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ALMOST DUAL FF-MODULES

By

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ABSTRACT

ALMOST DUAL FF-MODULES

By

Kimberly Ann Dyer

In this paper we consider a subgroup, L, of a finite group of local characteristic 2. The action of a maximal 2-local parabolic subgroup containing a non-normal large subgroup on its largest 2-reduced normal subgroup is considered in the quadratic L-Lemma and Structure Theorem [MSS]. They show $L/O_p(L) \cong SL_2(2), Sz(2)$, or D_{2r} and obtain a $2F$-offendor. The action which we are interested in can be determined by the Malle-Guralnick-Lawther Classification of $2F$-modules [GLM]. The Malle-Guralnick-Lawther papers depend on a $\mathcal{K}$-group assumption; that is, one needs to assume that all the simple sections of M are one of the known finite simple groups. In this paper we explore results that do not need a $\mathcal{K}$-group assumption and therefore do not use the classification of finite simple groups.

Let $\mathbb{F}$ be a finite field with $p := \operatorname{char} \mathbb{F} = 2$, G a finite group, and V a faithful, finite dimensional $\mathbb{F}G$-module such that there exists an elementary abelian p-subgroup, $A \leq G$ with $T_A := [V, A]$ and $R_A := [T_A, A]$. By considering whether or not A is a TI-set, that is if $A \cap A^g \subseteq \{1\}$ for all $g \in G \setminus N_G(A)$, we arrive at the various cases of the main theorem. The main theorem shows that we have $\mathbb{F} = 2$ and $4 \leq |A| \leq 16$, $|\mathbb{F}| > 2$ and $|A| = |\mathbb{F}|^2$, or $G_0/C_{G_0}(R_G) \cong SL_{\mathbb{K}}(N)/Z_0$ where $G_0 = \langle A^G \rangle$, $R_G = \sum_{B \in A^G} R_B$, N is a finite dimensional vector space over the finite field $\mathbb{K}$, and $Z_0 = \{k * \mathrm{id}_V \mid k \in \mathbb{K}, k^\sigma = k^{-1}\}$.

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Chapter 1

Motivating Introduction

Let p be a prime and G a finite group of local characteristic p. Suppose G has a large subgroup Q. Let M be a maximal p-local parabolic subgroup of G with $Q \leq M$ but $Q \ntrianglelefteq M$. Let $Y = Y_M$ be the largest p-reduced normal subgroup of M. In the Structure Theorem [MSS] the action of M on Y is determined. In the proof of the Structure Theorem one runs into the following situation.

There exists a subgroup L of G such that $Y \leq L$ but $Y \nleq O_p(L)$, $M \cap L$ is the unique maximal subgroup of L containing $YO_p(L)$, and L is of characteristic p. The quadratic L-Lemma [MSS] shows $L/O_p(L) \cong SL_2(q)$ where q is a power of p, $L/O_p(L) \cong Sz(q)$ where q is a power of p and $p = 2$, or $L/O_p(L) \cong D_{2r}$ where r is an odd prime power and $q = p = 2$.

In this situation one tries to obtain some information about the action of M on Y. For this, let $x \in L \setminus M$. Put $D = Y \cap O_p(L)$ and $A = D^x$. In the Structure Theorem, in this case, it was proved that

(a) $[Y, A]C_Y(A) = [y, A]C_Y(A) = D$ for all $y \in Y \setminus D$.

(b) $|D/C_D(A)| = |A/C_A(Y)| \geq q$.

(c) $|Y/D| = q$.

(d) $[D, A] \leq C_Y(A)$.

Notice that (a) implies $C_Y(A) = C_D(A)$ so we have $|Y/C_Y(A)| = |Y/C_D(A)| = |Y/D||D/C_D(A)| \overset{(c)}{=} q|D/C_D(A)| \overset{(b)}{\leq} |A/C_A(Y)||A/C_A(Y)| = |A/C_A(Y)|^2$ giving $|Y/C_Y(A)| \leq |A/C_A(Y)|^2$. Hence, A is a $2F$-offender on Y. The Malle-Guralnick-Lawther Classification of $2F$-modules [GLM] now allows us to determine the action of M on Y. The Malle-Guralnick-Lawther papers depend on a $\mathcal{K}$-group assumption; that is, one needs to assume that all the simple sections of M are one of the known finite simple groups. In this paper we would like to explore what can be said without making a $\mathcal{K}$-group assumption.

If there exist large enough quadratically acting subgroups, then this quadratic action can be used to determine Y and $M/C_M(Y)$. We consider what happens when there is not a large enough quadratically acting subgroup. Put $\overline{M} = M/C_M(Y)$ and $H = \langle A^M \rangle$.

Let $\mathbb{F}$ be a maximal subfield of the ring $\mathrm{End}_H(Y)$. Then Y is an $\mathbb{F}H$-module. We assume that there is no large quadratic action on Y in the following sense:

(e) If P is a p-subgroup of $\overline{M}$ and $1 \neq a \in P$ with $[Y, P, a] = 0$, then $|P| \leq |\mathbb{F}|$.

Let B_0 be a normal subgroup of L minimal with $B_0 \not\leq Z(L)$. If $p = 2$, it can be shown that there exists $B \leq B_0$ such that $[B, Y, B] = 0$ and $|B/C_B(Y)| \geq q$. So $[Y, B/C_B(Y), B/C_B(Y)] = 0$ and (e) now shows that $|\mathbb{F}| \geq q$. A acts $\mathbb{F}$-linearly on Y so both $[Y, A]$ and $C_Y(A)$ are $\mathbb{F}$-subspaces. Then (a) shows D is an $\mathbb{F}$-subspace of Y. By (c), $q = |Y/D| = |\mathbb{F}|^{\dim_{\mathbb{F}}(Y/D)}$. Since $|\mathbb{F}| \geq q$, we conclude that $|\mathbb{F}| = q$ and D is an $\mathbb{F}$-hyperplane of Y. Since we would prefer to work with a 1-dimensional $\mathbb{F}$-subspace than with a hyperplane, we consider the $\mathbb{F}$-dual, V, of Y and arrive at the following set of assumptions (where the G below is now taking the place of $M/C_M(Y)$ from above).

Hypothesis 1.1 *Let $\mathbb{F}$ be a finite field with $p := \operatorname{char}\mathbb{F} = 2$, G a finite group, and V a faithful, finite dimensional $\mathbb{F}G$-module such that there exists an elementary abelian p-subgroup, $A \leq G$, with $T_A := [V, A]$ and $R_A := [T_A, A]$ such that*

(i) R_A is 1-dimensional.

(ii) For $a \in A$, define $\phi_A(a) : T_A/R_A \to R_A, t + R_A \to [t, a]$. Then $\phi_A : A \to \operatorname{Hom}_{\mathbb{F}}(T_A/R_A, R_A), a \to \phi_A(a)$ is onto.

(iii) If $1 \neq a \in A$ and P is a p-subgroup of G with $[V, a, P] = 0$, then $|P| \leq |\mathbb{F}|$.

Chapter 2

Almost dual FF-modules

Definition 2.1 *Put* $\mathcal{A} = \mathcal{A}(G) = A^G$ *and* $\mathcal{R} = \mathcal{R}(G) = {R_A}^G$. *For* $H \leq G$, *let* $\mathcal{A}(H) = \{B \in \mathcal{A} \mid B \leq H\}$, $H_0 = \langle \mathcal{A}(H) \rangle$, $\mathcal{R}(H) = \{R_B \mid B \in \mathcal{A}(H)\}$, $R_H = \sum_{B \in \mathcal{A}(H)} R_B$, *and* $T_H = \sum_{B \in \mathcal{A}(H)} T_B$.

The goal of this paper is to prove the following theorem. To this end, we assume that Hypothesis 1.1 holds throughout the entire document.

Theorem 2.2 *Assume Hypothesis 1.1. Then* $C_G(A)/A$ *is a* p'*-group,* A *is a weakly closed subgroup of* G*, for* $A \neq B \in \mathcal{A}$, $R_A \neq R_B$ *and* $T_A \neq T_B$*, and one of the following holds:*

1. *Each of the following holds:*

   (a) $A \in \mathrm{Syl}_p(G)$.

   (b) $|A| = |\mathbb{F}|^2$.

   (c) $|\mathbb{F}| > 2$.

   (d) $G_0/C_{G_0}(R_G) \cong SL_2(\mathbb{F})$ *or* $G_0/C_{G_0}(R_G) \cong \Omega_4^+(\mathbb{F})$.

   (e) R_G *is a corresponding natural module for* G_0.

2. *Each of the following holds:*

(a) $|\mathbb{F}| = 2$.

(b) $4 \leq |A| \leq 16$.

(c) $|A \cap B| \leq 2$ *for* $A \neq B \in \mathcal{A}$.

3. $G_0/C_{G_0}(R_G) \cong SL_{\mathbb{K}}(N)/Z_0$, *where* N *is a finite dimensional vector space over the finite field* $\mathbb{K}$. *Moreover, there exists a* 1*-dimensional subspace,* C, *of* N *and a field automorphism,* σ, *of* $\mathbb{K}$ *of order two with* $C_{\mathbb{K}}(\sigma) = \mathbb{F}$, $\mathbb{K} \otimes_{\mathbb{F}} R_G \cong N \otimes_{\mathbb{K}} N^\sigma$, $Z_0 = \{k * \mathrm{id}_V \mid k \in \mathbb{K}, k^\sigma = k^{-1}\}$, *and the image of* A *in* $PSL_{\mathbb{K}}(N)$ *consists of the identity and all transvections with center* C.

Lemma 2.3

(a) $C_{T_A}(A) = R_A$.

(b) R_A *is contained in every non-zero* $\mathbb{F}A$*-submodule of* T_A.

Proof. Since R_A is a non-trivial p-group, $C_{R_A}(A) \neq 0$. $C_{R_A}(A) \leq R_A$ and R_A is 1-dimensional from 1.1(i) so $C_{R_A}(A) = R_A$. Then $R_A \leq C_V(A)$. Let $v \in T_A \setminus R_A$. Then there exists $\rho \in \mathrm{Hom}(T_A/R_A, R_A)$ with $\rho(v + R_A) \neq 0$. By 1.1(ii), ϕ_A is onto so there exists $a \in A$ with $\phi_A(a) = \rho$. Then $0 \neq \rho(v + R_A) = \phi_A(a)(v + R_A) = [v, a]$. Hence, $v \notin C_{T_A}(A)$. Then $T_A \setminus R_A \nsubseteq C_{T_A}(A)$ so $C_{T_A}(A) = C_{R_A}(A) = R_A$ and (a) is proven.

Let $0 \neq W$ be an $\mathbb{F}A$-submodule in T_A. Since A is a p-group, $C_W(A) \neq 0$. Thus by (a), $W \cap R_A = W \cap C_{T_A}(A)$ and $0 \neq C_W(A) \leq C_{T_A}(A)$ so $0 \neq W \cap C_{T_A}(A) = W \cap R_A$. As R_A is 1-dimensional, $R_A \leq W$ which proves (b). $\square$

Definition 2.4 *For* H *a group, let* X *and* Y *be* $\mathbb{F}H$*-modules. Define* $\mathrm{Hom}_{\mathbb{F}}(X, Y)$ *to be the set of* $\mathbb{F}$*-linear maps from* X *to* Y.

We remark that $\mathrm{Hom}_{\mathbb{F}}(X, Y)$ is an $\mathbb{F}$-space via $(f\alpha)(x) = f\alpha(x)$ and $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$. Also, $\mathrm{Hom}_{\mathbb{F}}(X, Y)$ is an $\mathbb{F}H$-module via $(\alpha^g)(x) = (\alpha(x^{g^{-1}}))^g$.

Lemma 2.5

(a) ϕ_A *is* G*-invariant.*

(b) $C_A(T_A) = 1$.

(c) ϕ_A *is an isomorphism of* $\mathbb{Z}N_G(A)$*-modules.*

(d) $\dim T_A/R_A \geq 2$.

Proof. Using the remark following 2.4, $\phi_A(a^g)(x + R_A) = [x, a^g] = [x^{g^{-1}}, a]^g = (\phi_A(a)(x^{g^{-1}} + R_A))^g = (\phi_A(a))^g(x + R_A)$ and (a) holds.

By 1.1(ii), ϕ_A is onto and $C_A(T_A)$ is the kernel of ϕ_A so

$$\Phi : A/C_A(T_A) \to \mathrm{Hom}_{\mathbb{F}}(T_A/R_A, R_A), a + C_A(T_A) \to \phi_A(a)$$

is an isomorphism of $\mathbb{Z}N_G(A)$-modules. Since R_A is 1-dimensional, it follows that $|A/C_A(T_A)| = |\mathbb{F}|^{\dim T_A/R_A}$. Suppose that $C_A(T_A) \neq 1$. Then there exists $1 \neq a \in A$ with $[T_A, a] = 0$. Since A is abelian, we get $[[V, a], A] = 0$ from the Three Subgroups Lemma [Gor, 2.2.3]. We can then apply 1.1(iii) to see that $|A| \leq |\mathbb{F}|$. Now

$$|A/C_A(T_A)| \leq |A| \leq |\mathbb{F}| \leq |\mathbb{F}|^{\dim T_A/R_A}.$$

But $|A/C_A(T_A)| = |\mathbb{F}|^{\dim T_A/R_A}$ from above so $|A/C_A(T_A)| = |A|$ and $C_A(T_A) = 1$ which proves (b). Then $\Phi = \phi_A$ is an isomorphism and (c) is proven.

Suppose that T_A/R_A is 1-dimensional. Let $a \in A$ with $[V, a] \nleq R_A$ so $T_A = [V, a] + R_A$.

$$[V, a, a] = (v^a - v)^a - (v^a - v) = v^{a^2} - v^a - v^a + v = 0$$

as $p = 2$ so $[V, a] \leq C_V(a)$. We also have $[R_A, a] = 0$ since $C_{R_A}(A) = R_A$. Then $T_A = [V, a] + R_A \leq C_V(a)$, but $T_A \nleq C_V(a)$ since $C_A(T_A) = 1$ from (b). This is a contradiction so (d) is proven. □

Lemma 2.6 *Let $U \leq T_A$ be an $\mathbb{F}$-subspace. Then $A/C_A(U) \cong ((U + R_A)/R_A)^*$ as $N_G(A) \cap C_G(R_A) \cap N_G(U)$-modules, where * denotes the dual space of an $\mathbb{F}$-module.*

Proof. $A \cong \mathrm{Hom}(T_A/R_A, R_A) \to \mathrm{Hom}((U + R_A)/R_A, R_A)$. The first map is an isomorphism by 2.5 and the second map is onto. Hence, the map $A \to \mathrm{Hom}((U + R_A)/R_A, R_A)$ is onto and its kernel is $C_A(U)$ so we have the result by the first isomorphism theorem. □

Lemma 2.7 *For $f \in \mathbb{F}$ and $a \in A$ define $fa := \phi_A^{-1}(f\phi_A(a))$. Then*

(a) ϕ_A is $\mathbb{F}$-linear.

(b) fa is the unique element in A with $[t, fa] = f[t, a]$ for all $t \in T_A$.

(c) A is an $\mathbb{F}$-space.

(d) $N_G(A)$ acts $\mathbb{F}$-linearly on A.

(e) $A \cong \mathrm{Hom}_{\mathbb{F}}(T_A/R_A, R_A)$ as $\mathbb{F}N_G(A)$-modules.

Proof. (a) This holds by the definition of fa.

(b) Recall $\phi_A(a)(t + R_A) = [t, a]$ so $\phi_A(fa)(t + R_A) = [t, fa]$. Also $f(\phi_A(a)(t + R_A)) = f[t, a]$ and $\phi_A(fa) = f\phi_A(a)$ by definition so $[t, fa] = f[t, a]$.

(c) Note that an $\mathbb{F}$-space is an $\mathbb{F}$-module. Consider the map $\mathbb{F} \times A \to A, (f, a) \to fa$. We have $f(a+\tilde{a}) = \phi_A{}^{-1}(f\phi_A(a+\tilde{a})) = \phi_A{}^{-1}(f\phi_A(a)+f\phi_A(\tilde{a})) = \phi_A{}^{-1}(f\phi_A(a))+\phi_A{}^{-1}(f\phi_A(\tilde{a})) = fa+f\tilde{a}$. Also, $(f\tilde{f})a = \phi_A^{-1}(f\tilde{f}\phi_A(a)) = \phi_A^{-1}(f\phi_A(\phi_A^{-1}(\tilde{f}\phi_A(a)))) = f\phi_A^{-1}(\tilde{f}\phi_A(a)) = f(\tilde{f}a)$.

(d) Let $g \in N_G(A)$. We have $\phi_A((fa)^g) \overset{2.5(a)}{=} (\phi_A(fa))^g \overset{2.7(a)}{=} (f\phi_A(a))^g = f(\phi_A(a))^g \overset{2.5(a)}{=} f\phi_A(a^g) \overset{2.7(a)}{=} \phi_A(fa^g)$. Then we get $(fa)^g = f(a^g)$ since ϕ_A is a bijection.

(e) This now follows from part (d) and 2.5(c). □

Lemma 2.8

(a) Let $U \subseteq T_A$. Then $C_A(U)$ is an $\mathbb{F}$-subspace of A. If U is an $\mathbb{F}$-subspace of T_A with $R_A \leq U$, then $\dim U + \dim C_A(U) = \dim T_A$ and $U = C_{T_A}(C_A(U))$. Also, $C_A(U) = C_A(\mathbb{F}U)$, and for any $\mathbb{F}$-subspace, Y, of T_A, $|A/C_A(Y)| = |Y + R_A/R_A|$.

(b) Let $X \subseteq A$. Then $C_{T_A}(X)$ is an $\mathbb{F}$-subspace of T_A containing R_A. If X is an $\mathbb{F}$-subspace of A, then $\dim X + \dim C_{T_A}(X) = \dim T_A$ and $X = C_A(C_{T_A}(X))$.

(c) Let $K \leq C_G(R_A)$. Then $C_A([T_A, K]) = C_A(K)$.

(d) Let $K \leq C_G(R_A)$. Then $[T_A, K] + R_A = C_{T_A}(C_A(K))$.

(e) Let $K \leq C_G(R_A)$. Then $C_{T_A}([A, K])/R_A = C_{T_A/R_A}(K)$.

Proof. (a) Since A is an $\mathbb{F}$-space from 2.7, $C_A(U)$ is an $\mathbb{F}$-subspace of A. If U is an $\mathbb{F}$-subspace of T_A with $R_A \leq U$, then 2.6 gives $A/C_A(U) \cong ((U + R_A)/R_A)^*$. Since $\dim A \overset{2.7(e)}{=} \dim T_A/R_A$, we have $\dim T_A/R_A - \dim C_A(U) = \dim((U + R_A)/R_A)$ so we get $\dim U + \dim C_A(U) = \dim T_A$. Also, $C_A(U) \leq C_A(C_{T_A}(C_A(U))) \leq C_A(U)$ and $\dim U = \dim T_A - \dim C_A(U) = \dim C_{T_A}(C_A(U))$ so we have $U = C_{T_A}(C_A(U))$. $C_{T_A}(C_A(U))$ is an $\mathbb{F}$-subspace so $\mathbb{F}U \subseteq C_{T_A}(C_A(U)) = U$. Hence, $C_A(U) \subseteq C_A(\mathbb{F}U) \subseteq C_A(U)$.

Let Y be an $\mathbb{F}$-subspace of T_A and put $U = Y + R_A$. Then $\dim U + \dim C_A(U) = \dim T_A$ gives us that $|U||C_A(U)| = |T_A| = |T_A/R_A||R_A| = |A||R_A|$. Hence, $|Y + R_A/R_A| = |U/R_A| = |A/C_A(U)| = |A/C_A(Y + R_A)| = |A/C_A(Y)|$.

(b) Define $\rho : T_A \to \mathrm{Hom}_{\mathbb{F}}(X, R_A)$ by $\rho(t)(x) = [t, x]$. Then $\mathrm{Ker}\ \rho = C_{T_A}(X)$ and $\mathrm{Im}\ \rho \leq X^*$. Hence, $\dim T_A/C_{T_A}(X) \leq \dim X^*$. By (a) applied with $U = C_{T_A}(X)$, $\dim X^* = \dim X \leq \dim C_A(C_{T_A}(X)) = \dim T_A - \dim C_{T_A}(X) = \dim T_A/C_{T_A}(X) \leq \dim X^*$. Thus, $\dim X + \dim C_{T_A}(X) = \dim T_A$ and since $X \leq C_A(C_{T_A}(X))$, $X = C_A(C_{T_A}(X))$.

(c) Let $X \le A$. Since $[T_A, X, K] \le [R_A, K] = 0$, the Three Subgroups Lemma shows that $[X, K, T_A] = 0$ if and only if $[K, T_A, X] = 0$. Since A acts faithfully on T_A, we conclude that $[X, K] = 0$ if and only if $[K, T_A, X] = 0$.

(d) We have $C_A(K) \overset{(c)}{=} C_A([T_A, K])$ so $C_{T_A}(C_A(K)) = C_{T_A}(C_A([T_A, K])) = [T_A, K] + R_A$.

(e) $v + R_A \in C_{T_A/R_A}(K)$ if and only if $[v, K] \in R_A$ if and only if $[v, K, A] = 0$ by 2.3(a). Since $K \in C_G(R_A)$, $[v, A, K] = 0$ so the Three Subgroups Lemma gives $[v, K, A] = 0$ if and only if $[A, K, v] = 0$ which holds if and only if $v \in C_{T_A}([A, K])$. Then $C_{T_A}([A, K])/R_A = C_{T_A/R_A}(K)$. □

Lemma 2.9 *Let $v \in V$ and define $s = s_v : A \times A \to R_A, (a, b) \to [v, a, b]$. Then s is symplectic and $\mathbb{F}$-bilinear. (Here symplectic means $s(a, a) = 0$ for all $a \in A$, and we will also see that s is alternating; that is $s(a, b) = -s(b, a)$, for all $a, b \in A$).*

Proof. By definition, $s(a, a) = [v, a, a] = 0$ which means s is symplectic. Also s is alternating since $s(a, b) = [v, a, b] = -[v, b, a] = -s(b, a)$ by the Three Subgroups Lemma [Gor, 2.2.3]. By 2.7(b), s is $\mathbb{F}$-linear in the second coordinate. Since s is alternating, it is also $\mathbb{F}$-linear in the first coordinate. □

Lemma 2.10 *Let $v \in V$ and let s be as defined in 2.9. Let U be an $\mathbb{F}$-subspace of T_A with $R_A \le U$ and put $D = C_A(U)$. Then*

(a) $\{a \in A \mid [v, a] \in U\} = D^{\perp} = C_A([v, D]) = C_A([v, D] + R_A)$ is an $\mathbb{F}$-subspace of A.

(b) $[v, D] + U$ is an $\mathbb{F}$-subspace of T_A.

(c) $[v, D] + U = C_{T_A}(C_D([v, D]))$.

(d) $\{a \in A \mid [v, a] \in R_A\} = \operatorname{rad} s_v = C_A([v, A]) = C_A([v, A] + R_A)$ is an $\mathbb{F}$-subspace of A.

(e) $([v, A] + R_A)/R_A$ is an $\mathbb{F}$-subspace of T_A/R_A.

(f) $[v, A] + R_A = C_{T_A}(C_A([v, A]))$.

Proof. (a) Let $a \in A$.

Claim: The following are equivalent.

1. $[v, a] \in U$.

2. $[v, a, C_A(U)] = 0$.

3. $s_v(a, b) = 0$ for all $b \in C_A(U)$.

4. $a \in C_A(U)^{\perp}$.

5. $s_v(b, a) = 0$ for all $b \in C_A(U)$.

6. $[v, C_A(U), a] = 0$.

Proof of Claim. $1 \iff 2$ since $U = C_{T_A}(C_A(U))$ from 2.8; $2 \Rightarrow 3$ by definition; $3 \Rightarrow 4$ again by definition; $4 \Rightarrow 5$ since s is alternating; $5 \Rightarrow 6$ by definition; $6 \Rightarrow 2$ by the Three Subgroups Lemma and since A is abelian. Then the claim is proven. $\boxdot$

Now $\{a \in A \mid [v, a] \in U\} = D^{\perp}$ by $1 \iff 4$ of the claim. And $D^{\perp} = C_A([v, D])$ by $4 \iff 6$ of the claim. Also, $C_A([v, D]) = C_A([v, D] + R_A)$ since $C_A(R_A) \overset{2.3}{=} A$. So $\{a \in A \mid [v, a] \in U\} = D^{\perp} = C_A([v, D]) = C_A([v, D] + R_A)$ is an $\mathbb{F}$-subspace of A from 2.7. Then (a) holds.

(b) $[V/U, A, A] = [T_A/U, A] = R_A/U = 0$, so A acts quadratically on V/U. Let $\alpha : D \longrightarrow ([v, D] + U)/U, a \longrightarrow [v, a] + U$. α is a homomorphism by quadratic action; it's onto and its kernel is $\{d \in D \mid [v, d] \in U\}$ so

$$|([v, D] + U)/U| = |D/\{d \in D \mid [v, d] \in U\}| \overset{(a)}{=} |D/C_D([v, D] + R_A)|$$

$$\overset{2.8(a)}{=} |D/C_D(\mathbb{F}[v, D] + R_A)|.$$

By the last statement in 2.8(a) applied with $Y = \mathbb{F}[v, D] + U$ we get $|A/C_A(\mathbb{F}[v, D] + U)| = |(\mathbb{F}[v, D] + U)/R_A|$. Also, $C_A(U) = D$ implies $C_A(\mathbb{F}[v, D] + U) = C_A(\mathbb{F}[v, D]) \cap$

$C_A(U) = C_D(\mathbb{F}[v, D]) = C_D(\mathbb{F}[v, D] + R_A)$. Then

$$|A/D||D/C_D(\mathbb{F}[v, D] + R_A)| = |A/C_D(\mathbb{F}[v, D] + R_A)| = |A/C_A(\mathbb{F}[v, D] + U)|$$
$$= |(\mathbb{F}[v, D] + U)/R_A| = |(\mathbb{F}[v, D] + U)/U||U/R_A|.$$

Since $|A/C_A(U)| = |U/R_A|$, we must have $|D/C_D(\mathbb{F}[v, D]+R_A)| = |(\mathbb{F}[v, D]+U)/U|$. Hence,

$$|([v, D] + U)/U| = |(\mathbb{F}[v, D] + U)/U|$$

and (b) holds.

(c) Since $[v, D] + U$ is an $\mathbb{F}$-subspace of T_A from (b), 2.8(a) gives $[v, D] + U = C_{T_A}(C_A([v, D]+U))$. Since $C_A([v, D]+U) = C_D([v, D])$, $[v, D]+U = C_{T_A}(C_D([v, D]))$.

If $U = R_A$, then $D = A$ and (d), (e), and (f) follow from (a), (b), and (c) respectively. □

Lemma 2.11 *Let $v \in V$ and let X be an $\mathbb{F}$-space of $[v, A] + R_A$ with $[v, A] + R_A = X \oplus R_A$. Define $q := q_{v,X} : A \to R_A$ so that $q(a)$ is the unique element of R_A such that $[v, a] - q(a) \in X$. Then for all $a, b \in A$,*

$$q(ab) = q(a) + s_v(a, b) + q(b).$$

Proof. Let $x(a) = [v, a] - q(a)$. $[v, a] = v^a - v$ so $v^a = v + x(a) + q(a)$ with $x(a) \in X$ and $q(a) \in R_A$. Also for any $b \in A$, $[x(a), b] = [v, a, b] - [q(a), b] = [v, a, b] = s_v(a, b)$ since $q(a) \in R_A$ implies $q(a)^b = q(a)$. Then $x(a)^b = x(a) + s_v(a, b)$. Hence,

$$\begin{aligned} v^{ab} &= (v^a)^b = (v + x(a) + q(a))^b \\ &= v^b + x(a)^b + q(a)^b = v + x(b) + q(b) + x(a) + s_v(a, b) + q(a) \\ &= v + (x(a) + x(b)) + (q(a) + s_v(a, b) + q(b)). \end{aligned}$$

It follows that $x(ab) = x(a) + x(b)$ and $q(ab) = q(a) + s_v(a, b) + q(b)$. □

We remark that q from the previous lemma does not need to be a quadratic form. In particular, we do not know whether $q(fa) = f^2 q(a)$ for all $f \in \mathbb{F}$ and $a \in A$.

Lemma 2.12 *Let $1 \neq a \in A$ and put $A_a = C_A([V, a])$. Then*

(a) $A_a = \mathbb{F}a$.

(b) $[V, A_a] = [V, a] = C_{T_A}(a)$ is a hyperplane of T_A and $R_A \leq [V, A_a]$.

(c) A_a is quadratic on V.

(d) $A_a \in Syl_p(C_G([V, a]))$.

(e) $A_a \trianglelefteq N_G([V, a])$.

(f) $C_G([V, a])/A_a$ is a p'-group.

Proof. From 2.3(b) we have $R_A \leq [V, a]$ and by 2.5(b), $T_A \neq C_{T_A}(a)$. Thus

$$(*) \qquad R_A \leq [V, a] \leq C_{T_A}(a) < T_A$$

since $[V, a, a] = 0$.

Observe that $[V, a]$ is an $\mathbb{F}$-subspace of V. Thus, $|A_a| = |C_A([V, a])| \overset{2.8(a)}{=} |T_A/[V, a]| = |\mathbb{F}|^n$, where n is a positive integer, as $T_A/[V, a]$ is an $\mathbb{F}$-space. Also, $[V, a, A_a] = 0$ so $|A_a| \leq |\mathbb{F}|$ by 1.1(iii). Hence, $|A_a| = |\mathbb{F}|$ and $A_a = \mathbb{F}a$. Then (a) is proven.

We have just shown $|\mathbb{F}| = |T_A/[V, a]|$ and thus $[V, a]$ is a hyperplane of T_A. $[V, a] \leq C_{T_A}(a) < T_A$ and since $[V, a]$ is a hyperplane of T_A, $[V, a] = C_{T_A}(a)$. Let $1 \neq b \in A_a$. Then $[V, a] \leq C_{T_A}(b) = [V, b]$ and again, since $[V, a]$ is already a hyperplane of T_A, we conclude that $[V, a] = [V, b] = [V, A_a]$. Since $R_A \leq [V, a]$ for any $1 \neq a \in A$, $R_A \leq [V, A_a]$. So (b) is proven.

Then $[V, A_a, A_a] = [V, a, A_a] = [V, A_a, a] = [V, a, a] = 0$ from (b). So A_a is quadratic on V and (c) is proven.

Let P be any p-subgroup of G with $A_a \le P$. Since $[V, a, A_a] = 0$, $A_a \le C_P([V, a])$. If $A_a \ne C_P([V, a])$, then $|A_a| < |C_P([V, a])| \le |\mathbb{F}|$ from 1.1(iii) which contradicts (a). So $C_P([V, a]) = A_a$ for any p-subgroup, P, of G with $A_a \le P$. Thus, $A_a \in Syl_p(C_G([V, a]))$ and (d) is proven.

We have

$$[V, \langle A_a^{N_G([V,a])} \rangle] = \langle [V, A_a]^{N_G([V,a])} \rangle = \langle [V, a]^{N_G([V,a])} \rangle = [V, a]$$

by (b). $[V, a, A_a] = 0$ so $[V, a, A_a^{N_G([V,a])}] = 0$ which means $[V, \langle A_a^{N_G([V,a])} \rangle] = [V, a] \le C_V(\langle A_a^{N_G([V,a])} \rangle)$. Therefore, $\langle A_a^{N_G([V,a])} \rangle$ is quadratic on V. Hence, $\langle A_a^{N_G([V,a])} \rangle$ is a p-group so (d) implies that $A_a = \langle A_a^{N_G([V,a])} \rangle$. Thus $A_a \trianglelefteq N_G([V, a])$ and (e) is proven.

$A_a \trianglelefteq N_G([V, a])$ and $A_a \in Syl_p(C_G([V, a]))$ imply that $C_G([V, a])/A_a$ is a p'-group and (f) is proven. □

Lemma 2.13 *$C_G(T_A)$ is a p'-group.*

Proof. Let P be a p-subgroup of $C_G(T_A)$ and let $1 \ne a \in A$. Then $P \le C_G(T_A) \le C_G([V, a])$. By 2.12(f), $C_G([V, a])/A_a$ is a p'-group. So $P \le A_a \le A$ which implies $P \le C_A(T_A)$ making $P = 1$ by 2.5. □

Lemma 2.14 *Let P be a p-subgroup of G with $A \le P \le N_G(T_A) \cap C_G(R_A)$. Then $C_P(T_A/R_A) = A$ and $C_G(V/T_A) \cap C_G(T_A/R_A) \cap C_G(R_A) = A$.*

Proof. Extend ϕ_A to $\Phi : C_P(T_A/R_A) \to \mathrm{Hom}(T_A/R_A, R_A), s \to (t + R_A \to [t, s])$ so that $\Phi|_A = \phi_A$. Since ϕ_A is onto, $C_P(T_A/R_A) = (\mathrm{Ker}\ \Phi)A$. This gives $C_P(T_A/R_A) = C_{C_P(T_A/R_A)}(T_A)A$. $C_G(T_A)$ is a p'-group from 2.13 so $C_{C_P(T_A/R_A)}(T_A) = 1$. Hence, $C_P(T_A/R_A) = A$.

We have $C_G(V/T_A) \cap C_G(T_A/R_A) \cap C_G(R_A)$ is a p-group by A.1, so using what we just proved,

$$C_G(V/T_A) \cap C_G(T_A/R_A) \cap C_G(R_A) = C_{C_G(V/T_A) \cap C_G(T_A/R_A) \cap C_G(R_A)}(T_A/R_A) = A.$$

$\square$

Lemma 2.15 *Let $A, B \in \mathcal{A}$ with $T_A = T_B$ and $R_A = R_B$. Then $A = B$.*

Proof. $A = C_G(V/T_A) \cap C_G(T_A/R_A) \cap C_G(R_A) = C_G(V/T_B) \cap C_G(T_B/R_B) \cap C_G(R_B) = B$ from 2.14. $\square$

Lemma 2.16 *$C_G(A)/A$ is a p'-group.*

Proof. Let P be a p-subgroup of $C_G(A)$ with $A \leq P \leq C_G(A)$. Now P centralizes A and so it also normalizes A. P normalizes V as V is an $\mathbb{F}G$-module. Then P normalizes $[V, A] = T_A$ and $[T_A, A] = R_A$. Since R_A is 1-dimensional and $C_{R_A}(P) \neq 0$ as both are p-groups, we have $C_{R_A}(P) = R_A$. Hence, P centralizes R_A. Therefore,

$$P \leq C_G(A) \cap C_G(R_A) \leq C_G(\mathrm{Hom}(T_A/R_A, R_A)) \cap C_G(R_A) \leq C_G(T_A/R_A)$$

from 2.7(e). Then $P = C_P(T_A/R_A) = A$ by 2.14 and $C_G(A)/A$ is a p'-group. $\square$

Definition 2.17 *Let H be a group and let $X \leq Y \leq H$. We say that X is* weakly closed *in Y with respect to H if for all $h \in H$ with $X^h \leq Y$, we have $X^h = X$. That is, if X is the only H-conjugate of X contained in Y. We say that X is a weakly closed subgroup of H if there exists a Sylow p-subgroup, P, of H such that X is weakly closed in P with respect to H.*

Lemma 2.18 *Let R be a p-subgroup of a finite group H. Then the following are equivalent:*

(a) R is a weakly closed subgroup of H.

(b) Any p-subgroup of H contains at most one conjugate of R.

(c) Let $h \in H$ with $[R, R^h] \leq R \cap R^h$. Then $R = R^h$.

(d) If $R \leq S \trianglelefteq X \leq H$ and S is a p-group, then $R \trianglelefteq X$.

Proof. ($a \Rightarrow b$) Every p-subgroup of H is contained in a Sylow p-subgroup of H which contains only one conjugate of R by the definition of weakly closed.

($b \Rightarrow c$) Since $[R, R^h] \leq R \cap R^h$, we have $R \leq N_G(R^h)$ and $R^h \leq N_G(R)$ so RR^h is a subgroup, in fact it's a p-group. We have $R \leq RR^h$ and $R^h \leq RR^h$. Then (b) gives $R = R^h$.

($c \Rightarrow d$) Let $x \in X$ and $R \leq S \trianglelefteq X \leq H$. If $R \trianglelefteq S$, then conjugating by x we get $R^x \trianglelefteq S$ since $S \trianglelefteq X$. So we have $[R, R^x] \leq R \cap R^x$. Then $R = R^x$ so $R \trianglelefteq X$. So we see that $R \trianglelefteq S \trianglelefteq X \Longrightarrow R \trianglelefteq X$. Consider $R \trianglelefteq N_S(R) \trianglelefteq N_S(N_S(R))$. Then $R \trianglelefteq N_S(N_S(R)) \leq N_S(R)$ which gives $N_S(R) = N_S(N_S(R)) = S$ and $R \trianglelefteq S$.

($d \Rightarrow a$) Let $S = X \in Syl_p(H)$ so $R \trianglelefteq S$. Thus, $S \in Syl_p(N_H(R))$. Assume $R^h \leq S$ for some $h \in H$. Conjugating $S \in Syl_p(N_H(R))$ by h we get, $S^h \in Syl_p(N_H(R^h))$. Also, $R \trianglelefteq S^{h^{-1}}$ by (d) so $R^h \trianglelefteq S$. Then $S \in Syl_p(N_H(R^h))$. Hence, $S = S^{ht}$ for some $t \in N_G(R^h)$. Then we have $R \leq S \trianglelefteq S\langle ht \rangle \leq H$ again from (d) so $R = R^{ht} = R^h$. $\square$

Lemma 2.19 *A is a weakly closed subgroup of G.*

Proof. Otherwise 2.18 implies that there exists $B \in \mathcal{A}$ with $[A, B] \leq A \cap B$ and $A \neq B$. By 2.16, $C_G(A)/A$ is a p'-group so $B \not\leq C_G(A)$. Hence, $[A, B] \neq 1$. Then $[V, [A, B]] \neq 0$. Since B normalizes A, B normalizes T_A and $T_A \cap T_B$. Similarly, A normalizes $T_A \cap T_B$.

As U and AB are p-groups, $C_U(AB) \neq 0$. So we have $0 \neq C_U(AB) \leq C_{T_A}(A) = R_A$ by 2.3. By symmetry, $C_U(AB) \leq R_B$ and since R_A is 1-dimensional, $C_U(AB) = R_A = R_B$.

Now if $T_A = T_B$, 2.15 gives a contradiction to $A \neq B$, so $T_A \neq T_B$. Let $1 \neq a \in A \cap B$. 2.12(b) states that $[V, a]$ is a hyperplane of T_A. Then $[V, a] \leq [V, A \cap B] \leq$

$U < T_A$ since $T_A \neq T_B$. We see that $[V, a] = [V, A \cap B] = U$ is a hyperplane of T_A and similarly of T_B. So $U = [V, a] \leq C_V(a)$ for all $a \in A \cap B$; hence $U \leq C_V(A \cap B)$. Then $A \cap B \leq C_A(U) = A_a$ and

$$A_a = C_A([V, a]) \leq C_{AB}([V, a]) \leq C_G([V, a]).$$

By 2.12(d), $C_G([V, a])/A_a$ is a p'-group so $C_{AB}([V, a]) = A_a$ and similarly $C_{AB}([V, a]) = B_a$.

Now $A \cap B \leq A_a = B_a \leq A \cap B$ and we have

$$A \cap B = A_a = C_{AB}([V, a]) = C_{AB}(U)$$

which has order $|\mathbb{F}|$ from 2.12(a). Also, $[U, AB] \leq [T_A, A][T_B, B] = R_A R_B = R_A = R_B$.

Define $\tau : AB \to Hom(U/R_A, R_A), l \to (u + R_A \to [u, l])$. Restricted to A, τ is onto from 1.1 so $AB = A(\mathrm{Ker}\ (\tau))$. We get $AB = A(C_{AB}(U)) = A(A \cap B) = A$ which contradicts $A \neq B$. $\square$

Lemma 2.20 *If $H \leq G$, then H acts transitively on $\mathcal{A}(H)$ and $\mathcal{A}(H) = C^H$ for any $C \in \mathcal{A}(H)$.*

Proof. Let $A \in \mathcal{A}(H)$. For any $C \in \mathcal{A}(H)$, $\langle A, C^h \rangle$ is a 2-group for some $h \in H$ by Sylow's theorem. Since A is a weakly closed subgroup of G by 2.19, 2.18(b) gives $A = C^g$. Then H acts transitively on $\mathcal{A}(H)$ and $\mathcal{A}(H) = C^H$. $\square$

Notation: From now on let $A \neq B \in \mathcal{A}$ and put $L := \langle A, B \rangle$, $R := O_p(L)$, $E := (A \cap R)(B \cap R)$, $Z := A \cap B$, $U := T_A \cap T_B$, and $W := R_A + (T_A \cap T_B) + R_B$.

Lemma 2.21 *Let $\tilde{L}$ be a finite group and $\tilde{A}$ be an elementary abelian weakly closed subgroup of $\tilde{L}$ where $\tilde{B} \in \tilde{A}^{\tilde{L}}$ and $\tilde{L} = \langle \tilde{A}, \tilde{B} \rangle$. Let $\tilde{R} = O_p(\tilde{L})$, $\tilde{E} = (\tilde{A} \cap O_p(\tilde{L}))(\tilde{B} \cap O_p(\tilde{L}))$, and $\tilde{C} := C_{\tilde{L}}(\tilde{A}^{\tilde{L}})$. Then* *(a)* $[\tilde{R}, \tilde{L}] \leq \tilde{E} \trianglelefteq \tilde{L}$.

(b) $\tilde{E}' \leq \tilde{A} \cap \tilde{B} \leq Z(\tilde{L}) \cap \tilde{E}$.

(c) $O_p(\tilde{L}) \leq \tilde{C}$.

(d) $\tilde{A} \cap \tilde{C} = \tilde{A} \cap \tilde{E}$.

(e) $\tilde{C} \trianglelefteq \tilde{L}$ *and* $[\tilde{L}, \tilde{C}] \leq \tilde{E} \leq O_p(\tilde{L})$.

(f) $\tilde{C}/O_p(\tilde{L})$ *is an abelian* p'*-group.*

(g) Moreover, if $\tilde{L}/\tilde{C} \cong SL_2(q), Sz(q)$ *with* q *a prime power of* p *larger than 2, or* $\tilde{L}/\tilde{C} \cong D_{2r}$ *with* r *an odd prime, then* $\tilde{C} = \tilde{E} = O_p(\tilde{L})$.

Proof. (a) Notice that $\tilde{A}\tilde{R}$ is a p-group. Since $\tilde{A}$ is a weakly closed subgroup of $\tilde{L}$, 2.18 shows that $\tilde{R}$ normalizes $\tilde{A}$. Similarly, $\tilde{R}$ normalizes $\tilde{B}$. Thus, $[\tilde{R}, \tilde{A}] \leq \tilde{A} \cap \tilde{R} \leq \tilde{E} \leq \tilde{R}$ and $[\tilde{R}, \tilde{B}] \leq \tilde{B} \cap \tilde{R} \leq \tilde{E} \leq \tilde{R}$. So $[\tilde{R}, \tilde{L}] \leq \tilde{E}$. Since $\tilde{E} \leq \tilde{R}$, $[\tilde{E}, \tilde{L}] \leq [\tilde{R}, \tilde{L}] \leq \tilde{E}$ and then $[\tilde{R}, \tilde{L}] \leq \tilde{E} \trianglelefteq \tilde{L}$.

(b) $\tilde{E}' = [\tilde{A} \cap \tilde{R}, \tilde{B} \cap \tilde{R}] \leq [\tilde{A}, \tilde{R}] \cap [\tilde{R}, \tilde{B}] \leq \tilde{A} \cap \tilde{B}$. Since $\tilde{A} \cap \tilde{B} \leq Z(\tilde{L})$, $\tilde{A} \cap \tilde{B} \leq \tilde{R}$ so $\tilde{A} \cap \tilde{B} \leq Z(\tilde{L}) \cap \tilde{E}$.

(c) Since $O_p(\tilde{L})\tilde{A}$ is a p-group and $\tilde{A}$ is a weakly closed subgroup of $\tilde{L}$, $\tilde{A} \trianglelefteq O_p(\tilde{L})\tilde{A}$ by 2.18(d) as $\tilde{A} \leq O_p(\tilde{L})\tilde{A} \trianglelefteq O_p(\tilde{L})\tilde{A} \leq G$. $\tilde{A}$ is normal in any p-subgroup so $O_p(\tilde{L}) \leq N_G(\tilde{A})$. Thus, $O_p(\tilde{L}) \leq \tilde{C}$.

(d) If $J \in \mathcal{A}(\tilde{L})$, then $\tilde{A} \cap \tilde{C}$ normalizes $J \cap \tilde{C}$ by definition of $\tilde{C}$. Let $\hat{E} := \langle D \cap \tilde{C} \mid D \in \mathcal{A}(\tilde{L}) \rangle$. $\hat{E}$ is a p-group since each D is a conjugate of $\tilde{A}$. Thus $\hat{E} \leq O_p(\tilde{L})$, yielding

$$\tilde{A} \cap O_p(\tilde{L}) \overset{(c)}{\leq} \tilde{A} \cap \tilde{C} \leq \tilde{A} \cap \hat{E} \leq \tilde{A} \cap O_p(\tilde{L}).$$

So $\tilde{A} \cap O_p(\tilde{L}) = \tilde{A} \cap \tilde{C}$. This, along with the definition of $\tilde{E}$, gives us $\tilde{A} \cap \tilde{E} \leq \tilde{A} \cap O_p(\tilde{L}) \leq \tilde{E}$. Hence, $\tilde{A} \cap \tilde{C} = \tilde{A} \cap O_p(\tilde{L}) = \tilde{A} \cap \tilde{E}$.

(e) Notice that $\tilde{C} \trianglelefteq \tilde{L}$ and $\tilde{A} \trianglelefteq \tilde{C}$ by definition. We have $[\tilde{A}, \tilde{C}] \le \tilde{A} \cap \tilde{C} \le \tilde{E}$ and similarly, $[\tilde{B}, \tilde{C}] \le \tilde{B} \cap \tilde{C} \le \tilde{E}$. Thus, $[\tilde{L}, \tilde{C}] \le \tilde{E} \le O_p(\tilde{L})$.

(f) As $[\tilde{L}, \tilde{C}] \le \tilde{E} \le O_p(\tilde{L})$, we have $\tilde{C}/O_p(\tilde{L}) \le Z(\tilde{L}/O_p(\tilde{L}))$. Since $O_p(\tilde{L}/O_p(\tilde{L})) = 1$, $\tilde{C}/O_p(\tilde{L})$ is an abelian p'-group.

(g) In addition, suppose now that $\tilde{L}/\tilde{C} \cong SL_2(q), Sz(q)$ or D_{2r}. Let $\tilde{A} \le S \in Syl_p(\tilde{L})$ and let $\overline{L} = \tilde{L}/\tilde{C}$. For notational simplicity let $\tilde{S} = S$. Define $\overline{M} = N_{\overline{L}}(\overline{S})$ and let $\tilde{M} := M$ be the inverse image of $\overline{M}$ in $\tilde{C}$. Since $[\tilde{C}, \tilde{S}] \overset{(e)}{\le} O_p(\tilde{L})$, $\tilde{S}$ is normal in $\tilde{C}\tilde{S}$ so $\tilde{S}$ is the unique Sylow p-subgroup of $\tilde{C}\tilde{S}$. As $\overline{S} \trianglelefteq \overline{M}$ since $\tilde{C}\tilde{S} \trianglelefteq \tilde{M}$, $\tilde{S}$ is the unique Sylow p-subgroup of $\tilde{M}$. Hence, $\tilde{S} = O_p(\tilde{M}) \trianglelefteq \tilde{M}$. Since $\tilde{A}$ is a weakly closed subgroup of $\tilde{L}$, $\tilde{A} \trianglelefteq \tilde{M}$.

Consider the $\tilde{L}/\tilde{C} \cong SL_2(q)$ case. We may assume $\overline{S} := \{\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) \mid s \in \mathbb{F}\}$ and is a Sylow p-subgroup of $SL_2(q)$. Let $\overline{A} := \tilde{A}/\tilde{C} \le \overline{S} \in Syl_p(\overline{L})$ and $N_{SL_2(q)}(\overline{S})$ correspond to $\left(\begin{smallmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{smallmatrix}\right)$. We have $\left(\begin{smallmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ * & \lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \lambda^{-2}\alpha & 1 \end{smallmatrix}\right)$. So $N_{SL_2(q)}(\overline{S})$ acts irreducibly on $\overline{S}$. Since $\tilde{A} \trianglelefteq \tilde{M}$, $\overline{A} = \overline{S}$.

For $\tilde{L}/\tilde{C} \cong Sz(q)$, it follows easily from [Huppert, Chapter XI Section 3] that $Z(\overline{S}) = \Omega_1(\overline{S})$, $\overline{M}$ acts irreducibly on $Z(\overline{S})$, and $\overline{M}$ acts irreducibly on $\overline{S}/Z(\overline{S})$. Since $\tilde{A}$ is elementary abelian, $\tilde{A} \le \Omega_1(\overline{S}) = Z(\overline{S})$. Since $\tilde{A} \trianglelefteq \tilde{M}$ and $\tilde{M}$ is irreducible on $Z(\overline{S})$, we get $\overline{A} = Z(\overline{S})$.

For $\tilde{L}/\tilde{C} \cong D_{2r}$ with r an odd prime, $|\overline{A}| = 2$. Since r is prime, $\tilde{A}$ is a maximal subgroup, so $\overline{A} = \overline{M}$.

Define $\hat{L} = \tilde{L}/\tilde{E}$ and for $\tilde{X} \le \tilde{L}$ let $\hat{X} = \tilde{X}\tilde{E}/\tilde{E}$. From $[\tilde{L}, \tilde{C}] \overset{(e)}{\le} \tilde{E}$ we have $[\hat{L}, \hat{C}] = 1$, so $\hat{C} \le Z(\hat{L})$. Also, $\tilde{A} \cap \tilde{C} \le \tilde{E}$ from (d), so $\hat{A} \cap \hat{C} = 1$.

Case 1: Suppose that $\tilde{L}/\tilde{C} \cong D_{2r}$. Then $|\overline{B}| = 2 = |\overline{A}| = |\tilde{A}/(\tilde{A} \cap \tilde{C})| \overset{(d)}{=} |\tilde{A}/(\tilde{A} \cap \tilde{E})| = |\tilde{A}\tilde{E}/\tilde{E}| = |\hat{A}|$. Hence, $\hat{L} = \langle \hat{A}, \hat{B} \rangle$ is also a dihedral group of order $2s$ for some s.

Since $\tilde{A}$ and $\tilde{B}$ are weakly closed subgroups of $\tilde{L}$, they are conjugate in $\tilde{L}$. If s is even, then $\langle a, b\rangle / \langle (ab)^2 \rangle \cong D_4 = C_2 \times C_2$ which contradicts that $\tilde{A}$ and $\tilde{B}$ are conjugate. So it follows that $s = \frac{|\hat{L}|}{2}$ is odd. Then $\hat{C} \leq Z(\hat{L}) = 1$ so $\hat{C} = 1$ and $\tilde{C} = \tilde{E}$. Also, $\tilde{E} \overset{(e)}{\leq} O_p(\tilde{L}) \overset{(c)}{\leq} \tilde{C}$ so $\tilde{C} = \tilde{E} = \tilde{R}$ and we are done in this case.

Case 2: Suppose now that $\tilde{L}/\tilde{C} \cong SL_2(q)$ or $Sz(q)$ with $q > 2$. $\tilde{M}$ acts irreducibly on $\overline{A}$ so $\tilde{A} \leq [\tilde{A}, \tilde{M}]\tilde{C}$. Hence, $\tilde{A} = [\tilde{A}, \tilde{M}](\tilde{A} \cap \tilde{C}) \overset{(d)}{=} [\tilde{A}, \tilde{M}](\tilde{A} \cap \tilde{E})$. So $\hat{A} = [\hat{A}, \tilde{M}] \leq \hat{L}$. Similarly, $\hat{B} \leq \hat{L}$. Thus, $\hat{L} = \hat{L}'$.

[Griess, Table 1] gives the orders of the Schur multipliers. Since the Schur multipliers of $SL_2(q)$ and $Sz(q)$ are p-groups, we conclude that $\hat{C}$ is a p-group and so $\tilde{C} = \tilde{R}$.

Claim: There exists a complement, $\hat{P}$, to $\hat{R}$ in $\hat{S}$.

Proof of Claim. Consider $\tilde{L}/\tilde{R} = \tilde{L}/\tilde{C} \cong SL_2(q)$. We have $\tilde{R} \leq Syl_p(\tilde{L})$. Also $\tilde{S}\tilde{C} = \tilde{A}\tilde{C}$ since (as proved above) $\overline{A} = \overline{S}$, so $\tilde{A}\tilde{R} = \tilde{S}$. Let $\hat{P} = \hat{A}$.

$$\hat{A} \cap \hat{R} = (\tilde{A}/\tilde{E}) \cap (\tilde{R}/\tilde{E}) = 1.$$

So $\hat{P} = \hat{A}$ is a complement to $\hat{R}$ in $\hat{S}$.

Now consider $\tilde{L}/\tilde{R} \cong Sz(q)$. Then $\tilde{M}$ acts irreducibly on $\tilde{S}/\tilde{A}\tilde{C}$ and $|\tilde{S}/\tilde{A}\tilde{C}| = q$. $\hat{A}\hat{C}/\hat{A} \leq Z(\hat{S}/\hat{A})$ and $[\hat{C}, \hat{L}] = 1$. Also, $\hat{S}/\hat{A}\hat{C}$ is elementary abelian so $\hat{S}' \leq \hat{A}\hat{C}$. Then $t^* : \hat{S}/\hat{A}\hat{C} \times \hat{S}/\hat{A}\hat{C} \to \hat{A}\hat{C}/\hat{A}, (\hat{a}\hat{T}, \hat{b}\hat{T}) \to [\hat{a}, \hat{b}]\hat{A}$ is a symplectic map.

Assume $\hat{S}/\hat{A}$ is not abelian so $(\hat{S}/\hat{A})' \neq 1$. Choose $H \leq (\hat{S}/\hat{A})'$ with $|(\hat{S}/\hat{A})'/H| = 2$. $t : \hat{S}/\hat{A}\hat{C} \times \hat{S}/\hat{A}\hat{C} \to (\hat{S}/\hat{A})'/H$ is a symplectic form over $\mathbb{F}_2$. If $\operatorname{rad} t = \hat{S}/\hat{A}\hat{C}$, then $\hat{S}/\hat{A}$ is abelian. If $\operatorname{rad} t = 1$, then t is non-degenerate and $\tilde{M}$-invariant on $\hat{S}/\hat{A}\hat{C}$. So $\hat{S}/\hat{A}\hat{C}$ has even dimension over $\mathbb{F}_2$. This is a contradiction since q is an odd power of 2 for the Suzuki groups, so we conclude that $\hat{S}/\hat{A}$ is abelian.

Let $P = [\tilde{S}, \tilde{M}]$. Since $\tilde{M}/\tilde{S}$ is a p'-group, we get $\hat{S}/\hat{A} = C_{\hat{S}/\hat{A}}(\tilde{M}) \times [\hat{S}/\hat{A}, \tilde{M}] = C_{\hat{S}/\hat{A}}(\tilde{M}) \times \hat{P}/\hat{A}$ from [Gor, 5.2.2.3]. Since $\tilde{M}$ acts irreducibly on $Z(\overline{S})$ and on $\overline{S}/Z(\overline{S})$, we have $C_{\overline{S}}(\tilde{M}) = 1$. Thus, $C_{\hat{S}}(\tilde{M}) \leq \hat{R}$. Also, $[\tilde{R}, \tilde{L}] \overset{(e)}{\leq} \tilde{E}$ so $\hat{R} \leq C_{\hat{S}}(\tilde{M})$ and $\hat{R} = C_{\hat{S}}(\tilde{M})$. Then $\hat{S} = \hat{R}\hat{P}$ and $\hat{S}/\hat{A} = \hat{A}\hat{R}/\hat{A} \times \hat{P}/\hat{A}$ gives $\hat{R} \cap \hat{P} \leq (\hat{A}\hat{R}) \cap \hat{P} \leq$

$\hat{A}$. So $\hat{R} \cap \hat{P} \leq \hat{R} \cap \hat{A} = 1$ since $\hat{A} = [\hat{A}, \hat{M}] \leq \hat{P}$. This completes the proof of the claim that there exists a complement to $\hat{R}$ in $\hat{S}$. $\boxdot$

Now that the claim is proven, Gaschütz' Theorem [Asch, 10.4] implies that there exists a complement $\hat{K}$ to $\hat{R}$ in $\hat{L}$. Then, $\hat{L} = \hat{R} \times \hat{K}$, $\hat{L} = \hat{L}' = [\hat{L}, \hat{L}] \leq [\hat{R}, \hat{R}][\hat{R}, \hat{K}][\hat{K}, \hat{K}] = [\hat{K}, \hat{K}]$ since $\hat{R} = \hat{C}$ is in the center of L. So $\hat{L} \leq \hat{K} \leq \hat{L}$ and we have $\hat{L} = \hat{K}$ and $\hat{R} = 1$. Thus $\tilde{C}/\tilde{E} = \tilde{R}/\tilde{E} = 1$ so $\tilde{C} = \tilde{E} = \tilde{R}$. $\square$

Lemma 2.22 $T_L = T_A + T_B = [V, L] = [V, O^p(L)] + U$.

Proof. We have

$$T_L = [V, L] = [V, A] + [V, B] = T_A + T_B.$$

Let $Y = [V, O^p(L)]$. We have $A \leq S_A$ and $B \leq S_B$ where S_A and S_B are in $Syl_p(L)$. So ${S_A}^x = S_B$ for $x \in L = S_A O^p(L)$. Then ${S_A}^l = {S_A}^x$ for $l \in O^p(L)$. So $A^l \leq {S_A}^l = S_B$. However, $B \leq S_B$ so $A^l = B$ since A is a weakly closed subgroup of G from 2.19. Then $T_L = T_A + T_A^l \leq T_A + [T_A, l] \leq T_A + Y$. $T_A \leq T_L$ and $Y \leq [V, L] = T_L$ so $T_L = T_A + Y$. Then A.2 gives $Y = [Y, L] = [Y, A] + [Y, B]$. Thus, $T_L = T_A + [Y, A] + [Y, B]$ and $[Y, A] \leq [V, A] = T_A$ so $T_L = T_A + [Y, B]$. $T_B = T_L \cap T_B$ since $T_B \leq T_L$ so

$$T_B = T_L \cap T_B = (T_A + [Y, B]) \cap T_B = T_A \cap T_B + [Y, B]$$

as $[Y, B] \leq T_B$. Similarly $T_A = [Y, A] + T_A \cap T_B$. Hence,

$$[V, L] = T_L = [Y, A] + (T_A \cap T_B) + [Y, B] = [Y, L] + U = Y + U = [V, O^p(L)] + U. \quad \square$$

Lemma 2.23 $R_A \neq R_B$.

Proof. Assume not and choose B such that $R_A = R_B$ with $A \neq B$ and, in addition, $L = \langle A, B \rangle$ minimal. By 2.19, any Sylow p-subgroup contains only one conjugate of A so L is not a p-group.

By the quadratic L-Lemma [MSS], we have $L/O_p(L) \cong SL_2(q)$, $Sz(q)$ or D_{2r}, where q is a power of p and r is an odd prime. Then by 2.21(e), $[L, C] \leq O_p(L)$ where $C = C_L(A^L)$. So $C_L(A^L)/O_p(L) \leq Z(L/O_p(L)) = 1$ as $L/O_p(L)$ is simple. Hence, $C_L(A^L) = R$.

If $T_A = T_B$, then by 2.15, $A = B$, yielding a contradiction, so $T_A \neq T_B$. Notice that $[V, Z] = [V, A \cap B] \leq U$ and $Z \overset{2.21(d)}{\leq} Z(L)$. Suppose that $Z \neq 1$. Then 2.12(b) implies that $[V, Z]$ contains a hyperplane of T_A and T_B. Since U is also a hyperplane of T_A and T_B, $U = [V, Z]$. Since $[U, L] \leq [T_A, A][T_B, B] = R_A R_B = R_A = R_B$, A.1 says $L/C_L(U)$ is a p-group. So $O^p(L) \leq C_L(U)$ by the definition of $O^p(L)$. Hence,

$$(*) \qquad [U, O^p(L)] = 0.$$

Thus $[V, Z, O^p(L)] = 0$, and the Three Subgroups Lemma gives $[V, O^p(L), Z] = 0$. Let $a \in Z$. Then $[V, a] = U$ from 2.12 and $[V, a] = [V, Z]$. $[V, a, a] = 0$ gives $[V, Z, a] = 0$. Hence, $[V, Z, Z] = 0$. Then

$$[V, L, Z] \overset{(2.22)}{=} [[V, O^p(L)] + U, Z] = [[V, O^p(L)] + [V, Z], Z] = 0.$$

Since $A \leq L$, we conclude $[V, A, Z] = 0$. Thus, $Z \leq A$ centralizes T_A and 2.5 yields a contradiction making $Z = 1$.

$E' = [A \cap R, B \cap R] \leq A \cap B = Z = 1$ so E is abelian. Recall that $R = E$ by 2.21. Then

$$E = \langle A \cap R \rangle^L = (A \cap R)[A \cap R, L] \leq (A \cap R)[E, L] = (A \cap R)[E, A][E, B]$$

$$= (A \cap R)[E, B] \leq E.$$

Thus, $E \cap B = [E, B](A \cap B \cap R) = [E, B]Z = [E, B]$ and similarly $E \cap A = [E, A]$, so $E = (A \cap R)[E, B] = (A \cap E)[E, B] = [E, A][E, B] = [E, L]$. Hence, $E = [E, O^p(L)]$

by A.4 and we have $R \leq O^p(L)$.

Let $a \in A \setminus R$. By minimality of L, $L = \langle a, B\rangle$. Thus, $T_L = \langle T_B^L \rangle = \langle T_B^{\langle a,B\rangle}\rangle = [T_B, a] + T_B$ and $T_A = T_A \cap T_L = [T_B, a] + (T_B \cap T_A) = [T_B, a] + U$. $R_A \leq C_{T_A}(a) = [V, a] < T_A$ by 2.12(b). If $U = R_A$, then $T_A = [T_B, a] + U \leq [V, a] < T_A$ by 2.12(b) again, a contradiction. This implies that $U \neq R_A$. Observe that $[T_A, a] = [T_B, a, a] + [U, a] = [U, a]$. However, 2.3 gives $[T_A, a] \neq 0$ and hence $[U, a] \neq 0$.

On the other hand, $[U, E] = [U, R] \leq [U, O^p(L)] \overset{(*)}{=} 0$ and so for any $d \in A$ we have $d \in E$ if and only if $[U, d] = 0$. Thus, $A \cap R = C_A(U)$ and $|U/R_A| \overset{2.8(a)}{=} |A/C_A(U)| = |A/(A \cap R)| = |AR/R|$ by the second isomorphism theorem. If $L/R \not\cong D_{2r}$, then L/R is simple so $L/R = O^p(L/R)$ and $L = O^p(L)R$. Since $R \leq O^p(L)$, we have $L = O^p(L)$. Then $[U, L] = [U, O^p(L)] = 0$ since $O^p(L) \leq C_L(U)$, and therefore $[U, a] = 0$, a contradiction. Hence, $L/R \cong D_{2r}$. Thus, $|AR/R| = 2$ and $|U/R_A| = 2$. Since $|U/R_A|$ is an $\mathbb{F}$-space, $|\mathbb{F}| = 2$ and therefore $|U| = 4$. Now let $1 \neq a \in A \cap E$. We have $[V, a] = C_{T_A}(a)$ from 2.12(b). Then $|[V, a]| = |C_{T_A}(a)| = |T_A/R_A| = |A|$ and so $|[V, a]/U| = \frac{|A|}{4}$. Since $R = E$, $|A/A \cap E| = |U/R_A| = 2$. We also have $|B \cap E| = \frac{|B|}{2} = \frac{|A|}{2}$. $[[V, a], B \cap E] \leq [T_A, B \cap E] \leq T_A \cap [V, B] = U$ since $E \leq C \leq N_L(A)$. Define $D := C_{B \cap E}([V, a]/R_A)$. Since $B \cap E$ centralizes U, $(B \cap E)/D$ embeds into $\mathrm{Hom}_{\mathbb{F}}([V, a]/U, U/R_A)$. So $|(B \cap E)/D| \leq |[V, a]/U| = \frac{|A|}{4}$. Since $|B \cap E| = \frac{|A|}{2}$, it must be the case that $D \neq 1$. As $Z = 1$, $D \not\leq A$ so $|AD| > |A|$. $[V, a, A] \leq R_A$ and $[V, a, D] \leq R_B = R_A$ so $[[V, a], AD] \leq R_A$. Then $|AD/C_{AD}([V, a])| \leq |\,\mathrm{Hom}([V, a]R_A/R_A, R_A)| = |[V, a]R_A/R_A| = |A|/2$ and since $|AD| > |A|$ we have $|C_{AD}([V, a])| \geq 4$. Observe that AD is a p-group and $[V, a, C_{AD}([V, a])] = 0$. So we can apply 1.1(iii) yielding $|C_{AD}([V, a])| \leq |\mathbb{F}| = 2$, a contradiction. Therefore, $R_A \neq R_B$. □

Lemma 2.24

(a) $N_G(A) = N_G(R_A)$.

(b) $C_L(\mathcal{A}(L)) = C_L(\mathcal{R}(L))$.

(c) $O_p(L) \le C_L(R_L) \le C_L(\mathcal{A}(L))$.

Proof. (a) $N_G(A) \le N_G(R_A)$. Let $l \in N_G(R_A)$. We have $R_{A^l} = R_A{}^l = R_A$ so $A^l = A$ from 2.23. Then $N_G(R_A) \le N_G(A)$.

(b) $C_L(\mathcal{A}(L)) = \bigcap_{D \in \mathcal{A}(L)} N_L(D) = \bigcap N_L(R_D) = C_L(\mathcal{R}(L))$ from (a).

(c) $O_p(L) \le C_L(A^L)$ by 2.21(c) so $O_p(L)$ normalizes each $D \in \mathcal{A}(L)$ and so centralizes each R_D. Thus $O_p(L) \le C_L(R_L)$. Also, $C_L(R_L) = C_L(\Sigma\mathcal{R}(L)) \le C_L(\mathcal{R}(L)) \overset{(b)}{=} C_L(\mathcal{A}(L))$. $\square$

Lemma 2.25 *If $R_A \le T_B$, then also $R_B \le T_A$.*

Proof. Observe that $C_B(R_A) = C_B(R_A + R_B)$, so by 2.8(a), $|B/C_B(R_A)| = |R_A + R_B/R_B| = |\mathbb{F}|$ and $C_{T_B}(C_B(R_A)) = R_A + R_B$.

We have $C_B(R_A) \le N_G(R_A) = N_G(A) \le N_G(T_A)$ by 2.24. Suppose that $R_B \not\le T_A$. Then $(R_A + R_B) \cap T_A = R_A \cap T_A = R_A$. Also,

$$[T_A \cap T_B, C_B(R_A)] \le [T_A, C_B(R_A)] \cap [T_B, C_B(R_A)] \le T_A \cap [T_B, B] \le T_A \cap R_B = 0$$

so $T_A \cap T_B \le C_{T_B}(C_B(R_A)) \cap T_A = (R_A + R_B) \cap T_A = R_A$. Now $[T_A, C_B(R_A)] \le T_A$ and $[T_A, C_B(R_A)] \le [V, B] = T_B$ so

$$(*) \qquad [T_A, C_B(R_A)] \le T_A \cap T_B \le R_A.$$

Let $P = AC_B(R_A)$. Then 2.14 implies $C_P(T_A/R_A) = A$, and thus $C_B(R_A) \overset{(*)}{\le} C_P(T_A/R_A) = A$. By 2.5(d), $|B| \ge |\mathbb{F}|^2$ and since $|B/C_B(R_A)| = |\mathbb{F}|$, we have $C_B(R_A) \ne 1$. Pick $1 \ne b \in C_B(R_A)$. Then by 2.12(b), $R_B \le [V, b] \le [V, C_B(R_A)] \le [V, A] = T_A$. $\square$

Chapter 3

The case where A is not a TI-set

Lemma 3.1 *Suppose that $R_A \leq T_B$. Then $R_L = R_A + R_B$, $E = C_L(R_L)$, $L/E \cong SL_2(\mathbb{F})$, $T_A \neq T_B$, and one of the following holds:*

(a) $|\mathbb{F}| > 2$, $|A| = |\mathbb{F}|^2$, *and* $|A \cap B| = |\mathbb{F}|$.

(b) $|\mathbb{F}| = 2$, $|A \cap B| = 2$, *and* $|A| \leq 2^4$.

Proof. From 2.25 we have that $R_B \leq T_A$. It follows that both A and B normalize $R_A + R_B$. Then $R_L = \langle R_A^L \rangle \leq R_A + R_B \leq R_L$ gives $R_L = R_A + R_B$ as $R_A + R_B$ is L-invariant. So R_L is 2-dimensional.

By 2.8(a), $|A/C_A(R_B)| = |R_A + R_B/R_A| = |\mathbb{F}|$. Then $AC_L(R_L)/C_L(R_L) = \{\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right) | * \in \mathbb{F}\}$ and $BC_L(R_L)/C_L(R_L) = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) | * \in \mathbb{F}\}$, so $L/C_L(R_L) \cong SL_2(\mathbb{F})$. Now 2.21 and 2.24(c) give $C_L(R_L) \leq C_L(A^L) = E = R \leq C_L(R_L)$. Hence, $E = C_L(R_L)$ and $L/E \cong SL_2(\mathbb{F})$. $|A| \geq |\mathbb{F}|^2$ by 2.5(d) so $|A \cap E| = |A \cap C_L(R_L)| = |C_A(R_B)| \geq |\mathbb{F}|$.

Since $Z \leq Z(L)$ by 2.21(b), $Z \trianglelefteq L$. Define $\overline{L} = L/Z$.

1° *Choose $b \in (B \cap E) \setminus Z$. Then $C_A(\bar{b}) = A \cap E$ and $|[A, b]Z/Z| = |A/C_A(\bar{b})| = |\mathbb{F}|$.*

Proof of (1°). Suppose $a \in A \setminus E$ with $[a, b] \in Z$. $[\bar{b}, \langle a, B\rangle] = [\bar{b}, a][\bar{b}, B] = 1$ since B is abelian and $[a, b] \in Z$. Then $\bar{b}$ is centralized by $\tilde{L} := \langle a, B\rangle$. Since

$L/E \cong SL_2(\mathbb{F}) \cong \tilde{L}E/E$, $\tilde{L}E = L$. In particular, there exists $l \in \tilde{L}$ with $AE = B^l E$. Since A is a weakly closed subgroup in G, $A = B^l$. Then $b \in bZ = b^l Z \leq B^l = A$ and $b \in A \cap B = Z$, which is a contradiction. Thus, $C_A(\bar{b}) \leq A \cap E$. $[A \cap E, B \cap E] \leq [A, E] \cap [E, B] \leq Z$ so $[A \cap E, b] \leq Z$ and we have $A \cap E \leq C_A(\bar{b})$. Hence, $A \cap E = C_A(\bar{b})$. Consider the commutator map which sends a to $[a, b]Z$; this gives $|[A, b]Z/Z| = |A/C_A(\bar{b})| = |A/A \cap E| = |\mathbb{F}|$. $\square$

2° $T_A \neq T_B$.

Proof of (2°). Suppose that $T_A = T_B$. Then $[V, L, A] \leq [T_A, A] + [T_B, A] = [T_A, A] = R_A \leq R_L$ so $[V, L, L] \leq R_A + R_B \leq R_L$ and hence $[V, L'] \leq R_L$ and

$$(*) \qquad [V, L', E] = 0$$

since $E = C_L(R_L)$. Assume that $E \neq Z$. By 2.21(b), $Z \leq E$ so $B \cap E \neq Z$ and there exists $b \in B \cap E \setminus Z$. Thus, (1°) implies that $[A, b] \neq 1$. Let $1 \neq a \in [A, b]$. $[A, b] \leq A$ since $b \in E = C_L(A^L) \leq N_L(A)$. $[A, b] \leq [L, L] = L'$ so $a \in A \cap L'$. By $(*)$, $[V, a, E] = 0$ so $E \leq C_G([V, a])$. $E = R$ is a p-group so by 2.12, $E \leq A_a \leq A$. Thus $b \in Z$ which contradicts (1°). Hence, $E = Z$.

We already have that $|A \cap E| \geq |\mathbb{F}|$, so $Z = E \neq 1$. Then $1 \neq [T_A, Z] \leq R_A$ and since R_A is 1-dimensional, $R_A = [T_A, A \cap B] = [T_B, A \cap B] = R_B$, a contradiction to 2.23. $\square$

3° $C_{\overline{E}}(A) = \overline{A \cap E}$ *and* $C_{\overline{E}}(O^p(L)) = 1$.

Proof of (3°). $C_A(\bar{b}) = A \cap E \neq A$ from (1°). So $C_A(\bar{b}) \neq A$ and $\bar{b} \notin C_{\overline{B \cap E}}(A)$ so $C_{\overline{B \cap E}}(A) = 1$. $\overline{E} = (\overline{A \cap E})(\overline{B \cap E})$ from 2.21. Since A is abelian, $\overline{A \cap E} \leq C_{\overline{E}}(A)$ and we conclude that $C_{\overline{E}}(A) = (\overline{A \cap E})((\overline{B \cap E}) \cap C_{\overline{E}}(A)) = \overline{A \cap E}$ and similarly $C_{\overline{E}}(B) = \overline{B \cap E}$. Also,

$$C_{\overline{E}}(L) = C_{\overline{E}}(A) \cap C_{\overline{E}}(B) = \overline{A \cap E} \cap \overline{B \cap E} = \overline{A \cap B} = \overline{Z} = 1.$$

Hence, $C_{\overline{E}}(O^p(L)) = 1$. $\boxdot$

4° $\quad Z \neq 1$.

Proof of (4°). Suppose that $Z = 1$ and let $1 \neq b \in B \cap E$. Then (1°) and $Z = 1$ imply $C_A(\bar{b}) = C_A(b) = A \cap E$. From 2.8(d) we have $[T_A, b] + R_A = C_{T_A}(C_A(b)) = C_{T_A}(A \cap E)$. By 2.8(b),

$$|C_{T_A}(A \cap E)| = |T_A/(A \cap E)| = |\mathbb{F}||A/(A \cap E)| \overset{(1°)}{=} |\mathbb{F}|^2.$$

Also, $A \cap E \leq E = C_L(R_L)$ so $A \cap E$ centralizes R_L and we have $R_L \leq C_{T_A}(A \cap E)$. Then, as they have the same order, $R_L = C_{T_A}(A \cap E)$. So we have $[T_A, b] + R_A = C_{T_A}(A \cap E) = R_L$. Now $[T_A, b] + R_A = R_L$ for all $b \in B \cap E$, so $[T_A, B \cap E] \leq R_L$ and hence also

$$(**) \qquad [T_A, E] = [T_A, A \cap E][T_A, B \cap E] \leq R_A R_L \leq R_L.$$

Let $n = \dim A$. Since $|A/A \cap E| = |\mathbb{F}|$, we have $|A \cap E| = |\mathbb{F}|^{n-1}$. $E = (A \cap E)(B \cap E)$ and $Z = 1$ give $|E| = |\mathbb{F}|^{2(n-1)}$.

Let $a \in A \cap E$ and put $Y = C_{T_A}(a) = [V, a]$. Then Y is a hyperplane of T_A from 2.12. $R_L \leq T_A$ and $[R_L, E] = [R_L, C_L(R_L)] = 0$, so $R_L \leq Y$. Then $\dim Y/R_L = (n+1) - 1 - 2 = n - 2$ and

$$(***) \qquad [Y, E] \leq [T_A, E] \overset{(**)}{\leq} R_L.$$

Consider the map $\gamma : E \to \mathrm{Hom}(Y/R_L, R_L)$ which takes $e \to (y + R_L \to [y, e])$. This gives $|E/C_E(Y)| \leq |\mathbb{F}|^{(\dim(Y/R_L))(\dim R_L)} = |\mathbb{F}|^{2(n-2)}$ and so $|C_E(Y)| \geq |\mathbb{F}|^2$. By 2.12, $C_E(Y) \leq A_a$ and $|A_a| = |\mathbb{F}|$ which is a contradiction. Hence, $Z \neq 1$. $\boxdot$

By (4°) we can choose $1 \neq z \in Z$. Then $[V, z] \leq T_A \cap T_B = U$. From (2°) and 2.12 we get that $U = [V, z]$ is a hyperplane of T_A and T_B and $U = C_{T_A}(z)$.

$Z \le C_Z([V, z]) \le A_z \cap B_z \le Z$ so $A_z \cap B_z = Z$. However, A_z is the unique Sylow p-subgroup of $C_G([V, z])$ so $A_z = B_z$ and $Z = A_z = \mathbb{F}z$. Then $|Z| = |\mathbb{F}|$.

$C_E(U)$ is a p-subgroup in $C_G(U)$ so by 2.12, $C_E(U) \le A_a = Z = C_A(U) \le C_E(U)$. Therefore, $C_E(U) = Z$. $|T_A/U| = |\mathbb{F}|$ and $|T_B/U| = |\mathbb{F}|$ as U is a hyperplane of T_A and T_B, so $|T_L/U| = |\mathbb{F}|^2$.

Let U_0 be an $\mathbb{F}$-subspace of U with $U = U_0 \oplus R_L$. Put $A_0 = C_A(U_0)$, $B_0 = C_B(U_0)$ and $L_0 = \langle A_0, B_0 \rangle$. Since $R_L \le U$, we have $C_L(U) \le C_L(R_L) = E$ and so $C_L(U) = C_E(U) = Z$. Notice that $[U_0, L_0] = 0$ by construction. $C_{L_0}(U) \le C_L(U) = Z \le C_{L_0}(U)$ so L_0/Z acts faithfully on U. $C_{L_0}(R_L) = C_{L_0}(R_L + U_0) = C_{L_0}(U) = Z$ so L_0/Z acts faithfully on R_L. By 2.8(b), $|A_0| = |T_A/(U_0 + R_A)| = |T_A/U||U/(U_0 + R_A)| = |T_A/U||(U_0 + R_A + R_B)/(U_0 + R_A)| = |\mathbb{F}|^2$. Hence, $|A_0/C_{L_0}(R_L)| = |A_0/Z| = |\mathbb{F}|$. Thus A_0 acts as $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and B_0 acts as $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ on R_L. Then A_0 induces $SL_2(\mathbb{F})$ on R_L and $L_0/Z \cong SL_2(\mathbb{F})$.

5° *The commutator map, $T_L/U \to R_L$, $t + U \to [t, z]$ is a well defined isomorphism.*

Proof of (5°). $T_L/C_{T_L}(z) \cong [T_L, z] = [T_A, z] + [T_B, z]$. Also, $1 \ne z \in A \cap B$ so by 2.5, $1 \ne [T_A, z] \le R_A$. Since R_A is 1-dimensional, $[T_A, z] = R_A$ and similarly $[T_B, z] = R_B$. Then $T_L/C_{T_L}(z) \cong [T_L, z] = [T_A, z] + [T_B, z] = R_A + R_B = R_L$. Now T_L/U and $T_L/C_{T_L}(z) \cong R_L$ are both 2-dimensional so $U \le C_{T_L}(z) \le T_L$ gives $U = C_{T_L}(z) = C_V(z) \cap T_L$ and $T_L/U \cong R_L$. $\surd$

6° $[C_V(z), L] \le U$, $[U, L] \le R_L$, *and* $[C_V(z), O^p(L)] \le R_L$.

Proof of (6°). Let $V_0 \le V$ be maximal with $[V_0, z] \le U_0$. $\alpha : V/C_V(z) \to U$, $v \to [v, z]$ is an isomorphism. $V_0/C_V(z)$ is the inverse image of U_0 under α, so $V_0/C_V(z) \cong U_0$ as L_0-modules. Then, as L_0 centralizes U_0, L_0 also centralizes $V_0/C_V(z)$. We have $[C_V(z), L] \le C_V(z) \cap [V, L] \le C_V(z) \cap T_L = U$. Also, $[U, L] \le [T_A, A] \cap [T_B, B] \le R_L$ so $[U/R_L, L_0] = 1$. L_0 stabilizes the series $R_L \le U \le$

$C_V(z) \le V_0$ and so also stabilizes the series $1 \le U/R_L \le C_V(z)/R_L \le V_0/R_L$. Then by A.1, $[V_0/R_L, O^p(L_0)] = 1$. Hence, $[V_0, O^p(L_0)] \le R_L$. $R_L \le U \le C_V(z)$ so $O^p(L)$ centralizes $C_V(z)/R_L$. Then, $[C_V(z), O^p(L)] \le R_L$. $\boxdot$

Suppose that $|\mathbb{F}| > p$. Since $L_0/Z \cong SL_2(\mathbb{F})$ and $O^p(SL_2(\mathbb{F})) = SL_2(\mathbb{F})$, we have $O^p(L_0/Z) = L_0/Z$. By A.3, $O^p(L_0/Z) = O^p(L_0)Z/Z$. Then $L_0/Z = O^p(L_0)Z/Z$ and $L_0 = O^p(L_0)Z$. So $A_0 = (O^p(L_0) \cap A_0)Z$ and since $|A_0/Z| = |\mathbb{F}|$, $A_0 \not\le Z$. Then $A_0 \not\le E$ since $Z = C_{L_0}(R_L) = C_L(R_L) \cap L_0 = E \cap L_0$. So there exists $a \in O^p(L_0) \cap A_0$ with $a \notin E$. Since $a \in O^p(L_0)$, $[V_0, a] \le R_L$. Since $p = 2$, $[V_0, a] \le C_{R_L}(a)$. $C_{L_0}(R_L) = Z$ and $a \notin Z$ so $[R_L, a] \ne 0$. Hence, $R_A \le C_{R_L}(a) < R_L$. R_L is 2-dimensional and R_A is 1-dimensional so $C_{R_L}(a) = R_A$.

$[V, z] = U_0 \oplus R_L$ so $[V/U_0, z] = (R_L + U_0)/U_0 \cong R_L \cong U/U_0$. Then $R_L \cong [V/U_0, z] \cong (V/U_0)/C_{V/U_0}(z) \cong (V/U_0)/(V_0/U_0) \cong V/V_0$. We know $R_L/[R_L, a]$ is 1-dimensional so $(V/V_0)/[V/V_0, a] \cong (V/V_0)/(([V, a] + V_0)/V_0) \cong V/([V, a] + V_0)$ is 1-dimensional. We have shown $[V_0, a] \le C_{R_L}(a) = R_A$ so $[[V, a] + V_0, a] \le R_A$. We have also shown $V/([V, a] + V_0)$ is 1-dimensional so $V = \mathbb{F}v + [V, a] + V_0$ for some $v \in V \backslash ([V, a] + V_0)$. Then $[V, a] \le \mathbb{F}[v, a] + R_A$. Hence, $[V, a]$ is at most 2-dimensional. $[V, a]$ is a hyperplane in T_A so it follows that T_A/R_A is at most 2-dimensional and 2.5 gives that T_A/R_A is exactly 2-dimensional so $|A| = |\mathbb{F}|^2$ and (a) holds in this case.

Suppose now that $|\mathbb{F}| = p$. If $\dim T_A/R_A = 2$, then $\dim A = 2$, so $|A| = 4$, $|A \cap B| = 2$, and (b) holds. So now we may assume that $\dim T_A/R_A > 2$. From (1°), we have $|A/A \cap E| = |\mathbb{F}|$ so $|A \cap E| > |\mathbb{F}|$. However, $|A \cap Z| = |Z| = |\mathbb{F}|$ so $E \ne Z$.

Let D be a normal subgroup of L in E minimal with respect to $D \not\le Z$. By (3°), $C_{E/Z}(O^p(L)) = 1$ so $[D, O^p(L)] \ne 1$. If $[D, O^p(L)] \le Z$, then $D/Z \le C_{D/Z}(O^p(L)) = 1$, but $D \not\le Z$ so $[D, O^p(L)] \not\le Z$. The minimality of D implies

$$(****) \qquad D = [D, O^p(L)] \le O^p(L).$$

Let $l \in L$. If $[D, A] \le Z$, then $[D, A^l] \le Z$, but $L = \langle A^L \rangle$ so $[D, L] \le Z$. Then

$[D, O^p(L)] \leq Z$ which is a contradiction. Hence, $[D, A] \not\leq Z$ and therefore there exists $a \in (A \cap D) \setminus Z$. $U/C_U(a) \cong [U, a] \leq R_A$ so $|U/C_U(a)| \leq |\mathbb{F}|$. Now $a \in D \leq E = C_L(R_L)$ so $R_L \leq C_U(a)$. Then

$$[C_U(a), L] \overset{(6^\circ)}{\leq} R_L \leq C_U(a)$$

so L normalizes $C_U(a)$. Since $a \in D \backslash Z$, $\langle a^L \rangle \not\leq Z$, but $\langle a^L \rangle \leq D$ since D is normal in L and so by minimality, $D = \langle a^L \rangle$. Then $[C_U(a), D] = [C_U(a), \langle a^L \rangle] = [C_U(a), a]^L = 1$.

Define $X/C_U(a) = C_{V/C_U(a)}(z)$. Observe that $C_V(z) \leq X$ so $C_V(z) = C_X(z)$. $C_U(a) \leq U = [V, z]$ so $[X, z] = C_U(a)$. Also, $C_U(a) = [X, z] \cong X/C_X(z) \cong X/C_V(z)$ as L-modules. $1 = [C_U(a), D] \cong [X/C_V(z), D]$ and so $[X, D] \leq C_V(z)$. Hence, $[X, D, O^p(L)] \leq [C_V(z), O^p(L)] \overset{(6^\circ)}{\leq} R_L \leq [T_L, a] + R_L$. Now $\dim(T_L/R_L)/(T_A/R_L) = \dim T_L/T_A = \dim(T_B + T_A)/T_A = \dim T_B/(T_A \cap T_B) = 1$ since $T_A \cap T_B$ is a hyperplane of T_B. So T_A is a hyperplane of T_L. $[T_A, a] \leq R_A \leq R_L$ so T_A/R_L is centralized by a, making $T_A/R_L \leq C_{T_L/R_L}(a)$. Let $\tilde{W} = [T_L, a] + R_L$. We have $\dim \tilde{W}/R_L = \dim[T_L/R_L, a] = \dim(T_L/R_L)/C_{T_L/R_L}(a) \leq \dim(T_L/R_L)/(T_A/R_L) = 1$. So $\tilde{W}/R_L$ is at most 1-dimensional and $\tilde{W}$ is at most 3-dimensional. Also, $1 = [R_L, E] \overset{(5^\circ)}{\cong} [T_L/U, E]$ gives $[T_L, E] \leq U$. $R_L \leq U$ so we have $R_L \leq \tilde{W} \leq U$. Then $[\tilde{W}, L] \leq [U, L] \overset{(6^\circ)}{\leq} R_L \leq \tilde{W}$ and so $\tilde{W}$ is L-invariant and $\tilde{W} = \langle [T_L, a]^L \rangle + R_L = [T_L, D] + R_L$. Thus, $[X, O^p(L), D] \leq [V, L, D] = [T_L, D] \leq \tilde{W}$.

The Three Subgroups Lemma gives $[X, [D, O^p(L)]] \leq \tilde{W}$. Then $[X, D] \leq \tilde{W}$ since $D \overset{(****)}{=} [D, O^p(L)]$. Thus, $[X, a] \leq \tilde{W}$. Observe that $[U, a] \leq [T_A, a] \leq R_A$ and $[U, a] \neq 1$ since $a \notin Z$. Therefore $[U, a] = R_A$ as R_A is 1-dimensional. So $U/C_U(a)$ is 1-dimensional. Then $V/X \cong (V/C_U(a))/(X/C_U(a)) = (V/C_U(a))/C_{V/C_U(a)}(a) \cong [V/C_U(a), a] = [V, a]/C_U(a)$ is 1-dimensional since $[V, a]$ and $[V, z] = U$ are both hyperplanes of T_A and $U/C_U(a)$ is 1-dimensional. So we have V/X is 1-dimensional and $\tilde{W}$ at most 3-dimensional. Since $X/\tilde{W} \leq C_{V/\tilde{W}}(a)$, $(V/\tilde{W})/(C_{V/\tilde{W}}(a))$ is a quotient of $(V/\tilde{W})/(X/\tilde{W})$ and therefore at most 1-dimensional. Hence, $[V/\tilde{W}, a]$ and therefore, $([V, a] + \tilde{W})/\tilde{W}$ is at most 1-dimensional. Then $[V, a]$ is at most 4-dimensional. So $|A| \leq 2^4$ and (b) holds in this case. $\square$

Lemma 3.2 *Let $A \leq S \in Syl_p(G)$ and $s \in S \setminus A$. Then* $\dim[A, s] \geq 2$. *In particular, if* $\dim A \leq 3$, *then* $A \in \mathrm{Syl}_p(G)$.

Proof. Suppose $\dim[A, s] < 2$. Since A is a weakly closed subgroup of G, $A \trianglelefteq S$. By 2.16, $C_S(A) = A$; thus $[A, s] \neq 1$. Pick $1 \neq a \in [A, s]$. Since $[A, s]$ is an $\mathbb{F}$-subspace of A, $[A, s] = \mathbb{F}a = A_a$. Then $C_{T_A/R_A}(s) \overset{2.8(c)}{=} C_{T_A}([A, s])/R_A = C_{T_A}(A_a)/R_A \overset{2.12}{=} [V, a]/R_A$. Hence, $[V, a, s] \leq R_A$. Since A induces $\mathrm{Hom}_{\mathbb{F}}([V, a]/R_A, R_A)$ on $[V, a]$, $s \in C_S([V, a])A = A_aA = A$ by 2.12. This is a contradiction to the choice of s. Therefore, $\dim[A, s] \geq 2$.

Assume $\dim A \leq 3$ and suppose $S \neq A$. $S \setminus A$ has an element of order 2, so choose $s \in S \setminus A$ with $s^2 \in A$. Then $[A, s] \leq C_A(s) \leq A$. So

$$2 * \dim[A, s] = \dim A/C_A(s) + \dim[A, s] \leq \dim A \leq 3.$$

Then $\dim[A, s] < 2$ which is a contradiction and therefore $S = A$. $\square$

Lemma 3.3 *Suppose that $R_A \leq T_B$, $|\mathbb{F}| > 2$, $|A| = |\mathbb{F}|^2$, and $|A \cap B| = |\mathbb{F}|$. Then one of the following holds:*

(a) $A \cap B \trianglelefteq G$, $R_G = [V, A \cap B]$ is 2-dimensional, $G_0/C_{G_0}(R_G) \cong SL_2(\mathbb{F})$, and R_G is the corresponding natural module.

(b) R_G is 4-dimensional, $G_0/C_{G_0}(R_G) \cong \Omega_4^+(\mathbb{F})$, and R_G is the corresponding natural module.

Proof.

1° *A is a Sylow p-subgroup of G and both $N_G(A)/A$ and $N_G(A)/C_G(T_A/R_A)$ are p'-groups.*

Proof of (1°). $A \in Syl_p(G)$ by 3.2. $A \overset{2.14}{\leq} C_G(T_A/R_A)$ so $N_G(A)/C_G(T_A/R_A)$ and $N_G(A)/A$ are p'-groups. $\boxdot$

Let $\mathcal{P}$ be the set of 1-dimensional subspaces of T_A/R_A. Observe that $R_L/R_A \in \mathcal{P}$.

2° *Let $1 \neq z \in Z$. Then $E = Z$ and $R_L = [V, z]$.*

Proof of (2°). E is a p-group and $E \leq O_p(L) \overset{2.24(c)}{\leq} C_L(\mathcal{A}(L)) \leq N_G(A)$, so $E \leq A$ by (1°). $E = (A \cap E)(B \cap E) \leq E(B \cap A) \leq A \cap B = Z$ so $E = Z$. $[T_A, z] = R_A$ and $[T_B, z] = R_B$ so $[T_L, z] = R_L$ and therefore,

$$(*) \qquad R_L = [T_L, z] = [T_L, Z] \leq [V, Z].$$

R_L and $[V, Z]$ are both 2-dimensional so $R_L = [V, Z] = [V, z]$. Then, since Z is 1-dimensional, $Z = A_a \overset{2.12(e)}{\trianglelefteq} N_G(R_L)$ so $N_G(R_L) = N_G(Z)$. $\boxdot$

3° *$N_G(A)$ has orbits of length 1, $|\mathbb{F}| - 1$, and 1 on $\mathcal{P}$. Also, $N_G(A) \leq N_G(Z)$.*

Proof of (3°). Let $1 \neq z \in Z$. $[R_L, Z] = 0$ so $R_L \leq C_{T_L}(Z) \leq T_L$. Now $T_L/C_{T_L}(z) \cong [T_L, z] \overset{(*)}{=} R_L$ as L-modules. Since A is 2-dimensional, T_A is 3-dimensional. $T_L/T_A = (T_B + T_A)/T_A \cong T_B/(T_B \cap T_A)$ and since $T_A \cap T_B$ is a hyperplane of T_B, T_L/T_A is 1-dimensional. Hence, T_L is 4-dimensional and T_L/R_L is 2-dimensional. Then both $T_L/C_{T_L}(Z)$ and T_L/R_L are 2-dimensional, so $C_{T_L}(Z) = R_L$ and $T_L/R_L \cong R_L$.

Let $\rho : T_L/R_L \longrightarrow R_L$ be an L-isomorphism. Then $C_{R_L}(A) \cong C_{T_L/R_L}(A)$, but $C_{R_L}(A) = R_A$ and $C_{T_L/R_L}(A) = T_A/R_L$ making $T_A/R_L \cong R_A$ as $N_L(A)$-modules.

Claim: $T_A/R_A \cong R_L$ as $\mathbb{F}H$-modules where $H = N_L(A) \cap N_L(B)$.

Proof of Claim. $R_L = R_A \oplus R_B$ so $R_L/R_A \cong_H (R_A + R_B)/R_A \cong_H R_B/(R_A \cap R_B) \cong_H R_B$. Then $R_L = R_A \oplus R_B \cong_H T_A/R_L \oplus R_L/R_A$. $N_L(A)/A$ is a p'-group by (1°). Maschke's theorem [Asch, 12.9] gives the existence of $Y/R_A \leq T_A/R_A$

with $T_A/R_A = R_L/R_A \oplus Y/R_A$. $Y/R_A \cong_H (T_A/R_A)/(R_L/R_A) \cong_H T_A/R_L$. Then $R_L \cong_H T_A/R_L \oplus R_L/R_A \cong_H Y/R_A \oplus R_L/R_A = T_A/R_A$. So the claim is proven.

We now calculate the orbits of H on R_L instead of on T_A/R_A. Let $K = \{\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right) \mid 0 \neq \lambda \in \mathbb{F}\}$. The lengths of the orbits of H on R_L are the same as the lengths of orbits of K on the 1-subspaces of $\mathbb{F} \times \mathbb{F}$.

Observe that $\mathbb{F}(1,0)$ and $\mathbb{F}(0,1)$ are fixed points of K. Let $x, y \in \mathbb{F} \setminus \{0\}$. Since the characteristic of $\mathbb{F}$ is 2, each element in $\mathbb{F}$ is a square so $xy^{-1} = \lambda^2$ for some $0 \neq \lambda \in \mathbb{F}$. Hence, H/Z corresponds to $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)(1,1) = (\lambda, \lambda^{-1}) = \lambda x^{-1}(x,y)$. Thus, K has three orbits whose lengths are 1, 1, and $\mathbb{F} - 1$. Then H has three orbits on T_A/R_A whose lengths are 1, 1, and $\mathbb{F} - 1$.

A/Z corresponds to $\{\left(\begin{smallmatrix}1 & 0\\ * & 1\end{smallmatrix}\right) | * \in \mathbb{F}\}$ and $N_L(A)/Z$ corresponds to $\{\left(\begin{smallmatrix}\lambda & 0\\ * & \lambda^{-1}\end{smallmatrix}\right) | * \in \mathbb{F}\}$ so $N_L(A) = HA$. A acts trivially on T_A/R_A so the orbits of $N_L(A)$ are the same as the orbits of H on $\mathcal{P}$, namely orbits of length 1, 1, and $|\mathbb{F}| - 1$ on $\mathcal{P}$. An orbit of $N_G(A)$ and $N_G(A) \cap N_G(R_L)$ is a union of orbits of $N_L(A)$ so the possible lengths of such orbits are:

1) 1, 1, $|\mathbb{F}| - 1$

2) $|\mathbb{F}| + 1$

3) 1, $|\mathbb{F}|$

4) 2, $|\mathbb{F}| - 1$.

Since $N_G(A)/A$ is a p'-group by (1°), the orbits of R_L/R_A have p'-length so options 3 and 4 are not viable.

Suppose for a contradiction that $N_G(A)$ is transitive on $\mathcal{P}$. Since $|\mathbb{F}| > 2$, $N_L(A)$ has exactly 2 fixed points: R_L/R_A and R_L^g/R_A for some $g \in N_G(A)$ with $R_L/R_A \neq R_L^g/R_A$. Then $N_L(A)^{g^{-1}}$ fixes $R_L^{g^{-1}}/R_A$ and R_L/R_A. So $N_L(A)^{g^{-1}} \leq N_G(A) \cap N_G(R_L)$.

From (1°), $N_G(R_L) \cap N_G(A)$ does not have an orbit of length $|\mathbb{F}|$ on $\mathcal{P}$ but it fixes R_L/R_A so it does have an orbit of length 1. This implies that $N_G(R_L) \cap N_G(A)$ has orbits of length 1, $|\mathbb{F}|-1$, and 1 on $\mathcal{P}$. Notice that this means that $N_G(R_L)\cap N_G(A)$ and $N_L(A)$ have the same orbits. So $N_G(R_L)\cap N_G(A)$ fixes R_L/R_A and R_L^g/R_A. Since we proved that $N_L(A)^{g^{-1}} \leq N_G(A)\cap N_G(R_L)$, this means that $N_L(A)^{g^{-1}}$ fixes R_L/R_A and R_L^g/R_A. Since the fixed points of $N_L(A)^{g^{-1}}$ are R_L/R_A and $R_L^{g^{-1}}/R_A$, this gives $R_L^g = R_L^{g^{-1}}$. Hence, $R_L^{g^2} = R_L$.

Notice that gA/A has odd order from (1°) so $\langle g\rangle A = \langle g^2\rangle A$. Thus $R_L^g = R_L$. This is a contradiction to the choice of g. So option 2 is not viable and the orbits of $N_G(A)$ and $N_G(A)\cap N_G(R_L)$ must be of length 1, 1, and $|\mathbb{F}|-1$.

Let Y/R_A and R_L/R_A be the fixed points of $N_G(A)$ on $\mathcal{P}$. Let $\tilde{Z} := C_A(Y)$.

$[R_L, Z] = 0$ so $Z \leq C_A(R_L) \leq A$. Since A is 2-dimensional, Z is 1-dimensional, and $A \neq C_A(R_L)$, we have $Z = C_A(R_L)$. $N_G(A)$ normalizes $C_A(R_L)$ so it normalizes Z and we have $N_G(A) \leq N_G(Z)$. $\tilde{Z} = C_A(Y)$ and since Y is a fixed point of $N_G(A)$, $N_G(A) \leq N_G(\tilde{Z})$. $\boxdot$

4° *Let $1 \neq t \in A$. Then one of the following holds:*

(a) $t \notin Z$ and A is the unique Sylow p-subgroup of $C_G(t)$.

(b) $t \in Z$ and $O^{p'}(C_G(t)) = C_G(t)_0 = L$.

(c) $t \in \tilde{Z}$ and there exists $A \neq \tilde{B} \in \mathcal{A}$ with $t \in A\cap\tilde{B}$. Moreover, for any such $\tilde{B}$, $\tilde{Z} = A\cap\tilde{B}$ and $\tilde{L} := \langle A, \tilde{B}\rangle = O^{p'}(C_G(t)) = C_G(t)_0$.

Proof of (4°). Notice that $\mathrm{Syl}_p(C_G(t)) = \mathcal{A}(C_G(t))$. If $\{A\} = \mathcal{A}(C_G(t))$, then (a) holds. Suppose from now on that $\{A\} \neq \mathcal{A}(C_G(t))$. Then there exists $\tilde{B} \in \mathcal{A}(C_G(t))$ with $A \neq \tilde{B}$. So $t\tilde{B}$ is a p-group and $\tilde{B}$ is a Sylow p-subgroup of $C_G(t)$. Thus $t \in \tilde{B}$ and $t \in A\cap\tilde{B}$.

If $t \in Z$, then $t \in A\cap B\cap\tilde{B}$ and $R_A + R_B + R_{\tilde{B}} \leq [V,t]$ by 2.3. $[V,t] = R_L$ from (2°). Therefore, $R_{\tilde{B}} \leq R_L$ so $R_{\tilde{B}} = R_A{}^l$ for some $l \in L$. Thus $\tilde{B} = A^l$ by 2.23 and $\mathcal{A}(C_G(t)) = A^L$. Then $C_G(t)_0 = L$. $O^{p'}(H) = \langle Syl_p(H)\rangle$ for any finite group so $O^{p'}(C_G(t)) = \langle\mathcal{A}(C_G(t))\rangle$ and (b) holds.

So suppose that $t \notin Z$ and put $\tilde{L} = \langle A, \tilde{B} \rangle$. Then $R_{\tilde{L}} = R_A + R_{\tilde{B}} \leq [V,t] \leq T_A \cap T_{\tilde{B}}$. So $\tilde{L}$ fulfills all the assumptions L does. Then we can apply (b) to get $O^{p'}(C_G(t)) = C_G(t)_0 = \tilde{L}$. We can also apply (3°) to see that $R_{\tilde{L}}/R_A$ is a fixed point of $N_G(A)$ on $\mathcal{P}$. Hence, $R_{\tilde{L}} = Y$ and $C_L(R_L) \overset{3.1}{=} E \overset{(2°)}{=} A \cap B$ applied to $\tilde{L}$ gives $\tilde{Z} = C_A(Y) = C_A(R_{\tilde{L}}) = A \cap \tilde{B}$. So (c) holds. $\boxdot$

5° *Let $g \in G$. Then $Z\tilde{Z}^g = Z \times \tilde{Z}^g \in \mathcal{A}(L)$ and $|Z^G \wedge D| = 1$ for all $D \in \mathcal{A}$ where $Z^G \wedge D := \{Z^g \mid g \in G, Z^g \subseteq D\}$.*

Proof of (5°). For $1 \neq z \in Z$ we have $O_p(C_G(z)) \leq O^{p'}(C_G(z)) \overset{(4°)(b)}{=} L$. Also, 3.1 gives $L/E \cong SL_2(\mathbb{F})$ so $O_p(L) = E$. Then $O_p(C_G(z)) \leq O_p(L) = E \overset{3.1}{\leq} C_G(R_L) \overset{(3°)}{=} C_G([V,z])$. Hence, $O_p(C_G(z)) \leq O_p(C_G([V,z])) = A_z$ since A_z is the unique Sylow p-subgroup of $C_G([V,z])$ from 2.12(f). $A_z \trianglelefteq C_G(z)$ so $A_z \leq O_p(C_G(z))$. Hence, $O_p(C_G(z)) = A_z$.

Recall that $N_G(A) \leq N_G(Z)$ by (3°). Let $h \in N_G(A)$. $z^h \in Z^h = Z$ so z is not conjugate in $N_G(A)$ to any involution in $A \setminus Z$. By [Gor, 7.1.1], since A is an abelian Sylow p-subgroup of G, z is not conjugate in G to any involution in $A \setminus Z$. Then $z^G \cap A \subseteq Z$, $Z^G \wedge A = \{Z\}$, and $|Z^G \wedge D| = 1$ for any $D \in \mathcal{A}$.

Let $z^h \in C_G(z)$. A is a Sylow p-subgroup of $C_G(z)$ so $z^h \in A^k$ for some $k \in C_G(z)$. Then $z^{hk^{-1}} \in A$ and $z^{hk^{-1}} \in Z$ since $z^G \subseteq A = \{Z\}$. So $z^h \in Z^k = Z$, $Z \trianglelefteq C_G(z)$, and $z^G \cap C_G(z) = \{z^g | z^g \in C_G(z)\} \subseteq Z$.

$g \in G$ and let $1 \neq c \in \tilde{Z}^g$ and $\tilde{z} := c^{g^{-1}}$. Recall $\tilde{Z} = A \cap \tilde{B} = C_A(Y)$. Since $\tilde{Z}$ corresponds to a different fixed point in T_A/R_A than Z does, $\tilde{Z} \neq Z$. Then z and c are not conjugate in G. Notice that $\langle z, c \rangle \cong D_{2n}$ where n is even, otherwise z and c would be conjugate by Sylow's Theorem. Since n is even and $\langle zc \rangle$ is a cyclic group of order n, there exists $t \in \langle zc \rangle$ with $|t| = 2$. z and c send each element to its inverse in D_{2n} so $t \in Z(\langle z, c \rangle)$. Then $t \in O^{p'}(C_G(z)) \overset{(4°)}{=} L$. Pick $X \in Syl_p(L)$ with $t \in X$. Then $Z \leq O_p(L) \leq X$. We can then apply (4°) to X in the place of A.

Suppose for a contradiction that (4°)(b) holds. Since $t \in Z(\langle z, c \rangle)$, $c \in O^{p'}(C_G(t))$

$= L$. Hence, $[z, c] = 1$ and $t \in zc$. So $c \in tz \in Z$. Therefore, $\tilde{z} = c^{g^{-1}} \in c^G \cap A \subseteq Z$. This is a contradiction to $\tilde{z} \in \tilde{Z}$ and $\tilde{Z} \cap Z = 1$.

Suppose (4°)(c) holds. The setup is symmetric in Z and $\tilde{Z}$ so we arrive at a similar contradiction to the (4°)(b) case. Thus, (4°)(a) holds. Then X is the unique Sylow p-subgroup of $C_G(t)$. $z \in Z \leq X$ and $c \in C_G(t)$ so $c \in X$. Thus, $[z, c] = 1$ and $c \in O^{p'}(C_G(t)) = L$. This holds for any $c \in \tilde{Z}^g$. Then $\tilde{Z}^g \leq L$, $Z \in Z(L)$, and $Z \cap \tilde{Z} = 1$. So $\langle Z, \tilde{Z}^g \rangle = Z \times \tilde{Z}^g$ and has order $|\mathbb{F}|^2$. Therefore, $Z\tilde{Z}^g \in Syl_p(L) = \mathcal{A}(L)$. $\boxdot$

In particular, $A = Z \times \tilde{Z}$. By 3.1, $L/E \cong SL_2(\mathbb{F})$ and by (3°), $E = Z$; hence $L \cong Z \times SL_2(\mathbb{F})$ by Gaschütz's Theorem [Asch, 10.4]. We know $Z \leq Z(L)$ and $SL_2(\mathbb{F})$ is perfect so $L' \cong SL_2(\mathbb{F})$. Since $N_G(A)$ normalizes Y, it also normalizes $C_A(Y) = \tilde{Z}$. So $N_L(A)$ normalizes $\tilde{Z}$. Then $[A, N_L(A)] = [Z \times \tilde{Z}, N_L(A)] \leq \tilde{Z}$. We have $A/Z = [A/Z, N_L(A)]$ inside $SL_2(\mathbb{F})$ as $Z \times \begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix}$ corresponds to $N_L(A) \leq N_G(A) \leq N_G(\tilde{Z})$ from (3°). So $\tilde{Z} \leq A = Z[A, N_L(A)]$ and as $[A, N_L(A)] \leq \tilde{Z}$, we get $\tilde{Z} = (Z \cap \tilde{Z})[A, N_L(A)] = [A, N_L(A)] \leq L'$. Hence, $\tilde{Z} \leq L' \cong SL_2(\mathbb{F})$. Then $\langle \tilde{Z}^L \rangle = L'$ since A/Z has order $|\mathbb{F}|$.

Recall $Z\tilde{Z}^g \in \mathcal{A}(L)$ from (5°) so $Z\tilde{Z}^g = A^l$ for some $l \in L$ making $\tilde{Z} \leq A^{lg^{-1}}$. Thus, $A = A^{lg^{-1}h}$ by 2.20. Therefore, $lg^{-1}h \in N_G(A) \overset{(3°)}{\leq} N_G(\tilde{Z})$. So $\tilde{Z} = \tilde{Z}^{lg^{-1}h}$ and $\tilde{Z} = \tilde{Z}^{h^{-1}} = \tilde{Z}^{lg^{-1}}$. Then $\tilde{Z}^g = \tilde{Z}^l$ and we get $\langle \tilde{Z}^G \rangle = \langle \tilde{Z}^L \rangle = L'$. Hence, $[\langle Z^G \rangle, \langle \tilde{Z}^G \rangle] = 1$. Then $G_0 = \langle \mathcal{A} \rangle = \langle A^G \rangle = \langle Z^G \rangle \times \langle \tilde{Z}^G \rangle$.

Case 1: If $Z \trianglelefteq G$, then $Z = Z^G \leq A^G = (Z\tilde{Z})^G = (Z\tilde{Z})^L = A^L$. $\mathcal{R}_G = {R_A}^L = \mathcal{R}_L$ so $R_G = R_L \overset{(3°)}{=} [V, Z]$ is 2-dimensional. As $Z \trianglelefteq G$, $G_0 = Z \times \langle \tilde{Z}^G \rangle$ so $G_0 = Z \times L' = L$. Also recall from 3.1 and 2.21 that $Z = C_L(R_L)$, so $Z = C_{G_0}(R_G)$ and $G_0/C_{G_0}(R_G) \cong L/Z \cong SL_2(\mathbb{F})$ and (a) holds.

Case 2: Suppose that $Z \neq Z^g$ for some $g \in G$. Then by (5°), $\tilde{B} := Z^g\tilde{Z} = (Z\tilde{Z}^{g^{-1}})^g \in \mathcal{A}$. Then $A \cap \tilde{B} = \tilde{Z}$ by (4°). $\langle Z^G \rangle \cong SL_2(\mathbb{F}) \cong \langle \tilde{Z}^G \rangle$. Hence, $G_0 = \langle Z^G \rangle \times \langle \tilde{Z}^G \rangle \cong SL_2(\mathbb{F}) \times SL_2(\mathbb{F}) \cong \Omega_4^+(\mathbb{F})$.

Let $\tilde{w} \in \langle \tilde{Z}^G \rangle \setminus N_G(A)$. We have $\langle \tilde{Z}^G \rangle = L'$, $\langle Z^G \rangle = \tilde{L}'$, and $L = ZL' = Z\langle \tilde{Z}^G \rangle$. Then $\langle R_A^{\langle \tilde{Z}^G \rangle} \rangle = \langle R_A^{Z\langle \tilde{Z}^G \rangle} \rangle = \langle R_A^L \rangle = R_L$ giving $R_G = R_{G_0} = \langle R_A^{G_0} \rangle = \langle R_A{}^{\langle \tilde{Z}^G \rangle \langle Z^G \rangle} \rangle = \langle R_L^{\langle Z^G \rangle} \rangle$. Hence, $R_L = R_A + R_A{}^{\tilde{w}}$ so $R_G = \langle R_L^{\langle Z^G \rangle} \rangle = R_A^{\langle Z^G \rangle} + R_A^{\tilde{w}\langle Z^G \rangle} = R_{\tilde{L}} + R_{\tilde{L}}{}^{\tilde{w}}$ which are both 2-dimensional so R_G is 4-dimensional. Both $R_{\tilde{L}}$ and $R_{\tilde{L}}{}^{\tilde{w}}$ are natural $SL_2(\mathbb{F})$-modules for $\langle \tilde{Z}^G \rangle = L'$.

Hence, as an $\mathbb{F}\langle Z^G \rangle$-module, $R_G = V_1 \oplus V_2$ and as an $\mathbb{F}\langle \tilde{Z}^G \rangle$-module, $R_G = \tilde{V}_1 \oplus \tilde{V}_2$, where V_i and $\tilde{V}_i$ are natural $SL_2(\mathbb{F})$-modules. Notice that $\mathrm{End}_{\langle Z^G \rangle}(V_1) \cong \mathbb{F}$ and so by [Asch, 27.14(5)] , $R_G \cong V_1 \otimes_{\mathbb{F}} J$ for some $\mathbb{F}\langle \tilde{Z}^G \rangle$-module, J. Then $R_G \cong J \oplus J$ as $\mathbb{F}\langle \tilde{Z}^G \rangle$-modules and so $J \cong \tilde{V}_1$ as $\mathbb{F}\langle \tilde{Z}^G \rangle$-modules. Hence, $R_G \cong V_1 \cong \tilde{V}_1$ as $\mathbb{F}G_0$-modules.

The same argument also shows that the natural $\Omega_4^+(\mathbb{F})$-module for G_0 is the tensor product of two natural $SL_2(\mathbb{F})$-modules and so R_G is a natural $\Omega_4^+(\mathbb{F})$-module. This gives (b). $\square$

Chapter 4

The case where A is a TI-set

Corollary 4.1 *If A is a TI-set, then $R_A \not\leq T_B$, D is a TI-set for any $D \in \mathcal{A}$, and $R_B \cap T_A = 0$.*

Proof. Assume $R_A \leq T_B$. Then by 3.1, $|A \cap B| \neq 1$ which contradicts the definition of TI-set, so $R_A \not\leq T_B$.

Let $D = A^h$ for some $h \in G$. If D is not a TI-set, then there exists $g \in G \setminus N_G(D)$ such that $A^h \cap A^{hg} \neq 1$. This implies $A \cap A^g \neq 1$ which contradicts that A is a TI-set.

If $0 \neq R_B \cap T_A \leq R_B$, then $R_B \cap T_A = R_B$ since R_B is 1-dimensional. But then $R_B \leq T_A$ which contradicts that B is a TI-set. □

Notation 4.2

(a) $W = R_A + (T_A \cap T_B) + R_B$.

(b) $A_1 = N_A(W)$, $B_1 = N_B(W)$, *and* $L_1 = \langle A_1, B_1 \rangle$.

(c) $A_0 = N_A(R_A + R_B)$, $B_0 = N_B(R_A + R_B)$, *and* $L_0 = \langle A_0, B_0 \rangle$.

If X is one of the symbols just defined, then we will sometimes write $X(A, B)$ in place of X, to indicate the dependence of X on A and B. Also if $C, D \in \mathcal{A}$ with $C \neq D$, then $X(C, D)$ is defined as above with C and D in place of A and B.

We also fix $0 \neq r_B \in R_B$ and put $s = s_{r_B}$ and $s_1 = s\,|_{A_1}$. (For the definition of s_{r_B} see 2.9). Finally, if $R_A \not\leq T_B$, pick $X \leq [r_B, A] + R_A$ with $T_A \cap T_B \leq X$ and $[r_B, A] + R_A = X \oplus R_A$, then put $q = q_{r_B, X}$ (see 2.11).

Lemma 4.3 *Suppose that A is a TI-set. Then $N_A(R_B) \leq N_A(B)$.*

Proof. Let $a \in N_A(R_B)$. Then $R_B = R_B^a = R_{B^a}$ and by 2.23, $B^a = B$ so $a \in N_A(B)$ and we have $N_A(R_B) \leq N_A(B)$. □

Lemma 4.4 *Let $0 \neq r_B \in R_B$.*

(a) $W \cap T_A = (T_A \cap T_B) + R_A$.

(b) A normalizes $W \cap T_A$.

(c) A_1 centralizes $W/W \cap T_A$, $(W \cap T_A)/R_A$, and R_A.

(d) $A_1 = \{a \in A \mid [r_B, a] \in W \cap T_A\}$.

(e) $A_0 = \{a \in A \mid [r_B, a] \in R_A\} = \operatorname{rad} s$.

Proof. (a) Clearly $(T_A \cap T_B) + R_A \leq T_A$. Also, $R_B \cap T_A \leq T_A \cap T_B$ and so $W \cap T_A = (R_A + (T_A \cap T_B) + R_B) \cap T_A = (R_A + (T_A \cap T_B)) + (R_B \cap T_A) = R_A + (T_A \cap T_B)$.

(b) $[W \cap T_A, A] \leq R_A \leq W \cap T_A$.

(c) $[W \cap T_A, A] \leq R_A \leq W \cap T_A$ also gives us that A_1 centralizes $(W \cap T_A)/R_A$. A_1 clearly centralizes R_A. Observe that A_1 normalizes W and $W \cap T_A$ and that $W/(W \cap T_A) \overset{(a)}{=} (R_B + (W \cap T_A))/(W \cap T_A)$ is at most 1-dimensional so A_1 centralizes $W/W \cap T_A$.

(d) Let $a \in A$. Suppose $[r_B, a] \in W \cap T_A$. Then (a) gives $W = \mathbb{F}r_B + W \cap T_A$ so $[W, a] = [\mathbb{F}r_B, a] + [W \cap T_A, a] \leq W \cap T_A \leq W$ since a normalizes $W \cap T_A$. Hence, $a \in N_A(W) = A_1$. Suppose $a \in A_1$. Then by (c), A_1 centralizes $W/W \cap T_A$ so $[W, A_1] \leq W \cap T_A$ and we get $[r_B, a] \in W \cap T_A$.

(e) Let $a \in A$. Suppose $[r_B, a] \in R_A$. Then $[R_A + R_B, a] = [R_A, a] + [R_B, a] = [R_A, a] + \mathbb{F}[r_B, a] \leq R_A \leq R_A + R_B$. Hence, $a \in N_A(R_A + R_B) = A_0$. Now suppose $a \in A_0$. $(R_A + R_B)/R_A$ is a 1-space normalized by A_0 so A_0 acts trivially on it and $[r_B, a] \in R_A$. Hence, $A_0 = \{a \in A \mid [r_B, a] \in R_A\}$ and 2.10 gives $\{a \in A \mid [r_B, a] \in R_A\} = \operatorname{rad} s$. $\square$

Lemma 4.5 *Suppose that A is a TI-set and $N_B(A) \neq 1$. Then*

(a) $N_A(B) \neq 1$.

(b) $A \cap E = N_A(B) = C_A(R_B) = C_A(W) = C_A(T_A \cap T_B)$.

(c) *$A \cap E$ is an $\mathbb{F}$-subspace of A.*

(d) $|A/A \cap E| = |T_A \cap T_B|$.

(e) $[T_A, B \cap E] = T_A \cap T_B$.

(f) $\dim T_A \cap T_B \geq 2$.

(g) $\dim A \geq 4$.

(h) *W is an L-submodule of V. In particular, $R_L \leq W$ and $L = L_1$.*

Proof. From 4.1, with A and B interchanged, we have $R_B \not\leq T_A$. $[T_A, N_B(A)] \leq [V, B] = T_B$ and $[T_A, N_B(A)] = [V, A, N_B(A)] \leq T_A$ so we get

$$(*) \qquad [T_A, N_B(A)] \leq T_A \cap T_B$$

and

$$(**) \qquad [T_A \cap T_B, N_B(A)] \leq T_A \cap R_B = 0.$$

1° *Then $1 \neq N_A(B) = A \cap E$.*

Proof of (1°). Let $x \in C_A(N_B(A))$. Then $1 \neq N_B(A) = N_B(A)^x \leq B \cap B^x$. As B is a TI-set this gives $B = B^x$. Hence, $1 \neq C_A(N_B(A)) \leq N_A(B)$. Then $N_A(B) \neq 1$ and (a) is proven. In particular, the setup is symmetric in A and B.

From $(*)$ and $(**)$ we have $[T_A, N_B(A), N_B(A)] \leq [T_A \cap T_B, N_B(A)] \leq 0$. So $N_B(A)$ acts quadratically on T_A and therefore on T_A/R_A. Then, by 2.5(c), $N_B(A)$ acts quadratically on A^* and therefore also on A. Since $N_B(A)$ acts quadratically on A, we have $[A, N_B(A)] \leq C_A(N_B(A)) \leq N_A(B)$. So $[N_A(B)N_B(A), A] \leq N_A(B) \leq N_A(B)N_B(A)$. Thus, A normalizes $N_A(B)N_B(A)$, and similarly B normalizes $N_A(B)N_B(A)$. Then $N_A(B)N_B(A) \trianglelefteq \langle A, B\rangle = L$. $N_A(B)N_B(A)$ is a p-group so it's in $O_p(L)$. So also $N_A(B) \leq O_p(L)$. Then $A \cap O_p(L) \leq N_A(B) \leq A \cap O_p(L)$. So $A \cap O_p(L) = N_A(B)$. Also, $A \cap O_p(L) \leq A \cap E \leq A \cap O_p(L)$ so $N_A(B) = A \cap E$ and symmetrically $N_B(A) = B \cap E$. □

$(**)$ also gives us that $N_B(A)$ centralizes R_A, R_B, and $T_A \cap T_B$ and therefore W. Thus, $[W, N_B(A)] = 0$ and so $N_B(A) \leq C_B(W) \leq C_B(R_A) \leq N_B(R_A) \overset{4.3}{\leq} N_B(A)$. Then using the symmetry of A and B we have $A \cap E = N_A(B) = C_A(R_B) = C_A(W)$. So (b) is proven except for the last equality.

We have $E = (A \cap O_p(L))(B \cap O_p(L)) = (A \cap E)(B \cap E)$. Since A is a TI-set, we have $A \cap B = 1$. Hence, E is abelian. Then $C_A(B \cap E) \leq A \cap E$ and $C_A(B \cap E) \geq A \cap E$ so $C_A(B \cap E) = A \cap E$. By 2.7(d) $N_G(A)$, and therefore $B \cap E$, acts $\mathbb{F}$-linearly on A. Therefore, $C_A(B \cap E) = A \cap E$ is an $\mathbb{F}$-subspace of A. Then (c) is proven. Also, 2.8(d) gives

$$(***) \qquad [T_A, B \cap E] + R_A = C_{T_A}(C_A(B \cap E)) = C_{T_A}(A \cap E).$$

2° $\quad T_A \cap T_B = [T_A, B \cap E]$.

Proof of (2°). $R_A \leq C_{T_A}(A \cap E)$ so

$$(T_A \cap T_B) + R_A \overset{(**)}{\leq} C_{T_A}(A \cap E) \overset{(***)}{=} [T_A, B \cap E] + R_A \overset{(*) and (\mathrm{b})}{\leq} (T_A \cap T_B) + R_A.$$

So equality holds everywhere. Then $T_A \cap T_B = [T_A, B \cap E] + (T_A \cap T_B \cap R_A) = [T_A, B \cap E]$. □ So (e) is proven.

3° $W = [T_A + T_B, E] = [T_L, E]$ *is a L-submodule and* $R_L \leq W$.

Proof of (3°). $[T_L, E] = [T_A + T_B, E] = [T_A, A \cap E] + [T_A, B \cap E] + [T_B, A \cap E] + [T_B, B \cap E] = R_A + (T_A \cap T_B) + R_B = W$ from (2°). Then W is a L-submodule. Since $R_A \leq W$, $R_L = \langle R_A^L \rangle \leq W^L = W$. ⊡

So L normalizes W giving $A = A_1$, $B = B_1$, and $L = L_1$. Then (h) is proven. Put $\overline{L} = L/C_L(W)$ and $\overline{A} = A/C_A(W)$.

4° $A \cap E = C_A(T_A \cap T_B)$ *and* $|\overline{A}| = |A/A \cap E| = |T_A \cap T_B|$.

Proof of (4°). $(***)$ and (2°) give $C_{T_A}(A \cap E) = [T_A, B \cap E] + R_A = (T_A \cap T_B) + R_A$. Then $C_A(T_A \cap T_B) = C_A(T_A \cap T_B + R_A) = C_A(C_{T_A}(A \cap E)) = A \cap E$ by 2.8(b). Now the proof of (b) is complete.

Also, 2.8(b) applied with $Y = T_A \cap T_B$ gives $|A/A \cap E| = |A/C_A(T_A \cap T_B)| = |((T_A \cap T_B) + R_A)/R_A| = |T_A \cap T_B|$ since $R_A \nleq T_A \cap T_B$. Then (d) is proven. We also have $|\overline{A}| = |A/C_A(R_A + (T_A \cap T_B) + R_B)| = |A/C_A(T_A \cap T_B)|$ since $A \cap E \leq C_A(R_B)$.
⊡

5° $\dim_{\mathbb{F}} T_A \cap T_B \geq 2$.

Proof of (5°). Observe that $N_B(A)A$ is a p-group and $N_B(A) \nleq A$ since $1 \neq N_B(A)$ and $N_B(A) \cap A = 1$ as A is a TI-set. Then A is not a Sylow p-subgroup. Choose $1 \neq s \in N_B(A)$. Then $s \notin A$ and $A\langle s \rangle$ is a p-group. $\dim T_A \cap T_B = \dim[A, N_B(A)] \geq \dim[A, s] \geq 2$ by 3.2 and (f) is proven. ⊡

If $\dim A < 4$, then 3.2 gives $A \in Syl_p(G)$ yielding a contradiction so $\dim A \geq 4$ and (g) is proven. □

Lemma 4.6 *Suppose A is a TI-set. Then $N_A(B) = A \cap E = C_A(W)$ and $A \cap E$ is an $\mathbb{F}$-subspace of A.*

Proof. If $N_A(B) \neq 1$, this follows from 4.5. So suppose $N_A(B) = 1$. By 2.21, $A \cap E = A \cap C_L(\mathcal{A}(L)) \leq N_A(B) = 1$.

From 4.3 we have $C_A(W) \leq C_A(R_B) \leq N_A(R_B) \leq N_A(B) = 1$. So $N_A(B) = A \cap E = C_A(W) = 1$. In particular, $A \cap E = 1$ is $\mathbb{F}$-subspace of A. $\square$

Lemma 4.7 *Suppose A is a TI-set.*

(a) If $A/(A \cap E)$ is even dimensional, then $A \cap E = \operatorname{rad} s = A_0$.

(b) If $A/(A \cap E)$ is odd dimensional, then $A_0/A \cap E$ is 1-dimensional.

Proof. By 4.6, we have $A \cap E = C_A(W) = C_A(R_B) = C_A(R_A + R_B)$. Clearly, $C_{A_0}(R_A + R_B) \leq C_A(R_A + R_B)$ as $A_0 \leq A$. Also, $C_A(R_A + R_B) \leq C_{A_0}(R_A + R_B)$ since $C_A(R_A + R_B) \leq N_A(R_A + R_B) = A_0$. Hence, $C_{A_0}(R_A + R_B) = C_A(R_A + R_B)$ and $A \cap E = C_{A_0}(R_A + R_B)$. Also, $A_0/C_{A_0}(R_A + R_B)$ is isomorphic to a 2-subgroup of $SL_2(\mathbb{F})$ making $|A_0/A \cap E| \leq |\mathbb{F}|$. Notice that both $A_0 \overset{4.4}{=} \operatorname{rad} s$ and $A \cap E$ are $\mathbb{F}$-spaces from 2.10 and 4.6. Thus, $A_0/A \cap E$ is an $\mathbb{F}$-space and either $A_0 = A \cap E$ or $A_0/A \cap E$ is 1-dimensional. Since $A_0 = \operatorname{rad} s$ is a non-degenerate symplectic space, A/A_0 is even dimensional. So if $A_0 = A \cap E$, then (a) holds and if $A_0/A \cap E$ is 1-dimensional, (b) holds. $\square$

Lemma 4.8 *Suppose A is a TI-set and $A/A \cap E$ is odd dimensional. Then L_0 induces $SL_2(\mathbb{F})$ on $R_A + R_B$. In particular, L_0 acts transitively on the 1-spaces in $R_A + R_B$.*

Proof. We have $|A_0/A \cap E| = |\mathbb{F}|$ from 4.7(b). Also,

$$A \cap E \leq C_{A_0}(R_A + R_B) = C_{A_0}(R_B) = N_{A_0}(R_B) \overset{4.3}{\leq} N_{A_0}(B) \overset{4.6}{\leq} A \cap E$$

as $[R_A, A_0] = 0$. So $|A_0/C_{A_0}(R_A + R_B)| = |\mathbb{F}|$. Thus, A_0 induces the full centralizer of R_A in $SL_{\mathbb{F}}(R_A + R_B)$ on $R_A + R_B$. A similar statement holds for B_0 and we conclude that $L_0 = \langle A_0, B_0 \rangle$ induces $SL_2(\mathbb{F})$ on $R_A + R_B$. $\square$

Lemma 4.9 *Suppose A is a TI-set. Then*

(a) A_1 is an $\mathbb{F}$-subspace of V.

(b) *Let* $a \in A_1$*. Then* $r_B - q(a) \in T_{Ba}$*. Moreover,* $q(a) = 0$ *if and only if* $a \in A \cap E$*.*

(c) *Suppose* $A_1 \neq A \cap E$ *and let* D *be* $\mathbb{F}$*-subspace of* A_1 *maximal with* $C_D(W) = 1$ *and* $s_1|_D = 0$*. Then* $q|_D$ *is onto and* $\dim D = 1$*.*

(d) *One of the following holds:*

1. $s_1 = 0$, $A_1 = \operatorname{rad} s_1$ *and* $\dim A_1/A \cap E \leq 1$.

2. $s_1 \neq 0$, $\operatorname{rad} s_1 = A \cap E$ *and* $\dim A_1/A \cap E = 2$.

Proof. (a) By 4.4, $A_1 = \{a \in A \mid [r_B, a] \in W \cap T_A\}$. Since $W \cap T_A$ is an $\mathbb{F}$-subspace of T_A, we put $U = W \cap T_A$ in 2.10(a) and conclude that A_1 is an $\mathbb{F}$-subspace of A.

(b) Let $a \in A_1$. By definition of q and X in 2.11, $[r_B, a] - q(a) \in X$. Also, $[r_B, a] \in W \cap T_A = (T_A \cap T_B) + R_A$ from 4.4. Since $q(a) \in R_A$, $[r_B, a] - q(a) \in ((T_A \cap T_B) + R_A) \cap X$. Since $T_A \cap T_B \leq X$, this means $[r_B, a] - q(a) \in (T_A \cap T_B) + (R_A \cap X) = T_A \cap T_B + 0 = T_A \cap T_B$. So $r_B^a - r_B - q(a) \in T_A \cap T_B$ and since $r_B \in R_B^\sharp \subseteq T_B$, $r_B^a - q(a) \in T_B$. Conjugation by a gives, $r_B - q(a) \in T_{Ba}$ since $q(a) \in R_A \leq C_V(a)$.

If $a \in N_A(B)$, then $r_B{}^a = r_B$. In this case $q(a) \in T_A \cap T_B$, and since $R_A \cap (T_A \cap T_B) = 0$, we must have $q(a) = 0$. If $q(a) = 0$, we conclude that $r_B \in T_{Ba}$. If $B^a \neq B$, then $R_B \leq T_{Ba}$ gives a contradiction to 4.1 so we must have $B = B^a$ and $a \in N_A(B)$. By 4.6, $N_A(B) = A \cap E$ and so (b) holds.

(c) Notice that $D \neq 0$ since s_1 vanishes on any 1-dimensional subspace, so $|D| \geq |\mathbb{F}|$. By 2.11, $q|_D$ is $\mathbb{Z}$-linear. By (b) and 4.6, $q(d) \neq 0$ for all $a \notin C_A(W)$ and so for all $d \in D^\sharp$ since $C_D(W) = 1$. Thus, $q|_D$ is one to one. Since $|q(D)| \leq |R_A| \leq |\mathbb{F}|$, we conclude that $|D| \leq |\mathbb{F}|$ and so $\dim D = 1$ and $q\,|_D$ is onto.

(d) Notice that $C_D(W) = 1$ is equivalent to $D \cap (A \cap E) = 1$ and observe that $(D(A \cap E))/(A \cap E)$ is a maximal isotropic subspace of $A_1/A \cap E$. Thus, $A \cap E \leq \operatorname{rad} s_1 \leq D(A \cap E)$. Then $\operatorname{rad} s_1 = (D(A \cap E)) \cap \operatorname{rad} s_1 = (D \cap \operatorname{rad} s_1)(A \cap E)$. Moreover, $D \operatorname{rad} s_1/\operatorname{rad} s_1$ is a maximal isotropic subspace of $A_1/\operatorname{rad} s_1$ and since the maximal isotropic subspaces of non-degenerate $2n$-dimensional symplectic spaces have dimension n, we conclude that $\dim A_1/\operatorname{rad} s_1 = 2 \cdot \dim(D \operatorname{rad} s_1/\operatorname{rad} s_1)$. This comes from [Asch, 19.15, 19.16, 20.8].

If $D \not\leq \operatorname{rad} s_1$, we get $\dim A_1/\operatorname{rad} s_1 = 2$. Also in this case, if $D \cap \operatorname{rad} s_1 \neq 0$, then $D = D \cap \operatorname{rad} s_1$ since $\dim D \leq 1$. Hence, $D \leq \operatorname{rad} s_1$ which is a contradiction to our assumption. So $D \cap \operatorname{rad} s_1 = 0$ and $\operatorname{rad} s_1 = (D \cap \operatorname{rad} s_1)(A \cap E) = A \cap E$ in this case. If $D \leq \operatorname{rad} s_1$, we get $\dim A_1/\operatorname{rad} s_1 = 0$, $A_1 = \operatorname{rad} s_1 = D(A \cap E)$ and $\dim A_1/A \cap E = \dim D(A \cap E)/(A \cap E) \leq 1$ since $\dim D = 1$. $\square$

Lemma 4.10 *Suppose A is a TI-set, W is 4-dimensional, and $[r_B, A_1] + R_A = W \cap T_A$ for some $0 \neq r_B \in R_B$. Then $\mathcal{A}(W) := \{D \in \mathcal{A} \mid R_D \leq W\} = A^{L_1} = \{A\} \cup B^{A_1} = \{B\} \cup A^{B_1}$, $|A_1/A \cap E| = |\mathbb{F}|^2$, $|\mathcal{A}(W)| = |\mathbb{F}|^2 + 1$, and L_1 acts doubly transitive on $\mathcal{A}(W)$. Also A and B are conjugate and $W = \langle R_A^{L_1} \rangle$. Furthermore, $T_A \cap W$ is the perp of R_A with respect to s_W, where s_W is the symmetric form associate to q_W, and there exists $q_W : W \to R_A \cong \mathbb{F}$, an L_1-invariant quadratic form of $-$-type, (the maximal singular subspaces of W with respect to q_W are 1-dimensional). $\{R_D \mid D \in \mathcal{A}, R_D \leq W\} = \{R_D \mid D \in \mathcal{A}(W)\}$ is the set of singular 1-spaces and L_1 induces $\Omega(W, q_W)$ on W.*

Proof. Observe that $A_1 \neq A \cap E$ since $[r_B, A_1] \neq 1$. Let $D \in \mathcal{A}(W)$ with $D \neq A$. By 4.1, $R_D \not\leq W \cap T_A$. By assumption we have $(W \cap T_A)/R_A = ([r_B, A_1] + R_A)/R_A$. Let H be a 1-space of W/R_A with $H \not\leq (W \cap T_A)/R_A$. As $(W \cap T_A)/R_A \overset{4.4(a)}{=} (R_A + (T_A \cap T_B))/R_A$ is a hyperplane of W/R_A, $H + (W \cap T_A)/R_A = W/R_A$. So there exists $x \in (W \cap T_A)/R_A$ and $e \in H$ with $e + x = \tilde{r}_B$ where $\tilde{r}_B = r_B + R_A$. Then $e - \tilde{r}_B = -x \in (W \cap T_A)/R_A$. Now $-x = [\tilde{r}_B, a]$ for some $a \in A_1$ since $-x \in (W \cap T_A)/R_A = ([r_B, A_1] + R_A)/R_A$. Then $\tilde{r}_B^a = \tilde{r}_B + [\tilde{r}_B, a] = \tilde{r}_B - x = e$. Hence, $H = (R_B^a + R_A)/R_A$ for some $a \in A_1$. So we see that A_1 acts transitively on the 1-spaces of W/R_A that are not in $(W \cap T_A)/R_A$. We know $(R_A + R_D)/R_A$ is a 1-space so there exists $a \in A_1$ with $R_D \leq R_A + R_B^a$. Replacing D by $D^{a^{-1}}$ we may assume that $R_D \leq R_A + R_B$.

Choose $s \in R_A$ with $s + r_B \in R_D$. From 4.9(c) we have $q\mid_{A_1}$ is onto. So there exists $d \in A_1$ with $q(d) = s$ so $s + r_B = q(d) + r_B \in T_B^d$ from 4.9(b). Thus, $R_D \leq T_B^d$ and by 4.1, $D = B^d$. Hence, $\mathcal{A}(W) = \{A\} \cup B^{A_1}$. By symmetry, $\mathcal{A}(W) = \{B\} \cup A^{B_1}$. Then $|B^{A_1}| = |A_1E/E|$ so $|\mathcal{A}(W)| = |A_1E/E| + 1$. Therefore, L_1 acts doubly transitive on $\mathcal{A}(W)$. So A and B are conjugate and there exists $g \in L_1$ with $A^g = B$. By hypothesis, $[r_B, A_1] + R_A = W \cap T_A \overset{4.4(a)}{=} T_A \cap T_B + R_A$ so $W \leq R_B + [r_B, A_1] + R_A \leq \langle R_B^{L_1}\rangle + R_A \leq \langle R_A^{L_1}\rangle \leq W$ since A and B are conjugate. So $W = \langle R_A^{L_1}\rangle$.

It remains to show that such a q_W exists.

Now let $a \in A_1 \setminus E$. Then by 4.9(d), $a \notin \operatorname{rad} s_1$ and so $[W, a, A_1] \neq 0$. Since A_1 normalizes but does not centralize $[W, a]$, we conclude that $[W, a]$ is at least 2-dimensional. Note that $W = R_B \oplus R_B^a \oplus (T_B \cap T_B^a)$. Since $[W, a]$ is at least 2-dimensional, we get that $[T_B \cap T_B^a, a] \neq 0$.

Since $a \in A_1 \setminus E$ and $A \cap E = N_A(B)$, $B^a \neq B$. We know $B = A^g$ for some $g \in L_1$. We've shown that A and $B = A^g$ are in $\mathcal{A}(W)$ and since L_1 is doubly transitive on $\mathcal{A}(W)$, $(B^a)^l = A$ and $B^l = B$ for some $l \in L_1$. So we have $B^{al} = A$. Conjugating by $a^l = l^{-1}al$ we get $B^{all^{-1}al} = A^{l^{-1}al}$. Then $B^l = A^{a^l}$. As $B^l = B$, we can let $\omega = a^l$ and see that there exists an $\omega \in L_1$ such that $A^\omega = B$ and $\omega^2 = 1$. Since $0 \neq [T_B \cap T_B^\omega, \omega]$, conjugating by l gives $0 \neq [T_B{}^l \cap T_B^{\omega^l}, \omega] = [T_B \cap T_B^\omega, \omega] = [T_B \cap T_A, \omega]$.

Choose $v_0 \in R_A^\sharp$ and $v_1 \in T_A \cap T_B$ with $[v_1, \omega] \neq 0$. Put $v_2 = v_1^\omega$ and $v_3 = v_0^\omega$. Observe that $v_3 \in R_B$. Then (v_1, v_2) is an $\mathbb{F}$-basis for $T_A \cap T_B$. Since $W = R_A + (T_A \cap T_B) + R_B$, (v_0, v_1, v_2, v_3) is an $\mathbb{F}$-basis for W. Let $\tilde{L}_1$ be the image of L_1 in $GL_4(\mathbb{F})$ and $\tilde{A}_1 = A_1/C_{A_1}(W) \overset{4.6}{=} A_1/(A \cap E)$. Since E acts trivially on both W and $\mathcal{A}(L)$, $\tilde{L}_1$ acts on both W and $\mathcal{A}(L)$.

Let $c \in \tilde{A}_1$. Now $[v_3, c] \in [W, A_1] \leq W \cap T_A = R_A + (T_A \cap T_B) = \mathbb{F}v_0 + \mathbb{F}v_1 + \mathbb{F}v_2$

so $[v_3, c] = x_0(c)v_0 + x_1(c)v_1 + x_2(c)v_2$ for some $x_i(c) \in \mathbb{F}$. Put $x(c) = (x_1(c), x_2(c))$. Since R_B is 1-dimensional, $v_3 = \lambda r_B$ for some $0 \neq \lambda \in \mathbb{F}$. Thus, $[v_3, A_1] + R_A = [\lambda r_B, A_1] + R_A = \lambda([r_B, A_1] + R_A]) = \lambda(W \cap T_A)$ by assumption. And $\lambda(W \cap T_A) = W \cap T_A$ so $[v_3, A_1] + R_A = W \cap T_A = \mathbb{F}v_1 + \mathbb{F}v_2 + R_A$. Then $x : \tilde{A}_1 \to \mathbb{F}^2$ is onto. From 4.9(d), $\dim A_1/A \cap E \leq 2$ so we conclude that x is a bijection and $|A_1/A \cap E| = |\mathbb{F}^2|$. Since A_1 acts quadratically on W/R_A, $x(cd) = x(c) + x(d)$ for all $c, d \in \tilde{A}_1$. So x is an isomorphism from $\tilde{A}_1$ to $(\mathbb{F}^2, +)$. Let $a : \mathbb{F}^2 \to \tilde{A}_1, t \to a(t)$ be inverse of $x : \tilde{A}_1 \to \mathbb{F}^2$. Define $q(t) = x_0(a(t))$. Since $x(a(t)) = t$, we have $[v_3, a(t)] = q(t)v_0 + t_1v_1 + t_2v_2$. Hence, $v_3^{a(t)} = q(t)v_0 + t_1v_1 + t_2v_2 + v_3$. Then $q(t) = 0$ if and only if $v_3^{a(t)} \in \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3$. Since $W \cap T_B = \mathbb{F}v_1 + \mathbb{F}v_2 + \mathbb{F}v_3$ and $\mathbb{F}v_3 = R_B$, this holds if and only if $R_B^{a(t)} \leq T_B$ which, by 4.1, holds if and only if $a(t) \in N_{\tilde{A}_1}(B)$. $N_{\tilde{A}_1}(B) = 1$ since $N_A(B) = C_A(W)$ from 4.6 and $C_{\tilde{A}_1}(W) = 1$. So $q(t) = 0$ if and only if $a(t) = 1$. Since $a : \mathbb{F}^2 \to \tilde{A}_1$ is 1-1, this holds if and only if $t = 0$. Then $q(t) = 0$ if and only if $t = 0$.

We have $\tilde{\omega} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix}$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $t = (t_1, t_2) \in \mathbb{F}^2$. We have shown that there exists $a(t) \in \tilde{A}_1$, and similarly $b(t) \in \tilde{B}_1$, such that

$$a(t) = \begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n_1(t_1, t_2) & 1 & 0 & 0 \\ n_2(t_1, t_2) & 0 & 1 & 0 \\ q(t_1, t_2) & t_1 & t_2 & 1 \end{pmatrix} \in \tilde{A}_1$$

and

$$b(t) = \begin{pmatrix} 1 & t & \tilde{q}(t) \\ 0 & I & \tilde{n}(t) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1 & t_2 & \tilde{q}(t_1, t_2) \\ 0 & 1 & 0 & \tilde{n}_1(t_1, t_2) \\ 0 & 0 & 1 & \tilde{n}_2(t_1, t_2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \tilde{B}_1.$$

Observe that $x(a(t)a(t')) = t + t'$ so applying a to both sides gives $a(t)a(t') = a(t+t')$. Then we get

$$n(t) + n(t') = n(t+t'). \tag{1}$$

and

$$q(t) + tn(t') + q(t') = q(t+t'). \tag{2}$$

Define a tilde function from $\mathbb{F}^2 \setminus \{0\} \to \mathbb{F}^2 \setminus \{0\}$ by $t \to \tilde{t}$ with $B^{a(t)} = A^{b(\tilde{t})} \in \mathcal{A}(W) \setminus \{A, B\}$. Then $\mathbb{F}(q(t) \quad t \quad 1) = \mathbb{F}(1 \quad \tilde{t} \quad \tilde{q}(\tilde{t}))$ and we see that

$$\tilde{t} = \frac{t}{q(t)} \tag{3}$$

and

$$\tilde{q}(\tilde{t}) = \frac{1}{q(t)}. \tag{4}$$

Now

$$\begin{aligned} a(t)^\omega &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} q(t) & t & 1 \\ n^*(t) & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t^* & q(t) \\ 0 & I & n^*(t) \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where $n^*(t) = \begin{pmatrix} n_2(t_1,t_2) \\ n_1(t_1,t_2) \end{pmatrix}$ and $t^* = (t_2, t_1)$. Notice that $a(t)^\omega \in \tilde{B}_1$ and $\tilde{A}_1$ and $\tilde{B}_1$ are conjugate by ω so $a(t)^\omega = b(t^*)$.

Then, since we have $a(t)^\omega = b(t^*)$, we get

$$\tilde{q}(t^*) = q(t) \tag{5}$$

and

$$\tilde{n}(t^*) = n^*(t). \tag{6}$$

Thus, (6) together with (3) gives

$$\tilde{n}(\tilde{t}) = n^*(\frac{t^*}{q(t)}).$$

Since $B^{a(t)} = A^{b(\tilde{t})}$, we have $A^{\omega a(t)} = A^{b(\tilde{t})}$ so $A^{\omega a(t) b(\tilde{t})} = A$. Therefore, $\omega a(t) b(\tilde{t}) \in N_{L_1}(R_A)$ and $N_{L_1}(A)$ so it normalizes $[W, A_1]$. Hence, $\omega a(t) b(\tilde{t})$ must be of the shape $\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$.

We already have

$$\begin{aligned}
\omega a(t) b(\tilde{t}) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \\ 0 & I & n^*(\frac{t^*}{q(t)}) \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} q(t) & t & 1 \\ n^*(t) & X & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \\ 0 & I & n^*(\frac{t^*}{q(t)}) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} q(t) & t+tI & 1+tn^*(\frac{t^*}{q(t)}+1 \\ n^*(t) & n^*(t)\frac{t}{q(t)}+X & \frac{n^*(t)}{q(t)}+n(\frac{t^*}{q(t)}) \\ 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \end{pmatrix}$$

$$= \begin{pmatrix} q(t) & 0 & tn^*(\frac{t^*}{q(t)}) \\ n^*(t) & n^*(t)\frac{t}{q(t)}+X & \frac{n^*(t)}{q(t)}+n(\frac{t^*}{q(t)}) \\ 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \end{pmatrix}.$$

Then we get

$$tn^*(\frac{t^*}{q(t)}) = 0 \tag{7}$$

and

$$\frac{n^*(t)}{q(t)} = n(\frac{t^*}{q(t)}). \tag{8}$$

From (3), (4), and (5) we obtain

$$\frac{1}{q(t)} = q(\frac{t^*}{q(t)}). \tag{9}$$

From (7) and (8) we obtain

$$tn(t) = 0. \tag{10}$$

This gives $(t+t')n(t+t') = 0$ so (1) and (10) yield

$$t'n(t) = tn(t'). \tag{11}$$

Since we showed $[W, a(t')]$ is 2-dimensional earlier, $x_2 \neq 0$. Consider $t' = (1, 0)$.

Then $n(t') = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ for some x_2 from (10). And $t'n(t) + tn(t') = 0$ from (11) so $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} + \begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$. Hence, $n_1(t) + t_2x_2 = 0$. Similarly, consider $t' = (0,1)$. Then $n(t') = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ for some x_1 from (10). This yields $n_2(t) + t_1x_1 = 0$. Thus, $n(t) = \begin{pmatrix} t_2x_2 \\ t_1x_1 \end{pmatrix}$.

Again from (10), $tn(t) = 0$ so we have $\begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} t_2x_2 \\ t_1x_1 \end{pmatrix} = t_1t_2x_2 + t_2t_1x_1 = 0$. Then $t_1t_2(x_2 + x_1) = 0$ and $x_2 = x_1$. Replacing v_1 by x_2v_1 and so also v_2 by x_2v_2 we can let $x_2 = x_1 = 1$ and discover

$$n(t) = \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}. \tag{12}$$

We calculate

$$\begin{aligned}
\omega(t) := a(t)b(\tilde{t})a(t) &= \begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \\ 0 & I & n^*(\frac{t^*}{q(t)}) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \\ n(t) & n(t)\frac{t}{q(t)} + I & \frac{n(t)}{q(t)} + In^*(\frac{t^*}{q(t)}) \\ q(t) & t + tI & 1 + tn^*(\frac{t^*}{q(t)}) + 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{t}{q(t)} & \frac{1}{q(t)} \\ n(t) & n(t)\frac{t}{q(t)} + I & 0 \\ q(t) & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 1 + \frac{t}{q(t)} n(t) + 1 & \frac{t}{q(t)} I + \frac{t}{q(t)} & \frac{1}{q(t)} \\ n(t) + n(t)\frac{t}{q(t)} n(t) + n(t) & n(t)\frac{t}{q(t)} I + I & 0 \\ q(t) & 0 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 0 & \frac{1}{q(t)} \\ 0 & n(t)\frac{t}{q(t)} + I & 0 \\ q(t) & 0 & 0 \end{pmatrix}.
$$

Now we can find

$$
h(t) := \omega(t)\omega = \begin{pmatrix} 0 & 0 & \frac{1}{q(t)} \\ 0 & n(t)\frac{t}{q(t)} + I & 0 \\ q(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{1}{q(t)} & 0 & 0 \\ 0 & n(t)\frac{t^*}{q(t)} + X & 0 \\ 0 & 0 & q(t) \end{pmatrix}.
$$

So $h(t) \in N_G(R_A) = N_G(A)$.

Which allows us to calculate

$$
a(t)^{h(k)} = (h(k))^{-1} a(t) h(k)
$$

$$
= \begin{pmatrix} q(k) & 0 & 0 \\ 0 & (n(k)\frac{k^*}{q(k)} + X)^{-1} & 0 \\ 0 & 0 & \frac{1}{q(k)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ n(t) & I & 0 \\ q(t) & t & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} \frac{1}{q(k)} & 0 & 0 \\ 0 & n(k)\frac{k^*}{q(k)} + X & 0 \\ 0 & 0 & q(k) \end{pmatrix}
$$

$$= \begin{pmatrix} q(k) & 0 & 0 \\ (n(k)\frac{k^*}{q(k)} + X)^{-1}n(t) & (n(k)\frac{k^*}{q(k)} + X)^{-1}I & 0 \\ \frac{q(t)}{q(k)} & \frac{t}{q(k)} & \frac{1}{q(k)} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{q(k)} & 0 & 0 \\ 0 & n(k)\frac{k^*}{q(k)} + X & 0 \\ 0 & 0 & q(k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ (n(k)\frac{k^*}{q(k)} + X)^{-1}\frac{n(t)}{q(k)} & I & 0 \\ \frac{q(t)}{q(k)^2} & \frac{t}{q(k)}(n(k)\frac{k^*}{q(k)} + X) & 1 \end{pmatrix}.$$

Now $h(t)$ normalizes a so we have $a(t)^{h(k)} = a(r)$ for some $r \in \mathbb{F} \times \mathbb{F}$ which depends on t and k. Then

$$\begin{aligned} r &= \frac{t}{q(k)}(n(k)\frac{k^*}{q(k)} + X) \\ &= \frac{1}{q(k)^2}\begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} n_1(k_1) \\ n_2(k_2) \end{pmatrix}\begin{pmatrix} k_2 & k_1 \end{pmatrix} + \frac{1}{q(k)^2}q(k)\begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\overset{(12)}{=} \frac{1}{q(k)^2}\begin{pmatrix} t_1 & t_2 \end{pmatrix}\begin{pmatrix} k_2 \\ k_1 \end{pmatrix}\begin{pmatrix} k_2 & k_1 \end{pmatrix} + \frac{1}{q(k)^2}q(k)\begin{pmatrix} t_2 & t_1 \end{pmatrix}. \end{aligned}$$

So $r = \frac{1}{q(k)^2}(t_1k_2 + t_2k_1)\begin{pmatrix} k_2 & k_1 \end{pmatrix} + \frac{1}{q(k)^2}q(k)\begin{pmatrix} t_2 & t_1 \end{pmatrix}$
$= \begin{pmatrix} \frac{t_1k_2^2 + t_2k_1k_2 + q(k)t_2}{q(k)^2} & \frac{t_1k_2k_1 + t_2k_1^2 + q(k)t_1}{q(k)^2} \end{pmatrix}.$

This gives

$$\frac{q(r)}{q(k)^2} = q(\frac{t_1k_2^2 + t_2k_1k_2 + t_2q(k)}{q(k)^2}, \frac{t_1k_2k_1 + t_2k_1^2 + t_1q(k)}{q(k)^2}).$$

Let $q_1(k) = q(k,0)$ and $q_2(k) = q(0,k)$. If $k_2 = 0$, then

$$\frac{q(t)}{q_1(k_1)^2} = q(\frac{t_2}{q_1(k_1)}, \frac{t_2 k_1^2 + t_1 q_1(k_1)}{q_1(k_1)^2}).$$

If $t_2 = 0$, then

$$\frac{q_1(t_1)}{q_1(k_1)^2} = q_2(\frac{t_1}{q_1(k_1)}).$$

Let $\alpha = q_2(1)$ and $q_1(k_1) = t_1$. Then $q_1(t_1) = q_1(k_1)^2 q_2(\frac{t_1}{q_1(k_1)}) = t_1^2 q_2(1) = t_1^2\alpha$. Similarly, $q_2(t_2) = t_2^2\alpha$. (2) and (12) give

$$q(t_1,t_2) = \alpha t_1^2 + t_1 t_2 + \alpha t_2^2. \tag{13}$$

Then q is a quadratic form with associated symplectic form $s((t_1,t_2),(s_1,s_2)) = t_1 s_2 + t_2 s_1$. Define $q_W(s_0 v_0 + s_1 v_1 + s_2 v_2 + s_3 v_3) = s_0 s_3 + \alpha s_1^2 + s_1 s_2 + \alpha s_2^2$. q_W is a quadratic form with associated symplectic form $s_W((t_1,t_2,t_3,t_4),(s_1,s_2,s_3,s_4)) = t_1 s_4 + t_2 s_3 + t_3 s_2 + t_4 s_1$. Recall $\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & X & 0 \\ 1 & 0 & 0 \end{pmatrix}$ so it switches s_0 and s_3 and switches s_1 and s_2 clearly making q_W ω-invariant.

Now recall $a(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n_1(t_1,t_2) & 1 & 0 & 0 \\ n_2(t_1,t_2) & 0 & 1 & 0 \\ q(t_1,t_2) & t_1 & t_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_2 & 1 & 0 & 0 \\ t_1 & 0 & 1 & 0 \\ q(t) & t_1 & t_2 & 1 \end{pmatrix}$. Then

$$\begin{aligned}
&q_W((s_0 v_0 + s_1 v_1 + s_2 v_2 + s_3 v_3)^{a(t_1,t_2)}) \\
&= q_W(s_0 v_0 + (s_1 t_2 v_0 + s_1 v_1) + (s_2 t_1 v_0 + s_2 v_2) + (s_3 q(t) v_0 + s_3 t_1 v_1 + s_3 t_2 v_2 + s_3 v_3)) \\
&= q_W((s_0 + s_1 t_2 + s_2 t_1 + s_3 q(t)) v_0 + (s_1 + s_3 t_1) v_1 + (s_2 + s_3 t_2) v_2 + s_3 v_3) \\
&= (s_0 + s_1 t_2 + s_2 t_1 + s_3 q(t)) s_3 + \alpha (s_1 + s_3 t_1)^2 \\
&\quad + (s_1 + s_3 t_1)(s_2 + s_3 t_2) + \alpha (s_2 + s_3 t_2)^2
\end{aligned}$$

$$
\begin{aligned}
&= s_0s_3 + s_1t_2s_3 + s_2t_1s_3 + s_3^2q(t) + \alpha s_1^2 + \alpha s_3^2t_1^2 + s_1s_2 + s_1s_3t_2 + s_3t_1s_2 \\
&\quad + s_3^2t_1t_2 + \alpha s_2^2 + \alpha s_3^2t_2^2 \\
&= s_0s_3 + s_1t_2s_3 + s_2t_1s_3 + s_3^2\alpha t_1^2 + s_3^2t_1t_2 + s_3^2\alpha t_2^2 + \alpha s_1^2 + \alpha s_3^2t_1^2 + s_1s_2 \\
&\quad + s_1s_3t_2 + s_3t_1s_2 + s_3^2t_1t_2 + \alpha s_2^2 + \alpha s_3^2t_2^2 \\
&= s_0s_3 + \alpha s_1^2 + s_1s_2 + \alpha s_2^2 \\
&= q_W(s_0v_0 + s_1v_1 + s_2v_2 + s_3v_3).
\end{aligned}
$$

So q_W is A_1-invariant as well as ω-invariant. $L_1 = \langle A_1, A_1^\omega\rangle = \langle A, \omega\rangle$ so q_W is also L_1-invariant.

Consider $R_A^\perp = v_0^\perp = \mathbb{F}v_0 + \mathbb{F}v_1 + \mathbb{F}v_2$. Observe that the definition of s_W shows that $T_A \cap W$ is the perp of R_A with respect to s_W. To be singular we must have $0 = q_W(t_0v_0 + t_1v_1 + t_2v_2) = \alpha t_1^2 + t_1t_2 + \alpha t_2^2 = q(t_1, t_2)$ which implies $t_1 = t_2 = 0$ from earlier. Hence, R_A is a maximal singular subspace and we see that the maximal singular subspaces are 1-dimensional and q_W is of $-$-type.

Due to the double transitivity we have $R_A^{L_1} = \{R_D \mid D \in \mathcal{A}(W)\}$. W has $|\mathbb{F}|^2 + 1$ singular 1-subspaces and $|R_A^{L_1}| = |\mathbb{F}|^2 + 1$ where each R_D is singular so $\{R_D \mid D \in \mathcal{A}(W)\}$ is exactly the set of singular 1-subspaces of W.

[Asch, Chap 7] gives us that $\Omega(W, q_W) \cong SL_2(\mathbb{K})$ where $\mathbb{K}$ is a quadratic extension of $\mathbb{F}$. $\tilde{A}_1$ is the image of A_1 in $O(W, q_W)$. Since $\tilde{A}_1$ centralizes $R_A^\perp/R_A$, $A_1 \leq \Omega(W, q_W)$ and $|\tilde{A}_1| = q^2$, which is the order of the Sylow subgroup in $SL_2(\mathbb{K})$. So A_1 and B_1 are sent to different Sylow subgroups and $SL_2(\mathbb{K})$ is generated by two Sylow subgroups so L_1 induces $\Omega(W, q_W)$. □

Lemma 4.11 *Suppose that A is a TI-set and $N_A(B) \neq 1$. Then $L = L_1$,* $\dim A/A\cap E$ *is even, and $W \cap T_A = [r_B, A] + R_A$.*

Proof. By 4.5(h), $L = L_1$ and therefore $s = s_1$. By 4.5(d) and (f), $\dim_{\mathbb{F}} A/A \cap E = \dim_{\mathbb{F}} T_A \cap T_B \geq 2$. Suppose for a contradiction that $\dim A/A \cap E$ is odd. Then 4.7

gives $\operatorname{rad} s \neq A \cap E$. Since $s = s_1$, 4.9(d) implies $A = \operatorname{rad} s$ and $\dim A/A \cap E = 1$, a contradiction to $\dim A/A \cap E \geq 2$. Thus, $\dim A/A \cap E$ is even.

Since A is quadratic on V/R_A, the map $\delta : A \to T_A/R_A, a \to [r_B, a] + R_A/R_A$ is a homomorphism. From 4.4(e), $A_0 = \{a \in A \mid [r_B, a] \in R_A\}$ so A_0 is the kernel of δ. Thus, $|A/A_0| = |[r_B, A] + R_A/R_A|$. This, along with $A_0 \overset{4.7}{=} A \cap E$ and $A_1 = A$, gives

$$\begin{aligned} |A/A \cap E| &= |A/A_0| = |[r_B, A] + R_A/R_A| \leq |W \cap T_A/R_A| \\ &\overset{4.4(a)}{=} |R_A + (T_A \cap T_B)/R_A| = |T_A \cap T_B| \overset{4.5(d)}{=} |A/A \cap E|. \end{aligned}$$

Hence, $[r_B, A] + R_A = W \cap T_A$. □

Lemma 4.12 *Suppose A is a TI set, $N_A(B) = 1$, and $\dim A/A \cap E$ is even. Then $T_A = [r_B, A] + R_A$, $W \cap T_A = [r_B, A_1] + R_A$, and $|A_1| = |T_A \cap T_B|$.*

Proof. We have $A_0 \overset{4.7}{=} A \cap E \overset{4.6}{=} N_A(B) = 1$. The map δ as given in 4.11 is still a homomorphism with kernel A_0. Then

$$|A| = |A/A_0| = |[r_B, A] + R_A/R_A| \leq |T_A/R_A| = |A|.$$

Therefore, $T_A = [r_B, A] + R_A$ and δ is an isomorphism. From 4.4(d) we have $A_1 = \{a \in A \mid [r_B, a] \in W \cap T_A\}$, so $A_1 = \delta^{-1}(W \cap T_A/R_A)$. We conclude that $|A_1| = |W \cap T_A/R_A| = |T_A \cap T_B|$ and $W \cap T_A = [r_B, A_1] + R_A$. □

Lemma 4.13 *Suppose that A is a TI set, $\dim A/A \cap E$ is even, and $T_A \cap T_B \neq 0$. Then $\dim W = 4$, $\dim T_A \cap T_B = 2$, and $[r_B, A_1] + R_A = W \cap T_A$.*

Proof. By 4.11 and 4.12, $[r_B, A_1] + R_A = W \cap T_A$. By 4.5(d) and 4.12, $\dim A_1/A \cap E = \dim T_A \cap T_B > 0$. By 4.9(d), $\dim A_1/A \cap E \leq 2$. If $\dim A_1/A \cap E = 2$, then $\dim T_A \cap T_B = 2$, $\dim W = 4$ and we are done in this case.

So suppose for a contradiction, that $\dim A_1/A \cap E = \dim T_A \cap T_B = 1$. Then $\dim W = 3$. If $N_A(B) \neq 1$, then 4.5(f) gives $\dim T_A \cap T_B > 1$, a contradiction.

Hence, $N_A(B) = A \cap E = 1$ and $\dim A_1 = 1$. Since $\dim A/A \cap E$ is even, 4.7(a) gives $A_0 = A \cap E = 1$.

Let $1 \neq a \in A_1$. Then $a \notin A_0 \overset{4.4(e)}{=} \{c \in A \mid [r_B, c] \in R_A\}$ and so $[R_B, a] \nleq R_A$. From 4.4(a) we see $(W \cap T_A)/R_A \cong T_A \cap T_B$ which is 1-dimensional in this case and $[W, A_1] \leq W \cap T_A$, this implies $W \cap T_A = [W, a] + R_A \leq C_W(a)$. Thus, A_1 centralizes $W \cap T_A = R_A + (T_A \cap T_B)$ and so also $T_A \cap T_B$. By symmetry, B_1 centralizes $T_A \cap T_B$ and so $T_A \cap T_B \leq C_W(\langle A_1, B_1 \rangle) = C_W(L_1)$. Also, A_1 acts faithfully on $\overline{W} := W/(T_A \cap T_B)$ and $|A_1| = |\mathbb{F}|$. Thus, A_1 induces the full centralizer of $\overline{R_A} = (R_A + (T_A \cap T_B))/(T_A \cap T_B)$ in $SL_{\mathbb{F}}(\overline{W})$ on $\overline{W}$. A similar statement holds for B_1 and we conclude that L_1 induces $SL_{\mathbb{F}}(\overline{W})$ on $\overline{W}$. In particular, L_1 acts transitively on the 1-spaces of $\overline{W}$, and B_1 acts transitively on the 1-spaces in $\overline{W}$ distinct from $\overline{R_B}$. Since $a \notin N_A(B)$, $R_B \neq R_B^a$ and so $R_B^a \nleq W \cap T_B$ and $\overline{R_B^a} \neq \overline{R_B} \neq \overline{R_A}$. Thus, $\overline{R_B{}^a} = \overline{R_A{}^b}$ for some $b \in B_1$. It follows that

$$R_B^a \leq R_A^b + (T_A \cap T_B) = (R_A + (T_A \cap T_B))^b \leq T_A^b$$

and by 4.1, $B^a = A^b$ and $R_B^a = R_A^b$.

Hence, $\langle B^a, B \rangle = \langle A^b, B^b \rangle = \langle A, B \rangle^b = L^b = L$ and

$$T_A + T_B \overset{2.22}{=} [V, L] = [V, L^b] = T_B^a + T_B = [T_B, a] + T_B.$$

Thus,

$$|([T_B, a] + T_B)/T_B| = |(T_A + T_B)/T_B| = |T_A/(T_A \cap T_B)| = |T_A|/|\mathbb{F}|.$$

By 2.12 and 2.3, $[V, a]$ is a hyperplane of T_A containing R_A and so $|[V, a]| = |T_A|/|\mathbb{F}|$. Then $|[T_B, a]| \leq |[V, a]| = |T_A|/|\mathbb{F}| = |([T_B, a] + T_B)/T_B| = |[T_B, a]/([T_B, a] \cap T_B)|$ and so $[T_B, a] \cap T_B = 0$. It follows that $[V, a] = [T_B, a]$ and $[V, a] \cap T_B = 0$. In particular, $([V,a]\cap W)\cap(T_B\cap W)=0$. Thus, $\dim([V,a]\cap W)=\dim([V,a]\cap W)+(W\cap T_B)/(W\cap T_B)\le \dim W/(W\cap T_B)=1$ since $W\cap T_B$ is a hyperplane of W. Therefore, $[V,a]\cap W$ is at most 1-dimensional. Since $R_A\le[V,a]\cap W$, this gives $[V,a]\cap W=R_A$. $R_B\le W$ so $[R_B,a]\le[W,a]\le W\cap[V,a]=R_A$ since $a\in A_1$. Hence, $a\in A_0=1$, a contradiction. □

Lemma 4.14 *Suppose A is a TI-set, W is 4-dimensional, and $W\cap T_A=[r_B,A_1]+R_A$. Then there exists an L_1-invariant quadratic $\mathbb{F}$-form, q_W, associated with the symplectic form s_W such that*

(a) L_1 induces $\Omega(W,q_W)$ on W.

(b) q_W is of $-$-type.

(c) $W\cap T_A=R_A^{\perp}$ with respect to s_W.

(d) Let R be a 1-dimensional subspace of W. Then $R\in\mathcal{R}$ if and only if $q_W(R)=0$ and if and only if $R\in R_A^{L_1}$.

(e) $C_{L_1}(W)=E$.

Proof. (a), (b), and (c) are proven in 4.10.

(d) This is shown at the end of the proof of 4.10.

(e) We first show that A_1 is a weakly closed subgroup of L_1. Recall that $C_W(A_1)=R_A$ so it's 1-dimensional. Let $g\in L_1\le N_G(W)$ with $[A_1,A_1^g]\le A_1$. Then $A_1^g\le N_G(C_W(A_1))=N_G(R_A)$. So $R_A\le C_W(A_1^g)=R_A^g$ and thus $R_A=R_A^g$. Then $A=A^g$ and $A_1=A_1^g$ so A_1 is a weakly closed subgroup of L_1. From 4.10 we have $L_1/C_{L_1}(R_{L_1})\cong\Omega(W,q_W)\cong SL_2(\tilde{\mathbb{F}})$. So we can apply 2.21(g). Hence, $E=C_1:=C_{L_1}(A_1^{L_1})$. Then $C_{L_1}(W)\overset{4.10}{=}C_{L_1}(R_A^{L_1})\overset{2.24(a)}{\le}C_1=E$. 4.6 gives $E\le C_{L_1}(W)$ so $E=C_{L_1}(W)$. □

Theorem 4.15 *Suppose A is a TI-set. Then one of the following holds:*

1. $\dim A/A \cap E$ *is even,* $L = L_1$*,* $\dim W = 4$*, and* $[r_B, A] + R_A = W \cap T_A$.

2. $|A| = 2^4$*,* $E = 1$*, and* $|\mathbb{F}| = 2$.

Proof. If $N_A(B) \neq 1$, then 4.5, 4.11, and 4.13 show that (1) holds.

So suppose $N_A(B) = 1$. Then 4.5 gives $N_B(A) = 1$ and $A \cap E = 1 = B \cap E$ making $E = 1$. By 2.5, $\dim A \geq 2$ and by 4.7, $\dim A_0 \leq 1$. Thus, $A \neq A_0$.

If $A_1 \neq A$, pick $a \in A \setminus A_1$. If $A = A_1$, pick $a \in A \setminus A_0$. Since $A_0 \leq A_1$, we have $a \notin A_0$ in either case. Put $\tilde{B} = B$, $\tilde{A} = B^a$, $\tilde{L} = \langle B, B^a \rangle = \langle \tilde{A}, \tilde{B} \rangle$, $\tilde{E} = (\tilde{A} \cap O_p(\tilde{L}))(\tilde{B} \cap O_p(\tilde{L}))$, and $\tilde{W} = R_B + (T_B \cap T_B^a) + R_B^a = W(\tilde{A}, \tilde{B})$. Since $|T_B/C_{T_B}(a)| = |T_B/(T_B \cap C_V(a))| = |(T_B + C_V(a))/C_V(a)| \leq |V/C_V(a)| = |[V, a]| = |T_A|/|\mathbb{F}| < |T_B|$, we have $C_{T_B}(a) \neq 0$. Since $C_{T_B}(a) \leq T_B \cap T_B^a$, this implies $T_B \cap T_B^a \neq 0$.

Suppose for a contradiction that $\dim \tilde{A}/\tilde{A} \cap \tilde{E}$ is odd. Then by 4.8, $\tilde{L}_0 := N_{\tilde{L}}(R_{\tilde{A}} + R_{\tilde{B}})$ acts transitively on the 1-spaces of $R_{\tilde{A}} + R_{\tilde{B}} = R_B + R_B^a$. Hence, there exists $g \in N_{\tilde{L}}(R_{\tilde{A}} + R_{\tilde{B}})$ with $R_B^g = [R_B, a]$. It follows that $R_B^g \leq T_A$ and so $R_B^g = R_A$ by 4.1. But then $[R_B, a] = R_A$ and $a \in A_0$, a contradiction to the choice of a.

Thus, $\dim \tilde{A}/\tilde{A} \cap \tilde{E}$ is even. Since $T_{\tilde{A}} \cap T_{\tilde{B}} \neq 0$, we can apply 4.13 to see that $\dim \tilde{W} = 4$ and $[r_{\tilde{B}}, \tilde{A}_1] + R_{\tilde{A}} = \tilde{W} \cap T_{\tilde{A}}$. Therefore, we can apply 4.14 to $\tilde{L}$. So $\tilde{L}_1$ induces $\Omega(\tilde{W}, q_{\tilde{W}})$ on $\tilde{W}$, where $q_{\tilde{W}}$ is a non-degenerate quadratic form of $-$-type. Notice that a normalizes $\tilde{W}$ and $\tilde{L}$ so $q_{\tilde{W}}$ is a-invariant.

Assume that $\tilde{W} \cap T_A$ contains a singular 1-space R. Then $R = R_{\tilde{A}}^g$ for some $g \in \tilde{L}_1$ from 4.14(d). So $R_{\tilde{A}}^g \leq T_A$ and hence, $R_{\tilde{A}}^g \overset{4.10}{=} R_A$ yielding $\tilde{A}^g = A$. Since $A \neq B$ and $\tilde{L}_1$ is doubly transitive on the singular 1-spaces in $\tilde{W}$, we may choose g such that $R_B^g = R_B$. Then $\tilde{B}^g = B^g = B$ and $\tilde{A}^g = A$. Thus, $W = \tilde{W}^g = \tilde{W}$ since $\tilde{L}_1$ normalizes $\tilde{W}$ and then $a \in N_A(W) = A_1$. By our choice of a this implies $A = A_1$ and so $L = L_1$ and (1) holds.

Next, assume that $\tilde{W} \cap T_A$ contains no singular 1-space. Let Y be a isotropic subspace of $\tilde{W} \cap T_A$. Let q be the quadratic form on $\tilde{W}$. Then $q|_Y$ is $\mathbb{Z}$-linear. Since

Y contains no singular 1-spaces, $q\,|_Y$ is one-to-one. Thus, $|Y| = |q(Y)| \leq |\mathbb{F}|$. This implies that $\tilde{W} \cap T_A$ contains no isotropic space of dimension greater than 1.

Since a acts quadratically on $\tilde{W}$, $[\tilde{W}, a]$ is an isotropic subspace of $\tilde{W}$. Thus, $[\tilde{W}, a]$ is a non-singular 1-space. Let X_1 and X_2 be 1-subspaces of $[\tilde{W}, a]^{\perp}$ with $[\tilde{W}, a]^{\perp} = X_1 + X_2 + [\tilde{W}, a]$. So $X_1 + [\tilde{W}, a] \neq X_2 + [\tilde{W}, a]$. For the following argument let $i \in \{1, 2\}$. Since X_i and $[\tilde{W}, a]$ are 1-dimensional, they are both isotropic. As $X_i \leq [\tilde{W}, a]^{\perp}$ we also have $[\tilde{W}, a] \leq X_i^{\perp}$. Thus, both X_i and $[\tilde{W}, a]$ are contained in $X_i^{\perp}$ and in $[\tilde{W}, a]^{\perp}$. So $X_i + [\tilde{W}, a] \leq X_i^{\perp} \cap [\tilde{W}, a]^{\perp} = (X_i + [\tilde{W}, a])^{\perp}$. Then $X_i + [\tilde{W}, a]$ is an isotropic 2-space in $\tilde{W}$ and therefore contains a singular 1-space Y_i.

Since Y_i is singular, $Y_i \neq [\tilde{W}, a]$. Then $Y_i + [\tilde{W}, a] = X_i + [\tilde{W}, a]$ and so $Y_1 \neq Y_2$. Also, $Y_1 = R_C$ and $Y_2 = R_D$ for some $C, D \in B^{\tilde{L}_1}$ with $C \neq D$ by 4.14(d). So $R_C + R_D \leq [\tilde{W}, a]^{\perp} = C_{\tilde{W}}(a)$ by [Asch, 22.1]. The doubly transitive action of $\tilde{L}_1$ on $B^{\tilde{L}_1}$ implies $\tilde{W} = W(C, D)$ and $\tilde{L} = \langle C, D\rangle$. Since $R_C + R_D \leq [\tilde{W}, a]^{\perp} = C_{\tilde{W}}(a)$, a centralizes R_C and R_D. So a normalizes C and D and we have $a \in N_A(C) \cap N_A(D)$. 4.6 states that $N_A(C)$ and $N_A(D)$ are $\mathbb{F}$-subspaces of A, so we conclude that $A_a = \mathbb{F}a \leq N_A(C) \cap N_A(D) \leq N_G(\tilde{W})$. Since $N_A(R_B) \overset{4.3}{\leq} N_A(B) = 1$, we have $N_{A_a}(R_B) = 1$ and so 4.6 gives $C_{A_a}(\tilde{W}) = 1$ as well. Also, $[A_a, a] = 1$. Then, since A_a normalizes $\tilde{W}$ and centralizes a, it normalizes the 1-space $[\tilde{W}, a]$ and so also centralizes it giving $[\tilde{W}, a] \leq C_{\tilde{W}}(A_a)$. This gives $C_{\tilde{W}}(A_a)^{\perp} \leq [\tilde{W}, a]^{\perp}$. Thus, $[\tilde{W}, A_a] \leq [\tilde{W}, a]^{\perp}$. Suppose that $[\tilde{W}, A_a] \neq [\tilde{W}, a]$ and let T be a 1-subspace of $[\tilde{W}, A_a]$ with $[\tilde{W}, a] \neq T$. Then $[\tilde{W}, a] + T$ is an isotropic 2-space in $\tilde{W} \cap T_A$, a contradiction. Therefore, $[\tilde{W}, A_a] = [\tilde{W}, a]$ is 1-dimensional. Notice that this, along with A.5 implies $|A_a| = 2$ and so $|\mathbb{F}| = 2$.

Since $\langle B, B^a\rangle = \tilde{L} = \langle C, D\rangle$, we have

$$T_B + T_B^a = [V, \tilde{L}] = T_C + T_D.$$

Let $L^* = \langle A, C\rangle$, $E^* = (A \cap O_p(L^*))(C \cap O_p(L^*))$, and $W^* = R_A + (T_A \cap T_C) + R_C$.

Since $a \in N_A(C)$, $N_A(C) \neq 1$. So by 4.5(f), $T_A \cap T_C \neq 1$. By 4.11, $\dim A/A \cap E^*$ is even. Then 4.13 gives $\dim W^* = 4$ and $\dim T_A \cap T_C = 2$. Thus, $\dim[T_C, a] \leq \dim T_C \cap T_A \leq 2$. Since $[\tilde{W}, a]$ is non-singular, $R_C \neq [\tilde{W}, a]$. Therefore, $R_C \cap [\tilde{W}, a] = 0$ and then $\tilde{W} = 0^\perp = (R_C \cap [\tilde{W}, a])^\perp = R_C^\perp + [\tilde{W}, a]^\perp = (\tilde{W} \cap T_C) + C_{\tilde{W}}(a)$. Hence,

$$[\tilde{W}, a] = [(\tilde{W} \cap T_C), a] + [C_{\tilde{W}}(a), a] \leq [T_C, a].$$

By symmetry, $\dim[T_D, a] \leq 2$ and $[\tilde{W}, a] \leq [T_D, a] \cap [T_C, a]$. Thus, $\dim[T_B + T_B^a, a] = \dim[T_C, a] + [T_D, a] \leq 3$. Also, $\dim T_A = \dim T_B^a$ and $\dim T_B \cap T_B^a = 2$ so we have

$$\begin{aligned}
\dim A - 1 = \dim T_A - 2 &= \dim T_B^a - \dim(T_B \cap T_B^a) \\
&= \dim(T_B^a/(T_B \cap T_B^a)) = \dim((T_B^a + T_B)/T_B) \\
&= \dim(([T_B, a] + T_B)/T_B) \leq \dim[T_B, a]/([T_B, a] \cap T_B) \\
&\leq \dim[T_B, a] \leq \dim[T_B + T_B^a, a] \leq 3.
\end{aligned}$$

Thus, $\dim A \leq 4$. Observe that $a \notin C$ as A is a TI-set. And since a normalizes C, $C\langle a \rangle$ is a 2-group. It follows that C, and so also A, is not a Sylow 2-subgroup of G. 3.2 now shows that $\dim A \not\leq 3$. Thus, $\dim A = 4$ and so $|A| = |\mathbb{F}|^4 = 2^4$. Then (2) holds. $\square$

Chapter 5

Identifying $L_n(q^2)$

Hypothesis 5.1 *In this chapter we assume that A is a TI-set,* $\dim A/A \cap E = 2$, $L = L_1$, $\dim W = 4$, *and* $[r_B, A] + R_A = W \cap T_A$ *for all* $A \neq B \in \mathcal{A}$.

Lemma 5.2 $W = R_L$.

Proof. Since $L = L_1$, 4.10 gives $W = \langle R_A^{L_1} \rangle = R_L$. $\square$

Definition 5.3 *A point is an element of $\mathcal{A}$. If A and B are distinct points, then $l(A, B) = \mathcal{A}(\langle A, B \rangle)$. Any set of points of the form $l(A, B)$ is called a line. $\mathcal{L}$ is the set of all lines. A point A is said to be incident to a line l (or lies on a line l) if $A \in l$. If l is a line, then $L_l = \langle A \mid A \in l \rangle$.*

A subset $\mathcal{B}$ of $\mathcal{A}$ is called a subspace of $\mathcal{A}$ if $l(A, B) \subseteq \mathcal{B}$ for all $A \neq B \in \mathcal{B}$. The subspace generated by $\mathcal{B}$ is the smallest subspace of $\mathcal{A}$ containing $\mathcal{B}$; that is, the intersection of all the subspaces containing $\mathcal{B}$. We denote this subspace by $\lceil \mathcal{B} \rceil$.

Lemma 5.4

(a) Let $\mathcal{S}$ be a subspace of $\mathcal{A}$ and $A \in \mathcal{S}$. Then A normalizes $\mathcal{S}$.

(b) Let $0 \neq \mathcal{B} \subseteq \mathcal{A}$. Then $\langle \mathcal{A}(\langle \mathcal{B} \rangle) \rangle = \langle \mathcal{B} \rangle$.

(c) Let $\mathcal{B} \subseteq \mathcal{A}$. Then $\lceil \mathcal{B} \rceil = \mathcal{A}(\langle \mathcal{B} \rangle)$ and $\langle \mathcal{B} \rangle = \langle \lceil \mathcal{B} \rceil \rangle$.

Proof. (a) Let $B \in \mathcal{S}$. If $A = B$, then A fixes B and so $B^A \subseteq S$. If $B \neq A$, then A normalizes $l(A, B)$. Since $\mathcal{S}$ is a subspace, $l(A, B) \subseteq \mathcal{S}$ and so again $B^A \subseteq \mathcal{S}$.

(b) $\langle \mathcal{B} \rangle \leq \langle \mathcal{A}(\langle \mathcal{B} \rangle) \rangle \leq \langle \mathcal{B} \rangle$.

(c) If $A, B \in \mathcal{A}(\langle \mathcal{B} \rangle)$, then $\langle A, B \rangle \leq \langle \mathcal{B} \rangle$ and so $l(A, B) \subseteq \mathcal{A}(\langle \mathcal{B} \rangle)$. Thus, $\mathcal{A}(\langle \mathcal{B} \rangle)$ is a subspace of $\mathcal{A}$. If $D \in \mathcal{B}$, then $D \leq \langle \mathcal{B} \rangle$ and so $D \in \mathcal{A}(\langle \mathcal{B} \rangle)$. Therefore, $\mathcal{B} \subseteq \mathcal{A}(\langle \mathcal{B} \rangle)$ and $\lceil \mathcal{B} \rceil \subseteq \mathcal{A}(\langle \mathcal{B} \rangle)$. By (a), every element in $\langle \lceil \mathcal{B} \rceil \rangle$ normalizes $\lceil \mathcal{B} \rceil$ so $\langle \lceil \mathcal{B} \rceil \rangle$ normalizes $\lceil \mathcal{B} \rceil$. Thus,

$$\mathcal{A}(\langle \mathcal{B} \rangle) \overset{2.20}{=} A^{\langle \mathcal{B} \rangle} \subseteq A^{\langle \lceil \mathcal{B} \rceil \rangle} \subseteq \lceil \mathcal{B} \rceil \subseteq \mathcal{A}(\langle \mathcal{B} \rangle).$$

Then

$$\mathcal{A}(\langle \mathcal{B} \rangle) = \lceil \mathcal{B} \rceil = \mathcal{A}^{\langle \lceil \mathcal{B} \rceil \rangle} = \mathcal{A}(\langle \lceil \mathcal{B} \rceil \rangle)$$

and

$$\langle \mathcal{B} \rangle \overset{(b)}{=} \langle \mathcal{A}(\langle \mathcal{B} \rangle) \rangle = \langle \mathcal{A}(\langle \lceil \mathcal{B} \rceil \rangle) \rangle = \langle \lceil \mathcal{B} \rceil \rangle.$$

□

Lemma 5.5 *Let $l = l(A, B)$ be a line.*

(a) $L_l / C_{L_l}(l) \cong \Omega_4^-(\mathbb{F}) \cong SL_2(\tilde{\mathbb{F}})$.

(b) L_l acts doubly transitively on l.

(c) If $C \in l$, then $C/C_C(l)$ acts regularly on $l \setminus \{C\}$.

(d) $L_l = \langle C, D \rangle$ and $l = l(C, D)$ for all $C \neq D \in l$.

Proof. We may assume $l = l(A, B)$ and so $L_l = L$. By Hypothesis 5.1, 4.15(1) holds. So we can apply 4.14. Thus, the map $\Phi : l \longrightarrow$ {the set of singular 1-spaces of W} which takes $C \rightarrow R_C$ is a L-equivariant bijection. Also, the action of $\Omega_4^-(\mathbb{F})$ on the singular 1-spaces is isomorphic to the action of $SL_2(\tilde{\mathbb{F}})$ on the 1-spaces of $\tilde{\mathbb{F}}^2$. So we have (a). 4.10 and 4.5 give us $\mathcal{A}(L) = B^A \cup \{A\}$ and the doubly transitive action.

We also have $N_A(B) \overset{4.6}{=} A \cap E = C_A(l)$ so we have (b) and (c). In fact, a line contains exactly $|B^A \cup \{A\}| = |\tilde{\mathbb{F}}| + 1$ points. By (b), $\langle A, B\rangle = \langle C, D\rangle$ and so also $l = l(A, B) = l(C, D)$, which gives us (d). $\square$

Lemma 5.6 *Any two distinct points lie on a unique common line.*

Proof. By 5.5(d), $l(A, B)$ is the unique line incident with A and B. $\square$

Lemma 5.7

(a) Let m be a line. Then m is A-invariant if and only if $A \in m$.

(b) Let $a \in A$ and m an a-invariant line. Then one of the following holds:

1. *$A \in m$ and A is the unique fixed-point of a on m.*
2. *a fixes all points on m.*

(c) Let $J \subseteq A$. Then $C_{\mathcal{A}}(J)$ is a subspace of $\mathcal{A}$.

Proof. (a) Suppose first that $A \in m$ and let D be a point that lies on m with $A \neq D$. Then 5.6 gives $m = l(A, D) = \mathcal{A}(\langle A, D\rangle)$ and so m is invariant under A.

Suppose next that m is A-invariant. Let C be a point incident to m. Then both A and C are contained in $N_G(m)$. $A = C^g \in m$ by 2.20.

(b) We may assume that there exists $D, D^a \in m$ with $D \neq D^a$ or else a fixes all points on m. Put $l = l(A, D)$. Since l is A-invariant by (a), $D^a \in l$. Then 5.5(d) says $l = l(D, D^a)$. Also, 5.5(c) says $A/C_A(l)$ acts regularly on $m \setminus \{A\}$ and so A is the only fixed point of a on l.

(c) Let $a \in J$. Let $C \neq D \in C_{\mathcal{A}}(J)$. Then $m = l(C, D)$ is invariant under a and a has at least two fixed-points on m. Thus by (b), $m \subseteq C_{\mathcal{A}}(a)$ so $m \subseteq C_{\mathcal{A}}(J)$. $\square$

Lemma 5.8 *Let A, B, C be non-collinear points. Then $W \cap T_C$ is at most two-dimensional and there exists $D \in \mathcal{A}(L)$ with $T_D \cap T_C \nleq W$.*

Proof. Suppose $\dim W \cap T_C \geq 3$. Then $W \cap T_C$ contains an isotropic 2-space and then also a singular 1-space. So $R_A^g \leq T_C$ for some $g \in L$. But then $C = A^g \in \mathcal{A}(L)$, contrary to the assumption that A, B, C are non-collinear. So $\dim W \cap T_C \leq 2$.

Suppose next $T_D \cap T_C \leq W$ for all $D \in \mathcal{A}(L)$. Since 4.13 states that $T_D \cap T_C$ is 2-dimensional, we conclude that $W \cap T_C = T_D \cap T_C \leq R_D^\perp$. But then $W \cap T_C \leq \langle R_D \mid D \in \mathcal{A}(L)\}^\perp = W^\perp = 0$ and $W \cap T_C = 0$, a contradiction. □

Lemma 5.9 *Suppose that $N_A(B) = 1$. Then $G_0 = L$, $R_G = W$, and R_G is natural $\Omega_4^-(\mathbb{F})$-module for G_0.*

Proof. Since $N_A(B) \overset{4.6}{=} A \cap E = 1$, we have $\dim A = \dim A/(A \cap E) = 2$. Thus, $\dim T_A = 3$ and $T_A = W \cap T_A \leq W$. Let $C \in \mathcal{A}$ and suppose that $C \notin \mathcal{A}(L)$. Then by 5.8, $T_D \cap T_C \not\leq W$ for some $D \in \mathcal{A}(L)$. So $D \in A^l$ for some $l \in L$ making $T_D = T_{A^l} \leq W^l = W$, a contradiction. Thus, $C \in \mathcal{A}(L)$. Hence, $\mathcal{A} = \mathcal{A}(L)$, $G_0 = L$, and $W = R_L = R_G$. We have R_G is natural $\Omega_4^-(\mathbb{F})$-module for G_0 from 4.10. □

Lemma 5.10 $E = C_L(W)$ *and* $L = L'$.

Proof. For all $A \in \mathcal{A}(L)$, $E \leq N_G(A)$ so $E \leq C_G(R_A)$. Then we have $E \leq C_L(W) \overset{5.2}{=} C_L(R_L) \overset{2.24}{\leq} C_L(\mathcal{A}(L)) \overset{2.21}{=} E$.

$L/C_L(W) \cong \Omega_4^-(\mathbb{F}) \cong SL_2(\tilde{\mathbb{F}})$ from 4.10. So $L/C_L(W)$ is simple, $L = L'C_L(W)$, and $L = L'E$. It remains to show that $E \leq L'$. Let $a \in A \setminus E$. Since $C_B(a) \leq B \cap B^a = 1$, we have

$$C_E(a) = C_{(A\cap E)(B\cap E)}(a) = (A \cap E)C_{B\cap E}(a) = A \cap E.$$

Thus, $|E/C_E(a)| = |E/A \cap E| = |B \cap E| = |A \cap E|$ and so also $|[E, a]| = |A \cap E|$. On the other hand, a acts quadratically on E and so $[E, a] \leq C_E(a) = A \cap E$. As they have the same order, $A \cap E = [E, a] \leq L'$. By symmetry, $B \cap E \leq L'$ and so $E = (A \cap E)(B \cap E) \leq L'$. □

Lemma 5.11 *Let A, B, C be non-collinear points. Then $A = N_A(B)N_A(C)$.*

Proof. Suppose $N_A(B) = 1$. Then 5.9 gives $\mathcal{A} = \mathcal{A}(L)$, a contradiction to A, B, C non-collinear. Hence, $N_A(B) \neq 1$ and similarly $N_A(C) \neq 1$.

Suppose first that $N_A(B) \cap N_A(C) = 1$. In 4.5, (g) gives $\dim A \geq 4$ while (b), (d), and (f), together with the fact that $\dim W = 4$ give $\dim A/N_A(B) = 2 = \dim A/N_A(C)$. So $4 \leq \dim A = \dim A/(N_A(B) \cap N_A(C)) \leq \dim A/N_A(B) + \dim A/N_A(C) = 4$. It follows that $\dim A = 4$, $\dim N_A(B) = \dim N_A(C) = 2$ and $A = N_A(B)N_A(C)$ in this case.

So now suppose that $N_A(B) \cap N_A(C) \neq 1$ and $A \neq N_A(B)N_B(C)$. Let $\mathcal{P} = \lceil A, B, C \rceil$. Notice that $N_A(B) \cap N_A(C)$ fixes A, B and C. Since $C_{\mathcal{A}}(N_A(B) \cap N_A(C))$ is a subspace of $\mathcal{A}$ from 5.7(c), we conclude that $N_A(B) \cap N_A(C)$ fixes all points in $\mathcal{P}$ and so

$$C_A(\mathcal{P}) = N_A(B) \cap N_B(C) \neq 1.$$

Put $H = \langle A, B, C \rangle$, $Q = C_A(\mathcal{P})C_B(\mathcal{P})C_C(\mathcal{P})$, and $Y = W(A,B)W(A,C)W(B,C)$. By $(***)$, (b), and (e) in 4.5 we have

$$C_{T_A}(N_A(B)) = T_A \cap T_B + R_A.$$

Since $N_A(B)N_A(C)$ is a proper $\mathbb{F}$-subspace of A, this means

$$C_{T_A}(N_A(B)) \cap C_{T_A}(N_B(C)) = C_{T_A}(N_A(B)N_A(C)) \overset{2.8}{\neq} R_A$$

and so

$$((T_A \cap T_B) + R_A) \cap ((T_A \cap T_C) + R_A) \neq R_A.$$

Thus, there exists a 1-subspace, J, of T_A with $J \neq R_A$ and $J \leq ((T_A \cap T_B) + R_A) \cap ((T_A \cap T_C) + R_A)$. So $J \leq T_B + R_A$ and therefore $J + R_A \leq T_B + R_A$. Since T_B is a hyperplane of $T_B + R_A$, $(J + R_A) \cap T_B$ is a hyperplane of $J + R_A$ and therefore

1-dimensional. Since A acts transitively on the 1-subspaces of $J + R_A$ different from R_A, there exists $x \in A$ with $J^x = (J + R_A) \cap T_B$ so $J^x \leq T_B$. Similarly there exists $y \in A$ with $J^y = (J + R_A) \cap T_C \leq T_C$. Replacing B by $B^{x^{-1}}$ and C by $C^{y^{-1}}$ we may assume that $J \leq T_B$ and $J \leq T_C$. So $J \leq T_A \cap T_B \cap T_C$.

Note that

$$\begin{aligned} Y :&= R_A + R_B + R_C + (T_A \cap T_B) + (T_A \cap T_C) + (T_B \cap T_C) \\ &= W + R_C + (T_A \cap T_C) + (T_B \cap T_C). \end{aligned}$$

Since $J \leq W$, $\dim((T_B \cap T_C) + W)/W = \dim(T_B \cap T_C)/(T_B \cap T_C \cap W) \leq \dim(T_B \cap T_C)/J \leq 1$ as $T_B \cap T_C$ is 2-dimensional. So both $((T_B \cap T_C) + W)/W$ and similarly $((T_A \cap T_C) + W)/W$ are at most 1-dimensional. Thus, $\dim Y/W \leq 3$.

We will now show that $R_H \leq Y$. By 5.4, $H = \langle \mathcal{P} \rangle$. In particular, $\mathcal{P}$ is H-invariant. Observe that $C_A(\mathcal{P}) \leq N_G(B)$ and so $[C_A(\mathcal{P}), B] \leq C_B(\mathcal{P})$. It follows that H normalizes Q. Since $C_A(\mathcal{P}) \neq 1$, we have $R_A \leq [T_A, C_A(\mathcal{P})]$ from 2.3. Also, $[T_B, C_A(\mathcal{P})] \leq T_A \cap T_B \leq W(A, B) \leq Y$ and so

$$R_A \leq [T_A T_B T_C, Q] \leq Y.$$

2.20 says $\mathcal{A}(H) = A^H$ so $R_H = \langle R_A^H \rangle$. Since H normalizes $T_A T_B T_C$ and Q, this implies that

$$R_H = \langle R_A^H \rangle \leq [T_A T_B T_C, Q] \leq Y.$$

As $L \leq H$, we conclude that L acts on R_H/W. Since $|L/C_L(W)| = |\Omega_4^-(\mathbb{F})|$, we have that $|\mathbb{F}|^2 + 1$ divides $|L/C_L(W)| \overset{5.10}{=} |L/E|$. On the other hand, $|\mathbb{F}|^2 + 1$ does not divide $|GL_3(\mathbb{F})|$. So if $K := C_L(R_H/W) \leq E$, then

$$|\mathbb{F}|^2 + 1$$

divides

$$|L/C_L(W)| = |L/E|$$

divides

$$|L/E||E/K| = |L/K|$$

divides

$$|GL(R_H/W)|$$

which divides

$$|GL_3(\mathbb{F})|$$

as $\dim R_H/W \leq \dim Y/W \leq 3$. This is a contradiction, so it follows that $K \not\leq E$. Since L/E is simple, this means $L = EK$. Since E is abelian, we get L/K is abelian, and since L is perfect, $L = K$. Thus, $[R_H, L] \leq W$. Let $D \in \mathcal{A}(L)$ with $D \neq C$. Then

$$T_D \cap T_C \leq W(D, C) \cap T_D = [R_C, D] + R_D \leq [R_H, L] + R_D \leq W.$$

But this contradicts 5.8. □

Corollary 5.12 *Let A, B and C be non-collinear. Then $N_A(B)$ fixes all points on $l(A, B)$, fixes all lines through A, and acts transitively on the points of $l(A, C)$ distinct from A.*

Proof. Since $N_A(B) \overset{4.6}{=} A \cap E$ and E fixes $\mathcal{A}(L)$, $N_A(B)$ fixes all points on $l(A, B)$. By 5.5, A fixes all lines through A and acts transitively on $l(A, C) \setminus \{A\}$. Also, $N_A(C)$ fixes all points in $l(A, C)$. As $A = N_A(B)N_A(C)$ from 5.11, this implies that $N_A(B)$ acts transitively on $l(A, C) \setminus \{A\}$. □

Lemma 5.13 *Let A, B, C be non-collinear points and $\mathcal{P}$ the subspace of $\mathcal{A}$ generated by A, B and C. Then $\mathcal{P}$ is a Moufang plane.*

Proof. Let $\mathcal{P}_A$ be set of points which lie on a line from A to a point on $l(B,C)$. Similarly, let $\mathcal{P}_B$ the set of points which lie on a line from B to a point on $l(A,C)$ and let $\mathcal{P}_C$ be the set of points which lie on a line from C to a point on $l(A,B)$.

1° $\quad \mathcal{P} = \mathcal{P}_A$

Proof of (1°). We will first show that $\mathcal{P}_A \subseteq \mathcal{P}_B$.

Let D be a point on $l(B,C)$. If $D = B$, then $l(A,D) \subseteq \mathcal{P}_B$. So suppose $D \neq B$. Then $l(D,B) = l(C,B) \neq l(A,B)$ and D, A, B are non-collinear. Let F be a point on $l(D,A)$. So F is an arbitrary element in $\mathcal{P}_A$. If $F = A$, then $F \in \mathcal{P}_B$, so we may assume that $F \neq A$. By 5.12 there exists $y \in N_A(B)$ with $D^y = F$. Since $y \in N_G(B)$ and D lies on $l(C,B)$, $F = D^y$ lies on $l(C^y, B)$. Since $y \in A$ and A normalizes $l(A,C)$, C^y lies on $l(A,C)$. So $F \in \mathcal{P}_B$. This completes the proof that $\mathcal{P}_A \subseteq \mathcal{P}_B$. By symmetry, $\mathcal{P}_B \subseteq \mathcal{P}_A$. Hence, $\mathcal{P}_A = \mathcal{P}_B$ and by symmetry, $\mathcal{P}_A = \mathcal{P}_B = \mathcal{P}_C$.

Since $\mathcal{P}_A$ is the set of points from a union of lines through A and A normalizes every line through A by 5.7(a), A normalizes $\mathcal{P}_A$. Similarly B normalizes $\mathcal{P}_B$ and C normalizes $\mathcal{P}_C$. It follows that $H := \langle A, B, C \rangle$ normalizes $\mathcal{P}_A = \mathcal{P}_B = \mathcal{P}_C$. Clearly $\mathcal{P}_A \subseteq \mathcal{P}$. By 5.4(c),

$$\mathcal{P} = \lceil A, B, C \rceil = \mathcal{A}(H) \overset{2.20}{=} A^H \subseteq \mathcal{P}_A$$

and so $\mathcal{P} = \mathcal{P}_A$ and (1°) holds. $\boxdot$

2° $\quad$ *Put $n = |\mathbb{F}|^2$. Then there are $n^2 + n + 1$ points and $n^2 + n + 1$ lines in $\mathcal{P}$.*

Proof of (2°). 4.10 gives $|A \cup B^A| = |\mathbb{F}|^2 + 1$ so every line contains $n + 1$ points. Therefore, there are $n + 1$ lines from A to a point on $l(B,C)$. Each of these lines contains n points other than A and so there are $(n+1)n + 1 = n^2 + n + 1$ points in $\mathcal{P}_A = \mathcal{P}$. There are $(n^2 + n + 1)(n^2 + n)$ pairs of points in $\mathcal{P}$. Each line contains $(n+1)n$ pairs of points and each pair of points uniquely determines a line, so there are

$$\frac{(n^2+n+1)(n^2+n)}{(n+1)n} = n^2+n+1$$

lines. So (2°) holds. □

We conclude from (2°) and [Ha, Theorem 20.8.1] that $\mathcal{P}$ is a projective plane. Let P be a point and l a line with $P \in l$. An elation on $\mathcal{P}$ with center P and axis l is an automorphism of $\mathcal{P}$ which fixes all points on l and all lines in $\mathcal{P}$ through P. Let q be a line in $\mathcal{P}$ through P distinct from l. By definition the projective plane $\mathcal{P}$ is a Moufang plane if for all such P, l and q, the group of elations with center P and axis l acts transitively on $q \setminus \{P\}$.

Let R be a point on l distinct from P. By 5.12, $N_P(R)$ acts as a group of elations with center P and axis l on $\mathcal{P}$. Moreover, $N_P(R)$ acts transitively on $q \setminus \{P\}$. So $\mathcal{P}$ is indeed a Moufang plane. □

Corollary 5.14 *$\mathcal{P}$ is isomorphic to the projective plane defined over $\tilde{\mathbb{F}}$.*

Proof. Since $\mathcal{P}$ is Moufang plane, [Ha, Theorem 20.5.3] shows that the ternary ring R associated to $\mathcal{P}$ is an alternate division ring. Since $\mathcal{P}$, and therefore R, is finite, [Ha, Theorem 20.6.2] shows that R is a field. Since [Ha] has also shown us that $|R|+1$ is the number of points on a line, we get $R = |\mathbb{F}|^2 = |\tilde{\mathbb{F}}|$. Any two finite fields of the same order are isomorphic and so $R \cong \tilde{\mathbb{F}}$. □

Proposition 5.15 *$\mathcal{A}$ is a projective space defined over $\tilde{\mathbb{F}}$.*

Proof. According to the Veblen-Young axioms a projective space is set of points and lines such that

- Any two distinct points lie on a unique common line; and
- If A, B and C are non-collinear points and D and E are distinct points such that A, C, E and B, C, D are collinear, then the line through D and E intersects the line though A and B in a point F.

The first statement we already have in 5.6. Let $P = \langle A, B, C\rangle$. For the second, B, C, D collinear gives us that D lies in P and A, C, E collinear gives us that E lies in P. So the line through D and E lies in P. The line through A and B also lies in P so it intersects the line through D and E in a point F.

Then $\mathcal{A}$ is a projective space. If $\mathcal{A}$ is 2-dimensional, then $\mathcal{A}$ is generated by three points and 5.14 gives us the result. So assume that $\mathcal{A}$ is at least 3-dimensional. It follows that $\mathcal{A}$ is Desarguesian and therefore a projective space defined over a field R. From 5.14 we see that $R \cong \tilde{\mathbb{F}}$ and the Proposition is proven. $\square$

Theorem 5.16 *Part 3 of Theorem 2.2 holds.*

Proof. For $H \leq G$ let $H^\dagger$ be the image of H in the automorphism group, $\mathrm{Aut}(\mathcal{A})$, of the projective space $\mathcal{A}$, and let $H^\ddagger$ be the image of H in $\mathrm{GL}_{\mathbb{F}}(R_G)$. Let I be the subgroup of $\mathrm{Aut}(\mathcal{A})$ consisting of the identity element and all transvections with center A.

Claim: $A^\dagger = I$.

Proof of Claim. Suppose for a contradiction that $A^\dagger \neq I$. Then there exists an $\mathbb{F}_2$-hyperplane, I_1, of I with $A^\dagger \leq I_1$. Notice that I_1 contains an $\tilde{\mathbb{F}}$-hyperplane, I_2, of I and that $I_2 = C_I(D)$ for some point $A \neq D \in \mathcal{A}$. Then $A^\dagger C_I(D) \leq I_1$ and so $|A/N_A(D)| = |A^\dagger/N_{A^\dagger}(D)| < |I/C_I(D)| = |\tilde{\mathbb{F}}|$. Hence, $|D^A| < |\tilde{\mathbb{F}}| = |l(A, D) \setminus \{A\}| \overset{5.5(c)}{=} |D^A|$, a contradiction. Thus, $A^\dagger = I$ and the claim holds.

Since G acts transitively on $\mathcal{A}$, we conclude that $G_0^\dagger$ is the subgroup of $\mathrm{Aut}(\mathcal{A})$ generated by the transvections. Hence, $G_0/C_{G_0}(\mathcal{A}) \cong G_0^\dagger \cong PSL_m(\tilde{\mathbb{F}})$, where $m-1$ is the dimension of the projective space $\mathcal{A}$. Let x be a p-element in $C_G(\mathcal{A})$. Then x centralizes all $D \in \mathcal{A}$ so we have $x \in C_G(R_G)$. Thus, $C_{G^\ddagger}(\mathcal{A}) = C_{G_0}(\mathcal{A})/C_G(R_G)$ is a p'-group. In particular, $[A, C_G(\mathcal{A})] \leq A \cap C_G(\mathcal{A}) \leq C_G(R_G)$ and $C_{G_0^\ddagger}(\mathcal{A}) \leq Z(G_0^\ddagger)$. [Griess] now shows that $G_0^\ddagger \cong SL_m(\tilde{\mathbb{F}})/Z_0$ for some subgroup $Z_0 \leq Z(SL_m(\tilde{\mathbb{F}}))$.

For an $\mathbb{F}$-subspace, X, of V put $\tilde{X} = \tilde{\mathbb{F}} \otimes_{\mathbb{F}} X$. From $C_V(A) \overset{2.3}{=} R_A$ we get $C_{\tilde{V}}(A) = \tilde{R}_A$. Let $\tilde{X}$ be a non-zero $\tilde{\mathbb{F}}G_0$-submodule of $\tilde{R}_G$. Then $0 \neq C_{\tilde{X}}(A) \leq \tilde{R}_A$, and since $\widetilde{R}_A$ is 1-dimensional over $\widetilde{\mathbb{F}}$, $\widetilde{R}_A = C_{\widetilde{X}}(A) \leq \widetilde{X}$. Thus, $\widetilde{R}_G = \langle \widetilde{R}_A^{G_0} \rangle \leq \widetilde{X}$. Hence, $\widetilde{R}_G$ is a simple $\widetilde{\mathbb{F}}G_0$-module. Let $A_0 < A_1 < \ldots < A_{m-1} = \mathcal{A}$ be a chain of subspaces of $\mathcal{A}$ with $\dim A_i = i$, $A_0 = \{A\}$, and $A_1 = l(A, B)$. Let $P_i = \bigcap\{N_{G_0}(A_j)^{\ddagger} \mid 0 \leq j < m-1, i \neq j\}$ and $L_i = O^{p'}(P_i)$. For $i > 0$, $P_i \leq N_{G^{\ddagger}}(A)$ so L_i centralizes $\tilde{R}_A$. Now L_0 normalizes $l(A, B)$ and so also L and R_L. It follows that $L_0 = L^{\ddagger}C_{L_0}(R_L)$ and so $\widetilde{R}_L \cong N_0 \otimes N_0^{\sigma}$ where N_0 is a natural $SL_2(\tilde{F})$-module for L_0 and σ is the field automorphism of order 2 of $\tilde{\mathbb{F}}$. Curtis' Lemma [MS] now shows that $\widetilde{R}_G$ is uniquely determined up to isomorphism as an $\widetilde{\mathbb{F}}SL_m(\widetilde{\mathbb{F}})$-module and that $\widetilde{R}_G \cong N \otimes_{\widetilde{\mathbb{F}}} N^{\sigma}$ for some natural $\widetilde{\mathbb{F}}SL_m(\widetilde{\mathbb{F}})$-module, N. Let $z \in Z(SL_m(\widetilde{\mathbb{F}}))$. Then $z = \lambda * \mathrm{id}$ for some $\lambda \in \tilde{\mathbb{F}}$ with $\lambda^m = 1$. Moreover, z acts as $\lambda\lambda^{\sigma} * \mathrm{id}$ on $N \otimes_{\widetilde{\mathbb{F}}} N^{\sigma}$ and so $z \in Z_0$ if and only if $\lambda^{\sigma} = \lambda^{-1}$. Thus $Z_0 = \{\lambda * \mathrm{id} \mid \lambda \in \tilde{\mathbb{F}}, \lambda^m = 1, \lambda^{\sigma} = \lambda^{-1}\}$ and all parts of the theorem are proved. $\square$

Chapter 6

Main Theorem

We are now able to prove our main theorem.

Proof of Theorem 2.2. We have $C_G(A)/A$ a p'-group from 2.16. Also, 2.19 states that A is a weakly closed subgroup of G. We have $R_A \neq R_B$ from 2.23. If $T_A = T_B$, then $R_A \leq T_B$ and 3.1 applies making $T_A \neq T_B$. So we have $T_A \neq T_B$.

Case 1: Suppose A is not a TI-set. Then there exists $A \neq B \in \mathcal{A}$ such that $|A \cap B| \neq 1$. Hence, $R_A = [T_A, A \cap B] \leq T_B$.

Case 1a: Suppose $|\mathbb{F}| > 2$. Then 3.1(a) holds. So $|A| = |\mathbb{F}|^2$ making $|A| \geq 4$, and $|A \cap B| = |\mathbb{F}|$. We are then able to apply 3.2 to get $A \in Syl_p(G)$. We can also apply 3.3 to see that $G_0/C_{G_0}(R_G) \cong SL_2(\mathbb{F})$ or $G_0/C_{G_0}(R_G) \cong \Omega_4^+(\mathbb{F})$ and in either case R_G is the corresponding natural module. So in this situation 2.2(1) holds.

Case 1b: Suppose $|\mathbb{F}| = 2$. Then 3.1(b) holds. So $|A \cap B| \leq 2$, and $|A| \leq 2^4$. In this situation 2.2(2) holds.

Case 2: Suppose A is a TI-set.

Case 2a: Suppose $N_A(B) = 1$ for some $A \neq B \in \mathcal{A}$. Then we can apply 4.15(2) to get $|\mathbb{F}| = 2$ and $|A| = 2^4$. This situation also gives 2.2(2).

Case 2b: Suppose $N_A(B) \neq 1$ for all $A \neq B \in \mathcal{A}$. Then we can apply 4.15(1) to get $\dim A/A \cap E = 2$, $L = L_1$, $\dim W = 4$, and $[r_B, A] + R_A = W \cap T_A$. So hypothesis

5.1 is fulfilled and we can apply 5.16. Hence, 2.2(3) holds in this case. $\square$

Appendix A

Background Lemmas

Lemma A.1 *Let P be a finite p-group and H a finite group acting on P. If H stabilizes a subnormal series on P, then $H/C_H(P)$ is a p-group and $[P, O^p(H)] = 1$. In particular, if $[P, H, H, \dots H] = 1$, then $H/C_H(P)$ is a p-group and $[P, O^p(H)] = 1$.*

Proof. [Gor, 5.3.3] gives the main result. In particular, if $[P, H, H, \dots H] = 1$, then we have a subnormal series $1 = [P, H, H, \dots H] \subseteq \cdots \subseteq [P, H, H] \subseteq [P, H] \subseteq P$ stabilized by $H/C_H(P)$. □

Lemma A.2 $[V, O^p(G), O^p(G)] = [V, O^p(G), G] = [V, O^p(G)]$.

Proof. $[V, O^p(G), O^p(G)] \leq [V, O^p(G)] \leq V$ is a subnormal series stabilized by $O^p(G)$ so it's centralized by $O^p(O^p(G)) = O^p(G)$. Then $[V, O^p(G)] \leq [V, O^p(G), O^p(G)] \leq [V, O^p(G), G] \leq [V, O^p(G)]$. □

Lemma A.3 *If $N \trianglelefteq G$, then $O^p(G/N) = O^p(G)N/N$.*

Proof. $(G/O^p(G))/(O^p(G)N/O^p(G)) \cong G/O^p(G)N \cong (G/N)/(O^p(G)N/N)$ by the third isomorphism theorem. Then $G/O^p(G)N$ is a p-group since $(G/O^p(G))$ is a p-group by definition. Also by definition, $O^p(G/N)$ is the smallest normal subgroup of G/N with a p-group as its quotient so $O^p(G/N) \leq O^P(G)N/N$. $O^P(G)N/N$ is generated by the p' elements so $O^P(G)N/N \leq O^P(G/N)$. □

Lemma A.4 *If $V = [V, L]$, then $V = [V, O^p(L)]$.*

Proof. Let $\bar{V} = V/[V, O^p(L)]$. Then $L/C_L(\bar{V})$ is a p-group so if $\bar{V} \neq 1$, then $C_{\bar{V}^*}(L) \neq 0$. Hence, $[\bar{V}, L] \neq \bar{V}$. This is a contradiction so $\bar{V} = 1$. □

Lemma A.5 *Let W be an orthogonal space. Let $X \leq W$ be a 1-dimensional non-singular subspace. Then there exists at most one $a \in O(W)$ with $[W, a] = X$.*

Proof. Let $w \in W \setminus X^{\perp}$ and $x \in X$ with $w^a = w + kx$ where $k \neq 0$. So $q(w) = q(w^a) = q(w) + ks(w, x) + k^2 q(x)$. Then $k(s(w, x) + kq(x)) = 0$. Since $k \neq 0$, $k = \frac{-s(w,x)}{q(x)}$. So $w^a = w - \frac{s(w,x)}{q(x)} x$ and we see that a is unique. □

Appendix B

Definitions

Let G be a finite group and p a prime for all the following definitions.

Definition B.1 *A group which is abelian and all nontrivial elements have order p is called an* **elementary abelian** *p-group.*

Definition B.2 *G has* **characteristic** *p if $C_G(O_p(G)) \leq O_p(G)$. G has* **local characteristic** *p if all the p-local subgroups have characteristic p.*

Definition B.3 *Q is a* **large p-subgroup** *of G if Q is a p-subgroup, $Q \trianglelefteq N_G(A)$ for all $1 \neq A \leq Z(Q)$, and $C_G(Q) \leq Q$.*

Definition B.4 *A subgroup P is a* **parabolic subgroup** *of G if P contains a Sylow p-subgroup of G. A subgroup P containing a Sylow p-subgroup of G is a* **p-parabolic** *subgroup of G, and P is a* **local p-parabolic** *subgroup if, in addition, $O_p(P) \neq 1$.*

Definition B.5 *A p-subgroup Y of G is called* **p-reduced** *(for G) if Y is elementary abelian and normal in G, and $O_p(G/C_G(Y)) = 1$. The largest p-reduced subgroup of G is denoted by Y_G.*

Definition B.6 *Let A be an elementary abelian p-group and V a finite dimensional $GF(p)A$-module. Then A is*

(a) **quadratic** *on V if $[V, A, A] = 0$.*

(b) *a* **$2F$-offender** *on* V *if* $|V/C_V(A)| \leq |A/C_A(V)|^2$.

(c) **non-trivial** *on* V *if* $[V, A] \neq 0$.

Definition B.7 *A homomorphism,* $\pi : G \longrightarrow S_\Omega$ *is an* **action** *of* G *on* Ω *defined by* $\alpha^g = \alpha^{g^\pi}$. *If the kernel of* π *is 1, then* G *acts* **faithfully** *on* Ω. *If* $Ker\pi = G$, *then* G *acts trivially on* Ω.

Definition B.8 *If the action is transitive and no element other than the identity fixes any other element, then the action is called* **regular**.

Definition B.9 *If* G *is faithful on an elementary abelian p-group,* V, *and there exists and elementary abelian p-group,* A *with* $1 \neq A \leq G$ *with* $|A||C_V(A)| \geq |V|$, *then* V *is called a failure of factorization module or* **FF-module** *for* G. *The subgroup* A *is called an offending subgroup.*

Definition B.10 $O_p(M)$ *is the largest normal p-subgroup of* M.

Definition B.11 $O^p(M)$ *is the smallest normal subgroup of* M *such that* $M/O^p(M)$ *is a p-group.*

Definition B.12 *For* $X \subseteq V$, $X^\perp = \{v \in V \mid x \perp v \text{ for all } x \in X\}$ *where* $x \perp v$ *if* $s(x, v) = 0$.

Definition B.13 *A subspace* U *is* **isotropic** *if the symplectic form vanishes on* U*; that is, if* $U \leq U^\perp$ $(s(u, u) = 0)$.

Definition B.14 *A vector* $v \in V$ *is* **singular** *if* v *is isotropic* $(s(v, v) = 0)$ *and* $q(v) = 0$ *where* V *is an orthogonal space and* q *is the quadratic form.*

Definition B.15 $T \subseteq G$ *is a* **TI-set** *if* $T \cap T^g \subseteq \{1\}$ *for all* $g \in G \setminus N_G(T)$.

Definition B.16 *Let* G *be a finite group with* $C' = G$. *If there exists a largest group* H *(unique up to isomorphism) such that* $H/Z(H) \cong G$ *with* $H = H'$, *then* $Z(H)$ *is the* **Schur multiplier**. *Note that the Schur multiplier is the largest perfect central extension.*

Definition B.17 *If we have G acting on Ω, then G_ω is* **transitive** *on $\Omega \setminus \omega$ and G is* **doubly transitive** *on $\{(\omega_1, \omega_2) | \omega_1 \neq \omega_2\}$. An action is transitive if there is only one orbit. The action is doubly transitive if some permutation takes any pair of elements to any other pair.*

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