you simply just double the elements, the numbers are following the same rule
E7c	[13] If this sequence went to 16, there'd be sixteen and eight numbers	K7a	[33] A is bigger, [because] 3 exists between 2 and 4 [denser]
E7d	[14] A…because there'd be more numbers if you went from one, from 2 to 8…seven numbers. But here, there's only four	K7b	[34] In the same way, comparing cardinal numbers is impossible
Eoa	[15] Neither because they're both continuing on forever…B is bigger because it's like B if infinitely two times bigger than A	Kma	[35] This one is also the same because they all are infinite
Eob	[16] B was bigger because it's increasing at twice the rate of A…they're both continuing on forever	Kmb	[36] Same...because there is a one-to-one correspondence by this one [A] as n and this [B] as 2n
Eoc	[17] B because it's going faster…B is the A function times two	Kmc	
Eod	[18] Neither [same] again…because they both go to infinity. B's increasing more rapidly…B would be bigger	Kmd	[37] A…this seems to be the same as the previous problem.
Eoe	[19] A because it's going all integers…A is all integers and B is just even integers…there's an infinite amount of numbers	Kme	[38] Because they are infinite, it's not possible to compare them, but there is a one-to-one correspondence
Eof	[20] Neither…same thing…because sets have no end…it continues	Kmf	[39] A because within A, B is included

Table 4.33 Distribution of processual and structural uses of infinity

Comparisons	Students	Processual use	Structural use	Chi-square		
				df	Value	Sig.
III. Integers and evens	E-interviewees	90 %	10 %			
	K-interviewees	25 %	75 %			
	E-participants	46.2 %	6.1 %	2	53.32	.000*
	K-participants	13.5 %	38.1 %			

Note. *Differences are significant ($p \leq .01$).

O. The phenomenon first observed in [G] returned. Many E-speakers spoke about infinite sets in terms of processes ("goes to infinity", "goes on and on", "continues forever", "keeps going on" etc.). Such reference was rare among K-speakers. Once again, the Korean responses were formulated in a more formal mathematical way (e.g., "there is one-to-one correspondence" (see [28] and [36])). K-speakers also explicitly referred to cardinality of sets ([22], [27], [29], and [34]). Such reference did not appear in the E-speakers' responses. According to chi-square statistics in Table 4.33, this difference between the E-group and K-group was also significant in the survey participants' written responses.

Item IV

In item IV, the relation between two infinite sets was unspecified. The wording in item IV was different from those in items II and III. The word *larger* in item IV was used to compare two infinite sets rather than *more* in item II or *bigger* in item III.

Table 4.34 Distribution of *routines* in comparison tasks in items III and IV

Routine	E-speakers				K-speakers			
	interview		Survey		Interview		survey	
	III	IV	III	IV	III	IV	III	IV
Rejected the possibility of inequality of infinite sets (claimed infinite sets must be equal or incomparable)	15 %	20 %	27.3 %	15.2 %	30 %	5 %	24.6 %	7.9 %
1) Comparing by counting	10 %	0 %	0.8 %	3.0 %	0 %	5 %	10.3 %	3.2 %
2) Comparing that involves treating size of infinite sets as subject to the same rules as that of finite sets	0 %	0 %	1.5 %	0.8 %	5 %	0 %	2.4 %	0 %
3) Comparing the number of elements in two sets without specifying how the numbers were found	0 %	15 %	0 %	13.6 %	0 %	0 %	0 %	1.6 %
4) Comparing the size of corresponding elements of the two sets	50 %	25 %	43.2 %	17.4 %	0 %	0 %	0.8 %	16.7 %
5) Comparing the size of corresponding partial sums of two sets	5 %	0 %	2.3 %	2.3 %	0 %	0 %	0 %	0 %
6) Checking two sets for inclusion of one of them in the other	20 %	15 %	15.9 %	16.7 %	55 %	75 %	42.1 %	48.4 %
7) Comparing the numerical range covered by the elements of the sets	0 %	25 %	2.3 %	5.3 %	0 %	5 %	0 %	4.8 %
8) Constructing one-to-one mapping between the sets	0 %	0 %	0.8 %	0.8 %	10 %	5 %	12.7 %	3.2 %
9) The use of the assumption that any two infinite sets are equal	15 %	20 %	23.5 %	5.3 %	5 %	0 %	3.2 %	0 %
10) The use of the assumption that infinite sets are incomparable	0 %	0 %	3.8 %	1.5 %	25 %	5 %	21.4 %	4.8 %
11) Non-numerical comparisons	0 %	0 %	0 %	0 %	0 %	0 %	0 %	0 %
12) Unidentified routine	0 %	0 %	1.5 %	15.9 %	0 %	0 %	2.4 %	8.8 %
13) No answer or I don't know	0 %	0 %	4.6 %	17.4 %	0 %	0 %	4.8 %	8.7 %

P. The K-speakers were more open toward the possibility that two infinite sets in item IV can be unequal. Of the K-participants, 82.5 % reported that they accepted the endorsement that the two sets A and B were infinite, but A was larger than B, and 67.4 % of the E-participants reported the same thing. Many E-speakers (23.5%) and K-speakers (21.4%) who used routines 9 and 10 in item III changed their routines in item IV because they accepted the claim that the infinite set of A was larger than the infinite set of B (see 5.3 % of the E-speakers in routine 9 and 4.8 % of the K-speakers in routine 10). With the prompt *larger* in item IV to compare two infinite sets, only 17.4 % of the E-speakers used routine 4, compared to 43.2 % of the E-speakers in item III. Interestingly, the use of routine 4 in the K-group increased from 0.8% in item III to 16.7 % in item IV.

Table 4.35 Distribution of the word use of *larger* in item IV

Students	More numbers	Higher values	Chi-square		
			df	Value	Sig.
E-interviewees	75 %	25 %			
K-interviewees	100 %	0 %			
E-participants	42.4 %	22.7 %	2	9.96	.007*
K-participants	65.9 %	13.5 %			

*Differences are significant ($p \leq .01$).

Q. Some respondents compared the sets with respect to the size of corresponding elements rather than with respect to the amount of elements. The majority of E-speakers (42.4%) and K-speakers (65.9%) reported that the word *larger* meant that there were more numbers in A than B. However, about 23 % of the E-speakers considered the meaning of *larger* as either increasing faster or larger sum in item IV, compared to 13.5 % of the K-speakers. Many participants often used the example in item III to explain that A was larger than B. The chi-square analysis of the comparison of the E-participants' and K-participants' responses about the word use of *larger* indicates a significant difference, as shown in Table 4.35.

Table 4.36 Comparisons between the two infinite sets A and B in item IV

IV. A student claims that each of the two sets A and B is infinite, but A is larger than B. What can this mean?			
E4a	[1] A is larger then it could have a smaller multiplier…when it gets up to infinity, they're be like more numbers within the set itself	K1a	[21] Because A is all natural numbers and there are even numbers within it, A is larger than B
E4b	[2] They can't be larger 'cause even if it's going by a small increment, there's an infinite amount of numbers. They would be the same	K1b	[22] [like] the previous problem…A and B are both infinite, but integers are larger than even numbers
E4c	[3] They're still equal if it goes to infinity	K2a	[23] A is larger because all was included [like problem III]
E4d	[4] It could get bigger like it can increase by more	K2b	[24] A has elements of B…like problem [III]
E5a	[5] The way I would call larger is if it has more numbers within the set. They can both be infinite functions	K3a	[25] Both are infinite, but if you take out B from A, there is going to be something left. That is why A is larger
E5b	[6] I feel like larger is like the one that turns out would be higher value but either way…	K3b	[26] A is natural numbers and B is even numbers…A is larger.
E5c	[7] It goes up by smaller increments and so it has more numbers	K5a	[27] Both are infinite, but the sizes are comparable. By the concept of size…because it includes…
E5d	[8] If you're going to infinity…more integers in set A than in set B	K5b	[28] Because even numbers are included within natural numbers
E6a	[9] Total numbers included in that set because it's increasing by one	K6a	[29] The condition includes more things. It's inclusive
E6b	[10] A would have more numbers than B…so A would be larger	K6b	[30] I seem to say two stories. One story about inclusive relationship and the story…I cannot count because they are infinite.
E7a	[11] Even though it's infinite, it'd contain more integers inside of it	K6c	[31] A and B are both infinite sets…one of them is included in the other
E7b	[12] Odd numbers is an infinite set but contained in all real numbers	K6d	
E7c	[13] If you put a limit on each set of numbers, then A would be more. But, numbers would continue on forever…so the same thing	K7a	[32] The concept of large…more cardinal numbers of elements
E7d	[14] It's larger because there's more numbers in A	K7b	[33] 'Each of A and B is infinite' describes the cardinal number. 'larger' means denser…not about cardinal number but about range
Eoa	[15] In terms of the previous question,…they increased at a constant rate but different rates at that	Kma	[34] Being large in terms of an inclusive relationship. The example of odd numbers and integers
Eob	[16] A has a larger coefficient or starts at a larger number…I was thinking about it in terms of the previous question	Kmb	[35] If they both are infinite, there does not exist for being small or big…I don't think you can compare sizes
Eoc	[17] The student is lying…because infinity equals infinity…and so how can one be larger than the other?	Kmc	[36] A is larger than B, if we consider that A is integers and B is even numbers, even though they both are infinite
Eod	[18] A goes off infinity at a faster rate than B…the only way…larger	Kmd	
Eoe	[19] A is larger because it's all integers…B is only the even integers	Kme	[37] If there is no one-to-one correspondence, I think the size comparison is possible
Eof	[20] B is contained in A	Kmf	[38] When A includes B in the same extension, A is larger than B

Table 4.37 Distribution of processual and structural uses of infinity

Comparisons	Students	Processual use	Structural use	Chi-square		
				df	Value	Sig.
Even numbers and integers	E-interviewees	75 %	25 %			
	K-interviewees	20 %	80 %			
	E-participants	35.6 %	2.3 %	2	69.50	.000*
	K-participants	4.8 %	36.5 %			

Note. *Differences are significant ($p \leq .01$).

R. The same as [G] and [O] above. K-speakers' discourse on infinite sets was seemingly more structural than that of E-speakers, in spite of the fact that only the latter used the noun *infinity*. K-speakers' talk was more formal mathematical. The majority of K-interviewees consistently focused on one-to-one correspondence as well as cardinality of sets. In contrast, the majority of E-interviewees employed *infinity* in conjunction with an infinite process to compare two infinite sets ([1-9], [12-13], [15-16], and [18-19]). The chi-square results of the comparison of the E-participants' and K-participants' responses about the use of infinity show a significant difference in Table 4.37.

Item VIII-a

Table 4.38: Responses to the question in item VIII-a

VIII (a). A student claims that a sequence a_1, a_2, a_3, goes to infinity. What does this mean?			
E4a	[1] It just will keep on getting bigger and bigger	K1a	[21] It means increasing continuously and endlessly with a regular rule
E4b	[2] It goes to like a sub like infinity. It just keeps going…goes up by normal numbers	K1b	[22] This sequence as geometric sequence…when a_1, a_2 equal to a_1r and increase…when [r is] more than 1, it increases to infinity
E4c	[3] It just keeps increasing	K2a	[23] I thought that $\{a_n\}$ becomes an infinite set
E4d	[4] There are infinite numbers in the sequence without stopping	K2b	[24] When the number of here [subscript] increases, its value is infinity
E5a	[5] It cannot really be both. The value of A keeps increasing	K3a	[25] The sizes of a_1, a_2, a_3…continue to increase or decrease
E5b	[6] It gets mutually exclusive like having a limit or going to infinity. [It's] increasing or decreasing	K3b	[26] The difference between a_n and a_{n+1} is fixed or increases…1,1,1,1 this sequence can be said infinite
E5c	[7] It keeps going on, whether it gets bigger or smaller	K5a	[27] It continues to increase without a converging value and goes to infinity
E5d	[8] A sequence that keeps getting bigger	K5b	[28] The value of a_k does not reach a fixed point…go to infinity
E6a	[9] [It] keeps on getting bigger and bigger. It has no limit	K6a	[29] The size of this [sequence] goes to infinity more and more
E6b	[10] It doesn't end…continuous…goes out to infinity	K6b	[30] It diverges to either plus infinity or minus infinity
E7a	[11] That'll go to infinity as the end of the sequence. Or infinite amount of terms…an endless concept	K6c	[31] There is not the last specific an because the sequence continues to go infinitely…As $\lim_E$ n goes to infinity, a_n is infinity
E7b	[12] It increases in the positive direction without bound…there's no limit…it's not a number…it's like a concept	K6d	[32] a_n goes to infinity…diverges like that
E7c	[13] There are an infinite amount of numbers…it goes on and on	K7a	[33] The sequence diverges
E7d	[14] There is no limit…they just get higher and higher	K7b	[34] The limiting value does not exist and toward unlimitedness
Eoa	[15] It will continue on forever	Kma	[35] a_n is gradually an increasing sequence, it goes to infinity
Eob	[16] The sequence never ends	Kmb	[36] Limit_E a_n is as n goes to infinity comes out infinity… continue to increase or decrease
Eoc	[17] There won't be any horizontal asymptotes. It's not bound because it goes to infinity	Kmc	[37] As n goes to infinity, $\lim_E$ a_n equal to infinity…numbers grows more and more and goes to infinity like that as the word itself
Eod	[18] It keeps going without bounds	Kmd	[38] This means is to grow endlessly…grows more and more
Eoe	[19] There is no limit…It keeps going	Kme	[39] It is an infinite sequence. The last term cannot be defined
Eof	[20] There's no limit…a sequence never stops and a would continue to approach infinity. It's going to have an infinite number of values	Kmf	[40] The difference between a_n and a_{n-1} continues to become wider. It's going to infinity

Table 4.39 Use of *infinity* in a sequence

Students	*Processual* use	*structural* use	Chi-square		
			df	Value	Sig.
E-interviewees	95 %	5 %			
K-interviewees	25 %	75 %			
E-participants	65.2 %	6.1 %	2	102.0	.000*
K-participants	11.9 %	57.1 %			

*Differences are significant ($p \leq .01$). Missing data were not included in the table.

S. E-speakers' responses were relatively consistent and in tune with the canonical interpretation of the word *infinity*. K-speakers' interpretations were more diverse and often inconsistent with the canonical interpretations. Almost all E-speakers translated the expression *sequence* a_1, a_2, ... *goes to infinity* into utterances saying that the sequence is in the process of unbounded or never-ending growth (see [1-3], [5], [8-9], [12], [14] and [18-20]). K-speakers' responses were much less consistent. Many of them made no requirements with respect to the size of the elements of the sequence. This use was clearly phrase-driven (see [25-26], [30], [32-34], [36], and [39]). They also often made specific assumptions about the sequence (e.g., "a regular rule" [21], "geometric sequence" [22], and "1, 1, 1, 1" [26]).

T. E-speakers provided processual explanations. K-speakers spoke almost exclusively in structural terms. Almost all E-speakers used processual expressions such as "increasing," "getting bigger," "keeps going on," and "gets higher." Many K-speakers did not mention any process, except simply repeating the phrase "goes to infinity," used in the formulation of the question. Instead, they spoke in structural terms of "sets," "number of elements," "sizes," "values," and "last term." The chi-square results in Table 4.39 show a significant difference between the E-group and K-group. One word for a process that appeared in their talk and was not found in E-speakers' responses was "diverge."

U. E-speakers formulated interpretations in their own words. K-speakers used more formal mathematical formulations, even though some of them were not fully versed in it. Many K-speakers were careful to distinguish between the index of a_n sequence element and the element itself (e.g. [2]). Some K-speakers showed awkward uses of mathematical formulations (e.g., "A limit_E n goes to infinity, a_n is infinity" [31]; see also [36-37]). They often provided formal language to define *going to infinity* such as "converge" and "limit_E" (see [27], [31-33], and [36-37]).

V. In particular, unlike in the previous items, K-speakers used the noun *infinity*, but they did it in phrase-driven way, by simply repeating the expression *goes to infinity* that appeared in the question. The phrase-driven rather than structural nature of their use was evident from the fact that the word *infinity* did not appear except in this phrase. The K-respondents clearly had difficulty providing an equivalent statement expressed in words that was different from the interpreted phrase itself.

The following Episodes 5 and 6 exemplify the differences in defining that a sequence $a_1, a_2, a_3, \ldots$ goes to infinity between the E-group and K-group.

Episode 5. E-pair in defining that a sequence $a_1, a_2, a_3, \ldots$ goes to infinity (item VIII-a)

Speaker	What was said	What was done
238. I	Ok. **[show the card about item VIII-a]** What about this problem? A student claims that a sequence goes to infinity. What does this mean?	show the card
239. E6b	Uh...I wasn't sure if that one was a trick question because it's pretty simple. I mean you just keep adding maybe it's the uh what are these things called. Can't remember the name of them but one of those sequences where numbers become...have the same equation and they become larger. What is the name of that? I can't remember. But maybe its like one and then one plus three and then two I mean you know four plus three or something like that but. Regardless the sequence continues forever	
240. E6a	Keeps on going and going and going getting bigger and bigger the sequence does not stop. It has no limit.	
241. I	Ok. what do you mean by you know the sequence is continuous?	
242. E6b	It means it...it doesn't end. There is no set ending or beginning its continuous it doesn't stop.	

243. I	Do you want to add something else?	
244. E6a	Like I mean if you had a sequence when...where you're adding numbers...	
245. I	Mm-hmm.	
246. E6a	like say this was one and that was one plus two and this was one plus three it just keeps on...keeps on adding more...	
247. I	Mm-hmm.	
248. E6a	and more and	
249. I	Mm-hmm.	
250. E6a	More and more actually so you keep on going to infinity so.	
251. I	Mm-hmm. Anything else?	
252. E6a	No.	
253. I	Do you want to add?	
254. E6b	It basically goes out to infinity.	
255. I	About uh go to infinity?	
256. E6b	Well that's what I meant by continuous. It doesn't have an ending	
257. I	Mm-hmm.	
258. E6b	You can't say it stops at a certain point it goes to infinity.	

Episode 6. K-pair in defining that a sequence a_1, a_2, a_3,... goes to infinity (item VIII-a)

159. I	Now then **[show the card about item VIII-a]** A student claims that a sequence goes to infinity. What does this mean?	Show the card
160. K2b	[...] I don't know well what kind of sequence this is...if list [sequence] by numbering like this, then here 1...when the number of here [subscript] increases...its value is infinity.	
161. K2a	[...] I also...same...	
162. I	When you answered last time, you seemed to write that it means $\{a_n\}$ is an infinite sequence	
163. K2a	Then as soon as I saw it, I thought so. When this is represented by a set, I thought that it becomes an infinite set and I solved it. When I heard what he said, I thought this was not...	
164. I	Can you explain this more?	
165. K2b	So, a_1 is a certain number and a_2 is some number and like this way. So, because the sequence is not known, if thought in this way, then a really big number of a is substituted, when to be listed...really...if listed from left, if picked up a number way to the right, then there is a certain number. If I go to right infinitely, I pick up this. This can be expressed like this [write a_∞]. Because this goes to infinity, this value also has infinity. Yeah.	
166. I	How far does it go to the right?	
167. K2b	It should go infinitely	
168. I	Here you wrote the symbol of a_∞. What is the meaning? Please explain it concretely	
169. K2b	In the case that this is not a finite sequence which has a limit (*Jae-han*), but an infinite sequence, so...there is an increase or decrease or a regularity which a sequence has. That this [sequence] is infinite means that it cannot go due to endlessness. If I pick [it = a_∞] up here...if I can pick [it] up, then it says that this [sequence] will become like this. So, it's approaching but reach...doesn't seem to reach it. Even though it [a_∞] cannot be picked up, if [I] go and pick it up, it's going to be infinity when seen within the rule in this. That kind of story...	
170. I	Anything else to add?	
171. K2b	A sequence is the array of numbers which have a rule. Then, I don't	

know how far a [term] should be....if the general term of this sequence is a_n, going to infinity means the term itself...the number of term [n] itself becomes the value of the sequence. So, if a_n is equal to n, then going to infinity means value also infinitely....in this way.

Summary: Discourse on infinity

Table 4.40 summarizes salient characteristics of the colloquial and mathematical discourses on infinity in the E-group and K-group.

Table 4.40 Salient properties of E-speakers' and K-speakers' discourse on *infinity*

Item: topic	Aspect	E-speakers	K-speakers
Item I: Creating sentences with given words	A. *Infinite* and *infinitely*: type of discourse	Used the adjective, but *not the adverb*. The use was mainly in *non-mathematical discourse*	Used both the adjective and the adverb. They used the adverb when they were asked to create a sentence with the noun *infinity* (either mathematical or colloquial). Their uses of *infinitely* and *infinite* was mainly in non-mathematical context (even more so than in the US group).
	B. *Infinite* and *infinitely*: context of use	The use was in the context of *large sets*: E-speakers used the adjective *infinite* almost always in conjunction with "number of" or "amount of."	K-speakers used the adjective *infinite* as descriptors of concrete things known for their large dimensions or of spiritual matters. They used the adverb *infinitely* as a descriptor of *many*, in the context of sets.
	C. *Infinity*	E-speakers' use of the word *infinity* was in association with processes, which were said to be *never ending*. The infinity itself was said to be *unreachable* or *impassable*. There was no association of infinity with largeness.	With just one exception, K-speakers did not use the noun *infinity*, either in its colloquial or mathematical version. When asked to construct a sentence with the $\text{infinity}_{\text{KColl}}$, they converted the noun into adjective (infinite) and when asked to construct a sentence with the $\text{infinity}_{\text{MColl}}$, they converted the noun into adverb (infinitely).
Item II: Comparing pairs of two sets	D. II-a: A=fingers; B=toes	Almost all E-speakers compared the two sets according to the number of elements, and if they did not choose the option A=B, it was only because of the fact that they interpreted the word *finger* as not referring to the thumb.	The 40% of K-speakers who chose an option other than A=B did so because their comparison was non-numerical.
	E. II-b: A=evens; B=odds	Only two fifths of E-speakers chose the canonical response A=B; other responses were distributed evenly between A<B and A>B.	Only half as many K-speakers chose the canonical option A=B. Other responses were distributed evenly between A<B, A>B, and "can't say".
	F. II-c: A=odds, B=integers	Only as few as 10%-15% of E-speakers chose the canonical response A=B; An overwhelming majority (80% in	The percentage of K-speakers who chose the canonical response A=B was even smaller than that of E-speakers (5% in the interview and 6.4% in the

		the interview and 61.4% in the survey) chose the option A<B.	survey); approximately 70% chose the option A<B.
	G. Discourse on infinite sets	E-speakers' use of *infinity* was phrase-driven rather than structural (it appeared in phrases such as "goes [on] to infinity").	K-speakers' discourse on infinite sets appeared more structural than that of E-speakers
	H. routine	Eleven routines for comparing sets were identified in the interviews and the survey	
	I. II-b: A=odds; B=evens	E-speakers tended to claim that two infinite sets are always equal (routine 9, 44%)	Only very few K-speakers used routine 9 (5% and 1.6% in the interview and in the survey, respectively). They relied more strongly than E-speakers on the endorsed narratives that *infinite sets cannot be compared* (routine 10; 22.2% for K-speakers versus 6.1% E-speakers).
	J. II-c: A=odds, B=integers	Both groups relied strongly on the endorsed narrative "proper subset is smaller/has less elements than the whole set" (or "infinite set that includes 'half as much elements' as another infinite set is smaller) (routine 6; E-speakers: 50%, E-speakers 67%)	
	K. II-b and II-c	In items II-b and II-c there were respondents who produced two contradictory answers as the result of applying two different routines of comparison.	
Item III: Comparing integers (A) with evens (B)	L. Changes in the question III: A=integers, B=evens	The changes in the question III in comparison to II-c did have an impact on the responses of E-participants	They did not have an impact on the responses of K-participants
	M. Change in wording	The change in wording influenced strongly the way E-participants implemented the comparison of sets	It did not change K-participants' choices
	N. Routine	A dominant *routine* in the E-group was changed from routine 6 in item II-c to routine 4 in item III.	A dominant comparison routine in the K-group was routine 6 in both item II-c and III
	O. The phenomenon first observed in [G] returned	Many E-speakers spoke about infinite sets in terms of processes ("goes to infinity", "goes on and on", 'continues forever', keeps going on" etc.).	Such reference was rare among K-speakers; once again, the Korean responses were formulated in a more formal mathematical way.
Item IV: Explaining that infinite set A is larger than infinite set B	P. IV: Two infinite sets A and B	67.4 % of the E-participants reported that they accepted the endorsement that the two sets A and B were infinite, but A was larger than B	Of the K-participants, 82.5 % reported the same thing.
	Q. Use of *larger*	Some respondents compared the sets with respect to the size of corresponding elements rather than with respect to the amount of elements.	
	R. The same as [G] and [O] above	The majority of E-speakers employed *infinity* in conjunction with an infinite process to compare two infinite sets	K-speakers' discourse on infinite sets was seemingly more structural than that of E-speakers. K-speakers' talk was more formal mathematical.
Item VIII-a: Defining that a sequence $a_1, a_2, a_3, \ldots$ goes to	S. Interpretation of infinity	E-speakers' responses were relatively consistent and in tune with the canonical interpretation of the word *infinity*.	K-speakers interpretations were more diverse and often inconsistent with the canonical interpretations.
	T. Use of	E-speakers provided processual	K-speakers spoke almost

infinity	infinity	explanations.	exclusively in structural terms.
	U. Definition of infinity	E-speakers formulated interpretations in their own words.	K-speakers used more formal mathematical formulations, even though some of them were not fully versed in it.
	V. Phrase-driven use of infinity		K-speakers used the noun *infinity*, but they did it in phrase-driven way, by simply repeating the expression *goes to infinity* that appeared in the question.

Discourse on *Limit*

Items I and

Table 4.41 Summary of interview responses using the word limited in items I and IX

Table 4.41 Summary of interview responses using the word *limited* in items I and IX

	I and IX. Create a sentence with the following word								
	E-speakers			K-speakers					
		Limited$_E$			Limited$_{KColl}$			Limited$_{KMath}$	
		Response	context		Response	context		response	context
Limited (adjective)	E4a	[1] I was limited by my handicap	Non	K1a	[18] Solve this problem within a limited range	Math		[31] He drove himself out into a limited (*gŭk-han-euh*) [= *extreme*] circumstance	Non
	E4b	[2] The height of my jump was limited by the strength of my legs	Non	K1b					
	E4c	[3] Limited…it's as far as you can go before you have to stop	Math	K2a	[19] Let's use limited resource efficiently	Non		[32] Within a limited [= *extreme*] loneliness, that man had endured	Non
	E4d			K2b	[20] Thought is limited	Non		[33] Find the limit value (limited in Korean)	Math
	E5a	[4] The function is limited	Math	K3a	[21] Some limited products are only on sale	Non		[34] Humans become stronger when they face limited [= *extreme*] situations	Non
	E5b			K3b					
	E5c	[5] My dorm room has limited space	Non	K5a	[22] The number of apples that we can buy with limited money	Non		[35] My anger has reached the limited [= *extreme*] point	Non
	E5d			K5b					
	E6a	[6] The domain of a certain function is limited	Math	K6a	[23] I couldn't buy because they were sold out due to limited quantity	Non		[36] I want to make myself strong to overcome the limited [= *extreme*] circumstance	Non
	E6b	[7] Fossil fuels are limited	Non	K6b					
	E7a	[8] Your ability to learn is not limited	Non	K6c	[24] There is limited water	Non		[37] The people exploring the South Pole had to fight the limited [= *extreme*] cold	Non
	E7b	[9] There's a limited number of people	Non	K6d	[25] We live in a limited place called Korea	Non			
	E7c	[10] There are a limited number of alphabets	Non	K7a	[26] A limited amount of materials can be distributed	Non		[38] It rose to limited [= *extreme*] situation	Non
	E7d	[11] Transportation was limited	Non	K7b	[27] Limited people can enter there	Non			
	Eoa	[12] Survey gives limited information… like a point that can't go any further	Non	Kma	[28] Here is the area in which exit and entrance is limited	Non		[39] The man overcame limited [= *extreme*] pain	Non
	Eob	[13] x is limited to the values	Math	Kmb					
	Eoc	[14] The function is like limited	Math	Kmc	[29] Let's use limited natural resource	Non		[40] Physical exercise to the limited [= *extreme*] practice.	Non
	Eod	[15] My free time seems more limited	Non	Kmd					
	Eoe	[16] You're limited to how much TV you can watch	Non	Kme	[30] Save limited natural resources well	Non		[41] A person who has endured in a limited [= *extreme*] circumstance is a real strong one	Non
	Eof	[17] The amount of food is limited	Non	Kmf					

Note. Math = mathematical context; Non = non-mathematical context

Table 4.42 Summary of interview responses using the word *limit* in items I and IX

		I and IX. Create a sentence with the following word					
		E-speakers		K-speakers			
		$Limit_E$		$Limit_{KColl}$		$Limit_{KMath}$	
		Response	context	response	context	Response	context
Limit (noun)	E4a	[42] I will limit myself to only three pies	Non	K1a	[58] The range of x is limited	Math	[72] A number p is the limit of the sequence	Math
	E4b	[43] There is a limit on how far I can go	Non	K1b				
	E4c E4d	[44] Limit, it has a certain point to where it just has to end	Math	K2a	[59] People have gathered much in limited space	Non	[73] When right-hand limit is the same with left-limit in graph, it is called convergence	Math
				K2b	[60] The limit of our country is blocked above	Non		
	E5a E5b	[45] The function has no limit	Math	K3a K3b	[61] This is the limit to what I can do	Non	[74] Find the limit value	Math
	E5c	[46] The speed limit is 35 miles per hour.	Non	K5a	[62] I experienced my limit	Non	[75] We need to be able to survive in the limited (*gŭk-han-euh*) [= *extreme*] environments	Non
	E5d	[47] Something you can't pass exceed	Non	K5b				
	E6a	[48] The limit of a constant function is the constant	Math	K6a	[63] There is a limit to x value	Math	[76] Find the following limit	Math
	E6b	[49] I try to limit my food	Non	K6b	[64] I feel the limit of physical strength	Non		
	E7a E7b	[50] Find the limit of a function	Math	K6c	[65] We faced a limit	Non	[77] When I first learned about the concept of limit, I was very confused.	Math
				K6d	[66] A limit to something means a challenge	Non		
	E7c E7d	[51] There's a limit to how many fingers you have [52] My mom places a limit on how much time I can watch TV.	Non Non	K7a	[67] Limit that person's action	Non	[78] [We] faced limit [= *extremity*] situation.	Non
				K7b	[68] It is my limit to here	Non		
	Eoa	[53] The limit as h approaches 0 equals…	Math	Kma Kmb	[69] We must overcome a limit to accomplish our goal	Non	[79] Find the limit of the following sequence	Math
	Eob	[54] The limit of f of x equals 2	Math					
	Eoc	[55] My friend got a ticket for breaking the speed limit	Non	Kmc Kmd	[70] Let's resolve it within a limited range	Math	[80] I experienced the limit [= *extremity*] of life	Non
	Eod	[56] I put the limit of x as x approaches…	Math					
	Eoe Eof	[57] Every person has their limit	Non	Kme Kmf	[71] That's your limit	Non	[81] The limit can be defined as the argument of using epsilon and delta	Math

Note. Math = mathematical context; Non = non-mathematical context

Table
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*Diff

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K-p

*Diff

Table 4.43 Distribution of the contexts of use of the words *limited* and the chi-square results of the comparison of difference between the E-group and K-group

Students	Words	Non-mathematical contexts	Mathematical contexts	None	Chi-square		
					df	Value	Sig.
E-interviewees	$Limited_{E}$	65 %	35 %				
K-interviewees	$Limited_{KColl}$	90 %	10 %				
E-participants	$Limited_{E}$	50.0 %	34.1 %	15.9 %	1	8.75	.003*
K-participants	$Limited_{KColl}$	73.8 %	21.4 %	4.8 %			

*Differences are significant ($p \leq .01$).

A. *Limited*: context of use. Both groups used the word *limited* in any version mainly in non-mathematical context. Among K-speakers the phenomenon was stronger than among E-speakers.

The E-interviewees used the word *limited* in both non-mathematical and mathematical contexts. However, the K-interviewees employed the word $limited_{KColl}$ predominantly in non-mathematical contexts (11 responses out of 14 for $limited_{KColl}$ in Table 4.41). The chi-square statistics in Table 4.43 indicate that the distribution of the word use of the E-participants and K-participants between non-mathematical and mathematical contexts is significantly different when the word *limited* is colloquial.

Table 4.44 Distribution of the use of the word $limited_{Coll}$ as an upper-bounded process

Students	The use as upper-bounded process	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	30 %			
K-interviewees (n = 20)	0 %			
E-participants (n = 132)	38.4 %			
K-participants (n =126)	41.6 %	1	.50	.479

Table 4.45 Distribution of the use of $limited_{KMath}$ as extreme

Students	The use of $limited_{KMath}$ as extreme	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	0 %			
K-interviewees (n = 20)	95 %			
E-participants (n = 132)	0 %	1	63.62	.000*
K-participants (n =126)	42.1 %			

*Differences are significant ($p \leq .01$).

B. *Limited*: type of use. Both groups used $limited_{Coll}$ in the sense of *finite* (not infinite – something that has an upper bound) or *restricted* – one that *could be larger*. K-speakers used $limited_{KMath}$ in the sense of *extreme*.

The majority of interviewees employed *limited* as something they could not go over in their daily experience ("height" [2], "as far as you can go" [3], "ability to learn" [8], "how much TV you can watch" [16], "food" [17], and money" [22] in Table 4.41). In the case of the Korean mathematical word $limited_{KMath}$, K-interviewees employed it in both non-mathematical and mathematical contexts. However, in non-mathematical contexts the word $limited_{KMath}$ was only applied to something extreme. This was not observed in the use of the Korean colloquial words (see [31-32], [34-41], [75], [78], and [80]). The E-speakers never used the words *limited* and *limit* in these ways. A chi-square test in Table 4.44 indicates that there is no significant difference between the E-group and K-group in the number of written survey responses in which the colloquial word *limited* is employed to refer to an upper-bounded process. All of the K-participants who employed the words $limited_{KMath}$ in non-mathematical contexts used it as the concept of *extreme*, as shown in Table 4.45.

Table 4.46 Distribution of the contexts of use of the word *limit* and the chi-square results of the comparison of difference between the E-group and K-group

Students	Words	Non-mathematical contexts	Mathematical contexts	None	Chi-square		
					df	Value	Sig.
E-interviewees	$Limit_E$	50 %	50 %				
K-interviewees	$Limit_{KColl}$	75 %	25 %				
	$Limit_{KColl}$	30 %	70 %				
E-participants	$Limit_E$	28.8 %	63.6%	7.6 %	1	76.17	.000*
K-participants	$Limit_{KColl}$	81.7 %	12.7%	5.6 %			
	$Limit_{KColl}$	30.2 %	52.4%	17.5 %	1	.02	.896

*Differences are significant ($p \leq .01$).

C. *Limit*: context of use. Both groups used the word, in whatever its version, in both mathematical and non-mathematical contexts. E-speakers used the word $limit_{Coll}$ mainly

in mathematical contexts, whereas K speakers employed it predominantly in non-mathematical contexts. The chi-square results in Table 4.46 show that the distribution of the word use of the E-participants and K-participants between mathematical and non-mathematical contexts is significantly different when the word *limit* is colloquial (χ^2 = 76.17, p = .000). However, there is no significant difference between the two groups when the word is mathematical (χ^2 = .02, p = .89).

Table 4.47 Distribution of the use of the word $limit_{Coll}$

Students	The use of *limit* as an upper bound	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	30 %			
K-interviewees (n = 20)	5 %			
E-participants (n = 132)	15.6 %	1	6.00	.014
K-participants (n =126)	15.4 %			

Table 4.48 Distribution of the use of and $limit_{KMath}$ as extremity

Students	The use of limit as extremity	Chi-square		
		df	Value	Sig.
E-interviewees (n = 20)	0 %			
K-interviewees (n = 20)	30 %			
E-participants (n = 132)	0 %	1	46.25	.000*
K-participants (n =126)	30.2 %			

D. *Limit*: type of use. Both groups used $limit_{Coll}$ to denote an *upper bound* – something that does not let a thing or process to get larger (see Table 4.47). The usage of $limit_{KMath}$ as the concept of extremity in non-mathematical contexts was also seen in the K-participants' written responses (see Table 4.48). However, the majority of K-speakers used $limit_{KMath}$ mainly in expressions that were formulated as a textbook instruction to do something or as textbook definition. In general, it seemed that K-speakers were not very proficient in creating sentences with $limit_{KMath}$. In several cases, the noun was converted into adjective. When it was used as a noun, the expression was personal (about a person rather than about mathematical objects) and appeared as if copied from a textbook (see

"find the limit" [74, 76, 79] for instruction; [73] and [78] for definition in Table 4.42). Finally, several pairs did not mange to produce more than one sentence.

Item V

In item V, the following table was shown to students along with the question "What do you think will happen later in the columns of this table?"

Table 4.49 The table shown in item V in the questionnaire

x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
0.01	0.099990
0.005	0.099995
0.001	0.099999
...	...

Table 4.50 Distribution of *endorsed narratives* in item V

Endorsed narratives	E-speakers		K-speakers	
	Interview	survey	Interview	survey
1) The sequence is increasing/decreasing (without the limit)	30 %	29.5 %	0 %	2.4 %
2) The sequence is getting close to (or approaches) 0.1	30 %	29.5 %	30 %	27.8 %
3) The sequence approaches 0.1 but never reach it	30 %	8.3 %	5 %	2.4 %
4) The sequence converges to or becomes 0.1	0 %	3.0 %	15 %	15.1 %
5) The limit of the sequence is 0.1	10 %	12.9 %	50 %	45.2 %
6) Other	0 %	10.6 %	0 %	4.0 %
7) No answer	0 %	6.1 %	0 %	3.2 %

Even though some students made a mistake and answered one or 0.01 instead of 0.1 as the limit, their responses were categorized as one of the five *endorsed narratives* because they showed the same idea as one of them. The "other" category in Table 4.50 is the case, for example, when the student writes 0.1 or any number without providing further information.

E. A *process* of the sequence dominated the idea of *limit* in the E-group, compared to an *object* as the *limit* of the sequence in the K-group. About 30 % of E-speakers conceptualized the idea of the limit in the sequence as an increasing or decreasing process

without the limit. However, in the E-speakers, 29.5 % reported it as an approaching process to the limit. Both E-speakers (29.5 %) and K-speakers (27.8 %) considered numbers of the sequence as an approaching process to the limit. The limit of the sequence as a number (15.1 % and 45.2 % in categories 4 and 5 respectively in Table 4.50) dominated the idea of limit in the K-group.

Table 4.51 Responses to the question about the limit of an infinite sequence in item V

V. What do you think will happen later in the columns of this table?			
E4a	[1] x column…getting smaller but hasn't reached zero, the second column…will just keep on going at that rate	K1a K1b	[21] As the values of x are decreasing, [it's] getting close to 0.1
E4b	[2] The limit as x approaches zero the function will approach .1 but it'll never actually reach point one		[22] This answer is going to be a form of 0 over 0… As x is getting close to 0, [the function] is getting close to 0.1
E4c	[3] One goes to maybe zero, and the other one to one tenth...never reaching…They are limited to a certain number appears to be 0 and .1	K2a	[23] As x decreases…because it's the same as 0… the answer will become 0.01…after taking $\lim_{E}$ …by rationalization…1 over 10
E4d	[4] They'll both go to infinity, but they'll never get there. I don't know why I wrote down infinity…it is zero and point one	K2b	[24] I also took $\lim_{E}$ …I divided denominator and numerator by x… calculating it…infinity… I make a wrong calculation
E5a	[5] It's gonna approach zero and point one. Eventually it will have to round up because they wouldn't make .099999 forever	K3a	[25] By simply looking at the numbers, x becomes close to 0 and the right side becomes close to 0.01
E5b	[6] That one would approach zero and it would just approach point one	K3b	[26] As x will become closer to 0, it won't pass 0.01…becomes closer
E5c	[7] x will become 0. This become .9999 infinite amount…round up to .1	K5a K5b	[27] This goes to 0 and this goes to 0.1…Let $\lim_{E} x$ go to 0 and I solved it by rationalizing the numerator
E5d	[8] Zero point one…it's getting closer to some…like an infinite function		
E6a	[9] This one keeps on getting closer and closer to zero and this one keeps on getting closer and closer to point one	K6a	[28] Because x decreases…If I multiply root x plus 25 plus 5, then the remainder is 1 over root x plus 25 plus 5…approach 0.1
E6b	[10] You can't say it'll eventually hit one…it never becomes an integer it just stays a decimal and increases in value	K6b	[29] As x approaches 0, a form of 0 over 0 comes out…if I use l'Hospital's rule, 0.1 comes out. I wrote that it converges to 0.1
E7a	[11] As *x* approaches zero, this becomes closer to one tenth since that is the limit to the problem…Infinitely reach one tenth without reaching it	K6c	[30] When you multiply the denominator by $\sqrt{x+25}+5$ …it reduces to x…by 0.1
E7b	[12] Zero in the *x* column and one tenth in *y* column…never reaching it	K6d	[31] If $\lim_{E}$ x goes to zero…after calculating…it was 1 over 10
E7c	[13] As *x* gets smaller, *y* gets bigger…divide by a small number, bigger	K7a	[32] If sent by 0…if I multiply $\sqrt{x+25}+5$ …that comes out 1/10
E7d	[14] *x* will get closer to zero and this will get closer to one but they're never gonna reach on	K7b	[33] *x* is closer to 0 and solved it to $\lim_{E}$ problem. I used l'Hospital's rule because it's a form of 0 over 0
Eoa	[15] It would always be approaching infinity, but never actually make it	Kma	[34] As x goes to 0, calculated $\lim_{E}$[rationalization], 0.1 came out
Eob	[16] Because *x* will never reach zero…[it] will never reach one	Kmb	[35] As x goes to 0, this value is getting closer to 0.01…table
Eoc	[17] Increasing towards 0 and 1…never reach it cause they're the limits	Kmc	[36] As x goes to 0, that value is toward 0.1…multiply $\sqrt{x+25}+5$
Eod	[18] Approach 0 and 1 but they will never reach it...square root of 25 is 5, so 5 minus 5 is 0…square root of *x* over *x*…so you have *x* over *x*	Kmd	[37] It converges to 0.1. I used l'Hospital's rule
Eoe	[19] The left one is decreasing and the right one is increasing	Kme Kmf	[38] *x* is approaching 0 infinitely…multiplying $[\sqrt{x+25}+5]$…it can be resolved
Eof	[20] It approaches 0.01, as x approach 0…the limit of the function		

Table 4.52 Distribution of the use of the limit of the infinite sequence

Students	Processual use	Structural use	Chi-square		
			df	Value	Sig.
E-interviewees	90 %	10 %			
K-interviewees	15 %	85 %			
E-participants	75.8 %	16.7 %	2	60.62	.000*
K-participants	33.3 %	63.5 %			

*Differences are significant ($p \leq .01$).

Table 4.53 Distribution of the use of two columns

Students	Separate processes	connected processes	Chi-square		
			Df	Value	Sig.
E-interviewees	80 %	20 %			
K-interviewees	15 %	85 %			
E-participants	62.9 %	31.8 %	2	46.25	.000*
K-participants	23.0 %	73.8 %			

*Differences are significant ($p \leq .01$).

F. K-speakers' discourse on the infinite sequence appeared more structural than that of E speakers. E-speakers used the infinite sequence in terms of infinite *processes*, whereas such reference was rare in the K-speakers' responses. Instead, K-speakers explicitly referred to mathematical algorithms associated with the function of the infinite sequence. In other words, the K-speakers' use of the sequence was structural rather than processual.

The *processual* use of the sequence in the E-interviewees was related to either that the limit is *unreachable* or that it is *rounding up* ([1-5], [10-12], and [14-18] for *unreachability*; [5] and [7] for *rounding up*). In contrast, the infinite sequence seemed to guide most K-interviewees to focus on finding its limit value according to known mathematical algorithms ("a form of 0 over 0" [22], "limit_E" [23, 24, 27, 31, 33], "multiply $\sqrt{x+25}+5$" [28, 30, 32, 34, 36, 38], and "use l'Hôpital's rule" [29, 33, 37]). In addition, most of the E-interviewees seemed to separate two processes in two columns (the x and function columns) of the table shown in item V (see [1], [3], [4], [5], [6], [7], [9], [12], [14], [17], [18], and [19]). In contrast, the task of finding the limit in the K-

group seemed to make two processes in these two columns connected ([21-23], [26-29], [31-32], and [34-36]). Thus, when the question "What will happen later in the columns of this table?" was asked, the K-speakers associated numbers of the infinite sequence with mathematical algorithms. The chi-square results in Table 4.52 show a significant difference between the E-group and K-group in the use of the sequence. Chi-square analysis of the distribution in the use of two columns also indicates a significant difference between these two groups, as shown in Table 4.53.

G. When finding the limit in a given infinite sequence, five routines were identified according to how to use limiting processes to find out the limit:

1. **Describing a process by looking at patterns of numbers in the sequence:** There were interviewees who looked at either increasing or decreasing patterns of numbers in the infinite sequence. They then endorsed that it was increasing or decreasing. Their discourse seemed to be mediated syntactically on the basis of changes of numbers in the sequence.
2. **Describing a process by characterizing the given function:** Some interviewees focused on a character of the given function, that is, that one divided by a smaller number was a bigger number. Then they concluded that the function kept increasing because x got smaller and smaller.
3. **Finding the limit by looking at patterns of numbers in the sequence:** Other interviewees checked the value which numbers in the sequence approached. When there was an approaching number, then they endorsed that the sequence approached the number.

4. **Rationalization:** Some other interviewees multiplied $\sqrt{x+25}+5$ to both numerator and denominator to find out the limit value in the infinite sequence. After calculating the limit value, they endorsed that it was 0.1, it converged to 0.1, or it got closer to 0.1.

5. **l'Hôpital's rule:** There were interviewees who seemed to first look at the function as a form of 0/0 when x approaches 0. Then they used l'Hôpital's rule to find out the limit value by differentiating both numerator and denominator. After getting the limit, they endorsed that it became 0.1, or that it converged to 0.1, or that it approached 0.1.

6. **Unidentified:** This is the case when students say 0.1 without giving a reason.

7. **No answer**

Table 4.54 Distribution of *routines* in item V

Routines	E-speakers		K-speakers	
	interview	survey	interview	Survey
1) Describing a process by looking at patterns of numbers in the sequence	20 %	27.3 %	0 %	1.6 %
2) Describing a process by characterizing the given function	5 %	6.1 %	0 %	0 %
3) Finding the limit by looking at patterns of numbers in the sequence	70 %	50.0 %	25 %	44.4 %
4) Rationalization	0 %	1.5 %	60 %	41.3 %
5) l'Hôpital's rule	0 %	0 %	15 %	7.1 %
6) Unidentified	5 %	10.6 %	0 %	2.4 %
7) No answer	0 %	4.5 %	0 %	3.2 %

H. E-speakers used *routines* based on *processual* use of *limit*. K-speakers employed routines on the basis of *structural* use of *limit*. About one third of the E-speakers in routines 1 and 2 of Table 4.54 and many E-speakers (70 % and 50 % in the interview and in the survey, respectively) in routine 3 appeared to employ their routines based on *processual* use of *limit*. In contrast, 48.4 % of the K-participants in routines 4 and 5 and many K-speakers in routine 3 used their routines on the basis of *structural* use of limit by considering the concept of limit as an object, that is, a number. In routine 3, only 34.8 %

of the E-participants (23 students out of 66) seemed to report two connected processes in the use of two columns on the table in item V. In contrast, 53.6 % of the K-participants (30 students out of 56) appeared to present two connected processes in the use of the two columns.

The following Episodes 7 and 8 are excerpts from the conversations with an E-pair and a K-pair respectively. These two episodes exemplify their routines and difference in the use of the infinite sequence.

Episode 7. E-pair in finding the limit of an infinite sequence

Speaker	What is said	What is done
190. I	[show the card with "what do you think will happen later in the columns of this table"] What about this one? What do you think will happen later in the columns of this table?	show the card
191. E5a	Um...	
192. E5b	I said it would just approach point one	
193. E5a	Yeah this one would	
194. E5b	Yeah and that one would approach zero.	
195. E5a	Yeah. I'm guessing because it just keeps getting point 09999	
196. E5b	Right.	
197. E5a	It's just going to keep getting nines.	
198. E5b	Yeah.	
199. E5a	I think it's gonna...it's gonna approach zero from the left side the right sides gonna approach point one.	
200. E5b	Mm-hmm.	
201. I	Why?	
202. E5a	Because it's just the way the trend is appearing to be this one keeps kind of moving over one decimal place well like every other one. And this one you keep adding more nines which means eventually it will have to run...round up so	
203. E5b	Mm-hmm.	
204. E5a	Because I don't think they would make the answer point 099999 forever [laughing] so it would eventually have to round up to point one.	
205. E5b	Yeah [laughing] Sounds good to me. That's what I though too.	

Episode 8. K-pair in finding the limit of an infinite sequence

71. I	[show the card with "what do you think will happen later in the columns of this table"] What do you think will happen later in the columns of this table?	Show the card
72. K1a	I looked at it only around here and I thought it is getting closer to 0.1	
73. K1b	I simply thought...This [first column] is decreasing by less units from 1...less and less...0.5, 0.1 and once again by a half of 1...0.05 and again 0.01 and so 5, 1, 5, 1 separately like this...following the trend, I thought that x is getting closer to 0 more and more	
74. K1a	To 0	
75. K1b	Yeah...so...this from 1 to 0...I thought the next one is going to be 0.0005	
76. K1a	Yeah	

77. K1b It continues to go on like the rule...rule...anyway...because I thought x is decreasing in that trend, as a result, x is not zero but is going to get to a number that is close to 0.
78. K1a Yeah
79. K1b So if x is 0, this answer is going to be in the form of 0 over 0.
80. K1a Yeah
81. K1b As x is getting closer to 0, I thought that root x plus 25 minus 5 over x is getting closer to 0.01...ah, 0.1
82. K1a Uh
83. K1b Just exactly...
84. K1a I just thought...as the values of x are decreasing, [it's] getting closer to 0.1
85. K1b Wordings seem to be just a little different.

Item VI

Item VI was designed to elicit students' mathematical discourse on *limit* by asking them "What will happen to the curve when x approaches 0 from left or from right?" in item VI-a ($\frac{1}{x}$) and "What will happen to the curve when x goes to positive infinity?" in items VI-b ($\frac{x^2}{1+x}$) and VI-c ($\frac{x^2}{(1+x)^2}$). These three different functions and their graphs were provided in the questionnaire.

Table 4.55 Distribution of *endorsed narratives* in items VI-a and VI-b

Endorsed narratives	E-speakers				K-speakers			
	Interview		survey		interview		Survey	
	VI-a	VI-b	VI-a	VI-b	VI-a	VI-b	VI-a	VI-b
1) It approaches 0	10 %	0 %	13.6 %	0.8 %	0 %	0 %	0 %	0 %
2) It is increasing (without the limit)	25 %	30 %	9.8 %	31.8 %	10 %	0 %	4.0 %	5.6 %
3) It approaches an asymptote	10 %	0 %	11.4 %	8.3 %	0 %	0 %	10.3 %	11.9 %
4) It goes to infinity	45 %	50 %	30.3 %	28.0 %	25 %	35 %	16.7 %	19.0 %
5) It diverges (to infinity)	0 %	0 %	0 %	0 %	25 %	20 %	23.8 %	20.6 %
6) (It becomes) infinity	10 %	10 %	26.5 %	17.4 %	35 %	35 %	35.7 %	39.7 %
7) Unidentified	0 %	0 %	6.8 %	6.8 %	5 %	0 %	7.1 %	0.8 %
8) No answer	0 %	0 %	1.5 %	6.8 %	0 %	0 %	2.4 %	2.4 %

Table 4.56 Distribution of *endorsed narratives* in item V-c

Endorsed narratives	E-speakers		K-speakers	
	interview	survey	interview	Survey
1) It approaches 0	5 %	3.8 %	0 %	0.8 %
2) It gets closer to 1	60 %	25.8 %	15 %	20.6 %
3) It converges to (becomes) 1	10 %	16.7 %	80 %	68.3 %
4) It is increasing (without the limit)	0 %	10.6 %	0 %	0.8 %
5) It approaches an asymptote	10 %	7.6 %	0 %	5.6 %
6) It goes to infinity	15 %	12.1 %	5 %	0.8 %
7) Unidentified	0 %	7.6 %	0 %	0.8 %
8) No answer	0 %	15.9 %	0 %	2.4 %

I. **VI-a:** $\frac{1}{x}$. About one third of E-speakers in the first highest percentage responded "it goes to infinity." In the next highest percentage, the E-speakers (26.5 %) replied "(it becomes) infinity." Other responses were distributed evenly among "it approaches 0," "it is increasing," and "it approaches an asymptote." More than one third of K-speakers in the first highest percentage answered "(it becomes) infinity." Other Korean responses were distributed evenly between "it goes to infinity" and "it approaches an asymptote." However, only K-speakers (23.8 %) replied "it diverges (to infinity)."

J. **VI-b:** $\frac{x^2}{1+x}$. About one third of E-speakers responded "it is increasing." In the next two highest percentages, 28 % and 17.4 % of E-speakers answered "it goes to infinity" and "(it becomes) infinity" respectively. Approximately 40 % of K-speakers replied "(it becomes) infinity." However, only K-speakers (20.6 %) responded "it diverges (to infinity)." Other Korean responses were distributed between "it goes to infinity" and "it approaches an asymptote."

K. **VI-c:** $\frac{x^2}{(1+x)^2}$. About 26 % of E-speakers answered "it gets closer to 1." Other responses were distributed almost evenly among "it converges to 1," "it is increasing," and "it goes to infinity." The percentage of K-speakers who responded "it gets closer to 1" was smaller than that of E-speakers (15 % in the interview and 20.6 % in the survey). An overwhelming majority of K-speakers (80 % in the interview and 68.3 % in the survey) replied "it converges to 1."

L. **The number of "no response" from item VI-a to item VI-c.** The number of E-speakers who did not answer questions increased considerably from item VI-a to item VI-

c. Comparatively, the number of K-speakers who did not respond to questions remained at the same rate.

Table 4.57 Responses to the question about the limit of $\frac{1}{x}$ in item VI-a

VI-a. What happens to the curve as x approaches 0 from left or from right?			
E4a	[1] As it gets smaller it'll go much higher because one divided by decimal will be a bigger number	K1a K1b	[19] This sentence made me think I can use limit_E. Let limit_E x go to 0 and because 1 over 0 is infinity
E4b	[2] As it approaches from the left it approaches negative infinity…because one over a small number is a large number		[20] When we send this limit_E…as a form of 1 over 0…because it's getting closer to 0 at the bottom, it diverges to infinity
E4c	[3] The limit would be zero because it's keep going but never actually equal…even though it's so big the decimal would be so small	K2a	[21] Same…I also thought so that the graph goes by this trend…the part that wasn't pictured is following the rule of the graph pictured
E4d	[4] They're both asymptotes so they'll go for infinity…they won't ever touch it	K2b	[22] If [x] approaches 0 from right, positive infinity. After looking at the shape…extended this [graph], I reached such conclusion
E5a	[5] Approach infinity…because one divided by a teeny tiny number	K3a	[23] I thought of it mathematically and answered plus infinity
E5b	gives you a really big number…it will never touch zero	K3b	[24] When sent to infinity, 1 over 0 is definitely infinity
E5c	[6] From left that's going to be negative infinity…there is an asymptote at zero because it can't be zero	K5a K5b	[25] If it approaches from right, it is infinity. If I use the concept of 0 plus epsil…delta value…because the numerator is a constant, if the denominator decreases, it goes to infinity
E5d	[7] From the left over here, as x keeps getting closer to 0, it keeps getting down…keep doing that forever.		
E6a	[8] It approaches infinity. *y*-axis acts as a vertical asymptote. But it never touches or crosses it	K6a	[26] It becomes almost a straight line but it's not a straight line.
E6b	[9] As x approaches zero, we divide one into smaller portions and it just gets larger but it never becomes nothing	K6b	[27] As x approaches zero from left, it goes to minus infinity…I answered plus infinity and minus infinity, if I look at the graph
E7a	[10] As you approach zero, it is reaching a positive infinite number. It can't equal 0…because the two would never meet	K6c K6d	[28] If the absolute value of x becomes smaller, the value of 1/x becomes larger…just by looking at the graph
E7b	[11] It'll approach infinity...the slope is becoming more and more vertical. It cannot ever cross the y-axis because there is an asymptote		[29] If we solve this as we learned….if it converges on the right…it goes to infinity
E7c	[12] As x approaches 0 from the left, *y* will go to negative infinity because there's an asymptote at 0 and it doesn't cross it	K7a	[30] As *x* approaches from right, it diverges to infinity by seeing just the graph. By seeing expression…because denominator
E7d	[13] It keeps getting higher but it never touches 0	K7b	approaches 0 and numerator is a natural number
Eoa	[14] One over *x*…from the right, the number gets infinitely large	Kma	[31] The graph…if x is getting close to 0…infinity or minus infinity
Eob	[15] It becomes extremely small…it's going to be a large number	Kmb	[32] I solved all by using limit (*gŭk-han*)…increasing infinitely
Eoc	[16] They approach infinity..1 over an infinitely small number is infinity	Kmc	[33] As x approaches 0, it diverges to infinity…a form of 1 over 0
Eod	[17] As you go from the left, it goes off to negative infinity	Kmd	[34] Because the graph is painted so
Eoe		Kme	[35] I also thought the same way
Eof	[18] It gets closer and closer to the y axis, but never exactly touches	Kmf	[36] If plugging in 1 over 10 and 1 over 100 and 1 over 1000… when approaching from the right, the curve grows toward infinity

Table 4.58 Responses to the question about the limit of $\frac{x^2}{1+x}$ in item VI-b

VI-b. What will happen to the curve when x goes to positive infinity?			
E4a	[37] It keeps on becoming a bigger number…this line becomes steeper	K1a K1b	[56] Because the increasing rate of x^2 is more than the increasing rate of 1+x, if x increase to infinity…it diverges to infinity
E4b	[38] It'd just keep going to infinity…approaches an oblique asymptote		
E4c	[39] When x goes to positive infinity, the curve will continue on and go to also positive infinity because it's approaching the slant asymptote	K2a	[57] Increasing speed in the degree of 2 is faster than that in the degree of 1. So finally, it goes to infinity
E4d	[40] It'll approach infinity forever and it won't get there	K2b	[58] If coefficient is bigger and $\lim_{E}$ is taken, [it's] toward infinity
E5a	[41] Because infinity squared divided by infinity…getting bigger.	K3a K3b	[59] Infinity…by looking at the curve it's hard to tell
E5b	[42] It would go to infinity…it's like infinity squared over infinity. It's gonna go forever		[60] The numerator increases faster…it will go toward positive infinity
E5c	[43] It goes to infinity…like it keeps going on there forever	K5a K5b	[61] It increases to positive infinity…diverges to infinity. The ratio of the numerator increasing is larger than that of the denominator
E5d	[44] Like an asymptote…as x gets bigger y gets bigger…y would go to positive infinity because the *x* here is bigger than the *x* here		
E6a	[45] As x gets larger, this keeps on getting larger…this oblique asymptote…they approach it but they never cross or touch it	K6a	[62] It increases toward infinity. I checked the changed value…when it goes from 4 to 5 and…from 5 to 6
E6b	[46] Because it's an oblique asymptote…*x* will just follow that line as it goes to positive infinity	K6b	[63] When x goes to infinity, a form of infinity over infinity comes out. If I use l'Hospital's rule, 2x…infinity comes out as a result
E7a	[47] Its slope will become a constant increase into infinity	K6c K6d	[64] Because degree [numerator] is larger, it diverges to infinity
E7b	[48] It'll be approaching infinity without end		[65] If divide both sides with x…it becomes x/(1/x)+1 and the denominator converges to 1…it goes to infinity
E7c	[49] As x goes to positive infinity, *y* goes to positive infinity	K7a	[66] It goes to infinity because increase rate of numerator is faster
E7d	[50] I agree both goes to infinity and because the top is higher than the bottom. There's an asymptote at negative 1	K7b	[67] It diverges to infinity… if denominator and numerator are divided by x…because x is remained and x goes to infinity
Eoa	[51] The curve will go to positive infinity from the graph	Kma Kmb	[68] This one is infinity…if following like this [in the graph]
Eob	[52] This will go up when *x* is positive infinity		[69] Me too…because the degree of denominator is much larger
Eoc	[53] It will reach infinity because the degree of the numerator is higher than the degree of the denominator	Kmc Kmd	[70] When x goes to infinity, if use l'Hospital's rule, it comes out 2*x*… toward infinity
Eod	[54] It goes off infinity, because infinity squared over 1 plus infinity		[71] I also used l'Hospital's rule
Eoe	[55] The power on top is larger than the power on bottom, so it's not going to get any smaller. It's only going to grow…the curve will continue to rise	Kme Kmf	[72] If dividing this numerator and denominator by x, the numerator is approaching 1 and the denominator is going to infinity
Eof			[73] Because disparity becomes wider by plugging in [numbers]

Table 4.59 Responses to the question about the limit of $\frac{x^2}{(1+x)^2}$ in item VI-c

VI-c. What will happen to the curve when x goes to positive infinity?			
E4a	[74] This gradually gets closer to one…same exponent…a little bigger	K1a	[93] After looking at the graph, I thought it's going to infinity
E4b	[75] It will approach y=1 but it'll never get there because it's a litter bigger on the bottom…it has the same degree on top as bottom	K1b	[94] I sent this graph by limit_E and a form of infinity over infinity came out. If…use l'Hospital's rule, it converges to one
E4c	[76] No asymptote...as *x* increases *y* goes to infinity in the process	K2a	[95] Their increasing speed are similar…solving limit (*gŭk-han*)
E4d	[77] This line will go to infinity because it's going to go forever	K2b	[96] It is going toward 1 over 1…if 1 is substituted for x…if 100...
E5a	[78] I kind of just went off the graph too	K3a	[97] The answer is 1. x is a big number…it is similar but different.
E5b	[79] It would approach one. I just judged from the graph. I guess you could divide by x and take the limits of all the parts.	K3b	[98] It can get closer to one… it can never be one…the bottom is always greater
E5c	[80] It'll become one but never really equal one…[bigger] denominator	K5a	[99] It converges to 1, because the speed of x^2 increasing will be bigger than the speed of x increasing…value of 2x can be ignored
E5d	[81] As x approaches positive infinity, it'll approach *y* equals one…it never equals one	K5b	
E6a	[82] As it goes to positive infinity, it will never become greater than one	K6a	[100] Let limit_E…If we compare their coefficients, the limit is one
E6b	so it approaches one…once x becomes a very large number, 2x+1 don't matter…kind of a horizontal asymptote	K6b	[101] I used l'Hospital's rule too. So 1 came out
E7a	[83] The curve will approach *y* value of one. But it will never cross it	K6c	[102] Because each coefficient is 1…so I thought it converges to 1
E7b	[84] The limit will be one…it can't ever reach that value…the greatest coefficients is the x squared over x squared, so it's one over one	K6d	[103] After considering limit_E, If I divide both sides by x^2…because both the numerator and the denominator converge to one
E7c	[85] As x goes to positive infinity, *y* goes to 1. You can tell there's an asymptote here. I was looking at the equation but I can't figure out why	K7a	[104] Toward one…because degree is the same and coefficients are one over one
E7d	[86] It gets closer to 1…there is an asymptote at one because of the way that they both curved	K7b	[105] This approaches one…if one compares two greatest degrees, it approaches one but it is less than 1 because…2x+1 in numerator
Eoa	[87] I feel the limit is going to be positive infinity because x is going to positive infinity	Kma	[106] Because two quantities [x^2 and $(x+1)^2$] are almost similar, if they grows a lot…toward one
Eob	[88] It has a limit but I didn't calculate that…a horizontal asymptote	Kmb	[107] Toward one…if x grows…the calculation method of Limit_E
Eoc	[89] It will approach 1 because there's an asymptote; the graph implies	Kmc	[108] It converges to one because the coefficients are the same
Eod	[90] If you actually plug infinity in the equation, it would go off to 0 because the bottom faster than the top	Kmd	[109] It converges to one if divide both numerator and denominator by the greatest term
Eoe	[91] It'll just keep growing…based on the graph…infinitely closer to 1	Kme	[110] One… both denominator and numerator divided by x square
Eof	[92] It will approach one but never touch it because that's the horizontal asymptote…the limit of the function…I looked at the graph	Kmf	[111] One…by plugging in [numbers]…the difference between numerator and denominator is decreasing

Table 4.60 Distribution of the use of the contexts in item VI

Functions	Students	Processual use	Structural use	Chi-square		
				df	Value	Sig.
VI-a $\frac{1}{x}$	E-interviewees	90 %	10 %			
	K-interviewees	45 %	55 %			
	E-participants	72.7 %	25.8 %	1	49.60	.000*
	K-participants	28.6 %	68.3 %			
VI-b $\frac{x^2}{1+x}$	E-interviewees	95 %	5 %			
	K-interviewees	15 %	85 %			
	E-participants	65.9 %	27.3 %	1	35.96	.000*
	K-participants	31.7 %	65.9 %			
VI-c $\frac{x^2}{(1+x)^2}$	E-interviewees	80 %	20 %			
	K-interviewees	5 %	95 %			
	E-participants	64.4 %	19.7 %	1	102.8	.000*
	K-participants	10.3 %	85.7 %			

Note. *Differences are significant ($p \leq .01$).

M. The phenomenon first observed in [F] returned. Many E-speakers spoke about the limit of a function in terms of processes. Once again, the Korean responses were formulated in a more formal mathematical way with the purpose of finding the limit.

The contexts, "x approaches 0 from left or from right" or "x goes to positive infinity," directed almost all E-interviewees to use either geometric features on the graph (such as slope and asymptote) or algebraic properties (such as comparison of degrees and increasing rates between numerator and denominator). These contextual uses led them to a processual use of the limit. This processual use of the limit in the E-group often seemed to be related to *unreachability* in the limit algebraically or geometrically ([3-6], [8-9], [10-13], and [18]; [40] and [48]; [75], [80-82], [84-86], [89], and [91-92]). Comparatively, the same contexts seemed to guide the majority of K-interviewees, with just two exceptions ([68] and [93]), to an algebraic reasoning to find the limits of three functions by using different mathematical algorithms ([19-20], [23-25], [27], and [29-33]; [56], [58], [61], [63-65], [67], and [70-72]; [94-95], [99-101], [104-105], [107], and [109-110]). When the question "What will happen to the curve?" was asked, almost all K-

interviewees took for granted the task of finding the limits, and then structuralized limiting processes with the concept of a limit.

This difference between the E-group and K-group in the use of the contexts was also salient in written survey responses. The chi-square results in Table 4.60 show a significant difference between the E-group and K-group in the use of the contexts.

N. In the task of finding the limit of a given function with its graph, eight routines were identified according to the aspects of limiting process and the limit that were used.

1. **Describing the limit by looking at the graph:** There were interviewees who looked at either increasing or decreasing features on the graph. Based on the graph, they endorsed that it was increasing or that it went to infinity or that it diverged to infinity or that it converged to one.
2. **Describing the limit by looking at an asymptote:** Some interviewees focused on an asymptote approached by a function or its graph. When there was a vertical asymptote, then they endorsed that it went to infinity. When there was a horizontal asymptote, then they endorsed that it approached the asymptote value.
3. **Finding the limit by assuming that infinity is a number:** Other interviewees substituted infinity for x to find the limit of a given function. Then, they interpreted infinity as a real number (e.g. $1/0 = \infty$ or one over an infinitely small number is infinity). For example, if the result after substitution was infinity squared over infinity, they concluded that it went to infinity because an infinity in both numerator and denominator was cancelled out.

4. **Finding the limit by comparing increasing rate in numerator with that in denominator:** There were interviewees who focused on a comparison between numerator and denominator to decide which one grew faster than the other. If numerator grew faster than denominator, then they concluded that it went to infinity. In the opposite case, they endorsed that it went to zero. In the case of $1/x$, because the numerator was fixed as one, they endorsed that it went to infinity when x got smaller. Also, if increasing rates between numerator and denominator were the same, they endorsed that it converged to one.

5. **Finding the limit by dividing the greatest term in numerator or by comparing the greatest terms in both numerator and denominator:** Some interviewees divided both numerator and denominator by the greatest term in denominator or compared the greatest terms in both numerator and denominator. Then they calculated the limit value of the function by direct substitution.

6. **Finding the limit by using l'Hôpital's rule:** Other interviewees seemed to first look at the function as a form of ∞/∞ when x approaches infinity. Then they used l'Hôpital's rule to find the limit value.

7. **Finding the limit without specifying the method**

8. **Undefined:** Other interviewees focused on the idea that the limits from the right and left were not equal and then endorsed that the limit did not exist.

9. **Unidentified:** This is the case, for example, when the student reported that it keeps going or increases with no other explanation.

10. **No answer**

Table 4.61 Distribution of *routines* in item V-a

Routines	E-speakers		K-speakers	
	interview	Survey	interview	survey
1) Describing a process by looking at the graph	30 %	18.9 %	45 %	41.3 %
2) Describing the limit by looking at an asymptote	35 %	29.5 %	0 %	11.1 %
3) Finding the limit by assuming that infinity is a number	0 %	1.5 %	10 %	3.2 %
4) Finding the limit by comparing numerator with denominator	35 %	9.8 %	25 %	20.6 %
5) Finding the limit by dividing the greatest term in numerator	0 %	0 %	0 %	0 %
6) Finding the limit by using l'Hôpital's rule	0 %	0 %	0 %	0 %
7) Finding the limit without specifying the method	0 %	12.1 %	0 %	4.8 %
8) Undefined	0 %	6.1%	0 %	0 %
9) Unidentified	0 %	20.5%	20 %	16.7 %
10) No answer	0 %	1.5%	0 %	2.4 %

Table 4.62 Distribution of *routines* in items VI-b and VI-c

Routines	E-speakers				K-speakers			
	Interview		Survey		interview		survey	
	VI-b	VI-c	VI-b	VI-c	VI-b	VI-c	VI-b	VI-c
1) Describing a process by looking at the graph	30 %	25 %	23.5 %	12.9 %	10 %	5 %	16.7 %	5.6 %
2) Describing the limit by looking at an asymptote	20 %	40 %	16.7 %	22.0 %	0 %	0 %	4.0 %	0.8 %
3) Finding the limit by assuming that infinity is a number	20 %	0 %	1.5 %	0 %	0 %	0 %	0.8 %	2.4 %
4) Finding the limit by comparing numerator with denominator	15 %	20 %	10.6 %	6.1 %	60 %	30 %	34.9 %	24.6 %
5) Finding the limit by dividing the greatest term in numerator	0 %	5 %	2.3 %	4.5 %	15 %	45 %	12.7 %	38.1 %
6) Finding the limit by using l'Hôpital's rule	0 %	0 %	6.1 %	0 %	15 %	10 %	3.2 %	4.0 %
7) Finding the limit without specifying the method	0 %	0 %	6.1 %	10.6 %	0 %	0 %	5.6 %	10.3 %
8) Undefined	0 %	0 %	0.8 %	0 %	0 %	0 %	0 %	0 %
9) Unidentified	10 %	10 %	31.8 %	29.5 %	0 %	10 %	19.0 %	11.1 %
10) No answer	0 %	0 %	6.8 %	13.6 %	0 %	0 %	2.4 %	2.4 %

O. In finding the limit of $\frac{1}{x}$ in item VI-a, dominant routines in both groups were based on the geometric representation (i.e., routines 1 and 2 in Table 4.61).

P. In finding the limits of the more complicated functions in items VI-b and VI-c, *routines* of E-speakers were based on the geometric representations, compared to those of K-speakers on algebraic representations. In other words, the use of geometric representations in routines 1 and 2 was still dominant in the E-group, whereas the use of algebraic reasoning (i.e., routines 4 and 5) dominated routines in the K-group.

As previously explained, visual mediators are visual means used by students for communication. In item VI, both algebraic symbols and graphs of three functions were provided for limit-finding tasks. In calculating the limit of each function, different visual mediators between the E-group and K-group were employed for the sake of scanning mathematical context.

Q. *Syntactic* mode in the E-group and *objectified* mode in the K-group. The first attempt in the responses of most E-speakers at scanning the written phrase of "when x goes to positive infinity," involved uses of the shape of graphs and geometric mediators such as asymptotes and slopes. In contrast, the same written phrase was scanned as the algebraic expression of each function in the responses of about 60 % of K-speakers (see Table 4.62). The scanned processes in the E-group may be called *syntactic* mode, as it requires only knowing the written phrases and given geometric mediators. However, most K-speakers seem to make a transition from *syntactic* mode to a different mode for finding limits. In other words, they seem to reify both original written phrases and geometric mediators with the purpose of finding limits. The flexibility in the scanned process of the written phrases and given geometric mediators can be described as better performed than the less flexible syntactic mode because the syntactic mode allows for very little interpretation of the visual mediators and few predictions.

The following Episodes 9 and 10 exemplify differences between E-group and K-group in finding the limits of three functions in item VI.

Episode 9. E-pair in finding the limits of three functions in item VI

Speaker	What is said	What is done
146. I	So, **[show the card about item VI-a]** what about this one? What happens to the curve as x approaches 0 from the left or right?	show the card
147. E7c	The curve goes to zero.	
148. E7d	The curve goes to neg... [both say "oh"] [laughing]. I was gonna say negative infinity, but that's 0. I don't know.	

149. E7c	Well, yeah it'd go, yeah it'd go to negative infinity and this is the asymptote would be 0. [...] So...		
150. I	So because?		
151. E7c	It depends on if you're looking at...		
152. I	Mm-hmm		
153. E7c	The *y* axis or if you're looking at the *x* axis.		
154. I	Mm-hmm		
155. E7d	But it says *x*		
156. E7c	Yeah as *x* approaches 0 from the left, so as *x* goes to 0, *y* will go to negative infinity.		
157. E7d	Mm-hmm		
158. I	Because?		
159. E7c	Because there's an asymptote at 0 and it doesn't cross it.		
160. E7d	[...] It just keeps getting higher and higher and higher but it never touches 0, so		
161. I	Mm-hmm		
162. E7d	Uh, yeah the x coordinate would be...		
163. I	Mm-hmm		
164. E7d	close to 0		
165. I	Mm-hmm		
166. E7c	Mm-hmm		
167. E7d	But the *y* ... *y* coordinate is gonna be infinity. It's like the same, that right, negative infinity.		
168. I	Anything else?		
169. E7d	No.		
170. I	**[show the card about item VI-b]** What about this one?	show the card	
171. E7c	As *x* goes to positive infinity, um, *y* goes to positive infinity.		
172. I	[..] Because?		
173. E7c	Um because here as *x* goes on to infinity, the *y* goes up to infinity, and gets closer to infinity.		
174. E7d	[....] Yes, I agree both goes to infinity and because the top is higher than the bottom		
175. I	Mm-hmm		
176. E7d	So, it's just going to keep becoming a larger number.		
177. I	OK, [...] anything else? Do you want to add?		
178. E7c	Um, it looks like...isn't there an asymptote here at negative 1?		
179. E7d	It looks like.		
180. E7c	Yeah, so um but this one for this side um, it doesn't go to positive infinity, it goes to negative 1. So, oh never mind [laughing]		
181. I	But, you know, when x goes to, you know, positive infinity?		
182. E7c	But...		
183. E7d	Yeah there's an asymptote at negative 1 and whatever		
184. I	[....] Do you want to add something?		
185. E7c	Oh no		
186. I	**[show the card about item VI-c]** What about this one? *x* squared over 1 plus *x* squared?	show the card	
187. E7c	As *x* goes to positive infinity		
188. I	Mm-hmm positive infinity.		
189. E7c	*Y* goes to 1		
190. E7d	That's what I said, it gets closer to 1, you can see that there would be an asymptote right here.		
191. E7c	Yeah		
192. E7d	And that would be at 1		
193. I	Can you explain why?		
194. E7d	Because of the way that these both curved		
195. I	Mm-hmm		

196. E7d They'll end...
197. E7c [...] Well
198. E7d And here, there's an asymptote at one, I think.
199. I Mm-hmm
200. E7c [.....] That's...that's what I have, y equals 1 as x goes to infinity...positive infinity
201. I Do you want to say why?
202. E7c Um, basically the same thing that you can tell there's an asymptote here.
203. I Mm-hmm
204. E7c And it doesn't cross it, I was looking at the equation but I can't figure out why
205. E7d Yeah
206. E7c Yeah

Episode 10. K-pair in finding the limits of three functions in item VI

191. I	Then...**[show the card about item VI-a]** let's solve the next problem. What happens to the curve as x approaches zero from left or from right?	show the card
192. Kmf	[.....] When approaching from the right, the curve infinitely upward ...grow toward plus infinity...	
193. I	Um...um...	
194. Kmf	When approaching from the left, the curve toward minus infinity infinitely...it does not intercept y axis...I thought it's going to that direction.	
195. I	The reason is...?	
196. Kmf	[...] in 1 over x...in 1 over x because it's not 0 but approaching plus 0...from a number which is a little bigger than 0...so it's positive...so it's growing to the upward direction on x axis...if plugging in 1 over 10 and 1 over 100 and 1 over 1000, it continues to increase...by becoming 10, 100, and 1000...Considering that way, because I thought that it's becoming infinity...without intersecting with y axis...because it's not 0.	
197. Kme	I also thought the same way.	
198. I	So...because it continues to increase by plugging in 1 over 10, 1 over 100, and 1 over 1000.	
199. Kme	Yes	
200. I	**[show the card about item VI-b]** When x goes to positive infinity, what happens to the curve?	show the card
201. Kmf	[...] I thought that this one is similarly going to become infinity. Here now what he and I said, speed which is similar to ratio comes out...Because the increasing speed of the numerator is bigger than the increasing speed of the denominator, the disparity between numerator and denominator is going to become wider by plugging in [numbers] ...	
202. I	Um...um	
203. Kmf	If this disparity decreases, it's converging. Because the disparity increases, I think that it's going to diverge.	
204. Kme	Thinking so is okay, but...that x grows to positive infinity means that [x] equal to 0 doesn't need to be thought. Because x equal to 0 doesn't need to be thought,...that...thinking that x is not 0....if dividing this numerator and denominator by x, the numerator is approaching 1 infinitely and the denominator is going to positive infinity because x is positive infinity....if by doing so, I think that it can be more easily calculated	
205. I	Instead, are you saying that x goes to infinity?	
206. Kme	Yes	
207. I	**[show the card about item VI-c]** When x goes to positive infinity, what happens to the curve?	show the card
208. Kmf	As I said before, the difference continues to decrease. The difference between numerator and denominator is big at first, but as x increases, the	

difference is decreasing.

209. I How did you know that the difference is decreasing?

210. Kmf By plugging in [numbers]...continues to do ...the difference continues to decrease little by little.

211. I Please tell me concrete numbers?

212. Kmf Yes. At first if plugging in 1, it comes out 1 over 4.

213. I Yes

214. Kmf Because it's 1 over 4, the difference between 1 and 4 is 3

215. I Um...um

216. Kmf If plugging in 2, um...it comes out 4 over 9. But ...this one becomes 5.

217. I Um...um

218. Kmf Just if continuing to plug in [numbers], because I finally thought that this difference is going to decrease little by little...

219. I How do you think?

220. Kme As I said before....uh....that denominator...so in the status that numerator is fixed, if denominator is increasing infinitely or decreasing infinitely, while thinking that value is approaching 0, both denominator and numerator were divided by x square. Then it can be calculated as in the case of the previous one.

221. I So, what's your answer?

222. Kmf One

223. Kme One

Item VII

In item VII, the question "What happens if you continue increasing the number of sides of a regular polygon inscribed in a circle?" was asked with the following concrete geometric representations in Figure 7.

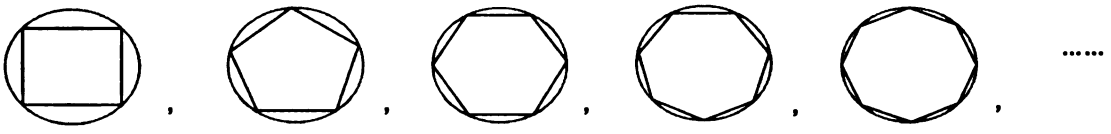


Figure 7. Geometric representations provided in item VII

Table 4.63 Distribution of *endorsed narratives* in item VII

Endorsed narratives	E-speakers		K-speakers	
	Interview	survey	interview	Survey
1) It approaches the circle	55 %	48.5 %	15 %	38.1 %
2) It gets closer to the circle but never reaches it	25 %	7.6 %	0 %	0 %
3) It converges to (or becomes) the circle	20 %	28.8 %	85 %	57.1 %
4) Unidentified	0 %	9.1 %	0 %	4.0 %
5) No answer	0 %	6.1 %	0 %	0.8 %

R. The majority of K-speakers reported that they accepted the endorsement that the geometric sequence of polygons became the circle, compared to less than one third of E-speakers. Only 28.8 % of E-speakers reported "it becomes the circle."

Howe

57.1

However, an overwhelming majority of K-speakers (category 3: 85 % in the interview 57.1 % in the survey) replied "it converges to (or becomes) the circle."

Table 4.64 Responses to the question about the limit of regular polygons inscribed in a circle in item VII

Table 4.64 Responses to the question about the limit of regular polygons inscribed in a circle in item VII

VII. What happens if you continue increasing the number of sides of a regular polygon inscribed in a circle? Explain your reasoning			
E4a	[1] As you increases the sides, it'd become more a circle…use it to figure our area of a circle, but it wouldn't be quite a circle	K1a	[21] Vertexes increases so that they are getting close to the circle
E4b	[2] If increasing the size of polygons, it'll be almost equivalent to the area of the circle. But you can't perfectly get it…approximating the area	K1b	[22] As vertex increases, this empty space decreases more and more… the vertex would form a circle
E4c	[3] It will tend to look like a circle but would never be a circle because there's infinite amount of sides…like a graph with a asymptote	K2a	[23] Because area is decreasing, if it meets at a point, it has the property of a circle
E4d	[4] You'll never fill up the circle because there's always sides	K2b	[24] If this continues to increase and become infinity…the circle.
E5a	[5] Because there's lots of little lines…look more and more like a circle but it can't become a circle	K3a K3b	[25] Because x increases to infinity in the x-gon, it will become a circle
E5b	[6] More and more like a circle…circles just like as polygons of infinite sides. So I guess that's why I think it becomes a circle		[26] It will become closer to the circle. The different areas between the circle and the figures will decrease
E5c	[7] If you could calculate the area…it'll get so small…it's basically zero and then you'd say that they're just two circles overlapping	K5a K5b	[27] Later the sides can eventually be seen as a point…because the lengths of the inscribed polygons are same, it becomes a circle
E5d	[8] The sides keep getting infinitely smaller…you'll get the point		
E6a	[9] The [between] area gets smaller…It'll never become a circle	K6a	[28] The number of sides increases…it's getting close to the circle
E6b	[10] The more sides you add the more it becomes like a circle. It won't actually be a circle it just extremely close to a circle	K6b	[29] My mathematics teacher divided the circle into triangles. If we add all triangles, this becomes the area of the circle
E7a	[11] It would become an infinitely large amount of sides…you're trying to get closer to it but you can't get the slope of that point.	K6c K6d	[30] If we assume n-gons, the angle [of a triangle] is $2\pi/n$. If I took limit to the value of cosine, it's 1. It's a circle if it goes to infinity
E7b	[12] As the number of sides increases, it wasn't a circle, but it would look close enough to a circle		[31] By looking at this picture, I thought that the area of the polygon seems to converge to π
E7c	[13] It's not rounded…many sides…it's hard to tell the difference	K7a	[32] If one think so [integral calculus], it seems to become a circle
E7d	[14] There are an infinite amount of sides…it would become a circle	K7b	[33] If it goes to infinity…it becomes a circle
Eoa	[15] Less area inside of the circle…getting closer to becoming a circle	Kma	[34] If sides increase infinitely, this [between-part] area converges to 0. So it becomes a circle
Eob	[16] That the area of the figure and the area of the circle would become closer and closer to each other, but it would never be the same	Kmb	[35] If [the side] becoming a point…finally they become a circle
Eoc	[17 The limit will be a circle with an infinite number of sides, because a circle has no sides. So, yeah, it won't ever be a circle	Kmc	[36] The smaller this distance [base of triangles] is, the closer shapes approach a circle [Sum of triangles converges to the circle]
Eod	[18] small lines...it's gonna be a circle, but never physically be a circle	Kmd	[37] As $\lim_{E}$ n goes to infinity, this area converges to πr^2
Eoe	[19] The area of the circle…outside of the polygon…it just got cut up into smaller and smaller pieces…the area stays the same in between…	Kme	[38] The lengths of sides are approaching 0…toward $\lim_{E}$ 0… the length of side is 0…become a circle
Eof	[20] The area between the circle and the polygon shrinks and can not prove at all	Kmf	[39] It's going to become a circle… because finally sharing points with the circle become infinitely many

Table 4.65 Distribution of the use of the limit in the sequence of regular polygons

Students	Processual use	Structural use	Chi-square		
			df	Value	Sig.
E-interviewees	65 %	35 %			
K-interviewees	15 %	85 %			
E-participants	60.6 %	33.3 %	1	6.26	.012
K-participants	48.4 %	50.8 %			

*Differences are significant ($p \leq .01$).

S. The same as [F] and [M] above. In the case of regular polygons inscribed in a circle in item VII, the context of *increasing the number of sides of a regular polygon* in the E-group seemed to be used only for its geometric features. Such uses of the context can be evidenced by the utterances of either increasing sides or decreasing areas (see "sides" [1-4, 6, 10, 11, 13-15, 18] and "lines" [15, 18] for increasing sides; "area" [7, 9, 16-17, 20] for decreasing areas). These geometric uses of the context appeared to direct them to think about a *processual* use of the limit of regular polygons ([1-12], [16-17], [19], and [20]). This processual use of the limit often was related to *unreachability* ([1-2] and [18]). In contrast, almost all K-interviewees continued to employ *structural* uses of limit. In the same context, many K-interviewees characterized mathematical features of polygons such as the lengths of sides, vertexes, and triangles ([27], [35] and [38] for the lengths of sides; [21-22] and [39] for vertexes; [29-30] and [36] for triangles). This *structural* use of the limit was related to *reachability*, that is, the sequence of regular polygons became the circle ([22-25], [27], [29-30], [31-35], and [37-39]). However, this difference between the E-group and K-group in the use of regular polygons did not appear clearly in written survey responses. The chi-square results in Table 4.65 show little difference between the E-group and K-group.

T. Two routines for finding the limit in item VII were identified in the interviews and the survey:

1. **Describing a process by looking at patterns of regular polygons in the sequence:** Some interviewees focused on the idea that the number of sides was increasing or that the areas between polygons and circle were decreasing. Then they endorsed that the sequence of polygons was getting closer to the circle.

2. **Finding the limit by looking at algebraic features of regular polygons in the sequence:** Other interviewees considered the concepts of limit with limiting processes of regular polygons. As for the concept of limit, they used the ideas that the length of the sides became a point (or zero), or that the number of sides went to infinity, or that the difference in area converges to zero. Then they concluded that the sequence of polygons became a circle.

3. **Unidentified:** This is the case when students said that it approached the circle or that it becomes the circle without giving a reason.

4. **No answer**

Table 4.66 Distribution of *routines* in item VII

Endorsed narratives	E-speakers		K-speakers	
	Interview	survey	interview	Survey
1) Describing a process by looking at patterns of regular polygons	0 %	56.1 %	0 %	31.0 %
2) Finding the limit by looking at algebraic features of regular polygons	0 %	20.5 %	0 %	49.2 %
3) Unidentified	0 %	17.4 %	0 %	19.0 %
5) No answer	0 %	6.1 %	0 %	0.8 %

U. *Routines* in the E-group were based on *syntactic mode* of regular polygons, compared to those based on their *objectified* mode in the K-group. The majority of E-speakers scanned the written phrase of "increasing the number of sides of a regular polygon" and then they used geometric features such as the sides of the polygon and the areas to give an answer. Comparatively, after the majority of K-speakers scanned the same written phrase, they focused on algebraic features of geometric mediators such that

the length of sides converges to zero or that the number of sides went to infinity. The former discourse in the E-group seemed to be mediated syntactically on the basis of changes of regular polygons in the sequence, whereas the latter in the K-group appeared to be based on its objectified mode.

The following Episodes 11 and 12 exemplify differences in their routines between E-group and K-group in finding the limit of the sequence of regular polygons in item VII.

Episode 11. E-pair in finding the limit of the sequence of regular polygons in item VII

Speaker	What is said	What is done
301. I	**[show the card about item VII]** What about, you know, this one, using the sequence, the square, pentagon, hexagon inscribed in the circle, what happens if you continue... increasing the number of sides of a regular polygon inscribed in the circle	Show the card
302. Eob	Alright, I...	
303. I	Explain your reasoning.	
304. Eob	I thought about this one in geometrical terms, which I don't know why I switched up right here, but I said that the difference in the area between, like, the area of the square in the circle and the area of, like, pentagon circle, the area, the difference in it would become smaller. But I don't think that's the...	
305. Eoa	I said...	
306. Eob	direction I should have gone [laughing] in.	
307. Eoa	I said that...	
308. Eob	It doesn't make...	
309. Eoa	There would be less area, less, or less area inside of the circle, like, I meant, like, pretty much exactly what you just said, I just said it a little differently.	
310. Eob	Ok, we both thought about it, like, geometrically	
311. Eoa	yeah, like, like, this area decreases each time, like, the area between the outside of the shape, like, outside...	
312. Eob	Yeah	
313. Eoa	of this sided figure and the circle just decreases, so every time you add another line so it's getting closer to becoming...	
314. Eob	a circle	
315. Eoa	the circle	
316. Eob	yeah.	
317. I	[...] So what-what-what's your final answers?	
318. Eob	That the area of the figure and the area of the circle would become closer and closer to each other, but it would never be the same.	

Episode 12. K-pair in finding the limit of the sequence of regular polygons in item VII

183. I	Then **[show the card about item VII]** examine the sequence of the square, pentagon, hexagon, heptagon..., which is inscribed in the circle. What happens if you continue increasing the number of sides of a regular polygon inscribed in a circle? Explain your reasoning.	Show the card
184. K5a	close to the circle...if it goes to infinity, we can say that it is the same as	

the circle.

185. K5b So, the meaning that the sides are increasing is that the length of the sides is getting shorter. Later the sides can eventually be seen as a point. Then, a circle is originally the collection of points in the same distance. Then, from the center, because the lengths of the inscribed polygons are same, it becomes a circle.

186. K5a Yes.

Item VIII-b

Table 4.67 Responses to the question in item VIII-b

VIII-b. A student claims that the number P is the limit of a sequence a_1, a_2, a_3 … What does this mean?			
E4a	[1] The number can approach it but can't go to it	K1a K1b	[21] $\text{Limit}_{\text{KMath}}$ is something $\text{limited}_{\text{KColl}}$…as having an end. It converges to P
E4b	[2] You can never reach A sub P but really close. Once you get to A sub P there's nothing after that		[22] When the ratio r is less than 1 [as geometric sequence], it [general term] converges to the certain number P
E4c	[3] As a_n approaches infinity, the limit equals P. It's going on forever because it can never reach that limit. P…graphs with asymptotes	K2a	[23] The value of infinitith term is P…like a graph it's approaching. It's not a number but if [it's] going to end, how it is going to be
E4d	[4] It's not infinite…P is the point where it has to stop	K2b	[24] After listed, if the infinitith element is picked up, the picked number [write a_∞] is going to be P
E5a	[5] That sequence will not go onto infinity because there's a limit…If P is zero, it can be point 00001. But it can't ever be zero	K3a K3b	[25] When n is sent to infinity, the numbers of the sequence is becoming closer to P
E5b	[6] It gets infinitely close to but can't touch P		[26] If the sequence is increasing, it gets endlessly closer to or becomes P
E5c	[7] Like certain asymptotes…it cannot equal a certain number	K5a	[27] The sequence is blocked below P…to a neighborhood of P
E5d	[8] It's gonna get closer to P but it won't reach it…like an asymptote	K5b	[28] If we continue with a1, a2, a3, it moves gradually towards P
E6a	[9] The sequence goes up until it reaches that number P…where it stops	K6a	[29] As n increases, the general term a_n is getting closer to P
E6b	[10] A limit is usually something from a function…x value at a function…a sequence of numbers doesn't really have a limit	K6b	[30] As n increases, a_n is getting close to a certain number P. Limit (*gŭk-han*) is either converging to a certain number or diverging.
E7a	[11] The number that it will approach but never actually be	K6c	[31] As limit n goes to infinity, a_n converges to P
E7b	[12] It can get to a number, but it will never reach it. That's estimation	K6d	[32] As limit n goes to infinity, a_n converges. Therefore, it means that it will have a certain value
E7c	[13] The sequence will end at P as a limit in general	K7a	[33] *lim* a_n = P. n is infinity…it's going for P but it's not P
E7d	[14] It never actually becomes P, but it gets closer and closer to the number P	K7b	[34] As n increases continuously, the value of the sequence a_n approaches P … it can be P here
Eoa	[15] It will never get to P…it will always be approaching P	Kma	[35] This sequence is getting close to infinitely to P
Eob	[16] The sequence will never reach/pass this number, but wouldn't end	Kmb	[36] These numbers are going toward P… getting close to P
Eoc	[17] It's bound by a function…the function at P is the limit	Kmc	[37] *lim* a_n = P. When n becomes infinity, it converges to P
Eod	[18] It goes off to P and no further. It's gonna stop at P if not before it	Kmd	[38] In the sequence of a_n, as n grows, it approaches a certain value
Eoe	[19] It just approaches P. It gets infinitely close to P	Kme Kmf	[39] If the difference between a_n and a_{n-1} decreases, numbers [are] being close to a fixed number [P]
Eof	[20] a is going to continue to approach P…never actually reach P		[40] If a term is later than a certain term…the difference between the term and P can be decreased

Table 4.68 Distribution of the use of the limit P in a sequence

Students	processual use	structural use	Chi-square		
			df	Value	Sig.
E-interviewees	80 %	15 %			
K-interviewees	30 %	70 %			
E-participants	40.2 %	5.3 %	2	62.76	.000*
K-participants	11.9 %	45.2 %			

*Differences are significant ($p \leq .01$).

V. E-speakers' responses were relatively consistent but not in tune with the canonical interpretation of the word *limit*. K-speakers' interpretations were more diverse but often consistent with the canonical interpretations. The majority of E-speakers used expressions such as "approaching," and "getting closer to" to explain *the number P is the limit of a sequence* ([1], [3], [6], [8], [11-12], [14-16], and [18-20]). K-speakers' responses were diverse. However, they often made utterances saying that the sequence a_n converges to P as n goes to infinity (see [25], [29-34], and [37-38]).

W. E-speakers provided processual explanations. K-speakers spoke about the limit P in structural terms. In explaining the meaning of "P is the limit of a sequence," most E-interviewees' responses referred to the word *limit* as processes in their definitions. This *processual* use seemed to be related to only sequence processes of a_n without considering processes of subscript *n*. The *processual* use often was connected with the idea of an asymptote ([3], [7], and [8]). In contrast, most of the K-interviewees focused on either a general term a_n or a relationship between two terms at first. Then either this general term or a relationship between two terms was characterized by specific mathematical features. The chi-square results in Table 4.68 show a significant difference between the E-group and K-group in written survey responses.

X. The idea of *reachability* in the K-group and the idea of *unreachability* in the E-group. This *processual* use in E-group was often characterized by *unreachability* to the

limit P ([1-3], [5-8], [11-12], [14-16], and [19-20]), that is, "P is the limit as an unreachable process." Comparatively, the K-speakers' ideas of "P is the limit" were dominated by the idea of reachability ([21-22], [30-32], and [37]).

Y. E-speakers formulated interpretations in their own words. K-speakers used more formal mathematical formulations. Many K-speakers clearly made a relationship between the two processes of a_n and n in the limit of a sequence. In order to define "P is the limit of a sequence," they often provided formal languages such as "converge" and "limit_E" ("converge" [21, 22, 31, 32, 37] and "limit_E" [33, 37]).

The following Episodes 13 and 14 exemplify differences in defining that P is the limit of a sequence $a_1, a_2, a_3, \ldots$ between the E-group and K-group.

Episode 13. E-pair in defining that P is the limit of a sequence $a_1, a_2, a_3, \ldots$(item VIII-b)

Speaker	What is said	What is done
145. I	**[show the card about problem VIII-b]** What about this one? A student claims that a number P is the limit of a sequence a1, a2, a3, ... What does this mean?	show the card
146. Eoe	Uh, it, this, uh, personally I interpreted this as, you know as, a1, a2, a3 it just keeps rising to a4, a5, etc., and it's, slowly, or quickly, approaching P, whatever P might be...	
147. Eof	Yeah, no matter what the value of the, of a is going to continue to approach P 'cause P is the limit.	
149. I	Is there anything else you want to say?	
150. Eoe	Not really, I mean that's-	
151. Eof	Yeah	
152. Eoe	This based, um, you know everything I've done on that's just my understanding of what a limit is	
153. I	Um-hmm	
154. Eoe	You know, numbers rising like that, it just approaches P. Its, it gets infinitely close to P.	
155. I	So do you, uh, understand the infinitely close to P? What he just said?	
156. Eof	Yeah, uh-huh.	
157. I	What is the meaning of infinitely close, to you?	
158. Eof	It means that, like, if you never actually touch P, never actually reach P, but it's going to keep approaching it, no matter how big a gets, it's going to keep going towards P.	
159. I	So do you agree with mainly what he said?	
160. Eoe	Yeah, more or less.	
161. I	OK	

Episode 14. K-pair in defining that P is the limit of a sequence a_1, a_2, a_3,...(item VIII-b)

175. I	Then **[show the card about problem VIII-b]** A student claims that the number P is the limit of a sequence. What does this mean?	Show the card
176. Kmd	In this case, I set up an expression in the same way ($\lim_{n\to\infty} a_n = P$)	
177. I	Then what is the meaning? If you explain the meaning more...	
178. Kmd	Um...so...seeing on the graph...no...rather than in the graph, when this becomes a_n... when n becomes infinity, I thought the meaning of that it converges to a number P...In case that graph is painted like that [root shape]...like the previous problem, it can converge to 1...I thought so [laughing]...yes, I thought so...	
179. I	What does the word "converge" mean concretely?	
180. Kmc	It approaches a certain value.	
181. Kmd	approaching as close as possible	
182. Kmc	If put 0.99999 as 1 equally, like that....the same as approaching endlessly	
183. I	How did you explain this?	
184. Kmc	I just...sequence...in the sequence...in the sequence of a_n, as n grows, it approaches a certain value and considered an example.	
185. I	What kind of example?	
186. Kmc	So, if there is the sequence of 1 minus 1 over n	
187. I	Um...um	
188. Kmc	As n grows endlessly, the value of sequence approaches 1. So, like that.	

Summary: Discourse on limit

Table 4.69 summarizes salient characteristics of the colloquial and mathematical discourses on *limit* in the E-group and K-group.

Table 4.69 Salient properties of E-speakers' and K-speakers' discourse on *limit*

Item: topic	Aspect	E-speakers	K-speakers
Item I: Creating sentences with given words	A. Limited: context of use	Both groups used the word *limited* in any version mainly in non-mathematical context. Among K-speakers the phenomenon was stronger than among E-speakers	
	B. *Limited*: type of use	Both groups used $limited_{Coll}$ in the sense of *finite* (not infinite – something that has an upper bound) or *restricted* – one that *could be larger*.	K-speakers used $limited_{KMath}$ in the sense of *extreme*.
	C. Limit: context of use	Both groups used the word, in whatever its version, in both mathematical and non-mathematical contexts	
	D. *Limit*: type of use	Both groups used $limit_{Coll}$ to denote an *upper bound* – something that does not let a thing or process to get larger	K-speakers used $limit_{KMath}$ mainly in expressions that were formulated as a textbook instruction to do something or as textbook definition. In general, it seemed that K-speakers were not very proficient in creating sentences with $limit_{KMath}$.
Item V: Finding the limit of an	E. V: Limit of a sequence	A *process* of the sequence dominated the idea of *limit*. About 30 % of E-speakers	An object as the limit of sequence dominated the idea of *limit*. The limit of the sequence as a number

		conceptualized the idea of the limit in the sequence as an increasing or decreasing process without the limit.	dominated the idea of limit in the K-group.
infinite sequence	F. Use of sequence	E-speakers used the infinite sequence in terms of processes	K-speakers explicitly referred to mathematical algorithms associated with the function of the infinite sequence. In other words, the K-speakers' use of the sequence was structural
	G. Routine	When finding the limit in a given infinite sequence, five routines were identified according to how to use limiting processes to find out the limit	
	H. Use of routines	E-speakers used routines based on processual use of limit	K-speakers' routines were based on structural use of limit by considering the concept of limit as a number.
Item VI: Finding the limit of a given function	I. VI-a: $\frac{1}{x}$	About one third of E-speakers in the first highest percentage responded "it goes to infinity." In the next highest percentage, the E-speakers (26.5 %) replied "(it becomes) infinity."	More than one third of K-speakers in the first highest percentage answered "(it becomes) infinity." However, only K-speakers (23.8 %) replied "it diverges (to infinity)."
	J. VI-b: $\frac{x^2}{1+x}$	About one third of E-speakers responded "it is increasing." In the next two highest percentages, 28 % and 17.4 % of E-speakers answered "it goes to infinity" and "(it becomes) infinity" respectively.	Approximately 40 % of K-speakers replied "(it becomes) infinity." However, only K-speakers (20.6 %) responded "it diverges (to infinity)."
	K. VI-c: $\frac{x^2}{(1+x)^2}$	About 26 % of E-speakers answered "it gets closer to 1." Other responses were distributed almost evenly among "it converges to 1," "it is increasing," and "it goes to infinity."	An overwhelming majority of K-speakers (80 % in the interview and 68.3 % in the survey) replied "it converges to 1."
	L. The rate of "no response"	The number of E-speakers who did not answer questions increased considerably from item VI-a to item VI-c.	The number of K-speakers who did not respond to questions stayed in the same rate.
	M. The phenomenon first observed in [F] returned	Many E-speakers spoke about the limit of function in terms of processes.	Korean responses were formulated in a more formal mathematical way with the purpose of finding the limit.
	N. Routine	In the task of finding the limit in a given function with its graph, eight routines were identified according to the aspects of limiting process and the limit that were used.	
	O. VI-a: $\frac{1}{x}$	Dominant routines in both groups were based on the geometric representation	
	P. VI-b and VI-c	The use of geometric representations was still dominant in the E-group.	The use of algebraic reasoning dominated routines in the K-group.
	Q. Visual Mediators	The scanned processes in the E-group may be called *syntactic* mode, as it requires only knowing the written phrases and	Most K-speakers seem to make a transition from *syntactic* mode to a different mode for finding limits in items VI-b and VI-c

		given geometric mediators.	
Item VII: Finding the limit of the sequence of regular polygons inscribed in a circle	R. Limit of regular polygons	Only 28.8 % of E-speakers reported "it becomes the circle."	An overwhelming majority of K-speakers replied "it converges to (or becomes) the circle."
	S. The same as [F] and [M] above	Geometric uses of the context appeared to direct them to think about a *processual* use of the limit of regular polygons	Almost all K-interviewees continued to employ *structural* uses of limit in the same context
	T. Routine	Two routines for finding the limit in item VII were identified in the interviews and the survey	
	U. Use of routine	*Routines* in the E-group were based on *syntactic mode* of regular polygons.	Those of K-speakers were used on the basis of *objectified* mode. In other words, they seem to reify both original written phrases and geometric mediators with the purpose of finding limits.
Item VIII-b: Defining that the number P is the limit of a sequence $a_1, a_2, a_3, \ldots$	V. Interpretation of limit	E-speakers' responses were relatively consistent but not in tune with the canonical interpretation of the word *limit*.	K-speakers' interpretations were more diverse but often consistent with the canonical interpretations
	W. Use of limit	E-speakers provided processual explanations.	K-speakers spoke about the limit P in structural terms.
	X. Idea of limit	This *processual* use in E-group was often characterized by *unreachability* to the limit P.	The K-speakers' ideas of "P is the limit" were dominated by the idea of *reachability*.
	Y. Definition of limit	E-speakers formulated interpretations in their own words.	K-speakers used more formal mathematical formulations.

CHAPTER V

SUMMARY, DISCUSSION AND CONCLUSIONS

This study sought to describe characteristics of university students' non-mathematical and mathematical discourses on *infinity* and *limit* using the communicational analysis framework. The primary motivation for this study was the difference between Korean and English speakers when it comes to the relation between their non-mathematical and mathematical discourses on *infinity* and *limit*. The mathematical words for *infinity* and *limit* in Korean are different from the Korean colloquial words that correspond most closely to the English colloquial words *infinity* and *limit*. It is known that colloquial uses of mathematical words influence students' mathematical discourses featuring these words. It was thus likely that mathematical discourse of E-speakers would be influenced by their colloquial discourse, whereas it was likely there would be no comparable influence of the colloquial discourse in the case of K-speakers. In other words, it was expected that there would be differences between K-speakers' and E-speakers' mathematical discourses that could be ascribed to this difference in colloquial influences.

In order to better understand possibilities about the specific sources of differences in mathematical discourses of the two groups, students' backgrounds which were likely to influence their mathematical discourses on *infinity* and *limit* were also investigated in this study. The research questions were as follows:

- QUESTION 1: What are the relationships between colloquial and mathematical uses of the English words infinity and limit?; What are the relationships between colloquial and mathematical uses of the corresponding Korean words?
- QUESTION 2: What are the similarities and differences between native-English and native-Korean speakers in their previous mathematics education

experiences, their learning habits and curricular experience with the words *infinity* and *limit*?

- QUESTION 3: What are the similarities and differences between non-mathematical and mathematical discourses on *infinity* and *limit* of native-English and native-Korean speakers?

The setting for the study was a calculus class for university students in the U.S. and Korea. Methodology involved surveys and interviews. The results of the study derived from two types of analyses: analysis of response patterns of the survey participants and detailed discourse analysis of interview transcripts. The survey responses were then re-examined to help validate findings from the interviews. A total of 132 English speakers and 126 Korean speakers participated in the survey. Within each linguistically distinct group, twenty representatives were selected from the survey participants for follow-up interviews in pairs. The detailed discourse analyses of the interview transcripts generated preliminary hypotheses in the interviewees' discourse on *infinity* and *limit*. Overall analyses of response patterns involved searching for frequencies and percentages of the survey participants' responses and comparing the proportions of the two groups through chi-square analysis to confirm the emerging hypotheses. Data from both sources were used to elaborate and verify hypotheses. Detailed findings are found in chapter 4. This chapter presents a summary of the detailed findings.

Summary of Findings

Question1: What are the relationships between colloquial and mathematical uses of the English words infinity and limit?; What are the relationships between colloquial and mathematical uses of the corresponding Korean words?

As explained in Chapter 3, the relation between the colloquial and mathematical words in the two languages can be presented as follows:

$$\text{Infinity}_{\text{EColl}} = \text{Infinity}_{\text{EMath}}, \text{ whereas } \text{Infinity}_{\text{KColl}} \neq \text{Infinity}_{\text{KMath}}$$

$$\text{Limit}_{\text{EColl}} = \text{Limit}_{\text{EMath}}, \text{ whereas } \text{Limit}_{\text{KColl}} \neq \text{Limit}_{\text{KMath}}$$

The equality symbol '=' means that the words *sound* the same. $\text{Infinity}_{\text{KColl}}$ and $\text{Limit}_{\text{KColl}}$ signify the colloquial Korean words that can count as rough translations of the colloquial English words $\text{Infinity}_{\text{EColl}}$ and $\text{Limit}_{\text{EColl}}$, respectively. $\text{Infinity}_{\text{KMath}}$ and $\text{Limit}_{\text{KMath}}$ do not have any previous meaning in colloquial Korean – they are words of Chinese origin. Therefore, there is a discontinuity in Korean and a continuity in English between the colloquial and mathematical discourses on *infinity* and *limit*. K-speakers have little experience with the colloquial Korean use of the mathematical terms *infinity* and *limit*, whereas E-speakers do have experience with the colloquial use of the English words *infinity* and *limit*.

Question 2: What are the similarities and differences between native-English and native-Korean speakers in their previous mathematics education experiences, their learning habits and curricular experience with the words *infinity* and *limit*?

The second research question about students' backgrounds is: What are the similarities and differences between native-English and native-Korean speakers in their former mathematics education and curricular experience with the words *infinity* and *limit*?

Similarities between the two groups relative to previous mathematics education experiences, learning habits and curricular experience with infinity and limit

To analyze different levels of students' former education with *infinity* and *limit*, *course* and *content* variables were considered. In the *course* variable, students' recollections about their first experience with *infinity* and *limit* in a mathematics course were analyzed. Even though some E-speakers and K-speakers reported that they first heard the words *infinity* and *limit* before 7th grade, the majority of the participants recalled their first experience with *infinity* and *limit* in a mathematics course between 7th

and 12th grade. According to the results of *t*-tests, there was no significant difference between the E-group and K-group in the range of grades for first experience with *infinity* in a mathematics course. This result may indicate that students in both groups first learned about infinity during similar time periods.

The survey included a list of seven topics related to infinity and limit and asked students to indicate whether they had learned about these topics. A *chi* square statistic was used to test for possible differences in the percentage of students who reported having learned the seven topics. According to the *chi* square analysis, there was no significant difference between the E-group and K-group in their reports about having learned the following six topics: 'infinite process,' comparison of infinite sets,' and 'infinitely small or infinitesimal' in the case of infinity; 'limits to infinity,' 'limit of a function numerically,' 'and limit of a function geometrically' in the case of limit. These results (especially with the comparison of both infinite sets and limit of a function used in the survey) can support comparability of these two groups.

In terms of curricular experience with *infinity* and *limit*, the majority of students in both groups (88.9 % of the K-speakers and 86.4% of the E-speakers) took either a pre-calculus or calculus course in high school. Students in both groups also reported that they listened to teachers' lecture-style presentation in their classrooms for more than half of the lessons. These are important similarities between the E-group and K-group for comparability of these two groups.

In addition to examining students' former mathematics education and curricular experience with the words *infinity* and *limit*, other aspects that were likely to influence their mathematical discourse on infinity and limit also were investigated in the survey.

These aspects are students' self report on their experiences of learning mathematics and attitudes toward mathematics. Both the K-speakers and E-speakers reported that they listened to teachers' lecture-style presentation for more than half of their lessons. Neither group reported small group work occurring very much.

Overall mean scores of both the K-speakers and E-speakers in their attitudes about learning mathematics and the role of mathematics in their lives were higher than a neutral response. These results imply that students in both groups showed positive attitudes about learning mathematics and they also agreed on the need for mathematics in their lives.

Differences between the two groups relative to previous mathematics education experiences, learning habits and curricular experience with infinity and limit

The survey asked about when students were first introduced to the concept of limit and infinity. The mean of the E-speakers' responses about the grade level at which they were first introduced to the concept was significantly higher than that of the K-speakers in the case of *limit*, indicating they were first introduced to limit later than the K-speakers. As for course sequences taken in high school, almost all K-speakers reported having taken Mathematics 10 (for 10^{th} grade), Mathematics I (for 11^{th} grade), and Mathematics II (for 12^{th} grade) course sequence through integrated curricula. In contrast, most E-speakers reported having taken a course sequence in which algebra and geometry are separated.

There was a significant difference between the E-group and K-group in self-reports about having learned the following four topics related to *infinity*: 'infinite decimal fractions,' 'infinite sequence or series,' 'sums of infinite geometric series,' and 'countable and uncountable sets.' The percentage of the K-speakers who reported having learned each of the four topics was significantly higher than that of the E-speakers. In the

case of limit, the percentage of the K-speakers reporting having learned related topics was significantly higher that that of the E-speakers in the following three topics: 'limit of a sequence or series,' 'limit and the definition of continuity,' and 'limit and the definition of derivative.' Interestingly, the percentage of the E-speakers, however, is significantly higher than that of the K-speakers in the topic of 'the ε-δ definition of limit.' These results indicate that the K-speakers seemed to have had generally more opportunities to learn concepts related to *infinity* and *limit* in high school than the E-speakers.

E-speakers reported that they used a variety of textbooks in high school, beyond the representative examples provided in the survey, whereas many K-speakers replied that they studied from one of the seven textbook examples, and in some cases more than two. That is, there was considerable variation in the textbooks that the E-speakers used in high school, in comparision to consistency in the textbooks used by the K-speakers. Due to this complexity, it was clear that analysis of textbooks, as a possible means of illuminating the different discourse experiences of students, would not be a productive endeavor., student representatives were selected and interviewed in terms of their high school textbooks instead of analyzing their textbooks.

There were differences in the two grops in their descriptions of how they studied calculus. During interviews, K-speakers reported that they had systematically planned study methods and periods of rest and followed that plan. In contrast, E-speakers seemed to have planned less and did not follow it too tightly even if they had a plan. Even if these responses were from a very small sample of students, they suggest that K-speakers more carefully organized their studying and they devoted more time to it than E-speakers.

Question 3: What are the similarities and differences between non-mathematical and mathematical discourses on *infinity* and *limit* of native-English and native-Korean speakers?

The third research question about students' discourse is: What are the similarities and differences between native-English and native-Korean speakers in their non-mathematical and mathematical discourses on *infinity* and *limit*?

Similarities between the two groups relative to non-mathematical and mathematical discourses on infinity

a. Discourse on *infinity*

- Non-mathematical uses of *infinite*, *infinity*, and *infinitely*

No systematic similarities were found in the non-mathematical discourse on *infinity* of both groups – Only E-group actively used the noun infinity.

- Mathematical uses of *infinity*

The mathematical discourse on *infinity* of both groups was not yet well developed, in that it was quite different from the canonical mathematical discourse on infinity. In the mathematical discourse on *infinity*, the following common characteristics in *routines* and *endorsed narratives* were observed in both groups.

Set-comparison routines: Many respondents in both groups made non-canonical choices of routines for comparison. Some respondents in both groups compared the sets with respect to the size of corresponding elements rather than with respect to the amount of elements. In addition, there were respondents in both groups who produced two contradicting answers as the result of applying two different routines of comparison.

Endorsed narratives: Many respondents in both groups endorsed the narrative "A proper subset is smaller/has less elements than the whole set". K-speakers were more likely to report that they accepted non-canonical properties of comparison.

b. Discourse on *limit*

- Non-mathematical uses of *limit* and *limited*

Both groups used the colloquial words *limited* and *limit* in both mathematical and non-mathematical contexts. The adjective *limited* was used in the sense of *finite* or *restricted* to describe something that has an upper bound or one that could be larger. The noun *limit* was employed to denote an upper bound – something that does not let a thing or process to get larger.

- Mathematical uses of *limit*

In the mathematical discourse on *limit*, the following common characteristics in *routines* and *endorsed narratives* were observed in both groups.

Limit-finding routines: Many respondents in both groups made non-canonical choices of routines for finding limits by looking at their graph. There were also some respondents in both groups who found limits by assuming that infinity is a number.

Endorsed narratives: Some respondents in both groups endorsed the narrative "limit is an upper boundary". E-speakers were statistically significantly more likely to report the non-canonical property of unreachability.

Differences between the two groups relative to non-mathematical and mathematical discourses

a. Discourse on *infinity*

- Non-mathematical uses of *infinite*, *infinity*, and *infinitely*

E-speakers had a relatively well developed non-mathematical discourse on *infinity*, which seemed to be building on their non-mathematical uses of the words *infinite* and *infinity*. The adjective *infinite* appeared mainly in the phrases "infinite number/amount of...," as a descriptor of sets, signaling these sets' large dimensions (see the property [B]

of *infinity* in Chapter 4). The noun *infinity* appeared mainly in the context of *processes* that do not end, as the target-point which these processes never reach (see the property [C] of *infinity* in Chapter 4).

K-speakers' non-mathematical discourse on *infinity* seemed less developed. They used the adjective $infinite_{KMath}$ in non-mathematical discourse as descriptors of objects known for their large size or of spiritual matters – not of sets or processes (see the property [B] of *infinity* in Chapter 4). They did not use the noun $infinity_{KMath}$ at all. When asked to use the non-mathematical version of the noun, they converted it to the adjective *infinite*. When given the mathematical word for *infinity*, they turned it into adverb *infinitely* and used it in conjunction with many (*infinitely many*), as a descriptor the cardinality of sets (see the property [C] of *infinity* in Chapter 4).

- Mathematical uses of *infinity*

There were several differences between mathematical discourses of the two groups in terms of *use of words*, *routines*, and *endorsed narratives*.

Use of words: There was not much difference between E-speakers' mathematical and non-mathematical use of the words *infinite* and *infinity* – they used the adjective with relation to sets and they used the noun within the same phrases reporting on unending processes: "goes to," "continues to," etc (see the properties [G], [O], and [R] of *infinity* in Chapter 4). In contrast, K-speakers' mathematical discourse on *infinity* was quite different from their non-mathematical discourse. In their mathematical talk $infinite_{KMath}$, which appeared not very frequently, was used as a descriptor of *cardinal numbers*. The word $infinity_{KMath}$ appeared only when explicitly mentioned in the question, and it then was used within the phrase 'goes to infinity', which also appeared in the question. It was

not elaborated. In K-speakers' mathematical discourse, there was almost never any reference to processes. The discourse used mathematical vocabulary and was structural rather than processual (see the property [T] of *infinity* in Chapter 4).

Routines of comparison: One of the findings was that E-speakers' choice of comparison routine was sensitive to the wording of the question and, more specifically, it depended on whether they were asked which set was *bigger* or which had *more* elements (see the property [N] of *infinity* in Chapter 4). K-speakers' choice of routine did not change with wording (see the property [M] of *infinity* in Chapter 4). Another finding was that the majority of E-speakers employed *infinity* in conjunction with an infinite process to compare two infinite sets. K-speakers' talk about $infinity_{KMath}$ in their comparison routine was more formally mathematical (see the property [R] of *infinity* in Chapter 4). The other finding was that E-speakers tended to claim that two infinite sets are always equal. K-speakers relied more strongly than E-speakers on the narrative that infinite sets are incomparable.

Endorsed narratives: E-speakers formulated interpretations about what *infinity* is in their own words. Their responses were relatively consistent and in tune with the canonical interpretation of the word *infinity*. In contrast, K-speakers used more formal mathematical formulations about the concept of *infinity*. Their interpretations were more diverse and often inconsistent with the canonical interpretations. They spoke almost exclusively in structural terms (see the property [S] of *infinity* in Chapter 4).

To sum up, E-speakers seemed to be quite proficient in the non-mathematical and mathematical discourse on *infinity* (e.g, they were able to answer questions using their own formulations, different from that in the question itself). Their mathematical use of

the words *infinite* and *infinity* was not much different from their non-mathematical use. *Routines* of E-speakers were sensitive to the wording of the question. The majority of E-speakers employed *infinity* in conjunction with an infinite process to compare two infinite sets. Their *endorsed narratives* were relatively consistent with the canonical interpretation of the word *infinity*.

In contrast, K-speakers' discourse on *infinity* was more formally mathematical and structural, but they were not yet very proficient in this discourse – in their responses, they tended to use phrases that appeared in the question. The mathematical discourse on *infinity* of K-speakers was quite different from their non-mathematical discourse on *infinity*. In the use of word *infinity*, there were few references to processes. Their *routines* did not change with wording. Comparison *routines* of K-speakers were more formally mathematical. They endorsed narratives exclusively in structural terms.

b. Discourse on *limit*

- Non-mathematical uses of *limit* and *limited*

K-speakers' non-mathematical discourse in $limited_{KMath}$ and $limit_{KMath}$ seemed less developed than that of E-speakers. They used the adjective $limited_{KMath}$ in non-mathematical discourse only in the sense of *extreme* which was not observed in the use of $limited_{KColl}$. They used the noun $limit_{KMath}$ mainly in expressions similar to a textbook formulation (see the property [D] of *limit* in Chapter 4). Generally, K-speakers seemed to be not very proficient in creating sentences with $limited_{KMath}$ and $limit_{KMath}$.

- Mathematical uses of *limit*

The mathematical discourse on *limit* of E-speakers was not yet well developed. In contrast, K-speakers appeared to provide more developed canonical mathematical

discourse on *limit* (see the property [V] of *limit* in Chapter 4). There were several differences between the mathematical discourses of the two groups in terms of *use of words*, *routines*, *endorsed narratives*, and *visual mediators*.

Use of words: There was little difference between E-speakers' mathematical and colloquial use of the words *limited* and *limit* – they used the adjective in the sense of finite and they used the noun to denote an upper bound. E-speakers' use was processual and they often spoke about limit as *unreachable*. K-speakers, in contrast, spoke about *limit* in structural terms. This structural use of *limit* was often related to the idea of limit as a number (see the properties [F], [M], [S], and [W] of *limit* in Chapter 4). In general, they used more formal mathematical formulations.

Limit-finding routines: E-speakers' choice of limit-finding routine was based on a processual use of *limit*. K-speakers' choice of routine was based on a structural use of limit by considering the concept of limit as a number. Routines in the E-group were employed on the basis of *syntactic mode* of context, as it requires only knowing the written phrases and given geometric mediators. Routines of K-speakers were used on the basis of *objectified mode* (see the property [U] of limit). Finally, the majority of E-speakers used geometric representations to find the limits of more complicated functions, whereas K-speakers employed algebraic representations with algebraic reasoning in their routines (see the property of [P] of limit).

Endorsed narratives: E-speakers formulated the idea of *limit* in their own words. Their responses were relatively consistent but not consistent with the canonical interpretation of the word *limit*. K-speakers interpreted the idea of $limit_{KMath}$ as an object by utilizing formal language as well as formal mathematical representation. Their interpretations

were more diverse but often consistent with the canonical interpretations. They used more formal mathematical formulations about the idea of *limit* (see the property [V] of *limit* in Chapter 4).

Visual mediators: E-speakers used a syntactic mode as they scanned processes for finding the limit of the sequence of regular polygons inscribed in a circle. K-speakers seemed to make a transition from syntactic mode to a different mode for finding the limit (see the property [U] of *limit* in Chapter 4).

To sum up, E-speakers did not develop canonical mathematical discourse on *limit*. They provided processual explanations of the noun *limit* and seemed to be quite proficient in this discourse (e.g, they formulated interpretations in their own words.). They often spoke about limit as unreachable. There was little difference between E-speakers' mathematical and colloquial uses of the words *limited* and *limit* as a process. *Routines* of E-speakers in limit-finding tasks were based on a processual use of limit. These *routines* were often employed on the basis of geometric representations. E-speakers endorsed narratives in their own words. Their *endorsed narratives* were not consistent with the canonical interpretation of the word *limit*. *Visual mediators* of E-speakers in finding the limit of the sequence of regular polygons were used syntactically.

In contrast, K-speakers used $limit_{KMath}$ in non-mathematical discourse only in the sense of *extremity* which was not observed in the E-group. In the mathematical discourse on *limit*, they spoke in expressions similar to a textbook formulation. They appeared to provide more developed canonical mathematical discourse on *limit* than E-speakers. K-speakers' choice of limit-finding routine was based on a structural use of limit. These *routines* were often employed with algebraic representations. Their *endorsed narratives*

were more diverse but often consistent with the canonical interpretations. *Visual mediators* of K-speakers in finding the limit of the sequence of regular polygons were changed from syntactic mode to a different mode.

Discussion of Findings

Reasons for Differences in Mathematical Discourse on *Infinity* and *Limit*

There is one main difference between the E-group and K-group in their discourses on *infinity* and *limit*. The main difference is in their processual and structural uses of *infinity* and *limit* in the E-group and K-group respectively. In the E-group, the use of *infinity* and *limit* as process seems to be related to the processual use of the words *infinity* and *limit* in mathematical discourse. Due to the processual nature of the colloquial uses of *infinity* and *limit* in English, the mathematical English use also seems be processual. This may be evidence for a connection between colloquial and mathematical discourses on *infinity* and *limit*. It seems that E-speakers build on their everyday experience with colloquial discourse in developing their mathematical discourse on *infinity* and *limit*.

In contrast, K-speakers' non-mathematical discourse in the Korean mathematical words *infinity* and *limit* is rare and less developed in their own formulations. Their uses appear to be formulated from textbook treatments? or definition. In the mathematical discourse on *infinity* and *limit*, their language is more formally mathematical and structural. Their mathematical discourse on *infinity* and *limit* is more consistent with, and possibly developed from, the structural discourse of mathematical textbooks, perhaps due to lack of proficiency or experience with these words in their non-mathematical discourse. As a matter of fact, the nouns *infinity* and *limit* in their mathematical versions are rarely used as themselves in colloquial Korean. In addition, the more rigorous structure of their

definitions may be additional evidence that these students were introduced to the discourse on *infinity* and *limit* through mathematical definitions taught at school rather than through non-mathematical use. There is therefore a clear disconnection between K-speakers' mathematical discourse on *infinity* and *limit* and their non-mathematical discourse. Their discourse on *infinity* and *limit* is consistent with the structural discourse of mathematical textbooks, and does not appear to draw on their previous everyday experience with discourse on $infinity_{KColl}$ and $limit_{KColl}$.

K-speakers exhibited the tendency to use mathematical-objectified discourse on *infinity* despite the fact that they did not have a good mastery of this discourse. One possible reason is the lack of alternatives. In other words, unlike E-speakers, K-speakers did not demonstrate a well developed non-mathematical discourse on *infinity* and thus kept using words from the mathematical discourse on *infinity* by employing phrases that appeared in the question. Another possible reason is they might have a stronger tendency than the E-speakers to remain in the discourse introduced by their interlocutor (interviewer). The results of item I (about creating a sentence) seem to confirm this. The question in item I framed the discourse as being about language or literature (note the request to *create sentences*) and the K-speakers' answers were mainly non-mathematical – much more so than those of E-speakers. The same may explain the results of II-a, which was about fingers and toes, where the routines of comparison chosen by K-speakers were often non-mathematical.

There are other reasons to speculate on a connection between E-speakers' mathematical and non-mathematical discourses on *infinity* and *limit* and a lack of connection in the K-speakers'. The change in wording for comparing two infinite sets

strongly influenced the way E-speakers implemented the comparison of sets, whereas it did not influence K-speakers' choices. This difference reinforces the claim about a greater disconnection between the non-mathematical and mathematical discourse in the K-group. It seems that the K-speakers, rather than gradually proceeding from more processual non-mathematical discourse on infinite processes to a structural discourse on infinity, have been directly exposed in their mathematics classes to the structural formal mathematical version. E-speakers' sensitivity to the wording of the question may be evidence for a connection between colloquial and mathematical discourses on *infinity* and *limit*.

In limit-finding tasks, the use of geometric representations was dominant in the E-group, compared to the use of algebraic representations in the K-group. The reasons for this difference are likely to be multifaceted. They may include discourse–related differences in the non-mathematical use of the word *limit*. Another may be differences in educational experiences prior to studying calculus. And, it is possible that the curricular emphasis on asymptotic behavior in the U.S. curriculum and the algebraic approach in Korea curriculum lead to a disposition to the use of a geometric approach that is more prominent in E-speakers. Another possible reason is the processual uses of E-speakers in non-mathematical discourse on *limit* and the structural uses of K-speakers in mathematical discourse. Additional research would be needed to understand this more fully.

Students' *routines* and *endorsed narratives*, as well as their *visual mediators* seem to be grounded in the characteristics of their *word use* (i.e., processual and structural uses of the words *infinity* and *limit*). For instance, E-speakers' canonical interpretation of the

word *infinity* is consistent with processual uses of infinity. Their *routines* were employed on the basis of a syntactic mode of context due to a processual use of limit. In contrast, *routines* of K-speakers were used on the basis of objectified mode with a structural use of limit. K-speakers often reified limit-finding tasks with the idea of limit as a number.

E-speakers' responses were consistent with the canonical interpretation of *infinity* with processual explanations, and K-speakers' interpretations were more consistent with the canonical interpretation of *limit*. The interplay between non-mathematical and mathematical discourses on *infinity* can be helpful for E-speakers to develop their canonical interpretations of *infinity*. However, the connection between non-mathematical and mathematical discourses on *limit* can be problematic because the non-mathematical and mathematical uses of the word *limit* are different and sometimes conflicting (i.e., The processual use was often characterized by unreachability). K-speakers seemed to differentiate mathematical discourse from non-mathematical discourse on both *infinity* and *limit*. This may help them develop the rigorous definitions and complicated structures that underlie the mathematical concept of *limit* without conflicts. However, the difference between non-mathematical and mathematical discourses on *infinity* may make it difficult for K-speakers to reify and conceive of infinity as a process, unlike E-speakers.

A salient property in routines is that many respondents in both the E-group and the K-group used different *routines* for the different cases. This may be explained on the basis of the fact that learning (*transfer*) does not happen automatically, but takes time and practice. One may conjecture that students use different routines in different mathematical contexts. In other words, *routines* seem to be highly context-dependent.

The results of this study can also provide insights about the relationship between students' self-concept in mathematics and their efforts in mathematics learning. Higher self-concept in mathematics may be related to more effort and better performance in some students, but may not in others. The E-speakers reported that they were more confident than the K-speakers about their capability of learning mathematics. This finding was consistent with previous studies (Chen & Stevenson, 1995; Stevenson, Chen, & Lee, 1993) that Chinese students were less confident than U.S. students about their mathematics learning. When students are working on mathematics homework, the Korean students' lower confidence in mathematics may cause them to devote more time and effort to it, as other studies (e.g., Fuligni & Stevenson, 1995) have suggested. In addition, the result from the follow-up questionnaire suggests that for K-speakers learning was more central because they reported systematically devoting more thought and more time to it compared with E-speakers.

In addition to the continuity in English and the discontinuity in Korean between the non-mathematical and mathematical discourses on *infinity* and *limit*, differences in the discourse opportunities between U.S. and Korean classrooms may be a possible reason which is likely to account for the differences in students' mathematical discourse on these words between the E-group and K-group. Because the students in this study reported that their calculus instructors used lecture-style presentation for more than half of their lessons, one may conjecture that due to the processual nature of E-teachers' mathematical talk about *infinity* and *limit* in their classrooms, E-students' mathematical discourse on these words may be processual. In contrast, K-students' mathematical discourse on the topics may be formally mathematical and structural perhaps because of K-teachers'

mathematical and structural language in their classrooms. An investigation of differences in the nature of classroom discourse between the two groups may provide more thorough information about reasons for differences in their mathematical discourse on *infinity* and *limit*.

The claim about the differences in the mathematics discourse on *infinity* and *limit* as caused by differences in the colloquial uses of these words, however, is supported with process-type causality argument rather than regularity-type causality. In other words, rather than trying to look for a statistical evidence for the co-appearance of the phenomena that are claimed to be in causal relation (similar uses of words infinity in limit in non-mathematical and mathematical discourses), I argued for the causal dependence by explaining *how* the non-mathematical discourse on *infinity* and *limit* leads to the mathematical discourse on these words. The results of this study confirmed the analytic argument: some English colloquial uses of *infinity* and *limit* reappeared in mathematical uses. In other words, E-speakers continued to use the words in the way they did so far. The absence of this type of use in Korean mathematical discourse on *infinity* and *limit* was also in accord with the claim about intra-discursive mechanisms: The unwanted types of use of the words *infinity* and *limit* which, in the English mathematical discourse, stemmed from the colloquial uses of these words did not appear in the K-speakers mathematical discourse. Considering all this, additional information on the classroom discourse in U.S. and Korea, although important, is of only secondary significance when it comes to substantiating the claim about the impact of colloquial discourses on their mathematical counterparts. The classroom discourse can strengthen or weaken this impact, but it is not its primary source.

Generalizability

The findings of this study may be generalizable to native-English speakers in the U.S. and native-Korean speakers in Korea who start to learn calculus in a university setting. However, generalizations from the present findings should be made with appropriate caution for several reasons. First, different geographical areas can be related to differences in students' former education and curricular experience with *infinity* and *limit* as important factors in mathematical discourse. In addition, characteristics in mathematical discourse can be determined by various other factors such as socioeconomic status (SES) and university contexts. Since SES was not taken into account, and university differences were not considered in this study, the findings in this study did not reflect this potential influence of additional variables.

Limitations of the Study

There are several limitations of this study. First, the 132 U.S. and 126 Korean survey participants were not randomly selected out of the total population in the research sites. They were recruited from several calculus classes on a voluntary basis and thus area sample of convenience. Thus the findings from this work cannot be generalized beyond this group of students, although they provide important focused hypotheses for subsequent work. Second, there are some limitations in the survey and interview instruments. The chi-square results in the survey questions should be interpreted with caution due to the possible ambiguous nature of some questions to the students. For instance, the interpretation of "extra lessons" in question 19[3] could be interpreted

[3] Question 19: During this school year, how often have you had extra lessons or tutoring in mathematics that is not part of your regular class?

differently, especially to the E-speakers. The meaning of "every day" in question 20[4] was also problematic because of the fact that the U.S. and Korean classes met three days and twice per week respectively.

Third, follow-up interviews, with 20 % of the survey participants, were conducted in pairs. A possible limitation of this approach is that different pairings such as friendship pairings and acquaintance pairings may cause different levels of performance in mathematical discourse (e.g., Kutnick & Kington, 2005). Variations in interviewees' friendships were not controlled in this study, even though one or two pairs were randomly selected within each setting. It may be useful to include the relationship between different pairings and their mathematical performances in a further study.

A fourth limitation is that the timing of data collection between in the U.S. and in Korea was different. The Korean data of this study were collected in May 2007, the second half of the first semester for university freshmen in Korea, compared to the U.S. data in September or October 2007, the first half of the first semester for university freshmen in the United States. Because of the differences in timing, it could be expected that K-speakers had more opportunities to learn about *limit* in their calculus class.

Finally, the K-speakers were exposed to English through the use of an English textbook in their calculus classrooms in college settings, even though the college lecturing was in Korean. However, the impact of the English calculus textbook may not be significant in terms of the effects of English on the K-speakers' mathematics learning because mathematics in precollege settings in Korea is taught in Korean and its textbooks are in Korean.

[4] How often does your instructor give you homework in mathematics?

Further Questions

One question that this study has raised concerns about is the generalizability of this study. Another question is about additional important variables which can affect mathematical discourse, such as socioeconomic status (SES) and social and organizational contexts. In the narrow sense of generalizability, the present study can be expanded by collecting data from other groups of students in other U.S. and Korean universities. Additional comparisons of relationships between non-mathematical and mathematical discourses on *infinity* and *limit* could provide more comprehensive information about differences and similarities.

In the broader sense of generalizability, the findings in the present study may not be generalizable to students in other East Asian countries, even though these East Asian languages share many common characteristics with Korean in terms of how mathematical words are represented and used. Further research could include students from different East Asian countries such as China, Japan, and Singapore, and could compare the results from the findings in those studies with those of the present study. In addition, groups of native speakers in East Asian countries could be compared with those of other native speakers in English-speaking countries. These comparisons could provide more thorough information about the phenomena caused by both continuity in English and discontinuity in East Asian languages between non-mathematical and mathematical discourses on *infinity* and *limit*.

As for other important variables related to mathematical discourse, many further questions are still interesting. For example, given the belief that social and organizational contexts are relevant to mathematical discourse, one might ask, do students from different

university contexts differ in their non-mathematical and mathematical discourses? Examining how students' SES is related to their mathematics performance, one might ask, do students from different SES levels differ in their non-mathematical and mathematical discourses? In addition, one might ask, is there a gender difference in non-mathematical and mathematical discourses?

Conclusions

Theoretical Conclusions

For the last several decades, most previous research on how students learn mathematics has been based on cognitive psychological perspectives, called cognitive theories. These cognitive theories are implicitly grounded in absolutism, that is, the assumptions they hold about the nature of mathematics are context-independent. In cognitive theories influenced by the work of Piaget, mathematical learning has also been considered as a process of active cognitive acquisition. The *acquisitionists* who rely on the acquisition of mathematical concepts as mathematics learning have explained that the idea of cognitive processes is based on uniform and orderly forms of rule-following (Harré & Gillett, 1995).

Static constructs in mathematics education such as concept images and concept structures are useful descriptions in mathematical discourse. For a better understanding of *how* and *why* students think mathematical concepts, however, there needs to be a new emphasis on dynamic variables in learning mathematics. The findings of this study add two new perspectives to theoretical perspectives on learning mathematics. First, mathematical thinking is words-based thinking due to the sensitivity of *word use* in such thinking. We notice that different students could decipher different meanings and thus

engage in different thinking, even with respect to the same word, *infinity*. Second, mathematical thinking is also a continuously dynamic process, rather than a one-way directional internalization from others to the individual learner because of the contextual dependence of *routines* as interwoven processes with *word use*. To become more skillful and articulate in a certain mathematical discourse, students should continuously and dynamically interpret what others who are more discursively advanced in mathematics say, word by word, with routines due to dynamic processes of *routines* with *word use*. If students misunderstand words or misuse routines in these communicative activities, their learning could be inapplicable and even wrong in the mode of learning.

Thus, it is warranted to move beyond the cognitive perspective and to examine continuous changes in the learning context through the study of *word use* and of *routines*. By not examining *word use* and *routines* in mathematical discourse, researchers may have a distorted view. To reveal situated learning difficulties, researchers need to investigate *thinking in context* rather than *thought* because of the sensitivity of *word use* and dynamic processes of *routines* based on *word use*. *Thinking* is an on-going process, whereas *thought* is the end product of thinking. Research methods that are sensitive to *word use* and dynamic processes of *routines* (by using *visual mediators* and *endorsed narratives*) need to be employed in order to explore questions about both *how* students know and *why*.

One difficulty in the discursive approach to student learning is the requirement for a common systematic and analytical framework. However, with a common framework of analyzing discourse that is well organized, discourse analysis is a theory-based research method including contextual sensitivity as well as addressing continuing processes in

learning mathematics. To improve mathematical learning in practice requires that research aids us not only in deepening our fundamental understanding, but also by indicating pragmatic processes for resolving student learning difficulties. Discourse analysis as a multi-lateral approach is a promising method to reveal the complex mechanisms of mathematical learning in contexts because it can be sensitive to *word use* as well as to the dynamic processes of *routines* as essential components in mathematical discourse.

Practical Implications

The results of the current study offer several implications for teaching and learning about *infinity* and *limit*. The findings indicate that how to use words may be the most fundamental and significant factor in determining the content students try to learn.

Word use

This study suggests that learning is highly words-dependent. Even though a teacher provides the same word several times on the same concept in class, students could grasp its unintended meaning because they decipher it from their own point of view. Often their interpretation might differ from what the teacher intended to explain. Thus, explicit dynamic dialogues between the teacher and students seem to be a promising pedagogical option to improve student learning difficulties. The most basic and critical perspective on teaching mathematics is to know the words which students use to communicate the meanings of mathematical concepts. Students' learning difficulties may come from packed words which explain mathematical concepts and structures. Reifying the dense words and their complex structures could assist students to more easily access

mathematical discourse from their own point of view. It seems that mathematical learning starts with mathematical words and continues to develop through them.

Using problem solving task for developing proficient word use and endorsed narratives

Problem solving in calculus should continue to address how to use words, but it should also attempt to ascertain students' *endorsed narratives*. In order to assess students' development in *word use* and *endorsed narratives*, instructors may need to use alternative assessment techniques such as oral exams and short-answer questions that ask students to discuss their understanding of mathematical words. Problem solving itself may not improve students' *word use* and *endorsed narratives* without explicitly mentioning the relations between them. For instance, discussing why the limit of the sequence 1,1,1, ... is one might help students shift their word use of *limit* from unreachability to reachability. With explicit clarifications of the relations between problem solving and *word use* (including *endorsed narratives*), each problem solving episode can be used to extend students' *word use* and *endorsed narratives* from non-canonical to canonical interpretations.

Making routines more precise

The importance of routines in promoting mathematical learning is established in the mathematics education literature. In addition to the emphasis on routines in mathematical discourse, it is desirable to make routines more precise. In other words, not only how to implement routines, but when to use those routines need to be explicitly expressed. The finding of this study – routines are highly context-dependent – supports such recommendations. In addition, differences between routines should be emphasized in terms of *when* and *how* to apply them for student understanding. Decompressing routines

to when and how to apply them step-by-step and differences between the routines would assist students' access to mathematical discourse.

Thought and language are inseparable in the process of human developmental pathways (Vygotsky, 1986). Mathematical learning involves thinking with and through everyday language, generalizing everyday concepts associated with mathematical words, and abstracting those concepts with schooling. According to the results of the current study, non-mathematical discourse seems to have an impact on mathematical discourse due to certain clear differences between the mathematical discourses on *infinity* and *limit* of the E-speakers and those of the K-speakers. These differences can be ascribed to the fact that only in English do the mathematical words *infinity* and *limit* appear in non-mathematical discourse.

Although some researchers (Cornu, 1992; Davis & Vinner, 1986) addressed the issue of everyday language conceptions on limit as a cognitive obstacle, they did not deal with how those conceptions were related to mathematical discourse on limit. Students' non-mathematical discourse may have an impact not only on the students' later use of the mathematical keywords, but also on other aspects of their mathematical discourse, such as endorsed narratives, routines, and visual mediators – a fact that should be kept in mind in both the research on advanced mathematical concepts and teaching advanced mathematics. I hope the practical and theoretical conclusions presented here will influence research in advanced mathematical thinking and improve students' learning of the notions of *infinity* and *limit*.

APPENDIX A

THE PILOT STUDY QUESTIONNAIRE

The Pilot Study Questionnaire

I. Create a sentence with the following word (term).

1. Small
2. Large
3. 158
4. –2
5. Triangle
6. Triangular
7. Limited
8. Limit
9. Diagonal
10. Infinite
11. Infinity

II. Say the same thing without using the underlined word.

12. Some arrow points have **triangular** forms.
13. Eyeglasses are for people with **limited** eyesight.
14. There are **infinitely** too many lawyers.
15. He has **infinite** potential.
16. My love for you is **infinite**.
17. The **limit** of $\frac{1}{x}$ is 0 as x approaches **infinity**.
18. In box A, there are **more** matches than in box B.

III. Which is a greater amount and how do you know?

19. A: Your fingers B: Your toes
20. A: Odd numbers B: Even numbers
21. A: Grains of sand in the world B: Size of the sky
22. A: Odd numbers B: Integers

IV. $\frac{1}{4} = 0.25, \frac{2}{8} = 0.25, \frac{3}{12} = 0.25, \ldots$

How many such equalities can you write?

V. What do you think will happen later in this table? How do you know?

x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
0.01	0.099990
0.005	0.099995
0.001	0.099999

VI. What is the limit of the following when x goes to infinity?

23. $\frac{1}{x}$

24. $\frac{x^2}{1+x}$

25. $\frac{x^2}{(1+x)^2}$

VII. Read aloud

26. $\lim_{x\to\infty} \frac{x^2+3x}{2x^2} = \frac{1}{2}$. Explain what it says.

VIII. What is limit?
What is infinity?

APPENDIX B

OTHERS

- Overview of Education Systems in the United States and Korea
- Development of Korean Words
- Representative Analysis Examples (in terms of *word use*, *endorsed narratives*, *visual mediators*, and *routines*)

Overview of Education Systems in the United States and Korea

Korea has a centralized educational system. Compulsory education in Korea begins with elementary education, when children are six years old. Elementary education has a duration of six years. Secondary education is divided into two stages: middle and high school education. Middle school lasts three years. High school education includes two separate programs: vocational and general education. Vocational education lasts three years and leads students to have a certificate for the labor market. General education also lasts three years and leads to the university. Postsecondary education in Korea includes both vocational and academic higher education. Vocational higher education usually lasts two years and results in an associate's degree. Academic higher education includes the four university programs that lead to a bachelor's, a master's, a professional degree in medicine, and a doctoral degree respectively. In all universities, semester system is used during the academic year. Spring semester, which is the first semester of the academic year, goes from March to June, whereas fall semester period goes from September to December.

Mathematics curricula in Korea are controlled by the Ministry of Education. The 7th School Curriculum is the latest published document (Korea Ministry of Education, 1997). According to the document, all students from grade 1 to 10 are required to take mathematics. From grade 11, students take mathematics elective courses. There are six elective courses: Mathematics I, Mathematics II, Applied Mathematics, Discrete mathematics, Calculus, and Probability and Statistics. Students who are going to study social science in the university are required to take only Mathematics I. In contrast, students who want to study science in the university take both Mathematics I and

Mathematics II as requirements. They are also required to take one of the last four elective courses.

The United States has a decentralized educational system. In other words, education is primarily a State and local responsibility (U.S. Department of Education, 2006). Primary education begins at age six in elementary schools and lasts for five or six years. Secondary education is divided into two levels. Lower secondary education has three or four years and is offered in grades six or seven to nine in middle school or junior high schools. Upper secondary education lasts three or four years and includes grades nine or ten to twelve in high schools. Postsecondary education in the United States is offered in community colleges and includes vocational certification programs. Vocational higher education is typically about two years in length and results in an associate's degree. Academic higher education includes four university programs that lead to the award of a bachelor's degree, a master's degree, a professional degree in fields such as medicine and law, and a Ph. D. Some universities in the U.S. are in a semester system, whereas other universities are in a quarter system. In a semester system, there are fall and spring semesters implemented as the first and second periods respectively. In a quarter system, there are four academic periods during the academic year.

Mathematics curricula in the U.S. are each state's responsibility. Thus, the standards and expectations to guide mathematics curriculum in Michigan have been developed by the Michigan Department of Education. The Michigan Department of Education's Office of School Improvement led the developments of both the K-8 Mathematics Grade Level Content Expectations (GLCE) in 2004 and High School Content Expectations (HSCE) in 2006. The HSCE extends the GLCE as appropriate for grades 9-12. According to the

GLCE, all students from grade 1 to 8 are required to take mathematics. Based on Michigan's HSCE, all high school students take four courses as requirements out of five choices. The three courses (Algebra I, Algebra II, and Geometry) are required and both Pre-calculus and Statistics and Probability are optional. The two different educational systems in the United States and Korea are summarized in Table 1.

Table 1: Overview of educational systems in the United States and Korea

	U. S.	Korea
System	Decentralized	Centralized
Compulsory education	Elementary, middle school, high school educations	Elementary, middle school, high school educations
Postsecondary Education	Vocational and academic higher educations	Vocational and academic higher educations
Mathematics curriculum	For grades 1 – 8, mathematics as a requirement. For grades 9 – 12, five elective courses: Algebra I, Algebra II, Geometry, Pre-calculus, and Statistics and Probability	For grades 1 – 10, mathematics as a requirement, For grades 11 and 12, six elective courses: Mathematics I, Mathematics II, Applied Mathematics, Discrete Mathematics, Calculus, and Probability and Statistics.

Development of Korean Words

Korean has borrowed a huge number of Chinese characters. The Korean lexicon consists mainly of pure Korean words and Korean words built on Chinese characters: about 30 percent native Korean words and 65 percent Chinese character words (Sohn, 2006). Those borrowed words and Chinese characters have become integral parts of the Korean language. Ancient Koreans learned both the Chinese characters and language. However, they also explored ways of recording their native language with the Chinese characters. In the middle of the 15th century, *Hunminjungeum* (*Hangŭl* - the Korean alphabet) was invented in order to bring literacy to the common people who were not intellectually elite and proficient in Literary Chinese (Lee & Ramsey, 2000). Until this time, there was no common way to read Chinese characters.

From this time, many words in Korean were constructed as terms parallel to the words conveyed by Chinese characters and numerous words have also been created by Koreans represented with Chinese characters. Chinese characters are usually associated with the meanings of the words in Korean. Pronunciations and writing systems of borrowed Chinese-character words in Korean have evolved independently in Korea. Nowadays, use of Hangŭl spellings has increased in newspapers and even scholarly books, and the use of Chinese characters is considerably limited (Sohn, 1999). To show meanings and etymologies of words more clearly, however, some professional writings, such as in law and in Korean history, continue to be written in Korean mixed with Chinese characters.

As Korean language, culture, and society were influenced by Chinese language and culture in the past, they have currently been influenced by their American counterparts

since 1945. Most borrowed words after 1945 are words from English with American socio-cultural concepts. For instance, over ninety percent of some 20,000 borrowed words currently used in Korean are from English, such as coffee, café, and hotel (Sohn, 2006).

Most mathematical terminologies in Korean (e.g., infinite, infinity, limit, element, and set) were borrowed directly from the Chinese characters, but in school mathematics they are written in native Korean spellings without explaining the words with Chinese characters. For example, the word 무한 in Korean terminology (without using the Chinese character of 無限 which means boundless or boundlessness) is written and read as *mu-han* in Korean pronunciations in a 7th grade mathematics textbook. According to a Korean dictionary, there are two Korean mathematical words for *infinity*: 무한 (*mu-han*) and 무한대 (*mu-han-dae*). The first Chinese character (無 - *mu*) of the two mathematical words means none and the meaning of the second character (限 - *han)* is bound and the last character (大-*dae*) means bigness. Thus, the word *mu-han* means "boundless or boundlessness," whereas the meaning of the word *mu-han-dae*, used only in mathematics, is "boundless greatness."

A Korean word comprises one or more syllables. Most Korean syllables borrowed from Chinese characters are used colloquially and mathematically. Mathematical and colloquial uses of Korean words are determined by cultural and historical uses of the words, rather than their syllables or characters. The existence of sharing the same syllable or character between two words does not mean necessarily their colloquial uses or mathematical uses. Whether each syllable is used as part of either a colloquial word or a mathematical word is decided on the basis of whether the Korean word (including the

syllable) is either a colloquial term or a mathematical term. Korean colloquial words for infinity and limit are often used in everyday language rather than the mathematical words of infinity and limit. For instance, 무수히 (*mu-su-hee*, countlessly) or 한없이 (*han-up-see*, endlessly) are used for "infinitely" in colloquial Korean, rather than the mathematical word 무한히 (*mu-han-hee*, infinitely). Instead of the mathematical word 극한 (*gŭk-han*) for limit, the Korean colloquial words 제한 (*Jae-han,* limitation) and 경계 (*kyung-gae*, boundary) are often employed. The Korean mathematical words for *infinity* and *limit* are summarized in Table 2.

Table 2: Summary of the Korean mathematical words for infinity and limit

English	Korean			
	Words	Pronunciation	Chinese characters	Meaning
Infinity	무한	*mu-han*	無限	Boundless or boundlessness
	무한대	*mu-han-dae*	無限大	Boundless greatness
Limit	극한	*gŭk-han*	極限	Utmost boundary
	극한값	*gŭk-han-gab*	極限값	A value to which a function approaches

Representative Analysis Examples

In the first example of the pilot study, students were asked to define limit. The definitions of *limit* given by the students are presented in Table 3.

Table 3: The summary of definitions of limit

Students		What is limit?
American	A_5	**[1]** Limit can go on and will stop at some point.
	A_7	**[2]** The limit is the value of a number
	A_{10}	**[3]** Limit is something that numbers can't go past. They have to stop at a certain point.
	A_U	**[4]** As not in mathematics, limit is the absolute ending of something like no more. In a mathematical sense, it would always be the ending of...like an answer, but sense... numbers are infinite. You can get to the answer but never reach the limit because there's infinite numbers...kind of opposite in a way because there isn't... there is no limit in infinity.
Korean	K_4	**[5]** Limit means like the *maximum.*
	K_7	**[6]** Limit is the furthest something to go... like where it stops...like boundary you cannot go on...I think infinity is the opposite of the limit.
	K_{10}	**[7]** Limit is a certain number that you can't reach but you get very close to.
	K_U	**[8]** Limit is a special number...for instance if I say limit four, then there is one, two, three, and four. Not over four.

1) Word use

Objectification: There is a considerable difference with age in ways of defining the word *limit*. In the group of the younger participants in both ethnic groups, the word *limit* was used without explicitly relating it to numbers. In contrast, the older participants specified *limit* with numbers. Evidence of such a use (as an objectification) can be seen only in the utterances of the 10^{th} graders and university students. In all such cases the entity was presented as a particular number or related to numbers ("something that numbers can't go past" [3], "the answer" [4], "a certain number" [7], and "a special number" [8]; the status of the utterance [2] is unwarranted, in the present context).

2) Endorsed narratives

The students' definitions of *limit* revealed the following endorsed narrative.

Limit is an upper boundary: In this endorsement, *limit* is used to describe something that limits the process from above. Some of the students held this view, as evidenced by expressions such as: "limit can go on and will stop at some point" [1], "numbers can't go past" [3], "you ... never reach the limit" [4, 7], "limit means like the maximum" [5], the "limit is the furthest something to go" [6], and "one, two, three, and four" [8].

In the second example, students were asked to find the limit of $\frac{1}{x}$ when x goes to infinity. The summaries of students' responses are presented in Table 4.

Table 4: Summary of responses about the limit of $\frac{1}{x}$

Students		**VI. (a) What is the limit of $\frac{1}{x}$ when x goes to infinity?**
American	A_5	**[9]** This one is zero...because last time we looked at it. One (problem) equals zero...x equals...infinity approaches... it's like zero.
	A_7	**[10]** I don't know what the limit is... That's gonna be one over infinity. When x goes on forever, one over forever.
	A_{10}	**[11]** The limit is always one over infinity...infinity does have no limit...something keep going on and on...I am not sure what the limit is.
	A_U	**[12]** One (problem) would be zero because the bigger x gets...it just be smaller and smaller decimal. It's a sense of going infinity, there is no ending which is gonna be very, very tiny.
Korean	K_4	**[13]** I don't get what it means by when x goes to infinity...
	K_7	**[14]** This would become zero...because the number one, you know one...this will be close thirty...It won't really be anything.
	K_{10}	**[15]** It will get close to zero...because x is two...point five...x...ten...it will get to keep smaller and...
	K_U	**[16]** Zero or one...I don't remember exactly why. But I know it's gonna be one or zero because I learned it. One over x goes to infinity...then I can say like...then I can think like one over infinity...if I divided one...one over one...but one over two is point five...that meaning is zero because infinity means too large number.

3) Visual mediators

For the purpose of this analysis, I differentiated between three visual mediators from my observations. The words *infinity* and *limit* in the task can be *syntactic* mediators used for the sake of scanning mathematical contexts. The second type is *concrete* mediators which students used in order to perform operations related to the task of finding the limit. The third type of mediators is *objectified*, which are used to reify those operations with the purpose of finding limits. As explained below, these mediators can be employed in different ways within the process of calculating the same limit. The American 7^{th} and 10^{th} graders scanned the word *limit* in a defined way. To calculate the limit of $\frac{1}{x}$, the word *infinity* was scanned as a process ("forever" [10], "keep going on and on" [11]) and the variable x in each function was replaced by *infinity*. In order to give an answer, they (A_7 and A_{10}) used algebraic expressions as mediators ("one over infinity" [11], [12]). This scanned process may be called *syntactic* mode, as it requires only knowing the words *infinity* and *limit* and algebraic expressions.

In contrast, the word *limit* was used in different ways in the utterances of the American undergraduate and the Korean students (K_7, K_{10}, and K_U). They (A_U, K_7, K_{10}, and K_U) manipulated concrete input values to calculate the limit of each function. Their first attempt at scanning the context of 'x goes to infinity' involved substituting a small number for x, performing the required operations ("you know one" [14], "two...point five" [15], "one over one" [16], A_U showed "the bigger x gets" [12] as the process of substituting a small number for x in the next problem) and then increasing the input values. In order to find the limit, they looked at the substituted values of each function and decided whether these values are increasing or decreasing to a number (approaching

a value). They turned spontaneously to concrete mode and then made a transition from concrete to objectified mode regarding the concept of limit as approaching a value.

4) Routines

The routines of the American 7^{th} and 10^{th} graders were to substitute infinity for x by alluding to an *operational* use of infinity ("when x goes on forever" [10] and "something keep going on and on" [11]). They seemed to be using just one mediational mode (*syntactic*) with little flexibility. This *syntactic* mode allows for very little interpretation of finding limits and no predictions.

In contrast, the Koreans (K_7, K_{10}, and K_U) and the American undergraduate substituted several increasing numbers for x ("the bigger x gets" [12], "the number one, you know one...this will be close thirty" [14], "x is two...point five...x...ten..." [15], and "if I divided one...but one over two...infinity means too large number" [16]) and checked whether the values of $\frac{1}{x}$ were increasing or decreasing to determine the limit. In the case of $\frac{1}{x}$, they (K_7, K_{10}, K_U, and A_U) made a few transitions from one mediational mode to another; from syntactic to concrete mode and then from concrete to objectified mode. Their flexibility from one mediational mode to another in the process of finding limits can be described as *objectified* discourse. This mediational flexibility provides more interpretations and predications regarding the concept of limit than only one mediational mode.

APPENDIX C

QUESTIONNAIRE

QUESTIONNAIRE

The purpose of this study is to investigate and compare how native-English and native-Korean speaking university students, who received their education respectively in the U.S. and in Korea, think about the concepts of *infinity* and *limit*.

Participation is voluntary, and you are free to discontinue your participation at any time. If you agree to participate in the study, you can agree to participate in Part I, or Parts I and II.

Part I: You will be asked several questions on this questionnaire. The questionnaire consists of two sections: *Section I, Background*; and *Section II, Discourse*. The questions on the questionnaire are dealing with your background regarding the mathematical concepts of infinity and limit, and questions about colloquial and mathematical discourses on these concepts. Based on your responses to these questions, you may be invited for an interview.

Part II: Some students will be asked to participate in a follow-up interview. The interviewer will ask pairs of students to think aloud about solving the same tasks presented in Part I. If you participate in the research study, you will be audio-taped, video-taped and transcribed in their entirely for further analysis. As compensation, each interviewee will receive 25 dollars.

There are no risks associated with the survey and interview. Any information obtained from this research will be kept strictly confidential. Only researchers will have access to these data. As soon as the investigators complete analysis, all video and audio data will be erased.

You are free to ask any questions concerning the procedure. If you have any questions about this study, please contact the investigators (*DJ Kim,884-1481, kimdon14@msu.edu*; *Dr. Joan Ferrini-Mundy, 432-1490, jferrini@msu.edu*; *Dr. Anna Sfard, 353-0881, annasfar@math.msu.edu*). If you have questions or concerns about your rights as a research participant, please feel free to contact Peter Vasilenko, Ph.D., Director of the Human Subject Protection Programs at Michigan State University: *(517) 355-2180, fax: (517) 432-4503, irb@msu.edu*, or regular mail: 202 Olds Hall, East Lansing, MI 48824.

Your signature below indicates your voluntary agreement to participate in this study to increase our understanding of student conceptions of infinity and limit.

Participant's Printed Name	PID
Participant's Signature	Date

Section I: Background

About You

1. Year:
   - ☐ Freshman
   - ☐ Sophomore
   - ☐ Junior
   - ☐ Senior

2. Native Language:
   - ☐ English
   - ☐ Korean
   - ☐ Other (Specify): _____

3. How often do you speak your native language at home?
   - ☐ Always
   - ☐ Almost always
   - ☐ Sometimes
   - ☐ Never

4. About how many books are there in your home? (Do not count magazines, newspapers, or your school books.)
   - ☐ None or very few (0-10 books)
   - ☐ Enough to fill one shelf (11-25 books)
   - ☐ Enough to fill one bookcase (26-100 books)
   - ☐ Enough to fill two bookcases (101-200 books)
   - ☐ Enough to fill three or more bookcases (more than 200 books)

Mathematics in School

5. Have you studied mathematics in any language other than your native language?

   - ☐ Yes
   - ☐ No

   If yes, specify the languages in which you learned mathematics:_____

6. Can you remember the first mathematics course in which you heard the word (mathematical) *infinity*?

   - ☐ Yes
   - ☐ No

7. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *infinity*.

Grade(s): _____ Course: _____

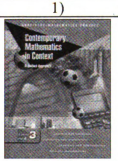
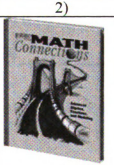
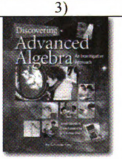
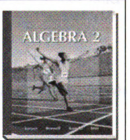
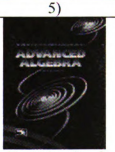
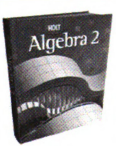
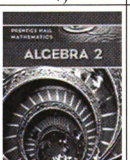
8. Can you remember the first mathematics course in which you heard the word (mathematical) *limit*?

- ❑ Yes
- ❑ No

9. If yes, specify in which grade(s) and in which course/class you heard the word (mathematical) *limit*.

Grade(s): _____ Course: _____

10. Please specify from which high school textbook(s) series you studied among the examples below. If you used textbooks other than those in the examples, please check the box for *Others*.

1)	2)	3)	4)
			
Core-Plus Mathematic s Project (CPMP)	Mathematics Connections	Key Curriculum Press	McDougal Littell
5)	6)	7)	*Others*
			
The University of Chicago School Mathematics Project (UCSMP)	Holt	Prentice Hall	

11. In high school, which course(s) in the series below did you take? Please check all the boxes that apply to you (under the textbook series you used).

CPMP	Math Connections	Key curriculum	McDougal Littell
❑ Course 1	❑ Math connections 1	❑ Algebra	❑ Algebra 1
❑ Course 2	❑ Math connections 2	❑ Geometry	❑ Geometry
❑ Course 3	❑ Math connections 3	❑ Advanced Algebra	❑ Algebra 2
❑ Course 4			

UCSMP	Holt	Prentice Hall	*Others*
❑ Transition Math	❑ Algebra 1	❑ Pre-Algebra	
❑ Algebra	❑ Geometry	❑ Algebra 1	
❑ Geometry	❑ Algebra 2	❑ Geometry	
❑ Advanced Algebra		❑ Algebra 2	

12. In high school, what is the most advanced mathematics course you took other than those in #11?

Course: _____

13. Please check the box(es) next to topics which you have studied. When and where did this study occur?

	When	Where
❑ Infinite processes		
❑ Infinite decimal fractions		
❑ Infinite sequence or series		
❑ The sum of infinite geometric series		
❑ Comparison of infinite sets		
❑ Countable and uncountable sets		
❑ Infinitely small or infinitesimal		
❑ Other topic (Describe):		

14. Please check the box(es) next to topics which you have studied. When and where did this study occur?

	When	Where
☐ Limits to infinity		
☐ The ε-δ (epsilon-delta) definition of limit		
☐ Limit of a sequence or series		
☐ Limit of a function numerically		
☐ Limit of a function geometrically		
☐ Limit and the definition of continuity		
☐ Limit and the definition of the derivative		
☐ Other topic (Describe):		

15. How much do you agree with these statements about learning mathematics?

		Check one box for each line			
		Agree a lot	Agree a little	Disagree a little	Disagree a lot
a)	I usually do well in mathematics.	☐	☐	☐	☐
b)	I would like to take more mathematics in school.	☐	☐	☐	☐
c)	Mathematics is more difficult for me than for many of my classmates.	☐	☐	☐	☐
d)	Sometimes, when I do not initially understand a new topic in mathematics, I know what I will never really understand it.	☐	☐	☐	☐
e)	Mathematics is not one of my strengths.	☐	☐	☐	☐
f)	I learn things quickly in mathematics.	☐	☐	☐	☐

16. How much do you agree with these statements about mathematics?

		Agree a lot	Agree a little	Disagree a little	Disagree a lot
a)	I think learning mathematics will help me in my daily life.	☐	☐	☐	☐
b)	I need mathematics to learn other school subjects.	☐	☐	☐	☐
c)	I need to do well in mathematics to get into the university of my choice.	☐	☐	☐	☐
d)	I would like a job that involved using mathematics.	☐	☐	☐	☐
e)	I need to do well in mathematics to get the job I want.	☐	☐	☐	☐

17. How often do you do these things in your mathematics lessons?

		Every or almost every lesson	About half the lessons	Some lessons	Never
a)	We work together in small groups.	☐	☐	☐	☐
b)	We relate what we are learning in mathematics to our daily lives.	☐	☐	☐	☐
c)	We explain our answers.	☐	☐	☐	☐
d)	We decide on our own procedures for solving complex problems.	☐	☐	☐	☐
e)	We review our homework.	☐	☐	☐	☐
f)	We listen to the teacher give a lecture-style presentation.	☐	☐	☐	☐
g)	We work problems on our own.	☐	☐	☐	☐
h)	We use calculators.	☐	☐	☐	☐

Things You Do Outside of School

18. On a normal school day, how much time do you spend before or after school doing each of these things?

		No time	Less than 1 hour	1-2 hours	More than 2 hours
a)	I watch television and videos.	☐	☐	☐	☐
b)	I play computer games.	☐	☐	☐	☐
c)	I play or talk with friends.	☐	☐	☐	☐
d)	I do jobs at home.	☐	☐	☐	☐
e)	I work at a paid job.	☐	☐	☐	☐
f)	I play sports.	☐	☐	☐	☐
g)	I read a book for enjoyment.	☐	☐	☐	☐
h)	I use the internet.	☐	☐	☐	☐
i)	I do homework.	☐	☐	☐	☐

19. During this school year, how often have you had extra lessons or tutoring in mathematics that is not part of your regular class?

- ☐ Every or almost every day
- ☐ Once or twice a week
- ☐ Sometimes
- ☐ Never or almost never

20. How often does your instructor give you homework in mathematics?

- ☐ Every day
- ☐ 3 or 4 times a week
- ☐ 1 or 2 times a week
- ☐ Less than once a week
- ☐ Never

21. When your instructor gives you mathematics homework, about how many minutes are you usually given?
    - ☐ Fewer than 15 minutes
    - ☐ 15-30 minutes
    - ☐ 31-60 minutes
    - ☐ 61-90 minutes
    - ☐ More than 90 minutes

Section II: Discourse

I. Create a sentence with the following word (term).

1. Limited: _____

2. Limit: _____

3. Infinite: _____

4. Infinity: _____

II. Of which are there more? Please check one of the boxes. How do you know?

5. ☐ Of your fingers or ☐ Of your toes

   Because: _____

6. ☐ Of odd numbers or ☐ Of even numbers

   Because: _____

7. ☐ Of odd numbers or ☐ Of integers

   Because: _____

III. Which of the two sets A and B is bigger? How do you know?

A = { 1, 2, 3, 4, 5, 6, 7, 8, ... }

B = { 2, 4, 6, 8, 10, 12, 14, 16, ... }

Answer: _____

Because: _____

IV. A student claims that each of the two sets A and B is infinite, but A is larger than B. What can this mean?

Response:

V. What do you think will happen later in the columns of this table?

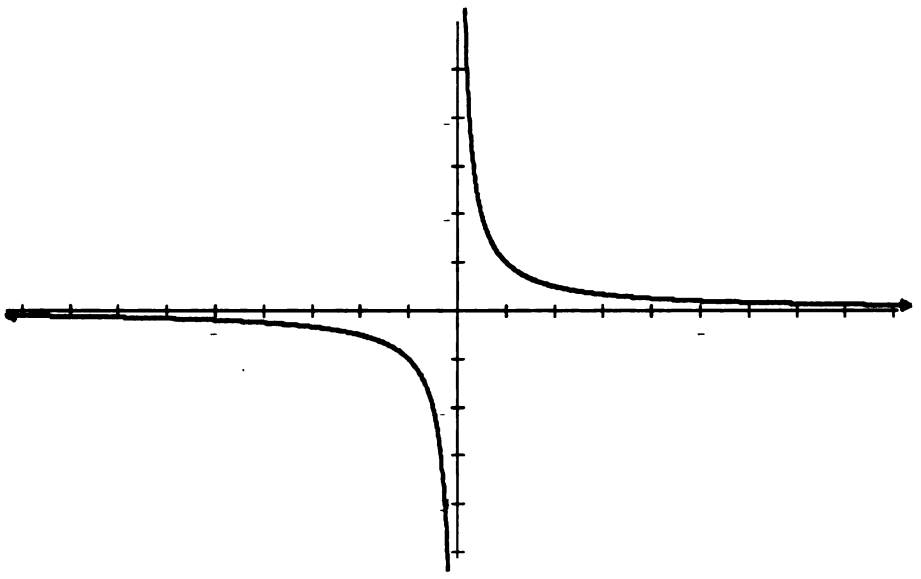
x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
0.01	0.099990
0.005	0.099995
0.001	0.099999
↓	↓

Answer:

Because:

VI. What happens to the curve as x approaches 0 from left or from right?

8. $\frac{1}{x}$

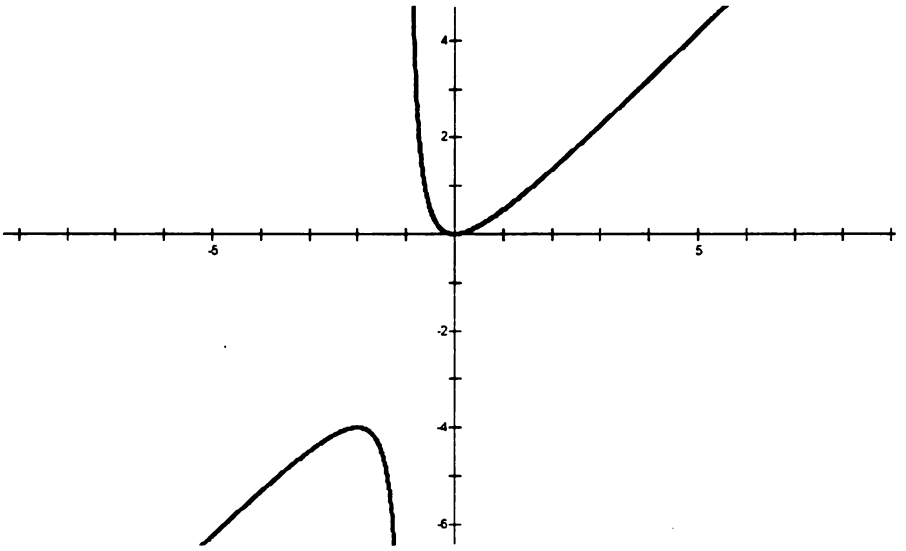


Answer: _____

Because: _____

What will happen to the curve in #9 and in # 10 when x goes to positive infinity?

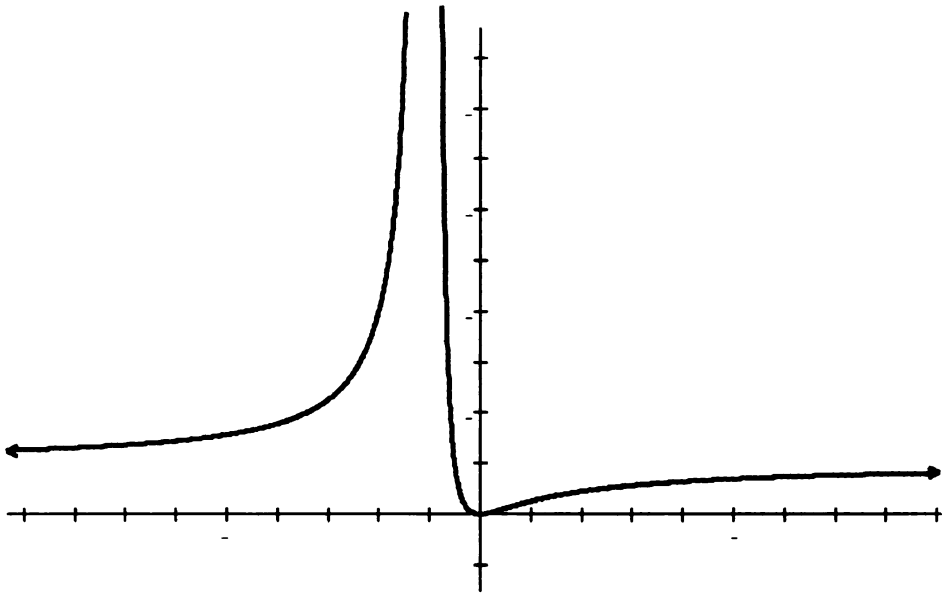
9. $\frac{x^2}{1+x}$



Answer: _____

Because: _____

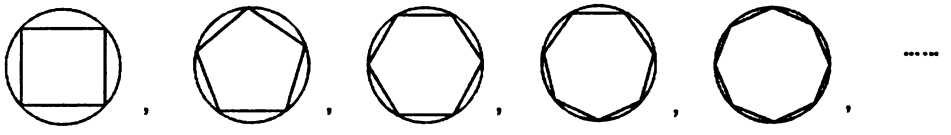
10. $\frac{x^2}{(1+x)^2}$



Answer:_____

Because: _____

VII. Examine the sequence of the square, the regular pentagon, hexagon (6 sides), heptagon (7 sides), ..., which is inscribed in the circle.



What happens if you continue increasing the number of sides of a regular polygon inscribed in a circle? Explain your reasoning.

Answer: _____

Because: _____

VIII. A student claims that a sequence $a_1, a_2, a_3, \ldots$ goes to infinity. What does this mean?

Response:

A student claims that the number P is the limit of a sequence $a_1, a_2, a_3, \ldots$ What does this mean?

Response:

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연구도중 어떤 질문도 자유럽게 물을수 있습니다. 이 연구에 대해서 질문이 있으면 다음의 연구자들에게 연락할수 있습니다 (김 동중, *(02)379-9328(Korea), (517)884-1481(USA), kimdon14@msu.edu*; *Dr. Joan Ferrini-Mundy, (517)432-1490 (USA), jferrini@msu.edu*; *Dr. Anna Sfard, 353-0881 (USA), annasfar@math.msu.edu*). 연구 참여자로서 권리에 대해 질문이나 염려하는 부분이 있으면 언제든지 다음 사람에게 연락하십시오 (Peter Vasilenko Ph.D., Director of the Human Subject Protection Programs at Michigan State University: *(517) 355-2180 (USA), fax: (517) 432-4503, irb@msu.edu*, or regular mail: 202 Olds Hall, East Lansing, MI 48824).

아래의 당신의 서명은 무한과 극한에 대한 학생들의 이해를 증가시키기위해 이 연구에 참여하겠다는 당신의 자발적인 동의를 나타내는 것입니다.

_____ _____

참여자 이름 학번

_____ _____

참여자 서명 날짜

첫번째 섹션: 배경

자신에 대해서

1. 학년:
   - ⌟ 1 학년생
   - ⌟ 2 학년생
   - ⌟ 3 학년생
   - ⌟ 4 학년생

2. 모국어:
   - ⌟ 영어
   - ⌟ 한국어
   - ⌟ 다른언어 (자세히 쓰세요): _____

3. 얼마자 자주 집에서 한국말을 합니까?
   - ⌟ 항상
   - ⌟ 거의 항상
   - ⌟ 가끔
   - ⌟ 결코 하지 않음

4. 대략 얼마나 많은 책들이 집에 있습니까? (잡지들이나 신문들 또는 학교책들은 세지마세요.)
   - ⌟ 없거나 매우 적음 (0-10 권)
   - ⌟ 책장 한칸을 채울정도(11-25 권)
   - ⌟ 책장 한개를 채울정도 (26-100 권)
   - ⌟ 책장 두개를 채울정도 (101-200 권)
   - ⌟ 책장 세개나 그 이상을 채울정도 (200 권 이상)

학교수학에 대해서

5. 모국어 이외에 다른 언어로 수학공부를 해 본적이 있습니까?

   - ⌟ 예
   - ⌟ 아니오

   만약 "예"면, 수학을 배웠던 언어들을 쓰십시오: _____

6. "무한"이라는 단어를 들었던 첫 수학과목을 기억합니까?

   - ⌟ 예
   - ⌟ 아니오

7. 만약 "예"면, 무한이라는 단어를 처음들었던 학년과 과목을 자세히 쓰십시오.

학년: _____ 과목: _____

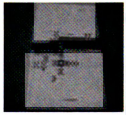
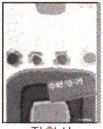
8. "극한"이라는 단어를 들었던 첫 수학과목을 기억합니까?

- ☐ 예
- ☐ 아니오

9. 만약 "예"면, 극한이라는 단어를 처음 들었던 학년과 과목을 자세히 쓰십시오.

학년: _____ 과목: _____

10. 다음 아래의 예제들중에서 여러분이 사용했던 고등학교 교과서를 표시하십시오. 만약 예제들과 다른 교과서를 사용했으면, 다른교과서 란에 표시하십시요.

1)	2)	3)	4)
 교학사	 금성 교과서	 대한 교과서	 두산동아
5)	**6)**	**7)**	**다른 교과서**
 중앙교육 진흥연구소	 지학사	 천재교육	

11. 고등학교때, 아래있는 어느과목(들)을 들었습니까? 자신이 사용했던 교과서 아래에 자신에게 적용되는 모든 과목들을 표시하세요.

교학사	금성 교과서	대한 교과서	두산동아
ㅋ 실용수학	ㅋ 실용수학	ㅋ 실용수학	ㅋ 실용수학
ㅋ 수학 10-가, 나	ㅋ 수학 10-가, 나	ㅋ 수학 10-가, 나	ㅋ 수학 10-가, 나
ㅋ 수학 I	ㅋ 수학 I	ㅋ 수학 I	ㅋ 수학 I
ㅋ 수학 II	ㅋ 수학 II	ㅋ 수학 II	ㅋ 수학 II
ㅋ 미분과 적분	ㅋ 미분과 적분	ㅋ 미분과 적분	ㅋ 미분과 적분

중앙교육 진흥연구소	지학사	천재교육	다른 교과서
ㅋ 실용수학	ㅋ 실용수학	ㅋ 실용수학	
ㅋ 수학 10-가, 나	ㅋ 수학 10-가, 나	ㅋ 수학 10-가, 나	
ㅋ 수학 I	ㅋ 수학 I	ㅋ 수학 I	
ㅋ 수학 II	ㅋ 수학 II	ㅋ 수학 II	
ㅋ 미분과 적분	ㅋ 미분과 적분	ㅋ 미분과 적분	

12. 문항 11 에 있는것과 달리 고등학교때 들었던 가장 고등수학 과목은 무엇입니까?

과목: _____

13. 여러분이 공부한 제목들을 표시하세요. 언제 어디서 배웠습니까?

	언제	어디서
ㅋ 무한과정		
ㅋ 무한 소수		
ㅋ 무한 수열 또는 급수		
ㅋ 무한 기하 급수의 합		
ㅋ 무한집합들의 비교		
ㅋ 셀수 있는 집합과 셀수 없는 집합		
ㅋ 무한소		
ㅋ 다른제목 (쓰십시오):		

14. 여러분이 공부한 제목들을 표시하세요. 언제 어디서 배웠습니까?

	언제	어디서
❑ 무한 극한		
❑ ε-δ (epsilon-delta)의 극한 정의		
❑ 수열과 급수의 극한		
❑ 대수적인 함수의 극한		
❑ 기하학적인 함수의 극한		
❑ 극한과 연속의 정의		
❑ 극한과 미분의 정의		
❑ 다른 제목(쓰십시오):		

15. 수학을 배우는것에 관한 다음의 진술들에 대해 얼마정도 동의 하십니까?

각 줄에서 한가지를 표시하세요

		많이 동의	조금 동의	조금 동의 않함	많이 동의 않함
a)	나는 일반적으로 수학을 잘한다	☐	☐	☐	☐
b)	나는 학교에서 수학과목을 더 듣고싶다.	☐	☐	☐	☐
c)	다른 많은 동기생들보다 수학이 나에게 더 어렵다.	☐	☐	☐	☐
d)	가끔씩 수학의 새로운 개념을 처음에 이해할수 없을때, 나는 내가 결코 이해할수 없을 것이라고 안다.	☐	☐	☐	☐
e)	수학은 나의 강점중 하나가 아니다.	☐	☐	☐	☐
f)	나는 수학을 빨리 배운다.	☐	☐	☐	☐

16. 수학에 관한 다음의 진술들에 대해 얼마정도 동의 하십니까?

		많이 동의	조금 동의	조금 동의 않함	많이 동의 않함
a)	나는 수학을 배우는것은 일상생활에 도움이 된다고 생각한다	☐	☐	☐	☐
b)	나는 다른 학교과목을 배우기 위해 수학이 필요하다.	☐	☐	☐	☐
c)	내가 선택한 대학에 들어가기 위해 나는 수학을 잘할 필요가 있다.	☐	☐	☐	☐
d)	나는 수학을 이용하는 직업을 좋아한다.	☐	☐	☐	☐
e)	나는 내가 원하는 직업을 얻기위해 수학을 잘할 필요가 있다.	☐	☐	☐	☐

17. 수학수업에서 다음의 것들을 얼마나 자주 합니까?

		(거의) 모든 수업에서	수업의 반 정도	수업의 조금	결코하지 않음
a)	우리는 소규모그룹에서 함께 공부한다.	☐	☐	☐	☐
b)	우리는 수학에서 배운것을 일상생활과 연관시킨다.	☐	☐	☐	☐
c)	우리의 대답을 설명한다.	☐	☐	☐	☐
d)	복잡한 문제를 해결하기위해 우리자신의 해결과정을 결정한다.	☐	☐	☐	☐
e)	우리 자신의 숙제를 복습한다.	☐	☐	☐	☐
f)	강의 위주의 수업을 듣는다.	☐	☐	☐	☐
g)	우리는 독자적으로 수학문제를 푼다.	☐	☐	☐	☐
h)	우리는 계산기를 사용한다.	☐	☐	☐	☐

학교외의 것들에 대해서

18. 평소 등교일에, 수업이전이나 이후에 다음 각각에 얼마정도의 시간을 씁니까?

		사용하지 않음	1 시간 미만	1-2 시간	2 시간 이상
a)	TV 나 비디오를 본다.	☐	☐	☐	☐
b)	컴퓨터 게임을 한다.	☐	☐	☐	☐
c)	친구와 놀거나 이야기 한다.	☐	☐	☐	☐
d)	집에서 일을 한다.	☐	☐	☐	☐
e)	돈을 받는 일을 한다.	☐	☐	☐	☐
f)	운동을 한다.	☐	☐	☐	☐
g)	즐기기 위해 책을 읽는다.	☐	☐	☐	☐
h)	인터넷을 사용한다.	☐	☐	☐	☐
i)	숙제를 한다.	☐	☐	☐	☐

19. 이번 학년동안, 정규수업외에 과외 수업이나 교습을 얼마나 자주 받았습니까?

- ☐ 매일 또는 거의 매일
- ☐ 일주일에 한 두번
- ☐ 가끔씩
- ☐ 전혀 또는 거의 전혀 받지 않음

20. 얼마나 자주 수학숙제가 주어집니까?

- ❑ 매일
- ❑ 일주일에 3 번이나 4 번
- ❑ 일주일에 한번이나 두번
- ❑ 일주일에 한번 미만
- ❑ 결코 없음

21. 수학숙제가 주어질때, 보통 몇분정도 길이로 주어집니까?

- ❑ 15 분 미만
- ❑ 15-30 분
- ❑ 31-60 분
- ❑ 61-90 분
- ❑ 90 분 이상

두번째 섹션: 담화

I. 주어진 단어를 이용해서 한 문장을 만드세요.

1. 한정된: _____

2. 제한: _____

3. 한없는: _____

4. 무수: _____

II. 어느쪽이 더 많습니까? 한가지를 표시하십시오. 어떻게 알았습니까?

5. ☐ 너의 손가락 또는 ☐ 너의 발가락

이유: _____

6. ☐ 홀수 또는 ☐ 짝수

이유: _____

7. ☐ 홀수 또는 ☐ 정수

이유: _____

III. A 와 B 집합들 중에서 어느쪽이 더 많습니까? 어떻게 알았습니까?

A = { 1, 2, 3, 4, 5, 6, 7, 8, ... }

B = { 2, 4, 6, 8, 10, 12, 14, 16, ... }

답: _____

이유: _____

IV. 한 친구가 "두 집합 **A**와 **B** 각각이 무한이지만 A는 B보다 크다"라고 주장했습니다. 이것은 무엇을 의미할수 있습니까?

답:

V. 이 테이블 각 열들에서 나중에 어떤 일이 발생할것이라고 생각합니까?

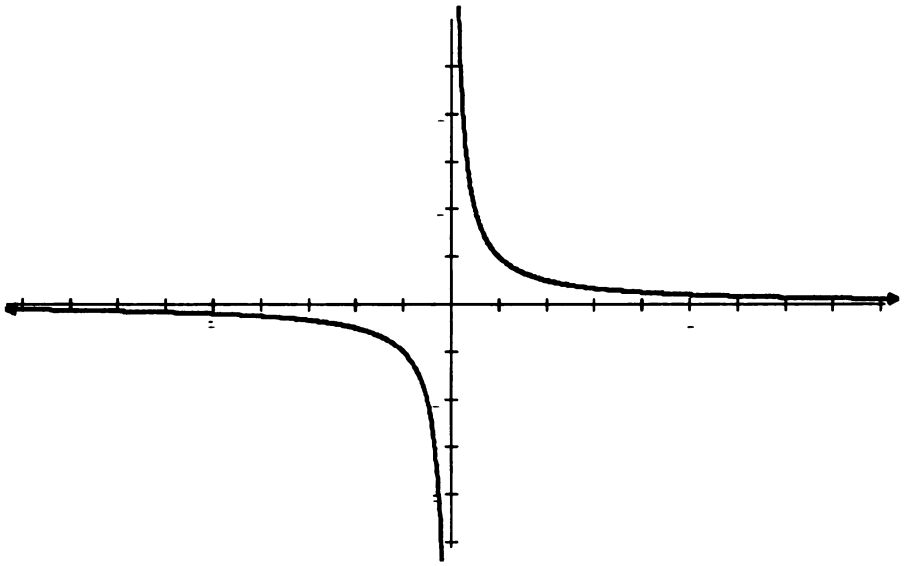
x	$\frac{\sqrt{x+25}-5}{x}$
1.0	0.099020
0.5	0.099505
0.1	0.099900
0.05	0.099950
0.01	0.099990
0.005	0.099995
0.001	0.099999
↓	↓

답:

이유:

VI. x 가 왼쪽에서나 오른쪽에서 0 으로 가까워질때, 다음의 곡선은 어떻게 됩니까?

8. $\frac{1}{x}$

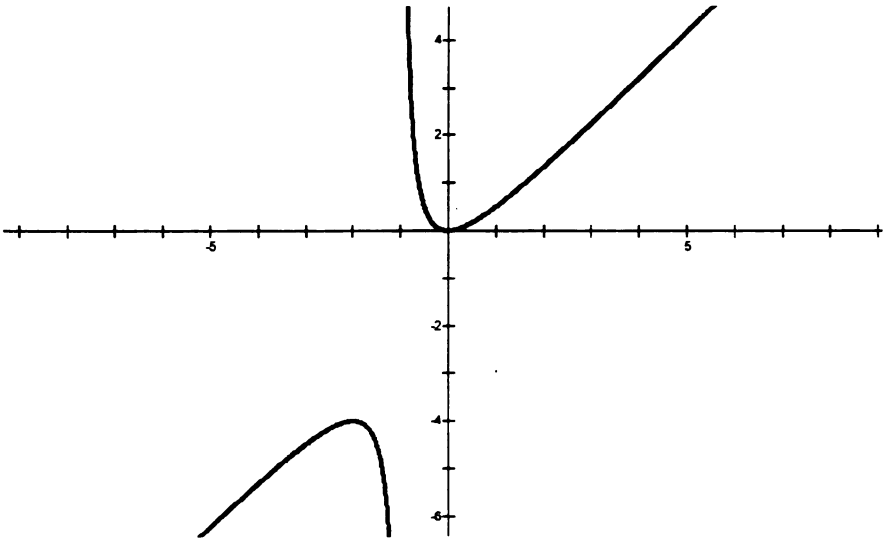


답: _____

이유: _____

x 가 양의무한대로 증가할 때, 문항 9와 10에 있는 곡선들은 어떻게 됩니까?

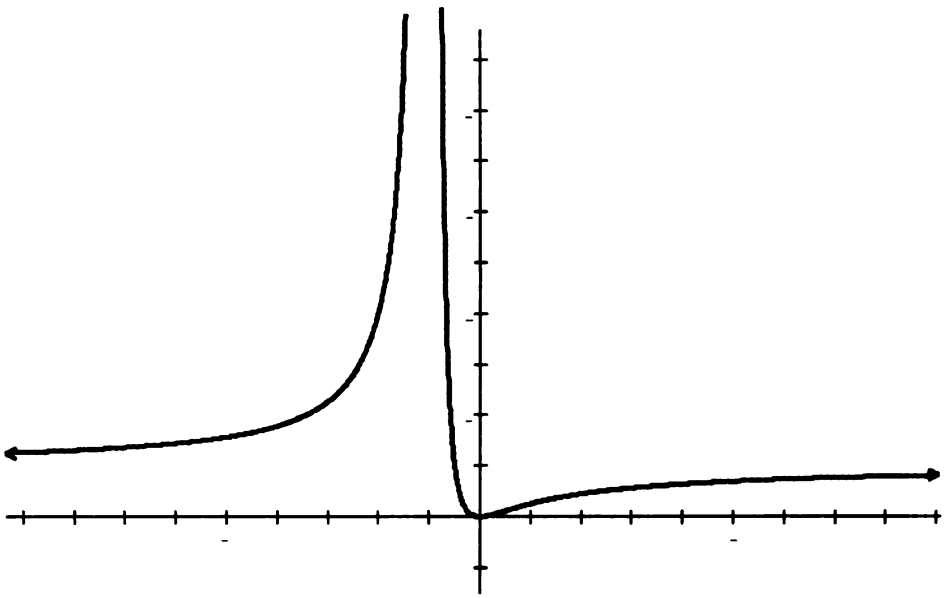
9. $\frac{x^2}{1+x}$



답: _____

이유: _____

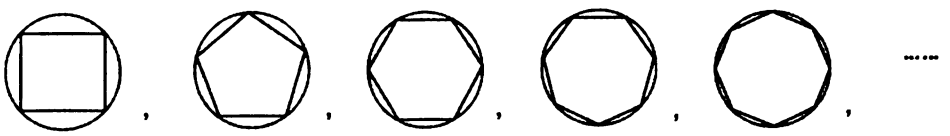
10. $\frac{x^2}{(1+x)^2}$



답: _____

이유: _____

VII. 같은 크기의 원 안에 내접시킨 정사각형, 정오각형, 정육각형, 정칠각형,…, 도형의 열을 생각해 보십시오.



이 도형열에서 내접 다각형의 변의 수가 계속 커지면 어떻게 됩니까? 여러분의 생각을 설명해 보십시오.

답: _____

이유: _____

VIII. 한 친구가 "수열 $a_1, a_2, a_3, \ldots$이 무한으로 간다"고 주장합니다. 이것은 무엇을 의미합니까?

답:

한 친구가 "수 **P** 는 수열 $a_1, a_2, a_3, \ldots$의 극한이다" 라고 주장합니다. 이것은 무엇을 의미합니까?

답:

IX. 주어진 단어를 이용해서 한 문장을 만드세요.

11. 극한의:

12. 극한:

13. 무한의:

14. 무한:

REFERENCES

REFERENCES

Aczel, A. D. (2000). *The mystery of the aleph: Mathematics, the kabbalah, and the search for infinity*. New York, NY: Washington Square Press.

Allen, G. D. (1999). *The History of Infinity: What is it? Where did it come from? How do we use it? Who are the inventors?* Retrieved May 10, 2004, from http://www.math.tamu.edu/~don.allen/masters/infinity/infinity.pdf.

Anderson, A., Anderson, J., & Shapiro, J. (2004). Mathematical discourse in shared storybook reading. *Journal for Research in Mathematics Education. 35*, 5-33.

Artigue, M. (1992). Analysis. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 167-198). Dordrecht, The Netherlands: Kluwer Academic.

Barwell, R. (2003). Patterns of attention in the interaction of a primary school mathematics student with English as an additional language. *Educational Studies in Mathematics,* 53, 35-59.

Barwell, R. (2005). Language in the mathematics classroom. *Language and Education, 19*, 96 -101.

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words or what ars students' access to arithmetical discourses. *Journal for Research in Mathematics Education*, *36*, 176-247.

Berlinghoff, W. P., Sloyer, C., & Hayden, R. W. (2000). Math Connections. A secondary Mathematics Core Curriculum (Book 1.a.) Armonk, NY: IT'S ABOUT TIME.

Bintz, W. P., & Moore, S. D. (2002). Using literature to support mathematical thinking in middle school. *Middle School Journal, 34*, 25-32.

Borasi, R. (1985). Errors in the enumeration of infinite sets. *Focus on Learning Problems in Mathematics, 7*, 77-89.

Boyer, C. (1949). *The history of the calculus and its conceptual development*. New York, NY: Dover.

Brownell, W. A. (1945). When is arithmetic meaningful? *Journal of Educational Research, 38*, 481-498.

Chen, C., & Stevenson, H. W. (1995). Motivation and mathematics achievement: A comparative study of Asian American, Caucasian-American, and East Asian high school students. *Child Development,* 66(4), 1215-1234.

Cobb, P. (1994). Where is the mind? Constructivist and socio-cultural approaches on mathematical development. *Educational Researcher, 23*, 13-20.

Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*, 362-385.

Corbin, J., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative Sociology, 13*, 3-21.

Cornu, B. (1992). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic.

Cotter, J. A. (2000). Using language and visualization to teach place value. *Teaching Children Mathematics, 7*, 108-114.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior, 15*, 167–192.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., Watkins, A. E. with Messenger, M. J., & Ritsema, B. (1997). *Contemporary mathematics in context: A unified approach* (Course 1, Course 2, Course 3 and Course 4). Columbus, OH: Glencoe/McGraw-Hill.

Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5*, 281-303.

Edwards, C. H. (1979). *The historical development of the calculus*. New York, NY: Springer-Verlag.

Edwards, D. (1997). *Discourse and cognition*. Thousand Oaks, CA: Sage Publications.

Ernest, P. (1994). The philosophy of mathematics and the didactics of mathematics. In R. Biehler, R. W. Scholz, R. Sträβer, and B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline*. Dordrecht, Boston: Kluwer Academic Publishers

Fischbein, E. (2001). Tacit models and infinity. *Educational Studies in Mathematics, 48*, 309-329.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics, 10*, 3-40.

Fischbein, E., Tirosh, D., & Melamed, U. (1981). Is it possible to measure the intuitive acceptance of a mathematical statement? *Educational Studies in Mathematics, 12*, 491-512.

Fuson, K. C., & Kwon, Y. (1992). Learning addition and subtraction: Effects of number word and other cultural tools. In J. Bideau, C. Meljac, &J. P. Fisher (Eds.), *Pathways to number* (pp. 351-374). Hillsdale, NJ: Erlbaum.

Gadakiri. (1992). *The process of problem solving and question analysis*. Seoul, Korea: Kyung-Mun company.

Gray, E., Pinto, M., Pitta, D., & Tall, D. (1999). Knowledge construction and diverging thinking in elementary & advanced mathematics. *Educational Studies in Mathematics, 38*, 111-133.

Gutierrez, R. (2002). Beyond essentialism: The complexity of language in teaching mathematics to Latina/o students. *American Educational Research Journal, 39*, 1047-1088.

Hairer, E., & Wanner, G. (1995). *Analysis by its history*. New York, NY: Springer-Verlag.

Han, Y., & Ginsburg, H. P. (2001). Chinese and English mathematics language: The relationship between linguistic clarity and mathematics performance. *Mathematical Thinking and Learning, 3*(2-3), 201-220.

Hersh, R. (1997). *What is mathematics, really?* New York, NY: Oxford University Press.

Jahnke, H. N. (2001). Cantor's cardinal and ordinal infinites: An epistemological and didactic view. *Educational Studies in mathematics, 48*, 175-197.

Kim, D. (2006). *Students' colloquial and mathematical discourses on infinity and limit: The case of American and Korean students*. Unpublished doctoral practicum, Michigan State University, East Lansing.

Kim, D., Sfard, A., & Ferrini-Mundy, J. (2005a). Students' colloquial and mathematical discourses on infinity. In Chick, H. L., & Vincent, J. L. (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 201-208). Melbourne, Australia: PME.

Kim, D., Sfard, A., & Ferrini-Mundy, J. (2005b). Students' colloquial and mathematical discourses on limit. In G.M. Lloyd, M.R. Wilson, J.L.M. Wilkins, & S.L. Behm(Eds.), *Proceedings of the 27th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* [CD-ROM]. Eugene, OR: All Academic.

Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics, 48*, 137-174.

Korea Ministry of Education (1997). *The school curriculum of the Republic of Korea.* Retrieved December 8, 2006, from http://www.moe.go.kr/search

Lakoff, G., & Nùñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Lave, J. (1988). *Life after school. Cognition in practice*. (pp. 45-75). Cambridge: Cambridge University Press.

Lee, I., & Ramsey, S. R. (2000). *The Korean language*. Albany, NY: State University of New York Press.

Leung, C. (2005). Mathematical vocabulary: Fixers of knowledge or points of exploration? *Language and Education, 19*, 126-134.

Li, C., & Nuttall, R. (2001). Writing Chinese and mathematics achievement: A study with Chinese-American undergraduates. *Mathematics Education Research Journal, 13*(1), 15-27.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics, 48*, 259-288.

Maor, E. (1987). *To infinity and beyond: A cultural history of the infinite*. Princeton: Princeton University Press.

Miura, I. T., Okamoto, Y., Kim, C. C., Chang, C. M., Steere, M., Fayol, M. (1994). Comparisons of children's cognitive representation of number: China, France, Japan, Korea, Sweden, and the United States. *International Journal of Behavioral Development, 17*, 401-411.

Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics, 11*, 20-24.

Monaghan, J. (2001). Young peoples' ideas of infinity. *Educational Studies in Mathematics, 48*, 239-257.

Mooney, C. G. (2000). *Theories of childhood: An introduction to Dewey, Montessori, Erickson, Piaget, & Vygotsky*. St. Paul, MN: Redleaf Press.

Moore, A.W. (1990). *The Infinite*. London: Routledge.

Moschikovich, J. (1996). Learning in two languages. In L. Puig & A. Gutierezz (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 27-34). Valencia: Universitat De Valencia.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and experimental curriculum. *Journal for Research in Mathematics Education, 30*, 122-147.

National Research Council. (2001). Adding it up: Helping children learn mathematics. J. Kilpatrick, J. Swafford, and B. Findell (Eds.). *Mathematics Learning Study Committee, Center for Education, Division of Behavior and social Science and Education*. Washington, DC: National Academy Press.

Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics*(pp. 1-27). Cambridge, MA: Cambridge University Press.

Pallant, J. (2005). *SPSS survival manual: A step by step guide to data analysis using SPSS version 12* (2nd ed.). Buckingham: Open University Press.

Pimm, D. (1984). Who is we? *Mathematics Teaching, 107*, 39-42.

Pirie, S., & Kieren, T. E. (1992). Watching Sandy's understanding grow. *Journal of Mathematical Behavior, 11*, 243-257.

Przenioslo, M. (2004). Image of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics, 55*, 103-132.

Raiker, A. (2002). Spoken language and mathematics. *Cambridge Journal of Education, 32*, 45-60.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York: Oxford University Press.

Rotman, B. (1993). *Ad Infinitum...the ghost in Turing's machine: Taking God out of mathematics and putting the body back in*. Stanford, CA: Stanford University Press.

Rubenstein, R. N., & Schwartz, R. K. (2000). Word histories: Melding mathematics and meanings. *Mathematics Teacher, 93*, 664-669.

Rucker, R. (1995). *Infinity, Infinity and the mind: The science and philosophy of the infinite* (pp. 1-52). Princeton, NJ: Princeton University Press.

Schwarzenberger, R., & Tall, D. (1978). Conflict in the learning of real numbers and limits. *Mathematics Teaching, 82*, 44-49.

Setati, M. (2002). Researching mathematics education and language in multilingual South Africa. *Mathematics Educator, 12*, 6-20.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*, 4-13.

Sfard, A. (2000). Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education. 31*, 296-327.

Sfard, A. (2001a). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of 21st Conference of PME-NA* (pp. 23-44). Columbus, Ohio: Clearing House for Science, Mathematics, and Environmental Education

Sfard, A. (2001b). There is more to discourse than meets the ears: Learning from mathematical communication things that we have not known before. *Educational Studies in Mathematics, 46*, 13-57.

Sfard, A. (2006). Participationist discourse on mathematical learning. In J. Maaβ & W. Schlöglmann (Eds.), *New mathematics education research and practice* (pp. 153-170). Rotterdam, The Netherlands: Sense Publisher.

Sfard A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? – Early numerical thinking revisited. *Cognition and Instruction, 23* (2), 237-309.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18*, 371-397.

Sierpinska, A. (1998). Three epistemologies, three views of classroom communication: Constructivism, sociocultural approaches, interactionism. In H. Steinbring, M.G. Bartolini Bussi & A. Sierpinska (Eds.), *Language and Communication in the Mathematics Classroom* (pp. 30–62). Reston, VA: NCTM.

Skemp, R. R. (1987). *The psychology of leaning mathematics* (Expanded American ed.). Hillsdale, NJ: Lawrence Erlbaum.

Smith III, J. P. (2002). The Development of Students' Knowledge of Fractions and Ratios. In B. Litwiller (Ed.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (pp. 3-17). Reston, VA: National Council of Teachers of Mathematics.

Sohn, H. M. (1999). The Korean language. Cambridge, UK: Cambridge University Press.

Sohn, H. M. (2006). Korean in contact with Chinese. In H. M. Sohn (Ed.), *Korean language in culture and society*. Honolulu, HI: University of Hawai'i Press.

SPSS for Windows, Rel. 16.0.1. 2007. Chicago: SPSS Inc.

Stevenson, H. W., Chen, C., & Lee, S. (1993). Motivation and achievement of gifted children in East Asia and the United States. *Journal for the Education of the Gifted, 16(3),* 223-250.

Tall, D. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics, 11*, 271-284.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-5 I 1). New York: Macmillan.

Tall, D. (2000). Cognitive development in advanced mathematics using technology. *Mathematics Education Research Journal, 12*, 210-230.

Tall, D. (2001a). A child thinking about infinity. *Journal of Mathematical Behaviour, 20*, 7- 19.

Tall, D. (2001b). Natural and formal infinities. *Educational Studies in Mathematics, 48*, 199-238.

Tall, D., & Tiroch, D. (2001). Infinity – The never-ending struggle. *Educational Studies in Mathematics, 48*, 129-136.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

TIMSS (2002). *Student Questionnaire: Grade 8*. Prepared by the International Study Center at Boston College. Chestnut Hill, MA: Boston College.

Tirosh, D. (1992). The role of students' intuitions of infinity in teaching Cantorian theory. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 199-214). Dordrecht, The Netherlands: Kluwer Academic.

Thompson, D. R., & Rubenstein, R. N. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. *Mathematics Teacher, 93*, 568-574.

Tsamir, P. (2001). When 'the same' is not perceived as such: The case of infinite sets. *Educational Studies in Mathematics, 48*, 289-307.

Tsamir, P., & Dreyfus, T. (2002). Comparing infinite sets – a process of abstraction: The case of Ben. *Journal of Mathematical Behaviour, 21*, 1-23.

Thomas, G. (2005). *Thomas' calculus* (11th ed. Revised by M. Weir, J. Hass, & F. Giordano.). NY: Addison Wesley.

U.S. Department of Education. (2006). *The federal role in education.* Retrieved October 8, 2006, from http://www.ed.gov/about/overview/fed/role.html

Vinner, S. (1992). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer Academic.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language*. Cambridge, M.A.: MIT press.

Wang, J., & Lin, E. (2005). Comparative studies on U.S. and Chinese mathematics learning and the implications for Standards-based mathematics teaching reform. *Educational Researcher, 34*(5), 3-13.

Weller, K., Brown, A., Dubinsky, E., McDonald, M., & Stenger, C. (2004). Intimations of infinity. *Notices of the American Mathematical Society, 51*, 741-750.

Whimbey A., & Lochhead, J. (1982). *Beyond problem solving and comprehension: An exploration of quantitative reasoning*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for research in Mathematics Education. 22*(3), 219-236.

Williams, S. R. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education, 32*(4), 343-367.

Wolfe, R. G. (1999). Measurement obstacles to international comparisons and the need for regional design and analysis in mathematics surveys. In G. Kaiser, E. Luna & I. Huntley (Eds.), *International comparisons in mathematics education* (pp. 225-240). Philadelphia, PA: Falmer Press

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