# THE EXCURSION PROBABILITY OF GAUSSIAN AND ASYMPTOTICALLY GAUSSIAN RANDOM FIELDS

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## ABSTRACT

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The purpose of this thesis is to develop the asymptotic approximation to excursion probability of Gaussian and asymptotically Gaussian random fields. It is composed of two parts. The first part is to study smooth Gaussian random fields. We extend the expected Euler characteristic approximation to a wide class of smooth Gaussian random fields with non-constant variances. Applying similar techniques, we also find that the joint excursion probability of vector-valued smooth Gaussian random fields can be approximated via the expected Euler characteristic of related excursion sets. As useful applications, the excursion probabilities over random intervals and infinite intervals are also investigated. The second part focuses on non-smooth Gaussian and asymptotically Gaussian random fields. We study the excursion probability of Gaussian random fields on the sphere and obtain an asymptotics based on the Pickands' constant. Using double sum method, we also derive the approximation, which involves the generalized Pickands' constant, to excursion probability of anisotropic Gaussian and asymptotically Gaussian random fields.

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# Chapter 1

# Introduction and Review of Existing Literature

## 1.1 Gaussian Random Fields

A real-valued random field is simply a stochastic process defined over a parameter space $T$, which could be a subset of $\mathbb{R}^N$ or even a manifold, etc. The following is the rigorous definition [cf. Adler and Taylor (2007)].

**Definition 1.1.1** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* $T$ *a topological space. Then a measurable mapping* $X : \Omega \to \mathbb{R}^T$ *(the space of all real-valued functions on* $T$*) is called a real-valued random field. Measurable mappings from* $\Omega$ *to* $(\mathbb{R}^T)^d$, $d \geq 1$, *are called vector-valued random fields.*

Thus, $X$ is a real-valued function $X(\omega, t)$, where $\omega \in \Omega$ and $t \in T$. For convenience, usually, we abbreviate $X(\omega, t)$ as $X(t)$ or $X$.

We define a real-valued *Gaussian (random) field* to be a real-valued random field $X$ on a parameter space $T$ for which the finite dimensional distributions of $(X(t_1), \ldots, X(t_n))$ are multivariate Gaussian ( i.e., multivariate Normal) for each $1 \leq n < \infty$ and each $(t_1, \ldots, t_n) \in T^n$. The functions $m(t) = \mathbb{E}\{X(t)\}$ and $C(t, s) = \mathbb{E}\{(X(t) - m(t))(X(s) - m(s))\}$ are called respectively the *mean* and *covariance functions* of $X$. If $m(t) \equiv 0$, we call $X$ a *centered*

Gaussian field. A vector-valued Gaussian field $X$ taking values in $\mathbb{R}^d$ is the random field for which $\langle \xi, X(t) \rangle$ is a real-valued Gaussian field for every $\xi \in \mathbb{R}^d$.

The following result is Theorem 1.4.1 in Adler and Taylor (2007), which gives a sufficient condition such that a Gaussian field $X$ is continuous and bounded.

**Theorem 1.1.2** *Let $\{X(t) : t \in T\}$ be a centered Gaussian field, where $T$ is a compact set of $\mathbb{R}^N$. If there exist positive constants $K$, $\alpha$ and $\eta$ such that*

$$\mathbb{E}\{|X(t) - X(s)|^2\} \leq K|\log \|t-s\||^{-1-\alpha}, \quad \forall \|t-s\| \leq \eta,$$

*then $X$ is continuous and bounded on $T$ with probability one.*

Note that the sufficient condition in the above theorem only depends on the covariance function of $X$. This is a huge advantage for studying centered Gaussian random fields: all of their properties only depend on the covariance structure. Similar sufficient conditions for the differentiability of Gaussian fields can also be obtained, see Chapter 1 in Adler and Taylor (2007) for more details.

## 1.2 Excursion Probability

The excursion probability above level $u > 0$ is defined as $\mathbb{P}\{\sup_{t \in T} X(t) \geq u\}$. Due to the wide applications in statistics and many other related areas, computing the excursion probability becomes a classical and very important problem in probability theory. However, usually, the exact probability is unable to obtain, instead, we try to find the asymptotic approximation as $u$ tends to infinity.

There is a classical result of Landau and Shepp (1970) and Marcus and Shepp (1972) that gives a logarithmic asymptotics for the excursion probability of a general centered Gaussian process. If we assume that $X(t)$ is a.s. bounded, then they showed that

$$\lim_{u\to\infty} u^{-2}\log \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = -\frac{1}{2\sigma_T^2}, \tag{1.2.1}$$

where $\sigma_T^2 = \sup_{t\in T}\mathrm{Var}(X(t))$.

We present here a non-asymptotic result due to Borell (1975) and Tsirelson, Ibragimov and Sudakov (TIS) (1976).

**Theorem 1.2.1 (Borell-TIS inequality).** *Let $\{X(t) : t \in T\}$ be a centered Gaussian field, a.s. bounded, where $T$ is a compact subset of $\mathbb{R}^N$. Then $\mathbb{E}\{\sup_{t\in T} X(t)\} < \infty$ and for all $u > 0$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) - \mathbb{E}\Big\{\sup_{t\in T} X(t)\Big\} \geq u\Big\} \leq e^{-u^2/(2\sigma_T^2)}.$$

It is evident to check that the Borell-TIS inequality implies (1.2.1). There are also several non-asymptotic bounds for the excursion probability of general (only assume continuity and boundedness a.s.) Gaussian fields, see Chapter 4 in Adler and Taylor (2007) for more details.

If assume $X$ to be stationary or locally stationary, then there is a famous approximation obtained by the double sum method. This technique was developed by Pickands (1969a, 1969b) for Gaussian processes, extended to Gaussian fields by Qualls and Watanabe (1973), and surveyed and developed in a monograph of Piterbarg (1996a).

**Theorem 1.2.2** *Let $T$ be a bounded Jordan measurable set in $\mathbb{R}^N$ such that* $\dim(T) = N$, *and let $\{X(t) : t \in T\}$ be a centered Gaussian field with covariance function $C(\cdot,\cdot)$ satisfying*

$$C(t,s) = 1 - \|t-s\|^{\alpha}(1+o(1)) \quad \text{as } \|t-s\| \to 0.$$

*Then as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = H_\alpha \mathrm{Vol}(T) u^{2N/\alpha} \Psi(u)(1+o(1)), \tag{1.2.2}$$

*where $H_\alpha$ is the Pickankds' constant and $\Psi(u) = (2\pi)^{-1/2}\int_u^\infty e^{-x^2/2}dx$.*

This result was developed further by Chan and Lai (2006) for Gaussian fields with a wider class of covariance structures. The coefficient $H_\alpha \mathrm{Vol}(T)$ above was generalized as $\int_T H_\alpha(t)dt$, where $H_\alpha(\cdot)$ is a function on $T$. Moreover, the result in Chan and Lai (2006) is applicable to certain asymptotically Gaussian random fields.

In Chapter 7, we investigate Gaussian random fields on the sphere and obtain Theorem 7.2.4, which is similar to Theorem 1.2.2. In Chapter 8, we extend the result in Chan and Lai (2006) to anisotropic and asymptotically anisotropic Gaussian random fields, see Theorem 8.1.1 and Theorem 8.2.6.

Can we get more accurate approximation to the excursion probability of "nicer" Gaussian random fields? The answer is yes. Sun (1993) used the tube method to find the approximation for Gaussian fields with finite Karhunen-Loève expansion. Also, many authors applied the Rice method to get accurate approximations for smooth Gaussian fields, see Piterbarg (1996a), Adler (2000) and Azaïs and Wschebor (2005, 2008, 2009), etc. Later on, these approximations were conjectured by statisticians that they should have close connection to the geometry of the excursion set $A_u = \{t \in T : X(t) \geq u\}$. Taylor, Takemura and Adler (2005) showed the rigorous proof that the expected Euler characteristic of the excursion set, denoted by $\mathbb{E}\{\varphi(A_u)\}$, can approximate the excursion probability very accurately. Their result is stated as follows.

**Theorem 1.2.3** *Let $X = \{X(t) : t \in T\}$ be a unit-variance smooth Gaussian random field* *parameterized on a manifold $T$. Under certain conditions on the regularity of $X$ and topology of $T$, the following approximation holds:*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\}(1 + o\big(e^{-\alpha u^2}\big)), \quad \text{as } u \to \infty, \tag{1.2.3}$$

*where $\alpha$ is some positive constant.*

Moreover, $\mathbb{E}\{\varphi(A_u)\}$ can be computed by the Kac-Rice formula, see Adler and Taylor (2007),

$$\mathbb{E}\{\varphi(A_u)\} = C_0\Psi(u) + \sum_{j=1}^{\dim(T)} C_j u^{j-1} e^{-u^2/2}, \tag{1.2.4}$$

where $C_j$, $j = 0, 1, \ldots, \dim(T)$, are constants depending on $X$ and $T$. Here is a simple example. Let $X$ be a smooth isotropic Gaussian field with unit variance and $T = [0, L]^N$, then

$$\mathbb{E}\{\varphi(A_u)\} = \Psi(u) + \sum_{j=1}^{N} \frac{\binom{N}{j} L^j \lambda^{j/2}}{(2\pi)^{(j+1)/2}} H_{j-1}(u) e^{-u^2/2},$$

where $\lambda = \mathrm{Var}(\frac{\partial X}{\partial t_i}(t))$ and $H_{j-1}(u)$ are Hermite polynomials of order $j-1$. It is worth mentioning here that if $X$ is not centered or not stationary, then $\mathbb{E}\{\varphi(A_u)\}$ becomes complicated to compute. In the recent monograph Adler and Taylor (2007), the authors only considered centered Gaussian random fields with constant variance. In Chapter 4 here, we study non-centered stationary Gaussian fields and derive exact formulae for computing $\mathbb{E}\{\varphi(A_u)\}$.

Comparing (1.2.3) and (1.2.4) with (1.2.2), we see that the approximation in (1.2.2) only uses one of the terms, which involves $u^{N-1}e^{-u^2/2}$, in $\mathbb{E}\{\varphi(A_u)\}$. Also, we note that the error term in (1.2.2) is only $o(1)$, and the expected Euler characteristic approximation in (1.2.3) is much more accurate since the error is exponentially smaller than the major term

$\mathbb{E}\{\varphi(A_u)\}$.

The requirement of "constant variance" on the Gaussian random fields in Theorem 1.2.3 is too restrictive for many applications. However, the original proof in Taylor, Takemura and Adler (2005) relies on this requirement heavily. If the constant variance condition is not satisfied, little had been known on whether the approximation (1.2.3) still holds. In a recent paper Azaïs and Wschebor (2008, Theorem 5), the authors proved (1.2.3) for a special case when the variance of the Gaussian field attains its maximum only in the interior of $T$. But this special case excludes many important Gaussian fields in which we are interested.

As a major contribution in this thesis, we shall use the Rice method to show (1.2.3) for more general smooth Gaussian fields without constant-variance. In Chapter 2, we study smooth Gaussian random fields with stationary increments and obtain the desired results in Theorem 2.3.7 and Theorem 2.3.8. Meanwhile, we provide a specific formula for computing $\mathbb{E}\{\varphi(A_u)\}$ in Theorem 2.2.2. To develop the theory further, we show in Chapter 3 that the expected Euler characteristic approximation also holds for a large class of smooth Gaussian random fields with non-constant variances. When computing $\mathbb{E}\{\varphi(A_u)\}$, we also find that it can be simplified in certain sense depending on the variance function of $X$.

As useful applications, we study the excursion probabilities of Gaussian processes over random intervals and infinite intervals in Chapter 5 and Chapter 6. The approximations we derived are also more accurate than the existing ones, since the errors are super-exponentially small.

Lastly, Chapter 9 is on a new topic: the excursion probability for vector-valued Gaussian random fields. There has been little research on this. The only exceptions are Piterbarg and Stamatovic (2005) and Debicki et al. (2010) who obtained some logarithmic asymptotics, and Ladneva and Piterbarg (2000) and Anshin (2006) who obtained certain asymptotics for non-smooth vector-valued Gaussian random fields with special covariance functions.

Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field, where $T$ and $S$ are rectangles in $\mathbb{R}^N$. Define the excursion set

$$A_u(X,T) \times A_u(Y,S) = \{(t,s) \in T \times S : X(t) \geq u, Y(s) \geq u\}.$$

We show in Theorem 9.1.9 that under certain smoothness and regularity conditions, as $u \to \infty$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s \in S} Y(s) \geq u\Big\} \\ &= \mathbb{E}\{\varphi(A_u(X,T) \times A_u(Y,S))\} + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T,S)} - \alpha u^2\Big\}\Big).\end{aligned}$$

where $\rho(T,S) = \sup_{t\in T, s\in S} \mathbb{E}\{X(t)Y(s)\}$.

Let $\{(X(t), Y(t)) : t \in T\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian process, where $T = [a,b]$ is a finite interval in $\mathbb{R}$. Define the excursion set

$$A_u(T, X \wedge Y) = \{t \in T : (X \wedge Y)(t) \geq u\}.$$

We show in Theorem 9.2.5 that under certain smoothness and regularity conditions, as $u \to \infty$,

$$\begin{aligned}&\mathbb{P}\{\exists t \in T \text{ such that } X(t) \geq u, Y(t) \geq u\} = \mathbb{P}\Big\{\sup_{t\in T}(X \wedge Y)(t) \geq u\Big\} \\ &= \mathbb{E}\{\varphi(A_u(T, X \wedge Y))\} + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T)} - \alpha u^2\Big\}\Big),\end{aligned}$$

where $\rho(T) = \sup_{t\in T} \mathbb{E}\{X(t)Y(t)\}$.

# Chapter 2

# Smooth Gaussian Random Fields with Stationary Increments

## 2.1 Gaussian Fields with Stationary Increments

Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments. We assume that $X$ has continuous covariance function $C(t, s) = \mathbb{E}\{X(t)X(s)\}$ and $X(0) = 0$. Then it is known [cf. Yaglom (1957)] that

$$C(t, s) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1)(e^{-i\langle s,\lambda\rangle} - 1)\, F(d\lambda) + \langle t,\, \Theta s\rangle \tag{2.1.1}$$

where $\langle x, y\rangle$ is the ordinary inner product in $\mathbb{R}^N$, $\Theta$ is an $N \times N$ non-negative definite (or positive semidefinite) matrix and $F$ is a non-negative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ which satisfies

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1 + \|\lambda\|^2}\, F(d\lambda) < \infty. \tag{2.1.2}$$

Similarly to stationary random fields, the measure $F$ and its density (if it exists) $f(\lambda)$ are called the *spectral measure* and *spectral density* of $X$, respectively.

By (2.1.1) we see that $X$ has the following stochastic integral representation

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1) W(d\lambda) + \langle \mathbf{Y}, t\rangle, \tag{2.1.3}$$

where $\mathbf{Y}$ is an $N$-dimensional Gaussian random vector and $W$ is a complex-valued Gaussian random measure (independent of $\mathbf{Y}$) with $F$ as its control measure. It is known that many probabilistic, analytic and geometric properties of a Gaussian field with stationary increments can be described in terms of its spectral measure $F$ and, on the other hand, various interesting Gaussian random fields can be constructed by choosing their spectral measures appropriately. See Xiao (2009), Xue and Xiao (2011) and the references therein for more information.

For simplicity we assume that $\mathbf{Y} = 0$. It follows from (2.1.1) that the *variogram* $\nu$ of $X$ is given by

$$\nu(h) := \mathbb{E}(X(t+h) - X(t))^2 = 2\int_{\mathbb{R}^N} (1 - \cos\langle h, \lambda\rangle) F(d\lambda). \tag{2.1.4}$$

Mean-square directional derivatives and sample path differentiability of Gaussian random fields have been well studied. See, for example, Adler (1981), Adler and Taylor (2007), Potthoff (2010), Xue and Xiao (2011). In particular, general sufficient conditions for a Gaussian random field to have a modification whose sample functions are in $C^k$ are given by Adler and Taylor (2007). For a Gaussian random field $X = \{X(t) : t \in \mathbb{R}^N\}$ with stationary increments, Xue and Xiao (2011) provided conditions for its sample path differentiability in terms of the spectral density function $f(\lambda)$. Similar arguments can be applied to give the spectral condition for the sample functions of $X$ to be in $C^k(\mathbb{R}^N)$.

**Definition 2.1.1** [Adler and Taylor (2007, p.22)]. *Let*

$$t, v_1, \ldots, v_k \in \mathbb{R}^N; \quad \mathbf{v} = (v_1, \ldots, v_k) \in \otimes^k \mathbb{R}^N.$$

*We say $X$ has a $k$th-order $L^2$ partial derivative at $t$, in the direction $\mathbf{v}$, which we denote by $D^{\mathbf{v}}_{L^2}X(t)$, if the limit*

$$D^{\mathbf{v}}_{L^2}X(t) := \lim_{h_1,\ldots,h_k\to 0} \frac{1}{\prod_{i=1}^k h_i} G_X\Big(t, \sum_{i=1}^k h_i v_i\Big)$$

*exists in $L^2$, where $G_X(t, \sum_{i=1}^k h_i v_i)$ is the symmetrized difference*

$$G_X\Big(t, \sum_{i=1}^k h_i v_i\Big) = \sum_{s\in\{0,1\}^k} (-1)^{k-\sum_{i=1}^k s_i} X\Big(t + \sum_{i=1}^k s_i h_i v_i\Big). \tag{2.1.5}$$

**Remark 2.1.2** Recall the fact that a sequence of random variables $\xi_n$ converges in $L^2$ if and only if $\mathbb{E}\{\xi_n \xi_m\}$ converges to a constant as $n, m \to \infty$. It follows immediately that $D^{\mathbf{v}}_{L^2}X(t)$ exists in $L^2$ if and only if

$$\lim_{h_1,\ldots,h_k,\hat{h}_1,\ldots,\hat{h}_k\to 0} \frac{1}{\prod_{i=1}^k h_i \hat{h}_i} \mathbb{E}\Big\{G_X\Big(t, \sum_{i=1}^k h_i v_i\Big) G_X\Big(t, \sum_{i=1}^k \hat{h}_i v_i\Big)\Big\} \tag{2.1.6}$$

exists.

Let $e_1, e_2, \ldots, e_N$ be the standard orthonormal basis of $\mathbb{R}^N$. If the direction $\mathbf{v}$ consists of $k_i$ many $e_i$, $1 \le i \le N$, and $k = \sum_{i=1}^N k_i$, then we write $D^{\mathbf{v}}_{L^2}X(t)$ simply as $\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}}$.

**Lemma 2.1.3** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and let $k = \sum_{i=1}^N k_i$. Then $\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}}$ exists in $L^2$ if and only if $\frac{\partial^{2k} \nu(0)}{\partial t_1^{2k_1} \cdots \partial t_N^{2k_N}}$ exists.*

**Proof** To simplify the notations, we only show the proof for $k = 2$ and the proof for general

$k$ will be similar. By the definition of the symmetric difference $G_X$ in (2.1.5),

$$
\begin{aligned}
&\frac{1}{h_1 h_2 \hat{h}_1 \hat{h}_2} \mathbb{E}\{G_X(t, h_1 e_i + h_2 e_j) G_X(t, \hat{h}_1 e_i + \hat{h}_2 e_j)\} \\
&\quad = \frac{1}{h_1 h_2 \hat{h}_1 \hat{h}_2} \mathbb{E}\{[X(t + h_1 e_i + h_2 e_j) - X(t + h_1 e_i) - X(t + h_2 e_j) + X(t)] \\
&\qquad\qquad\qquad \times [X(t + \hat{h}_1 e_i + \hat{h}_2 e_j) - X(t + \hat{h}_1 e_i) - X(t + \hat{h}_2 e_j) + X(t)]\}.
\end{aligned} \tag{2.1.7}
$$

Expanding the product above and applying the variogram $\nu$ defined in (2.1.4), we obtain that (2.1.7) becomes

$$
\begin{aligned}
&\frac{-1}{2h_1 h_2 \hat{h}_1 \hat{h}_2} \{\nu(h_1 e_i + h_2 e_j - \hat{h}_1 e_i - \hat{h}_2 e_j) - \nu(h_1 e_i + h_2 e_j - \hat{h}_1 e_i) \\
&\quad - \nu(h_1 e_i + h_2 e_j - \hat{h}_2 e_j) + \nu(h_1 e_i + h_2 e_j) - \nu(h_1 e_i - \hat{h}_1 e_i - \hat{h}_2 e_j) \\
&\quad + \nu(h_1 e_i - \hat{h}_1 e_i) + \nu(h_1 e_i - \hat{h}_2 e_j) - \nu(h_1 e_i) - \nu(h_2 e_j - \hat{h}_1 e_i - \hat{h}_2 e_j) \\
&\quad + \nu(h_2 e_j - \hat{h}_1 e_i) + \nu(h_2 e_j - \hat{h}_2 e_j) - \nu(h_2 e_j) + \nu(-\hat{h}_1 e_i - \hat{h}_2 e_j) \\
&\quad - \nu(-\hat{h}_1 e_i) - \nu(-\hat{h}_2 e_j) + \nu(0)\} \\
&= \frac{-1}{2h_1 h_2 (-\hat{h}_1)(-\hat{h}_2)} G_\nu(0, h_1 e_i + h_2 e_j + (-\hat{h}_1) e_i + (-\hat{h}_2) e_j).
\end{aligned} \tag{2.1.8}
$$

Note that as $h_1, h_2, \hat{h}_1, \hat{h}_2 \to 0$, the limit (if it exists) of the last term in (2.1.8) is just $-\frac{\partial^4 \nu(0)}{\partial t_i^2 \partial t_j^2}$, together with Remark 2.1.2, we obtain the desired result. $\square$

**Proposition 2.1.4** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and let $k_i$ $(1 \le i \le N)$ be non-negative integers. If there is a constant $\varepsilon > 0$ such that*

$$
\int_{\|\lambda\|>1} \prod_{i=1}^{N} |\lambda_i|^{2k_i+\varepsilon} F(d\lambda) < \infty, \tag{2.1.9}
$$

*then $X$ has a modification $\widetilde{X}$ such that the partial derivative $\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1}\cdots\partial t_N^{k_N}}$ is continuous on $\mathbb{R}^N$ almost surely, where $k = \sum_{i=1}^N k_i$. Moreover, $\forall T > 0$ and $\eta \in (0, \varepsilon \wedge 1)$, there exists a constant $\kappa$ such that*

$$\mathbb{E}\bigg(\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1}\cdots\partial t_N^{k_N}} - \frac{\partial^k \widetilde{X}(s)}{\partial s_1^{k_1}\cdots\partial s_N^{k_N}}\bigg)^2 \le \kappa\|t-s\|^\eta, \quad \forall t,s \in [-T,T]^N.$$

**Proof** Applying the dominated convergence theorem,

$$\begin{aligned}\frac{\partial^{2k}\nu(0)}{\partial t_1^{2k_1}\cdots\partial t_N^{2k_N}} &= \int_{\mathbb{R}^N} \lambda_1^{2k_1}\cdots\lambda_N^{2k_N}F(d\lambda)\\ &= \int_{\|\lambda\|\le 1} \lambda_1^{2k_1}\cdots\lambda_N^{2k_N}F(d\lambda) + \int_{\|\lambda\|>1} \lambda_1^{2k_1}\cdots\lambda_N^{2k_N}F(d\lambda)\\ &\le \int_{\|\lambda\|\le 1} \|\lambda\|^2F(d\lambda) + \int_{\|\lambda\|>1} \lambda_1^{2k_1}\cdots\lambda_N^{2k_N}F(d\lambda) < \infty,\end{aligned} \tag{2.1.10}$$

where the last inequality is due to the requirement (2.1.2) and condition (2.1.9). By Lemma 2.1.3, the partial derivative $\frac{\partial^k X(t)}{\partial t_1^{k_1}\cdots\partial t_N^{k_N}}$ exists in $L^2$.

Next, we show that for any $\eta \in (0, \varepsilon \wedge 1)$, there exists a constant $\kappa$ such that

$$\mathbb{E}\bigg(\frac{\partial^k X(t)}{\partial t_1^{k_1}\cdots\partial t_N^{k_N}} - \frac{\partial^k X(s)}{\partial s_1^{k_1}\cdots\partial s_N^{k_N}}\bigg)^2 \le \kappa\|t-s\|^\eta, \quad \forall t,s \in [-T,T]^N. \tag{2.1.11}$$

Recall that

$$\begin{aligned}C(t,s) &= \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle}-1\big)\big(e^{-i\langle s,\lambda\rangle}-1\big)F(d\lambda)\\ &= \int_{\mathbb{R}^N} (\cos\langle t-s,\lambda\rangle - \cos\langle t,\lambda\rangle - \cos\langle s,\lambda\rangle + 1)F(d\lambda),\end{aligned} \tag{2.1.12}$$

taking the derivative gives

$$\frac{\partial^{2k} C(t,s)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N} \partial s_1^{k_1} \cdots \partial s^{k_N}} = \int_{\mathbb{R}^N} \lambda_1^{2k_1} \cdots \lambda_N^{2k_N} \cos\langle t-s, \lambda\rangle F(d\lambda).$$

It follows that

$$\begin{aligned}
&\mathbb{E}\Big(\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}} - \frac{\partial^k X(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}}\Big)^2 \\
&= \mathbb{E}\Big(\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}}\Big)^2 + \mathbb{E}\Big(\frac{\partial^k X(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}}\Big)^2 - 2\mathbb{E}\Big(\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}} \frac{\partial^k X(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}}\Big) \\
&= 2\int_{\mathbb{R}^N} \lambda_1^{2k_1} \cdots \lambda_N^{2k_N} \big(1 - \cos\langle t-s, \lambda\rangle\big) F(d\lambda).
\end{aligned}$$

Let $\hat{s}_0 = t$, $\hat{s}_1 = (s_1, t_2, \ldots, t_N)$, $\hat{s}_2 = (s_1, s_2, t_3 \ldots, t_N)$, ..., $\hat{s}_{N-1} = (s_1, \ldots, s_{N-1}, t_N)$ and $\hat{s}_N = s$. Let $h = s - t := (h_1, \ldots, h_N)$. Then, by Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}\Big(\frac{\partial^k X(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}} - \frac{\partial^k X(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}}\Big)^2 \\
&\le N \sum_{j=1}^N \mathbb{E}\Big(\frac{\partial^k X(\hat{s}_j)}{\partial s_1^{k_1} \cdots \partial s_j^{k_j} \partial t_{j+1}^{k_{j+1}} \cdots \partial t_N^{k_N}} - \frac{\partial^k X(\hat{s}_{j-1})}{\partial s_1^{k_1} \cdots \partial s_{j-1}^{k_{j-1}} \partial t_j^{k_j} \cdots \partial t_N^{k_N}}\Big)^2 \\
&= 2N \sum_{j=1}^N \int_{\mathbb{R}^N} \big(1 - \cos(h_j \lambda_j)\big) \prod_{i=1}^N |\lambda_i|^{2k_i} F(d\lambda) \\
&\le 2N \sum_{j=1}^N \int_{\|\lambda\| \le 1} \big(1 - \cos(h_j \lambda_j)\big) \prod_{i=1}^N |\lambda_i|^{2k_i} F(d\lambda) \\
&\quad + 2N \sum_{j=1}^N \int_{\|\lambda\| > 1} \big(1 - \cos(h_j \lambda_j)\big) \prod_{i=1}^N |\lambda_i|^{2k_i} F(d\lambda) \\
&:= I_1 + I_2.
\end{aligned} \tag{2.1.13}$$

Combining the result in (2.1.10) with the elementary inequality $1 - \cos x \le x^2$ yields

$$I_1 \le 2N\Big(\sum_{j=1}^{N} |h_j|^2\Big) \int_{\|\lambda\|\le 1} \|\lambda\|^2 F(d\lambda) \le c_1 \|t-s\|^2 \tag{2.1.14}$$

for some positive constant $c_1$.

To bound the $j$th integral in $I_2$, we note that, when $\|\lambda\| > 1$, either $|\lambda_j| > 1/\sqrt{N}$ or there is $j_0 \ne j$ such that $\lambda_{j_0} > 1/\sqrt{N}$. We break the integral according to these two possibilities.

$$\begin{aligned}
&\int_{\|\lambda\|>1} \big(1-\cos(h_j\lambda_j)\big) \prod_{i=1}^{N} |\lambda_i|^{2k_i} F(d\lambda) \\
&\le \int_{|\lambda_j|>1/\sqrt{N}} \big(1-\cos(h_j\lambda_j)\big) \prod_{i=1}^{N} |\lambda_i|^{2k_i} F(d\lambda) \\
&\quad + \sum_{j_0\ne j} \int_{|\lambda_j|\le 1, |\lambda_{j_0}|>1/\sqrt{N}} \big(1-\cos(h_j\lambda_j)\big) \prod_{i=1}^{N} |\lambda_i|^{2k_i} F(d\lambda) \\
&:= I_3 + I_4.
\end{aligned} \tag{2.1.15}$$

Combining condition (2.1.9) with the elementary inequality $1 - \cos x \le x^2$ yields

$$\begin{aligned}
I_3 &\le \int_{1/\sqrt{N}<|\lambda_j|\le 1/|h_j|} \frac{1-\cos(h_j\lambda_j)}{|\lambda_j|^{\varepsilon}} \Big(|\lambda_j|^{\varepsilon} \prod_{i=1}^{N} |\lambda_i|^{2k_i}\Big) F(d\lambda) \\
&\quad + \int_{|\lambda_j|>1/|h_j|} \frac{1}{|\lambda_j|^{\varepsilon}} \Big(|\lambda_j|^{\varepsilon} \prod_{i=1}^{N} |\lambda_i|^{2k_i}\Big) F(d\lambda) \\
&\le c_2 |h_j|^{\varepsilon}
\end{aligned} \tag{2.1.16}$$

for some positive constant $c_2$. Similarly, it is evident to check that $I_4 \le c_3 |h_j|^2$ for some positive constant $c_3$. Therefore, the Höder condition for $L^2$ partial derivative in (2.1.11) holds, and then the desired result follows from Kolmogorov's continuity theorem. □

For simplicity we will not distinguish $X$ from its modification $\widetilde{X}$. As a consequence of Proposition 2.1.4, we see that, if $X = \{X(t) : t \in \mathbb{R}^N\}$ has a spectral density $f(\lambda)$ which satisfies

$$f(\lambda) = O\bigg(\frac{1}{\|\lambda\|^{N+2k+H}}\bigg) \quad \text{as } \|\lambda\| \to \infty, \tag{2.1.17}$$

for some integer $k \geq 1$ and $H \in (0,1)$, then the sample functions of $X$ are in $C^k(\mathbb{R}^N)$ a.s.

When $X(\cdot) \in C^2(\mathbb{R}^N)$ almost surely, we write $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$. Denote by $\nabla X(t)$ and $\nabla^2 X(t)$ the column vector $(X_1(t), \ldots, X_N(t))^T$ and the $N \times N$ matrix $(X_{ij}(t))_{i,j=1,\ldots,N}$, respectively. It follows from (2.1.1) that for every $t \in \mathbb{R}^N$,

$$\lambda_{ij} := \int_{\mathbb{R}^N} \lambda_i \lambda_j F(d\lambda) = \frac{\partial^2 C(t,s)}{\partial t_i \partial s_j}\Big|_{s=t} = \mathbb{E}\{X_i(t) X_j(t)\}. \tag{2.1.18}$$

Define the $N \times N$ matrix $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,N}$, then (2.1.18) shows that $\Lambda = \mathrm{Cov}(\nabla X(t))$ for all $t$. In particular, the distribution of $\nabla X(t)$ is independent of $t$. Let

$$\lambda_{ij}(t) := \int_{\mathbb{R}^N} \lambda_i \lambda_j \cos\langle t, \lambda\rangle \, F(d\lambda), \quad \Lambda(t) := (\lambda_{ij}(t))_{i,j=1,\ldots,N}.$$

Then we have

$$\lambda_{ij}(t) - \lambda_{ij} = \int_{\mathbb{R}^N} \lambda_i \lambda_j (\cos\langle t, \lambda\rangle - 1) \, F(\lambda) = \frac{\partial^2 C(t,s)}{\partial t_i \partial t_j}\Big|_{s=t} = \mathbb{E}\{X(t) X_{ij}(t)\},$$

or equivalently, $\Lambda(t) - \Lambda = \mathbb{E}\{X(t) \nabla^2 X(t)\}$.

Let $T = \prod_{i=1}^N [a_i, b_i]$ be a closed rectangle on $\mathbb{R}^N$, where $a_i < b_i$ for all $1 \leq i \leq N$ and $0 \notin T$ (the case of $0 \in T$ will be discussed in Remark 2.4.1). In addition to the stationary increments, we will make use of the following conditions on $X$:

($\mathbf{H}1$). $X(\cdot) \in C^2(T)$ almost surely and its second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants $L, \eta > 0$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(s))^2 \leq L\|t - s\|^{2\eta}, \quad \forall t, s \in T,\ i, j = 1, \ldots, N. \tag{2.1.19}$$

($\mathbf{H}2$). For every $t \in T$, the matrix $\Lambda - \Lambda(t)$ is non-degenerate.

($\mathbf{H}3$). For every pair $(t, s) \in T^2$ with $t \neq s$, the Gaussian random vector

$$(X(t), \nabla X(t), X_{ij}(t),\ X(s), \nabla X(s), X_{ij}(s), 1 \leq i \leq j \leq N)$$

is non-degenerate.

($\mathbf{H}3'$). For every $t \in T$, $(X(t), \nabla X(t), X_{ij}(t), 1 \leq i \leq j \leq N)$ is non-degenerate.

Clearly, by Proposition 2.1.4, condition ($\mathbf{H}1$) is satisfied if (2.1.17) holds for $k = 2$. Also note that ($\mathbf{H}3$) implies ($\mathbf{H}3'$). We shall use conditions ($\mathbf{H}1$), ($\mathbf{H}2$) and ($\mathbf{H}3$) to prove Theorems 2.3.7 and 2.3.8. Condition ($\mathbf{H}3'$) will be used for computing $\mathbb{E}\{\varphi(A_u)\}$ in Theorem 2.2.2.

The following lemma shows that for Gaussian fields with stationary increments, ($\mathbf{H}2$) is equivalent to $\Lambda - \Lambda(t)$ being positive definite.

**Lemma 2.1.5** *For every $t \neq 0$, $\Lambda - \Lambda(t)$ is non-negative definite. Hence, under* ($\mathbf{H}2$), *$\Lambda - \Lambda(t)$ is positive definite.*

**Proof** Let $t \neq 0$ be fixed. For any $(a_1, \ldots, a_N) \in \mathbb{R}^N \backslash \{0\}$,

$$\sum_{i,j=1}^{N} a_i a_j (\lambda_{ij} - \lambda_{ij}(t)) = \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N} a_i \lambda_i \right)^2 (1 - \cos \langle t, \lambda \rangle)\, F(\lambda). \tag{2.1.20}$$

Since $(\sum_{i=1}^N a_i\lambda_i)^2(1-\cos\langle t,\lambda\rangle)\geq 0$ for all $\lambda\in\mathbb{R}^N$, (2.1.20) is always non-negative, which implies $\Lambda-\Lambda(t)$ is non-negative definite. If (**H**2) is satisfied, then all the eigenvalues of $\Lambda-\Lambda(t)$ are positive. This completes the proof. $\square$

It follows from (2.1.20) that, if the spectral measure $F$ is carried by a set of positive Lebesgue measure (i.e., there is a set $B\subset\mathbb{R}^N$ with positive Lebesgue measure such that $F(B)>0$), then (**H**2) holds. Hence, (**H**2) is in fact a very mild condition for smooth Gaussian fields with stationary increments.

Lemma 2.1.5 and the following two lemmas indicate some significant properties of Gaussian fields with stationary increments. They will play important roles in later sections.

**Lemma 2.1.6** *For each $t$, $X_i(t)$ and $X_{jk}(t)$ are independent for all $i,j,k$; and $\mathbb{E}\{X_{ij}(t)X_{kl}(t)\}$ is symmetric in $i$, $j$, $k$, $l$.*

**Proof** By (2.1.1), one can verify that for $t,s\in\mathbb{R}^N$,

$$\begin{aligned}\mathbb{E}\{X_i(t)X_{jk}(s)\}&=\frac{\partial^3C(t,s)}{\partial t_i\partial s_j\partial s_k}=\int_{\mathbb{R}^N}\lambda_i\lambda_j\lambda_k\sin\langle t-s,\lambda\rangle\,F(d\lambda),\\ \mathbb{E}\{X_{ij}(t)X_{kl}(s)\}&=\frac{\partial^4C(t,s)}{\partial t_i\partial t_j\partial s_k\partial s_l}=\int_{\mathbb{R}^N}\lambda_i\lambda_j\lambda_k\lambda_l\cos\langle t-s,\lambda\rangle\;F(d\lambda).\end{aligned}\tag{2.1.21}$$

Letting $s=t$ we obtain the desired results. $\square$

It follows immediately from Lemma 2.1.6 that the following result holds.

**Lemma 2.1.7** *Let $A=(a_{ij})_{1\leq i,j\leq N}$ be a symmetric matrix, then*

$$\mathcal{S}_t(i,j,k,l)=\mathbb{E}\{(A\nabla^2X(t)A)_{ij}(A\nabla^2X(t)A)_{kl}\}$$

*is a symmetric function of $i$, $j$, $k$, $l$.*

## 2.2 The Mean Euler Characteristic

The rectangle $T = \prod_{i=1}^{N}[a_i, b_i]$ can be decomposed into several faces of lower dimensions. We use the same notations as in Adler and Taylor (2007, p.134).

A face $J$ of dimension $k$, is defined by fixing a subset $\sigma(J) \subset \{1, \dots, N\}$ of size $k$ and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0, 1\}^{N-k}$ of size $N - k$, so that

$$
\begin{aligned}
J = \{t = (t_1, \dots, t_N) \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J), \\
t_j = (1 - \varepsilon_j)a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}.
\end{aligned}
$$

Denote by $\partial_k T$ the collection of all $k$-dimensional faces in $T$, then the interior of $T$ is given by $\overset{\circ}{T} = \partial_N T$ and the boundary of $T$ is given by $\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J$. For $J \in \partial_k T$, denote by $\nabla X_{|J}(t)$ and $\nabla^2 X_{|J}(t)$ the column vector $(X_{i_1}(t), \dots, X_{i_k}(t))^T_{i_1,\dots,i_k \in \sigma(J)}$ and the $k \times k$ matrix $(X_{mn}(t))_{m,n \in \sigma(J)}$, respectively.

If $X(\cdot) \in C^2(\mathbb{R}^N)$ and it is a Morse function a.s. [cf. Definition 9.3.1 in Adler and Taylor (2007)], then according to Corollary 9.3.5 or page 211-212 in Adler and Taylor (2007), the Euler characteristic of the excursion set $A_u = \{t \in T : X(t) \geq u\}$ is given by

$$
\varphi(A_u) = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J) \tag{2.2.1}
$$

with

$$
\begin{aligned}
\mu_i(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \operatorname{index}(\nabla^2 X_{|J}(t)) = i, \\
\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},
\end{aligned} \tag{2.2.2}
$$

where $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues. We also define

$$\widetilde{\mu}_i(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = i\}. \tag{2.2.3}$$

Let $\sigma_t^2 = \text{Var}(X(t))$ and let $\sigma_T^2 = \sup_{t\in T} \sigma_t^2$ be the maximum variance. For Gaussian fields with stationary increments, it follows from (2.1.4) that $\nu(t) = \sigma_t^2$. For $t \in J \in \partial_k T$, where $k \geq 1$, let

$$\begin{aligned}
&\Lambda_J = (\lambda_{ij})_{i,j\in\sigma(J)}, \quad \Lambda_J(t) = (\lambda_{ij}(t))_{i,j\in\sigma(J)}, \\
&\theta_t^2 = \text{Var}(X(t)|\nabla X_{|J}(t)), \quad \gamma_t^2 = \text{Var}(X(t)|\nabla X(t)), \\
&\{J_1, \ldots, J_{N-k}\} = \{1, \ldots, N\} \backslash \sigma(J), \\
&E(J) = \{(t_{J_1}, \ldots, t_{J_{N-k}}) \in \mathbb{R}^{N-k} : t_j \varepsilon_j^* \geq 0, j = J_1, \ldots, J_{N-k}\}.
\end{aligned} \tag{2.2.4}$$

Then for all $t \in J$,

$$\Lambda_J = \text{Cov}(\nabla X_{|J}(t)), \quad \Lambda_J(t) - \Lambda_J = \mathbb{E}\{X(t)\nabla^2 X_{|J}(t)\}. \tag{2.2.5}$$

Note that $\theta_t^2 \geq \gamma_t^2$ for all $t \in T$ and $\theta_t^2 = \gamma_t^2$ if $t \in \partial_N T$. For $\{t\} \in \partial_0 T$, then $\nabla X_{|J}(t)$ is not defined, in this case we set $\theta_t^2$ as $\sigma_t^2$ by convention. Let $C_j(t)$ be the $(1, j+1)$ entry of $(\text{Cov}(X(t), \nabla X(t)))^{-1}$, i.e. $C_j(t) = M_{1,j+1}/\text{detCov}(X(t), \nabla X(t))$, where $M_{1,j+1}$ is the cofactor of the $(1, j+1)$ entry, $\mathbb{E}\{X(t)X_j(t)\}$, in the covariance matrix $\text{Cov}(X(t), \nabla X(t))$.

Denote by $H_k(x)$ the Hermite polynomial of order $k$, i.e., $H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2})$. Then the following identity holds [cf. Adler and Taylor (2007, p.289)]:

$$\int_u^\infty H_k(x) e^{-x^2/2}\, dx = H_{k-1}(u) e^{-u^2/2}, \tag{2.2.6}$$

where $u > 0$ and $k \geq 1$. For a matrix $A$, $|A|$ denotes its determinant. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$ and $\Psi(u) = (2\pi)^{-1/2} \int_u^\infty e^{-x^2/2} dx$.

The following lemma is an analogue of Lemma 11.7.1 in Adler and Taylor (2007). It provides a key step for computing the mean Euler characteristic in Theorem 2.2.2, meanwhile, it has close connection with Theorem 2.3.7.

**Lemma 2.2.1** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments satisfying* $(\mathbf{H}1)$, $(\mathbf{H}2)$ *and* $(\mathbf{H}3')$. *Then for each $J \in \partial_k T$ with $k \geq 1$,*

$$\mathbb{E}\left\{ \sum_{i=0}^{k} (-1)^i \widetilde{\mu}_i(J) \right\} = \frac{(-1)^k}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)} dt. \quad (2.2.7)$$

**Proof** Let $\mathcal{D}_i$ be the collection of all $k \times k$ matrices with index $i$. Recall the definition of $\widetilde{\mu}_i(J)$ in (2.2.3), thanks to $(\mathbf{H}1)$ and $(\mathbf{H}3')$, we can apply the Kac-Rice metatheorem [cf. Theorem 11.2.1 or Corollary 11.2.2 in Adler and Taylor (2007)] to get that the left hand side of (2.2.7) becomes

$$\int_J p_{\nabla X_{|J}(t)}(0) dt \sum_{i=0}^{k} (-1)^i \mathbb{E}\{|\mathrm{det} \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_i\}} \mathbb{1}_{\{X(t) \geq u\}} | \nabla X_{|J}(t) = 0\}. \quad (2.2.8)$$

Note that on the event $\mathcal{D}_i$, the matrix $\nabla^2 X_{|J}(t)$ has $i$ negative eigenvalues, which implies $(-1)^i |\mathrm{det} \nabla^2 X_{|J}(t)| = \mathrm{det} \nabla^2 X_{|J}(t)$. Also, $\cup_{i=0}^k \{\nabla^2 X_{|J}(t) \in \mathcal{D}_i\} = \Omega$ a.s., hence (2.2.8) equals

$$\begin{aligned}
&\int_J p_{\nabla X_{|J}(t)}(0)dt\, \mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{X(t)\geq u\}}|\nabla X_{|J}(t)=0\} \\
&\quad = \int_J \frac{e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}dt \int_u^\infty dx\, \mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t)=x, \nabla X_{|J}(t)=0\}.
\end{aligned} \tag{2.2.9}$$

Now we turn to computing $\mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t)=x, \nabla X_{|J}(t)=0\}$. By Lemma 2.1.5, under (**H**2), $\Lambda - \Lambda(t)$ and hence $\Lambda_J - \Lambda_J(t)$ are positive definite for every $t \in J$. Thus there exists a $k \times k$ positive definite matrix $Q_t$ such that

$$Q_t(\Lambda_J - \Lambda_J(t))Q_t = I_k, \tag{2.2.10}$$

where $I_k$ is the $k \times k$ identity matrix. By (2.2.5),

$$\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\} = -(Q_t(\Lambda_J - \Lambda_J(t))Q_t)_{ij} = -\delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta function. One can write

$$\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t)=x, \nabla X_{|J}(t)=0\} = \mathbb{E}\{\det\Delta(t,x)\}, \tag{2.2.11}$$

where $\Delta(t,x) = (\Delta_{ij}(t,x))_{i,j\in\sigma(J)}$ with all elements $\Delta_{ij}(t,x)$ being Gaussian variables. To study $\Delta(t,x)$, we only need to find its mean and covariance. Note that $\nabla X(t)$ and $\nabla^2 X(t)$

are independent by Lemma 2.1.6, then we apply Lemma 2.5.1 to obtain

$$\begin{aligned}\mathbb{E}\{\Delta_{ij}(t,x)\} &= \mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}|X(t)=x, \nabla X_{|J}(t)=0\} \\ &= (\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \ldots, 0)^T \\ &= (-\delta_{ij}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \ldots, 0)^T = -\frac{x}{\theta_t^2}\delta_{ij},\end{aligned} \tag{2.2.12}$$

where the last equality comes from the fact that the $(1,1)$ entry of $(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}$ is $\det\mathrm{Cov}(\nabla X_{|J}(t))/\det\mathrm{Cov}(X(t), \nabla X_{|J}(t)) = 1/\theta_t^2$. For the covariance, applying Lemma 2.5.1 again gives

$$\begin{aligned}&\mathbb{E}\{(\Delta_{ij}(t,x) - \mathbb{E}\{\Delta_{ij}(t,x)\})(\Delta_{kl}(t,x) - \mathbb{E}\{\Delta_{kl}(t,x)\})\} \\ &\quad= \mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}(Q_t\nabla^2 X_{|J}(t)Q_t)_{kl}\} - (\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\}, 0, \ldots, 0) \\ &\qquad\cdot (\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{kl}\}, 0, \ldots, 0)^T \\ &\quad= \mathcal{S}_t(i,j,k,l) - (-\delta_{ij}, 0, \ldots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(-\delta_{kl}, 0, \ldots, 0)^T \\ &\quad= \mathcal{S}_t(i,j,k,l) - \frac{\delta_{ij}\delta_{kl}}{\theta_t^2},\end{aligned}$$

where $\mathcal{S}_t$ is a symmetric function of $i$, $j$, $k$, $l$ by applying Lemma 2.1.7 with $A$ replaced by $Q_t$. Therefore (2.2.11) becomes

$$\mathbb{E}\left\{\frac{1}{\theta_t^k}\det(\theta_t Q_t(\nabla^2 X_{|J}(t))Q_t)\Big|X(t)=x, \nabla X_{|J}(t)=0\right\} = \frac{1}{\theta_t^k}\mathbb{E}\left\{\det\left(\widetilde{\Delta}(t) - \frac{x}{\theta_t}I_k\right)\right\},$$

where $\widetilde{\Delta}(t) = (\widetilde{\Delta}_{ij}(t))_{i,j\in\sigma(J)}$ and all $\widetilde{\Delta}_{ij}(t)$ are Gaussian variables satisfying

$$\mathbb{E}\{\widetilde{\Delta}_{ij}(t)\} = 0, \quad \mathbb{E}\{\widetilde{\Delta}_{ij}(t)\widetilde{\Delta}_{kl}(t)\} = \theta_t^2\mathcal{S}_t(i,j,k,l) - \delta_{ij}\delta_{kl}.$$

By Corollary 11.6.3 in Adler and Taylor (2007), (2.2.11) is equal to $(-1)^k\theta_t^{-k}H_k(x/\theta_t)$, hence

$$
\begin{aligned}
&\mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t)=x, \nabla X_{|J}(t)=0\}\\
&\quad= \mathbb{E}\{\det(Q_t^{-1}Q_t\nabla^2 X_{|J}(t)Q_tQ_t^{-1})|X(t)=x, \nabla X_{|J}(t)=0\}\\
&\quad= |\Lambda_J-\Lambda_J(t)|\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t)=x, \nabla X_{|J}(t)=0\}\\
&\quad= \frac{(-1)^k}{\theta_t^k}|\Lambda_J-\Lambda_J(t)|H_k\Big(\frac{x}{\theta_t}\Big).
\end{aligned}
$$

Plugging this into (2.2.9) and applying (2.2.6), we obtain the desired result. □

**Theorem 2.2.2** *Let $X=\{X(t):t\in\mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that* ($\mathbf{H}1$), ($\mathbf{H}2$) *and* ($\mathbf{H}3'$) *are fulfilled. Then*

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u)\} &= \sum_{\{t\}\in\partial_0 T}\mathbb{P}(X(t)\geq u, \nabla X(t)\in E(\{t\})) + \sum_{k=1}^N\sum_{J\in\partial_k T}\frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}}\\
&\quad\times\int_J dt\int_u^\infty dx\int\cdots\int_{E(J)}dy_{J_1}\cdots dy_{J_{N-k}}\frac{|\Lambda_J-\Lambda_J(t)|}{\gamma_t^k}\\
&\quad\times H_k\Big(\frac{x}{\gamma_t}+\gamma_t C_{J_1}(t)y_{J_1}+\cdots+\gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\Big)\\
&\quad\times p_{X(t),X_{J_1}(t),\ldots,X_{J_{N-k}}(t)}(x,y_{J_1},\ldots,y_{J_{N-k}}|\nabla X_{|J}(t)=0).
\end{aligned}
\tag{2.2.13}
$$

**Proof** According to Corollary 11.3.2 in Adler and Taylor (2007), ($\mathbf{H}1$) and ($\mathbf{H}3'$) imply that $X$ is a Morse function a.s. It follows from (2.2.1) that

$$
\mathbb{E}\{\varphi(A_u)\} = \sum_{k=0}^N\sum_{J\in\partial_k T}(-1)^k\mathbb{E}\bigg\{\sum_{i=0}^k(-1)^i\mu_i(J)\bigg\}. \tag{2.2.14}
$$

If $J\in\partial_0 T$, say $J=\{t\}$, it turns out $\mathbb{E}\{\mu_0(J)\}=\mathbb{P}(X(t)\geq u, \nabla X(t)\in E(\{t\}))$. If $J\in\partial_k T$ with $k\geq 1$, we apply the Kac-Rice metatheorem to obtain that the expectation on the right hand side of (2.2.14) becomes

$$\int_J p_{\nabla X_{|J}(t)}(0)dt \sum_{i=0}^{k}(-1)^i \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in \mathcal{D}_i\}} \mathbb{1}_{\{(X_{J_1}(t),\dots,X_{J_{N-k}}(t))\in E(J)\}}$$

$$\times \mathbb{1}_{\{X(t)\geq u\}}|\nabla X_{|J}(t)=0\}$$

$$= \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1}\cdots dy_{J_{N-k}}$$

$$\times \mathbb{E}\{\det \nabla^2 X_{|J}(t)|X(t)=x, X_{J_1}(t)=y_{J_1},\dots,X_{J_{N-k}}(t)=y_{J_{N-k}}, \nabla X_{|J}(t)=0\}$$

$$\times p_{X(t),X_{J_1}(t),\dots,X_{J_{N-k}}(t)}(x,y_{J_1},\dots,y_{J_{N-k}}|\nabla X_{|J}(t)=0). \tag{2.2.15}$$

For fixed $t$, let $Q_t$ be the positive definite matrix in (2.2.10). Then, similarly to the proof in Lemma 2.2.1, we can write

$$\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t)=x, X_{J_1}(t)=y_{J_1},\dots,X_{J_{N-k}}=y_{J_{N-k}}, \nabla X_{|J}(t)=0\}$$

as $\mathbb{E}\{\det\overline{\Delta}(t,x)\}$, where $\overline{\Delta}(t,x)$ is a matrix consisting of Gaussian entries $\overline{\Delta}_{ij}(t,x)$ with mean

$$\begin{aligned}
&\mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}|X(t)=x, X_{J_1}(t)=y_{J_1},\dots,X_{J_{N-k}}=y_{J_{N-k}}, \nabla X_{|J}(t)=0\}\\
&\quad = (-\delta_{ij},0,\dots,0)(\mathrm{Cov}(X(t),X_{J_1}(t),\dots,X_{J_{N-k}}(t),\nabla X_{|J}(t)))^{-1}\\
&\quad\quad \cdot (x,y_{J_1},\dots,y_{J_{N-k}},0,\dots,0)^T\\
&\quad = -\frac{\delta_{ij}}{\gamma_t^2}(x+\gamma_t^2 C_{J_1}(t)y_{J_1}+\cdots+\gamma_t^2 C_{J_{N-k}}(t)y_{J_{N-k}}),
\end{aligned} \tag{2.2.16}$$

and covariance

$$\mathbb{E}\{(\overline{\Delta}_{ij}(t,x)-\mathbb{E}\{\overline{\Delta}_{ij}(t,x)\})(\overline{\Delta}_{kl}(t,x)-\mathbb{E}\{\overline{\Delta}_{kl}(t,x)\})\}=\mathcal{S}_t(i,j,k,l)-\frac{\delta_{ij}\delta_{kl}}{\gamma_t^2}.$$

Following the same procedure in the proof of Lemma 2.2.1, we obtain that the last conditional expectation in (2.2.15) is equal to

$$\frac{(-1)^k|\Lambda_J-\Lambda_J(t)|}{\gamma_t^k}H_k\Big(\frac{x}{\gamma_t}+\gamma_t C_{J_1}(t)y_{J_1}+\cdots+\gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\Big). \tag{2.2.17}$$

Plug this into (2.2.15) and (2.2.14), yielding the desired result. □

**Remark 2.2.3** Usually, for nonstationary (including constant-variance) Gaussian field $X$ on $\mathbb{R}^N$, its mean Euler characteristic involves at least the third-order derivatives of the covariance function. For Gaussian random fields with stationary increments, as shown in Lemma 2.1.6, $\mathbb{E}\{X_{ij}(t)X_k(t)\}=0$ and $\mathbb{E}\{X_{ij}(t)X_{kl}(t)\}$ is symmetric in $i$, $j$, $k$, $l$, so the mean Euler characteristic becomes relatively simpler, contains only up to the second-order derivatives of the covariance function. In various practical applications, (2.2.13) could be simplified with only an exponentially smaller difference, see the discussions in Section 2.4.

## 2.3 Excursion Probability

As in Section 3.1, we decompose $T$ into several faces as $T = \bigcup_{k=0}^{N} \partial_k T = \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} J$. For each $J \in \partial_k T$, define the number of *extended outward maxima* above level $u$ as

$$M_u^E(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = k,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\}.$$

In fact, $M_u^E(J)$ is the same as $\mu_k(J)$ defined in (2.2.2) with $i = k$. We will make use of the following lemma.

**Lemma 2.3.1** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a Gaussian random field satisfying* (**H**1) *and* (**H**3$'$)*, then for any $u > 0$,*

$$\Big\{ \sup_{t\in T} X(t) \geq u \Big\} = \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.}$$

**Proof** By the definition of $M_u^E(J)$, it is clear that

$$\Big\{ \sup_{t\in T} X(t) \geq u \Big\} \supset \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.}$$

Suppose $\sup_{t\in T} X(t) \geq u$, since $X(t) \in C^2(\mathbb{R}^N)$ a.s., there exists $t_0 \in T$ such that $X(t_0) = \sup_{t\in T} X(t)$. Without loss of generality, assume $t_0 \in J \in \partial_k T$. Note that $t_0$ is a local maximum restricted on $J$, thus $\nabla X_{|J}(t_0) = 0$ and $\nabla^2 X_{|J}(t_0)$ is non-positive definite. Due to (**H**1) and (**H**3$'$), we apply Lemma 11.2.11 in Adler and Taylor (2007) to obtain that almost surely, $\det(\nabla^2 X_{|J}(t_0)) \neq 0$ and hence $\mathrm{index}(\nabla^2 X_{|J}(t_0)) = k$. If $\varepsilon_j^* X_j(t_0) < 0$ for some $j \notin \sigma(J)$, then we can find $t_1 \in T$ such that $X(t_1) > X(t_0)$, which contradicts

$X(t_0) = \sup_{t\in T} X(t)$. Hence $\varepsilon_j^* X_j(t_0) \geq 0$ for all $j \notin \sigma(J)$. These indicate $M_u^E(J) \geq 1$, therefore

$$\Big\{\sup_{t\in T} X(t) \geq u\Big\} \subset \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.,}$$

completing the proof. □

It follows from Lemma 2.3.1 that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \leq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} \leq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\}. \tag{2.3.1}$$

On the other hand, by the Bonferroni inequality,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} - \sum_{J\neq J'} \mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\}.$$

Let $p_i = \mathbb{P}\{M_u^E(J) = i\}$, then $\mathbb{P}\{M_u^E(J) \geq 1\} = \sum_{i=1}^\infty p_i$ and $\mathbb{E}\{M_u^E(J)\} = \sum_{i=1}^\infty ip_i$, it follows that

$$\begin{aligned}\mathbb{E}\{M_u^E(J)\} - \mathbb{P}\{M_u^E(J) \geq 1\} &= \sum_{i=2}^{\infty}(i-1)p_i \\ &\leq \sum_{i=2}^{\infty} \frac{i(i-1)}{2} p_i = \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}.\end{aligned}$$

Together with the obvious bound $\mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\} \leq \mathbb{E}\{M_u^E(J)M_u^E(J')\}$, we obtain the following lower bound for the excursion probability,

$$\begin{aligned} \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} &\Big(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}\Big) \\ &- \sum_{J\neq J'} \mathbb{E}\{M_u^E(J)M_u^E(J')\}. \end{aligned} \tag{2.3.2}$$

Define the number of *local maxima* above level $u$ as

$$M_u(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = k\},$$

then obviously $M_u(J) \geq M_u^E(J)$, and $M_u(J)$ is the same as $\widetilde{\mu}_k(J)$ defined in (2.2.3) with $i = k$. It follows similarly that

$$\begin{aligned} &\sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u(J)\} \geq \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\ &\geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \Big(\mathbb{E}\{M_u(J)\} - \frac{1}{2}\mathbb{E}\{M_u(J)(M_u(J)-1)\}\Big) - \sum_{J\neq J'} \mathbb{E}\{M_u(J)M_u(J')\}. \end{aligned} \tag{2.3.3}$$

We will use (2.3.1) and (2.3.2) to estimate the excursion probability for the general case, see Theorem 2.3.8. Inequalities in (2.3.3) provide another method to approximate the excursion probability in some special cases, see Theorem 2.3.7. The advantage of (2.3.3) is that the principal term induced by $\sum_{k=0}^{N}\sum_{J\in\partial_k T}\mathbb{E}\{M_u(J)\}$ is much easier to compute compared with the one induced by $\sum_{k=0}^{N}\sum_{J\in\partial_k T}\mathbb{E}\{M_u^E(J)\}$.

The following two lemmas provide the estimations for the principal terms in approximating the excursion probability.

**Lemma 2.3.2** *Let $X$ be a Gaussian field as in Theorem 2.2.2. Then for each $J \in \partial_k T$ with $k \geq 1$, there exists some constant $\alpha > 0$ such that*

$$\mathbb{E}\{M_u(J)\} = \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)}\, dt(1 + o(e^{-\alpha u^2})). \tag{2.3.4}$$

**Proof** Following the notations in the proof of Lemma 2.2.1, we obtain similarly that

$$\begin{aligned}\mathbb{E}\{M_u(J)\} &= \int_J p_{\nabla X_{|J}(t)}(0)dt\, \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)|\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}\mathbb{1}_{\{X(t)\geq u\}}|\nabla X_{|J}(t) = 0\}\\ &= \int_J dt \int_u^\infty dx \frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}\\ &\qquad \times \mathbb{E}\{\mathrm{det}\nabla^2 X_{|J}(t)\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}|X(t) = x, \nabla X_{|J}(t) = 0\}.\end{aligned} \tag{2.3.5}$$

Recall $\nabla^2 X_{|J}(t) = Q_t^{-1} Q_t \nabla^2 X_{|J}(t) Q_t Q_t^{-1}$ and we can write (2.2.12) as

$$\mathbb{E}\{Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0\} = -\frac{x}{\theta_t^2} I_k.$$

Make change of variables

$$V(t) = Q_t \nabla^2 X_{|J}(t) Q_t + \frac{x}{\theta_t^2} I_k,$$

where $V(t) = (V_{ij}(t))_{1\leq i,j\leq k}$. Then $(V(t)|X(t) = x, \nabla X_{|J}(t) = 0)$ is a Gaussian matrix whose mean is 0 and covariance is the same as that of $(Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0)$. Denote the density of Gaussian vectors $((V_{ij}(t))_{1\leq i\leq j\leq k}|X(t) = x, \nabla X_{|J}(t) = 0)$ by

$h_t(v)$, $v = (v_{ij})_{1\le i\le j\le k} \in \mathbb{R}^{k(k+1)/2}$, then

$$\begin{aligned}
&\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}|X(t)=x, \nabla X_{|J}(t)=0\}\\
&\quad= \mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)\mathbb{1}_{\{Q_t\nabla^2 X_{|J}(t)Q_t\in\mathcal{D}_k\}}|X(t)=x, \nabla X_{|J}(t)=0\}\\
&\quad= \int_{v:(v_{ij})-\frac{x}{\theta_t^2}I_k\in\mathcal{D}_k} \det\Big((v_{ij})-\frac{x}{\theta_t^2}I_k\Big)h_t(v)\,dv,
\end{aligned}\tag{2.3.6}$$

where $(v_{ij})$ is the abbreviation of matrix $(v_{ij})_{1\le i,j\le k}$. Since $\{\theta_t^2 : t\in T\}$ is bounded, there exists a constant $c>0$ such that

$$(v_{ij})-\frac{x}{\theta_t^2}I_k\in\mathcal{D}_k, \quad \forall\|(v_{ij})\| := \Bigg(\sum_{i,j=1}^k v_{ij}^2\Bigg)^{1/2} < \frac{x}{c}.$$

Thus we can write (2.3.6) as

$$\begin{aligned}
&\int_{\mathbb{R}^{k(k+1)/2}} \det\Big((v_{ij})-\frac{x}{\theta_t^2}I_k\Big)h_t(v)dv - \int_{v:(v_{ij})-\frac{x}{\theta_t^2}I_k\notin\mathcal{D}_k} \det\Big((v_{ij})-\frac{x}{\theta_t^2}I_k\Big)h_t(v)\,dv\\
&\quad= \mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t)=x, \nabla X_{|J}(t)=0\} + Z(t,x),
\end{aligned}\tag{2.3.7}$$

where $Z(t,x)$ is the second integral in the first line of (2.3.7) and it satisfies

$$|Z(t,x)| \le \int_{\|(v_{ij})\|\ge\frac{x}{c}} \Big|\det\Big((v_{ij})-\frac{x}{\theta_t^2}I_k\Big)\Big|h_t(v)dv.$$

Denote by $G(t)$ the covariance matrix of $((V_{ij}(t))_{1\le i\le j\le k}|X(t)=x, \nabla X_{|J}(t)=0)$, then by Lemma 2.5.2 in the Appendix, the eigenvalues of $G(t)$ and hence those of $(G(t))^{-1}$ are bounded for all $t\in T$. It follows that there exists some constant $\alpha'>0$ such that

$h_t(v) = o(e^{-\alpha' \|(v_{ij})\|^2})$ and hence $|Z(t,x)| = o(e^{-\alpha x^2})$ for some constant $\alpha > 0$ uniformly for all $t \in T$. Combine this with (2.3.5), (2.3.6), (2.3.7) and the proof of Lemma 2.2.1, yielding the result. $\square$

**Lemma 2.3.3** *Let $X$ be a Gaussian field as in Theorem 2.2.2. Then for each $J \in \partial_k T$ with $k \geq 1$, there exists some constant $\alpha > 0$ such that*

$$\begin{aligned}\mathbb{E}\{M_u^E(J)\} = {} & \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \\ & \times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t) y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t) y_{J_{N-k}}\right) \\ & \times p_{X(t), X_{J_1}(t), \ldots, X_{J_{N-k}}(t)}(x, y_{J_1}, \ldots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0)(1 + o(e^{-\alpha u^2})).\end{aligned} \tag{2.3.8}$$

**Proof** Under the notations in the proof of Theorem 2.2.2, applying the Kac-Rice formula, we see that $\mathbb{E}\{M_u^E(J)\}$ equals

$$\begin{aligned}& \int_J p_{\nabla X_{|J}(t)}(0) dt\, \mathbb{E}\{|\mathrm{det} \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} \mathbb{1}_{\{X(t) \geq u\}} \\ & \quad \times \mathbb{1}_{\{(X_{J_1}(t), \cdots, X_{J_{N-k}}(t)) \in E(J)\}} | \nabla X_{|J}(t) = 0\} \\ & = \frac{(-1)^k}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \\ & \quad \mathbb{E}\{\mathrm{det} \nabla^2 X_{|J}(t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} | X(t) = x, X_{J_1}(t) = y_{J_1}, \cdots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \\ & \quad \nabla X_{|J}(t) = 0\} p_{X(t), X_{J_1}(t), \cdots, X_{J_{N-k}}(t)}(x, y_{J_1}, \cdots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0).\end{aligned}$$

Recall $\nabla^2 X_{|J}(t) = Q_t^{-1} Q_t \nabla^2 X_{|J}(t) Q_t Q_t^{-1}$ and we can write (2.2.16) as

$$\begin{aligned}
&\mathbb{E}\{Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, X_{J_1}(t) = y_{J_1}, \cdots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\} \\
&= -\left(\frac{x}{\gamma_t^2} + C_{J_1}(t) y_{J_1} + \cdots + C_{J_{N-k}}(t) y_{J_{N-k}}\right) I_k.
\end{aligned}$$

Make change of variables

$$W(t) = Q_t \nabla^2 X_{|J}(t) Q_t + \frac{x}{\gamma_t^2} I_k,$$

where $W(t) = (W_{ij}(t))_{1 \le i,j \le k}$. Denote the density of

$$((W_{ij}(t))_{1 \le i \le j \le k} | X(t) = x, X_{J_1}(t) = y_{J_1}, \cdots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \nabla X_{|J}(t) = 0)$$

by $f_{t, y_{J_1}, \cdots, y_{J_{N-k}}}(w), w = (w_{ij})_{1 \le i \le j \le k} \in \mathbb{R}^{k(k+1)/2}$. Similarly to the proof in Lemma 2.3.2, to estimate

$$\mathbb{E}\left\{ \det \nabla^2 X_{|J}(t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} \middle| \begin{subarray}{c} X(t)=x, \nabla X_{|J}(t)=0, \\ X_{J_1}(t)=y_{J_1}, \cdots, X_{J_{N-k}}(t)=y_{J_{N-k}} \end{subarray} \right\},$$

we will get an expression similar to (2.3.7) with $Z(t,x)$ replaced by $\widetilde{Z}(t,x,y_{J_1},\cdots,y_{J_{N-k}})$. Then, similarly, we have

$$
\begin{aligned}
I(t,x) := \int\cdots\int_{E(J)} & dy_{J_1}\cdots dy_{J_{N-k}}\, p_{X(t),X_{J_1}(t),\cdots,X_{J_{N-k}}(t)}(x,y_{J_1},\cdots,y_{J_{N-k}}| \\
& |\nabla X_{|J}(t)=0)|\widetilde{Z}(t,x,y_{J_1},\cdots,y_{J_{N-k}})| \\
\le & \int\cdots\int_{E(J)} dy_{J_1}\cdots dy_{J_{N-k}}\, p_{X(t),X_{J_1}(t),\cdots,X_{J_{N-k}}(t)}(x,y_{J_1},\cdots,y_{J_{N-k}}| \\
& |\nabla X_{|J}(t)=0)|\int_{\|(w_{ij})\|\ge\frac{x}{c}}\left|\det\left((w_{ij})-\frac{x}{\gamma_t^2}I_k\right)\right| f_{t,y_{J_1},\cdots,y_{J_{N-k}}}(w)dw \\
\le & \, p_{X(t)}(x|\nabla X_{|J}(t)=0)\int_{\|(w_{ij})\|\ge\frac{x}{c}}\left|\det\left((w_{ij})-\frac{x}{\gamma_t^2}I_k\right)\right| f_t(w)dw,
\end{aligned}
$$

where the last inequality comes from replacing the integral region $E(J)$ by $\mathbb{R}^{N-k}$, and $f_t(w)$ is the density of $((W_{ij}(t))_{1\le i\le j\le k}|X(t)=x,\nabla X_{|J}(t)=0)$. Hence by the same discussions in the proof of Lemma 2.3.2, $I(t,x)=o(e^{-\alpha u^2-u^2/(2\sigma_T^2)})$ uniformly for all $t\in T$ and some constant $\alpha>0$. Combining the proofs of Lemma 2.3.2 and Theorem 2.2.2, we obtain the result. $\square$

We call a function $h(u)$ *super-exponentially small* (when compared with $\mathbb{P}(\sup_{t\in T}X(t)\ge u)$), if there exists a constant $\alpha>0$ such that $h(u)=o(e^{-\alpha u^2-u^2/(2\sigma_T^2)})$ as $u\to\infty$.

The following lemma is Lemma 4 in Piterbarg (1996b). It shows that the factorial moments are usually super-exponentially small.

**Lemma 2.3.4** *Let $\{X(t):t\in\mathbb{R}^N\}$ be a centered Gaussian field satisfying* (**H**1) *and* (**H**3). *Then for any $\varepsilon>0$, there exists $\varepsilon_1>0$ such that for any $J\in\partial_k T$ and $u$ large enouth,*

$$
\mathbb{E}\{M_u(J)(M_u(J)-1)\}\le e^{-u^2/(2\beta_J^2+\varepsilon)}+e^{-u^2/(2\sigma_J^2-\varepsilon_1)},
$$

*where* $\beta_J^2 = \sup_{t\in J}\sup_{e\in\mathbb{S}^{k-1}} \mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e)$ *and* $\sigma_J^2 = \sup_{t\in J}\mathrm{Var}(X(t))$. *Here* $\mathbb{S}^{k-1}$ *is the* $(k-1)$*-dimensional unit sphere.*

**Corollary 2.3.5** *Let* $X = \{X(t) : t \in \mathbb{R}^N\}$ *be a centered Gaussian random field with stationary increments satisfying* $(\mathbf{H}1)$, $(\mathbf{H}2)$ *and* $(\mathbf{H}3)$. *Then for all* $J \in \partial_k T$, $\mathbb{E}\{M_u(J)(M_u(J)-1)\}$ *and* $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}$ *are super-exponentially small.*

**Proof** Since $M_u^E(J) \le M_u(J)$, we only need to show that $\mathbb{E}\{M_u(J)(M_u(J)-1)\}$ is super-exponentially small. If $k = 0$, then $M_u(J)$ is either 0 or 1 and hence $\mathbb{E}\{M_u(J)(M_u(J)-1)\} = 0$. If $k \ge 1$, then, thanks to Lemma 2.3.4, it suffices to show that $\beta_J^2$ is strictly less than $\sigma_T^2$.

Clearly, $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) \le \sigma_T^2$. Applying Lemma 2.5.1 yields that

$$\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) = \sigma_T^2 \Rightarrow \mathbb{E}\{X(t)(\nabla^2 X_{|J}(t)e)\} = 0.$$

Note that the right hand side above is equivalent to $(\Lambda_J(t)-\Lambda_J)e = 0$. By $(\mathbf{H}2)$, $\Lambda_J(t)-\Lambda_J$ is negative definite, which implies $(\Lambda_J(t)-\Lambda_J)e \ne 0$ for all $e \in \mathbb{S}^{k-1}$, so that

$$\sup_{e\in\mathbb{S}^{k-1}} \mathrm{Var}(X(t)|\nabla_{|J}X(t), \nabla_{|J}^2 X(t)e) < \sigma_T^2.$$

Therefore $\beta_J^2 < \sigma_T^2$ by continuity. $\square$

The following lemma shows that the cross terms in (2.3.2) and (2.3.3) are super-exponentially small if the two faces are not adjacent. For the case when the faces are adjacent, the proof is more technical, see the proofs in Theorems 2.3.7 and 2.3.8.

**Lemma 2.3.6** *Let* $X = \{X(t) : t \in \mathbb{R}^N\}$ *be a centered Gaussian random field with stationary increments satisfying* $(\mathbf{H}1)$ *and* $(\mathbf{H}3)$. *Let* $J$ *and* $J'$ *be two faces of* $T$ *such that their* *distance is positive, i.e.,* $\inf_{t\in J, s\in J'} \|s-t\| > \delta_0$ *for some* $\delta_0 > 0$*, then* $\mathbb{E}\{M_u(J)M_u(J')\}$ *is super-exponentially small.*

**Proof** We first consider the case when $\dim(J) = k \geq 1$ and $\dim(J') = k' \geq 1$. By the Kac-Rice metatheorem for higher moments (the proof is the same as that of Theorem 11.5.1 in Adler and Taylor (2007)),

$$\begin{aligned}
\mathbb{E}\{M_u(J)M_u(J')\} &= \int_J dt \int_{J'} ds\, \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)|\mathbb{1}_{\{X(t)\geq u, X(s)\geq u\}} \\
&\quad\quad \times \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k, \nabla^2 X_{|J'}(s)\in\mathcal{D}_{k'}\}} |X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, \\
&\quad\quad \nabla X_{|J'}(s)=0\} p_{X(t),X(s),\nabla X_{|J}(t),\nabla X_{|J'}(s)}(x,y,0,0) \\
&\leq \int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy\, \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)| \\
&\quad\quad |X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} p_{X(t),X(s)}(x,y) \\
&\quad\quad \times p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0|X(t)=x, X(s)=y).
\end{aligned} \tag{2.3.9}$$

Note that the following two inequalities hold: for constants $a_i$ and $b_j$,

$$\prod_{i=1}^k |a_i| \prod_{j=1}^{k'} |b_j| \leq \frac{1}{k+k'}\left(\sum_{i=1}^k |a_i|^{k+k'} + \sum_{j=1}^{k'} |b_j|^{k+k'}\right);$$

and for any Gaussian variable $\xi$ and positive integer $l$,

$$\mathbb{E}|\xi|^l \leq \mathbb{E}(|\mathbb{E}\xi| + |\xi - \mathbb{E}\xi|)^l \leq 2^l(|\mathbb{E}\xi|^l + \mathbb{E}|\xi-\mathbb{E}\xi|^l) \leq 2^l(|\mathbb{E}\xi|^l + C_l(\mathrm{Var}(\xi))^{l/2}),$$

where the constant $C_l$ depends only on $l$. Combining these two inequalities with Lemma 2.5.1, we get that there exist some positive constants $C_1$ and $N_1$ such that for large $x$ and $y$,

$$\begin{aligned}\sup_{t\in J, s\in J'} \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y,\\ \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} \leq C_1 x^{N_1} y^{N_1}.\end{aligned} \tag{2.3.10}$$

Also, there exists a positive constant $C_2$ such that

$$\begin{aligned}&\sup_{t\in J, s\in J'} p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0|X(t)=x, X(s)=y)\\ &\quad\leq \sup_{t\in J, s\in J'} (2\pi)^{-(k+k')/2}[\det\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)|X(t)=x, X(s)=y)]^{-1/2} \leq C_2.\end{aligned} \tag{2.3.11}$$

Let $\rho(\delta_0) = \sup_{\|s-t\|>\delta_0} \frac{\mathbb{E}\{X(t)X(s)\}}{\sigma_t\sigma_s}$ which is strictly less than 1 due to (**H**3), then $\forall\varepsilon>0$, there exists a positive constant $C_3$ such that for all $t\in J$, $s\in J'$ and $u$ large enough,

$$\begin{aligned}&\int_u^\infty\int_u^\infty x^{N_1}y^{N_1}p_{X(t),X(s)}(x,y)dxdy = \mathbb{E}\{[X(t)X(s)]^{N_1}\mathbb{1}_{\{X(t)\geq u, X(s)\geq u\}}\}\\ &\quad\leq \mathbb{E}\{[X(t)+X(s)]^{2N_1}\mathbb{1}_{\{X(t)+X(s)\geq 2u\}}\} \leq C_3\exp\left(\varepsilon u^2 - \frac{u^2}{(1+\rho(\delta_0))\sigma_T^2}\right).\end{aligned} \tag{2.3.12}$$

Combine (2.3.9) with (2.3.10), (2.3.11) and (2.3.12), yielding that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.

When only one of the faces, say $J$, is a singleton, then let $J = \{t_0\}$ and we have

$$\mathbb{E}\{M_u(J)M_u(J')\} \le \int_{J'} ds \int_u^\infty dx \int_u^\infty dy\, p_{X(t_0),X(s),\nabla X_{|J'}(s)}(x,y,0) \\ \times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)||X(t_0)=x, X(s)=y, \nabla X_{|J'}(s)=0\}. \tag{2.3.13}$$

Following the previous discussions yields that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small.

Finally, if both $J$ and $J'$ are singletons, then $\mathbb{E}\{M_u(J)M_u(J')\}$ becomes the joint probability of two Gaussian variables exceeding level $u$ and hence is trivial. $\square$

**Theorem 2.3.7** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that* (**H**1)*,* (**H**2) *and* (**H**3) *are fulfilled. Suppose that for any face $J$,*

$$\{t \in J : \nu(t) = \sigma_T^2, \nu_j(t) = 0 \text{ for some } j \notin \sigma(J)\} = \emptyset. \tag{2.3.14}$$

*Then there exists some constant $\alpha > 0$ such that*

$$\begin{aligned} \mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} &= \sum_{k=0}^N \sum_{J\in\partial_k T} \mathbb{E}\{M_u(J)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}) \\ &= \sum_{\{t\}\in\partial_0 T} \Psi\Big(\frac{u}{\sigma_t}\Big) + \sum_{k=1}^N \sum_{J\in\partial_k T} \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \\ &\quad \times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)}\, dt + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}). \end{aligned} \tag{2.3.15}$$

**Proof** Since the second equality in (2.3.15) follows from Lemma 2.3.2 directly, we only need to prove the first one. By (2.3.3) and Corollary 2.3.5, it suffices to show that the last term in (2.3.3) is super-exponentially small. Thanks to Lemma 2.3.6, we only need to consider the case when the distance of $J$ and $J'$ is 0, or $I := \bar{J} \cap \bar{J}' \neq \emptyset$. Without loss of generality, assume

$$\sigma(J) = \{1, \ldots, m, m+1, \ldots, k\}, \quad \sigma(J') = \{1, \ldots, m, k+1, \ldots, k+k'-m\}, \tag{2.3.16}$$

where $0 \le m \le k \le k' \le N$ and $k' \ge 1$. If $k = 0$, we consider $\sigma(J) = \emptyset$ by convention. Under such assumption, $J \in \partial_k T$, $J' \in \partial_{k'} T$ and $\dim(I) = m$.

Case 1: $k = 0$, i.e. $J$ is a singleton, say $J = \{t_0\}$. If $\nu(t_0) < \sigma_T^2$, then by (2.3.13), it is trivial to show that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small. Now we consider the case $\nu(t_0) = \sigma_T^2$. Due to (2.3.14), $\mathbb{E}\{X(t_0)X_1(t_0)\} \neq 0$ and hence by continuity, there exists $\delta > 0$ such that $\mathbb{E}\{X(s)X_1(s)\} \neq 0$ for all $\|s - t_0\| \le \delta$. It follows from (2.3.13) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded from above by

$$\begin{aligned}
&\int_{s \in J' : \|s-t_0\| > \delta} ds \int_u^\infty dx \int_u^\infty dy\ \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| | X(t_0) = x, X(s) = y, \nabla X_{|J'}(s) = 0\} \\
&\qquad\qquad\qquad\qquad \times p_{X(t_0), X(s), \nabla X_{|J'}(s)}(x, y, 0) \\
&+ \int_{s \in J' : \|s-t_0\| \le \delta} ds \int_u^\infty dy\ \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| | X(s) = y, \nabla X_{|J'}(s) = 0\} p_{X(s), \nabla X_{|J'}(s)}(y, 0) \\
&:= I_1 + I_2.
\end{aligned}$$

Following the proof of Lemma 2.3.6 yields that $I_1$ is super-exponentially small. We apply Lemma 2.5.1 to obtain that there exists $\varepsilon_0 > 0$ such that

$$\sup_{s \in J' : \|s-t_0\| \le \delta} \mathrm{Var}(X(s) | \nabla X_{|J'}(s)) \le \sup_{s \in J' : \|s-t_0\| \le \delta} \mathrm{Var}(X(s) | X_1(s)) \le \sigma_T^2 - \varepsilon_0.$$

Then $I_2$ and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are super-exponentially small.

Case 2: $k \geq 1$. For all $t \in I$ with $\nu(t) = \sigma_T^2$, by assumption (2.3.14), $\mathbb{E}\{X(t)X_i(t)\} \neq 0$, $\forall\ i = m+1, \ldots, k+k'-m$. Note that $I$ is a compact set, by Lemma 2.5.1 and the uniform continuity of conditional variance, there exist $\varepsilon_1, \delta_1 > 0$ such that

$$\sup_{t \in B, s \in B'} \mathrm{Var}(X(t)|X_{m+1}(t), \ldots, X_k(t), X_{k+1}(s), \ldots, X_{k+k'-m}(s)) \leq \sigma_T^2 - \varepsilon_1, \tag{2.3.17}$$

where $B = \{t \in J : \mathrm{dist}(t, I) \leq \delta_1\}$ and $B' = \{s \in J' : \mathrm{dist}(s, I) \leq \delta_1\}$. It follows from (2.3.9) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$\begin{aligned}
&\int\int_{(J\times J')\backslash(B\times B')} dtds \int_u^\infty dx \int_u^\infty dy\, p_{X(t),X(s),\nabla X_{|J}(t),\nabla X_{|J'}(s)}(x,y,0,0) \\
&\quad \times \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} \\
&+ \int\int_{B\times B'} dtds \int_u^\infty dx\, p_{X(t)}(x|\nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0) p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0) \\
&\quad \times \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} \\
&:= I_3 + I_4.
\end{aligned}$$

Note that

$$(J\times J')\backslash(B\times B') = \Big((J\backslash B)\times B'\Big)\bigcup\Big(B\times(J\backslash B)\Big)\bigcup\Big((J\backslash B)\times(J\backslash B)\Big). \tag{2.3.18}$$

Since each product set on the right hand side of (2.3.18) consists of two sets with positive distance, following the proof of Lemma 2.3.6 yields that $I_3$ is super-exponentially small.

For $I_4$, taking into account (2.3.17), one has

$$\sup_{t \in B, s \in B'} \mathrm{Var}\big(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)\big) \leq \sigma_T^2 - \varepsilon_1. \tag{2.3.19}$$

To estimate

$$p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0) = (2\pi)^{-(k+k')/2}(\det\mathrm{Cov}(\nabla X_{|J}(t),\nabla X_{|J'}(s)))^{-1/2}, \tag{2.3.20}$$

we write the determinant on the right hand side of (2.3.20) as

$$\begin{aligned}&\det\mathrm{Cov}(X_{m+1}(t),\dots,X_k(t),X_{k+1}(s),\dots,X_{k+k'-1}(s)|X_1(t),\dots,X_m(t),X_1(s),\dots,X_m(s))\\ &\quad\times\det\mathrm{Cov}(X_1(t),\dots,X_m(t),X_1(s),\dots,X_m(s)),\end{aligned} \tag{2.3.21}$$

where the first determinant in (2.3.21) is bounded away from zero due to (**H**3). By (**H**1), as shown in Piterbarg (1996b), applying Taylor's formula, we can write

$$\nabla X(s) = \nabla X(t) + \nabla^2 X(t)(s-t)^T + \|s-t\|^{1+\eta}Y_{t,s}, \tag{2.3.22}$$

where $Y_{t,s} = (Y_{t,s}^1,\dots,Y_{t,s}^N)^T$ is a Gaussian vector field with bounded variance uniformly for all $t \in J$, $s \in J'$. Hence as $\|s-t\| \to 0$, the second determinant in (2.3.21) becomes

$$\begin{aligned}&\det\mathrm{Cov}(X_1(t),\dots,X_m(t),X_1(t)+\langle\nabla X_1(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^1,\dots,\\ &\qquad\qquad X_m(t)+\langle\nabla X_m(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^m)\\ &=\det\mathrm{Cov}(X_1(t),\dots,X_m(t),\langle\nabla X_1(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^1,\dots,\\ &\qquad\qquad \langle\nabla X_m(t),s-t\rangle+\|s-t\|^{1+\eta}Y_{t,s}^m)\\ &=\|s-t\|^{2m}\det\mathrm{Cov}(X_1(t),\dots,X_m(t),\langle\nabla X_1(t),e_{t,s}\rangle,\dots,\langle\nabla X_m(t),e_{t,s}\rangle)(1+o(1)),\end{aligned} \tag{2.3.23}$$

where $e_{t,s} = (s-t)^T/\|s-t\|$ and due to (**H**3), the last determinant in (2.3.23) is bounded away from zero uniformly for all $t \in J$ and $s \in J'$. It then follows from (2.3.21) and (2.3.23) that

$$\det\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)) \geq C_1 \|s-t\|^{2m} \tag{2.3.24}$$

for some constant $C_1 > 0$. Similarly to (2.3.10), there exist constants $C_2, N_1 > 0$ such that

$$\begin{aligned} \sup_{t \in J, s \in J'} & \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)||X(t) = x, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\} \\ & \leq C_2(1 + x^{N_1}). \end{aligned} \tag{2.3.25}$$

Combining (2.3.19) with (2.3.20), (2.3.24) and (2.3.25), and noting that $m < k'$ implies $1/\|s-t\|^m$ is integrable on $J \times J'$, we conclude that $I_4$ and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are finite and super-exponentially small. $\square$

**Theorem 2.3.8** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that* (**H**1)*,* (**H**2) *and* (**H**3) *are fulfilled. Then there exists some constant $\alpha > 0$ such that*

$$\begin{aligned} \mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} &= \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}) \\ &= \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}), \end{aligned} \tag{2.3.26}$$

*where $\mathbb{E}\{\varphi(A_u)\}$ is formulated in Theorem 2.2.2.*

It is worth mentioning here that the main idea for the proof of Theorem 2.3.8 comes from Azaïs and Delmas (2002) (especially Theorem 4). Before showing the proof, we list the following two lemmas.

**Lemma 2.3.9** *Under* (**H**2), *there exists a constant* $\alpha_0 > 0$ *such that*

$$\langle e, (\Lambda - \Lambda(t))e\rangle \geq \alpha_0, \quad \forall\, t \in T, e \in \mathbb{S}^{N-1}.$$

**Proof** Let $M_{N\times N}$ be the set of all $N \times N$ matrices. Define a mapping $\phi : \mathbb{R}^N \times M_{N\times N} \to \mathbb{R}$ by $(\xi, A) \mapsto \langle \xi, A\xi\rangle$, then $\phi$ is continuous. Since $\Lambda - \Lambda(t)$ is positive definite, $\phi(e, \Lambda - \Lambda(t)) > 0$ for each $t \in T$ and $e \in \mathbb{S}^{N-1}$. On the other hand, $\{(e, \Lambda - \Lambda(t)) : t \in T, e \in \mathbb{S}^{N-1}\}$ is a compact subset of $\mathbb{R}^N \times M_{N\times N}$ and $\phi$ is continuous, completing the proof. □

**Lemma 2.3.10** *Let* $\{\xi_1(t) : t \in T_1\}$ *and* $\{\xi_2(t) : t \in T_2\}$ *be two Gaussian random fields. Let*

$$\sigma_i^2(t) = \mathrm{Var}(\xi_i(t)), \quad \overline{\sigma}_i = \sup_{t\in T_i} \sigma_i(t), \quad \underline{\sigma}_i = \inf_{t\in T_i} \sigma_i(t),$$
$$\rho(t, s) = \frac{\mathbb{E}\{\xi_1(t)\xi_2(s)\}}{\sigma_1(t)\sigma_2(s)}, \quad \overline{\rho} = \sup_{t\in T_1, s\in T_2} \rho(t, s), \quad \underline{\rho} = \inf_{t\in T_1, s\in T_2} \rho(t, s),$$

*and assume* $0 < \underline{\sigma}_i \leq \overline{\sigma}_i < \infty$, *where* $i = 1, 2$. *If* $0 < \underline{\rho} \leq \overline{\rho} < 1$, *then for any* $N_1, N_2 > 0$, *there exists some* $\alpha > 0$ *such that as* $u \to \infty$,

$$\sup_{t\in T_1, s\in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t)\geq u, \xi_2(s)<0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

*Similarly, if* $-1 < \underline{\rho} \leq \overline{\rho} < 0$, *then*

$$\sup_{t\in T_1, s\in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t)\geq u, \xi_2(s)>0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

**Proof** We only prove the first case, since the second case follows from the first one. By elementary computation on the joint density of $\xi_1(t)$ and $\xi_2(s)$, we obtain

$$
\begin{aligned}
&\sup_{t\in T_1, s\in T_2} \mathbb{E}\{(1+|\xi_1(t)|^{N_1}+|\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t)\geq u, \xi_2(s)<0\}}\} \\
&\leq \frac{1}{2\pi\underline{\sigma}_1\underline{\sigma}_2(1-\overline{\rho}^2)^{1/2}}\int_u^\infty \exp\left\{-\frac{x_1^2}{2\overline{\sigma}_1^2}\right\} dx_1 \\
&\quad \int_{-\infty}^0 (1+|x_1|^{N_1}+|x_2|^{N_2})\exp\left\{-\frac{1}{2\overline{\sigma}_2^2(1-\underline{\rho}^2)}\left(x_2-\frac{\underline{\sigma}_2\underline{\rho}x_1}{\overline{\sigma}_1}\right)^2\right\} dx_2 \\
&= o\left(\exp\left\{-\frac{u^2}{2\overline{\sigma}_1^2}-\frac{\underline{\sigma}_2^2\underline{\rho}^2u^2}{2\overline{\sigma}_2^2(1-\underline{\rho}^2)\overline{\sigma}_1^2}+\varepsilon u^2\right\}\right),
\end{aligned}
$$

as $u\to\infty$, for any $\varepsilon>0$. $\square$

**Proof of Theorem 2.3.8** Note that the second equality in (2.3.26) follows from Theorem 2.2.2 and Lemma 2.3.3, and similarly to the proof in Theorem 2.3.7, we only need to show that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small when $J$ and $J'$ are neighboring. Let $I := \bar{J}\cap\bar{J}'\neq\emptyset$. We follow the assumptions in (2.3.16) and assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, which implies $E(J)=\mathbb{R}_+^{N-k}$ and $E(J')=\mathbb{R}_+^{N-k'}$.

We first consider the case $k \geq 1$. By the Kac-Rice metatheorem, $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded from above by

$$\begin{aligned}
&\int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_{k+1} \cdots \int_0^\infty dz_{k+k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k \\
&\quad \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, X_{k+1}(t)=z_{k+1}, \\
&\qquad \ldots, X_{k+k'-m}(t)=z_{k+k'-m}, \nabla X_{|J'}(s)=0, X_{m+1}(s)=w_{m+1}, \ldots, X_k(s)=w_k\} \\
&\quad \times p_{t,s}(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k) \\
&:= \int\int_{J\times J'} A(t,s)\, dtds,
\end{aligned} \tag{2.3.27}$$

where $p_{t,s}(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k)$ is the density of

$$(X(t), X(s), \nabla X_{|J}(t), X_{k+1}(t), \ldots, X_{k+k'-m}(t), \nabla X_{|J'}(s), X_{m+1}(s), \ldots, X_k(s))$$

evaluated at $(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k)$.

Let $\{e_1, e_2, \ldots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For $t \in J$ and $s \in J'$, let $e_{t,s} = (s-t)^T/\|s-t\|$ and let $\alpha_i(t,s) = \langle e_i, (\Lambda - \Lambda(t))e_{t,s}\rangle$, then

$$(\Lambda - \Lambda(t))e_{t,s} = \sum_{i=1}^N \langle e_i, (\Lambda - \Lambda(t))e_{t,s}\rangle e_i = \sum_{i=1}^N \alpha_i(t,s)e_i. \tag{2.3.28}$$

By Lemma 2.3.9, there exists some $\alpha_0 > 0$ such that

$$\langle e_{t,s}, (\Lambda - \Lambda(t))e_{t,s}\rangle \geq \alpha_0 \tag{2.3.29}$$

for all $t$ and $s$. Under the assumptions (2.3.16) and that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, we have the following representation,

$$t = (t_1, \dots, t_m, t_{m+1}, \dots, t_k, b_{k+1}, \dots, b_{k+k'-m}, 0, \dots, 0),$$

$$s = (s_1, \dots, s_m, b_{m+1}, \dots, b_k, s_{k+1}, \dots, s_{k+k'-m}, 0, \dots, 0),$$

where $t_i \in (a_i, b_i)$ for all $i \in \sigma(J)$ and $s_j \in (a_j, b_j)$ for all $j \in \sigma(J')$. Therefore,

$$\begin{aligned} &\langle e_i, e_{t,s}\rangle \geq 0, \quad \forall\ m+1 \leq i \leq k, \\ &\langle e_i, e_{t,s}\rangle \leq 0, \quad \forall\ k+1 \leq i \leq k+k'-m, \\ &\langle e_i, e_{t,s}\rangle = 0, \quad \forall\ k+k'-m < i \leq N. \end{aligned} \tag{2.3.30}$$

Let

$$\begin{aligned} D_i &= \{(t,s) \in J \times J' : \alpha_i(t,s) \geq \beta_i\}, \quad \text{if } m+1 \leq i \leq k, \\ D_i &= \{(t,s) \in J \times J' : \alpha_i(t,s) \leq -\beta_i\}, \quad \text{if } k+1 \leq i \leq k+k'-m, \\ D_0 &= \left\{(t,s) \in J \times J' : \sum_{i=1}^{m} \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \geq \beta_0\right\}, \end{aligned} \tag{2.3.31}$$

where $\beta_0, \beta_1, \dots, \beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0$. It follows from (2.3.30) and (2.3.31) that, if $(t,s)$ does not belong to any of $D_0, D_m, \dots, D_{k+k'-m}$, then by (2.3.28),

$$\langle(\Lambda - \Lambda(t))e_{t,s}, e_{t,s}\rangle = \sum_{i=1}^{N} \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \leq \beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0,$$

which contradicts (2.3.29). Thus $D_0 \cup \cup_{i=m+1}^{k+k'-m} D_i$ is a covering of $J \times J'$, by (2.3.27),

$$\mathbb{E}\{M_u^E(J)M_u^E(J')\} \leq \int\int_{D_0} A(t,s)\,dtds + \sum_{i=m+1}^{k+k'-m} \int\int_{D_i} A(t,s)\,dtds.$$

We first show that $\int\int_{D_0} A(t,s)\,dtds$ is super-exponentially small. Similarly to the proof of Theorem 2.3.7, applying (2.3.20), (2.3.24) and (2.3.25), we obtain

$$\begin{aligned}
&\int\int_{D_0} A(t,s)\,dtds \\
&\quad \leq \int\int_{D_0} dtds \int_u^\infty dx\, p_{\nabla X_{|J}(t), \nabla X_{|J'}(s)}(0,0) p_{X(t)}(x|\nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0) \\
&\qquad \times \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t) = x, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\} \\
&\quad \leq C_1' \int\int_{D_0} dtds \int_u^\infty dx(1 + x^{N_1})\|s-t\|^{-m} p_{X(t)}(x|\nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0),
\end{aligned} \tag{2.3.32}$$

for some positive constants $C_1'$ and $N_1$. Due to Lemma 2.3.6, we only need to consider the case when $\|s-t\|$ is small. It follows from Taylor's formula (2.3.22) that as $\|s-t\| \to 0$,

$$\begin{aligned}
&\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)) \leq \mathrm{Var}(X(t)|X_1(t), \ldots, X_m(t), X_1(s), \ldots, X_m(s)) \\
&\quad = \mathrm{Var}(X(t)|X_1(t), \ldots, X_m(t), X_1(t) + \langle \nabla X_1(t), s-t\rangle + \|s-t\|^{1+\eta} Y_{t,s}^1, \ldots, \\
&\qquad\qquad X_m(t) + \langle \nabla X_m(t), s-t\rangle + \|s-t\|^{1+\eta} Y_{t,s}^m) \\
&\quad = \mathrm{Var}(X(t)|X_1(t), \ldots, X_m(t), \langle \nabla X_1(t), e_{t,s}\rangle + \|s-t\|^{\eta} Y_{t,s}^1, \ldots, \\
&\qquad\qquad \langle \nabla X_m(t), e_{t,s}\rangle + \|s-t\|^{\eta} Y_{t,s}^m) \\
&\quad \leq \mathrm{Var}(X(t)|\langle \nabla X_1(t), e_{t,s}\rangle + \|s-t\|^{\eta} Y_{t,s}^1, \ldots, \langle \nabla X_m(t), e_{t,s}\rangle + \|s-t\|^{\eta} Y_{t,s}^m) \\
&\quad = \mathrm{Var}(X(t)|\langle \nabla X_1(t), e_{t,s}\rangle, \ldots, \langle \nabla X_m(t), e_{t,s}\rangle) + o(1).
\end{aligned} \tag{2.3.33}$$

By Lemma 2.5.2, the eigenvalues of $[\mathrm{Cov}(\langle \nabla X_1(t), e_{t,s}\rangle, \dots, \langle \nabla X_m(t), e_{t,s}\rangle)]^{-1}$ are bounded uniformly in $t$ and $s$. Note that $\mathbb{E}\{X(t)\langle \nabla X_i(t), e_{t,s}\rangle\} = -\alpha_i(t,s)$. Applying these facts and Lemma 2.5.1 to the last line of (2.3.33), we see that there exist constants $C_2 > 0$ and $\varepsilon_0 > 0$ such that for $\|s-t\|$ sufficiently small,

$$\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)) \le \sigma_T^2 - C_2 \sum_{i=1}^m \alpha_i^2(t,s) + o(1) < \sigma_T^2 - \varepsilon_0, \tag{2.3.34}$$

where the last inequality is due to the fact that $(t,s) \in D_0$ implies

$$\sum_{i=1}^m \alpha_i^2(t,s) \ge \sum_{i=1}^m \alpha_i^2(t,s)|\langle e_i, e_{t,s}\rangle|^2 \ge \frac{1}{m}\bigg(\sum_{i=1}^m \alpha_i(t,s)\langle e_{t,s}, e_i\rangle\bigg)^2 \ge \frac{\beta_0^2}{m}.$$

Plugging (2.3.34) into (2.3.32) and noting that $1/\|s-t\|^m$ is integrable on $J \times J'$, we conclude that $\int\int_{D_0} A(t,s)\,dtds$ is finite and super-exponentially small.

Next we show that $\int\int_{D_i} A(t,s)\,dtds$ is super-exponentially small for $i = m+1, \dots, k$. It follows from (2.3.27) that $\int\int_{D_i} A(t,s)\,dtds$ is bounded by

$$\begin{aligned}
&\int\int_{D_i} dtds \int_u^\infty dx \int_0^\infty dw_i\, p_{X(t), \nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)}(x, 0, w_i, 0)\\
&\times \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t) = x, \nabla X_{|J}(t) = 0, X_i(s) = w_i, \nabla X_{|J'}(s) = 0\}.
\end{aligned} \tag{2.3.35}$$

We can write

$$\begin{aligned}
p_{X(t),X_i(s)}(x, w_i|X_i(t)=0) &= \frac{1}{2\pi\sigma_1(t)\sigma_2(t,s)(1-\rho^2(t,s))^{1/2}}\\
&\quad \times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\bigg)\bigg\},
\end{aligned}$$

where

$$\sigma_1^2(t) = \mathrm{Var}(X(t)|X_i(t)=0), \quad \rho(t,s) = \frac{\mathbb{E}\{X(t)X_i(s)|X_i(t)=0\}}{\sigma_1(t)\sigma_2(t,s)},$$
$$\sigma_2^2(t,s) = \mathrm{Var}(X_i(s)|X_i(t)=0) = \frac{\mathrm{detCov}(X_i(s),X_i(t))}{\lambda_{ii}},$$

and $\rho^2(t,s) < 1$ due to (**H**3). Therefore,

$$\begin{aligned}
&p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)\\
&\quad = p_{\nabla X_{|J'}(s),X_1(t),\ldots,X_{i-1}(t),X_{i+1}(t),\ldots,X_k(t)}(0|X(t)=x, X_i(s)=w_i, X_i(t)=0)\\
&\qquad \times p_{X(t),X_i(s)}(x,w_i|X_i(t)=0)p_{X_i(t)}(0) \qquad\qquad (2.3.36)\\
&\quad \le C_3 \exp\left\{-\frac{1}{2(1-\rho^2(t,s))}\left(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\right)\right\}\\
&\qquad \times (\mathrm{detCov}(X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)))^{-1/2}
\end{aligned}$$

for some positive constant $C_3$. Also, by similar arguments in the proof of Theorem 2.3.7, there exist positive constants $C_4, C_5, C_6, C_7, N_2$ and $N_3$ such that

$$\mathrm{detCov}(\nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)) \ge C_4\|s-t\|^{2(m+1)}, \qquad (2.3.37)$$

$$C_5\|s-t\|^2 \le \sigma_2^2(t,s) \le C_6\|s-t\|^2, \qquad (2.3.38)$$

and

$$
\begin{aligned}
&\mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, X_i(s)=w_i, \nabla X_{|J'}(s)=0\}\\
&\quad = \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0,\\
&\qquad\quad \langle \nabla X_i(t), e_{t,s}\rangle = w_i/\|s-t\|+o(1), \nabla X_{|J'}(s)=0\}\\
&\quad \le C_7(x^{N_2}+(w_i/\|s-t\|)^{N_3}+1).
\end{aligned}
\tag{2.3.39}
$$

Combining (2.3.35) with (2.3.36), (2.3.37) and (2.3.39), and making change of variable $w = w_i/\|s-t\|$, we obtain that for some positive constant $C_8$,

$$
\begin{aligned}
&\int\int_{D_i} A(t,s)\, dtds\\
&\quad \le C_8 \int\int_{D_i} dtds \|s-t\|^{-m-1}\int_u^\infty dx \int_0^\infty dw_i (x^{N_2}+(w_i/\|s-t\|)^{N_3}+1)\\
&\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)}+\frac{w_i^2}{\sigma_2^2(t,s)}-\frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\bigg)\bigg\}\\
&\quad = C_8 \int\int_{D_i} dtds \|s-t\|^{-m}\int_u^\infty dx \int_0^\infty dw (x^{N_2}+w^{N_3}+1)\\
&\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)}+\frac{w^2}{\widetilde{\sigma}_2^2(t,s)}-\frac{2\rho(t,s)xw}{\sigma_1(t)\widetilde{\sigma}_2(t,s)}\bigg)\bigg\},
\end{aligned}
\tag{2.3.40}
$$

where $\widetilde{\sigma}_2(t,s) = \sigma_2(t,s)/\|s-t\|$ is bounded by (2.3.38). Applying Taylor's formula (2.3.22) to $X_i(s)$ and noting that $\mathbb{E}\{X(t)\langle \nabla X_i(t), e_{t,s}\rangle\} = -\alpha_i(t,s)$, we obtain

$$
\begin{aligned}
\rho(t,s) &= \frac{1}{\sigma_1(t)\sigma_2(t,s)}\bigg(\mathbb{E}\{X(t)X_i(s)\} - \frac{1}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(s)X_i(t)\}\bigg)\\
&= \frac{\|s-t\|}{\sigma_1(t)\sigma_2(t,s)}\bigg(-\alpha_i(t,s)+\|s-t\|^\eta \mathbb{E}\{X(t)Y_{t,s}^i\}\\
&\qquad\qquad - \frac{\|s-t\|^\eta}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(t)Y_{t,s}^i\}\bigg).
\end{aligned}
\tag{2.3.41}
$$

By (2.3.38) and the fact that $(t,s) \in D_i$ implies $\alpha_i(t,s) \geq \beta_i > 0$ for $i = m+1, \ldots, k$, we conclude that $\rho(t,s) \leq -\delta_0$ for some $\delta_0 > 0$ uniformly for $t \in J$, $s \in J'$ with $\|s-t\|$ sufficiently small. Then applying Lemma 2.3.10 to (2.3.40) yields that $\int\int_{D_i} A(t,s)\, dtds$ is super-exponentially small.

It is similar to prove that $\int\int_{D_i} A(t,s)\, dtds$ is super-exponentially small for $i = k+1, \ldots, k+k'-m$. In fact, in such case, $\int\int_{D_i} A(t,s)\, dtds$ is bounded by

$$
\begin{aligned}
&\int\int_{D_i} dtds \int_u^\infty dx \int_0^\infty dz_i\, p_{X(t),\nabla X_{|J}(t),X_i(t),\nabla X_{|J'}(s)}(x,0,z_i,0) \\
&\times \mathbb{E}\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, X_i(t)=z_i, \nabla X_{|J'}(s)=0\}.
\end{aligned}
$$

We can follow the proof in the previous stage by exchanging the positions of $X_i(s)$ and $X_i(t)$ and replacing $w_i$ with $z_i$. The details are omitted since the procedure is very similar.

If $k=0$, then $m=0$ and $\sigma(J') = \{1, \ldots, k'\}$. Since $J$ becomes a singleton, we may let $J = \{t_0\}$. By the Kac-Rice metatheorem, $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$
\begin{aligned}
&\int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_1 \cdots \int_0^\infty dz_{k'}\, p_{t_0,s}(x,y,z_1,\ldots,z_{k'},0) \\
&\times \mathbb{E}\{|\det\nabla^2 X_{|J'}(s)||X(t_0)=x, X(s)=y, X_1(t_0)=z_1, \ldots, X_{k'}(t_0)=z_{k'}, \nabla X_{|J'}(s)=0\} \\
&:= \int_{J'} \widetilde{A}(t_0,s)\, ds,
\end{aligned}
$$

where $p_{t_0,s}(x,y,z_1,\ldots,z_{k'},0)$ is the density of $(X(t_0), X(s), X_1(t_0), \ldots, X_{k'}(t_0), \nabla X_{|J'}(s))$ evaluated at $(x,y,z_1,\ldots,z_{k'},0)$. Similarly, $J'$ could be covered by $\cup_{i=1}^{k'} \widetilde{D}_i$ with $\widetilde{D}_i = \{s \in$

$J' : \alpha_i(t_0, s) \le -\widetilde{\beta}_i\}$ for some positive constants $\widetilde{\beta}_i$, $1 \le i \le k'$. On the other hand,

$$\int_{\widetilde{D}_i} \widetilde{A}(t_0, s)\, ds \le \int_{\widetilde{D}_i} ds \int_u^\infty dx \int_0^\infty dz_i\, p_{X(t_0), X_i(t_0), \nabla X_{|J'}(s)}(x, z_i, 0)$$
$$\times\, \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J'}(s)||X(t_0) = x, X_i(t_0) = z_i, \nabla X_{|J'}(s) = 0\}.$$

By similar discussions, we obtain that $\mathbb{E}\{M_u^E(J) M_u^E(J')\}$ is super-exponentially small and hence complete the proof. $\square$

## 2.4 Further Remarks and Examples

**Remark 2.4.1** (The case when $T$ contains the origin). We now show that Theorem 2.3.7 and Theorem 2.3.8 still hold when $T$ contains the origin. In such case, (**H**3) is actually not satisfied since $X(0) = 0$ is degenerate. However, we may construct a small open cube $T_0$ containing 0 such that $\sup_{t\in T_0} \sigma_t^2$ is sufficiently small, then according to the Borell-TIS inequality, $\mathbb{P}\{\sup_{t\in T_0} X(t) \ge u\}$ is super-exponentially small. Let $\widehat{T} = T\backslash T_0$, then

$$\mathbb{P}\Big\{\sup_{t\in \widehat{T}} X(t) \ge u\Big\} \le \mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} \le \mathbb{P}\Big\{\sup_{t\in \widehat{T}} X(t) \ge u\Big\} + \mathbb{P}\Big\{\sup_{t\in T_0} X(t) \ge u\Big\}. \quad (2.4.1)$$

To estimate $\mathbb{P}\{\sup_{t\in\widehat{T}} X(t) \ge u\}$, similarly to the rectangle $T$, we decompose $\widehat{T}$ into several faces by lower dimensions such that $\widehat{T} = \cup_{k=0}^N \partial_k \widehat{T} = \cup_{k=0}^N \cup_{L\in\partial_k\widehat{T}} L$. Then we can get the bounds similar to (2.3.3) with $T$ replaced with $\widehat{T}$ and $J$ replaced with $L$. Following the proof of Theorem 2.3.7 yields

$$\mathbb{P}\Big\{\sup_{t\in \widehat{T}} X(t) \ge u\Big\} = \sum_{k=0}^N \sum_{L\in\partial_k\widehat{T}} \mathbb{E}\{M_u(L)\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}).$$

Due to the fact that $\sup_{t\in T_0}\sigma_t^2$ is sufficiently small, $\mathbb{E}\{M_u(L)\}$ are super-exponentially small for all faces $L$ such that $L \subset \partial_k \bar{T}_0$ with $0 \le k \le N-1$ (note that $\bar{T}_0$ is a closed rectangle). The same reason yields that for $1 \le k \le N$, $L \in \partial_k \widehat{T}$, $J \in \partial_k T$ such that $L \subset J$, the difference between $\mathbb{E}\{M_u(L)\}$ and $\mathbb{E}\{M_u(J)\}$ is super-exponentially small. Hence we obtain

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in\widehat{T}} X(t) \ge u\Big\} = \sum_{\{t\}\in\partial_0 T}\Psi\Big(\frac{u}{\sigma_t}\Big) + \sum_{k=1}^{N}\sum_{J\in\partial_k T}\frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \\ \times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)}\,dt + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}).\end{aligned} \tag{2.4.2}$$

Here, by convention, if $\theta_t = 0$, we regard $e^{-u^2/(2\theta_t^2)}$ as 0. Combining (2.4.1) with (2.4.2), we conclude that Theorem 2.3.7 still holds when $T$ contains the origin. The arguments for Theorem 2.3.8 are similar.

**Example 2.4.2** (Refinements of Theorem 2.3.7). Let Gaussian field $X$ be as in Theorem 2.3.7. Suppose that $\nu(t_0) = \sigma_T^2$ for some $t_0 \in J \in \partial_k T$ ($k \ge 0$) and $\nu(t) < \sigma_T^2$ for all $t \ne t_0$.

(i). If $k = 0$, then, due to (2.3.14), $\sup_{t\in T\setminus\{t_0\}}\theta_t^2 \le \sigma_T^2 - \varepsilon_0$ for some $\varepsilon_0 > 0$. This implies that $\mathbb{E}\{M_u(J')\}$ are super-exponentially small for all faces $J'$ other than $\{t_0\}$. Therefore,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} = \Psi\Big(\frac{u}{\sigma_T}\Big) + o(e^{-u^2/(2\sigma_T^2)+\alpha u^2}), \quad \text{as } u \to \infty. \tag{2.4.3}$$

For example, let $Y$ be a stationary Gaussian field with covariance $\rho(t) = e^{-\|t\|^2}$ and define $X(t) = Y(t) - Y(0)$, then $X$ is a smooth Gaussian field with stationary increments satisfying conditions (**H**1)-(**H**3). Let $T = [0,1]^N$, then we can apply (2.4.3) to approximate the excursion probability of $X$ with $t_0 = (1,\dots,1)$.

(ii). If $k \geq 1$, then similarly, $\mathbb{E}\{M_u(J')\}$ are super-exponentially small for all faces $J' \neq J$. It follows from Theorem 2.3.7 that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \frac{u^{k-1}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^{2k-1}} e^{-u^2/(2\theta_t^2)}\, dt(1+o(1)).$$

Let $\tau(t) = \theta_t^2$, then $\forall i \in \sigma(J)$, $\tau_i(t_0) = 0$, since $t_0$ is a local maximum point of $\tau$ restricted on $J$. Assume additionally that the Hessian matrix

$$\Theta_J(t_0) := (\tau_{ij}(t_0))_{i,j\in\sigma(J)} \tag{2.4.4}$$

is negative definite, then the Hessian matrix of $1/(2\theta_t^2)$ at $t_0$ restricted on $J$,

$$\widetilde{\Theta}_J(t_0) = -\frac{1}{2\tau^2(t_0)}(\tau_{ij}(t_0))_{i,j\in\sigma(J)} = -\frac{1}{2\sigma_T^4}\Theta_J(t_0),$$

is positive definite. Let $g(t) = |\Lambda_J - \Lambda_J(t)|/\theta_t^{2k-1}$ and $h(t) = 1/(2\theta_t^2)$, applying Lemma 2.5.3 with $T$ replaced with $J$ gives us that as $u \to \infty$,

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} &= \frac{u^{k-1}|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_{t_0}^{2k-1}} \frac{(2\pi)^{k/2}}{u^k|\widetilde{\Theta}_J(t_0)|^{1/2}} e^{-u^2/(2\theta_{t_0}^2)}(1+o(1)) \\ &= \frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}| - \Theta_J(t_0)|^{1/2}} \Psi\Big(\frac{u}{\sigma_T}\Big)(1+o(1)).\end{aligned} \tag{2.4.5}$$

**Example 2.4.2** (**Continued**: the cosine field). We consider the *cosine random field* on $\mathbb{R}^2$:

$$Z(t) = \frac{1}{\sqrt{2}}\sum_{i=1}^{2}(\xi_i \cos t_i + \xi_i' \sin t_i), \quad t = (t_1, t_2) \in \mathbb{R}^2,$$

where $\xi_1, \xi_1', \xi_2, \xi_2'$ are independent, standard Gaussian variables. $Z$ is a well-known centered, unit-variance and smooth stationary Gaussian field [cf. Adler and Taylor (2007, p.382)]. Note that $Z$ is periodic and $Z(t) = -Z_{11}(t) - Z_{22}(t)$. To avoid such degeneracy, let $X(t) = \xi_0 + Z(t) - Z(0)$, where $t \in T \subset [0, 2\pi)^2$ and $\xi_0$ is a standard Gaussian variable independent of $Z$. Then $X$ is a centered and smooth Gaussian field with stationary increments. The variance and covariance of $X$ are given respectively by

$$\begin{aligned} \nu(t) = \sigma_t^2 &= 3 - \cos t_1 - \cos t_2, \\ C(t,s) &= 2 + \frac{1}{2}\sum_{i=1}^{2}[\cos(t_i - s_i) - \cos t_i - \cos s_i]. \end{aligned} \tag{2.4.6}$$

Therefore, $X$ satisfies conditions (**H**1), (**H**2) and (**H**3) [though $X_{12}(t) \equiv 0$, it can be shown that this does not affect the validity of Theorems 2.3.7 and 2.3.8]. Taking the partial derivatives of $C$ gives us that

$$\begin{aligned} \mathbb{E}\{X(t)\nabla X(t)\} = \frac{1}{2}(\sin t_1, \sin t_2)^T, \quad \Lambda = \mathrm{Cov}(\nabla X(t)) = \frac{1}{2}I_2, \\ \Lambda - \Lambda(t) = -\mathbb{E}\{X(t)\nabla^2 X(t)\} = \frac{1}{2}[I_2 - \mathrm{diag}(\cos t_1, \cos t_2)], \end{aligned} \tag{2.4.7}$$

where $I_2$ is the $2 \times 2$ unit matrix and diag denotes the diagonal matrix.

(i). Let $T = [0, \pi/2]^2$. Then by (2.4.6), $\nu$ attains its maximum 3 only at the corner $(\pi/2, \pi/2)$, where both partial derivatives of $\nu$ are positive. Applying the result (i) in Example 2.4.2, we obtain $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = \Psi(u/\sqrt{3})(1 + o(e^{-\alpha u^2}))$.

(ii). Let $T = [0, 3\pi/2] \times [0, \pi/2]$. Then $\nu$ attains its maximum 4 only at the boundary point $t^* = (\pi, \pi/2)$, where $\nu_2(t^*) > 0$ so that the condition (2.3.14) is satisfied. In this case, $t^* \in J = (0, 3\pi/2) \times \{\pi/2\}$. By (2.4.7), we obtain $\Lambda_J = \frac{1}{2}$ and $\Lambda_J - \Lambda_J(t^*) = \frac{1}{2}(1 - \cos t_1^*) =$

1. On the other hand, for $t \in J$, by Lemma 2.5.1 and (2.4.7),

$$\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X_1(t)) = 3 - \cos t_1 - \cos t_2 - \frac{1}{2}\sin^2 t_1, \tag{2.4.8}$$

therefore $\Theta_J(t^*) = \tau_{11}(t^*) = -2$. Plugging these into (2.4.5) with $k = 1$ gives us that $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = \sqrt{2}\Psi(u/2)(1 + o(1))$.

(iii). Let $T = [0, 3\pi/2]^2$. Then $\nu$ attains its maximum 5 only at the interior point $t^* = (\pi, \pi)$. In this case, $t^* \in J = (0, 3\pi/2)^2$. By (2.4.7), we obtain $\Lambda_J = \frac{1}{2}I_2$ and $\Lambda_J - \Lambda_J(t^*) = I_2$. On the other hand, for $t \in J$, by Lemma 2.5.1 and (2.4.7),

$$\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X_1(t), X_2(t)) = 3 - \cos t_1 - \cos t_2 - \frac{1}{2}\sin^2 t_1 - \frac{1}{2}\sin^2 t_2, \tag{2.4.9}$$

therefore $\Theta_J(t^*) = (\tau_{ij}(t^*))_{i,j=1,2} = -2I_2$. Plugging these into (2.4.5) with $k = 2$ gives us that $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = 2\Psi(u/\sqrt{5})(1 + o(1))$.

**Example 2.4.3** (Refinements of Theorem 2.3.8). Let $X$ be a Gaussian field as in Theorem 2.3.8. Suppose $t_0 \in J \in \partial_k T$ is the only point in $T$ such that $\nu(t_0) = \sigma_T^2$. Assume $\sigma(J) = \{1, \ldots, k\}$, all elements in $\varepsilon(J)$ are 1, $\nu_{k'}(t_0) = 0$ for all $k + 1 \leq k' \leq N$. Then by Theorem 2.3.8,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{M_u^E(J)\} + \sum_{k'=k+1}^{N} \sum_{J'\in\partial_{k'}T, \bar{J}'\cap\bar{J}\neq\emptyset} \mathbb{E}\{M_u^E(J')\} + o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}). \tag{2.4.10}$$

Lemma 2.3.3 indicates $\mathbb{E}\{M_u^E(J)\} = (-1)^k\mathbb{E}\{\sum_{i=0}^k(-1)^i\mu_i(J)\}(1+o(e^{-\alpha x^2}))$, therefore

$$\begin{aligned}
\mathbb{E}\{M_u^E(J)\} &= (-1)^k\int_J p_{\nabla X_{|J}(t)}(0)dt\,\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t),\ldots,X_N(t))\in\mathbb{R}_+^{N-k}\}}\\
&\qquad\qquad\times\mathbb{1}_{\{X(t)\geq u\}}|\nabla X_{|J}(t)=0\}(1+o(e^{-\alpha x^2}))\\
&= \int_u^\infty dx\int_J dt\,\frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t),\ldots,X_N(t))\in\mathbb{R}_+^{N-k}\}}\\
&\qquad\qquad|X(t)=x,\nabla X_{|J}(t)=0\}(1+o(e^{-\alpha u^2}))\\
&:= \int_u^\infty A_J(x)dx(1+o(e^{-\alpha u^2})),
\end{aligned}\tag{2.4.11}$$

and similarly,

$$\begin{aligned}
\mathbb{E}\{M_u^E(J')\} &= \int_u^\infty dx\int_{J'} dt\,\frac{(-1)^{k'} e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k'+1)/2}|\Lambda_{J'}|^{1/2}\theta_t}\mathbb{E}\{\det\nabla^2 X_{|J'}(t)\\
&\quad\times\mathbb{1}_{\{(X_{J'_1}(t),\ldots,X_{J'_{N-k'}}(t))\in\mathbb{R}_+^{N-k'}\}}|X(t)=x,\nabla X_{|J'}(t)=0\}(1+o(e^{-\alpha u^2})).
\end{aligned}$$

(i). First we consider the case $k\geq 1$. We shall follow the notations $\tau(t)$, $\Theta_J(t)$ and $\widetilde{\Theta}_J(t)$ in Example 2.4.2. Let $h(t) = 1/(2\theta_t^2)$ and

$$\begin{aligned}
g_x(t) = \frac{(-1)^k}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t}&\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{(X_{k+1}(t),\ldots,X_N(t))\in\mathbb{R}_+^{N-k}\}}\\
&\qquad|X(t)=x,\nabla X_{|J}(t)=0\}.
\end{aligned}$$

Note that $\sup_{t\in T}|g_x(t)| = o(x^{N_1})$ for some $N_1>0$ as $x\to\infty$, which implies that the growth of $g_x(t)$ can be dominated by the exponential decay $e^{-x^2h(t)}$, hence both Lemma 2.5.3 and 2.5.4 are still applicable. Applying Lemma 2.5.3 with $T$ replaced by $J$ and $u$ replaced by $x^2$, we obtain that as $x \to \infty$,

$$A_J(x) = \frac{(2\pi)^{k/2}}{x^k(\det\widetilde{\Theta}_J(t_0))^{1/2}} g_x(t_0) e^{-x^2/(2\sigma_T^2)}(1+o(1)). \tag{2.4.12}$$

On the other hand, it follows from (2.2.17) that

$$\begin{aligned} g_x(t) = &\frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \int \cdots \int_{\mathbb{R}_+^{N-k}} dy_{k+1} \cdots dy_N \\ &\times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\Big(\frac{x}{\gamma_t} + \gamma_t C_{k+1}(t) y_{k+1} + \cdots + \gamma_t C_N(t) y_N\Big) \\ &\times p_{X_{k+1}(t),\ldots,X_N(t)}(y_{k+1},\ldots,y_N | X(t) = x, \nabla X_{|J}(t) = 0). \end{aligned}$$

Note that $X(t_0)$ and $\nabla X(t_0)$ are independent, and $C_j(t_0) = 0$ for all $1 \le j \le N$. Therefore,

$$\begin{aligned} g_x(t_0) = &\frac{|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\sigma_T^{k+1}} H_k\Big(\frac{x}{\sigma_T}\Big) \\ &\times \mathbb{P}\{(X_{k+1}(t_0),\ldots,X_N(t_0)) \in \mathbb{R}_+^{N-k} | \nabla X_{|J}(t_0) = 0\}. \end{aligned}$$

Plugging this and (2.4.12) into (2.4.11), we obtain

$$\begin{aligned} \mathbb{E}\{M_u^E(J)\} = &\frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}| - \Theta_J(t_0)|^{1/2}} \Psi\Big(\frac{u}{\sigma_T}\Big) \\ &\times \mathbb{P}\{(X_{k+1}(t_0),\ldots,X_N(t_0)) \in \mathbb{R}_+^{N-k} | \nabla X_{|J}(t_0) = 0\}(1+o(1)). \end{aligned} \tag{2.4.13}$$

For $J' \in \partial_{k'} T$ with $\bar{J}' \cap \bar{J} \neq \emptyset$, similarly, applying Lemma 2.5.4 with $T$ replaced by $J'$, we obtain that

$$\mathbb{E}\{M_u^E(J')\} = \frac{2^{k'/2}|\Lambda_{J'} - \Lambda_{J'}(t_0)|}{|\Lambda_{J'}|^{1/2}|-\Theta_{J'}(t_0)|^{1/2}}\Psi\Big(\frac{u}{\sigma_T}\Big)\mathbb{P}\{Z_{J'}(t_0) \in \mathbb{R}_-^{k'-k}\}$$
$$\times \mathbb{P}\{(X_{J_1'}(t_0), \ldots, X_{J'_{N-k'}}(t_0)) \in \mathbb{R}_+^{N-k'} | \nabla X_{|J'}(t_0) = 0\}(1 + o(1)), \tag{2.4.14}$$

where $Z_{J'}(t_0)$ is a centered $(k'-k)$-dimensional Gaussian vector with covariance matrix $-(\tau_{ij})_{i,j\in\sigma(J')\setminus\sigma(J)}$. Plugging (2.4.13) and (2.4.14) into (2.4.10), we obtain the asymptotic result.

(ii). $k = 0$, say $J = \{t_0\}$. Note that $X(t_0)$ and $\nabla X(t_0)$ are independent, therefore

$$\mathbb{E}\{M_u^E(J)\} = \Psi\Big(\frac{u}{\sigma_T}\Big)\mathbb{P}\{\nabla X(t_0) \in \mathbb{R}_+^N\}. \tag{2.4.15}$$

For $J' \in \partial_{k'}T$ with $\bar{J}' \cap \bar{J} \neq \emptyset$, then $\mathbb{E}\{M_u^E(J')\}$ is given by (2.4.14) with $k = 0$. Plugging (2.4.15) and (2.4.14) into (2.4.10), we obtain the asymptotic formula for the excursion probability.

**Example 2.4.3** (**Continued**: the cosine field). We consider the Gaussian field $X$ defined in the continued part of Example 2.4.2.

(i). Let $T = [0,\pi]^2$. Then $\nu$ attains its maximum 5 only at the corner $t^* = (\pi,\pi)$, where $\nabla\nu(t^*) = 0$ so that the condition (2.3.14) is not satisfied. Instead, we will use the result (ii) in Example 2.4.3 with $J = \{t^*\}$ and $k = 0$. Let $J' = (0,\pi)\times\{\pi\}$, $J'' = \{\pi\}\times(0,\pi)$. Combining the results in the continued part of Example 2.4.2 with (2.4.15) and (2.4.14), and noting that $\Lambda = \frac{1}{2}I_2$ implies $X_1(t)$ and $X_2(t)$ are independent for all $t$, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \frac{1}{4}\Psi(u/\sqrt{5}), \quad \mathbb{E}\{M_u^E(\partial_2 T)\} = \frac{1}{2}\Psi(u/\sqrt{5})(1+o(1)),$$
$$\mathbb{E}\{M_u^E(J')\} = \mathbb{E}\{M_u^E(J'')\} = \frac{\sqrt{2}}{4}\Psi(u/\sqrt{5})(1+o(1)).$$

Summing these up, we have $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = [(3+2\sqrt{2})/4]\Psi(u/\sqrt{5})(1+o(1))$.

(ii). Let $T = [0, 3\pi/2] \times [0, \pi]$. Then $\nu$ attains its maximum 5 only at the boundary point $t^* = (\pi, \pi)$, where $\nu_2(t^*) = 0$. Applying the result (i) in Example 2.4.3 with $J = (0, 3\pi/2) \times \{\pi\}$ and $k = 1$, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \frac{\sqrt{2}}{2}\Psi(u/\sqrt{5}), \quad \mathbb{E}\{M_u^E(\partial_2 T)\} = \Psi(u/\sqrt{5})(1+o(1)),$$

which implies $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\} = [(2+\sqrt{2})/2]\Psi(u/\sqrt{5})(1+o(1))$.

**Remark 2.4.4** Note that we only provide the first-order approximation for the examples in this section. However, as shown in the theory of the approximations of integrals (see e.g. Wong (2001)), the integrals in (2.3.15) and (2.3.26) can be expanded with more terms once the covariance function of the Gaussian field is smooth enough. Hence for the examples above, higher-order approximation is available. Since the procedure is similar and the computation is tedious, we omit such arguments here.

## 2.5 Some Auxiliary Facts

The following lemma is well-known and is quoted here for reader's convenience.

**Lemma 2.5.1** *Let $Y$ and $Z$ be two Gaussian random vectors of dimension $p$ and $q$, respectively. Then $Y|Z = z$ is still a $p$-dimensional Gaussian random vector having the following* *mean and covariance:*

$$\mathbb{E}\{Y|Z=z\} = \mathbb{E}Y + \mathbb{E}\{(Y-\mathbb{E}Y)(Z-\mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}(z-\mathbb{E}Z),$$
$$\mathrm{Cov}(Y|Z=z) = \mathrm{Cov}(Y) - \mathbb{E}\{(Y-\mathbb{E}Y)(Z-\mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}\mathbb{E}\{(Z-\mathbb{E}Z)(Y-\mathbb{E}Y)^T\}.$$

*In particular, if $p=q=1$ and $\mathbb{E}Y=\mathbb{E}Z=0$, then*

$$\mathbb{E}\{Y|Z=z\} = \frac{z\mathbb{E}(YZ)}{\mathrm{Var}(Z)}, \quad \mathrm{Var}(Y|Z=z) = \mathrm{Var}(Y) - \frac{(\mathbb{E}(YZ))^2}{\mathrm{Var}(Z)}.$$

Using similar arguments in the proof of Lemma 2.3.9, we can obtain the following result.

**Lemma 2.5.2** *Let $\{A(t) = (a_{ij}(t))_{1\le i,j\le N} : t \in T\}$ be a family of positive definite matrices such that all elements $a_{ij}(\cdot)$ are continuous. Denote by $\underline{x}$ and $\overline{x}$ the infimum and supremum of the eigenvalues of $A(t)$ over $t \in T$ respectively, then $0 < \underline{x} \le \overline{x} < \infty$.*

The following two formulas state the results on the Laplace method approximation. Lemma 2.5.3 can be found in many books on the approximations of integrals, here we refer to Wong (2001). Lemma 2.5.4 can be derived by following similar arguments in the proof of Laplace method for the case of boundary point in Wong (2001).

**Lemma 2.5.3** (Laplace method for interior point). *Let $t_0$ be an interior point of $T$. Suppose the following conditions hold: (i) $g(t) \in C(T)$ and $g(t_0) \ne 0$; (ii) $h(t) \in C^2(T)$ and attains its unique minimum at $t_0$; and (iii) $\nabla^2 h(t_0)$ is positive definite. Then as $u \to \infty$,*

$$\int_T g(t)e^{-uh(t)}dt = \frac{(2\pi)^{N/2}}{u^{N/2}(\det\nabla^2 h(t_0))^{1/2}} g(t_0)e^{-uh(t_0)}(1+o(1)).$$

**Lemma 2.5.4** (Laplace method for boundary point). *Let $t_0 \in J \in \partial_k T$ with $0 \le k \le N-1$. Suppose that conditions (i), (ii) and (iii) in Lemma 2.5.3 hold, and additionally $\nabla h(t_0) = 0$.*

*Then as $u \to \infty$,*

$$\int_T g(t)e^{-uh(t)}dt = \frac{(2\pi)^{N/2}\mathbb{P}\{Z_J(t_0) \in (-E(J))\}}{u^{N/2}(\det\nabla^2 h(t_0))^{1/2}} g(t_0)e^{-uh(t_0)}(1 + o(1)),$$

*where $Z_J(t_0)$ is a centered $(N-k)$-dimensional Gaussian vector with covariance matrix $(h_{ij}(t_0))_{J_1 \le i,j \le J_{N-k}}$, $-E(J) = \{x \in \mathbb{R}^N : -x \in E(J)\}$, and the definitions of $J_1, \ldots, J_{N-k}$ and $E(J)$ are in (2.2.4).*

# Chapter 3

# Smooth Gaussian Random Fields with Non-constant Variances

## 3.1 Gaussian Fields on Rectangles

Let $\{X(t) : t \in \mathbb{R}^N\}$ be a smooth centered Gaussian random field with non-constant variance and let $T = \prod_{i=1}^N [a_i, b_i]$ be a closed rectangle in $\mathbb{R}^N$. In this Chapter, we study the excursion probability of $X$ over $T$.

Let $\nu(t) := \sigma_t^2 = \mathrm{Var}(X(t))$ and assume $\sup_{t\in T} \nu(t) = 1$. A matrix is called *negative semidefinite* if all of its eigenvalues are nonpositive. In addition to conditions (**H**1) and (**H**3) in the previous chapter, we will make use of the following condition.

(**H**4). $\forall t \in J \in \partial_k T$ such that $\nu(t) = 1$ and $0 \le k \le N-2$, $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is negative definite, where $\zeta(t) = \{n : \nu_n(t) = 0, 1 \le n \le N\}$.

**Proposition 3.1.1** *Let $X(\cdot) \in C^2(\mathbb{R}^N)$ a.s. If $(\nu_{ij}(t))_{i,j\in\zeta(t)}$ is negative semidefinite for each $t \in J \in \partial_k T$ such that $\nu(t) = 1$ and $0 \le k \le N-2$, then* (**H**4) *holds.*

**Proof** Since $\frac{1}{2}\nu_{ij}(t) = \mathbb{E}\{X(t)X_{ij}(t)\} + \mathbb{E}\{X_i(t)X_j(t)\}$,

$$(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)} = \frac{1}{2}(\nu_{ij}(t))_{i,j\in\zeta(t)} - (\mathbb{E}\{X_i(t)X_j(t)\})_{i,j\in\zeta(t)}. \tag{3.1.1}$$

But $(\nu_{ij}(t))_{i,j\in\zeta(t)}$ is negative semidefinite and $(\mathbb{E}\{X_i(t)X_j(t)\})_{i,j\in\zeta(t)}$ is positive definite, it follows that $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is negative definite and hence (**H**4) holds. $\square$

**Remark 3.1.2** In (**H**4), $\nu(t) = 1$ implies $\nu_n(t) = 0$ for all $n \in \sigma(J)$ and thus $\zeta(t) \supset \sigma(J)$. Additionally, we only consider $t \in J \in \partial_k T$ with $0 \le k \le N-2$, this is because for $N-1 \le k \le N$, $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is automatically negative definite due to $\nu(t) = 1$, as shown below.

(i). If $k = N$, then $t$ becomes a maximum point of $\nu$ in the interior of $T$, and $\zeta(t) = \sigma(J) = \{1, \cdots, N\}$, hence $(\nu_{ij}(t))_{i,j\in\zeta(t)}$ is always negative semidefinite. By (3.1.1), we see that $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is negative definite.

(ii). If $k = N-1$, we distinguish two cases. If $\zeta(t) = \sigma(J)$, then $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is negative by the same arguments in the previous step. If $\zeta(t) = \{1, \cdots, N\}$, it follows from Taylor's formula that

$$\begin{aligned}\nu(s) &= \nu(t) + \nabla\nu(t)(s-t)^T + (s-t)\nabla^2\nu(t)(s-t)^T + o(\|s-t\|^2)\\ &= \nu(t) + (s-t)\nabla^2\nu(t)(s-t)^T + o(\|s-t\|^2),\end{aligned} \tag{3.1.2}$$

for all $s \in T$ such that $\|s-t\|$ is small enough. Since $t \in J \in \partial_{N-1}T$, $\{\pm\frac{s-t}{\|s-t\|} : s \in T\}$ contains all the directions $e \in \mathbb{S}^{N-1}$. Note that $\nu(t) = 1$, $\nabla^2\nu(t)$ does not have any positive eigenvalue and hence $(\nu_{ij}(t))_{i,j\in\zeta(t)} = \nabla^2\nu(t)$ is negative semidefinite, then $(\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\zeta(t)}$ is negative by (3.1.1).

If $D$ is a subset (not necessary open) of $J \in \partial_k T$, we define

$$\begin{aligned}M_u^E(D) = \#\{t \in D\backslash\partial D : X(t) \ge u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = k,\\ \varepsilon_j^* X_j(t) > 0 \text{ for all } j \notin \sigma(J)\}.\end{aligned}$$

For $t \in J \in \partial_k T$, let

$$
\begin{aligned}
&\Lambda(t) = (\mathbb{E}\{X_i(t)X_j(t)\})_{1\le i,j\le N}, \quad \Lambda_J(t) = (\mathbb{E}\{X_i(t)X_j(t)\})_{i,j\in\sigma(J)},\\
&\Sigma(t) = (\mathbb{E}\{X(t)X_{ij}(t)\})_{1\le i,j\le N}, \quad \Sigma_J(t) = (\mathbb{E}\{X(t)X_{ij}(t)\})_{i,j\in\sigma(J)},\\
&\{1,\cdots,N\}\backslash\sigma(J) = \{J_1,\cdots,J_{N-k}\},\\
&E(J) = \{(t_{J_1},\cdots,t_{J_{N-k}}) \in \mathbb{R}^{N-k} : t_j\varepsilon_j^* \ge 0, j = J_1,\cdots,J_{N-k}\}.
\end{aligned}
\tag{3.1.3}
$$

Let $C_j(t)$ be the $(1, j+1)$ entry of $(\mathrm{Cov}(X(t), \nabla X(t)))^{-1}$, i.e. $C_j(t) = \frac{M_{1,j+1}}{\det\mathrm{Cov}(X(t),\nabla X(t))}$, where $M_{1,j+1}$ is the cofactor of the $(1, j+1)$ element, $\mathbb{E}\{X(t)X_j(t)\}$, in the covariance matrix $\mathrm{Cov}(X(t), \nabla X(t))$.

Note that the notations $\Lambda(t)$ and $\Lambda_J(t)$ are different from those defined in Chapter 2.

The result Lemma 3.1.3 below follows immediately from similar argumentss in the proof of Proposition 3.1.1.

**Lemma 3.1.3** *If $t_0 \in J \in \partial_k T$ satisfies $\nu(t_0) = 1$, where $k \ge 1$, then $\mathbb{E}\{X(t_0)X_i(t_0)\} = 0$ for all $i \in \sigma(J)$ and $\Sigma_J(t_0)$ is negative definite.*

**Corollary 3.1.4** *Let $\{X(t) : t \in \mathbb{R}^N\}$ be a Gaussian random field satisfying* (**H**1), (**H**3) *and* (**H**4), *then there exists some constant $\alpha > 0$ such that*

$$
\sum_{k=0}^{N}\sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\} = o(e^{-\alpha u^2 - u^2/2}).
$$

**Proof** Due to Lemma 2.3.4, it suffices to show that $\beta_J^2 < 1$ for each $J \in \partial_k T$, which is equivalent to $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) < 1$ for all $t \in \bar{J} = J \cup \partial J$ and $e \in \mathbb{S}^{k-1}$ by continuity.

(i). Suppose $\mathrm{Var}(X(t_0)|\nabla X_{|J}(t_0), \nabla^2 X_{|J}(t_0)e) = 1$ for some $t_0 \in J$, then

$$1 = \mathrm{Var}(X(t_0)|\nabla X_{|J}(t_0), \nabla^2 X_{|J}(t_0)e) \le \mathrm{Var}(X(t_0)|\nabla^2 X_{|J}(t_0)e) \le \mathrm{Var}(X(t_0)) \le 1.$$

Note that

$$\begin{aligned}
&\mathrm{Var}(X(t_0)|\nabla^2 X_{|J}(t_0)e) = \mathrm{Var}(X(t_0)) \\
&\Leftrightarrow \mathbb{E}\{X(t_0)(\nabla^2 X_{|J}(t_0)e)\} = 0 \\
&\Leftrightarrow \Sigma_J(t_0)e = 0.
\end{aligned}$$

Since $t_0$ is a maximum point, by Lemma 3.1.3, $\Sigma_J(t_0)$ is negative definite and hence $\Sigma_J(t_0)e \neq 0$ for all $e \in \mathbb{S}^{k-1}$, which is a contradiction.

(ii). Suppose $\mathrm{Var}(X(t_1)|\nabla X_{|J}(t_1), \nabla^2 X_{|J}(t_1)e) = 1$ for some $t_1 \in \partial J$. It then follows from similar arguments in step (i) that $\mathrm{Var}(X(t_1)|\nabla X_{|J}(t_1)) = 1$ and hence $\nu_i(t_1) = 0$ for all $i \in \sigma(J)$, which implies $(\mathbb{E}\{X(t_1)X_{ij}(t_1)\})_{i,j\in\sigma(J)}$ is negative definite by (**H**4). Thus there will be a contradiction as in step (i), completing the proof. □

**Lemma 3.1.5** *Let $\{X(t) : t \in \mathbb{R}^N\}$ be a Gaussian random field satisfying* (**H**1), (**H**3) *and* (**H**4). *Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}). \tag{3.1.4}$$

**Proof** Due to (2.2.1), we only need to show that for each $k \in \{0, 1, \dots, N\}$ and $J \in \partial_k T$,

$$\mathbb{E}\{M_u^E(J)\} = (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} + o(e^{-\alpha u^2 - u^2/2}). \tag{3.1.5}$$

Without loss of generality, let $\sigma(J) = \{1, \cdots, k\}$ and assume all elements in $\varepsilon(J)$ are 1. Let

$$O(\bar{J}) = \{t \in J : \nu(t) = 1\} \cup \{t \in \partial J : \nu(t) = 1, \nu_i(t) = 0, \forall 1 \le i \le k\}.$$

Our aim is to find an open neighborhood of $O(\bar{J})$ restricted on $J$, say $U_\delta(J) = \{t \in J : d(t, O(\bar{J})) < \delta\}$, such that as $u \to \infty$,

$$\begin{aligned} \mathbb{E}\{M_u^E(J)\} &= \mathbb{E}\{M_u^E(U_\delta(J))\} + o(e^{-\alpha u^2 - u^2/2}) \\ &= (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(U_\delta(J))\} + o(e^{-\alpha u^2 - u^2/2}) \\ &= (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} + o(e^{-\alpha u^2 - u^2/2}). \end{aligned} \tag{3.1.6}$$

For $n = k+1, \ldots, N$, let

$$O_n(\bar{J}) = \{t \in J : \nu(t) = 1, \nu_n(t) = 0\} \cup \{t \in \partial J : \nu(t) = 1, \nu_n(t) = 0, \nu_i(t) = 0, \forall 1 \le i \le k\}.$$

Firstly, we consider the subset $U^1(\bar{J}) = \cap_{n=k+1}^N O_n(\bar{J})$ and define its open neighborhood in $J$, $U_{\delta_1}^1(J) = \{t \in J : d(t, U^1(\bar{J})) < \delta_1\}$, where $\delta_1$ is a small positive number to be specified. Then, by the Kac-Rice metatheorem, $\mathbb{E}\{M_u^E(U_{\delta_1}^1(J))\}$ becomes

$$\begin{aligned} &\int_{U_{\delta_1}^1(J)} \frac{(-1)^k}{(2\pi)^{k/2}(\det\Lambda_J(t))^{1/2}}\, dt \int_u^\infty \int_{\mathbb{R}_+^{N-k}} \mathbb{E}\{\det\nabla^2 X_{|J}(t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}}| \\ &\quad |X(t) = x, \nabla X_{|J}(t) = 0, X_{k+1}(t) = y_{k+1}, \ldots, X_N(t) = y_N\} \\ &\quad \times p_{X(t), X_{k+1}(t), \ldots, X_N(t)}(x, y_{k+1}, \ldots, y_N | \nabla X_{|J}(t) = 0)\, dx dy_{k+1} \cdots dy_N, \end{aligned} \tag{3.1.7}$$

where $\mathcal{D}_k$ is the collection of all $k \times k$ matrices with $k$ negative eigenvalues.

Due to (**H**4) and continuity, we can choose $\delta_1$ small enough such that $\Sigma_J(t)$ are negative definite for all $t \in U^1_{\delta_1}(J)$. Write $\nabla^2 X_{|J}(t) = Q^{-1}_{t,J} Q_{t,J} \nabla^2 X_{|J}(t) Q_{t,J} Q^{-1}_{t,J}$, where $Q_{t,J}(-\Sigma_J(t))Q_{t,J} = I_k$. Let $a^l_{ij}(t) = \mathbb{E}\{X_l(t)(Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J})_{ij}\}$ for $l = 1, \cdots, N$, then

$$
\begin{aligned}
&\mathbb{E}\{(Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J})_{ij} | X(t) = x, \nabla X_{|J}(t) = 0, X_{k+1}(t) = y_{k+1}, \cdots, X_N(t) = y_N\} \\
&\quad = (\mathbb{E}\{X(t)(Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J})_{ij}\}, a^1_{ij}(t), \cdots, a^N_{ij}(t))(\mathrm{Cov}(X(t), \nabla X(t)))^{-1} \\
&\qquad \cdot (x, 0, \cdots, 0, y_{k+1}, \cdots, y_N)^T \\
&\quad = (-\delta_{ij}, a^1_{ij}(t), \cdots, a^N_{ij}(t))(\mathrm{Cov}(X(t), \nabla X(t)))^{-1}(x, 0, \cdots, 0, y_{k+1}, \cdots, y_N)^T.
\end{aligned}
$$

Make change of variables $W(t) = (W_{ij}(t))_{1\le i,j\le k}$, where

$$
W_{ij}(t) = (Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J})_{ij} - \bigg( -\frac{x}{\gamma_t^2}\delta_{ij} + \sum_{l=1}^N a^l_{ij}(t)C_l(t)x \bigg),
$$

i.e.,

$$
\begin{aligned}
Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J} &= W(t) - \frac{x}{\gamma_t^2} I_k + x\bigg( \sum_{l=1}^N a^l_{ij}(t)C_l(t) \bigg)_{1\le i,j\le k} \\
&= W(t) - xB(t),
\end{aligned}
$$

where $B(t) = \frac{1}{\gamma_t^2} I_k - (\sum_{l=1}^N a^l_{ij}(t)C_l(t))_{1\le i,j\le k}$. Denote the density of

$$
((W_{ij}(t))_{1\le i\le j\le k} | X(t) = x, \nabla X_{|J}(t) = 0, X_{k+1}(t) = y_{k+1}, \cdots, X_N(t) = y_N)
$$

by $g_t(w), w = (w_{ij} : 1 \le i, j \le k) \in \mathbb{R}^{k(k+1)/2}$, then $g_t(w)$ is independent of $x$. Let $(w_{ij})$ be the abbreviation of the $k \times k$ symmetric matrix $(w_{ij})_{1\le i\le j\le k}$, then

$$\begin{aligned}
&\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\mathcal{D}_k}(\nabla^2 X_{|J}(t))|X(t)=x, \nabla X_{|J}(t)=0, X_{k+1}(t)=y_{k+1},\cdots,X_N(t)=y_N\}\\
&\quad=\det(-\Sigma_J(t))\mathbb{E}\{\det(Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J})\mathbb{1}_{\{Q_{t,J}\nabla^2 X_{|J}(t)Q_{t,J}\in\mathcal{D}_k\}}|\\
&\qquad\qquad\qquad |X(t)=x, \nabla X_{|J}(t)=0, X_{k+1}(t)=y_{k+1},\cdots,X_N(t)=y_N\}\\
&\quad=\det(-\Sigma_J(t))\int_{(w_{ij}):(w_{ij})-xB(t)\in\mathcal{D}_k}\det((w_{ij})-xB(t))g_t(w)\,dw.
\end{aligned}$$

Since $\nu_n(t)=0$ for all $t\in U^1(\bar{J})$ and $n=k+1,\cdots,N$, we can find $\delta_1$ small enough such that $C_l(t)$ are close to 0 for all $l=1,\cdots,N$ and $t\in U^1_{\delta_1}(J)$. Together with the fact $\{\gamma_t^2 : t\in J\}$ is bounded, there exists a constant $c_1>0$ such that

$$(w_{ij})-xB(t)\in\mathcal{D}_k,\quad \forall\|(w_{ij})\|<\frac{x}{c_1}.$$

It then follows from similar arguments in Lemma 2.3.2 and Lemma 2.3.3 that

$$\mathbb{E}\{M_u^E(U^1_{\delta_1}(J))\}=(-1)^k\sum_{i=0}^k(-1)^i\mathbb{E}\{\mu_i(U^1_{\delta_1}(J))\}+o(e^{-\alpha u^2-u^2/2}).$$

Next, we consider the subset $U^2(\bar{J})=(\cap_{n=k+1}^{N-1}O_n(\bar{J}))\setminus U^1_{\delta_1}(J)$, and define its neighborhood $U^2_{\delta_2}(J)=\{t\in J: d(t,U^2(\bar{J}))<\delta_2\}\setminus U^1_{\delta_1}(J)$, where $\delta_2$ is a small positive number to be specified. Then we can write $\mathbb{E}\{M_u^E(U_{\delta_2}^2(J))\}$ as

$$\begin{aligned}
&\int_{U_{\delta_2}^2(J)} p_{\nabla X_{|J}(t)}(0)\, dt\, \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} \mathbb{1}_{\{X(t)\geq u\}} \\
&\qquad\qquad\qquad\qquad \times \mathbb{1}_{\{X_{k+1}(t)>0,\ldots,X_{N-1}(t)>0\}} |\nabla X_{|J}(t) = 0\} \\
&- \int_{U_{\delta_2}^2(J)} p_{\nabla X_{|J}(t)}(0)\, dt\, \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} \mathbb{1}_{\{X(t)\geq u\}} \\
&\qquad\qquad\qquad\qquad \times \mathbb{1}_{\{X_{k+1}(t)>0,\ldots,X_{N-1}(t)>0, X_N(t)\leq 0\}} |\nabla X_{|J}(t) = 0\}.
\end{aligned} \tag{3.1.8}$$

The second term in (3.1.8) is bounded by

$$\begin{aligned}
&\int_{U_{\delta_2}^2(J)} dt \int_u^{\infty} dx \int_{-\infty}^0 \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| | X(t) = x, X_N(t) = y_N, \nabla X_{|J}(t) = 0\} \\
&\qquad\qquad\qquad\qquad \times p_{X(t), X_N(t), \nabla X_{|J}(t)}(x, y_N, 0, \cdots, 0) dy_N.
\end{aligned} \tag{3.1.9}$$

Note that the conditional expectation in (3.1.9) can be bounded by $c_2(|x|^{N_1} + |y_N|^{N_2})$ when $u$ is large, for some positive constants $c_2$, $N_1$ and $N_2$; and

$$\begin{aligned}
&p_{X(t), X_N(t), \nabla X_{|J}(t)}(x, y_N, 0, \cdots, 0) \\
&\quad = p_{\nabla X_{|J}(t)}(0, \cdots, 0 | X(t) = x, X_N(t) = y_N) p_{X(t), X_N(t)}(x, y_N) \\
&\quad \leq c_3 p_{X(t), X_N(t}(x, y_N)
\end{aligned}$$

for some positive constant $c_3$. On the other hand, $U^2(\bar{J})$ is a compact set and for all $t \in U^2(\bar{J})$, $\nu_N(t) \neq 0$ which implies $\nu_N(t) > 0$ due to $\nu(t) = 1$, thus we can choose $\delta_2$ sufficiently small such that $\mathbb{E}\{X(t)X_N(t)\} > \delta_0$ for all $t \in U_{\delta_2}^2(J)$ and some $\delta_0 > 0$. Hence

(3.1.9) is super-exponentially small by Lemma 2.3.10. Similar arguments give

$$
\begin{aligned}
&(-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(U^2_{\delta_2}(J))\} \\
&\quad = (-1)^k \int_{U^2_{\delta_2}(J)} p_{\nabla X_{|J}(t)}(0)\, dt\, \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{X(t)\geq u\}} \\
&\qquad \times \mathbb{1}_{\{X_{k+1}(t)>0,\ldots,X_{N-1}(t)>0\}} | \nabla X_{|J}(t) = 0\} + o(e^{-\alpha u^2 - u^2/2}).
\end{aligned}
$$

Combining this with (3.1.8), and following the same arguments to simplify $\mathbb{E}\{M_u^E(U^1_{\delta_1}(J))\}$, we obtain

$$
\mathbb{E}\{M_u^E(U^2_{\delta_2}(J))\} = (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(U^2_{\delta_2}(J))\} + o(e^{-\alpha u^2 - u^2/2}).
$$

Continue this procedure at most finite many times, and take the union of those disjoint neighborhoods ($U^1_{\delta_1}(J)$, $U^2_{\delta_2}(J)$, $\ldots$), we can find $U_\delta(J) = \{t \in J : d(t, O(\bar{J})) < \delta\}$ inside the union for some $\delta > 0$ such that the second equality in (3.1.6) holds. On the other hand, By the Kac-Rice metatheorem,

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(J)\} - \mathbb{E}\{M_u^E(U_\delta(J))\} = \mathbb{E}\{M_u^E(J \setminus U_\delta(J))\} \\
&\leq \frac{1}{(2\pi)^{k/2}} \int_{J\setminus U_\delta(J)} \frac{1}{(\det \Lambda_J(t))^{1/2}}\, dt \int_u^\infty p_{X(t)}(x | \nabla X_{|J}(t) = 0) \\
&\qquad \times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| | X(t) = x, \nabla X_{|J}(t) = 0\}\, dx.
\end{aligned} \tag{3.1.10}
$$

But, by the definition of $U_\delta(J)$ and continuity,

$$
\sup_{t \in J\setminus U_\delta(J)} \mathrm{Var}(X(t) | \nabla X_{|J}(t) = 0) < 1 - \varepsilon_0
$$

for some $\varepsilon_0 > 0$, hence the first equality in (3.1.6) holds and the third equality in (3.1.6) follows similarly. We finish the proof. $\square$

Following the same proof in Lemma 2.3.6, we obtain the following result.

**Lemma 3.1.6** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field satisfying* (**H**1) *and* (**H**3). *Let $J$ and $J'$ be two faces of $T$ such that their distance is positive, i.e., $\inf_{t\in J, s\in J'} \|s - t\| > \delta_0$ for some $\delta_0 > 0$, then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.*

The next result is an extension of Theorem 2.3.7.

**Lemma 3.1.7** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field satisfying* (**H**1) *and* (**H**3). *Let $J$ and $J'$ be two neighboring faces, that is $\bar{J} \cap \bar{J}' \neq \emptyset$. Suppose*

$$\{t \in \bar{J} \cap \bar{J}' : \nu(t) = 1, \nu_j(t) = 0, \forall j \in \sigma(J) \cup \sigma(J')\} = \emptyset, \tag{3.1.11}$$

*then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.*

**Proof** Condition (3.1.11) implies that there exists $\varepsilon_0 > 0$ such that

$$\sup_{t\in U(\delta)} \mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(t)) \leq 1 - \varepsilon_0,$$

where $U(\delta) = \{t \in J \cup J' : d(t, \bar{J} \cap \bar{J}') \leq \delta\}$ and $\delta$ is a sufficiently small positive number. Following the same proof in Theorem 2.3.7 yields our desired result. $\square$

The next result follows from similar arguments in the proof of Theorem 2.3.8.

**Lemma 3.1.8** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field satisfying* ($\mathbf{H}1$), ($\mathbf{H}3$) *and* ($\mathbf{H}4$). *Let $J$ and $J'$ be two neighboring faces, that is $\bar{J} \cap \bar{J}' \neq \emptyset$. Then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.*

**Proof** Let $I = \bar{J} \cap \bar{J}'$. We follow the assumptions in (2.3.16) and assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, which implies $E(J) = \mathbb{R}_+^{N-k}$ and $E(J') = \mathbb{R}_+^{N-k'}$.

If condition (3.1.11) is satisfied, then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small by Lemma 3.1.7. So we will focus on the alternative case, which is

$$I_0 := \{t \in I : \nu(t) = 1, \nu_j(t) = 0, \forall 1 \leq j \leq k + k' - m\} \neq \emptyset.$$

Let $B(I_0, \delta) = \{t \in J \cup J' : d(t, I_0) < \delta\}$, where $\delta$ is a small number to be specified.

Note that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ can be written as

$$\begin{aligned}
&\mathbb{E}\{[M_u^E(J \cap B(I_0, \delta)) + M_u^E(J \cap B^c(I_0, \delta))][M_u^E(J' \cap B(I_0, \delta)) + M_u^E(J' \cap B^c(I_0, \delta))]\} \\
&\quad = \mathbb{E}\{M_u^E(J \cap B(I_0, \delta))M_u^E(J' \cap B(I_0, \delta))\} + o(e^{-\alpha u^2 - u^2/2}),
\end{aligned}$$

since the rest terms are super-exponentially small by the same arguments in Lemma 3.1.7. Therefore, to prove the result, we may estimate $\mathbb{E}\{M_u^E(J \cap B(I_0, \delta))M_u^E(J' \cap B(I_0, \delta))\}$ instead of $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ itself, with only a super-exponentially small difference. By ($\mathbf{H}4$),

$$\Sigma_{J \cup J'}(t) := \mathbb{E}\{X(t)X_{ij}(t)\}_{i,j=1,\ldots,k+k'-m}$$

are negative definite for all $t \in I_0$, so that by continuity (similar to Lemma 2.3.9), we can choose $\delta$ small enough such that

$$\langle \widetilde{e}, -\Sigma_{J\cup J'}(t)\widetilde{e}\rangle \geq \alpha_0, \quad \forall t \in B(I_0,\delta), \widetilde{e} \in \mathbb{S}^{k+k'-m-1} \tag{3.1.12}$$

for some constant $\alpha_0 > 0$.

We first consider the case $k \geq 1$. By the Kac-Rice metatheorem,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(J\cap B(I_0,\delta))M_u^E(J'\cap B(I_0,\delta))\} \\
&\quad \leq \int_{J\cap B(I_0,\delta)} dt \int_{J'\cap B(I_0,\delta)} ds \int_u^\infty dx \int_u^\infty dy \\
&\qquad \int_0^\infty dz_{k+1} \cdots \int_0^\infty dz_{k+k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k \\
&\qquad \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, X_{k+1}(t)=z_{k+1}, \\
&\qquad\quad \ldots, X_{k+k'-m}(t)=z_{k+k'-m}, \nabla X_{|J'}(s)=0, X_{m+1}(s)=w_{m+1},\ldots,X_k(s)=w_k\} \\
&\qquad \times p_{t,s}(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k) \\
&\quad := \int\int_{(J\cap B(I_0,\delta))\times(J'\cap B(I_0,\delta))} A(t,s)\,dtds,
\end{aligned} \tag{3.1.13}$$

where $p_{t,s}(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k)$ is the density of

$$(X(t), X(s), \nabla X_{|J}(t), X_{k+1}(t), \ldots, X_{k+k'-m}(t), \nabla X_{|J'}(s), X_{m+1}(s), \ldots, X_k(s))$$

evaluated at $(x,y,0,z_{k+1},\ldots,z_{k+k'-m},0,w_{m+1},\ldots,w_k)$.

Let $\{\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_{k+k'-m}\}$ be the standard orthonormal basis of $\mathbb{R}^{k+k'-m}$. For $t \in J$ and $s \in J'$, let $\widetilde{e}_{t,s}$ be the projection of $(s-t)^T/\|s-t\|$ on span$\{J, J'\}$, and let $\alpha_i(t,s) =$

$\langle \widetilde{e}_i, -\Sigma_{J\cup J'}(t)\widetilde{e}_{t,s}\rangle$, then

$$-\Sigma_{J\cup J'}(t)\widetilde{e}_{t,s} = \sum_{i=1}^{k+k'-m} \langle \widetilde{e}_i, -\Sigma_{J\cup J'}(t)\widetilde{e}_{t,s}\rangle \widetilde{e}_i = \sum_{i=1}^{k+k'-m} \alpha_i(t,s)\widetilde{e}_i. \tag{3.1.14}$$

It follows from (3.1.12) that

$$\langle \widetilde{e}_{t,s}, -\Sigma_{J\cup J'}(t)\widetilde{e}_{t,s}\rangle \geq \alpha_0 \tag{3.1.15}$$

for all $t \in J \cap B(I_0, \delta)$ and $s \in J' \cap B(I_0, \delta)$. Let

$$\begin{aligned}
D_i &= \{(t,s) \in (J\cap B(I_0,\delta)) \times (J' \cap B(I_0,\delta)) : \alpha_i(t,s) \geq \beta_i\}, \ \text{if } m+1 \leq i \leq k,\\
D_i &= \{(t,s) \in (J\cap B(I_0,\delta)) \times (J' \cap B(I_0,\delta)) : \alpha_i(t,s) \leq -\beta_i\}, \ \text{if } k+1 \leq i \leq k+k'-m,\\
D_0 &= \left\{(t,s) \in (J\cap B(I_0,\delta)) \times (J' \cap B(I_0,\delta)) : \sum_{i=1}^{m} \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \geq \beta_0\right\},
\end{aligned} \tag{3.1.16}$$

where $\beta_0, \beta_1, \ldots, \beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0$. As in the proof of Theorem 2.3.8, we see that $D_0 \cup \cup_{i=m+1}^{k+k'-m} D_i$ is a covering of $(J \cap B(I_0,\delta)) \times (J' \cap B(I_0,\delta))$. By (3.1.13),

$$\begin{aligned}
&\mathbb{E}\{M_u^E(J\cap B(I_0,\delta))M_u^E(J'\cap B(I_0,\delta))\}\\
&\quad \leq \int\int_{D_0} A(t,s)\,dtds + \sum_{i=m+1}^{k+k'-m} \int\int_{D_i} A(t,s)\,dtds.
\end{aligned}$$

Following the same arguments in the proof of Theorem 2.3.8, we obtain that $\int\int_{D_0} A(t,s)\,dtds$ and $\int\int_{D_i} A(t,s)\,dtds$ are all super-exponentially small for $i = m+1, \ldots, k+k'-m$, completing the proof. □

**Theorem 3.1.9** *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field satisfying* (**H**1)*,* (**H**3) *and* (**H**4)*. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** The result follows from the combination of (2.3.1), (2.3.2), Corollary 3.1.4, Lemma 3.1.5, Lemma 3.1.6, Lemma 3.1.7 and Lemma 3.1.8. □

## 3.2 Applications for Gaussian Fields with a Unique Maximum Point of the Variance

In this section, we consider the case when $\nu(t)$ attains its maximum 1 at a unique point $t_0 \in J \in \partial_k T$ such that $\nu_j(t_0) \neq 0$ for all $j \notin \sigma(J)$.

**Lemma 3.2.1** *Let $X$ be as in Theorem 3.1.9. Suppose $\nu(t_0) = 1$, where $t_0 \in J \in \partial_k T$ and $k \geq 1$, then as $x \to \infty$,*

$$\mathbb{E}\{\det\nabla^2 X_{|J}(t_0)|X(t_0) = x, \nabla X_{|J}(t_0) = 0\} = |\Sigma_J(t_0)|x^k(1 + o(1)).$$

**Proof** Since $\nu(t_0) = 1$, $\Sigma_J(t_0)$ is negative definite. Let $Q_{t_0,J}$ be the $k \times k$ positive definite matrix such that $Q_{t_0,J}(-\Sigma_J(t_0))Q_{t_0,J} = I_k$. Then we can write

$$\nabla^2 X_{|J}(t_0) = Q^{-1}_{t_0,J} Q_{t_0,J} \nabla^2 X_{|J}(t_0) Q_{t_0,J} Q^{-1}_{t_0,J},$$

and therefore,

$$
\begin{aligned}
&\mathbb{E}\{\det\nabla^2 X_{|J}(t_0)|X(t_0)=x, \nabla X_{|J}(t_0)=0\} \\
&\quad = |-\Sigma_J(t_0)|\mathbb{E}\{\det(Q_{t_0,J}\nabla^2 X_{|J}(t_0)Q_{t_0,J})|X(t_0)=x, \nabla X_{|J}(t_0)=0\}.
\end{aligned} \tag{3.2.1}
$$

Since $X(t_0)$ and $\nabla X_{|J}(t_0)$ are independent,

$$
\mathbb{E}\{Q_{t_0,J}\nabla^2 X_{|J}(t_0)Q_{t_0,J}|X(t_0)=x, \nabla X_{|J}(t_0)=0\} = -xI_k.
$$

It follows that

$$
\mathbb{E}\{\det(Q_{t_0,J}\nabla^2 X_{|J}(t_0)Q_{t_0,J})|X(t_0)=x, \nabla X_{|J}(t_0)=0\} = \mathbb{E}\{\det(\Delta(t_0)-xI_k)\}, \tag{3.2.2}
$$

where $\Delta(t_0)=(\Delta_{ij}(t_0))_{i,j\in\sigma(J)}$ is a $k\times k$ Gaussian random matrix such that $\mathbb{E}\{\Delta(t_0)\}=0$ and its covariance matrix is independent of $x$. By the Laplace expansion of the determinant,

$$
\det(\Delta(t_0)-xI_k) = (-1)^k[x^k - S_1(\Delta(t_0))x^{k-1} + S_2(\Delta(t_0))x^{k-2} + \cdots + (-1)^k S_k(\Delta(t_0))],
$$

where $S_i(\Delta(t_0))$ is the sum of the $\binom{k}{i}$ principle minors of order $i$ in $\Delta(t_0)$. Taking the expectation above, we see that as $x\to\infty$,

$$
\mathbb{E}\{\det(\Delta(t_0)-xI_k)\} = (-1)^k x^k(1+o(1)).
$$

Combining this with (3.2.1) and (3.2.2) completes the proof. $\square$

Let $\tau(t) = \theta_t^2$, then $\forall i \in \sigma(J)$, $\tau_i(t_0) = 0$, since $t_0$ is a local maximum point of $\tau$ restricted on $J$. Assume additionally that the Hessian matrix

$$\Theta_J(t_0) := (\tau_{ij}(t_0))_{i,j\in\sigma(J)}$$

is negative definite, then the Hessian matrix of $1/(2\theta_t^2)$ at $t_0$ restricted on $J$,

$$\widetilde{\Theta}_J(t_0) = -\frac{1}{2\tau^2(t_0)}(\tau_{ij}(t_0))_{i,j\in\sigma(J)} = -\frac{1}{2\sigma_T^4}\Theta_J(t_0), \tag{3.2.3}$$

is positive definite. Let $g(t) = |\Lambda_J - \Lambda_J(t)|/\theta_t^{2k-1}$ and $h(t) = 1/(2\theta_t^2)$, applying Lemma 2.5.3 with $T$ replaced with $J$ gives us that as $u \to \infty$,

**Proposition 3.2.2** *Let $X$ be as in Theorem 3.1.9. Suppose that $\nu(t)$ attains its maximum 1 at a unique point $t_0 \in J \in \partial_k T$ such that $\nu_j(t_0) \neq 0$ for all $j \notin \sigma(J)$. If $\Theta_J(t_0)$ is negative definite, then as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \frac{2^{k/2}|-\Sigma_J(t_0)|}{|\Lambda_J(t_0)|^{1/2}|-\Theta_J(t_0)|^{1/2}}\Psi(u)(1+o(1)).$$

**Proof** Since $t_0$ is the only point attaining the maximum variance, and also, $\nu_j(t_0) \neq 0$ for all $j \notin \sigma(J)$, similarly to Example 2.4.2, we obtain that as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} + o(e^{-\alpha u^2 - u^2/2})$$

for some $\alpha > 0$. Note that

$$\begin{aligned}
&(-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} \\
&= (-1)^k \int_J p_{\nabla X_{|J}(t)}(0) \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{X(t) \geq u\}} | \nabla X_{|J}(t) = 0\} dt \\
&= \frac{(-1)^k}{(2\pi)^{k/2}} \int_T \frac{1}{|\Lambda_J(t)|^{1/2}} dt \int_u^\infty \mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\qquad\qquad \times p_{X(t)}(x | \nabla^2 X_{|J}(t) = 0) dx \\
&= \frac{(-1)^k}{(2\pi)^{(k+1)/2}} \int_u^\infty dx \int_T \frac{1}{\theta_t |\Lambda_J(t)|^{1/2}} \mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x, \nabla X_{|J}(t) = 0\} e^{-x^2/2\theta_t^2} dt.
\end{aligned}$$

Now we apply the Laplace method in Lemma 2.5.3 with

$$\left.\begin{aligned}
g(t) &= \frac{1}{\theta_t |\Lambda_J(t)|^{1/2}} \mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x, \nabla X_{|J}(t) = 0 \\
h(t) &= \frac{1}{2\theta_t^2}, \quad u = x^2,
\end{aligned}\right\}, \tag{3.2.4}$$

and obtain

$$\begin{aligned}
&(-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} \\
&= \frac{(-1)^k (2\pi)^{k/2}}{(2\pi)^{(k+1)/2} |\Lambda_J(t_0)|^{1/2} |\widetilde{\Theta}_J(t_0)|^{1/2}} \\
&\quad \times \int_u^\infty \mathbb{E}\{\det \nabla^2 X_{|J}(t_0) | X(t_0) = x, \nabla X_{|J}(t_0) = 0\} x^{-k} e^{-x^2/2} (1 + o(1)) dx \qquad (3.2.5) \\
&= \frac{(-1)^k (2\pi)^{k/2} |\Sigma_J(t_0)|}{(2\pi)^{k/2} |\Lambda_J(t_0)|^{1/2} |\widetilde{\Theta}_J(t_0)|^{1/2}} \Psi(u)(1 + o(1)) \\
&= \frac{2^{k/2} |-\Sigma_J(t_0)|}{|\Lambda_J(t_0)|^{1/2} |-\Theta_J(t_0)|^{1/2}} \Psi(u)(1 + o(1)),
\end{aligned}$$

where the second equality is due to Lemma 3.2.1 and the last line is due to (3.2.3). □

If $\dim(T) = 1$, then the result in Proposition 3.2.2 becomes much simpler as stated in the following.

**Corollary 3.2.3** *Let $T \subset \mathbb{R}$ and let $X$ be as in Theorem 3.1.9. Suppose $\nu(t)$ attains its maximum 1 at a unique interior point $t_0$, and additionally,*

$$\mathrm{Var}(X'(t_0)) + \mathbb{E}\{X(t_0)X''(t_0)\} \neq 0. \tag{3.2.6}$$

*Then as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) > u\Big\} = \left(\frac{\mathrm{Var}(X'(t_0))}{\mathbb{E}\{X(t_0)X''(t_0)\}} + 1\right)^{-1/2} \Psi(u)(1+o(1)).$$

**Proof** Note that, under our assumptions, $k = 1$,

$$\Sigma(t_0) = \mathbb{E}\{X(t_0)X''(t_0)\}, \quad \Lambda(t_0) = \mathrm{Var}(X'(t_0)).$$

Also, $\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X'(t))$ implies

$$\Theta(t_0) = \tau''(t_0) = -\frac{2\mathbb{E}\{X(t_0)X''(t_0)\}(\mathrm{Var}(X'(t_0)) + \mathbb{E}\{X(t_0)X''(t_0)\})}{\mathrm{Var}(X'(t_0))}.$$

Applying Proposition 3.2.2 gives

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} &= \frac{2^{1/2}|-\Sigma(t_0)|}{|\Lambda(t_0)|^{1/2}|-\Theta(t_0)|^{1/2}}\Psi(u)(1+o(1)) \\ &= \left(\frac{\mathbb{E}\{X(t_0)X''(t_0)\}}{\mathrm{Var}(X'(t_0)) + \mathbb{E}\{X(t_0)X''(t_0)\}}\right)^{1/2}\Psi(u)(1+o(1)),\end{aligned}$$

completing the proof. $\square$

## 3.3 Gaussian Fields on Manifolds without Boundary

In this section, we assume that $T$ is an $N$-dimensional smooth manifold without boundary ($\mathbb{S}^N$ for example). Let $\{\partial/\partial x^i\}_{1\le i\le N}$ be a natural coordinate vector field and let $X$ be a smooth function on $T$. Define

$$\nabla X = \Big(\frac{\partial X}{\partial x^1}, \dots, \frac{\partial X}{\partial x^N}\Big),$$

and let $(\frac{\partial^2 X}{\partial x^i \partial x^j}(t))$ be the abbreviation of the $N \times N$ matrix $(\frac{\partial^2 X}{\partial x^i \partial x^j}(t))_{i,j=1,\dots,N}$.

If $X$ is a Morse function, then according to Corollary 9.3.5 or page 211-212 in Adler and Taylor (2007), the Euler characteristic of the excursion set $A_u = \{t \in T : X(t) \ge u\}$ is given by

$$\varphi(A_u) = (-1)^N \sum_{k=0}^{N} (-1)^k \mu_k(T),$$

where

$$\mu_k(T) := \#\Big\{t \in T : X(t) \ge u, \nabla X(t) = 0, \text{index}\Big(\frac{\partial^2 X}{\partial x^i \partial x^j}(t)\Big) = k\Big\}.$$

We also define the number of local maxima above level $u$ as

$$M_u(T) := \#\Big\{t \in T : X(t) \ge u, \nabla X(t) = 0, \text{index}\Big(\frac{\partial^2 X}{\partial x^i \partial x^j}(t)\Big) = N\Big\}.$$

Since $T$ has no boundary, we have the following much simpler bounds for the excursion probability.

$$\mathbb{E}\{M_u(T)\} \ge \mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} \ge \mathbb{E}\{M_u(T)\} - \frac{1}{2}\mathbb{E}\{M_u(T)(M_u(T)-1)\}. \tag{3.3.1}$$

We assume again $\sup_{t\in T} \mathrm{Var}(X(t)) = 1$.

**Lemma 3.3.1** *Let $T$ be an oriented, compact $C^3$ manifold without boundary. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that $X \in C^3(T)$ a.s. and $(\mathbf{H}3)$ is fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{E}\{M_u(T)(M_u(T) - 1)\} = o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** Since $T$ is compact it has a finite atlas. Let $(U, \varphi)$ be one of its charts and consider

$$\overline{X} := X \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^N \to \mathbb{R}.$$

Then it follows immediately from the definition of $M_u$ that

$$M_u(X, U) \equiv M_u(\overline{X}, \varphi(U)).$$

Since $X \in C^3(M)$, the condition $(\mathbf{H}1)$ holds for $\overline{X}$. Applying Lemma 2.3.4 yields

$$\mathbb{E}\{M_u(\overline{X}, \varphi(U))(M_u(\overline{X}, \varphi(U)) - 1)\} = o(e^{-\alpha u^2 - u^2/2}).$$

This verifies the desired result. $\square$

**Lemma 3.3.2** *Let $T$ be an oriented, compact $C^3$ manifold without boundary. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that $X \in C^3(T)$ a.s. and $(\mathbf{H}3)$ is fulfilled. Then* *there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{E}\{M_u(T)\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** The result follows from Lemma 3.1.5 and the arguments in the proof of Lemma 3.3.1. $\square$

**Theorem 3.3.3** *Let $T$ be an oriented, compact $C^3$ manifold without boundary. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that $X \in C^3(T)$ a.s. and $(\mathbf{H}3)$ is fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}),$$

*where $\mathbb{E}\{\varphi(A_u)\}$ is formulated by*

$$(-1)^N \int_T \mathbb{E}\Big\{\det\Big(\frac{\partial^2 X}{\partial x^i \partial x^j}(t)\Big)\mathbb{1}_{\{X(t)\geq u\}}\Big|\nabla X(t) = 0\Big\}p_{\nabla X(t)}(0)\partial x^1 \wedge \cdots \wedge \partial x^N.$$

**Proof** The result follows immediately from the combination of (3.3.1), Lemma 3.3.1, Lemma 3.3.2 and the Kac-Rice metatheorem on manifolds [cf. Theorem 12.1.1 in Adler and Taylor (2007)]. $\square$

## 3.4 Gaussian Fields on Convex Sets with Smooth Boundary

Let $T$ be a compact, convex, $N$-dimensional subset of $\mathbb{R}^N$ with smooth boundary $\partial T$. Morse theorem gives

$$\varphi(A_u) = (-1)^N \sum_{k=0}^{N} (-1)^k \mu_k(\overset{\circ}{T}) + (-1)^{N-1} \sum_{k=0}^{N-1} (-1)^k \mu_k(\partial T),$$

where

$$\mu_k(\overset{\circ}{T}) = \#\{t \in \overset{\circ}{T} : X(t) \geq u, \nabla X(t) = 0, \mathrm{index}(\nabla^2 X(t)) = k\},$$

$$\mu_k(\partial T) = \#\{t \in \partial T : X(t) \geq u, \nabla X_{|\partial T}(t) = 0, \langle \nabla X(t), n(t) \rangle \geq 0, \mathrm{index}(\nabla^2 X_{|\partial T}(t)) = k\},$$

and $n(t)$ is the unit normal vector pointing outwards. We also define the number of extended outward local maxima above level $u$ as

$$M_u^E(\overset{\circ}{T}) = \#\{t \in \overset{\circ}{T} : X(t) \geq u, \nabla X(t) = 0, \mathrm{index}(\nabla^2 X(t)) = N\},$$

$$M_u^E(\partial T) = \#\{t \in \partial T : X(t) \geq u, \nabla X_{|\partial T}(t) = 0, \langle \nabla X(t), n(t) \rangle \geq 0,$$
$$\mathrm{index}(\nabla^2 X_{|\partial T}(t)) = N - 1\}.$$

Note that $T$ can be stratified into $\overset{\circ}{T} \cup \partial T$. We have the following bounds for the excursion probability.

$$\begin{aligned}
\mathbb{E}\{M_u^E(\mathring{T})\} + \mathbb{E}\{M_u^E(\partial T)\} &\geq \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\
&\geq \mathbb{E}\{M_u^E(\mathring{T})\} + \mathbb{E}\{M_u^E(\partial T)\} - \mathbb{E}\{M_u^E(\mathring{T})M_u^E(\partial T)\} \\
&\quad - \frac{1}{2}\mathbb{E}\{M_u^E(\mathring{T})(M_u^E(\mathring{T})-1)\} - \frac{1}{2}\mathbb{E}\{M_u^E(\partial T)(M_u^E(\partial T)-1)\}.
\end{aligned} \tag{3.4.1}$$

Since $\partial T$ is a hypersurface. We may consider $\partial T$ as an $(N-1)$-dimensional submanifold embedded on $\mathbb{R}^N$. Similarly, we have the following result.

**Lemma 3.4.1** *Let $T$ be a compact, convex, $N$-dimensional subset of $\mathbb{R}^N$ with smooth boundary $\partial T$. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that* (**H**1) *and* (**H**3) *are fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}
\mathbb{E}\{M_u^E(\mathring{T})(M_u^E(\mathring{T})-1)\} &= o(e^{-\alpha u^2 - u^2/2}), \\
\mathbb{E}\{M_u^E(\partial T)(M_u^E(\partial T)-1)\} &= o(e^{-\alpha u^2 - u^2/2}), \\
\mathbb{E}\{M_u^E(\mathring{T})\} + \mathbb{E}\{M_u^E(\partial T)\} &= \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).
\end{aligned}$$

The next lemma shows that the crossing term is also super-exponentially small.

**Lemma 3.4.2** *Let $T$ be a compact, convex, $N$-dimensional subset of $\mathbb{R}^N$ with smooth boundary $\partial T$. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that* (**H**1) *and* (**H**3) *are fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{E}\{M_u^E(\mathring{T})M_u^E(\partial T)\} = o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** By the Kac-Rice metatheorem,

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(\overset{\circ}{T})M_u^E(\partial T)\}\\
&\quad\le \int_{\overset{\circ}{T}} dt \int_{\partial T} ds \int_u^{\infty} dx \int_u^{\infty} dy\ \mathbb{E}\{|\mathrm{det}\nabla^2 X(t)||\mathrm{det}\nabla^2 X_{|\partial T}(s)||X(t)=x,\\
&\qquad\quad X(s)=y, \nabla X(t)=0, \nabla X_{|\partial T}(s)=0\}p_{X(t),X(s),\nabla X(t),\nabla X_{|\partial T}(s)}(x,y,0,0).
\end{aligned}
$$

By similar arguments in Theorem 2.3.7, if $\nabla X(s)\neq 0$ for all $s\in\partial T$ such that $\nu(s)=1$, then $\mathbb{E}\{M_u^E(\overset{\circ}{T})M_u^E(\partial T)\}$ is super-exponentially small. Hence we will consider the alternative case when

$$
I_0 := \{s\in\partial T : \nu(s)=1, \nabla X(s)=0\}\neq\emptyset.
$$

Let $B(I_0,\delta)=\{t\in T : d(t,I_0)<\delta\}$, where $\delta$ is a small positive number to be specified. As discussed in Lemma 3.1.8, $\mathbb{E}\{M_u^E(\overset{\circ}{T})M_u^E(\partial T)\}$ can be reduced to $\mathbb{E}\{M_u^E(\overset{\circ}{T}\cap B(s_0,\delta))M_u^E(\partial T\cap B(I_0,\delta))\}$ with only a super-exponentially small difference. Due to the compactness of $T$, it suffice to show that $\mathbb{E}\{M_u^E(\overset{\circ}{T}\cap B(s_0,\delta))M_u^E(\partial T\cap B(s_0,\delta))\}$ is super-exponentially small for some $s_0\in I_0$ and $\delta>0$, where

$$
B(s_0,\delta)=\{t\in T : d(t,s_0)<\delta\}.
$$

Notice another fact that for all $s\in\partial T$ such that $\nu(s)=1$, $\nabla^2\nu(s)$ are negative semidefinite and hence $\Sigma(s)=\mathbb{E}\{X(s)\nabla^2 X(s)\}$ are negative definite. Therefore by continuity, we may choose $\delta$ small enough such that

$$
\langle -\Sigma(t)e_{t,s}, e_{t,s}\rangle \ge \alpha_0, \quad \forall t\in\overset{\circ}{T}\cap B(s_0,\delta), s\in\partial T\cap B(s_0,\delta), \tag{3.4.2}
$$

for some positive constant $\alpha_0$, where $e_{t,s} = (s-t)/\|s-t\|$.

For $s \in \partial T \cap B(s_0, \delta)$, let $Z(s) = \langle \nabla X(s), n(s) \rangle$, and denote by $\Pi_s$ the projection onto the tangent space of $\partial T$ at $s$, so that $\nabla X_{|\partial T}(s) = \Pi_s \nabla X(s)$. Then

$$\begin{aligned}
&\mathbb{E}\{M_u^E(\overset{\circ}{T} \cap B(s_0,\delta)) M_u^E(\partial T \cap B(s_0,\delta))\} \\
&= \int_{\overset{\circ}{T} \cap B(s_0,\delta))} dt \int_{\partial T \cap B(s_0,\delta)} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz \; \mathbb{E}\{|\det \nabla^2 X(t)||\det \nabla^2 X_{|\partial T}(s)| \\
&\qquad |X(t) = x, X(s) = y, \nabla X(t) = 0, \nabla X_{|\partial T}(s) = 0, Z(s) = z\} \\
&\quad \times p_{X(t), X(s), \nabla X(t), \nabla X_{|\partial T}(s), Z(s)}(x, y, 0, 0, z) \\
&\le \int_{\overset{\circ}{T} \cap B(s_0,\delta))} dt \int_{\partial T \cap B(s_0,\delta)} ds \int_u^\infty dx \int_0^\infty dz \; \mathbb{E}\{|\det \nabla^2 X(t)||\det \nabla^2 X_{|\partial T}(s)||X(t) = x, \\
&\qquad \nabla X(t) = 0, \nabla X_{|\partial T}(s) = 0, Z(s) = z\} p_{X(t), \nabla X(t), \nabla X_{|\partial T}(s), Z(s)}(x, 0, 0, z) \\
&:= \int\int_{(\overset{\circ}{T} \cap B(s_0,\delta)) \times (\partial T \cap B(s_0,\delta))} A(t,s) dt ds.
\end{aligned} \tag{3.4.3}$$

We can bound the integral in (3.4.3) as the following.

$$\int\int_{(\overset{\circ}{T} \cap B(s_0,\delta)) \times (\partial T \cap B(s_0,\delta))} A(t,s) dt ds \le \int\int_{D_1} A(t,s) dt ds + \int\int_{D_2} A(t,s) dt ds, \tag{3.4.4}$$

where

$$\begin{aligned}
D_1 &= \{t \in \overset{\circ}{T} \cap B(s_0,\delta), s \in \partial T \cap B(s_0,\delta) : \langle -\Sigma(t) e_{t,s}, n(s) \rangle \ge b_1\}, \\
D_2 &= \{t \in \overset{\circ}{T} \cap B(s_0,\delta), s \in \partial T \cap B(s_0,\delta) : \sum_{i=1}^{N-1} \langle -\Sigma(t) e_{t,s}, E_i(s) \rangle \langle e_{t,s}, E_i(s) \rangle \ge b_2\},
\end{aligned}$$

$b_1$ and $b_2$ are positive numbers such that $b_1 + b_2 < \alpha_0$, $\{E_1(s), \cdots, E_{N-1}(s)\}$ is the orthonormal basis of the tangent space of $\partial T$ at $s$. This is because, if $(t,s)$ does not belong to

$D_1$ nor $D_2$, then

$$
\begin{aligned}
&\langle -\Sigma(t)e_{t,s}, e_{t,s}\rangle \\
&= \langle -\Sigma(t)e_{t,s}, n(s)\rangle\langle e_{t,s}, n(s)\rangle + \sum_{i=1}^{N-1}\langle -\Sigma(t)e_{t,s}, E_i(s)\rangle\langle e_{t,s}, E_i(s)\rangle \\
&< b_1 + b_2 < \alpha_0,
\end{aligned}
$$

where we use the fact that the convexity of $T$ implies $\langle e_{t,s}, n(s)\rangle \geq 0$. But this conflicts (3.4.2), hence $D_1 \cup D_2$ is a covering of $(\overset{\circ}{T} \cap B(s_0,\delta)) \times (\partial T \cap B(s_0,\delta))$.

We first show that $\int\int_{D_2} A(t,s)dtds$ is super-exponentially small. By similar arguments in the proof for Gaussian fields on rectangle, we see that there exists positive constants $C_1$, $C_2$ and $N_1$ such that

$$
\begin{aligned}
&\mathbb{E}\{|\mathrm{det}\nabla^2 X(t)||\mathrm{det}\nabla^2 X_{|\partial T}(s)||X(t)=x, \nabla X(t)=0, \nabla X_{|\partial T}(s)=0\} \leq C_1(x^{N_1}+1), \\
&\mathrm{detCov}(\nabla X(t), \nabla X_{|\partial T}(s)) \geq C_2\|s-t\|^{2(N-1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A(t,s) \leq &\int_u^\infty \mathbb{E}\{|\mathrm{det}\nabla^2 X(t)||\mathrm{det}\nabla^2 X_{|\partial T}(s)||X(t)=x, \nabla X(t)=0, \nabla X_{|\partial T}(s)=0\} \\
&\times p_{X(t)}(x|\nabla X(t)=0, \nabla X_{|\partial T}(s)=0)p_{\nabla X(t), \nabla X_{|\partial T}(s)}(0,0)dx \qquad (3.4.5) \\
\leq &\, C_3\|s-t\|^{1-N}\int_u^\infty (1+x^{N_1})p_{X(t)}(x|\nabla X(t)=0, \nabla X_{|\partial T}(s)=0)dx
\end{aligned}
$$

for some positive constant $C_3$.

On the other hand, as $\|s-t\| \to 0$,

$$\begin{aligned}
&\mathrm{Var}(X(t)|\nabla X(t), \nabla X_{|\partial T}(s)) = \mathrm{Var}(X(t)|\nabla X(t), \Pi_s \nabla X(s)) \\
&\quad = \mathrm{Var}(X(t)|\nabla X(t), \Pi_s(\nabla X(s) - \nabla X(t))/\|s-t\|) \\
&\quad = \mathrm{Var}(X(t)|\nabla X(t), \Pi_s(\nabla^2 X(t) e_{t,s})) + o(1) \\
&\quad \le \mathrm{Var}(X(t)|\Pi_s(\nabla^2 X(t) e_{t,s})) + o(1) \\
&\quad \le 1 - (\Pi_s(\Sigma(t)e_{t,s}))[\mathrm{Cov}(\Pi_s(\nabla^2 X(t) e_{t,s}))]^{-1}(\Pi_s(\Sigma(t)e_{t,s}))^T + o(1),
\end{aligned}$$

where the third equality is due to Taylor's formula. Note that $\mathrm{Cov}(\Pi_s(\nabla^2 X(t)e_{t,s}))$ is bounded away from 0 because of the regularity condition (**H**3). Also, by the definition of $D_2$, the vectors $\Pi_s(\Sigma(t)e_{t,s})$ are not vanishing for all $(t,s) \in D_2$, thus there exists a constant $\varepsilon_1 > 0$ such that

$$\mathrm{Var}(X(t)|\nabla X(t), \nabla X_{|\partial T}(s)) \le 1 - \varepsilon_1, \quad \forall (t,s) \in D_2.$$

Combining this with (3.4.5), and noting that $\|s-t\|^{1-N}$ is integrable on $(\overset{\circ}{T} \cap B(s_0,\delta))) \times (\partial T \cap B(s_0,\delta))$, we conclude that $\int\int_{D_2} A(t,s)dtds$ is super-exponentially small.

Now we turn to estimating $\int\int_{D_1} A(t,s)dtds$. For $(t,s) \in D_1$, we have

$$\begin{aligned}
&p_{X(t),\nabla X(t),\nabla X_{|\partial T}(s),Z(s)}(x,0,0,z) \\
&\quad = p_{\Pi_s(\nabla X(t)),\nabla X_{|\partial T}(s)}(0,0|X(t)=x, Z(s)=z, \langle \nabla X(t), n(s)\rangle = 0) \\
&\qquad \times p_{X(t),Z(s)}(x,z|\langle \nabla X(t), n(s)\rangle = 0) p_{\langle \nabla X(t), n(s)\rangle}(0) \\
&\quad \le C_4 (\det\mathrm{Cov}(X(t),\nabla X(t),\nabla X_{|\partial T}(s), Z(s)))^{-1/2} \\
&\qquad \times \exp\bigg\{ -\frac{1}{2(1-\rho(t,s)^2)}\bigg(\frac{x^2}{\sigma_1^2(t,s)} + \frac{z^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xz}{\sigma_1(t,s)\sigma_2(t,s)}\bigg)\bigg\}
\end{aligned} \tag{3.4.6}$$

for some positive constant $C_4$, where

$$\sigma_1^2(t,s) = \mathrm{Var}(X(t)|\langle \nabla X(t), n(s)\rangle) = \frac{\det\mathrm{Cov}(X(t), \langle \nabla X(t), n(s)\rangle)}{\mathrm{Var}(\langle \nabla X(t), n(s)\rangle)},$$
$$\sigma_2^2(t,s) = \mathrm{Var}(Z(s)|\langle \nabla X(t), n(s)\rangle) = \frac{\det\mathrm{Cov}(Z(s), \langle \nabla X(t), n(s)\rangle)}{\mathrm{Var}(\langle \nabla X(t), n(s)\rangle)},$$
$$\rho(t,s) = \frac{\mathbb{E}\{X(t)Z(s)|\langle \nabla X(t), n(s)\rangle = 0\}}{\sigma_1(t,s)\sigma_2(t,s)}.$$

Recall $Z(s) = \langle \nabla X(s), n(s)\rangle$, similarly to the rectangle case, one can check that there exits positive constants $C_5$ and $C_6$ such that

$$C_5\|s-t\|^2 \le \sigma_2^2(t,s) \le C_6\|s-t\|^2. \tag{3.4.7}$$

Applying Taylor formula, we obtain

$$\begin{aligned}
\rho(t,s) &= \frac{1}{\sigma_1(t,s)\sigma_2(t,s)}\bigg(\mathbb{E}\{X(t)Z(s)\} - \frac{\mathbb{E}\{X(t)\langle \nabla X(t), n(s)\rangle\}\mathbb{E}\{Z(s)\langle \nabla X(t), n(s)\rangle\}}{\mathrm{Var}(\langle \nabla X(t), n(s)\rangle)}\bigg)\\
&= \frac{1}{\sigma_1(t,s)\sigma_2(t,s)}\bigg(\mathbb{E}\{X(t)\langle \nabla X(t) + \nabla^2 X(t)(s-t) + \|s-t\|^{1+\eta}Y_{t,s}, n(s)\rangle\}\\
&\quad - \frac{\mathbb{E}\{X(t)\langle \nabla X(t), n(s)\rangle\}}{\mathrm{Var}(\langle \nabla X(t), n(s)\rangle)}\mathbb{E}\{\langle \nabla X(t), n(s)\rangle\langle \nabla X(t) + \nabla^2 X(t)(s-t)\\
&\quad + \|s-t\|^{1+\eta}Y_{t,s}, n(s)\rangle\}\bigg)\\
&= \frac{\|s-t\|}{\sigma_1(t,s)\sigma_2(t,s)}\bigg(\mathbb{E}\{X(t)\langle \nabla^2 X(t)e_{t,s} + \|s-t\|^{\eta}Y_{t,s}, n(s)\rangle\}\\
&\quad - \frac{\mathbb{E}\{X(t)\langle \nabla X(t), n(s)\rangle\}}{\mathrm{Var}(\langle \nabla X(t), n(s)\rangle)}\mathbb{E}\{\langle \nabla X(t), n(s)\rangle\langle \nabla^2 X(t)e_{t,s} + \|s-t\|^{\eta}Y_{t,s}, n(s)\rangle\}\bigg).
\end{aligned}$$

By our assumption, $\mathbb{E}\{X(s_0)\nabla X(s_0)\}$=0, therefore if $\delta$ is sufficiently small, $\mathbb{E}\{X(t)\nabla X(t)\}$

gets close to 0 for $t \in \overset{\circ}{T} \cap B(s_0, \delta)$. Thus, as $\|s-t\| \to 0$,

$$
\begin{aligned}
\rho(t,s) &= \frac{\|s-t\|}{\sigma_1(t,s)\sigma_2(t,s)}(\langle \Sigma(t) e_{t,s}, n(s)\rangle - o(1)) \\
&\le \frac{\|s-t\|}{\sigma_1(t,s)(C_5\|s-t\|^2)^{1/2}}(-b_1 - o(1)) \\
&< -\varepsilon_2
\end{aligned}
\tag{3.4.8}
$$

for some positive constant $\varepsilon_2$, where the second inequality comes from (3.4.7) and the definition of $D_1$.

By similar arguments in the proof for Gaussian fields on rectangle, we see that there exists positive constants $C_7$, $C_8$, $N_2$ and $N_3$ such that

$$
\begin{aligned}
&\mathbb{E}\{|\mathrm{det}\nabla^2 X(t)||\mathrm{det}\nabla^2 X_{|\partial T}(s)||X(t)=x, \nabla X(t)=0, \nabla X_{|\partial T}(s)=0, Z(s)=z\} \\
&\le C_7(x^{N_2} + (z/\|s-t\|)^{N_3} + 1)
\end{aligned}
$$

and

$$
\mathrm{detCov}(X(t), \nabla X(t), \nabla X_{|\partial T}(s), Z(s)) \ge C_8 \|s-t\|^{2N}.
$$

Combining this with (3.4.6), and making change of variable $\tilde{z} = z/\|s-t\|$ and $\tilde{\sigma}_2(t,s) = \sigma_2(t,s)/\|s-t\|$, we obtain

$$
\begin{aligned}
A(t,s) &\le C_9\|s-t\|^{-N}\int_u^\infty dx \int_0^\infty dz\ (x^{N_2} + (z/\|s-t\|)^{N_3} + 1) \\
&\quad \times \exp\left\{-\frac{1}{2(1-\rho(t,s)^2)}\left(\frac{x^2}{\sigma_1^2(t,s)} + \frac{z^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xz}{\sigma_1(t,s)\sigma_2(t,s)}\right)\right\} \\
&\le C_9\|s-t\|^{1-N}\int_u^\infty dx \int_0^\infty d\tilde{z}\ (x^{N_2} + \tilde{z}^{N_3} + 1) \\
&\quad \times \exp\left\{-\frac{1}{2(1-\rho(t,s)^2)}\left(\frac{x^2}{\sigma_1^2(t,s)} + \frac{\tilde{z}^2}{\tilde{\sigma}_2^2(t,s)} - \frac{2\rho(t,s)x\tilde{z}}{\sigma_1(t,s)\tilde{\sigma}_2(t,s)}\right)\right\}
\end{aligned}
\tag{3.4.9}
$$

for some positive constant $C_9$. Applying Lemma 2.3.10 yields that $\int\int_{D_1} A(t,s)dtds$ is super-exponentially small. $\square$

**Theorem 3.4.3** *Let $T$ be a compact, convex, $N$-dimensional subset of $\mathbb{R}^N$ with smooth boundary $\partial T$. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that* ($\mathbf{H}1$) *and* ($\mathbf{H}3$) *are fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** The result follows immediately from applying (3.4.1), Lemma 3.4.1 and Lemma 3.4.2. $\square$

## 3.5 Gaussian Fields on Convex Sets with Piecewise Smooth Boundary

Let $T$ be an $N$-dimensional compact and convex set in $\mathbb{R}^N$. Suppose it has piecewise smooth boundary and can be stratified as $T = \cup_{i=0}^N \partial_i T$, where $\partial_i T$ is the *$i$-dimensional boundary* of $T$ made up of the disjoint union of a finite number of $i$-dimensional manifolds without boundary.

Define the *support cone* [cf. Adler and Taylor (2007, p.188)] of $T$ at $t$ as

$$\begin{aligned} \mathcal{S}_t T := \{\xi \in \mathbb{R}^N : \exists \delta > 0, C^1 \text{ curve } \gamma : (-\delta, \delta) \to \mathbb{R}^d \text{ such that} \\ \gamma(0) = t, \nabla\gamma(0) = \xi, \gamma(s) \in T \text{ for all } s \in [0,\delta)\}. \end{aligned}$$

It is easy to check that $\mathcal{S}_t T$ contains the tangent space of $t$.

Define the *normal cone* [cf. Adler and Taylor (2007, p.189)] of $T$ at $t$ as

$$\mathcal{N}_t T := \{z \in \mathbb{R}^N : \langle z, \xi \rangle \leq 0 \text{ for all } \xi \in \mathcal{S}_t T\}.$$

So $t \in T$ is called *extended outward critical point* if $\nabla X(t) \in \mathcal{N}_t T$.

Morse theorem gives

$$\varphi(A_u) = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J),$$

where

$$\begin{aligned}\mu_i(J) &:= \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = i, \nabla X(t) \in \mathcal{N}_t T\} \\ &= \#\{t \in J : X(t) \geq u, \nabla X(t) \in \mathcal{N}_t T, \mathrm{index}(\nabla^2 X_{|J}(t)) = i\},\end{aligned}$$

and the last line above is due to the fact that $\nabla X(t) \in \mathcal{N}_t T$ implies $\nabla X_{|J}(t) = 0$ for all $t \in J$.

We will need a modified version of $(\mathbf{H}4)$, say $(\mathbf{H}4')$, as the following.

$(\mathbf{H}4')$. $\forall t \in J \in \partial_k T$ such that $\nu(t) = 1$ and $0 \leq k \leq N-2$, $(\mathbb{E}\{X(t)\nabla^2 X(t)\})|_L$ is negative definite, where $L$ is the largest subspace of $\mathbb{R}^N$ such that $(\nabla \nu(t))|_L = 0$.

Here, $(\mathbb{E}\{X(t)\nabla^2 X(t)\})|_L$ and $(\nabla \nu(t))|_L$ are the projections of $\mathbb{E}\{X(t)\nabla^2 X(t)\}$ and $\nabla \nu(t)$ onto the subspace $L$, respectively.

Similar to Proposition 3.1.1, we have the following result.

**Proposition 3.5.1** *Let $X(t) \in C^2(\mathbb{R}^N)$ a.s. and let $L$ be the largest subspace of $\mathbb{R}^N$ such that $(\nabla \nu(t))|_L = 0$. If $(\nabla^2 \nu(t))|_L$ is negative semidefinite for each $t \in J \in \partial_k T$ such that $\nu(t) = 1$ and $0 \leq k \leq N-2$, then $(\mathbf{H}4')$ holds.*

For $J \in \partial_k T$, we also define the number of extended outward maxima above level $u$ as

$$\begin{aligned} M_u^E(J) &:= \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = k, \nabla X(t) \in \mathcal{N}_t T\} \\ &= \#\{t \in J : X(t) \geq u, \nabla X(t) \in \mathcal{N}_t T, \text{index}(\nabla^2 X_{|J}(t)) = k\}. \end{aligned}$$

Then similarly to Lemma 2.3.1, one has

$$\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\} \text{ a.s.}$$

Thus by similar discussions, we get

$$\begin{aligned} \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} &\geq \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\ &\geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \left(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}\right) \\ &\quad - \sum_{J\neq J'} \mathbb{E}\{M_u^E(J)M_u^E(J')\}. \end{aligned} \tag{3.5.1}$$

**Theorem 3.5.2** *Let $T$ be a compact, convex, $N$-dimensional subset of $\mathbb{R}^N$ with piecewise smooth boundary. Let $\{X(t) : t \in T\}$ be a Gaussian random field such that $(\mathbf{H}1)$, $(\mathbf{H}3)$ and $(\mathbf{H}4')$ are fulfilled. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).$$

**Proof** Similar to the arguments in proving the smooth boundary case, we only need to show that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small for neighboring faces $J \in \partial_k T$ and $J' \in \partial_{k'} T$. Moreover, similarly, it suffices to show that $\mathbb{E}\{M_u^E(J\cap B(s_0,\delta))M_u^E(J'\cap B(s_0,\delta))$

is super-exponentially small for $s_0 \in I = \bar{J} \cap \bar{J}'$ such that $\nu(s_0) = 1$ and $(\nabla \nu(s_0))|_L = 0$, where $L = \mathrm{span}\{\mathcal{S}_{s_0}\bar{J} \cup \mathcal{S}_{s_0}\bar{J}'\}$, $B(s_0, \delta) = \{t \in T : d(t, s_0) < \delta\}$ and $\delta$ is a small positive number to be specified. Without loss of generality, we assume $k \geq k'$ and $k \geq 1$.

Denote by $\{E_1, \cdots, E_m\}$ the normal basis on $I$. Since $J$ is a $k$-dimensional face, there are $(k-m)$ many $(m+1)$-dimensional faces which belong to the closure of $J$ and are adjacent to $I$ as well. Denote these $(m+1)$-dimensional faces by $\{J_{m+1}, \cdots, J_k\} \subset \partial_{m+1}T$. Now, for each $i = m+1, \ldots, k$, we may view $I$ as an $m$-dimensional boundary of $J_i$, and let $E_i(s_0)$ be the unit normal vector pointing outwards at $s_0$, i.e., $E_i(s_0) \in \mathcal{N}_{s_0}\bar{J}_i$. In such way, we have a smooth frame (not necessary orthogonal) $\{E_1(t), \cdots, E_m(t), E_{m+1}(t), \cdots, E_k(t)\}$ on $\bar{J} \cap B(s_0, \delta)$.

Similarly, since $J'$ is a $k'$-dimensional face, there are $(k'-m)$ many $(m+1)$-dimensional faces which belong to the closure of $J'$ and are adjacent to $I$ as well. Denote these $(m+1)$-dimensional faces by $\{J'_{m+1}, \cdots, J'_{k'}\} \subset \partial_{m+1}T$. Now, for each $i = m+1, \ldots, k'$, we may view $I$ as an $m$-dimensional boundary of $J'_i$, and let $E'_i(s_0)$ be the unit normal vector pointing outwards at $s_0$, i.e., $E'_i(s_0) \in \mathcal{N}_{s_0}\bar{J}'_i$. In such way, we have a smooth frame (not necessary orthogonal) $\{E_1(s), \cdots, E_m(s), E'_{m+1}(s), \cdots, E'_{k'}(s)\}$ on $\bar{J}' \cap B(s_0, \delta)$.

By ($\mathbf{H}4'$), $\Sigma_L(t) := (\mathbb{E}\{X(t)\nabla^2 X(t)\})_L$ is negative definite at $t = s_0$. If $\delta$ is small enough, by continuity,

$$\langle -\Sigma_L(t)e_{t,s}, e_{t,s}\rangle \geq \alpha_0, \quad \forall t \in J \cap B(s_0, \delta), s \in J' \cap B(s_0, \delta), \tag{3.5.2}$$

for some positive constant $\alpha_0$, where $e_{t,s}$ is the projection of $(s-t)/\|s-t\|$ on $L$ (in fact, the projection only removes the vanishing components of $(s-t)/\|s-t\|$ such that the number of components in $e_{t,s}$ is the same as $\dim(L)$). For $(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$, let

$$\alpha_i(t,s) = \langle -\Sigma_L(t)e_{t,s}, E_i(t)\rangle, \quad i = 1,\ldots,m,m+1,\ldots,k,$$
$$\alpha'_j(t,s) = \langle -\Sigma_L(t)e_{t,s}, E'_j(s)\rangle, \quad j = m+1,\ldots,k'.$$

We claim that there exist positive constants $b_0$, $b_{m+1}, \cdots, b_k$, $b'_{m+1}, \cdots, b'_{k'}$, whose property needs to be specified later, such that $D_0 \cup (\cup_{i=m+1}^{k} D_i) \cup (\cup_{j=m+1}^{k'} D'_j)$ is a covering of $(J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$, where

$$\begin{aligned} D_i &= \{(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta)) : \alpha_i(t,s) \ge b_i\} \quad \text{if } m+1 \le i \le k; \\ D'_j &= \{(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta)) : \alpha'_j(t,s) \le -b'_j\} \quad \text{if } m+1 \le j \le k'; \\ D_0 &= \left\{(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta)) : \sum_{i=1}^{m} |\alpha_i(t,s)|^2 \ge b_0\right\}. \end{aligned} \tag{3.5.3}$$

Note that as $\delta$ gets smaller, $(J \cap B(s_0,\delta)) \cup (J' \cap B(s_0,\delta))$ becomes more similar to two intersecting flat faces, and $e_{t,s}$ will be around the convex cone created by vectors

$$\{\pm E_1(t),\ldots,\pm E_m(t), E_{m+1}(t),\ldots,E_k(t), -E'_{m+1}(s),\ldots,-E'_{k'}(s)\}.$$

Hence due to the convexity of $T$, there exists $\varepsilon_0 > 0$, whose property needs to be specified later, such that for all $(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$ and $\delta$ sufficiently small, the following representation holds:

$$e_{t,s} = \sum_{i=1}^{k} \langle e_{t,s}, E_i(t)\rangle E_i(t) + \sum_{j=m+1}^{k'} \langle e_{t,s}, E'_j(s)\rangle E'_j(s), \tag{3.5.4}$$

where $\langle e_{t,s}, E_i(t)\rangle \in \mathbb{R}$ for $i = 1, \dots, m$, and

$$\langle e_{t,s}, E_i(t)\rangle \geq -\varepsilon_0, \quad \forall i = m+1, \dots, k,$$
$$\langle e_{t,s}, E'_j(s)\rangle \leq \varepsilon_0, \quad \forall j = m+1, \dots, k'.$$

By continuity, there exists a universal positive constant $r$ such that

$$\sup_{(t,s)\in J\times J'} \max\{|\alpha_i(t,s)|, |\alpha'_j(t,s)|\} \leq r,$$

for all $i = m+1, \dots, k$ and $j = m+1, \dots, k'$.

If $(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$ does not belong to any of sets in (3.5.3), then by (3.5.4),

$$\begin{aligned}\langle -\Sigma_L(t)e_{t,s}, e_{t,s}\rangle &= \sum_{i=1}^{m} \alpha_i(t,s)\langle e_{t,s}, E_i(t)\rangle + \sum_{i=m+1}^{k} \alpha_i(t,s)\langle e_{t,s}, E_i(t)\rangle \\ &\quad + \sum_{j=m+1}^{k'} \alpha'_j(t,s)\langle e_{t,s}, E'_j(s)\rangle \\ &\leq b_0 + \sum_{i=m+1}^{k} \max\{b_i,\, r\varepsilon_0\} + \sum_{j=m+1}^{k'} \max\{b'_j,\, r\varepsilon_0\},\end{aligned} \tag{3.5.5}$$

where the last inequality we use the facts that for $i = m+1, \dots, k$, if $(t,s) \notin D_i$ and $\langle e_{t,s}, E_i(t)\rangle \geq 0$, then $\alpha_i(t,s)\langle e_{t,s}, E_i(t)\rangle \leq b_i$; and for $j = m+1, \dots, k'$, if $(t,s) \notin D'_j$ and $\langle e_{t,s}, E'_j(s)\rangle \leq 0$, then $\alpha'_j(t,s)\langle e_{t,s}, E'_j(s)\rangle \leq b'_j$.

Now, we choose the positive constants $b_0, b_{m+1}, \cdots, b_k, b'_{m+1}, \cdots, b'_{k'}$ and $\varepsilon_0$ such that the last line of (3.5.5) is strictly less than $\alpha_0$. Then $\langle -\Sigma_L(t)e_{t,s}, e_{t,s}\rangle < \alpha_0$ conflicts (3.5.2). This verifies our claim that $D_0 \cup (\cup_{i=m+1}^{k} D_i) \cup (\cup_{j=m+1}^{k'} D'_j)$ is a covering of $(J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$.

Due to the convexity of $T$, if $\delta$ is small enough, then for each $(t,s) \in (J \cap B(s_0,\delta)) \times (J' \cap B(s_0,\delta))$,

$$\nabla X(t) \in \mathcal{N}_t T \Leftrightarrow \nabla X_{|J}(t) = 0 \text{ and } \langle \nabla X(t), E_j'(s)\rangle \geq 0, \quad \forall j = m+1, \ldots, k';$$

and for each $s \in J' \cap B(s_0,\delta)$,

$$\nabla X(s) \in \mathcal{N}_s T \Leftrightarrow \nabla X_{|J'}(s) = 0 \text{ and } \langle \nabla X(s), E_i(t)\rangle \geq 0, \quad \forall i = m+1, \ldots, k.$$

By the Kac-Rice metatheorem,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(J \cap B(s_0,\delta)) M_u^E(J' \cap B(s_0,\delta))\\
&\leq \int_{J\cap B(s_0,\delta)} dt \int_{J'\cap B(s_0,\delta)} ds \int_u^\infty dx \int_u^\infty dy\\
&\quad \int_0^\infty dz_{m+1} \cdots \int_0^\infty dz_{k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k\\
&\quad \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y,\\
&\qquad \nabla X_{|J}(t) = 0, \langle \nabla X(t), E_{m+1}'(s)\rangle = z_{m+1}, \cdots, \langle \nabla X(t), E_{k'}'(s)\rangle = z_{k'},\\
&\qquad \nabla X_{|J'}(s) = 0, \langle \nabla X(s), E_{m+1}(t)\rangle = w_{m+1}, \cdots, \langle \nabla X(s), E_k(t)\rangle = w_k\}\\
&\qquad \times p_{t,s}(x, y, 0, z_{m+1}, \cdots, z_{k'}, 0, w_{m+1}, \cdots, w_k)\\
&:= \int\int_{(J\cap B(s_0,\delta))\times(J'\cap B(s_0,\delta))} A(t,s)\, dtds,
\end{aligned}$$

where $p_{t,s}$ is the density of

$$\begin{aligned}
&(X(t), X(s), \nabla X_{|J}(t), \langle \nabla X(t), E_{m+1}'(s)\rangle, \cdots, \langle \nabla X(t), E_{k'}'(s)\rangle,\\
&\nabla X_{|J'}(s), \langle \nabla X(s), E_{m+1}(t)\rangle, \cdots, \langle \nabla X(s), E_k(t)\rangle).
\end{aligned}$$

Moreover, due to the covering discussed before, this integral can be bounded as

$$
\begin{aligned}
&\int\int_{(J\cap B(s_0,\delta))\times(J'\cap B(s_0,\delta))} A(t,s)\,dtds \\
&\quad\le \sum_{i=m+1}^{k}\int\int_{D_i} A(t,s)\,dtds + \sum_{j=m+1}^{k'}\int\int_{D'_j} A(t,s)\,dtds + \int\int_{D_0} A(t,s)\,dtds.
\end{aligned}
$$

We first show that $\int\int_{D_0} A(t,s)\,dtds$ is super-exponentially small. By similar arguments in the proof for Gaussian fields on rectangle, we see that there exist positive constants $C_1$, $C_2$ and $N_1$ such that

$$
\begin{aligned}
&\mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} \le C_1(x^{N_1}+1),\\
&\mathrm{detCov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)) \ge C_2\|s-t\|^{2m}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A(t,s) &\le \int_u^\infty \mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\}\\
&\qquad \times p_{X(t)}(x|\nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0)p_{\nabla X_{|J}(t), \nabla X_{|J'}(s)}(0,0)dx\\
&\le C_3\|s-t\|^m \int_u^\infty (1+x^{N_1})p_{X(t)}(x|\nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0)dx
\end{aligned}
\tag{3.5.6}
$$

for some positive constant $C_3$.

On the other hand, let $\Pi_t$ be the projection onto span$\{E_1(t), \cdots, E_m(t)\}$, then as $\delta \to 0$,

$$
\begin{aligned}
&\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)) \le \mathrm{Var}(X(t)|\Pi_t \nabla X_{|J}(t), \Pi_t \nabla X_{|J'}(s)) \\
&\quad = \mathrm{Var}(X(t)|\Pi_t \nabla X(t), \Pi_t \nabla X(s)) + o(1) \\
&\quad = \mathrm{Var}(X(t)|\Pi_t \nabla X(t), \Pi_t(\nabla X(s) - \nabla X(t))/\|s-t\|) + o(1) \\
&\quad = \mathrm{Var}(X(t)|\Pi_t \nabla X(t), \Pi_t(\nabla^2 X(t) e_{t,s})) + o(1) \\
&\quad \le \mathrm{Var}(X(t)|\Pi_t(\nabla^2 X(t) e_{t,s})) + o(1) \\
&\quad \le 1 - (\Pi_t(\Sigma_L(t) e_{t,s}))[\mathrm{Cov}(\Pi_t(\nabla^2 X(t) e_{t,s}))]^{-1}(\Pi_t(\Sigma_L(t) e_{t,s}))^T + o(1),
\end{aligned}
$$

where the third equality is due to Taylor's formula. Note that $\mathrm{Cov}(\Pi_t(\nabla^2 X(t) e_{t,s}))$ is bounded away from 0 because of the regularity condition (**H**3). Also, by the definition of $D_0$, the vectors $\Pi_t(\Sigma_L(t) e_{t,s})$ are not vanishing for all $(t,s) \in D_0$, thus there exists a constant $\varepsilon_1 > 0$ such that

$$
\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)) \le 1 - \varepsilon_1, \quad \forall (t,s) \in D_0.
$$

Combining this with (3.5.6), and noting that $\|s-t\|^m$ is integrable on $(J \cap B(s_0,\delta))) \times (J' \cap B(s_0,\delta))$, we conclude that $\int\int_{D_0} A(t,s)\, dtds$ is super-exponentially small.

Now we turn to estimating $\int\int_{D_i} A(t,s) dtds$, $i = m+1, \ldots, k$. Let $\widetilde{\Pi}_t$ be the projection onto span$\{E_1(t), \ldots, E_m(t), E_{m+1}(t), \ldots, E_{i-1}(t), E_{i+1}(t), \ldots, E_k(t)\}$, then for $(t,s) \in D_i$, we have

$$
\begin{aligned}
&p_{X(t),\nabla X_{|J}(t),\nabla X_{|J'}(s),\langle \nabla X(s),E_i(t)\rangle}(x,0,0,w)\\
&\quad = p_{\widetilde{\Pi}_t(\nabla X(t)),\nabla X_{|J'}(s)}(0,0|X(t)=x,\langle \nabla X(s),E_i(t)\rangle = w,\langle \nabla X(t),E_i(t)\rangle = 0)\\
&\qquad \times p_{X(t),\langle \nabla X(s),E_i(t)\rangle}(x,w|\langle \nabla X(t),E_i(t)\rangle = 0)p_{\langle \nabla X(t),E_i(t)\rangle}(0) \qquad (3.5.7)\\
&\quad \le C_4[\mathrm{detCov}(X(t),\nabla X_{|J}(t),\nabla X_{|J'}(s),\langle \nabla X(s),E_i(t)\rangle)]^{-1/2}\\
&\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho(t,s)^2)}\bigg(\frac{x^2}{\sigma_1^2(t,s)}+\frac{w^2}{\sigma_2^2(t,s)}-\frac{2\rho(t,s)xw}{\sigma_1(t,s)\sigma_2(t,s)}\bigg)\bigg\}
\end{aligned}
$$

for some positive constant $C_4$, where

$$
\begin{aligned}
\sigma_1^2(t,s) &= \mathrm{Var}(X(t)|\langle \nabla X(t),E_i(t)\rangle) = \frac{\mathrm{detCov}(X(t),\langle \nabla X(t),E_i(t)\rangle)}{\mathrm{Var}(\langle \nabla X(t),E_i(t)\rangle)},\\
\sigma_2^2(t,s) &= \mathrm{Var}(\langle \nabla X(s),E_i(t)\rangle|\langle \nabla X(t),E_i(t)\rangle) = \frac{\mathrm{detCov}(\langle \nabla X(s),E_i(t)\rangle,\langle \nabla X(t),E_i(t)\rangle)}{\mathrm{Var}(\langle \nabla X(t),E_i(t)\rangle)},\\
\rho(t,s) &= \frac{\mathbb{E}\{X(t)\langle \nabla X(s),E_i(t)\rangle|\langle \nabla X(t),E_i(t)\rangle = 0\}}{\sigma_1(t,s)\sigma_2(t,s)}.
\end{aligned}
$$

Similarly to the rectangle case, one can check that there exits positive constants $C_5$ and $C_6$ such that

$$C_5\|s-t\|^2 \le \sigma_2^2(t,s) \le C_6\|s-t\|^2. \qquad (3.5.8)$$

Applying Taylor formula, we obtain

$$
\begin{aligned}
\rho(t,s) &= \frac{1}{\sigma_1(t,s)\sigma_2(t,s)}\Bigg(\mathbb{E}\{X(t)\langle\nabla X(s), E_i(t)\rangle\} \\
&\quad - \frac{\mathbb{E}\{X(t)\langle\nabla X(t), E_i(t)\rangle\}\mathbb{E}\{\langle\nabla X(s), E_i(t)\rangle\langle\nabla X(t), E_i(t)\rangle\}}{\mathrm{Var}(\langle\nabla X(t), E_i(t)\rangle)}\Bigg) \\
&= \frac{1}{\sigma_1(t,s)\sigma_2(t,s)}\Bigg(\mathbb{E}\{X(t)\langle\nabla X(t) + \nabla^2 X(t)(s-t) + \|s-t\|^{1+\eta}Y_{t,s}, E_i(t)\rangle\} \\
&\quad - \frac{\mathbb{E}\{X(t)\langle\nabla X(t), E_i(t)\rangle\}}{\mathrm{Var}(\langle\nabla X(t), E_i(t)\rangle)}\mathbb{E}\{\langle\nabla X(t), E_i(t)\rangle\langle\nabla X(t) + \nabla^2 X(t)(s-t) \\
&\quad + \|s-t\|^{1+\eta}Y_{t,s}, E_i(t)\rangle\}\Bigg) \\
&= \frac{\|s-t\|}{\sigma_1(t,s)\sigma_2(t,s)}\Bigg(\mathbb{E}\{X(t)\langle\nabla^2 X(t)e_{t,s} + \|s-t\|^{\eta}Y_{t,s}, E_i(t)\rangle\} \\
&\quad - \frac{\mathbb{E}\{X(t)\langle\nabla X(t), E_i(t)\rangle\}}{\mathrm{Var}(\langle\nabla X(t), E_i(t)\rangle)}\mathbb{E}\{\langle\nabla X(t), E_i(t)\rangle\langle\nabla^2 X(t)e_{t,s} + \|s-t\|^{\eta}Y_{t,s}, E_i(t)\rangle\}\Bigg).
\end{aligned}
$$

By our assumption, $(\mathbb{E}\{X(s_0)\nabla X(s_0)\})|_L=0$, therefore $\mathbb{E}\{X(t)\langle\nabla X(t), E_i(t)\rangle\}$ gets close to 0 for $t \in J \cap B(s_0,\delta)$ and $\delta$ small enough. Thus, as $\|s-t\| \to 0$,

$$
\begin{aligned}
\rho(t,s) &= \frac{\|s-t\|}{\sigma_1(t,s)\sigma_2(t,s)}(\langle\Sigma_J(t)e_{t,s}, E_i(t)\rangle - o(1)) \\
&\le \frac{\|s-t\|}{\sigma_1(t,s)(C_5\|s-t\|^2)^{1/2}}(-b_i - o(1)) \\
&< -\varepsilon_2
\end{aligned}
\tag{3.5.9}
$$

for some positive constant $\varepsilon_2$, where the second inequality comes from (3.5.8) and the definition of $D_i$.

By similar arguments in the proof for Gaussian fields on rectangle, we see that there exists positive constants $C_7$, $C_8$, $N_2$ and $N_3$ such that

$$\begin{aligned}&\mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)||\mathrm{det}\nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, \nabla X_{J'}(s)=0, \langle \nabla X(s), E_i(t)\rangle = w\}\\ &\quad \leq C_7(x^{N_2}+(w/\|s-t\|)^{N_3}+1)\end{aligned}$$

and

$$\mathrm{detCov}(X(t), \nabla X_{|J}(t), \nabla X_{|J'}(s), \langle \nabla X(s), E_i(t)\rangle) \geq C_8\|s-t\|^{2(m+1)}.$$

Combining this with (3.5.7), and making change of variable $\tilde{w}=w/\|s-t\|$ and $\tilde{\sigma}_2(t,s)=\sigma_2(t,s)/\|s-t\|$, we obtain

$$\begin{aligned}A(t,s) &\leq C_9\|s-t\|^{-(m+1)}\int_u^\infty dx\int_0^\infty dw\ (x^{N_2}+(w/\|s-t\|)^{N_3}+1)\\ &\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho(t,s)^2)}\bigg(\frac{x^2}{\sigma_1^2(t,s)}+\frac{w^2}{\sigma_2^2(t,s)}-\frac{2\rho(t,s)xw}{\sigma_1(t,s)\sigma_2(t,s)}\bigg)\bigg\}\\ &\leq C_9\|s-t\|^{-m}\int_u^\infty dx\int_0^\infty d\tilde{w}\ (x^{N_2}+\tilde{w}^{N_3}+1)\\ &\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho(t,s)^2)}\bigg(\frac{x^2}{\sigma_1^2(t,s)}+\frac{\tilde{w}^2}{\tilde{\sigma}_2^2(t,s)}-\frac{2\rho(t,s)x\tilde{w}}{\sigma_1(t,s)\tilde{\sigma}_2(t,s)}\bigg)\bigg\}\end{aligned} \tag{3.5.10}$$

for some positive constant $C_9$. Applying Lemma 2.3.10 yields that $\int\int_{D_i} A(t,s)dtds$ is super-exponentially small.

Estimating $\int\int_{D'_j} A(t,s)dtds$ for $j=m+1,\ldots,k'$ is similar. And the proof for the case when $k=0$ is also similar. Now we obtain that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small, completing the proof. □

**Remark 3.5.3** Our proof in Theorem 3.5.2 only focuses on the neighborhood of $s_0$, and therefore the proof is also valid for the case when $T$ is *locally convex* [cf. Adler and Taylor (2007, p.189, Definition 8.2.1)].

# Chapter 4

# The Expected Euler Characteristic of Non-centered Stationary Gaussian Fields

It has been shown that the expected Euler characteristic of the excursion set, denoted by $\mathbb{E}\{\varphi(A_u)\}$, can be used to approximate the excursion probability very accurately. Now we turn to the computation of $\mathbb{E}\{\varphi(A_u)\}$. In the monograph Adler and Taylor (2007), the authors considered centered Gaussian fields with constant variance and they obtained very general formulae [cf. Theorem 12.4.1 and Theorem 12.4.2 therein] for $\mathbb{E}\{\varphi(A_u)\}$ involving the so called Lipschitz-Killing curvatures. Usually, it is very hard to simplify these Lipschitz-Killing curvatures. As a consequence, for general centered smooth Gaussian fields with constant variance, $\mathbb{E}\{\varphi(A_u)\}$ is difficult to compute. Therefore, $\mathbb{E}\{\varphi(A_u)\}$ would become even more complicated for general smooth Gaussian fields with non-constant variances.

However, for some relatively simple models and nice parameter space $T$, for example centered stationary Gaussian fields on rectangles, $\mathbb{E}\{\varphi(A_u)\}$ can be simplified a lot [cf. Theorem 11.7.2 and Corollary 11.7.3 in Adler and Taylor (2007)]. The results there rely heavily on the zero mean function. If the Gaussian field is stationary, but the mean function is varying, then the computation for $\mathbb{E}\{\varphi(A_u)\}$ will become more complicated. In this chapter, we will show the formulae of $\mathbb{E}\{\varphi(A_u)\}$ for stationary Gaussian fields with varying mean functions, and also for isotropic Gaussian fields on the sphere with varying mean functions.

## 4.1 Preliminary Gaussian Computations

The following result is Lemma 11.6.1 in Adler and Taylor (2007).

**Lemma 4.1.1 (Wick formula).** *Let $Z_1$, $Z_2$, ..., $Z_N$ be a set of real-valued random variables having a joint Gaussian distribution and zero means. Then for any integer $k$,*

$$\begin{aligned} \mathbb{E}\{Z_1 Z_2 \cdots Z_{2k+1}\} &= 0, \\ \mathbb{E}\{Z_1 Z_2 \cdots Z_{2k}\} &= \sum \mathbb{E}\{Z_{i_1} Z_{i_2}\} \cdots \mathbb{E}\{Z_{i_{2k-1}} Z_{i_{2k}}\}, \end{aligned} \tag{4.1.1}$$

*where the sum is taken over the $(2k)!/(k!2^k)$ different ways of grouping $Z_1$, ..., $Z_{2k}$ into $k$ pairs.*

Let $\Delta_N$ be a symmetric $N \times N$ matrix with elements $\Delta_{ij}$ such that each $\Delta_{ij}$ is a zero-mean normal variable with arbitrary variance but such that the following relationship holds:

$$\mathbb{E}\{\Delta_{ij}\Delta_{kl}\} = \mathcal{E}(i,j,k,l) - \delta_{ij}\delta_{kl}, \tag{4.1.2}$$

where $\mathcal{E}$ is a symmetric function of $i$, $j$, $k$, $l$, and $\delta_{ij}$ is the Kronecker delta function. Write $|\Delta_N|$ for the determinant of $\Delta_N$.

Let $\widetilde{\Delta}_N$ be a symmetric $N \times N$ matrix with elements $\widetilde{\Delta}_{ij}$ such that each $\widetilde{\Delta}_{ij}$ is a zero-mean normal variable with arbitrary variance but such that the following relationship holds:

$$\mathbb{E}\{\widetilde{\Delta}_{ij}\widetilde{\Delta}_{kl}\} = \widetilde{\mathcal{E}}(i,j,k,l) + \delta_{ij}\delta_{kl}, \tag{4.1.3}$$

where $\widetilde{\mathcal{E}}$ is a symmetric function of $i$, $j$, $k$, $l$, and $\delta_{ij}$ is the Kronecker delta function.

Let $\Delta'_N$ be a symmetric $N \times N$ matrix with elements $\Delta'_{ij}$ such that each $\Delta'_{ij}$ is a zero-mean normal variable with arbitrary variance but such that the following relationship holds:

$$\mathbb{E}\{\Delta'_{ij}\Delta'_{kl}\} = \mathcal{E}'(i,j,k,l), \tag{4.1.4}$$

where $\mathcal{E}'$ is a symmetric function of $i$, $j$, $k$, $l$.

**Lemma 4.1.2** *Let $B_N = (B_{ij})_{1\le i,j\le N}$ be a real symmetric $N \times N$ matrix and let $n$ be a positive integer. Then for $\Delta_n$ satisfying (4.1.2),*

$$\mathbb{E}\{|\Delta_n + B_n|\} = \sum_{k=0}^{\lfloor n/2 \rfloor} G_{2k} S_{n-2k}(B_n), \tag{4.1.5}$$

*where $G_{2j} = (-1)^j (2j)!/(j!2^j)$, $S_k(B_l)$ is the sum of the $\binom{l}{k}$ principle minors of order $k$ in $B_l$, and $G_0 = S_0(B_n) = 1$ in convention. Similarly, for $\widetilde{\Delta}'_n$ satisfying (4.1.3) and $\Delta'_n$ satisfying (4.1.4),*

$$\begin{aligned} \mathbb{E}\{|\widetilde{\Delta}_n + B_n|\} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{G}_{2k} S_{n-2k}(B_n), \\ \mathbb{E}\{|\Delta'_n + B_n|\} &= |B_n|, \end{aligned} \tag{4.1.6}$$

*where* $\widetilde{G}_{2j} = (2j)!/(j!2^j)$.

**Proof** We first consider the case when $n$ is even, say $n = 2l$, then

$$|\Delta_{2l} + B_{2l}| = \sum_{P} \eta(p)(\Delta_{1i_1} + B_{1i_1}) \cdots (\Delta_{2l,i_{2l}} + B_{2l,i_{2l}}), \tag{4.1.7}$$

where $p = (i_1, i_2 \cdots, i_{2l})$ is a permutation of $(1, 2, \cdots, 2l)$, $P$ is the set of the $(2l)!$ such permutations, and $\eta(p)$ equals $+1$ or $-1$ depending on the order of the permutation $p$. Then

$$\mathbb{E}\{|\Delta_{2l} + B_{2l}|\} = \sum_{P} \eta(p)\mathbb{E}\{(\Delta_{1i_1} + B_{1i_1}) \cdots (\Delta_{2l,i_{2l}} + B_{2l,i_{2l}})\}. \tag{4.1.8}$$

It follows from Lemma 4.1.1 that for $k \leq l$, $\mathbb{E}\{\Delta_{1i_1} \cdots \Delta_{2k+1,i_{2k+1}}\} = 0$ and

$$\begin{aligned}\mathbb{E}\{\Delta_{1i_1} \cdots \Delta_{2k,i_{2k}}\} = \sum_{Q_{2k}} &\{\mathcal{E}(1, i_1, 2, i_2) - \delta_{1i_1}\delta_{2i_2}\} \times \cdots \\ &\times \{\mathcal{E}(2k-1, i_{2k-1}, 2k, i_{2k}) - \delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}}\},\end{aligned}$$

where $Q_{2k}$ is the set of the $(2k)!/(k!2^k)$ ways of grouping $(i_1, i_2, \cdots, i_{2k})$ into pairs without regard to order, keeping them paired with the first index. Let $\widetilde{P}$ be the set of all the permutations of $(2k+1,\ldots,2l)$, then

$$\begin{aligned}
&\sum_{P}\eta(p)\mathbb{E}\{\Delta_{1i_1}\cdots\Delta_{2k,i_{2k}}\}B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&=\sum_{P}\eta(p)\Big(\sum_{Q_{2k}}\{\mathcal{E}(1,i_1,2,i_2)-\delta_{1i_1}\delta_{2i_2}\}\times\cdots\\
&\quad\times\{\mathcal{E}(2k-1,i_{2k-1},2k,i_{2k})-\delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}}\}\Big)B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&=\sum_{P}\eta(p)\sum_{Q_{2k}}(-1)^k(\delta_{1i_1}\delta_{2i_2})\cdots(\delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}})B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&=\frac{(-1)^k(2k)!}{k!2^k}\sum_{\widetilde{P}}\eta(\widetilde{p})B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&=\frac{(-1)^k(2k)!}{k!2^k}|(B_{ij})_{2k+1\le i,j\le 2l}|,
\end{aligned}$$

where the second equality is due to the fact that all products involving at least one $\mathcal{E}$ term will cancel out because of their symmetry property, and the third equality comes from noting that for only one permutation in $P$ is the product of the delta functions nonzero. Thus

$$\mathbb{E}\{|\Delta_{2l}+B_{2l}|\}=|B_{2l}|+G_2S_{2l-2}(B_{2l})+\cdots+G_{2l-2}S_2(B_{2l})+G_{2l}.$$

Similarly, we obtain that

$$\mathbb{E}\{|\Delta_{2l+1}+B_{2l+1}|\}=|B_{2l+1}|+G_2S_{2l-1}(B_{2l+1})+\cdots+G_{2l}S_1(B_{2l+1}).$$

Then we obtain (4.1.5). The proof for (4.1.6) follows similarly. □

**Corollary 4.1.3** *Let $\Delta_N$, $\widetilde{\Delta}_N$, $\Delta'_N$, $B_N$, $G_{2j}$, $\widetilde{G}_{2j}$ and $S_k(B_l)$ be as in Lemma 4.1.2. Let*

*$I_N$ be the $N \times N$ unit matrix, and $x \in \mathbb{R}$. Then*

$$\mathbb{E}\{|\Delta_N + B_N - xI_N|\} = (-1)^N \sum_{n=0}^{N} (-1)^n \Bigg( \sum_{k=0}^{\lfloor n/2 \rfloor} G_{2k} S_{n-2k}(B_N) \Bigg) x^{N-n}, \tag{4.1.9}$$

*and similarly,*

$$\begin{aligned} \mathbb{E}\{|\widetilde{\Delta}_N + B_N - xI_N|\} &= (-1)^N \sum_{n=0}^{N} (-1)^n \Bigg( \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{G}_{2k} S_{n-2k}(B_N) \Bigg) x^{N-n}, \\ \mathbb{E}\{|\Delta'_N + B_N - xI_N|\} &= (-1)^N \sum_{n=0}^{N} (-1)^n S_n(B_N) x^{N-n}. \end{aligned} \tag{4.1.10}$$

**Proof** It follows from the usual Laplace expansion of the determinant that

$$|\Delta_N + B_N - xI_N| = (-1)^N \sum_{n=0}^{N} (-1)^n S_n(\Delta_N + B_N) x^{N-n}. \tag{4.1.11}$$

It follows from Lemma 4.1.2 that

$$\mathbb{E}\{S_n(\Delta_N + B_N)\} = \sum_{k=0}^{\lfloor n/2 \rfloor} G_{2k} S_{n-2k}(B_N), \tag{4.1.12}$$

and hence we obtain (4.1.9). (4.1.10) follows similarly. $\square$

## 4.2 Stationary Gaussian Fields on Rectangles

Consider a centered stationary Gaussian random field $Z = \{Z(t), t \in \mathbb{R}^N\}$. It has representation

$$Z(t) = \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle} W(d\lambda)$$

and covariance

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle}\nu(d\lambda),$$

where $W$ is a complex-valued Gaussian random measure and $\nu$ is the spectral measure satisfying $\nu(\mathbb{R}^N) = C(0) = \sigma^2$. We introduce the second-order spectral moments

$$\lambda_{ij} = \int_{\mathbb{R}^N} \lambda_i\lambda_j\nu(d\lambda),$$

and denote $\Lambda = (\lambda_{ij})_{1\le i,j\le N}$. Denoting also differentiation via subscripts, so that $Z_i = \partial Z/\partial t_i$, $Z_{ij} = \partial^2 Z/\partial t_i\partial t_j$, etc., we have

$$\mathbb{E}\{Z_i(t)Z_j(t)\} = \lambda_{ij} = -C_{ij}(0) = -\mathbb{E}\{Z(t)Z_{ij}(t)\}.$$

The covariances among the second-order derivatives can be similarly defined. However, all we shall need is that

$$\mathcal{E}_0(i,j,k,l) := \mathbb{E}\{Z_{ij}(t)Z_{kl}(t)\} = \int_{\mathbb{R}^N} \lambda_i\lambda_j\lambda_k\lambda_l\nu(d\lambda) \tag{4.2.1}$$

is a symmetric function of $i$, $j$, $k$, $l$. Also note that for any fixed $t$, $Z_i(t)$ is independent of both $Z(t)$ and $Z_{kl}(t)$.

Let $X(t) = Z(t) + m(t)$, where $m(\cdot) \in C^2(\mathbb{R}^N)$ is a real-valued deterministic function. Let $T = \prod_{i=1}^N [a_i, b_i]$, $-\infty < a_i < b_i < \infty$.

**Theorem 4.2.1** *Let $X(t) = Z(t) + m(t)$, where $Z(\cdot) \in C^2(\mathbb{R}^N)$ a.s. is a stationary Gaussian field and $m(\cdot) \in C^2(\mathbb{R}^N)$ is a real-valued deterministic function. Suppose also that $Z$*

*satisfies* ($\mathbf{H}3'$). *Then we have*

$$\begin{aligned}
&\mathbb{E}\{\varphi(A_u(X,T))\}\\
&= \sum_{\{t\}\in\partial_0 T} \mathbb{P}\{\nabla X(t)\in E(\{t\})\}\Psi\bigg(\frac{u-m(t)}{\sigma}\bigg) + \sum_{k=1}^{N}\sum_{J\in\partial_k T}\frac{|\Lambda_J|^{1/2}}{(2\pi)^{(k+1)/2}\sigma^{k+1}}\\
&\quad\times\int_J dt\int_u^\infty dx\ \exp\bigg\{-\frac{1}{2}(\nabla m_{|J}(t))^T\Lambda_J^{-1}\nabla m_{|J}(t)-\frac{1}{2\sigma^2}(x-m(t))^2\bigg\}\\
&\quad\times\mathbb{P}\{(X_{J_1}(t),\cdots,X_{J_{N-k}}(t))\in E(J)|\nabla X_{|J}(t)=0\}\\
&\quad\times\Bigg[\sum_{j=0}^{k}(-1)^j\bigg(\sum_{i=0}^{\lfloor j/2\rfloor}G_{2i}S_{j-2i}(\sigma Q_J\nabla^2 m_J(t)Q_J)\bigg)\Big(\frac{x}{\sigma}\Big)^{k-j}\Bigg],
\end{aligned}\tag{4.2.2}$$

*where $G$ and $S$ are defined in Lemma 4.1.2 and $Q_J=\Lambda_J^{-1/2}$.*

**Proof** If $J=\{t\}\in\partial_0 T$, then

$$\begin{aligned}
\mathbb{E}\{\mu_0(J)\} &= \mathbb{P}\{X(t)\geq u, \varepsilon_j^* X_j(t)\geq 0 \text{ for all } 1\leq j\leq N\}\\
&= \mathbb{P}\{\nabla X(t)\in E(\{t\})\}\Psi\bigg(\frac{u-m(t)}{\sigma}\bigg),
\end{aligned}\tag{4.2.3}$$

where the last equality is due to the independence of $X(t)$ and $\nabla X(t)$ for each fixed $t$.

Now we consider $J\in\partial_k T$ for some $k\geq 1$. Let $\mathcal{D}_i$ be the collection of all $k\times k$ matrices with index $i$. Applying Kac-Rice formula [cf. Theorem 11.2.1 or Corollary 11.3.2 in Adler and Taylor (2007)], together with the definition (2.2.2), we obtain

$$\begin{aligned}
\mathbb{E}\bigg\{\sum_{i=0}^{k}(-1)^i\mu_i(J)\bigg\} &= \int_J p_{\nabla X_{|J}(t)}(0)dt\sum_{i=0}^{k}(-1)^i\mathbb{E}\{|\mathrm{det}\nabla^2 X_{|J}(t)|\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\}}\\
&\qquad\times\mathbb{1}_{\{X(t)\geq u\}}\mathbb{1}_{\{(X_{J_1}(t),\cdots,X_{J_{N-k}}(t))\in E(J)\}}|\nabla X_{|J}(t)=0\}.
\end{aligned}\tag{4.2.4}$$

Note that on the event $\mathcal{D}_i$, the matrix $\nabla^2 X_{|J}(t)$ has $i$ negative eigenvalues which implies

$(-1)^i|\det\nabla^2 X_{|J}(t)| = \det\nabla^2 X_{|J}(t)$. Also, $\cup_{i=0}^k\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\} = \{\nabla^2 X_{|J}(t)\in\mathbb{R}^k\}$, and $\nabla X(t)$ is independent of both $X(t)$ and $\nabla^2 X(t)$ for each fixed $t$, thus (4.2.4) becomes

$$
\begin{aligned}
&\int_J p_{\nabla X_{|J}(t)}(0)dt\,\mathbb{E}\{\det\nabla^2 X_{|J}(t)\mathbb{1}_{\{X(t)\geq u\}}\mathbb{1}_{\{(X_{J_1}(t),\cdots,X_{J_{N-k}}(t))\in E(J)\}}|\nabla X_{|J}(t)=0\}\\
&=\frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\sigma}\int_J dt\int_u^\infty dx\, e^{-\frac{1}{2}(\nabla m_{|J}(t))^T\Lambda_J^{-1}\nabla m_{|J}(t)}e^{-\frac{1}{2\sigma^2}(x-m(t))^2}\\
&\quad\times\mathbb{P}\{(X_{J_1}(t),\cdots,X_{J_{N-k}}(t))\in E(J)|\nabla X_{|J}(t)=0\}\mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t)=x\}.
\end{aligned}
\tag{4.2.5}
$$

Now we turn to computing $\mathbb{E}\{\det\nabla^2 X_{|J}(t)|X(t)=x\}$. Since $\Lambda_J$ is positive definite, there exists a unique $k\times k$ positive definite matrix $Q_J$ (called principal square root of $\Lambda_J^{-1}$, also denoted as $\Lambda_J^{-1/2}$) such that

$$Q_J\Lambda_J Q_J = I_k,$$

where $I_k$ is the $k\times k$ identity matrix. Hence

$$\mathbb{E}\{Z(t)(Q_J\nabla^2 Z_{|J}(t)Q_J)_{ij}\} = -(Q_J\Lambda_J Q_J)_{ij} = -\delta_{ij}, \tag{4.2.6}$$

where $\delta_{ij}$ is the Kronecker delta function. One can write

$$
\begin{aligned}
&\mathbb{E}\{\det(Q_J\nabla^2 X_{|J}(t)Q_J)|X(t)=x\}\\
&\quad=\mathbb{E}\{\det(Q_J\nabla^2 Z_{|J}(t)Q_J+Q_J\nabla^2 m_{|J}(t)Q_J)|X(t)=x\}\\
&\quad=\mathbb{E}\{\det(\Delta(x)+Q_J\nabla^2 m_J(t)Q_J)\},
\end{aligned}
$$

where $\Delta(x)=(\Delta_{ij}(x))_{i,j\in\sigma(J)}$ with all elements $\Delta_{ij}(x)$ being Gaussian variables. To study $\Delta(x)$, we only need to find its mean and covariance. Applying Lemma 2.5.1 and (4.2.6), we obtain

$$\mathbb{E}\{\Delta_{ij}(x)\} = \mathbb{E}\{(Q_J\nabla^2 Z_{|J}(t)Q_J)_{ij}|X(t)=x\} = -\frac{x}{\sigma^2}\delta_{ij}$$

and

$$\begin{aligned}
&\mathbb{E}\{(\Delta_{ij}(x) - \mathbb{E}\{\Delta_{ij}(x)\})(\Delta_{kl}(x) - \mathbb{E}\{\Delta_{kl}(x)\})\} \\
&\quad = \mathbb{E}\{(Q_J\nabla^2 Z_{|J}(t)Q_J)_{ij}(Q_J\nabla^2 Z_{|J}(t)Q_J)_{kl}\} - \frac{\delta_{ij}\delta_{kl}}{\sigma^2} \\
&\quad = \mathcal{E}(i,j,k,l) - \frac{\delta_{ij}\delta_{kl}}{\sigma^2},
\end{aligned}$$

where $\mathcal{E}$ is a symmetric function of $i,j,k,l$ by Lemma 2.1.7 with $A$ replaced by $Q_J$. Then we have

$$\begin{aligned}
&\mathbb{E}\{\det(Q_J\nabla^2 X_{|J}(t)Q_J)|X(t)=x\} \\
&\quad = \mathbb{E}\left\{\frac{1}{\sigma^k}\det(\sigma Q_J(\nabla^2 X_{|J}(t))Q_J)\Big|X(t)=x\right\} \\
&\quad = \frac{1}{\sigma^k}\mathbb{E}\left\{\det\left(\Delta + \sigma Q_J\nabla^2 m_J(t)Q_J - \frac{x}{\sigma}I_k\right)\right\},
\end{aligned}$$

where $\Delta = (\Delta_{ij})_{i,j\in\sigma(J)}$ and all $\Delta_{ij}$ are Gaussian variables satisfying

$$\mathbb{E}\{\Delta_{ij}\} = 0, \qquad \mathbb{E}\{\Delta_{ij}\Delta_{kl}\} = \sigma^2\mathcal{E}(i,j,k,l) - \delta_{ij}\delta_{kl}.$$

Applying Corollary 4.1.3, we get

$$\begin{aligned}
&\mathbb{E}\{\det(Q_J\nabla^2 X_{|J}(t)Q_J)|X(t)=x\} \\
&\quad = \frac{(-1)^k}{\sigma^k}\sum_{j=0}^{k}(-1)^j\left(\sum_{i=0}^{\lfloor j/2\rfloor} G_{2i}S_{j-2i}(\sigma Q_J\nabla^2 m_J(t)Q_J)\right)\left(\frac{x}{\sigma}\right)^{k-j}.
\end{aligned} \tag{4.2.7}$$

It follows from (4.2.7) that

$$
\begin{aligned}
&\mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x\} \\
&\quad = \mathbb{E}\{\det(Q_t^{-1} Q_J \nabla^2 X_{|J}(t) Q_J Q_t^{-1}) | X(t) = x\} \\
&\quad = |\Lambda_J| \mathbb{E}\{\det(Q_J \nabla^2 X_{|J}(t) Q_J) | X(t) = x\} \\
&\quad = \frac{(-1)^k}{\sigma^k} |\Lambda_J| \sum_{j=0}^{k} (-1)^j \bigg( \sum_{i=0}^{\lfloor j/2 \rfloor} G_{2i} S_{j-2i}(\sigma Q_J \nabla^2 m_J(t) Q_J) \bigg) \Big(\frac{x}{\sigma}\Big)^{k-j}.
\end{aligned}
$$

Plugging this into (4.2.5), we obtain

$$
\begin{aligned}
&\mathbb{E}\bigg\{ \sum_{i=0}^{k} (-1)^i \mu_i(J) \bigg\} \\
&= \frac{(-1)^k |\Lambda_J|^{1/2}}{(2\pi)^{(k+1)/2} \sigma^{k+1}} \int_J dt \int_u^\infty dx\, e^{-\frac{1}{2}(\nabla m_{|J}(t))^T \Lambda_J^{-1} \nabla m_{|J}(t)} e^{-\frac{1}{2\sigma^2}(x-m(t))^2} \\
&\quad \times \mathbb{P}\{(X_{J_1}(t), \cdots, X_{J_{N-k}}(t)) \in E(J) | \nabla X_{|J}(t) = 0\} \\
&\quad \times \bigg[ \sum_{j=0}^{k} (-1)^j \bigg( \sum_{i=0}^{\lfloor j/2 \rfloor} G_{2i} S_{j-2i}(\sigma Q_J \nabla^2 m_J(t) Q_J) \bigg) \Big(\frac{x}{\sigma}\Big)^{k-j} \bigg].
\end{aligned}
\tag{4.2.8}
$$

Combining (4.2.3), (4.2.8) and the definition (2.2.1) yields the desired result. □

**Corollary 4.2.2** *Let $Z$ be an isotropic Gaussian random field with* $\mathrm{Var}(Z_1(t)) = \gamma^2$*, then* *under the conditions in Theorem 4.2.1, we have*

$$
\begin{aligned}
&\mathbb{E}\{\varphi(A_u(X,T))\}\\
&= \sum_{\{t\}\in\partial_0 T} \mathbb{P}\{\nabla X(t)\in E(\{t\})\}\Psi\left(\frac{u-m(t)}{\sigma}\right) + \sum_{k=1}^{N}\sum_{J\in\partial_k T}\frac{\gamma^k}{(2\pi)^{(k+1)/2}\sigma^{k+1}}\\
&\quad\times\int_J dt\int_u^\infty dx\,\exp\left\{-\frac{1}{2\gamma^2}\|\nabla m_{|J}(t)\|^2-\frac{1}{2\sigma^2}(x-m(t))^2\right\} \qquad (4.2.9)\\
&\quad\times\mathbb{P}\{(X_{J_1}(t),\cdots,X_{J_{N-k}}(t))\in E(J)\}\\
&\quad\times\left[\sum_{j=0}^{k}(-1)^j\left(\sum_{i=0}^{\lfloor j/2\rfloor}G_{2i}S_{j-2i}(\sigma\gamma^{-2}\nabla^2 m_J(t))\right)\left(\frac{x}{\sigma}\right)^{k-j}\right].
\end{aligned}
$$

**Proof** The result follows by applying Theorem 4.2.1 and noting that $\Lambda_J=\gamma^2 I_k$ and hence $Q_J=\gamma^{-1}I_k$. $\square$

**Corollary 4.2.3** *Under the conditions in Theorem 4.2.1, assume that $t_0$, an interior point in $T$, is the unique maximal point of $m(t)$ and that $\nabla^2 m(t_0)$ is nondegenerate. Then as $u\to\infty$,*

$$\mathbb{E}\{\varphi(A_u(X,T))\}=\frac{|\Lambda|^{1/2}u^{N/2}}{\sigma^N|-\nabla^2 m(t_0)|^{1/2}}\Psi\left(\frac{u-m(t_0)}{\sigma}\right)(1+o(1)).$$

**Proof** By Theorem 4.2.1,

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u(X,T))\} &= \frac{|\Lambda|^{1/2}}{(2\pi)^{(N+1)/2}\sigma^{N+1}}\int_u^\infty dx\int_J e^{-\frac{1}{2}(\nabla m(t))^T\Lambda^{-1}\nabla m(t)}\\
&\quad\times e^{-\frac{1}{2\sigma^2}(x-m(t))^2}\left(\frac{x}{\sigma}\right)^N dt(1+o(1)).
\end{aligned} \qquad (4.2.10)
$$

Applying Laplace method, we obtain that as $x \to \infty$,

$$\begin{aligned}
&\int_J e^{-\frac{1}{2}(\nabla m(t))^T \Lambda^{-1} \nabla m(t)} e^{-\frac{1}{2\sigma^2}(x-m(t))^2} dt \\
&\quad = \frac{(2\pi)^{N/2}\sigma^N}{x^{N/2}|-\nabla^2 m(t_0)|^{1/2}} e^{-\frac{1}{2\sigma^2}(x-m(t_0))^2}(1+o(1)).
\end{aligned} \tag{4.2.11}$$

Thus as $u \to \infty$,

$$\begin{aligned}
\mathbb{E}\{\varphi(A_u(X,T))\} &= \frac{|\Lambda|^{1/2}}{(2\pi)^{1/2}\sigma^{N+1}|-\nabla^2 m(t_0)|^{1/2}} \int_u^\infty x^{N/2} e^{-\frac{1}{2\sigma^2}(x-m(t_0))^2} dx(1+o(1)) \\
&= \frac{|\Lambda|^{1/2} u^{N/2}}{\sigma^N |-\nabla^2 m(t_0)|^{1/2}} \Psi\left(\frac{u-m(t_0)}{\sigma}\right)(1+o(1)).
\end{aligned}$$

□

## 4.3 Isotropic Gaussian Random Fields on Sphere

We consider isotropic Gaussian random fields on $N$-dimensional unit sphere $\mathbb{S}^N$. For $x = (x_1, \cdots, x_{N+1}) \in \mathbb{S}^N$, we shall use the spherical coordinates as follows.

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\vdots \\
x_N &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \cos\theta_N, \\
x_{N+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \sin\theta_N,
\end{aligned} \tag{4.3.1}$$

where $0 \le \theta_i \le \pi$ for $1 \le i \le N-1$ and $0 \le \theta_N < 2\pi$. Let $\theta = (\theta_1, \cdots, \theta_N)$. Accordingly, let $y = (y_1, \cdots, y_{N+1})$ be another point in $\mathbb{S}^N$, we use $\varphi = (\varphi_1, \cdots, \varphi_N)$ to denote its corresponding spherical coordinates.

Let $\|\cdot\|, \langle\cdot,\cdot\rangle$ be the Euclidean norm and the inner product respectively. Denote by $d(\cdot,\cdot)$ the distance function in $\mathbb{S}^N$, i.e., $d(x,y) = \arccos\langle x, y\rangle$, $\forall x, y \in \mathbb{S}^N$.

The following theorem by Schoenberg (1942) characterizes the covariance function of isotropic Gaussian field on sphere.

**Theorem 4.3.1** *A real continuous function C(d) is a valid covariance on unit sphere* $\mathbb{S}^N$ *for every dimension* $N \ge 1$ *if and only if it has the form*

$$C(d) = \sum_{n=0}^{\infty} b_n \cos^n d, \quad d \in [0, \pi],$$

*where* $b_n \ge 0$ *and* $\sum_{n=0}^{\infty} b_n < \infty$.

Recall $d(x,y) = \arccos\langle x, y\rangle, \forall x, y \in \mathbb{S}^N$. It follows from the above theorem that a function $C(x,y)$, which is the covariance function of an isotropic Gaussian field on $\mathbb{S}^N$ for every dimension $N \ge 1$, has the form

$$C(x,y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \quad \forall x, y \in \mathbb{S}^N, \tag{4.3.2}$$

where where $b_n \ge 0$ and $\sum_{n=0}^{\infty} b_n < \infty$.

Let $\varphi(A_u(X, \mathbb{S}^N))$ be the Euler characteristic of excursion set $A_u(X, \mathbb{S}^N) = \{x \in \mathbb{S}^N : X(x) \ge u\}$. Then according to Corollary 9.3.5 in Adler and Taylor (2007),

$$\varphi(A_u(X, \mathbb{S}^N)) = (-1)^N \sum_{i=0}^{N} (-1)^i \mu_i(\mathbb{S}^N) \tag{4.3.3}$$

with

$$\mu_i(\mathbb{S}^N) := \#\{x \in \mathbb{S}^N : X(x) \geq u, \nabla X(x) = 0, \text{index}(\nabla^2 X(x)) = i\}, \tag{4.3.4}$$

where $\nabla X$ and $\nabla^2 X$ are the gradient and Hessian on manifold respectively.

Let $X(x) = Z(x) + m(x)$, $x \in \mathbb{S}^N$, where $Z$ is a centered, unit-variance smooth Gaussian random field on $\mathbb{S}^N$ with covariance function $C(\cdot, \cdot)$ and $m(\cdot) \in C^2(\mathbb{S}^N)$ is a real-valued deterministic function. Under the spherical coordinate, let $\overline{X}(\theta) = X(x)$, $\overline{Z}(\theta) = Z(x)$, $\overline{m}(\theta) = m(x)$, $\overline{C}(\theta, \varphi) = C(x, y)$.

**Lemma 4.3.2** *Let $h(x, y) = \langle x, y\rangle^n$, $x, y \in \mathbb{S}^N$, where $n$ is a nonnegative integer, and let $\overline{h}(\theta, \varphi)$ be its spherical version. Then $\overline{h}(\theta, \theta) = 1$ and*

$$\begin{aligned}
\frac{\partial \overline{h}(\theta, \varphi)}{\partial \theta_i}|_{\theta=\varphi} &= \frac{\partial^3 \overline{h}(\theta, \varphi)}{\partial \theta_i \partial \varphi_j \partial \varphi_k}|_{\theta=\varphi} = 0, \\
\frac{\partial^2 \overline{h}(\theta, \varphi)}{\partial \theta_i \partial \varphi_j}|_{\theta=\varphi} &= -\frac{\partial^2 \overline{h}(\theta, \varphi)}{\partial \theta_i \partial \theta_j}|_{\theta=\varphi} = n\delta_{ij}, \\
\frac{\partial^4 \overline{h}(\theta, \varphi)}{\partial \theta_i \partial \theta_j \partial \varphi_k \partial \varphi_l}|_{\theta=\varphi} &= n(n-1)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + n\delta_{ij}\delta_{kl}.
\end{aligned} \tag{4.3.5}$$

Let $\Theta = \{\theta \in \mathbb{R}^N : 0 \leq \theta_N < 2\pi, 0 \leq \theta_i \leq \pi, \forall 1 \leq i \leq N-1\}$ and let $d\sigma(\theta)$ be the volume element on the sphere, i.e.,

$$d\sigma(\theta) = \left(\prod_{i=1}^{N-1} \sin^{N-i}\theta_i\right) d\theta, \quad \forall \theta \in \Theta.$$

Then we can state our result as follows.

**Theorem 4.3.3** *Let $X = \{X(x) = Z(x) + m(x) : x \in \mathbb{S}^N\}$, where $Z$ is a Gaussian field on $\mathbb{S}^N$ satisfying $(\mathbf{H}3')$ and $m(\cdot) \in C^2(\mathbb{S}^N)$ is a real-valued deterministic function. Suppose* *also that $X$ has the covariance function $C(\cdot,\cdot)$ such that*

$$C(x,y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \quad \forall x, y \in \mathbb{S}^N, \tag{4.3.6}$$

*where $b_n \geq 0$, $\sum_{n=0}^{\infty} b_n = 1$ and $\sum_{n=1}^{\infty} n^4 b_n < \infty$. Let $\beta = \sum_{n=1}^{\infty} n b_n$, $\overline{m}(\theta) = m(x)$, and let $G$, $\widetilde{G}$ and $S$ be as in Lemma 4.1.2. Then for $\beta > 1$,*

$$\begin{aligned}\mathbb{E}\{\varphi(A_u(X, \mathbb{S}^N))\} = \frac{(\beta-1)^{N/2}}{(2\pi)^{N/2}} \int_\Theta d\sigma(\theta) \int_u^\infty dw \, \exp\Big\{ -\frac{1}{2\beta}\|\nabla \overline{m}(\theta))\|^2 \Big\} \\ \times \bigg[ \sum_{j=0}^{N} (-1)^j \bigg( \sum_{i=0}^{\lfloor j/2 \rfloor} G_{2i} S_{j-2i}\Big(\frac{\nabla^2 \overline{m}(\theta)}{\sqrt{\beta^2-\beta}}\Big)\bigg) \Big(\frac{\beta w}{\sqrt{\beta^2-\beta}}\Big)^{N-j} \bigg];\end{aligned}$$

*for $\beta < 1$,*

$$\begin{aligned}\mathbb{E}\{\varphi(A_u(X, \mathbb{S}^N))\} = \frac{(1-\beta)^{N/2}}{(2\pi)^{N/2}} \int_\Theta d\sigma(\theta) \int_u^\infty dw \, \exp\Big\{ -\frac{1}{2\beta}\|\nabla \overline{m}(\theta))\|^2 \Big\} \\ \times \bigg[ \sum_{j=0}^{N} (-1)^j \bigg( \sum_{i=0}^{\lfloor j/2 \rfloor} \widetilde{G}_{2i} S_{j-2i}\Big(\frac{\nabla^2 \overline{m}(\theta)}{\sqrt{\beta-\beta^2}}\Big)\bigg) \Big(\frac{\beta w}{\sqrt{\beta-\beta^2}}\Big)^{N-j} \bigg];\end{aligned}$$

*and for $\beta = 1$,*

$$\begin{aligned}\mathbb{E}\{\varphi(A_u(X, \mathbb{S}^N))\} = \frac{1}{(2\pi)^{N/2}} \int_\Theta d\sigma(\theta) \int_u^\infty dw \, \exp\Big\{ -\frac{1}{2}\|\nabla \overline{m}(\theta)\|^2 \Big\} \\ \times \bigg[ \sum_{j=0}^{N} (-1)^j S_j(\nabla^2 \overline{m}(\theta)) w^{N-j} \bigg].\end{aligned}$$

**Remark 4.3.4** It is easy to check that the condition $\sum_{n=1}^{\infty} n^4 b_n < \infty$ makes $C(\cdot,\cdot) \in C^4(\mathbb{S}^N \times \mathbb{S}^N)$ and hence $X(\cdot) \in C^2(\mathbb{S}^N)$.

**Proof** Case 1: $\beta > 1$. Let $\kappa = \sum_{n=2}^{\infty} n(n-1)b_n$. Then by Lemma 4.3.2,

$$\begin{aligned} \mathbb{E}\{\overline{Z}(\theta)\overline{Z}_i(\theta)\} &= \mathbb{E}\{\overline{Z}_i(\theta)\overline{Z}_{jk}(\theta)\} = 0, \\ \mathbb{E}\{\overline{Z}_i(\theta)\overline{Z}_j(\theta)\} &= -\mathbb{E}\{\overline{Z}(\theta)\overline{Z}_{ij}(\theta)\} = \beta\delta_{ij}, \\ \mathbb{E}\{\overline{Z}_{ij}(\theta)\overline{Z}_{kl}(\theta)\} &= \kappa(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \beta\delta_{ij}\delta_{kl}. \end{aligned} \tag{4.3.7}$$

By the Kac-Rice metatheorem,

$$\begin{aligned} &\mathbb{E}\{\varphi(A_u(X, \mathbb{S}^N))\} \\ &\quad = (-1)^N \int_\Theta p_{\nabla\overline{X}(\theta)}(0)\mathbb{E}\{\det\nabla^2\overline{X}(\theta)\mathbb{1}_{\{\overline{X}(\theta)\geq u\}}|\nabla\overline{X}(\theta) = 0\}d\sigma(\theta) \\ &\quad = (-1)^N \int_\Theta d\sigma(\theta)\int_u^\infty p_{\nabla\overline{X}(\theta)}(0)\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta) = w\}dw \\ &\quad = \frac{(-1)^N}{(2\pi)^{N/2}\beta^{N/2}}\int_\Theta d\sigma(\theta)\int_u^\infty e^{-\frac{1}{2\beta}\|\nabla\overline{m}(\theta))\|^2}\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta) = w\}dw. \end{aligned} \tag{4.3.8}$$

Now we turn to computing $\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta) = w\}$. Note that

$$\begin{aligned} &\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta) = w\} \\ &\quad = \mathbb{E}\{\det(\nabla^2\overline{Z}(\theta) + \nabla^2\overline{m}(\theta))|\overline{X}(\theta) = w\} \\ &\quad = (\beta^2-\beta)^{N/2}\mathbb{E}\{\det((\beta^2-\beta)^{-1/2}\nabla^2\overline{Z}(\theta) + (\beta^2-\beta)^{-1/2}\nabla^2\overline{m}(\theta))|\overline{X}(\theta) = w\} \\ &\quad = (\beta^2-\beta)^{N/2}\mathbb{E}\{\det(\Delta + (\beta^2-\beta)^{-1/2}\nabla^2\overline{m}(\theta) - \beta(\beta^2-\beta)^{-1/2}wI_N)\}, \end{aligned} \tag{4.3.9}$$

where $\Delta = (\Delta_{ij})_{1\leq i,j\leq N}$ and all $\Delta_{ij}$ are centered Gaussian variables satisfying

$$\begin{aligned} \mathbb{E}\{\Delta_{ij}\Delta_{kl}\} &= (\beta^2-\beta)^{-1}\mathbb{E}\{\overline{Z}_{ij}(\theta)\overline{Z}_{kl}(\theta)|\overline{X}(\theta) = w\} \\ &= (\beta^2-\beta)^{-1}\{\kappa(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \beta\delta_{ij}\delta_{kl} - \beta^2\delta_{ij}\delta_{kl}\} \\ &= \overline{\mathcal{E}}(i,j,k,l) - \delta_{ij}\delta_{kl}, \end{aligned} \tag{4.3.10}$$

where $\overline{\mathcal{E}}(i,j,k,l) = (\beta^2-\beta)^{-1}\kappa(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Applying Corollary 4.1.3, we get

$$\begin{aligned}
&\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta)=w\}\\
&=(-1)^N(\beta^2-\beta)^{N/2}\sum_{j=0}^{N}(-1)^j\bigg(\sum_{i=0}^{\lfloor j/2\rfloor}G_{2i}S_{j-2i}\bigg(\frac{\nabla^2\overline{m}(\theta)}{\sqrt{\beta^2-\beta}}\bigg)\bigg)\bigg(\frac{\beta w}{\sqrt{\beta^2-\beta}}\bigg)^{N-j}.
\end{aligned}\tag{4.3.11}$$

Then

$$\begin{aligned}
\mathbb{E}\{\varphi(A_u(X,\mathbb{S}^N))\} = {}& \frac{(\beta-1)^{N/2}}{(2\pi)^{N/2}}\int_\Theta d\sigma(\theta)\int_u^\infty dw\, e^{-\frac{1}{2\beta}\|\nabla\overline{m}(\theta))\|^2}\\
&\times\bigg[\sum_{j=0}^{N}(-1)^j\bigg(\sum_{i=0}^{\lfloor j/2\rfloor}G_{2i}S_{j-2i}\bigg(\frac{\nabla^2\overline{m}(\theta)}{\sqrt{\beta^2-\beta}}\bigg)\bigg)\bigg(\frac{\beta w}{\sqrt{\beta^2-\beta}}\bigg)^{N-j}\bigg].
\end{aligned}$$

Case 2: $\beta<1$. Then (4.3.9) becomes

$$\begin{aligned}
&\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta)=w\}\\
&\quad=\mathbb{E}\{\det(\nabla^2\overline{Z}(\theta)+\nabla^2\overline{m}(\theta))|\overline{X}(\theta)=w\}\\
&\quad=(\beta-\beta^2)^{N/2}\mathbb{E}\{\det((\beta-\beta^2)^{-1/2}\nabla^2\overline{Z}(\theta)+(\beta-\beta^2)^{-1/2}\nabla^2\overline{m}(\theta))|\overline{X}(\theta)=w\}\\
&\quad=(\beta-\beta^2)^{N/2}\mathbb{E}\{\det(\widetilde{\Delta}+(\beta-\beta^2)^{-1/2}\nabla^2\overline{m}(\theta)-\beta(\beta-\beta^2)^{-1/2}wI_N)\},
\end{aligned}$$

where $\widetilde{\Delta}=(\widetilde{\Delta}_{ij})_{1\le i,j\le N}$ and all $\widetilde{\Delta}_{ij}$ are centered Gaussian variables satisfying

$$\begin{aligned}
\mathbb{E}\{\widetilde{\Delta}_{ij}\widetilde{\Delta}_{kl}\} &= (\beta-\beta^2)^{-1}\mathbb{E}\{\overline{Z}_{ij}(\theta)\overline{Z}_{kl}(\theta)|\overline{X}(\theta)=w\}\\
&=(\beta-\beta^2)^{-1}\{\kappa(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\beta\delta_{ij}\delta_{kl}-\beta^2\delta_{ij}\delta_{kl}\}\\
&=\widetilde{\mathcal{E}}(i,j,k,l)+\delta_{ij}\delta_{kl},
\end{aligned}$$

where $\widetilde{\mathcal{E}}(i,j,k,l) = (\beta-\beta^2)^{-1}\kappa(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Applying Corollary 4.1.3, we get

$$
\begin{aligned}
&\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta)=w\}\\
&= (-1)^N(\beta-\beta^2)^{N/2}\sum_{j=0}^{N}(-1)^j\bigg(\sum_{i=0}^{\lfloor j/2\rfloor}\widetilde{G}_{2i}S_{j-2i}\Big(\frac{\nabla^2\overline{m}(\theta)}{\sqrt{\beta-\beta^2}}\Big)\bigg)\Big(\frac{\beta w}{\sqrt{\beta-\beta^2}}\Big)^{N-j}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u(X,\mathbb{S}^N))\} &= \frac{(1-\beta)^{N/2}}{(2\pi)^{N/2}}\int_\Theta d\sigma(\theta)\int_u^\infty dw\, e^{-\frac{1}{2\beta}\|\nabla\overline{m}(\theta))\|^2}\\
&\quad\times\bigg[\sum_{j=0}^{N}(-1)^j\bigg(\sum_{i=0}^{\lfloor j/2\rfloor}\widetilde{G}_{2i}S_{j-2i}\Big(\frac{\nabla^2\overline{m}(\theta)}{\sqrt{\beta-\beta^2}}\Big)\bigg)\Big(\frac{\beta w}{\sqrt{\beta-\beta^2}}\Big)^{N-j}\bigg].
\end{aligned}
$$

Case 3: $\beta = 1$. Then (4.3.9) becomes

$$
\begin{aligned}
\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta)=w\} &= \mathbb{E}\{\det(\nabla^2\overline{Z}(\theta)+\nabla^2\overline{m}(\theta))|\overline{X}(\theta)=w\}\\
&= \mathbb{E}\{\det(\Delta'+\nabla^2\overline{m}(\theta)-wI_N)\},
\end{aligned}
$$

where $\Delta' = (\Delta'_{ij})_{1\le i,j\le N}$ and all $\Delta'_{ij}$ are centered Gaussian variables satisfying

$$
\begin{aligned}
\mathbb{E}\{\Delta'_{ij}\Delta'_{kl}\} &= \mathbb{E}\{\overline{Z}_{ij}(\theta)\overline{Z}_{kl}(\theta)|\overline{X}(\theta)=w\}\\
&= \kappa(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\beta\delta_{ij}\delta_{kl}-\beta^2\delta_{ij}\delta_{kl}\\
&= \mathcal{E}'(i,j,k,l),
\end{aligned}
$$

where $\mathcal{E}'(i,j,k,l) = \kappa(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Applying Corollary 4.1.3, we get

$$
\mathbb{E}\{\det\nabla^2\overline{X}(\theta)|\overline{X}(\theta)=w\} = (-1)^N\sum_{j=0}^{N}(-1)^jS_j(\nabla^2\overline{m}(\theta))w^{N-j}.
$$

Then

$$\mathbb{E}\{\varphi(A_u(X,\mathbb{S}^N))\} = \frac{1}{(2\pi)^{N/2}} \int_\Theta d\sigma(\theta) \int_u^\infty dw\, e^{-\frac{1}{2}\|\nabla\overline{m}(\theta)\|^2} \times \left[ \sum_{j=0}^{N} (-1)^j S_j(\nabla^2 \overline{m}(\theta)) w^{N-j} \right].$$

$\square$

# Chapter 5

# Excursion Probability of Smooth Gaussian Processes over Random Intervals

Let $\{X(t) : t \in [0, \infty)\}$ be a Gaussian process and let $T > 0$ be a fixed number, the tail probability $\mathbb{P}\{\sup_{0 \le t \le T} X(t) \ge u\}$ for large $u$ has been extensively studied in the literature. However, the supremum over the a fixed domain $[0, T]$ is not adequate in certain applications [cf. Kozubowski et al. (2004, 2006)], instead, one needs to consider the asymptotics for $\sup_{0 \le t \le \mathcal{T}} X(t)$ where $\mathcal{T}$ is a non-negative random variable independent of $X$.

To study $\mathbb{P}\{\sup_{0 \le t \le \mathcal{T}} X(t) \ge u\}$, we have to take into account the behaviors of both $X$ and $\mathcal{T}$. Therefore, some interesting phenomena arise due to the connection between the Gaussian process and the random interval. Recently, M. Arendarczyk and K. Debicki (2011, 2012) considered the case when the Gaussian process $X$ is non-smooth (i.e. the sample path is not twice differentiable), and obtained the following result under certain conditions:

$$\mathbb{P}\Big\{ \sup_{0 \le t \le \mathcal{T}} X(t) \ge u \Big\} = g_1(u)(1 + o(1)), \quad \text{as } u \to \infty, \tag{5.0.1}$$

where $g_1(u)$ is a function depending on $X$ and $\mathcal{T}$.

In the theory of approximating $\mathbb{P}\{\sup_{0\le t\le T} X(t) \ge u\}$ for a fixed domain $[0, T]$, the asymptotic results for smooth Gaussian processes [cf. Adler and Taylor (2007) and Azaïs and M. Wschebor (2009)] are much more accurate than those for non-smooth Gaussian processes [cf. Piterbarg (1996a)]. More specifically, under certain smoothness condition, one can get a higher-order approximation such that the error term decays exponentially faster than the principle term. Motivated by this, one may expect that for smooth Gaussian processes over random interval $[0, \mathcal{T}]$, the following approximation holds under certain conditions:

$$\mathbb{P}\Big\{\sup_{0\le t\le \mathcal{T}} X(t) \ge u\Big\} = g_2(u)(1 + o(e^{-\alpha u^2})), \quad \text{as } u \to \infty, \tag{5.0.2}$$

for some $\alpha > 0$, where $g_2(u)$ is a function depending on $X$ and $\mathcal{T}$. Obviously, compared with (5.0.1), (5.0.2) provides a much more accurate approximation. In this chapter, we apply the Rice method [cf. Azaïs and Delmas (2002) and Azaïs and M. Wschebor (2009)] to prove our main results Theorem 5.1.6 and Theorem 5.2.5 which are of the form as (5.0.2).

## 5.1 Stationary Gaussian Processes

Let $X = \{X(t) : t \in \mathbb{R}_+\}$ be a centered stationary Gaussian process with $\mathrm{Var}(X(0)) = 1$. Define

$$r(t) := \mathbb{E}\{X(t)X(0)\}, \quad \lambda^2 := \mathrm{Var}(X'(0)).$$

We will impose conditions (**H**1), (**H**3) and the following regularity condition (**A**1) on $X$.

(**A**1). For any fixed $\delta > 0$, $\sup_{t\ge\delta} r(t) < 1$.

The number of maxima above level $u$ over $[0,T]$ becomes

$$M_u(0,T) = \#\{t \in (0,T) : X(t) \geq u, X'(t) = 0, X''(t) < 0\}. \tag{5.1.1}$$

Note that for each fixed $t$, $X(t)$ and $X'(t)$ are independent, by (2.3.1), we have the following upper bound for the excursion probability,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{0\leq t\leq T} X(t) \geq u\Big\} \\
&\quad \leq \mathbb{P}\{X(0) \geq u, X'(0) \leq 0\} + \mathbb{P}\{X(T) \geq u, X'(T) \geq 0\} + \mathbb{E}\{M_u(0,T)\} \\
&\quad = \Psi(u) + \mathbb{E}\{M_u(0,T)\}.
\end{aligned} \tag{5.1.2}$$

Similarly, by (2.3.2), the lower bound becomes

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{0\leq t\leq T} X(t) \geq u\Big\} \\
&\quad \geq \Psi(u) + \mathbb{E}\{M_u(0,T)\} - \frac{1}{2}\mathbb{E}\{M_u(0,T)(M_u(0,T)-1)\} \\
&\qquad - \mathbb{P}\{X(0) \geq u, X'(0) \leq 0, X(T) \geq u, X'(T) \geq 0\} \\
&\qquad - \mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(0)\geq u, X'(0)\leq 0\}}\} - \mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(T)\geq u, X'(T)\geq 0\}}\}.
\end{aligned} \tag{5.1.3}$$

**Lemma 5.1.1** *Let $X$ be a centered stationary Gaussian process satisfying* ($\mathbf{H}1$) *and* ($\mathbf{H}3$). *Then there exists some universal $\alpha > 0$ such that for all $T > 0$, as $u \to \infty$,*

$$\mathbb{E}\{M_u(0,T)\} = \frac{\lambda T}{2\pi} e^{-u^2/2}(1 + o(e^{-\alpha u^2})).$$

**Proof** Due to (**H**1) and (**H**3), one can use the Kac-Rice metatheorem and get

$$\begin{aligned}
\mathbb{E}\{M_u(0,T)\} &= \int_0^T p_{X'(t)}(0)\mathbb{E}\{|X''(t)|\mathbb{1}_{\{X(t)\geq u, X''(t)<0\}}|X'(t)=0\}dt \\
&= -\int_0^T \frac{1}{2\pi\lambda}dt \int_u^\infty \mathbb{E}\{X''(t)\mathbb{1}_{\{X''(t)<0\}}|X(t)=x, X'(t)=0\}e^{-x^2/2}dx \\
&= -\frac{T}{2\pi\lambda}dt \int_u^\infty \mathbb{E}\{X''(0)\mathbb{1}_{\{X''(0)<0\}}|X(0)=x\}e^{-x^2/2}dx,
\end{aligned} \tag{5.1.4}$$

where the second equality is due to the independence of $X(t)$ and $X'(t)$, the last equality is due to the stationarity of $X$. Note that $\mathbb{E}\{X(0)X''(0)\} = -\lambda^2$, by Lemma 2.5.1,

$$\mathbb{E}\{X''(0)|X(0)=x\} = -\lambda^2 x.$$

Make change of variable $V = X''(0) + \lambda^2 x$, then $V|X(0)=x$ is a Gaussian variable with mean 0 and variance $\kappa^2 = \mathrm{Var}(X''(0)|X(0))$. Let us denote its density by $g(v)$, then

$$\begin{aligned}
&\mathbb{E}\{X''(0)\mathbb{1}_{\{X''(0)<0\}}|X(0)=x\} \\
&= \mathbb{E}\{X''(0)|X(0)=x\} - \mathbb{E}\{X''(0)\mathbb{1}_{\{X''(0)\geq 0\}}|X(0)=x\} \\
&= -\lambda^2 x - \int_{v\geq \lambda^2 x}(v-\lambda^2 x)g(v)dv.
\end{aligned} \tag{5.1.5}$$

But the last integral in (5.1.5) is non-negative and bounded by

$$\int_{v\geq\lambda^2 x} vg(v)dv = \int_{v\geq\lambda^2 x}\frac{v}{\sqrt{2\pi}\kappa}e^{-\frac{v^2}{2\kappa^2}}dv = \frac{\kappa}{\sqrt{2\pi}}e^{-\frac{\lambda^4 x^2}{2\kappa^2}}.$$

Since $\lambda^2$ and $\kappa^2$ are both constants not depending on $T$, plugging (5.1.5) into (5.1.4), we obtain the desired result by choosing some $\alpha \in (0, \lambda^4/(2\kappa^2))$. $\square$

**Lemma 5.1.2** *Let $X$ be a centered stationary Gaussian process satisfying* $(\mathbf{H}1)$, $(\mathbf{H}3)$ *and* $(\mathbf{A}1)$. *For $t_2 > t_1 > 0$, let*

$$f(t_1, t_2) := \min\Big\{1, \inf_{t_1 \le t \le t_2} \text{detCov}(X(0), X'(0), X(t), X'(t))\Big\}. \tag{5.1.6}$$

*Then for any $\varepsilon > 0$, there exist positive constants $C$, $\delta$ and $\varepsilon_1$ such that for all $(0, T) \subset \mathbb{R}$ and $u$ large enough,*

$$\begin{aligned} &\mathbb{E}\{M_u(0,T)(M_u(0,T)-1)\} \\ &\quad \le CT \exp\Big\{-\frac{u^2}{2\beta^2+\varepsilon}\Big\} + T^2(f(\delta,T))^{-5/2}\exp\Big\{-\frac{u^2}{2-\varepsilon_1}\Big\}, \end{aligned}$$

*where $C, \delta$ and $\varepsilon_1$ do not depend on $T$ and $\beta^2 = \text{Var}(X(0)|X''(0)) < 1$.*

**Remark 5.1.3** Note that $X$ is stationary, hence for any fixed $t \ge 0$, $X'(t)$ is independent of both $X(t)$ and $X''(t)$, thus

$$\begin{aligned} \text{detCov}(X(0), X'(0), X(t), X'(t)) &= \det \begin{pmatrix} 1 & 0 & r(t) & r'(t) \\ 0 & \lambda^2 & -r'(t) & -r''(t) \\ r(t) & -r'(t) & 1 & 0 \\ r'(t) & -r''(t) & 0 & \lambda^2 \end{pmatrix} \\ &= [\lambda^4 - (r''(t))^2][1 - r^2(t)] + (r'(t))^2[(r'(t))^2 - 2r(t)r''(t) - 2\lambda^2]. \end{aligned}$$

**Proof** Let $b > a \ge 0$ and $b - a \le 2\delta$ for some $\delta > 0$. Due to $(\mathbf{H}1)$ and $(\mathbf{H}3)$, one can use the Kac-Rice formula for factorial moments [cf. Theorem 11.5.1 in Adler and Taylor (2007)], thus

$$
\begin{aligned}
&\mathbb{E}\{M_u(a,b)(M_u(a,b)-1)\} \\
&\quad = \int_a^b dt \int_a^b ds\, p_{X'(t),X'(s)}(0,0) \\
&\qquad \times \mathbb{E}\{|X''(t)X''(s)| \mathbb{1}_{\{X(t)\geq u, X''(t)<0\}} \mathbb{1}_{\{X(s)\geq u, X''(s)<0\}} | X'(t)=X'(s)=0\} \qquad (5.1.7) \\
&\quad \leq \int_a^b dt \int_a^b ds \int_u^\infty dx\, p_{X(t)}(x|X'(t)=X'(s)=0) p_{X'(t),X'(s)}(0,0) \\
&\qquad \times \mathbb{E}\{|X''(t)X''(s)| | X(t)=x, X'(t)=X'(s)=0\}.
\end{aligned}
$$

Let $E(t,s) = \mathbb{E}\{|X''(t)X''(s)||X(t)=x, X'(t)=X'(s)=0\}$. By Taylor's formula,

$$
X'(s) = X'(t) + X''(t)(s-t) + |s-t|^{1+\eta} Y_{t,s}, \qquad (5.1.8)
$$

where $Y_{t,s}$ is a centered Gaussian variable. In particular, for $s > t$,

$$
Y_{t,s} = \frac{X'(s) - X'(t) - X''(t)(s-t)}{|s-t|^{1+\eta}} = \frac{\int_t^s (X''(v) - X''(t))dv}{(s-t)^{1+\eta}},
$$

and thus by ($\mathbf{H}1$), $\mathrm{Var}(Y_{t,s}) \leq L^2$. Due to (5.1.8), we have

$$
\begin{aligned}
E(t,s) &= \mathbb{E}\{|X''(t)X''(s)||X(t)=x, X'(t)=0, X''(t)(s-t) = -|s-t|^{1+\eta} Y_{t,s}\} \\
&= |s-t|^{\eta} \mathbb{E}\{|Y_{t,s} X''(s)||X(t)=x, X'(t)=0, X''(t)(s-t) = -|s-t|^{1+\eta} Y_{t,s}\}.
\end{aligned}
\qquad (5.1.9)
$$

By stationarity and (**H**1),

$$
\begin{aligned}
\operatorname{Var}(X''(t)|X(t)=x, X'(t)=X'(s)=0) &\le \operatorname{Var}(X''(t)) \le C_1,\\
\operatorname{Var}(Y_{t,s}|X(t)=x, X'(t)=X'(s)=0) &\le \operatorname{Var}(Y_{t,s}) \le L^2,
\end{aligned}
\tag{5.1.10}
$$

where $C_1$ is a positive constant. On the other hand, for $|s-t|$ small enough,

$$
\begin{aligned}
&|\mathbb{E}\{X''(s)|X(t)=x, X'(t)=X'(s)=0\}|\\
&\quad = |\mathbb{E}\{X''(s)|X(t)=x, X'(t)=0, X''(t)+|s-t|^{\eta}Y_{t,s}=0\}|\\
&\quad = |\mathbb{E}\{X''(s)|X(t)=x, X'(t)=0, X''(t)=0\}|(1+o(1))\\
&\quad \le C_2|x|,
\end{aligned}
\tag{5.1.11}
$$

and similarly,

$$
|\mathbb{E}\{Y_{t,s}|X(t)=x, X'(t)=X'(s)=0\}| \le C_3|x|, \tag{5.1.12}
$$

for some $C_2, C_3 > 0$. Note that for any Gaussian variables $\xi_1$ and $\xi_2$,

$$
\mathbb{E}|\xi_1\xi_2| \le \mathbb{E}\xi_1^2 + \mathbb{E}\xi_2^2 = (\mathbb{E}\xi_1)^2 + \operatorname{Var}(\xi_1) + (\mathbb{E}\xi_2)^2 + \operatorname{Var}(\xi_2). \tag{5.1.13}
$$

Applying (5.1.13) and plugging (5.1.10), (5.1.11) and (5.1.12) into (5.1.9), we obtain that there exists some $C_4 > 0$ such that

$$
E(t,s) \le C_4|s-t|^{\eta}(1+x^2). \tag{5.1.14}
$$

By Taylor's formula (5.1.8),

$$
\begin{aligned}
&\mathrm{Var}(X(t)|X'(t), X'(s)) \\
&\quad = \mathrm{Var}(X(t)|X'(t), X'(t) + X''(t)(s-t) + |s-t|^{1+\eta}Y_{t,s}) \\
&\quad = \mathrm{Var}(X(t)|X'(t), X''(t) \pm |s-t|^{\eta}Y_{t,s}) \qquad (5.1.15)\\
&\quad = \mathrm{Var}(X(t)|X'(t), X''(t))(1+o(1)) \\
&\quad = \mathrm{Var}(X(0)|X''(0))(1+o(1)),
\end{aligned}
$$

where the last equality is due to the fact that $X'(t)$ is independent of both $X(t)$ and $X''(t)$. Hence for any $\varepsilon > 0$, if $|b-a|$ is small enough and $u$ is large enough, then

$$
\int_u^\infty (1+x^2)p_{X(t)}(x|X'(t) = X'(s) = 0)dx \le e^{-\frac{u^2}{2\beta^2+\varepsilon}}. \tag{5.1.16}
$$

Note that

$$
p_{X'(t),X'(s)}(0,0) \le \frac{1}{2\pi\sqrt{\mathrm{detCov}(X'(t), X'(s))}}
$$

and by Taylor's formula (5.1.8),

$$
\begin{aligned}
\mathrm{detCov}(X'(t), X'(s)) &= \mathrm{detCov}(X'(t), X'(t) + X''(t)(s-t) \pm |s-t|^{1+\eta}Y_{t,s}) \\
&= |s-t|^2 \mathrm{detCov}(X'(t), X''(t) \pm |s-t|^{\eta}Y_{t,s}) \qquad (5.1.17)\\
&= |s-t|^2 \mathrm{detCov}(X'(t), X''(t))(1+o(1)),
\end{aligned}
$$

as $|s-t| \to 0$ uniformly. Thus there exists some $C_5 > 0$ such that for $|s-t|$ sufficiently small,

$$
p_{X'(t),X'(s)}(0,0) \le \frac{C_5}{|s-t|}. \tag{5.1.18}
$$

Plugging (5.1.14), (5.1.16) and (5.1.18) into (5.1.7) we obtain that for any $\varepsilon > 0$, if $\delta$ is small enough, then there exists $C_6 > 0$ such that for large u,

$$\begin{aligned}
\mathbb{E}\{M_u(a,b)(M_u(a,b)-1)\} &\le C_4 C_5 e^{-\frac{u^2}{2\beta^2+\varepsilon}} \int_a^b \int_a^b |s-t|^{\eta-1} dt ds \\
&\le C_6 (b-a)^{1+\eta} e^{-\frac{u^2}{2\beta^2+\varepsilon}} \\
&\le C_6 (b-a) e^{-\frac{u^2}{2\beta^2+\varepsilon}}.
\end{aligned} \tag{5.1.19}$$

The set $[0,T]$ may be covered by congruent intervals $I_i = [a_i, a_{i+1}]$ with the same length $\delta$ and disjoint interiors. Then

$$\begin{aligned}
&\mathbb{E}\{M_u(0,T)(M_u(0,T)-1)\} \le \mathbb{E}\Big\{ \sum_i M_u(I_i) \Big( \sum_i (M_u(I_i)-1) \Big) \Big\} \\
&= \mathbb{E}\Big\{ \sum_i M_u(I_i) \sum_j M_u(I_j) - \sum_i M_u(I_i) \Big\} \\
&= \sum_i \mathbb{E}\{M_u(I_i)^2\} + \sum_{i \ne j} \mathbb{E}\{M_u(I_i) M_u(I_j)\} - \sum_i \mathbb{E}\{M_u(I_i)\} \\
&= \sum_i \mathbb{E}\{M_u(I_i)(M_u(I_i)-1)\} + \sum_{i \ne j} \mathbb{E}\{M_u(I_i) M_u(I_j)\}.
\end{aligned} \tag{5.1.20}$$

If $I_i$ and $I_j$ are neighboring, say $j = i+1$, we have

$$\begin{aligned}
&\mathbb{E}\{M_u(I_i \cup I_{i+1})(M_u(I_i \cup I_{i+1}) - 1)\} \\
&\quad = \mathbb{E}\{(M_u(I_i) + M_u(I_{i+1}))(M_u(I_i) + M_u(I_{i+1}) - 1)\} \\
&\quad = 2\mathbb{E}\{M_u(I_i) M_u(I_{i+1})\} + \mathbb{E}\{M_u(I_i)(M_u(I_i)-1)\} + \mathbb{E}\{M_u(I_{i+1})(M_u(I_{i+1})-1)\}.
\end{aligned} \tag{5.1.21}$$

It follows from (5.1.19) and (5.1.21) that for any $\varepsilon > 0$, if $\delta$ is small enough and $u$ is large enough, then

$$\begin{aligned}
\mathbb{E}\{M_u(I_i)M_u(I_{i+1})\} &\le \frac{1}{2}\mathbb{E}\{M_u(I_i \cup I_{i+1})(M_u(I_i \cup I_{i+1}) - 1)\} \\
&= \frac{1}{2}\mathbb{E}\{M_u(a_i, a_{i+2})(M_u(a_i, a_{i+2}) - 1)\} \\
&\le \frac{C_6}{2}(a_{i+2} - a_i)e^{-\frac{u^2}{2\beta^2+\varepsilon}}
\end{aligned} \tag{5.1.22}$$

and hence

$$\sum_i \mathbb{E}\{M_u(I_i)(M_u(I_i) - 1)\} + \sum_{i\neq j, I_i \cap I_{i+1} \neq \emptyset} \mathbb{E}\{M_u(I_i)M_u(I_{i+1})\} \le 2C_6 T e^{-\frac{u^2}{2\beta^2+\varepsilon}}. \tag{5.1.23}$$

Next we consider the case when $I_i = [a_i, a_{i+1}]$ and $I_j = [a_j, a_{j+1}]$ are non-neighboring, which implies $a_j - a_{i+1} \ge \delta$. By the Kac-Rice formula for higher moments [cf. the proof is the same as that of Theorem 11.5.1 in Adler and Taylor (2007)],

$$\begin{aligned}
&\mathbb{E}\{M_u(I_i)M_u(I_j)\} \\
&= \int_{a_i}^{a_{i+1}} dt \int_{a_j}^{a_{j+1}} ds\, p_{X'(t),X'(s)}(0,0) \\
&\quad \times \mathbb{E}\{|X''(t)X''(s)| \mathbb{1}_{\{X(t)\ge u, X''(t)<0\}} \mathbb{1}_{\{X(s)\ge u, X''(s)<0\}} | X'(t) = X'(s) = 0\} \\
&\le \int_{a_i}^{a_{i+1}} dt \int_{a_j}^{a_{j+1}} ds \int_u^\infty dx \int_u^\infty dy\, p_{X'(t),X'(s)}(0,0|X(t) = x, X(s) = y) p_{X(t),X(s)}(x,y) \\
&\quad \times \mathbb{E}\{|X''(t)X''(s)| | X(t) = x, X(s) = y, X'(t) = X'(s) = 0\}.
\end{aligned} \tag{5.1.24}$$

By Lemma 2.5.1 and the stationarity of $X$, there exists some $C_7 > 0$ such that

$$
\begin{aligned}
&|\mathbb{E}\{X''(t)|X(t) = x, X(s) = y, X'(t) = X'(s) = 0\}| \\
&\quad \le \frac{C_7(|x| + |y|)}{\det\mathrm{Cov}(X(t), X(s), X'(t), X'(s))}
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathbb{E}\{X''(s)|X(t) = x, X(s) = y, X'(t) = X'(s) = 0\}| \\
&\quad \le \frac{C_7(|x| + |y|)}{\det\mathrm{Cov}(X(t), X(s), X'(t), X'(s))}.
\end{aligned}
$$

Together with (5.1.13), similarly to (5.1.14), we obtain that

$$
\begin{aligned}
&\mathbb{E}\{|X''(t)X''(s)||X(t) = x, X(s) = y, X'(t) = X'(s) = 0\} \\
&\quad \le C_8\bigg(1 + \frac{x^2 + y^2}{[\det\mathrm{Cov}(X(t), X(s), X'(t), X'(s))]^2}\bigg),
\end{aligned}
\tag{5.1.25}
$$

for some $C_8 > 0$. On the other hand,

$$
\begin{aligned}
&p_{X'(t),X'(s)}(0, 0|X(t) = x, X(s) = y) \\
&\quad \le \frac{1}{2\pi\sqrt{\det\mathrm{Cov}(X'(t), X'(s)|X(t), X(s))}} \\
&\quad = \frac{1}{2\pi}\sqrt{\frac{\det\mathrm{Cov}(X(t), X(s))}{\det\mathrm{Cov}(X(t), X(s), X'(t), X'(s))}} \\
&\quad \le \frac{C_9}{\sqrt{\det\mathrm{Cov}(X(t), X(s), X'(t), X'(s))}},
\end{aligned}
\tag{5.1.26}
$$

for some $C_9 > 0$. Plugging (5.1.25) and (5.1.26) into (5.1.24), we obtain that for large u,

$$
\begin{aligned}
&\mathbb{E}\{M_u(I_i)M_u(I_j)\} \\
&\quad \le 2(a_{i+1}-a_i)(a_{j+1}-a_j)(f(\delta,T))^{-5/2}\int_u^\infty\int_u^\infty (x^2+y^2)p_{X(t),X(s)}(x,y)dxdy.
\end{aligned} \tag{5.1.27}
$$

Let $R(\delta) := \sup_{t\ge\delta} r(t)$ which is strictly less than 1 by (**H**3), then for sufficiently large $u$,

$$
\begin{aligned}
&\sup_{|s-t|\ge\delta}\int_u^\infty\int_u^\infty (x^2+y^2)p_{X(t),X(s)}(x,y)dxdy \\
&\quad \le \sup_{|s-t|\ge\delta}\mathbb{E}\{(X(t)X(s))^2\mathbb{1}_{\{X(t)\ge u, X(s)\ge u\}}\} \\
&\quad \le \sup_{|s-t|\ge\delta}\mathbb{E}\{(X(t)+X(s))^4)\mathbb{1}_{\{X(t)+X(s)\ge 2u\}}\} \\
&\quad \le u^4 e^{-u^2/(1+R(\delta))}.
\end{aligned} \tag{5.1.28}
$$

Let $\varepsilon_1 > 1-R(\delta)$, by (5.1.27) and (5.1.28), we obtain that for sufficiently large $u$,

$$
\sum_{i\ne j, I_i\cap I_j=\emptyset}\mathbb{E}\{M_u(I_i)M_u(I_j)\} \le T^2(f(\delta,T))^{-5/2}e^{-u^2/(2-\varepsilon_1)}. \tag{5.1.29}
$$

Combining (5.1.20) and (5.1.23) with (5.1.29), we obtain the desired result. $\square$

**Lemma 5.1.4** *Let $X$ be a centered stationary Gaussian process satisfying* (**H**1), (**H**3) *and* (**A**1). *Then there exist some universal positive constants $\delta$ and $\alpha$ such that for all $T > 0$ and $u$ large enough,*

$$
\mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(0)\ge u, X'(0)\le 0\}}\} \le T[f(\delta,T)]^{-3/2}e^{-\alpha u^2 - u^2/2},
$$

$$
\mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(T)\ge u, X'(T)\ge 0\}}\} \le T[f(\delta,T)]^{-3/2}e^{-\alpha u^2 - u^2/2},
$$

*where $f(\delta, T)$ is defined in (5.1.6).*

**Proof** We shall only prove the first inequality, since the proof for the second one is similar. By Kac-Rice formula,

$$
\begin{aligned}
&\mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(0)\ge u, X'(0)\le 0\}}\} \\
&= \int_0^T dt \int_u^\infty dx \int_u^\infty dy \int_{-\infty}^0 dz\, \mathbb{E}\{|X''(t)|\mathbb{1}_{\{X''(t)<0\}}|X(t)=x, X(0)=y, \\
&\quad X'(t)=0, X'(0)=z\}p_{X(t),X(0),X'(t),X'(0)}(x,y,0,z) \\
&\le \int_0^T dt \int_u^\infty dx \int_u^\infty dy \int_{-\infty}^0 dz\, \mathbb{E}\{|X''(t)||X(t)=x, X(0)=y, X'(t)=0, X'(0)=z\} \\
&\quad \times p_{X(t),X(0),X'(t),X'(0)}(x,y,0,z) \\
&:= \int_0^T A_u(t)dt.
\end{aligned}
$$

Let $\delta$ be a positive constant to be specified. We first consider

$$
\begin{aligned}
\int_\delta^T A_u(t)dt \le \int_\delta^T dt \int_u^\infty dx \int_u^\infty dy\, &\mathbb{E}\{|X''(t)||X(t)=x, X(0)=y, X'(t)=0\} \\
&\times p_{X'(t)}(0|X(t)=x, X(0)=y)p_{X(t),X(0)}(x,y).
\end{aligned}
\tag{5.1.30}
$$

Note that $\mathbb{E}|\xi| \le \mathbb{E}|\xi - \mathbb{E}\xi| + |\mathbb{E}\xi| \le \sqrt{\mathrm{Var}(\xi)} + |\mathbb{E}\xi|$ for any random variable $\xi$,

$$
\mathrm{Var}(X''(t)|X(t), X(0), X'(t)) \le \mathrm{Var}(X''(t)) = \mathrm{Var}(X''(0)).
$$

By Lemma 2.5.1 and the stationarity of $X$, there exists some $C_1 > 0$ such that

$$
\begin{aligned}
&\sup_{\delta \le t \le T} |\mathbb{E}\{X''(t)|X(t) = x, X(0) = y, X'(t) = 0\}| \\
&\quad \le \sup_{\delta \le t \le T} \frac{C_1(|x| + |y|)}{\det\mathrm{Cov}(X(t), X(0), X'(t))} \\
&\quad = \sup_{\delta \le t \le T} \frac{C_1(|x| + |y|)\mathrm{Var}(X'(0)|X(t), X(0), X'(t))}{\det\mathrm{Cov}(X(t), X(0), X'(t), X'(0))} \\
&\quad \le C_1\lambda^2[f(\delta, T)]^{-1}(|x| + |y|).
\end{aligned}
$$

It follows that there exists some $C_2 > 0$ such that

$$
\mathbb{E}\{|X''(t)||X(t) = x, X(0) = y\} \le C_2(1 + [f(\delta, T)]^{-1})(|x| + |y|). \tag{5.1.31}
$$

Similarly, there exists some $C_3 > 0$ such that

$$
\begin{aligned}
&\sup_{\delta \le t \le T} p_{X'(t)}(0|X(t) = x, X(0) = y) \\
&\quad \le \sup_{\delta \le t \le T} \frac{1}{\sqrt{2\pi\mathrm{Var}(X'(t)|X(t), X(0))}} \le C_3[f(\delta, T)]^{-1/2}.
\end{aligned} \tag{5.1.32}
$$

On the other hand, for sufficiently large $u$,

$$
\begin{aligned}
&\sup_{t \ge \delta} \int_u^\infty dx \int_u^\infty dy(x + y)p_{X(t),X(0)}(x, y) \\
&\quad = \sup_{t \ge \delta} \mathbb{E}\{(X(t) + X(0))\mathbb{1}_{\{X(t) \ge u, X(0) \ge u\}}\} \\
&\quad \le \sup_{t \ge \delta} \mathbb{E}\{(X(t) + X(0))\mathbb{1}_{\{X(t) + X(0) \ge 2u\}}\} \le ue^{-u^2/(1+R(\delta))}.
\end{aligned} \tag{5.1.33}
$$

Plugging (5.1.31), (5.1.32) and (5.1.33) into (5.1.30), we obtain that for all $T > 0$, if $u$ is sufficiently large, then

$$\int_\delta^T A_u(t)dt \le T[f(\delta,T)]^{-3/2} e^{-\frac{u^2}{2-\varepsilon_1}},$$

where $\varepsilon_1 > 1 - R(\delta)$.

Next we consider

$$\int_0^\delta A_u(t)dt \le \int_0^\delta dt \int_u^\infty dy \int_{-\infty}^0 dz \mathbb{E}\{|X''(t)||X(0) = y, X'(0) = z, X'(t) = 0\}$$
$$\times p_{X(0),X'(0),X'(t)}(y,z,0).$$

Note that

$$\begin{aligned} &p_{X(0),X'(0),X'(t)}(y,z,0) \\ &= p_{X(0)}(y|X'(0) = z, X'(t) = 0) p_{X'(0)}(z|X'(t) = 0) p_{X'(t)}(0) \\ &\le (2\pi)^{-3/2}[\det\mathrm{Cov}(X(0), X'(0), X'(t))]^{-1/2} e^{-\frac{(y-\mu_{t,z})^2}{2\sigma_t^2}} e^{-\frac{z^2}{2\gamma_t^2}}, \end{aligned}$$

where

$$\begin{aligned} \mu_{t,z} &= \mathbb{E}\{X(0)|X'(0) = z, X'(t) = 0\}, \\ \sigma_t^2 &= \mathrm{Var}(X(0)|X'(0), X'(t)), \\ \gamma_t^2 &= \mathrm{Var}(X'(0)|X'(t)). \end{aligned}$$

By (**H**1) and Taylor's formula, we can write

$$X'(t) = X'(0) + X''(0)t + Y_{0,t}t^{1+\eta},$$

where $Y_{0,t}$ is a centered Gaussian variable. We find that for $t \in (0, \delta)$ with $\delta$ sufficiently small,

$$\begin{aligned}\mathbb{E}\{X(0)X'(t)\} &= t(\mathbb{E}\{X(0)X''(0)\} + t^{\eta}\mathbb{E}\{X(0)Y_{0,t}\}) \\ &= t(-\lambda^2 + t^{\eta}\mathbb{E}\{X(0)Y_{0,t}\}) \leq 0.\end{aligned}$$

Since $z < 0$, it follows from Lemma 2.5.1 that $\mu_{t,z} \leq 0$. If $\delta$ is sufficiently small, we also have, similarly to (5.1.15),

$$\sigma_t^2 = \mathrm{Var}(X(0)|X''(0))(1 + o(1)) < 1 - \varepsilon_0,$$

and similarly to (5.1.17),

$$C_4 t^2 \leq \gamma_t^2 \leq C_5 t^2,$$

$$\mathrm{detCov}(X(0), X'(0), X'(t)) \geq C_6 t^2,$$

where $\varepsilon_0, C_4, C_5$ and $C_6$ are some positive constants. Together with the fact that

$$\begin{aligned}&\mathbb{E}\{|X''(t)||X(0) = y, X'(0) = z, X'(t) = 0\} \\ &\quad = \mathbb{E}\{|X''(t)||X(0) = y, X'(t) - tX''(t) + t^{1+\eta}Y_{t,0} = z, X'(t) = 0\} \\ &\quad = \mathbb{E}\{|X''(t)||X(0) = y, X''(t) - t^{\eta}Y_{t,0} = z/t, X'(t) = 0\} \\ &\quad \leq C_7(y + |z/t| + 1)\end{aligned}$$

for some $C_7 > 0$, where the first equality is due to Taylor's formula, we obtain that for $\delta$ sufficiently small and $u$ sufficiently large,

$$\begin{aligned}\int_0^\delta A_u(t)dt &\le (2\pi)^{-3/2}C_6^{-1/2}\int_0^\delta \frac{1}{t}dt\int_u^\infty dy\int_{-\infty}^0 dz(y+|z/t|+1)e^{-\frac{(y-\mu_{t,z})^2}{2\sigma_t^2}}e^{-\frac{z^2}{2\gamma_t^2}}\\ &\le (2\pi)^{-3/2}C_6^{-1/2}\int_0^\delta \frac{1}{t}dt\int_u^\infty dy\int_{-\infty}^0 dz(y+|z/t|+1)e^{-\frac{y^2}{2\sigma_t^2}}e^{-\frac{z^2}{2C_5t^2}}\\ &= (2\pi)^{-3/2}C_6^{-1/2}\int_0^\delta dt\int_u^\infty dy\int_{-\infty}^0 dz(y+|z|+1)e^{-\frac{y^2}{2\sigma_t^2}}e^{-\frac{z^2}{2C_5}}\\ &\le \delta e^{-\frac{u^2}{2(1-\varepsilon_0)}}.\end{aligned} \tag{5.1.34}$$

Combining (5.1.30) with (5.1.34), we obtain that there exist some universal $\delta, \alpha > 0$ such that for all $T > 0$ and $u$ large enough,

$$\mathbb{E}\{M_u(0,T)\mathbb{1}_{\{X(0)\ge u, X'(0)\le 0\}}\} \le T[f(\delta,T)]^{-3/2}e^{-\alpha u^2 - u^2/2}.$$

This completes the proof. $\square$

**Lemma 5.1.5** *Let $X$ be a centered stationary Gaussian process satisfying* (**H**1)*,* (**H**3) *and* (**A**1)*. Then there exists some universal $\alpha > 0$ such that for all $T > 0$ and $u$ large enough,*

$$\mathbb{P}\{X(0)\ge u, X'(0)\le 0, X(T)\ge u, X'(T)\ge 0\} \le e^{-\alpha u^2 - u^2/2}.$$

**Proof** Let $\delta > 0$, then similarly to (5.1.33), we obtain that for sufficiently large $u$,

$$\begin{aligned}&\sup_{T\geq\delta}\mathbb{P}\{X(0)\geq u, X'(0)\leq 0, X(T)\geq u, X'(T)\geq 0\}\\&\quad\leq \sup_{T\geq\delta}\mathbb{P}\{X(0)\geq u, X(T)\geq u\}\leq e^{-u^2/(1+R(\delta))}.\end{aligned}\tag{5.1.35}$$

For $T\in(0,\delta)$, by Taylor's formula,

$$X'(T) = X'(0) + X''(0)T + Y_{0,T}T^{1+\eta},$$

it follows that

$$\begin{aligned}&\mathbb{P}\{X(0)\geq u, X'(0)\leq 0, X(T)\geq u, X'(T)\geq 0\}\\&\quad\leq \mathbb{P}\{X(0)\geq u, X'(0)\leq 0, X'(T)\geq 0\}\\&\quad= \mathbb{P}\{X(0)\geq u, X'(0)\leq 0, X'(0)+X''(0)T+Y_{0,T}T^{1+\eta}\geq 0\}\\&\quad\leq \mathbb{P}\{X(0)\geq u, X''(0)+Y_{0,T}T^{\eta}\geq 0\}.\end{aligned}\tag{5.1.36}$$

Let $\xi(T) = X''(0)+Y_{0,T}T^{\eta}$, $\kappa^2(T) = \mathrm{Var}(\xi(T))$ and $\rho(T) = \mathbb{E}\{X(0)\xi(T)\}/\kappa(T)$, $T\in(0,\delta)$. Since $\mathbb{E}\{X(0)X''(0)\} = -\lambda^2$, if $\delta$ is sufficiently small, $\sup_{0\leq T\leq\delta}\rho(T) < -\varepsilon_0$ for some $\varepsilon_0 > 0$. Let

$$\begin{aligned}\overline{\kappa} &= \sup_{0\leq T\leq\delta}\kappa(T), \quad \underline{\kappa} = \inf_{0\leq T\leq\delta}\kappa(T),\\ \overline{\rho} &= \sup_{0\leq T\leq\delta}\rho(T), \quad \underline{\rho} = \inf_{0\leq T\leq\delta}\rho(T),\end{aligned}$$

then $0 < \underline{\kappa} \le \overline{\kappa} < \infty$ and $-1 < \underline{\rho} \le \overline{\rho} < -\varepsilon_0$. We obtain that as $u \to \infty$,

$$
\begin{aligned}
&\sup_{0\le T\le\delta} \mathbb{P}\{X(0)\ge u, X''(0)+Y_{0,T}T^{\eta}\ge 0\} \\
&\quad = \sup_{0\le T\le\delta} \frac{1}{2\pi\kappa(T)(1-\rho^2(T))^{1/2}} \int_u^\infty dx_1 \int_0^\infty dx_2 \\
&\qquad \times \exp\left\{ -\frac{1}{2(1-\rho^2(T))}\left( x_1^2 + \frac{x_2^2}{\kappa^2(T)} - \frac{2\rho(T)x_1x_2}{\kappa(T)} \right) \right\} \\
&\quad = \sup_{0\le T\le\delta} \frac{1}{2\pi\kappa(T)(1-\rho^2(T))^{1/2}} \int_u^\infty dx_1 \exp\{-x_1^2/2\} \qquad\qquad (5.1.37) \\
&\qquad \times \int_0^\infty \exp\left\{ -\frac{(x_2-\kappa(T)\rho(T)x_1)^2}{2\kappa^2(T)(1-\rho^2(T))} \right\} dx_2 \\
&\quad \le \frac{1}{2\pi\underline{\kappa}(1-\underline{\rho}^2)^{1/2}} \int_u^\infty dx_1 \exp\{-x_1^2/2\} \int_0^\infty \exp\left\{ -\frac{(x_2-\underline{\kappa}\overline{\rho}x_1)^2}{2\overline{\kappa}^2(1-\overline{\rho}^2)} \right\} dx_2 \\
&= o\left( \exp\left\{ -\frac{u^2}{2} - \frac{\underline{\kappa}^2\overline{\rho}^2u^2}{2\overline{\kappa}^2(1-\overline{\rho}^2)} + \varepsilon u^2 \right\} \right),
\end{aligned}
$$

for any $\varepsilon > 0$. Combining (5.1.35) and (5.1.36) with (5.1.37) yields the result. $\square$

**Theorem 5.1.6** *Let $\{X(t) : t \in \mathbb{R}_+\}$ be a centered stationary Gaussian process satisfying* (**H**1)*,* (**H**3) *and* (**A**1)*, and let $\mathcal{T}$ be a non-negative random variable independent of $X$. If $\mathbb{E}\mathcal{T}^2(f(\delta,\mathcal{T}))^{-5/2} < \infty$ for any fixed $\delta > 0$, then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$
\mathbb{P}\Big\{ \sup_{0\le t\le\mathcal{T}} X(t) \ge u \Big\} = \Psi(u) + \frac{\lambda\mathbb{E}\mathcal{T}}{2\pi} e^{-u^2/2} + o(e^{-\alpha u^2 - u^2/2}).
$$

**Proof** Let $F_{\mathcal{T}}$ be the cumulative distribution function of $\mathcal{T}$. Note that

$$
\mathbb{P}\Big\{ \sup_{0\le t\le\mathcal{T}} X(t) \ge u \Big\} = \int_0^\infty \mathbb{P}\Big\{ \sup_{0\le t\le T} X(t) \ge u \Big\} F_{\mathcal{T}}(dT), \qquad (5.1.38)
$$

combining (5.1.2), (5.1.3), Lemma 5.1.1, Lemma 5.1.2 and Lemma 5.1.4 with Lemma 5.1.5, we obtain the result. □

**Example 5.1.7** Let $X$ be a centered stationary Gaussian process with covariance function $r(t) = e^{-\frac{t^2}{2}}$. Then $\mathrm{Var}(X'(0)) = \mathrm{Var}(X(0)) = 1, \mathbb{E}\{X'(t)X(0)\} = -\mathbb{E}\{X(t)X'(0)\} = r'(t) = -te^{-\frac{t^2}{2}}$ and $\mathbb{E}\{X'(t)X'(0)\} = -r''(t)$, thus

$$\det\mathrm{Cov}(X(t), X'(t), X(0), X'(0)) = (1 - e^{-t^2})^2, \tag{5.1.39}$$

which is increasing in $t > 0$. Hence if $\mathbb{E}\mathcal{T}^2 < \infty$, then

$$\mathbb{P}\Big\{\sup_{0\le t\le\mathcal{T}} X(t) \ge u\Big\} = \Psi(u) + \frac{\mathbb{E}\mathcal{T}}{2\pi}e^{-u^2/2} + o(e^{-\alpha u^2 - u^2/2}).$$

## 5.2 Gaussian Processes with Increasing Variance

In this section, we consider a Gaussian process $\{X(t) : t \in \mathbb{R}_+\}$ with increasing variance at infinity. Let

$$\sigma_t^2 = \mathrm{Var}(X(t)), \quad \lambda_t^2 = \mathrm{Var}(X'(t)), \quad \theta_t^2 = \mathrm{Var}(X(t)|X'(t)). \tag{5.2.1}$$

Let $\mathcal{T}$ be a non-negative random variable satisfying

$$\mathbb{P}\{\mathcal{T} \ge t\} = \exp\{-\beta t^\alpha(1 + o(1))\} \quad \text{as } t \to \infty, \tag{5.2.2}$$

where $\alpha, \beta > 0$. We write $\mathcal{T} \in \mathcal{E}(\alpha, \beta)$ if $\mathcal{T}$ satisfies (5.2.2). Note that (5.2.2) implies that the corresponding cumulative distribution function $F_{\mathcal{T}}(t)$ is continuous when $t$ is sufficiently large.

In additional to conditions (**H**1) and (**H**3), we will impose the following two conditions (**A**2) and (**A**3) on $X$.

(**A**2). There exist $\alpha_\infty > 0$ and $D_1 > D_2 > 0$ such that as $t \to \infty$,

$$\sigma_t^2 = D_1 t^{\alpha_\infty}(1 + o(1)), \quad \theta_t^2 = D_2 t^{\alpha_\infty}(1 + o(1)).$$

(**A**3). There exists $N_1 > 0$ such that as $t \to \infty$,

$$\max\{\lambda_t^2, \mathrm{Var}(X''(t)), (\det\mathrm{Cov}(X(t), X'(t)))^{-1}\} = O(t^{N_1}).$$

We will make use of the following inequality to estimate the excursion probability over each fixed interval $[0, T]$:

$$\mathbb{P}\{X(T) \geq u\} \leq \mathbb{P}\Big\{\sup_{0 \leq t \leq T} X(t) \geq u\Big\} \leq \mathbb{P}\{X(T) \geq u\} + \mathbb{P}\{X(0) \geq u\} + \mathbb{E}\{M_u(0, T)\}. \tag{5.2.3}$$

The following result is Lemma 6.2 in Arendarczyk and Decicki (2011), which is analogous to the Laplace method.

**Lemma 5.2.1** *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ *and* $a(u) = u^{(1-\delta)\alpha_1/(\alpha_1+\alpha_2)}$, $A(u) = u^{(1+\delta)\alpha_1/(\alpha_1+\alpha_2)}$, *where* $0 < \delta < \alpha_2/\alpha_1$. *Then as* $u \to \infty$,

$$\int_{a(u)}^{A(u)} \exp\Big\{-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\Big\} dx = \exp\{-\beta_3 u^{\alpha_3}(1 + o(1))\},$$

*where*

$$\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$
$$\beta_3 = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)}\beta_2^{\alpha_1/(\alpha_1+\alpha_2)}\Big[\Big(\frac{\alpha_1}{\alpha_2}\Big)^{\alpha_2/(\alpha_1+\alpha_2)} + \Big(\frac{\alpha_2}{\alpha_1}\Big)^{\alpha_1/(\alpha_1+\alpha_2)}\Big].$$

Now we prove a lemma similar to Lemma 2.1 in Arendarczyk and Decicki (2011).

**Lemma 5.2.2** *Let $X \in \mathcal{E}(\alpha_1,\beta_1), Y \in \mathcal{E}(\alpha_2,\beta_2)$ be independent non-negative random variables. Then $XY \in \mathcal{E}(\alpha,\beta)$ with*

$$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$
$$\beta = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)}\beta_2^{\alpha_1/(\alpha_1+\alpha_2)}\Big[\Big(\frac{\alpha_1}{\alpha_2}\Big)^{\alpha_2/(\alpha_1+\alpha_2)} + \Big(\frac{\alpha_2}{\alpha_1}\Big)^{\alpha_1/(\alpha_1+\alpha_2)}\Big].$$

**Proof** Let $a(u) = u^{(1-\delta)\alpha_1/(\alpha_1+\alpha_2)}$, $A(u) = u^{(1+\delta)\alpha_1/(\alpha_1+\alpha_2)}$, where $0 < \delta < \alpha_2/\alpha_1$. Then

$$\begin{aligned}
\mathbb{P}\{XY \geq u\} &= \int_0^\infty \mathbb{P}\{X \geq u/y\} dF_Y(y),\\
&= \int_0^{a(u)} \mathbb{P}\{X \geq u/y\} dF_Y(y) + \int_{a(u)}^{A(u)} \mathbb{P}\{X \geq u/y\} dF_Y(y)\\
&\quad + \int_{A(u)}^\infty \mathbb{P}\{X \geq u/y\} dF_Y(y)\\
&= I_1(u) + I_2(u) + I_3(u).
\end{aligned}$$

For any $\varepsilon > 0$ and $u$ large enough, we see that

$$\begin{aligned}
I_1(u) &\leq \mathbb{P}\{X \geq u/a(u)\} \leq \exp\{-(\beta_1-\varepsilon)[u/a(u)]^{\alpha_1}\}\\
&\leq \exp\{-(\beta_1-\varepsilon)u^{\alpha_1\alpha_2/(\alpha_1+\alpha_2)+\delta\alpha_1^2/(\alpha_1+\alpha_2)}\} = o(\exp\{-u^{\alpha_3+\varepsilon_0}\})
\end{aligned}$$

and

$$I_3(u) \le \mathbb{P}\{Y \ge A(u)\} \le \exp\{-(\beta_2 - \varepsilon)(A(u))^{\alpha_2}\}$$
$$\le \exp\{-(\beta_2 - \varepsilon)u^{(1+\delta)\alpha_1\alpha_2/(\alpha_1+\alpha_2)}\} = o(\exp\{-u^{\alpha_3+\varepsilon_0}\}).$$

Next we estimate $I_2$. Note that both $u/a(u)$ and $u/A(u)$ tend to $\infty$, hence for any $\varepsilon > 0$ and $u$ large enough, we have

$$\begin{aligned}
I_2(u) &\ge \int_{a(u)}^{A(u)} \exp\{-(\beta_1 + \varepsilon)(u/y)^{\alpha_1}\} dF_Y(y) \\
&= \int_{a(u)}^{A(u)} \frac{\partial}{\partial y} \exp\{-(\beta_1 + \varepsilon)(u/y)^{\alpha_1}\} \mathbb{P}\{Y \ge y\} dy \\
&\quad + \exp\{-(\beta_1 + \varepsilon)[u/a(u)]^{\alpha_1}\} \mathbb{P}\{Y \ge a(u)\} \\
&\quad - \exp\{-(\beta_1 + \varepsilon)[u/A(u)]^{\alpha_1}\} \mathbb{P}\{Y \ge A(u)\} \\
&\ge \int_{a(u)}^{A(u)} \exp\{-(\beta_1 + \varepsilon)(1 + \varepsilon)(u/y)^{\alpha_1}\} \exp\{-(\beta_2 + \varepsilon)u^{\alpha_2}\} dy \\
&\quad + \exp\{-(\beta_1 + \varepsilon)[u/a(u)]^{\alpha_1}\} \exp\{-(\beta_2 + \varepsilon)(a(u))^{\alpha_2}\} \\
&\quad - \exp\{-(\beta_1 + \varepsilon)[u/A(u)]^{\alpha_1}\} \exp\{-(\beta_2 - \varepsilon)(A(u))^{\alpha_2}\} \\
&= \underline{I}_2(u, \varepsilon) + \underline{R}_1(u, \varepsilon) - \underline{R}_2(u, \varepsilon),
\end{aligned}$$

and similarly,

$$\begin{aligned}
I_2(u) &\le \int_{a(u)}^{A(u)} \exp\{-(\beta_1 - \varepsilon)(1 - \varepsilon)(u/y)^{\alpha_1}\} \exp\{-(\beta_2 - \varepsilon)u^{\alpha_2}\} dy \\
&\quad + \exp\{-(\beta_1 - \varepsilon)[u/a(u)]^{\alpha_1}\} \exp\{-(\beta_2 - \varepsilon)(a(u))^{\alpha_2}\} \\
&\quad - \exp\{-(\beta_1 - \varepsilon)[u/A(u)]^{\alpha_1}\} \exp\{-(\beta_2 + \varepsilon)(A(u))^{\alpha_2}\} \\
&= \overline{I}_2(u, \varepsilon) + \overline{R}_1(u, \varepsilon) - \overline{R}_2(u, \varepsilon).
\end{aligned}$$

Applying Lemma 5.2.1, we obtain that for any $\varepsilon > 0$, as $u \to \infty$

$$\underline{I}_2(u,\varepsilon) = \exp\{-\underline{\beta}_3(\varepsilon)u^{\alpha_3}(1+o(1))\}, \quad \overline{I}_2(u,\varepsilon) = \exp\{-\overline{\beta}_3(\varepsilon)u^{\alpha_3}(1+o(1))\},$$

where

$$\begin{aligned}
\alpha_3 &= \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \\
\underline{\beta}_3(\varepsilon) &= [(\beta_1+\varepsilon)(1+\varepsilon)]^{\alpha_2/(\alpha_1+\alpha_2)}(\beta_2+\varepsilon)^{\alpha_1/(\alpha_1+\alpha_2)} \\
&\quad \times \Big[\Big(\frac{\alpha_1}{\alpha_2}\Big)^{\alpha_2/(\alpha_1+\alpha_2)} + \Big(\frac{\alpha_2}{\alpha_1}\Big)^{\alpha_1/(\alpha_1+\alpha_2)}\Big], \\
\overline{\beta}_3(\varepsilon) &= [(\beta_1-\varepsilon)(1-\varepsilon)]^{\alpha_2/(\alpha_1+\alpha_2)}(\beta_2-\varepsilon)^{\alpha_1/(\alpha_1+\alpha_2)} \\
&\quad \times \Big[\Big(\frac{\alpha_1}{\alpha_2}\Big)^{\alpha_2/(\alpha_1+\alpha_2)} + \Big(\frac{\alpha_2}{\alpha_1}\Big)^{\alpha_1/(\alpha_1+\alpha_2)}\Big].
\end{aligned}$$

Together with the fact that there exists some $\varepsilon_0 > 0$ such that all $I_1(u)$, $I_3(u)$, $\underline{R}_1(u,\varepsilon)$, $\underline{R}_2(u,\varepsilon)$, $\overline{R}_1(u,\varepsilon)$, $\overline{R}_2(u,\varepsilon)$ are $o(\exp\{-u^{\alpha_3+\varepsilon_0}\})$, we obtain the desired result. $\square$

**Lemma 5.2.3** *Let $X$ be a Gaussian process satisfying* (**A**2) *and* (**A**3) *and let $\mathcal{T} \in \mathcal{E}(\alpha,\beta)$ be a non-negative random variable independent of $X$. Then $X(\mathcal{T}) \in \mathcal{E}(\widetilde{\alpha},\widetilde{\beta}_1)$ with*

$$\begin{aligned}
\widetilde{\alpha} &= \frac{2\alpha}{\alpha+\alpha_\infty}, \\
\widetilde{\beta}_1 &= \beta^{\alpha_\infty/(\alpha+\alpha_\infty)}\Big(\frac{1}{2D_1}\Big)^{\alpha/(\alpha+\alpha_\infty)}\Big[\Big(\frac{\alpha}{\alpha_\infty}\Big)^{\alpha_\infty/(\alpha+\alpha_\infty)} + \Big(\frac{\alpha_\infty}{\alpha}\Big)^{\alpha/(\alpha+\alpha_\infty)}\Big].
\end{aligned} \tag{5.2.4}$$

**Proof** Let $\mathcal{N}$ be the standard Normal random variable and let $\nu(\cdot)$ be the standard deviation function of $X$, i.e. $\nu(t) = \sigma_t$. Note that

$$\mathbb{P}\{X(\mathcal{T}) \geq u\} = \mathbb{P}\{\nu(\mathcal{T})\mathcal{N} \geq u\}. \tag{5.2.5}$$

On the other hand, as $u \to \infty$,

$$\mathbb{P}\{\nu(\mathcal{T}) \geq u\} = \mathbb{P}\{\mathcal{T} \geq \nu^{-1}(u)\} = \exp\{-\beta(\nu^{-1}(u))^{\alpha}(1+o(1))\}$$

and

$$\nu^{-1}(u) = D_1^{-1/\alpha_\infty} u^{2/\alpha_\infty}(1+o(1)),$$

thus

$$\mathbb{P}\{\nu(\mathcal{T}) \geq u\} = \exp\{-\beta D_1^{-\alpha/\alpha_\infty} u^{2\alpha/\alpha_\infty}(1+o(1))\},$$

i.e., $\nu(\mathcal{T}) \in \mathcal{E}(2\alpha/\alpha_\infty, \beta D_1^{-\alpha/\alpha_\infty})$. Note that $\mathcal{N} \in \mathcal{E}(2, 1/2)$, applying Lemma 5.2.2 in (5.2.5), we conclude the result. $\square$

**Lemma 5.2.4** *Let $X$ be a Gaussian process satisfying* (**H**1), (**H**3), (**A**2) *and* (**A**3), *and let $\mathcal{T} \in \mathcal{E}(\alpha, \beta)$ be a non-negative random variable independent of $X$. Then for any $\varepsilon > 0$,*

$$\int_0^\infty \mathbb{E}\{M_u(0,T)\} F_{\mathcal{T}}(dT) = o(\exp\{-(\widetilde{\beta}_2 - \varepsilon)u^{\widetilde{\alpha}}\}) \quad \text{as } u \to \infty,$$

*where*

$$\begin{aligned} \widetilde{\alpha} &= \frac{2\alpha}{\alpha+\alpha_\infty}, \\ \widetilde{\beta}_2 &= \beta^{\alpha_\infty/(\alpha+\alpha_\infty)} \Big(\frac{1}{2D_2}\Big)^{\alpha/(\alpha+\alpha_\infty)} \Big[\Big(\frac{\alpha}{\alpha_\infty}\Big)^{\alpha_\infty/(\alpha+\alpha_\infty)} + \Big(\frac{\alpha_\infty}{\alpha}\Big)^{\alpha/(\alpha+\alpha_\infty)}\Big] > \widetilde{\beta}_1. \end{aligned} \tag{5.2.6}$$

**Proof** By the Kac-Rice formula,

$$
\begin{aligned}
&\mathbb{E}\{M_u(0,T))\} \\
&= \int_0^T dt \int_u^\infty dx\, \mathbb{E}\{|X''(t)| \mathbb{1}_{\{X''(t)<0\}} | X(t)=x, X'(t)=0\} p_{X(t),X'(t)}(x,0) \\
&\le \int_0^T dt \int_u^\infty dx\, \mathbb{E}\{|X''(t)| | X(t)=x, X'(t)=0\} p_{X(t)}(x|X'(t)=0) p_{X'(t)}(0) \\
&= \int_0^T \frac{1}{2\pi\lambda_t} dt \int_u^\infty dx\, \mathbb{E}\{|X''(t)| | X(t)=x, X'(t)=0\} \frac{1}{\theta_t} e^{-x^2/2\theta_t^2}
\end{aligned}
$$

Note that $\mathbb{E}|\xi| \le \mathbb{E}|\xi - \mathbb{E}\xi| + |\mathbb{E}\xi| \le \sqrt{\mathrm{Var}(\xi)} + |\mathbb{E}\xi|$ for any random variable $\xi$, and

$$
\mathrm{Var}(X''(t)|X(t), X'(t)) \le \mathrm{Var}(X''(t)),
$$

$$
\begin{aligned}
&\mathbb{E}\{X''(t)|X(t)=x, X'(t)=0\} \\
&\quad= \frac{\mathbb{E}\{X''(t)X(t)\}\lambda_t^2 - \mathbb{E}\{X''(t)X'(t)\}\mathbb{E}\{X'(t)X(t)\}}{\det\mathrm{Cov}(X(t), X'(t))} x.
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mathbb{E}\{|X''(t)| | X(t)=x, X'(t)=0\} \\
&\quad\le \sqrt{\mathrm{Var}(X''(t))} + \frac{|\mathbb{E}\{X''(t)X(t)\}\lambda_t^2 - \mathbb{E}\{X''(t)X'(t)\}\mathbb{E}\{X'(t)X(t)\}|}{\det\mathrm{Cov}(X(t), X'(t))} x,
\end{aligned}
$$

Now let

$$
h(t) \triangleq \frac{1}{\lambda_t\theta_t}\left( \sqrt{\mathrm{Var}(X''(t))} + \frac{|\mathbb{E}\{X''(t)X(t)\}\lambda_t^2 - \mathbb{E}\{X''(t)X'(t)\}\mathbb{E}\{X'(t)X(t)\}|}{\det\mathrm{Cov}(X(t), X'(t))} \right).
$$

By (**A**3), there is some $N_2 > 0$ such that

$$
h(t) = o(t^{N_2}) \quad \text{as } t \to \infty.
$$

Let $T_0$ be a large number such that for $t > T_0$, $h(t) \leq t^{N_1}$, $\theta_t^2$ is increasing and $\theta_t^2 \leq t^{\alpha_1+1}$. Let $A$ be a large number such that $\sup_{0\leq t\leq T_0} h(t) \leq A$ and $\sup_{0\leq t\leq T_0} \theta_t^2 \leq A$, then for $u$ large enough,

$$\begin{aligned}
\mathbb{E}\{M_u(0,T))\} &\leq \int_0^{T_0} \frac{1}{2\pi} h(t)dt \int_u^\infty dx e^{-x^2/2\theta_t^2} + \int_{T_0}^{T} \frac{1}{2\pi} h(t)dt \int_u^\infty dx e^{-x^2/2\theta_t^2} \\
&\leq \frac{T_0}{2\pi} A e^{-u^2/(2A)} + \int_{T_0}^{T} \frac{1}{2\pi} t^{N_1} dt \int_u^\infty dx e^{-x^2/2\theta_T^2} \\
&\leq \frac{T_0}{2\pi} A e^{-u^2/(2A)} + \frac{1}{\sqrt{2\pi}} T^{N_1+\alpha_1/2+3/2} \int_u^\infty dx \frac{1}{\sqrt{2\pi}\theta_T} e^{-x^2/2\theta_T^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_0^\infty \mathbb{E}\{M_u(0,T)\} dF_{\mathcal{T}}(T) \\
&\quad\leq \frac{T_0}{2\pi} A e^{-u^2/(2A)} + \int_0^\infty T^{N_1+\alpha_1+2} dF_{\mathcal{T}}(T) \int_u^\infty \frac{1}{\sqrt{2\pi}\theta_T} e^{-x^2/2\theta_T^2} dx, \\
&\quad= I_1(u) + I_2(u).
\end{aligned}$$

Let $\widehat{\mathcal{T}}$ be a non-negative random variable with cumulative distribution function satisfying $dF_{\widehat{\mathcal{T}}}(t) = t^{N_1+\alpha_1+2} dF_{\mathcal{T}}(t)$, then $\widehat{\mathcal{T}} \in \mathcal{E}(\alpha,\beta)$. Let $\{\widehat{X}(t) : t \in \mathbb{R}_+\}$ be a Gaussian process with $\mathrm{Var}(\widehat{X}(t)) = \theta_t^2$, then by Lemma 5.2.3,

$$\widehat{X}(\widehat{\mathcal{T}}) \in \mathcal{E}(\widetilde{\alpha}, \widetilde{\beta}_2),$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}_2$ are as shown in (5.2.6). Note that

$$I_2(u) = \mathbb{P}\{\widehat{X}(\widehat{\mathcal{T}}) \geq u\}$$

and $2 > \widetilde{\alpha}$ hence $I_1(u) = o(\exp\{-u^{\widetilde{\alpha}+\delta}\})$ for any $\delta \in (0, 2-\widetilde{\alpha})$. Thus both $I_1(u)$ and $I_2(u)$

are $o(\exp\{-(\widetilde{\beta}_2 - \varepsilon)u^{\widetilde{\alpha}}\})$ for any $\varepsilon > 0$. The proof is completed. $\square$

**Theorem 5.2.5** *Let $\{X(t) : t \in \mathbb{R}_+\}$ be a Gaussian process satisfying* (**H**1), (**H**3), (**A**2) *and* (**A**3), *and let $\mathcal{T} \in \mathcal{E}(\alpha, \beta)$ be a non-negative random variable independent of $X$. Then $X(\mathcal{T}) \in \mathcal{E}(\widetilde{\alpha}, \widetilde{\beta}_1)$ and as $u \to \infty$,*

$$\begin{aligned}\mathbb{P}\Big\{\sup_{0\le t\le \mathcal{T}} X(t) \ge u\Big\} &= \mathbb{P}\{X(\mathcal{T}) \ge u\} + o(\exp\{-(\widetilde{\beta}_2 - \varepsilon)u^{\widetilde{\alpha}}\}) \\ &= \mathbb{P}\{X(\mathcal{T}) \ge u\}(1 + o(\exp\{-(\widetilde{\beta}_2 - \widetilde{\beta}_1 - \varepsilon)u^{\widetilde{\alpha}}\})\end{aligned}$$

*for any $\varepsilon > 0$, where $\widetilde{\alpha}$, $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ are as shown in (5.2.4) and (5.2.6).*

**Proof** Note that

$$\mathbb{P}\Big\{\sup_{0\le t\le \mathcal{T}} X(t) \ge u\Big\} = \int_0^\infty \mathbb{P}\Big\{\sup_{0\le t\le T} X(t) \ge u\Big\} F_{\mathcal{T}}(dT),$$

combining (5.2.3) and Lemma 5.2.3 with Lemma 5.2.4, we obtain the result. $\square$

**Example 5.2.6** Let $X(t) = \int_0^t \int_0^s B(v)dvds$, where $B(v)$ is the standard Brownian motion. Then one has

$$\begin{aligned}&\sigma_t^2 = \frac{t^5}{20}, \quad \lambda_t^2 = \frac{t^3}{3}, \quad \mathrm{Var}(X''(t)) = t, \\ &\mathbb{E}\{X(t)X'(t)\} = \frac{t^4}{8}, \quad \mathbb{E}\{X(t)X''(t)\} = \frac{t^3}{6}, \quad \mathbb{E}\{X'(t)X''(t)\} = \frac{t^2}{2}, \\ &\theta_t^2 = \mathrm{Var}(X(t)|X'(t)) = \frac{t^5}{320}.\end{aligned}$$

**Example 5.2.7** Let $X(t) = \int_0^t \int_0^s Z(v)dvds$, where $Z(v)$ is a continuous stationary Gaussian process with covariance function $R(t)$ such that $R(0) = 1$ and

$$R(t) = Dt^{\alpha_\infty - 4}(1 + o(1)) \quad \text{as } t \to \infty,$$

where $D > 0, 2 < \alpha_\infty < 4$. Then

$$\sigma_t^2 = \frac{2D}{\alpha_\infty(\alpha_\infty - 2)(\alpha_\infty - 3)} t^{\alpha_\infty}(1 + o(1))$$

and

$$\begin{aligned}
\theta_t^2 &= \sigma_t^2 - [\mathbb{E}\{X(t)X'(t)\}]^2/\mathrm{Var}(X'(t)) \\
&= \frac{(4 - \alpha_\infty)D}{2\alpha_\infty(\alpha_\infty - 2)(\alpha_\infty - 3)} t^{\alpha_\infty}(1 + o(1)).
\end{aligned}$$

# Chapter 6

# Ruin Probability of a Certain Class of Smooth Gaussian Processes

Let $\{X(t) : t \geq 0\}$ be a centered smooth Gaussian process with variance $t^{2\gamma}$ for some $\gamma > 2$. We consider the probability $\mathbb{P}\{\sup_{t\geq 0}(X(t) - ct^{\beta}) \geq u\}$ as $u \to \infty$, where $c > 0$ and $\beta > \gamma$. We derive some asymptotic approximations to such probability which refine the result of Hüsler and Piterbarg (1999).

## 6.1 Self-similar Processes

Let $\{X(t) : t \geq 0\}$ be a centered smooth Gaussian process with variance $t^{2\gamma}$ for some $\gamma > 2$. We say $X$ is self similar if its covariance function $C(t, s)$ satisfies

$$C(at, as) = a^{2\gamma}C(t, s), \quad \forall t, s \geq 0, a > 0. \tag{6.1.1}$$

Let $Y(t) = \frac{X(t)}{1+ct^\beta}$, then

$$\begin{aligned}
\mathbb{P}\Big\{ \sup_{t\ge 0}(X(t) - ct^\beta) \ge u \Big\} &= \mathbb{P}\{X(t) \ge u + ct^\beta \text{ for some } t \ge 0\} \\
&= \mathbb{P}\{X(u^{1/\beta}t) \ge u + c(u^{1/\beta}t)^\beta \text{ for some } t \ge 0\} \\
&= \mathbb{P}\{u^{\gamma/\beta}X(t) \ge u(1+ct^\beta) \text{ for some } t \ge 0\} \qquad (6.1.2) \\
&= \mathbb{P}\Big\{ \sup_{t\ge 0} \frac{X(t)}{1+ct^\beta} \ge u^{1-\gamma/\beta} \Big\} \\
&= \mathbb{P}\Big\{ \sup_{t\ge 0} Y(t) \ge u^{1-\gamma/\beta} \Big\},
\end{aligned}$$

where the third equality is due to the self-similarity (6.1.1). Note that $\mathrm{Var}(Y(t)) = \frac{t^{2\gamma}}{(1+ct^\beta)^2}$, as a function of $t$, attains its maximum

$$\sigma^2 = \left(\frac{\gamma}{c(\beta-\gamma)}\right)^{2\gamma/\beta} \left(\frac{\beta}{\beta-\gamma}\right)^{-2}$$

at the unique point

$$t_0 = \left(\frac{\gamma}{c(\beta-\gamma)}\right)^{1/\beta}.$$

**Theorem 6.1.1** *Let $\{X(t) : t \ge 0\}$ be a centered self-similar Gaussian process with variance $t^{2\gamma}$ for some $\gamma > 2$. Let $\beta > \gamma$ and $c > 0$. Suppose $X$ satisfies ($\mathbf{H}1$) and ($\mathbf{H}3$). Then there exists some $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}
&\mathbb{P}\Big\{ \sup_{t\ge 0}(X(t) - ct^\beta) \ge u \Big\} = \mathbb{P}\Big\{ \sup_{t\ge 0} Y(t) \ge u^{1-\gamma/\beta} \Big\} \\
&= -\int_{t_0/2}^{2t_0} dt \int_{u^{1-\gamma/\beta}}^\infty \frac{\mathbb{E}\{Y''(t)|Y(t)=x, Y'(t)=0\}}{2\pi\sqrt{\mathrm{detCov}(Y(t),Y'(t))}} e^{-x^2/2\theta_t^2}\, dx \qquad (6.1.3) \\
&\quad + o\bigg( \exp\Big\{ -\frac{u^{2-2\gamma/\beta}}{2\sigma^2} - \alpha u^{2-2\gamma/\beta} \Big\} \bigg),
\end{aligned}$$

*where* $Y(t) = \frac{X(t)}{1+ct^\beta}$ *and* $\theta_t^2 = \mathrm{Var}(Y(t)|Y'(t))$.

**Proof** The first equality in (6.1.3) is the result in (6.1.2). Note that

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t_0/2\le t\le 2t_0} Y(t) \ge u^{1-\gamma/\beta}\Big\} \le \mathbb{P}\Big\{\sup_{t\ge 0} Y(t) \ge u^{1-\gamma/\beta}\Big\} \\
&\quad \le \mathbb{P}\Big\{\sup_{t_0/2\le t\le 2t_0} Y(t) \ge u^{1-\gamma/\beta}\Big\} + \mathbb{P}\Big\{\sup_{0\le t\le t_0/2} Y(t) \ge u^{1-\gamma/\beta}\Big\} \qquad (6.1.4)\\
&\qquad + \mathbb{P}\Big\{\sup_{t\ge 2t_0} Y(t) \ge u^{1-\gamma/\beta}\Big\},
\end{aligned}$$

where the last two terms are super-exponentially small due to the Borell-TIS inequality [cf. Theorem 2.1.1 in Adler and Taylor (2007)]. On the other hand, by Theorem 3.1.9,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t_0/2\le t\le 2t_0} Y(t) \ge u^{1-\gamma/\beta}\Big\} \\
&= -\int_{t_0/2}^{2t_0} dt \int_{u^{1-\gamma/\beta}}^{\infty} \frac{\mathbb{E}\{Y''(t)|Y(t)=x, Y'(t)=0\}}{2\pi\sqrt{\det\mathrm{Cov}(Y(t),Y'(t))}} e^{-x^2/2\theta_t^2} dx \\
&\quad + o\bigg(\exp\Big\{-\frac{u^{2-2\gamma/\beta}}{2\sigma^2} - \alpha u^{2-2\gamma/\beta}\Big\}\bigg).
\end{aligned}$$

Combining this with (6.1.4) yields the desired result. □

**Corollary 6.1.2** *Under the assumptions in Theorem 6.1.1, as* $u\to\infty$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\ge 0}(X(t)-ct^\beta) \ge u\Big\} = \mathbb{P}\Big\{\sup_{t\ge 0} Y(t) \ge u^{1-\gamma/\beta}\Big\} \\
&= \bigg(\frac{\mathrm{Var}(Y'(t_0))}{\mathbb{E}\{Y(t_0)Y''(t_0)\}} + 1\bigg)^{-1/2} \Psi\Big(\frac{u^{1-\gamma/\beta}}{\sigma}\Big)(1+o(1)).
\end{aligned}$$

**Proof** One can check that the second derivative of the variance function of $Y$, $\mathrm{Var}(Y(t)) = \frac{t^{2\gamma}}{(1+ct^\beta)^2}$, at $t_0$ is not equal to 0. This implies that the condition in (3.2.6) holds. Applying

Corollary 3.2.3 and Theorem 6.1.1, we obtain the result. □

## 6.2 Integrated Fractional Brownian Motion

In this section, we show the application to a typical example, the double integrated fractional Brownian motion.

Let $X(t) = \int_0^t \int_0^s B_H(u)duds$, where $B_H$ is fractional Brownian motion with Hurst index $H$, i.e. $\mathrm{Cov}(B_H(t)B_H(s)) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$. Then $X$ satisfies (6.1.1) with $\gamma = H+2$, it also satisfies (**H**1) and (**H**3), and

$$\mathrm{Var}(X(t)) = \frac{t^{2H+4}}{2(2H+1)(2H+4)}. \tag{6.2.1}$$

Let $\beta > H+2$ and $Y(t) = \frac{X(t)}{1+ct^\beta}$, we consider the probability

$$\mathbb{P}\Big\{\sup_{t\geq 0}(X(t)-ct^\beta) \geq u\Big\} = \mathbb{P}\Big\{\sup_{t\geq 0} Y(t) \geq u^{1-(H+2)/\beta}\Big\}. \tag{6.2.2}$$

We see that

$$\mathrm{Var}(Y(t)) = \frac{t^{2H+4}}{2(2H+1)(2H+4)(1+ct^\beta)^2}, \tag{6.2.3}$$

which attains the maximum at the unique point

$$t_0 = \left(\frac{H+2}{c(\beta-H-2)}\right)^{1/\beta}. \tag{6.2.4}$$

Note that

$$\mathbb{E}(Y(t)Y'(t)) = \frac{t^{2H+3}}{2(2H+1)(1+ct^\beta)^2}\left\{\frac{1}{2} - \frac{c\beta t^\beta}{(2H+4)(1+ct^\beta)}\right\},$$

$$\mathrm{Var}(Y'(t)) = \frac{t^{2H+2}}{(1+ct^\beta)^2}\left\{\frac{1}{2H+2} + \frac{c^2\beta^2 t^{2\beta}}{2(2H+1)(2H+4)(1+ct^\beta)^2} - \frac{c\beta t^\beta}{2(2H+1)(1+ct^\beta)}\right\},$$

and

$$\begin{aligned}
&\mathbb{E}\{Y(t)Y''(t)\} \\
&\quad = \frac{t^{2H+2}}{2(2H+1)(1+ct^\beta)^2}\left\{\frac{2H^2+H+1}{2H+2} + \frac{c^2\beta(\beta+1)t^{2\beta} - c\beta(\beta-1)t^\beta}{(2H+4)(1+ct^\beta)^2} - \frac{c\beta t^\beta}{1+ct^\beta}\right\},
\end{aligned}$$

it follows that

$$\frac{\mathrm{Var}(Y'(t_0))}{\mathbb{E}\{Y(t_0)Y''(t_0)\}} = \frac{H^2-H}{(\beta-2)(H+1)-2H^2}.$$

Thus by Corollary 6.1.2,

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\geq 0}(X(t)-ct^\beta)\geq u\right\} &= \mathbb{P}\left\{\sup_{t\geq 0} Y(t) \geq u^{1-(H+2)/\beta}\right\} \\
&\sim \left(\frac{\mathrm{Var}(Y'(t_0))}{\mathbb{E}\{Y(t_0)Y''(t_0)\}}+1\right)^{-1/2}\Psi\left(\frac{u^{1-(H+2)/\beta}}{\sqrt{\mathrm{Var}(Y(t_0))}}\right) \qquad (6.2.5)\\
&\sim \left(\frac{(\beta-2)(H+1)-2H^2}{(\beta-H-2)(H+1)}\right)^{1/2}\Psi\left(\frac{u^{1-(H+2)/\beta}}{\sqrt{\mathrm{Var}(Y(t_0))}}\right).
\end{aligned}$$

Now let $H=1/2, \beta=3$ and $c=1$. By the discussions above,

$$\mathbb{P}\left\{\sup_{t\geq 0}(X(t)-ct^\beta)\geq u\right\} \sim \sqrt{4/3}\,\Psi\left(\frac{\sqrt{144}u^{1/6}}{5^{1/3}}\right). \qquad (6.2.6)$$

However, applying the Laplace approximation of higher order to (6.1.3), we will get a more accurate approximation:

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\geq 0}(X(t)-ct^{\beta})\geq u\Big\}\\&\sim\frac{1}{2\pi}\int_{u^{1/6}}^{\infty}\Big\{\sqrt{\pi}c_0+\Big(\frac{\sqrt{\pi}}{2}c_2+\widetilde{c}_0\Big)\frac{1}{x^2}\Big\}\exp\Big(-\frac{72x^2}{5^{2/3}}\Big)dx,\end{aligned}\tag{6.2.7}$$

where

$$c_0=\frac{8\sqrt{30}5^{1/6}}{5},\,c_2=\frac{19\sqrt{30}5^{5/6}}{2700},\,\widetilde{c}_0=\frac{\sqrt{15}}{5}.$$

## 6.3 More General Gaussian Processes

Assume that $\{X(t):t\geq 0\}$ is a centered smooth Gaussian process with variance $t^{2\gamma}$ for some $\gamma>2$. Let $X_u(t)=\frac{X(u^{1/\beta}t)}{u^{\gamma/\beta}(1+ct^{\beta})}$, then

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\geq 0}(X(t)-ct^{\beta})\geq u\Big\}&=\mathbb{P}\{X(t)\geq u+ct^{\beta}\text{ for some }t\geq 0\}\\&=\mathbb{P}\{X(u^{1/\beta}t)\geq u+c(u^{1/\beta}t)^{\beta}\text{ for some }t\geq 0\}\\&=\mathbb{P}\Big\{\frac{X(u^{1/\beta}t)}{u^{\gamma/\beta}(1+ct^{\beta})}\geq u^{1-\gamma/\beta}\text{ for some }t\geq 0\Big\}\\&=\mathbb{P}\Big\{\sup_{t\geq 0}X_u(t)\geq u^{1-\gamma/\beta}\Big\}.\end{aligned}\tag{6.3.1}$$

Note that $\mathrm{Var}(X_u(t))=\frac{t^{2\gamma}}{(1+ct^{\beta})^2}$, as a function of $t$, attains its maximum

$$\sigma^2=\Big(\frac{\gamma}{c(\beta-\gamma)}\Big)^{2\gamma/\beta}\Big(\frac{\beta}{\beta-\gamma}\Big)^{-2}$$

at the unique point

$$t_0=\Big(\frac{\gamma}{c(\beta-\gamma)}\Big)^{1/\beta}.$$

We see that neither $\sigma^2$ nor $t_0$ depend on $u$.

Let $r(t,s) = \frac{\mathbb{E}\{X(t)X(s)\}}{\sqrt{\mathrm{Var}(X(t))\mathrm{Var}(X(s))}}$, we will make use of the following condition.

($\mathbf{A}1'$). For any fixed $\delta > 0$, $R(\delta) := \sup_{|t-s|\geq\delta} r(t,s) < 1$.

Let $M_{u^{1-\gamma/\beta}}(X_u, (t_0/2, 2t_0))$ be the number of local maximum points $t \in (t_0/2, 2t_0)$ such that $X_u(t)$ exceeding level $u^{1-\gamma/\beta}$.

**Lemma 6.3.1** *Let $\{X(t) : t \geq 0\}$ be a centered Gaussian process with variance $t^{2\gamma}$ for some $\gamma > 2$. Assume $X \in C^2(\mathbb{R}_+)$ a.s. and that $X$ satisfies the regularity conditions ($\mathbf{H}3$) and ($\mathbf{A}1'$). Suppose there exist positive constants $C_0$, $N_0$ and $\eta_0$ such that for all $t \geq 0$,*

$$\begin{aligned} \mathrm{Var}(X''(t)) &\leq C_0(t^{N_0} + 1), \\ [\det\mathrm{Cov}(X(t), X'(t), X''(t))]^{-1} &\leq C_0(t^{N_0} + 1); \end{aligned} \tag{6.3.2}$$

*for all $t \neq s$,*

$$\mathbb{E}(X''(t) - X''(s))^2 \leq C_0[(t+s)^{N_0} + 1](t-s)^{2\eta_0}; \tag{6.3.3}$$

*and all $|t-s| \geq \delta_0$, where $\delta_0 > 0$ is some fixed number,*

$$[\det\mathrm{Cov}(X(t), X'(t), X(s), X'(s))]^{-1} \leq C_0(t+s)^{N_0}. \tag{6.3.4}$$

*Then there exists some $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned} &\mathbb{E}\{M_{u^{1-\gamma/\beta}}(X_u, (t_0/2, 2t_0))[M_{u^{1-\gamma/\beta}}(X_u, (t_0/2, 2t_0)) - 1]\} \\ &\quad = o\bigg(\exp\bigg\{-\frac{u^{2-2\gamma/\beta}}{2\sigma^2} - \alpha u^{2-2\gamma/\beta}\bigg\}\bigg). \end{aligned}$$

**Proof** By the Kac-Rice metatheorem, one has

$$\begin{aligned}
&\mathbb{E}\{M_{u^{1-\gamma/\beta}}(X_u,(t_0/2,2t_0))[M_{u^{1-\gamma/\beta}}(X_u,(t_0/2,2t_0))-1]\} \\
&\quad\le \int_{t_0/2}^{2t_0} dt \int_{t_0/2}^{2t_0} ds \int_{u^{1-\gamma/\beta}}^{\infty} dx \mathbb{E}\{|X_u''(t)X_u''(s)||X_u(t)=x, X_u'(t)=X_u'(s)=0\} \qquad (6.3.5)\\
&\quad\times p_{X_u(t)}(x|X_u'(t)=X_u'(s)=0)p_{X_u'(t),X_u'(s)}(0,0).
\end{aligned}$$

Let $E_u(t,s) := \mathbb{E}\{|X_u''(t)X_u''(s)||X_u(t)=x, X_u'(t)=X_u'(s)=0\}$. By Taylor's formula,

$$X_u'(s) = X_u'(t) + X_u''(t)(s-t) + |s-t|^{1+\eta} Y_{t,s,u}, \qquad (6.3.6)$$

where $Y_{t,s,u}$ is a centered Gaussian variable. In particular, for $s > t$,

$$Y_{t,s,u} = \frac{X_u'(s) - X_u'(t) - X_u''(t)(s-t)}{(s-t)^{1+\eta}} = \frac{\int_t^s (X_u''(v) - X_u''(t))dv}{(s-t)^{1+\eta}}.$$

Differentiating $\mathrm{Var}(X(t)) = t^{2\gamma}$ with respective to $t$ twice, we see that

$$2(\mathrm{Var}(X'(t))) + \mathbb{E}\{X(t)X''(t)\} = 2\gamma(2\gamma-1)t^{2\gamma-2}.$$

Since $|\mathbb{E}\{X(t)X''(t)\}| \le \mathrm{Var}(X(t))\mathrm{Var}(X''(t)) = t^{2\gamma}\mathrm{Var}(X''(t))$, together with condition (6.3.2), we get

$$\mathrm{Var}(X'(t)) \le C_1(t^{N_1}+1)$$

for some positive constants $C_1$ and $N_1$. Combining this fact with conditions (6.3.2) and (6.3.3), we obtain

$$\sup_{t_0/2 \le t < s \le 2t_0} \mathrm{Var}(Y_{t,s,u}) \le C_2 u^{N_2}$$

for some positive constants $C_2$ and $N_2$.

Applying (6.3.6), we have

$$
\begin{aligned}
E_u(t,s) &= \mathbb{E}\{|X''_u(t)X''_u(s)||X_u(t)=x, X'_u(t)=0, X''_u(t)(s-t)=-|s-t|^{1+\eta}Y_{t,s,u}\} \\
&= |s-t|^{\eta}\mathbb{E}\{|Y_{t,s,u}X''_u(s)||X_u(t)=x, X'_u(t)=0, X''_u(t)(s-t)=-|s-t|^{1+\eta}Y_{t,s,u}\}.
\end{aligned}
\tag{6.3.7}
$$

For any Gaussian variables $\xi_1, \xi_2$, the following inequality holds,

$$
\mathbb{E}|\xi_1\xi_2| \le \mathbb{E}\xi_1^2 + \mathbb{E}\xi_2^2 = (\mathbb{E}\xi_1)^2 + \mathrm{Var}(\xi_1) + (\mathbb{E}\xi_2)^2 + \mathrm{Var}(\xi_2). \tag{6.3.8}
$$

We have

$$
\begin{aligned}
\mathrm{Var}(X''_u(t)|X_u(t)=x, X'_u(t)=X'_u(s)=0) &\le \mathrm{Var}(X''_u(t)) \le C_3u^{N_3}, \\
\mathrm{Var}(Y_{t,s,u}|X_u(t)=x, X'_u(t)=X'_u(s)=0) &\le \mathrm{Var}(Y_{t,s,u}) \le C_2u^{N_2},
\end{aligned}
$$

for some positive constants $C_3$ and $N_3$.

On the other hand, for $s>t$ and $|s-t|\to 0$, there exist positive constants $C_4$, $C_5$, $N_4$ and $N_5$ such that for large $x$ and $u$,

$$
\begin{aligned}
&|\mathbb{E}\{X''_u(t)|X_u(t)=x, X'_u(t)=X'_u(s)=0\}| \\
&\quad = |\mathbb{E}\{X''_u(t)|X_u(t)=x, X'_u(t)=0, X''_u(t)+|s-t|^{\eta}Y_{t,s,u}=0\}| \\
&\quad \le |\mathbb{E}\{X''_u(t)|X_u(t)=x, X'_u(t)=0, X''_u(t)=0\}| + o(1)u^{N_4}|x| \\
&\quad \le \frac{C_4|x|}{\det\mathrm{Cov}(X_u(t), X'_u(t), X''_u(t))} + o(1)u^{N}|x| \le C_5|x|u^{N_5},
\end{aligned}
$$

where the last line is due to condition (6.3.2). In fact, let $(\xi_1, \xi_2, \xi_3)$ be a non-degenerate

Gaussian vector, then

$$\det\mathrm{Cov}(\xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3) = \det\mathrm{Cov}(\xi_1, \xi_2, \xi_3).$$

By using this identity, we see that condition (6.3.2) implies that there exist positive constants $C_0'$ and $N_0'$ such that for large $u$,

$$\sup_{t_0/2 \le t < s \le 2t_0} [\det\mathrm{Cov}(X_u(t), X_u'(t), X_u''(t))]^{-1} \le C_0' u^{N_0'}.$$

Similarly we obtain that as $|s-t| \to 0$, there exist positive constants $C_6$ and $N_6$ such that for large $x$ and $u$,

$$|\mathbb{E}\{Y_{t,s,u} | X_u(t) = x, X_u'(t) = X_u'(s) = 0\}| \le C_6 |x| u^{N_6}.$$

Combining these results with (6.3.7) and (6.3.8), we get

$$E_u(t,s) \le |s-t|^\eta (C_3 u^{N_3} + C_4 u^{N_4} + C_5^2 |x|^2 u^{2N_5} + C_6^2 |x|^2 u^{2N_6}). \tag{6.3.9}$$

By Taylor's formula (6.3.6), as $|s-t| \to 0$,

$$\begin{aligned}
&\mathrm{Var}(X_u(t) | X_u'(t), X_u'(s)) \\
&\quad = \mathrm{Var}(X_u(t) | X_u'(t), X_u'(t) + X_u''(t)(s-t) + |s-t|^{1+\eta} Y_{t,s,u}) \\
&\quad = \mathrm{Var}(X_u(t) | X_u'(t), X_u''(t) \pm |s-t|^\eta Y_{t,s,u}) \\
&\quad = \mathrm{Var}(X_u(t) | X_u'(t), X_u''(t))(1 + o(1)).
\end{aligned}$$

Let

$$\kappa^2 := \sup_{u>0, t_0/2 \le t \le 2t_0} \mathrm{Var}(X_u(t) | X_u'(t), X_u''(t)).$$

One can check that the second derivative of the variance function of $X_u(t)$, $\mathrm{Var}(X_u(t)) = \frac{t^{2\gamma}}{(1+ct^\beta)^2}$, at $t_0$ is not equal to 0. Therefore $\sup_{u>0} \mathbb{E}\{X_u(t_0)X_u''(t_0)\} < 0$ and moreover, $\kappa^2 < \sigma^2$.

For any $\varepsilon > 0$, if $|s-t|$ is sufficiently small, then for large $u$,

$$\int_{u^{1-\gamma/\beta}}^{\infty} x^2 p_{X_u(t)}(x | X_u'(t) = X_u'(s) = 0) dx \le e^{-\frac{u^{2-2\gamma/\beta}}{2\kappa^2+\varepsilon}}.$$

Note that

$$p_{X_u'(t), X_u'(s)}(0,0) \le \frac{1}{2\pi\sqrt{\mathrm{detCov}(X_u'(t), X_u'(s))}},$$

and by the Taylor expansion, for $|s-t| \to 0$ and large $u$,

$$\begin{aligned}
&\mathrm{detCov}(X_u'(t), X_u'(s)) \\
&\quad = \mathrm{detCov}(X_u'(t), X_u'(t) + X_u''(t)(s-t) + |s-t|^{1+\eta} Y_{t,s,u}) \\
&\quad = |s-t|^2 \mathrm{detCov}(X_u'(t), X_u''(t) + |s-t|^{\eta} Y_{t,s,u}) \\
&\quad = |s-t|^2 \mathrm{detCov}(X_u'(t), X_u''(t))(1 + u^{N_7} o(1)),
\end{aligned}$$

where $N_7$ is some positive constant. Note that

$$\mathrm{detCov}(X_u'(t), X_u''(t)) = \frac{\mathrm{detCov}(X_u(t), X_u'(t), X_u''(t))}{\mathrm{Var}(X_u(t) | X_u'(t), X_u''(t))}.$$

Thus by conditions (6.3.2) and (6.3.3), there exists positive constant $N_8$ such that for small $|s-t|$ and large $u$,

$$p_{X'_u(t),X'_u(s)}(0,0) \le \frac{u^{N_8}}{|s-t|}.$$

Note that when $u$ tends to infinity, the polynomials of $u$ will be killed by the exponential decay of $u$. Plugging these results into (6.3.5), we obtain that for any $\varepsilon > 0$, there exists $\delta > 0$ small enough, such that for large u,

$$\begin{aligned}
&\mathbb{E}\{M_{u^{1-\gamma/\beta}}(X_u,(t_0-\delta,t_0+\delta))[M_{u^{1-\gamma/\beta}}(X_u,(t_0-\delta,t_0+\delta))-1]\} \\
&\le e^{-\frac{u^{2-2\gamma/\beta}}{2\kappa^2+\varepsilon}} \int_{t_0-\delta}^{t_0+\delta}\int_{t_0-\delta}^{t_0+\delta} |s-t|^{\eta-1} dtds \\
&\le C_7 \delta \exp\left\{ -\frac{u^{2-2\gamma/\beta}}{2\kappa^2+\varepsilon} \right\}
\end{aligned}$$

for some positive constant $C_7$.

The set $[t_0/2, 2t_0]$ may be covered by congruent intervals $I_i = [a_i, a_{i+1}]$ with disjoint interiors such that the lengths are less than $\delta/2$. By similar discussions in Lemma 5.1.2, we only need to consider non-neighboring $I_i = [a_i, a_{i+1}]$ and $I_j = [a_j, a_{j+1}]$, say $a_j - a_{i+1} \ge \delta/2$. Then

$$\begin{aligned}
&\mathbb{E}\{M_{u^{1-\gamma/\beta}}(X_u, I_i) M_{u^{1-\gamma/\beta}}(X_u, I_j)\} \\
&= \int_{a_i}^{a_{i+1}}\int_{a_j}^{a_{j+1}} dtds \int_{u^{1-\gamma/\beta}}^{\infty}\int_{u^{1-\gamma/\beta}}^{\infty} dxdy \mathbb{E}\{|X''_u(t)X''_u(s)| |X_u(t)=x, X_u(s)=y, \\
&\qquad X'_u(t)=X'_u(s)=0\} p_{X'_u(t),X'_u(s)}(0,0|X_u(t)=x,X_u(s)=y) p_{X_u(t),X_u(s)}(x,y).
\end{aligned}$$

Similarly to (6.3.9), by conditions (6.3.2) and (6.3.3), there exists a positive constant $N_9$

such that for large $u$,

$$
\begin{aligned}
&\mathbb{E}\{|X_u''(t)X_u''(s)||X_u(t)=x, X_u(s)=y, X_u'(t)=X_u'(s)=0\} \\
&\quad \le \frac{u^{N_9}(x^2+y^2)}{[\det\mathrm{Cov}(X_u(t), X_u(s), X_u'(t), X_u'(s))]^2};
\end{aligned}
$$

and also,

$$
\begin{aligned}
&p_{X_u'(t), X_u'(s)}(0,0|X_u(t)=x, X_u(s)=y) \\
&\quad \le \frac{1}{2\pi\sqrt{\det\mathrm{Cov}(X_u'(t), X_u'(s)|X_u(t), X_u(s))}} \\
&\quad = \frac{1}{2\pi}\sqrt{\frac{\det\mathrm{Cov}(X_u(t), X_u(s))}{\det\mathrm{Cov}(X_u'(t), X_u'(s), X_u(t), X_u(s))}} \\
&\quad \le \frac{u^{N_9}}{\sqrt{\det\mathrm{Cov}(X_u'(t), X_u'(s), X_u(t), X_u(s))}}.
\end{aligned}
$$

Thus by condition (6.3.4), there exists a positive constant $N_{10}$ such that for large $u$,

$$
\begin{aligned}
&\mathbb{E}\{M_u(I_i)M_u(I_j)\} \\
&\quad \le u^{2N_9}(a_{i+1}-a_i)(a_{j+1}-a_j)\int_{u^{1-\gamma/\beta}}^{\infty}\int_{u^{1-\gamma/\beta}}^{\infty} dxdy(x^2+y^2)p_{X_u(t),X_u(s)}(x,y) \\
&\qquad \times \left[\inf_{t,s\in[t_0/2,2t_0]:|t-s|\ge\delta/2} \det\mathrm{Cov}(X_u(t), X_u(s), X_u'(t), X_u'(s))\right]^{-5/2} \\
&\quad \le u^{N_{10}}(a_{i+1}-a_i)(a_{j+1}-a_j)\int_{u^{1-\gamma/\beta}}^{\infty}\int_{u^{1-\gamma/\beta}}^{\infty} dxdy(x^2+y^2)p_{X_u(t),X_u(s)}(x,y).
\end{aligned}
$$

By $(\mathbf{A}1')$, $R(\delta) = \sup_{|s-t|\geq\delta} r(t,s)$ is strictly less than 1 hence for $u$ sufficiently large,

$$
\begin{aligned}
&\sup_{|s-t|\geq\delta/2}\int_{u^{1-\gamma/\beta}}^{\infty}\int_{u^{1-\gamma/\beta}}^{\infty} dxdy(x^2+y^2)p_{X_u(t),X_u(s)}(x,y)\\
&\quad\leq \sup_{|s-t|\geq\delta/2}\mathbb{E}\{(X_u(t)X_u(s))^2 I(X_u(t)\geq u^{1-\gamma/\beta}, X_u(s)\geq u^{1-\gamma/\beta})\}\\
&\quad\leq \sup_{|s-t|\geq\delta/2}\mathbb{E}\{(X_u(t)+X_u(s))^4) I(X_u(t)+X_u(s)\geq 2u^{1-\gamma/\beta})\}\\
&\quad\leq u^4\exp\Big\{-\frac{u^{2-2\gamma/\beta}}{1+R(\delta/2)}\Big\}.
\end{aligned}
$$

Combining the results completes the proof. $\square$

**Theorem 6.3.2** *Suppose the assumptions in Lemma 6.3.1 hold. Then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\geq 0}(X(t)-ct^{\beta})\geq u\Big\} = \mathbb{P}\Big\{\sup_{t\geq 0}X_u(t)\geq u^{1-\gamma/\beta}\Big\}\\
&= -\int_{t_0/2}^{2t_0}dt\int_{u^{1-\gamma/\beta}}^{\infty}\frac{\mathbb{E}\{X_u''(t)|X_u(t)=x, X_u'(t)=0\}}{2\pi\sqrt{\det\mathrm{Cov}(X_u(t),X_u'(t))}}e^{-x^2/(2\theta_u^2(t))}dx\\
&\quad + o\bigg(\exp\Big\{-\frac{u^{2-2\gamma/\beta}}{2\sigma^2}-\alpha u^{2-2\gamma/\beta}\Big\}\bigg),
\end{aligned}
\tag{6.3.10}
$$

*where $X_u(t) = \frac{X(u^{1/\beta}t)}{u^{\gamma/\beta}(1+ct^{\beta})}$ and $\theta_u^2(t) = \mathrm{Var}(X_u(t)|X_u'(t))$.*

**Proof** The first equality in (6.3.10) is the result in (6.3.1). Note that

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t_0/2\leq t\leq 2t_0}X_u(t)\geq u^{1-\gamma/\beta}\Big\}\leq \mathbb{P}\Big\{\sup_{t\geq 0}X_u(t)\geq u^{1-\gamma/\beta}\Big\}\\
&\quad\leq \mathbb{P}\Big\{\sup_{t_0/2\leq t\leq 2t_0}X_u(t)\geq u^{1-\gamma/\beta}\Big\}+\mathbb{P}\Big\{\sup_{0\leq t\leq t_0/2}X_u(t)\geq u^{1-\gamma/\beta}\Big\}\\
&\qquad+\mathbb{P}\Big\{\sup_{t\geq 2t_0}X_u(t)\geq u^{1-\gamma/\beta}\Big\},
\end{aligned}
\tag{6.3.11}
$$

where the last two terms are super-exponentially small due to the Borell-TIS inequality [cf. Theorem 2.1.1 in Adler and Taylor (2007)]. On the other hand, by Lemma 6.3.1 and the bounds in (2.3.3),

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t_0/2\le t\le 2t_0} X_u(t) \ge u^{1-\gamma/\beta}\Big\}\\
&= \mathbb{E}\{M_{u^{1-\gamma/\beta}}(X_u,(t_0/2,2t_0))\} + o\bigg(\exp\bigg\{-\frac{u^{2-2\gamma/\beta}}{2\sigma^2}-\alpha u^{2-2\gamma/\beta}\bigg\}\bigg)\\
&= -\int_{t_0/2}^{2t_0} dt \int_{u^{1-\gamma/\beta}}^{\infty} \frac{\mathbb{E}\{X_u''(t)|X_u(t)=x, X_u'(t)=0\}}{2\pi\sqrt{\det\mathrm{Cov}(X_u(t),X_u'(t))}} e^{-x^2/(2\theta_u^2(t))}dx\\
&\quad + o\bigg(\exp\bigg\{-\frac{u^{2-2\gamma/\beta}}{2\sigma^2}-\alpha u^{2-2\gamma/\beta}\bigg\}\bigg),
\end{aligned}
$$

where the last equality comes from the combination of similar discussions in Lemma 2.3.2 and Lemma 6.3.1. Combining this with (6.3.11) yields the desired result. □

Applying the Laplace method, we obtain the following result.

**Corollary 6.3.3** *Under the assumptions in Theorem 6.3.2, one has that as $u\to\infty$,*

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\ge 0}(X(t)-ct^\beta)\ge u\Big\} = \mathbb{P}\Big\{\sup_{t\ge 0} X_u(t) \ge u^{1-\gamma/\beta}\Big\}\\
&= \bigg(\frac{\mathrm{Var}(X_u'(t_0))}{\mathbb{E}\{X_u(t_0)X_u''(t_0)\}}+1\bigg)^{-1/2}\Psi\Big(\frac{u^{1-\gamma/\beta}}{\sigma}\Big)(1+o(1)).
\end{aligned}
$$

**Example 6.3.4** Let $\beta > H > 2$, $X(t) = t^H Z(t)$, $X_u(t) = \frac{X(tu^{1/\beta})}{u^{H/\beta}(1+ct^\beta)} = \frac{t^H Z(tu^{1/\beta})}{1+ct^\beta}$, where $Z$ is a smooth stationary Gaussian process with covariance $r(t)$ and $r(0)=1$. Then $\mathrm{Var}(X(t)) = t^{2H}$, and

$$
\mathbb{P}\Big\{\sup_{t>0}(X(t)-ct^\beta)\ge u\Big\} = \mathbb{P}\Big\{\sup_{t>0} X_u(t)\ge u^{1-H/\beta}\Big\}.
$$

Notice that $\mathrm{Var}(X_u(t))$ attains it maximum

$$\mathrm{Var}(X_u(t_0)) = \left(\frac{H}{c(\beta - H)}\right)^{2H/\beta}\left(\frac{\beta}{\beta - H}\right)^{-2}$$

at the unique point

$$t_0 = \left(\frac{H}{c(\beta - H)}\right)^{1/\beta}.$$

By tedious computations, we get

$$X_u'(t) = \frac{Ht^{H-1} - c(\beta - H)t^{H+\beta-1}}{(1+ct^\beta)^2} Z(tu^{1/\beta}) + \frac{t^H u^{1/\beta}}{1+ct^\beta} Z'(tu^{1/\beta}),$$

and

$$\begin{aligned} X_u''(t) = {} & \frac{H(H-1)t^{H-2} + c((2H-\beta)(H-1) - \beta^2)t^{H+\beta-2} + c^2(\beta - H)(1-H)t^{2\beta+H-2}}{(1+ct^\beta)^3} \\ & \times Z(tu^{1/\beta}) + \frac{2[Ht^{H-1}(1+ct^\beta) - c\beta t^{H+\beta-1}]u^{1/\beta}}{(1+ct^\beta)^2} Z'(tu^{1/\beta}) + \frac{t^H u^{2/\beta}}{1+ct^\beta} Z''(tu^{1/\beta}). \end{aligned}$$

Notice that $\mathbb{E}\{Z(t)Z'(t)\} = 0$ and $\mathrm{Var}(Z'(t)) = -\mathbb{E}\{Z(t)Z''(t)\} = r''(0)$ for all $t$, we obtain

$$\begin{aligned} \mathrm{Var}(X_u'(t)) &= \frac{(Ht^{H-1} - c(\beta - H)t^{H+\beta-1})^2}{(1+ct^\beta)^4} - \frac{t^{2H}u^{2/\beta}}{(1+ct^\beta)^2} r''(0) \\ &= \frac{t^{2H-2}}{(1+ct^\beta)^2}\left(\frac{(H - c(\beta - H)t^\beta)^2}{(1+ct^\beta)^2} - t^2 u^{2/\beta} r''(0)\right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\{X_u(t)X_u''(t)\} = {} & \left(\frac{(H(H-1) + c((2H-\beta)(H-1) - \beta^2)t^\beta + c^2(\beta - H)(1-H)t^{2\beta}}{(1+ct^\beta)^2}\right. \\ & \left. + t^2 u^{2/\beta} r''(0)\right) \frac{t^{2H-2}}{(1+ct^\beta)^2}. \end{aligned}$$

It follows that

$$\begin{aligned}\frac{\mathbb{E}\{X_u(t_0)X_u''(t_0)\}}{\mathrm{Var}(X_u'(t_0))+\mathbb{E}\{X_u(t_0)X_u''(t_0)\}} &= \frac{-H(\beta-H)+t_0^2u^{2/\beta}r''(0)}{-H(\beta-H)}\\ &= 1-\frac{H^{2/\beta-1}u^{2/\beta}r''(0)}{c^{2/\beta}(\beta-H)^{2/\beta+1}},\end{aligned}$$

and thus

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t>0}(X(t)-ct^\beta)>u\Big\} = \mathbb{P}\Big\{\sup_{t>0}X_u(t)>u^{1-H/\beta}\Big\}\\ &\sim \left(\frac{\mathbb{E}\{X_u(t_0)X_u''(t_0)\}}{\mathrm{Var}(X_u'(t_0))+\mathbb{E}\{X_u(t_0)X_u''(t_0)\}}\right)^{1/2}\Psi\left(\frac{u^{1-H/\beta}}{\mathrm{Var}(X_u(t_0))^{1/2}}\right)\\ &= \left(1-\frac{H^{2/\beta-1}u^{2/\beta}r''(0)}{c^{2/\beta}(\beta-H)^{2/\beta+1}}\right)^{1/2}\Psi\left(\frac{u^{1-H/\beta}}{\mathrm{Var}(X_u(t_0))^{1/2}}\right).\end{aligned}$$

# Chapter 7

# Excursion Probability of Gaussian Random Fields on Sphere

In this chapter, we consider a real-valued Gaussian random field $X = \{X(x) : x \in \mathbb{S}^N\}$ indexed on the $N$-dimensional unit sphere $\mathbb{S}^N$. The approximations to excursion probability of the field $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \geq u\}$, as $u \to \infty$, are obtained for two cases: (i) $X$ is locally isotropic and the sample path is non-smooth; (ii) $X$ is isotropic and the sample path is twice differentiable. For the first case, it is shown that the asymptotics is similar to Pickands' approximation on Euclidean space which involves Pickands' constant; while for the second case, we use the expected Euler characteristic method to obtain a more precise approximation such that the error is super-exponentially small.

## 7.1 Notations

For $x = (x_1, \ldots, x_{N+1}) \in \mathbb{S}^N$, its corresponding spherical coordinate $\theta = (\theta_1, \ldots, \theta_N)$ is defined as follows.

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\vdots \\
x_N &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \cos\theta_N, \\
x_{N+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \sin\theta_N,
\end{aligned}$$

where $0 \le \theta_i \le \pi$ for $1 \le i \le N-1$ and $0 \le \theta_N < 2\pi$.

Throughout this chapter, for two points $x = (x_1, \ldots, x_{N+1})$ and $y = (y_1, \ldots, y_{N+1})$ on $\mathbb{S}^N$, we always denote by $\theta = (\theta_1, \ldots, \theta_N)$ the spherical coordinate of $x$ and by $\varphi = (\varphi_1, \ldots, \varphi_N)$ the spherical coordinate of $y$ respectively.

Let $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ be Euclidean norm and inner product respectively. Denote by $d(\cdot,\cdot)$ the distance function in $\mathbb{S}^N$, i.e., $d(x, y) = \arccos\langle x, y\rangle$, $\forall x, y \in \mathbb{S}^N$.

## 7.2 Non-smooth Gaussian Fields on Sphere

### 7.2.1 Locally Isotropic Gaussian Fields on Sphere

Let $X = \{X(x) : x \in \mathbb{S}^N\}$ be a centered Gaussian random field with covariance function $C$ satisfying

$$C(x, y) = 1 - cd^\alpha(x, y)(1 + o(1)) \quad \text{as } d^\alpha(x, y) \to 0, \tag{7.2.1}$$

for some constants $c > 0$ and $\alpha \in (0, 2]$.

Covariance functions satisfying (7.2.1) behave like isotropic in local sense, hence they fall under the general category of locally isotropic covariance. Also, there are many examples of covariances of isotropic Gaussian fields on $\mathbb{S}^N$ satisfying (7.2.1). For instance, $C(x, y) = e^{-cd^{\alpha}(x,y)}$, where $c > 0$ and $\alpha \in (0, 1]$.

Recall the spherical coordinate representation, we define $\widetilde{X}(\theta) := X(x)$ and denote by $\widetilde{C}$ the covariance function of $\widetilde{X}$ accordingly.

**Lemma 7.2.1** *Let $x, y \in \mathbb{S}^N$ and let $x$ be fixed. Then as $d(y, x) \to 0$,*

$$d^2(y, x) \sim (\varphi_1 - \theta_1)^2 + (\sin^2 \theta_1)(\varphi_2 - \theta_2)^2 + \cdots + \left(\prod_{i=1}^{N-1} \sin^2 \theta_i\right)(\varphi_N - \theta_N)^2,$$

*where $\theta = (\theta_1, \ldots, \theta_N)$ and $\varphi = (\varphi_1, \ldots, \varphi_N)$ are the spherical coordinates of $x$ and $y$ respectively.*

**Proof** Note that $x, y \in \mathbb{S}^N$ implies $\|x\|^2 = \|y\|^2 = 1$, hence as $d(y, x) \to 0$, $\|x - y\| \to 0$ and

$$\cos \|y - x\| \sim 1 - \frac{1}{2}\|y - x\|^2 = \langle y, x \rangle.$$

Applying the spherical coordinates, we obtain that as $d(y, x) \to 0$, or equivalently $\|\varphi - \theta\| \to 0$ (There is an exception for $\theta$ with $\theta_N = 0$, since for those $\varphi$ such that $d(y, x) \to 0$ and $\varphi_N$ tending to $2\pi$, $\|\varphi - \theta\|$ does not tend to 0. In such case, we may treat $\theta_N$ to be $2\pi$ instead of 0 and this does not affect the result thanks to the periodicity.),

$$
\begin{aligned}
d^2(y,x) &= \arccos^2 \langle y, x\rangle \sim \|y-x\|^2 \\
&= (\cos\varphi_1 - \cos\theta_1)^2 + (\sin\varphi_1\cos\varphi_2 - \sin\theta_1\cos\theta_2)^2 + \cdots \\
&\quad + (\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{N-1}\cos\varphi_N - \sin\theta_1\sin\theta_2\cdots\sin\theta_{N-1}\cos\theta_N)^2 \\
&\quad + (\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{N-1}\sin\varphi_N - \sin\theta_1\sin\theta_2\cdots\sin\theta_{N-1}\sin\theta_N)^2 \\
&= 2 - 2\cos(\varphi_1-\theta_1) + 2(\sin\varphi_1\sin\theta_1)[1-\cos(\varphi_2-\theta_2)] \\
&\quad + \cdots + 2\bigg(\prod_{i=1}^{N-1}\sin\varphi_i\sin\theta_i\bigg)[1-\cos(\varphi_N-\theta_N)].
\end{aligned}
$$

It then follows from Taylor's expansion that

$$
\begin{aligned}
d^2(y,x) &\sim (\varphi_1-\theta_1)^2 + (\sin\varphi_1\sin\theta_1)(\varphi_2-\theta_2)^2 + \cdots + \left(\prod_{i=1}^{N-1}\sin\varphi_i\sin\theta_i\right)(\varphi_N-\theta_N)^2 \\
&\sim (\varphi_1-\theta_1)^2 + (\sin^2\theta_1)(\varphi_2-\theta_2)^2 + \cdots + \left(\prod_{i=1}^{N-1}\sin^2\theta_i\right)(\varphi_N-\theta_N)^2,
\end{aligned}
$$

completing the proof. $\square$

Next, we need some existing results on the approximations to excursion probability of Gaussian random fields over Euclidean space.

Let $0 < \alpha \le 2$ and let $\{W_t(s) : t \in \mathbb{R}^N, s \in [0,\infty)^N\}$ ba a Gaussian random field such that

$$
\begin{aligned}
\mathbb{E}W_t(s) &= -\|s\|^\alpha r_t(s/\|s\|), \\
\mathrm{Cov}(W_t(s), W_t(v)) &= \|s\|^\alpha r_t(s/\|s\|) + \|v\|^\alpha r_t(v/\|v\|) \qquad (7.2.2) \\
&\quad - \|s-v\|^\alpha r_t((s-v)/\|s-v\|),
\end{aligned}
$$

where $r_t(\cdot) : \mathbb{S}^{N-1} \to \mathbb{R}_+$ is a continuous function satisfying

$$\sup_{v \in \mathbb{S}^{N-1}} |r_t(v) - r_s(v)| \to 0 \quad \text{as } s \to t. \tag{7.2.3}$$

Define

$$H_\alpha^r(t) = \lim_{K \to \infty} K^{-N} \int_0^\infty e^u \mathbb{P}\Big\{ \sup_{s \in [0,K]^N} W_t(s) \geq u \Big\} du. \tag{7.2.4}$$

Denote by $H_\alpha$ the usual Pickands' constant, that is

$$H_\alpha = \lim_{K \to \infty} K^{-N} \int_0^\infty e^u \mathbb{P}\Big\{ \sup_{s \in [0,K]^N} Z(s) \geq u \Big\} du,$$

where $\{Z(s) : s \in [0, \infty)^N\}$ is a Gaussian random field such that

$$\mathbb{E}Z(s) = -\|s\|^\alpha,$$

$$\mathrm{Cov}(Z(s), Z(v)) = \|s\|^\alpha + \|v\|^\alpha - \|s - v\|^\alpha.$$

It is clear that $H_\alpha^r(t)$ becomes $H_\alpha$ when $r_t \equiv 1$.

Let $D \subset \mathbb{R}^N$ be a bounded $N$-dimensional Jordan measurable set. Let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian field such that the covariance function $C_Y$ satisfies

$$C_Y(t, t+s) = 1 - \|s\|^\alpha r_t(s/\|s\|)(1 + o(1)) \quad \text{as } \|s\| \to 0, \tag{7.2.5}$$

for some constant $\alpha \in (0, 2]$, uniformly over $t \in \bar{D}$.

We will make use of the following theorem of Chan and Lai (2006).

**Theorem 7.2.2** *[Theorem 2.1 in Chan and Lai (2006)] Suppose the Gaussian random field $\{Y(t) : t \in \mathbb{R}^N\}$ satisfies condition (7.2.5), in which $r_t(\cdot) : \mathbb{S}^{N-1} \to \mathbb{R}_+$ is a continuous* *function such that the convergence (7.2.3) is uniformly in* $\bar{D}$ *and* $\sup_{t\in\bar{D},v\in\mathbb{S}^{N-1}} r_t(v) < \infty$. *Then as* $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{t\in D} Y(t) \ge u\Big\} \sim u^{2N/\alpha}\Psi(u)\int_D H_\alpha^r(t)dt.$$

**Lemma 7.2.3** *Let* $\{W_t(s) : t \in \mathbb{R}^N, s \in [0,\infty)^N\}$ *be a Gaussian random field satisfying (7.2.2) with*

$$r_t(v) = \|M_t v\|^\alpha, \quad \forall v \in \mathbb{S}^{N-1},$$

*where* $M_t$ *are non-degenerate* $N \times N$ *matrices. Then for each* $t \in \mathbb{R}^N$,

$$H_\alpha^r(t) = |\det M_t| H_\alpha.$$

**Proof** Let $\widetilde{W}_t(s) = W_t(M_t^{-1}s)$, $\forall s \in [0,\infty)^N$. Then under the above conditions, $\widetilde{W}_t$ satisfies

$$\mathbb{E}\widetilde{W}_t(s) = -\|s\|^\alpha,$$
$$\mathrm{Cov}(\widetilde{W}_t(s), \widetilde{W}_t(v)) = \|s\|^\alpha + \|v\|^\alpha - \|s-v\|^\alpha.$$

Let $B_K = [0,K]^N$ and define $M_tB_K = \{s \in \mathbb{R}^N : \exists v \in B_K \text{ such that } s = M_t v\}$. Note that $\mathrm{Vol}(M_tB_K) = |\det M_t|\mathrm{Vol}(B_K)$ and $\sup_{s\in B_K} W_t(s) = \sup_{s\in M_tB_K}\widetilde{W}_t(s)$, it follows from (7.2.4) that

$$\begin{aligned}
H_\alpha^r(t) &= \lim_{K\to\infty}\frac{1}{\mathrm{Vol}(B_K)}\int_0^\infty e^u\mathbb{P}\Big\{\sup_{s\in B_K} W_t(s) \ge u\Big\}du \\
&= \lim_{K\to\infty}\frac{\mathrm{Vol}(M_tB_K)}{\mathrm{Vol}(B_K)}\frac{1}{\mathrm{Vol}(M_tB_K)}\int_0^\infty e^u\mathbb{P}\Big\{\sup_{s\in M_tB_K}\widetilde{W}_t(s) \ge u\Big\}du \\
&= |\det M_t|\lim_{K\to\infty}\frac{1}{\mathrm{Vol}(M_tB_K)}\int_0^\infty e^u\mathbb{P}\Big\{\sup_{s\in M_tB_K}\widetilde{W}_t(s) \ge u\Big\}du.
\end{aligned}$$

By modifying the proofs in Qualls and Watanabe (1973), we can check that

$$H_\alpha = \lim_{K\to\infty} \frac{1}{\mathrm{Vol}(M_t B_K)} \int_0^\infty e^u \mathbb{P}\Big\{ \sup_{s\in M_t B_K} \widetilde{W}_t(s) \geq u \Big\} du,$$

completing the proof. $\square$

Now we can prove our main result.

**Theorem 7.2.4** *Let $\{X(x) : x \in \mathbb{S}^N\}$ be a centered Gaussian random field satisfying condition (7.2.1) and let $T \subset \mathbb{S}^N$ be an $N$-dimensional Jordan measurable set. Then as $u \to \infty$,*

$$\mathbb{P}\Big\{ \sup_{x\in T} X(x) \geq u \Big\} \sim c^{N/\alpha} \mathrm{Area}(T) H_\alpha u^{2N/\alpha} \Psi(u),$$

*where* Area$(T)$ *denotes the spherical area of $T$.*

**Proof** Let $M_\theta = c^{1/\alpha}\mathrm{diag}(1, \sin\theta_1, \ldots, \prod_{i=1}^{N-1}\sin\theta_i)$. If $N = 1$, we set $M_\theta = c^{1/\alpha}$. By Lemma 7.2.1, condition (7.2.1) becomes

$$\widetilde{C}(\theta, \theta+\xi) = 1 - \|\xi\|^\alpha r_\theta(\xi/\|\xi\|)(1+o(1)) \quad \text{as } \|\xi\| \to 0,$$

where $r_\theta(\tau) = \|M_\theta \tau\|^\alpha$, $\forall \tau \in \mathbb{S}^{N-1}$. Denote by $D$ the domain of $T$ under spherical coordinates. Then by Theorem 7.2.2, as $u \to \infty$,

$$\mathbb{P}\Big\{ \sup_{x\in T} X(x) \geq u \Big\} = \mathbb{P}\Big\{ \sup_{\theta\in D} \widetilde{X}(\theta) \geq u \Big\} \sim u^{2N/\alpha}\Psi(u) \int_D H_\alpha^r(\theta) d\theta. \tag{7.2.6}$$

It follows from Lemma 7.2.3 that for any $\theta$ such that $M_\theta$ is non-degenerate( i.e., $\prod_{i=1}^{N-1}\sin\theta_i \neq$

0),

$$H^r_\alpha(\theta) = c^{N/\alpha}\bigg(\prod_{i=1}^{N-1} \sin^{N-i}\theta_i\bigg)H_\alpha.$$

Note that $(\prod_{i=1}^{N-1} \sin^{N-i}\theta_i)d\theta$ is the spherical area element and $M_\theta$ are non-degenerate for $\theta \in D$ almost everywhere, we obtain

$$\int_D H^r_\alpha(\theta)d\theta = c^{N/\alpha}\mathrm{Area}(T)H_\alpha.$$

Plugging this into (7.2.6), we finish the proof. $\square$

### 7.2.2 Standardized Spherical Fractional Brownian Motion

Theorem 7.2.4 is an application of Lemma 7.2.1 and Theorem 7.2.2, and it provides a nice formula since (7.2.1) has a simple form. More generally, the local behavior of covariance function may be more complicated, but we can still apply Lemma 7.2.1 to find the corresponding local behavior of covariance function under spherical coordinates and then apply Theorem 7.2.2 to obtain the asymptotics for the excursion probability. Here, we present an example about spherical fractional Brownian motion on sphere.

Let $o$ be a fixed point on $\mathbb{S}^N$. The Spherical Fractional Brownian Motion $B_\beta(x)$ is defined as a centered real-valued Gaussian random field such that

$$B(o) = 0$$
$$\mathbb{E}(B(x) - B(y))^2 = d^{2\beta}(x, y) \quad \forall x, y \in \mathbb{S}^N,$$

where $\beta \in (0, 1/2]$. It follows immediately that

$$\mathrm{Cov}(B(x), B(y)) = \frac{1}{2}\big(d^{2\beta}(x, o) + d^{2\beta}(y, o) - d^{2\beta}(x, y)\big).$$

Without loss of generality, we take $o = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$, whose corresponding spherical coordinate is $(0, \ldots, 0) \in \mathbb{R}^N$. Define

$$X(x) = \frac{B(x)}{d^\beta(x, o)}, \quad \forall x \in \mathbb{S}^N \backslash \{o\}.$$

Then the covariance is

$$C(x, y) = \mathrm{Cov}(X(x), X(y)) = \frac{d^{2\beta}(x, o) + d^{2\beta}(y, o) - d^{2\beta}(x, y)}{2d^\beta(x, o)d^\beta(y, o)}.$$

Note that under the spherical coordinates, $d(x, o) = \theta_1$ and $d(y, o) = \varphi_1$, together with Lemma 7.2.1, we obtain that as $d(x, y) \to 0$,

$$\begin{aligned}\widetilde{C}(\theta, \varphi) = \mathrm{Cov}(\widetilde{X}(\theta), \widetilde{X}(\varphi)) = 1 - (1 + o(1))\frac{1}{2\theta_1^{2\beta}}\Bigg[&(\varphi_1 - \theta_1)^2 + (\sin^2\theta_1)(\varphi_2 - \theta_2)^2 \\ &+ \cdots + \bigg(\prod_{i=1}^{N-1} \sin^2\theta_i\bigg)(\varphi_N - \theta_N)^2\Bigg]^\beta.\end{aligned}$$

Let

$$\begin{aligned} M_\theta &= \frac{1}{2^{1/(2\beta)}\theta_1}\mathrm{diag}\bigg(1, \sin\theta_1, \ldots, \prod_{i=1}^{N-1}\sin\theta_i\bigg), \\ r_\theta(\tau) &= \|M_\theta \tau\|^{2\beta}, \quad \forall \tau \in \mathbb{S}^{N-1}. \end{aligned}$$

Then as $\|\xi\| = \|\varphi - \theta\| \to 0$,

$$\widetilde{C}(\theta, \theta + \xi) = 1 - \|\xi\|^{2\beta} r_\theta(\xi/\|\xi\|)(1 + o(1)).$$

Let $T \subset \mathbb{S}^N$ be an $N$-dimensional Jordan measurable set such that $o \notin \bar{T}$, and denote its domain under spherical coordinates by $D$. Then by Theorem 7.2.2, as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} = \mathbb{P}\Big\{\sup_{\theta\in D} \widetilde{X}(\theta) \geq u\Big\} \sim u^{N/\beta}\Psi(u)\int_D H_{2\beta}^r(\theta)d\theta.$$

It follows from Lemma 7.2.3 that for any $\theta$ such that $M_\theta$ is non-degenerate( i.e., $\prod_{i=1}^{N-1}\sin\theta_i \neq 0$),

$$H_{2\beta}^r(\theta) = \frac{1}{2^{N/(2\beta)}\theta_1^N}\bigg(\prod_{i=1}^{N-1}\sin^{N-i}\theta_i\bigg)H_{2\beta}.$$

Therefore,

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} \sim u^{N/\beta}\Psi(u)2^{-N/(2\beta)}H_{2\beta}\int_D \theta_1^{-N}\bigg(\prod_{i=1}^{N-1}\sin^{N-i}\theta_i\bigg)d\theta.$$

## 7.3 Smooth Isotropic Gaussian Fields on Sphere

In this section we consider the excursion probabilities for smooth isotropic Gaussian fields on sphere.

### 7.3.1 Preliminaries

Given $\lambda > 0$ and an integer $n \geq 0$, the function $P_n^\lambda(t)$ is defined by the expansion

$$(1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(t), \quad t \in [-1, 1],$$

and $P_n^\lambda(t)$ is called the *ultraspherical polynomial* (or *Gegenbauer polynomial*) of degree $n$. If $\lambda = 0$, we follow Schoenberg (1942) and set $P_n^0(t) = \cos(n \arccos t) = T_n(t)$, where $T_n$, $n \geq 0$, are *Chebyshev polynomials of the first kind* defined by the expansion

$$\frac{1 - rt}{1 - 2rt + r^2} = \sum_{n=0}^{\infty} r^n T_n(t), \quad t \in [-1, 1].$$

For reference later on, we need the following formulae on $P_n^\lambda$.

(i). For all $n \geq 0$, $P_n^0(1) = 1$, and if $\lambda > 0$ [cf. Szegö (1975, p.80)],

$$P_n^\lambda(1) = \binom{n + 2\lambda - 1}{n}. \tag{7.3.1}$$

(ii). For all $n \geq 0$,

$$\frac{d}{dt} P_n^0(t) = n P_{n-1}^1(t), \tag{7.3.2}$$

and if $\lambda > 0$ [cf. Szegö (1975, p.81)],

$$\frac{d}{dt} P_n^\lambda(t) = 2\lambda P_{n-1}^{\lambda+1}(t). \tag{7.3.3}$$

The following theorem by Schoenberg (1942) characterizes the covariance function of an isotropic Gaussian field on sphere [see also Gneiting (2012)].

**Theorem 7.3.1** *Let $N \geq 1$, then a continuous function $C(\cdot,\cdot) : \mathbb{S}^N \times \mathbb{S}^N \to \mathbb{R}$ is the covariance of an isotropic Gaussian field on $\mathbb{S}^N$ if and only if it has the form*

$$C(x, y) = \sum_{n=0}^{\infty} a_n P_n^{\lambda}(\langle x, y\rangle), \quad x, y \in \mathbb{S}^N,$$

*where $\lambda = (N-1)/2$, $a_n \geq 0$, $\sum_{n=0}^{\infty} a_n P_n^{\lambda}(1) < \infty$.*

**Remark 7.3.2** Note that for $N = 1$, $\lambda = 0$ and $\sum_{n=0}^{\infty} a_n P_n^0(1) < \infty$ is equivalent to $\sum_{n=0}^{\infty} a_n < \infty$; while for $N \geq 2$, $\lambda = (N-1)/2$ and by (7.3.1), $\sum_{n=0}^{\infty} a_n P_n^{\lambda}(1) < \infty$ is equivalent to $\sum_{n=0}^{\infty} n^{N-2} a_n < \infty$.

When $N = 2$, $\lambda = 1/2$ and $P_n^{\lambda}$ become *Legendre polynomials.* For more results on isotropic Gaussian fields on $\mathbb{S}^2$, we refer to a recent monograph by Marinucci and Peccati (2011).

The following statement (**S**) is a smoothness condition for Gaussian fields on sphere. In Lemma 7.3.3 below, we show that it implies $C(\cdot,\cdot) \in C^4(\mathbb{S}^N \times \mathbb{S}^N)$.

(**S**). The covariance $C(\cdot,\cdot)$ of $\{X(x) : x \in \mathbb{S}^N\}$ satisfies

$$C(x, y) = \sum_{n=0}^{\infty} a_n P_n^{\lambda}(\langle x, y\rangle), \quad x, y \in \mathbb{S}^N,$$

where $\lambda = (N-1)/2$, $a_n \geq 0$, and $\sum_{n=1}^{\infty} n^8 a_n < \infty$ if $N = 1$, $\sum_{n=1}^{\infty} n^{N+6} a_n < \infty$ if $N \geq 2$.

**Lemma 7.3.3** *Let $\{X(x) : x \in \mathbb{S}^N\}$ be an isotropic Gaussian field such that* (**S**) *is fulfilled. Then the covariance $C(\cdot,\cdot) \in C^4(\mathbb{S}^N \times \mathbb{S}^N)$ and hence $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s.*

**Proof** We first consider $N \geq 2$. By Theorem 7.3.1, each $P_n^\lambda(\langle t, s\rangle)$ is the covariance of an isotropic Gaussian field on $\mathbb{S}^N$ and hence

$$|P_n^\lambda(\langle x, y\rangle)| \leq P_n^\lambda(\langle x, x\rangle) = P_n^\lambda(1), \quad \forall x, y \in \mathbb{S}^N. \tag{7.3.4}$$

Combining (**S**) with (7.3.1), (7.3.3) and (7.3.4), together with the fact $P_0^\lambda(t) \equiv 1$, we obtain that there exist positive constants $M_1$ and $M_2$ such that

$$\sup_{t\in[-1,1]} \sum_{n=0}^{\infty} a_n \Big| \Big( \frac{d^4}{dt^4} P_n^\lambda(t) \Big) \Big| \leq M_1 \sum_{n=4}^{\infty} a_n P_{n-4}^{\lambda+4}(1) \leq M_2 \sum_{n=1}^{\infty} n^{N+6} a_n < \infty.$$

This gives $C^4(\mathbb{S}^N \times \mathbb{S}^N)$. The proof for $N = 1$ is similar once we apply both (7.3.2) and (7.3.3). □

By Schoenberg (1942) or Gneiting (2012), $C(\cdot, \cdot)$ is a covariance function on $\mathbb{S}^N$ for every $N \geq 1$ if and only if it has the form

$$C(x, y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \quad x, y \in \mathbb{S}^N,$$

where $b_n \geq 0$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then we may state (**S′**) below as another form of smoothness condition for Gaussian fields on sphere.

(**S′**). The covariance $C(\cdot, \cdot)$ of $\{X(x) : x \in \mathbb{S}^N\}$ satisfies

$$C(x, y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \quad x, y \in \mathbb{S}^N,$$

where $b_n \geq 0$ and $\sum_{n=0}^{\infty} n^4 b_n < \infty$.

We obtain an analogue of Lemma 7.3.3 below. Since the proof is similar, it is omitted.

**Lemma 7.3.4** *Let $\{X(x) : x \in \mathbb{S}^N\}$ be an isotropic Gaussian field such that* $(\mathbf{S}')$ *is fulfilled. Then the covariance $C(\cdot,\cdot) \in C^4(\mathbb{S}^N \times \mathbb{S}^N)$ and hence $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s.*

### 7.3.2 Excursion Probability

Let $\chi(A_u(X, \mathbb{S}^N))$ be the Euler characteristic of excursion set $A_u(X, \mathbb{S}^N) = \{x \in \mathbb{S}^N : X(x) \geq u\}$. Denote by $H_j(x)$ the Hermite polynomial, i.e.,

$$H_j(x) = (-1)^j e^{x^2/2} \frac{d^j}{dx^j}\big(e^{-x^2/2}\big).$$

Let $\omega_j := \mathrm{Area}(\mathbb{S}^j)$, where $\mathbb{S}^j$ is the $j$-dimensional unit sphere.

**Lemma 7.3.5** *Let $\{X(x) : x \in \mathbb{S}^N\}$ be a centered, unit-variance, isotropic Gaussian field satisfying* $(\mathbf{S})$*. Suppose also that the joint distribution of $(X(x), \nabla X(x), \nabla^2 X(x))$ is non-degenerate for each $x \in \mathbb{S}^N$. Then*

$$\mathbb{E}\{\chi(A_u(X, \mathbb{S}^N))\} = \sum_{j=0}^{N} (C')^{j/2} \mathcal{L}_j(\mathbb{S}^N) \rho_j(u),$$

*where*

$$C' = \begin{cases} (N-1)\sum_{n=1}^{\infty} \binom{n+N-1}{N} a_n & \text{if } N \geq 2, \\ \sum_{n=1}^{\infty} n^2 a_n & \text{if } N = 1, \end{cases} \tag{7.3.5}$$

*$\rho_0(u) = \Psi(u)$, $\rho_j(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2}$ with Hermite polynomials $H_{j-1}$ for*

$j \geq 1$ *and, for* $j = 0, \ldots, N$,

$$\mathcal{L}_j(\mathbb{S}^N) = \begin{cases} 2\binom{N}{j}\frac{\omega_N}{\omega_{N-j}} & \text{if } N-j \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{7.3.6}$$

*are the Lipschitz-Killing curvatures of* $\mathbb{S}^N$.

**Remark 7.3.6** In Lemma 7.3.5, if condition (**S**) is replaced by (**S′**), then it can be seen from the proof below that $C'$ would be changed to a much simpler form

$$C' = \sum_{n=1}^{\infty} n b_n. \tag{7.3.7}$$

**Proof** Due to Theorem 12.4.1 in Adler and Taylor (2007), we only need to show that the Lipschitz-Killing curvatures induced by $X$ on $\mathbb{S}^N$ is $\mathcal{L}_j(X, \mathbb{S}^N) = (C')^{j/2}\mathcal{L}_j(\mathbb{S}^N)$.

The Riemannian structure induced by $X$ on $\mathbb{S}^N$ is defined as [cf. Adler and Taylor (2007, p.305)]

$$g_{x_0}^{X,\mathbb{S}^N}(\xi_{x_0}, \sigma_{x_0}) := \mathbb{E}\{(\xi_{x_0}X)\cdot(\sigma_{x_0}X)\} = \xi_{x_0}\sigma_{x_0}C(x,y)|_{x=y=x_0}, \quad \forall x_0 \in \mathbb{S}^N,$$

where $\xi_{x_0}, \sigma_{x_0} \in T_{x_0}\mathbb{S}^N$, the tangent space of $\mathbb{S}^N$ at $x_0$. We may choose two smooth curves on $\mathbb{S}^N$, say $\gamma(t)$, $\tau(s)$, $t, s \in [0,1]$, such that $\gamma(0) = \tau(0) = x_0$ and $\gamma'(0) = \xi_{x_0}, \tau'(0) = \sigma_{x_0}$.

We first consider $N \geq 2$, then

$$
\begin{aligned}
&\xi_{x_0}\sigma_{x_0}C(x,y)|_{x=y=x_0} \\
&\quad = \frac{\partial}{\partial t}\frac{\partial}{\partial s}C(\gamma(t),\tau(s))|_{t=s=0} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\sum_{n=0}^{\infty} a_n P_n^{\lambda}(\langle\gamma(t),\tau(s)\rangle)|_{t=s=0} \\
&\quad = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}(\langle\gamma(t),x_0\rangle)\langle\gamma(t),\sigma_{x_0}\rangle|_{t=0} \\
&\quad = \sum_{n=2}^{\infty} a_n (N-1)(N+2) P_{n-2}^{\lambda+2}(\langle x_0,x_0\rangle)\langle\xi_{x_0},x_0\rangle\langle x_0,\sigma_{x_0}\rangle \\
&\qquad + \sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}(\langle x_0,x_0\rangle)\langle\xi_{x_0},\sigma_{x_0}\rangle \\
&\quad = \left(\sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}(1)\right)\langle\xi_{x_0},\sigma_{x_0}\rangle = C'\langle\xi_{x_0},\sigma_{x_0}\rangle,
\end{aligned}
$$

where the third and fourth equalities are from (7.3.3), the fifth equality comes from $\langle x_0, x_0\rangle = 1$ and $\langle\xi_{x_0}, x_0\rangle = \langle\sigma_{x_0}, x_0\rangle = 0$, since the vector $x_0$ is always orthogonal to its tangent space. The case of $N = 1$ can be proved similarly once we apply (7.3.2) instead of (7.3.3).

Hence the induced metric is

$$g_{x_0}^{X,\mathbb{S}^N}(\xi_{x_0},\sigma_{x_0}) = C'\langle\xi_{x_0},\sigma_{x_0}\rangle, \quad \forall x_0 \in \mathbb{S}^N.$$

By the definition of Lipschitz-Killing curvatures, one has

$$\mathcal{L}_j(X,\mathbb{S}^N) = (C')^{j/2}\mathcal{L}_j(\mathbb{S}^N),$$

where $\mathcal{L}_j(\mathbb{S}^N)$ are the original Lipschitz-Killing curvatures of $\mathbb{S}^N$ given by (7.3.6). We finish the proof. $\square$

**Theorem 7.3.7** *Suppose the conditions in Lemma 7.3.5 hold. Then, under the notations therein, there exists $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{ \sup_{x\in\mathbb{S}^N} X(x) \geq u \Big\} = \sum_{j=0}^{N} (C')^{j/2} \mathcal{L}_j(\mathbb{S}^N)\rho_j(u) + o(e^{-\alpha u^2 - u^2/2}). \tag{7.3.8}$$

**Remark 7.3.8** Under the conditions in Theorem 7.3.7, the covariance function $C$ satisfies (7.2.1) with $\alpha = 2$. Also note that when $\alpha = 2$, Pickands' constant $H_2 = \pi^{-N/2}$. Then one can check that the approximation in Theorem 7.2.4 only provides the leading term of the approximation in Theorem 7.3.7. This also affects the errors in two approximations: the error in the former one is only $o(1)$, while the error in the latter one is $o(e^{-\alpha u^2})$.

**Proof** The result is an immediate consequence of Lemma 7.3.5 and Theorem 14.3.3 in Adler and Taylor (2007). □

If the set $\mathbb{S}^N$ is replaced by a more general subset $T \subset \mathbb{S}^N$, by simply revising Lemma 7.3.5 and applying Theorem 14.3.3 in Adler and Taylor (2007) again, we obtain the following corollary.

**Corollary 7.3.9** *Suppose the conditions in Lemma 7.3.5 hold. Let $T \subset \mathbb{S}^N$ be a $k$-dimensional, locally convex, regular stratified manifold [cf. Adler and Taylor (2007)], then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{ \sup_{x\in T} X(x) \geq u \Big\} = \sum_{j=0}^{k} (C')^{j/2} \mathcal{L}_j(T)\rho_j(u) + o(e^{-\alpha u^2 - u^2/2}),$$

*where $\mathcal{L}_j(T)$ are the Lipschitz-Killing curvatures of $\mathbb{S}^N$ [cf. Adler and Taylor (2007)], $C'$ and $\rho_j(u)$ are as in Lemma 7.3.5.*

**Example 7.3.10** Canonical field on $\mathbb{S}^N$, whose covariance structure is given by $C(x,y) = \langle x, y\rangle$. Since $C(x,y) = \cos d(x,y)$, it satisfies

$$C(x,y) = 1 - \frac{1}{2}d^2(x,y)(1+o(1)), \quad \text{as } d(x,y) \to 0.$$

Applying Theorem 7.2.4, one can get an approximation to the excursion probability. However, by applying Theorem 7.3.7, we will get a more precise approximation for $N \geq 2$.

**Example 7.3.11** Consider the Hamiltonian of the pure $p$-spin model on $\mathbb{S}^{N-1}$

$$H_{N,p}(x) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} x_{i_1} \cdots x_{i_p}, \quad \forall x = (x_1, \cdots, x_N) \in \mathbb{S}^{N-1},$$

where $J_{i_1,\ldots,i_p}$ are independent standard Gaussian random variables. Then

$$\mathbb{E}\{H_{N,p}(x)H_{N,p}(y)\} = \frac{1}{N^{p-1}}\langle x, y\rangle^p.$$

Let

$$X(x) = \sum_{p=2}^{\infty} b_p H_{N,p}(x),$$

where $(b_p)_{p\geq 2}$ is a sequence of positive numbers such that $\sum_{p=2}^{\infty} 2^p b_p < \infty$, and $H_{N,p}$ and $H_{N,p'}$ are independent for any $p \neq p'$. Then $X$ is a smooth Gaussian random field on sphere with covariance

$$C(x,y) = \sum_{p=2}^{\infty} \frac{b_p^2}{N^{p-1}}\langle x, y\rangle^p.$$

**Example 7.3.12** We show how to apply Corollary 7.3.9. If $T$ is the semisphere of dimension one, then $\mathcal{L}_0(T) = 1$ and $\mathcal{L}_1(T) = \pi$. If $T$ is the semisphere of dimension two, then $\mathcal{L}_0(T) = 1$, $\mathcal{L}_1(T) = \pi$ and $\mathcal{L}_2(T) = 2\pi$. Basically, $\mathcal{L}_0(T)$ is the Euler characteristic, $\mathcal{L}_k(T)$ is the volume and $\mathcal{L}_{k-1}(T)$ is half of the surface area. Usually, one may use Steiner's formula [Adlar and Taylor (2007, p.142)] to compute the Lipschitz-Killing curvatures exactly.

**Example 7.3.13** Consider the covariance structure $C(x, y) = 1 - \frac{2}{\pi} d(x, y)$, which can be verified to be a valid covariance on sphere. Since $d(x, y) = \arccos \langle x, y \rangle$, we can write

$$\begin{aligned} C(x, y) = 1 - \arccos \langle x, y \rangle &= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} \langle x, y \rangle^{2n+1} \\ &:= \sum_{n=0}^{\infty} b^n \langle x, y \rangle^n, \end{aligned}$$

it is easy to check that $\sum_{n=0}^{\infty} n b^n = \infty$, hence Theorem 7.3.7 is not applicable. In fact, $C(x, y)$ is not smooth neither. Instead, we may use Theorem 7.2.2 to get the approximation to excursion probability.

# Chapter 8

# Excursion Probability of Anisotropic Gaussian and Asymptotically Gaussian Random Fields

## 8.1 Preliminaries

For vectors $u, v \in \mathbb{R}^d$, the relation $u \le v$ means $u_i \le v_i$ for all $i$ and $u < v$ means $u_i < v_i$ for all $i$, also we let $uv := (u_1v_1, \cdots, u_dv_d)$. For $t = (t_1, \cdots, t_d) \in \mathbb{R}^d$ and $\zeta = (\zeta_1, \cdots, \zeta_d) > 0$, define

$$I_{t,\zeta} = \prod_{i=1}^{d} [t_i, t_i + \zeta_i).$$

Let $\|\cdot\|$ be the Euclidean norm of a vector, $\lfloor \cdot \rfloor$ be the greatest integer function, $\mu(\cdot)$ be the volume of set. For bounded and Jordan measurable set $D \subset \mathbb{R}^d$ and $\delta > 0$, define

$$[D]_\delta = \{t + u : t \in D, \|u\| \le \delta\}.$$

Let $0 < \alpha \leq 1$, $p = (p_1, \cdots, p_d)$ with $0 < p_i \leq 2$ for all $1 \leq i \leq d$ and let $\{W_t(u) : u \in [0, \infty)^d\}$ be a Gaussian random field such that $W_t(0) = 0$ and

$$
\begin{aligned}
\mathbb{E}(W_t(u)) &= -\left(\sum_{i=1}^d u_i^{p_i}\right)^\alpha r_t\left(\frac{u_1^{p_1}}{\sum_{i=1}^d u_i^{p_i}}, \cdots, \frac{u_d^{p_d}}{\sum_{i=1}^d u_i^{p_i}}\right), \\
\mathrm{Cov}(W_t(u), W_t(v)) &= \left(\sum_{i=1}^d u_i^{p_i}\right)^\alpha r_t\left(\frac{u_1^{p_1}}{\sum_{i=1}^d u_i^{p_i}}, \cdots, \frac{u_d^{p_d}}{\sum_{i=1}^d u_i^{p_i}}\right) \\
&\quad + \left(\sum_{i=1}^d v_i^{p_i}\right)^\alpha r_t\left(\frac{v_1^{p_1}}{\sum_{i=1}^d v_i^{p_i}}, \cdots, \frac{v_d^{p_d}}{\sum_{i=1}^d v_i^{p_i}}\right) \\
&\quad - \left(\sum_{i=1}^d |u_i - v_i|^{p_i}\right)^\alpha r_t\left(\frac{|u_1 - v_1|^{p_1}}{\sum_{i=1}^d |u_i - v_i|^{p_i}}, \cdots, \frac{|u_d - v_d|^{p_d}}{\sum_{i=1}^d |u_i - v_i|^{p_i}}\right),
\end{aligned} \tag{8.1.1}
$$

where $r_t : \mathcal{S} = \{v \in [0, \infty)^d : \sum_{i=1}^d v_i = 1\} \to \mathbb{R}_+$ is a continuous function satisfying

$$
\sup_{v \in \mathcal{S}} |r_t(v) - r_s(v)| \to 0, \qquad \text{as } \|t - s\| \to 0. \tag{8.1.2}
$$

In particular, we define

$$
\begin{aligned}
H_K(t) &= \int_0^\infty e^y \mathbb{P}\left\{\sup_{0 \leq u_i \leq K, \forall i} W_t(u) > y\right\} dy, \\
H(t) &= \lim_{K \to \infty} K^{-d} H_K(t).
\end{aligned} \tag{8.1.3}
$$

Let $L$ be a slowly varying function, define

$$
\Delta_{c,i} = \min\{x > 0 : x^{\alpha p_i} L(x) = c^{-2}\}, \qquad \forall 1 \leq i \leq d, \tag{8.1.4}
$$

and $\Delta_c = (\Delta_{c,1}, \cdots, \Delta_{c,d})$. For example, if $L(x) \equiv 1$, then $\Delta_c = (c^{-\frac{2}{\alpha p_1}}, \cdots, c^{-\frac{2}{\alpha p_d}})$. Our goal is to investigate the asymptotic property of centered Gaussian fields satisfying the following condition:

$$\begin{aligned} &\mathbb{E}(X(t)X(t+u)) \\ &\quad = 1 - (1+o(1))\Big(\sum_{i=1}^{d} |u_i|^{p_i}\Big)^{\alpha} L\Big(\sum_{i=1}^{d} |u_i|^{p_i}\Big) r_t\Big(\frac{|u_1|^{p_1}}{\sum_{i=1}^{d} |u_i|^{p_i}}, \cdots, \frac{|u_d|^{p_d}}{\sum_{i=1}^{d} |u_i|^{p_i}}\Big), \end{aligned} \tag{8.1.5}$$

as $\|u\| \to 0$, uniformly over $t \in [D]_\delta$.

**Theorem 8.1.1** *Suppose Gaussian random field $X$ satisfies condition (8.1.5), where $0 < \alpha \leq 1$, $p = (p_1, \cdots, p_d)$ with $0 < p_i \leq 2$ for all $1 \leq i \leq d$, and $r_t : \mathcal{S} \to \mathbb{R}_+$ is a continuous function such that the convergence in (8.1.2) is uniform in $t \in [D]_\delta$ and $\sup_{t \in [D]_\delta, v \in \mathcal{S}} r_t(v) < \infty$. Then as $c \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t \in D} X(t) > c\Big\} \sim \Psi(c)\Big(\prod_{i=1}^{d} \Delta_{c,i}^{-1}\Big) \int_D H(t)dt.$$

## 8.2 Asymptotically Gaussian Random Fields

For $c > 0$, let $X_c$ be random fields such that $\mathbb{E}X_c(t) = 0$, $\mathbb{E}X_c^2(t) = 0$ for all $c$ and $t$. Define $\rho_c(t, u) = \mathbb{E}(X_c(t)X_c(u))$. We impose the following conditions for $X_c$.

(**C**). There exist $0 < \alpha \leq 1$, $p = (p_1, \cdots, p_d)$ with $0 < p_i \leq 2$ for all $1 \leq i \leq d$ and a slowly varying function $L$ such that as $\|u\| \to 0$,

$$\rho(t, t+u) = 1 - (1+o(1))\Big(\sum_{i=1}^{d} |u_i|^{p_i}\Big)^{\alpha} L\Big(\sum_{i=1}^{d} |u_i|^{p_i}\Big) r_t\Big(\frac{|u_1|^{p_1}}{\sum_{i=1}^{d} |u_i|^{p_i}}, \cdots, \frac{|u_d|^{p_d}}{\sum_{i=1}^{d} |u_i|^{p_i}}\Big)$$

uniformly over $t \in [D]_\delta$ and $c > 0$.

(**B**1). As $c \to \infty$,

$$\mathbb{P}\{X_c(t) > c - y/c\} \sim \Psi(c - y/c)$$

uniformly over $t \in [D]_\delta$ and positive, bounded values of $y$.

(**B**2). The convergence in (8.1.2) is uniform over $t \in [D]_\delta$, with $\sup_{t \in [D]_\delta, v \in \mathcal{S}} r_t(v) < \infty$. Moreover, for any $a > 0$, $\mathbf{a} = \{a^{1/p_1}, \cdots, a^{1/p_d}\}$ and positive integers $m_i$, as $c \to \infty$,

$$\{c[X_c(t + \mathbf{a}k\Delta_c) - X_c(t)] : 0 \le k_i < m_i\} | X_c(t) = c - y/c$$
$$\Rightarrow \{W_t(\mathbf{a}k) : 0 \le k_i < m_i\}$$

uniformly over positive, bounded values of $y$.

(**B**3). There exists a positive function $h$ such that $\lim_{y \to \infty} h(y) = 0$ and

$$\mathbb{P}\{X_c(t + u\Delta_c) > c - \gamma/c, X_c(t) \le c - y/c\} \le h(y)\Psi(c),$$

for all $u \ge 0$ ($u$ is a vector) and $\gamma > 0$.

(**B**4). Let $p_{i_0} = \min\{p_i, 1 \le i \le d\}$, $\mathbf{a} = (a^{1/p_1}, \cdots, a^{1/p_d})$. There exist nonincreasing functions $N_a$ on $\mathbb{R}_+$ and positive constants $\gamma_a$ such that $\gamma_a \to 0$ and $N_a(\gamma_a) + \int_1^\infty \omega^s N_a(\gamma_a +$

$\omega)d\omega = o(a^{d/p_{i_0}})$ as $a \to 0$, and

$$\mathbb{P}\Big\{ \sup_{v \in I_{t,\mathbf{a}\Delta_c}} X_c(v) > c, X_c(t) \le c - y/c \Big\} \le N_a(\gamma)\Psi(c),$$

for all $\gamma_a \le \gamma \le c$ and $s > 0$.

($\mathbf{B}$5). There exists a nonincreasing function $f : [0, \infty) \to \mathbb{R}_+$ such that $f(y) = O(e^{-y^{q'}})$ for some $q' > 0$ and for all $\gamma \ge 0$ and $c$ sufficiently large,

$$\mathbb{P}\{X_c(t) > c - \gamma/c, X_c(t + u\Delta_c) > c - \gamma/c\} \le \Psi(c - \gamma/c) f\Big( \sum_{i=1}^{d} |u_i|^{p_i} \Big)$$

uniformly in $t$ and $t + u\Delta_c$ belonging to $[D]_\delta$.

For $K > 0$ and $a > 0$, let $A_t = (A_t(K, a, c)) \triangleq \{t + \mathbf{a}k\Delta_c : 0 \le k_i < m_i, k \in \mathbb{Z}^d\}$, where $m_i = \lfloor K/a^{1/p_i} \rfloor$ and $\mathbf{a} = (a^{1/p_1}, \cdots, a^{1/p_d})$. As a discrete set, $A_t$ will be used to approximate $I_{t,K\Delta_c}$.

**Lemma 8.2.1** *Under* ($\mathbf{C}$) *and* ($\mathbf{B}$1)-($\mathbf{B}$3),

$$H_{K,a}(t) \triangleq \int_0^\infty e^y \mathbb{P}\Big\{ \sup_{0 \le k_i < m_i, \forall i} W_t(\mathbf{a}k) > y \Big\} dy$$

*is uniformly continuous in* $t \in [D]_\delta$ *and* $\sup_{t \in [D]_\delta} H_{K,a}(t) < \infty$. *Moreover, for* $\gamma \ge 0$, *as* $c \to \infty$,

$$\mathbb{P}\Big\{ \sup_{u \in A_t} X_c(u) > c - \gamma/c \Big\} \sim \Psi(c - \gamma/c)(1 + H_{K,a}(t)) \tag{8.2.1}$$

*uniformly in* $t \in [D]_\delta$.

**Proof** Let $\varepsilon > 0$. By (**B**3), there exists $y^* > \gamma$ such that $h(y^*) < \varepsilon/(\prod_{i=1}^d m_i)$ and

$$\begin{aligned} 0 &\le \mathbb{P}\left\{\sup_{u\in A_t} X_c(u) > c - \gamma/c\right\} - \mathbb{P}\{X_c(t) > c - \gamma/c\} \\ &\quad - \mathbb{P}\left\{\sup_{u\in A_t} X_c(u) > c - \gamma/c, c - y^*/c < X_c(t) \le c - \gamma/c\right\} \\ &= \mathbb{P}\left\{\sup_{u\in A_t} X_c(u) > c - \gamma/c, X_c(t) \le c - y^*/c\right\} \\ &\le \left(\prod_{i=1}^d m_i\right) h(y^*)\Psi(c) < \varepsilon\Psi(c), \end{aligned} \tag{8.2.2}$$

since $\mathrm{card}(A_t) = \prod_{i=1}^d m_i$. By (**B**1), there exists $\xi_c \to 0$ such that

$$|\mathbb{P}\{X_c(t) > c - y/c\}/\Psi(c - y/c) - 1| = O(\xi_c^2) \tag{8.2.3}$$

uniformly for $\gamma \le y \le y^*$; we can also assume that $\xi_c^{-1}(y^* - \gamma) \in \mathbb{Z}$. Since $e^{\xi_c} = 1 + \xi_c + O(\xi_c^2)$ and $\Psi(c - y/c) \sim e^y\Psi(c)$, (8.2.3) implies

$$\begin{aligned} &\mathbb{P}\{c - (y + \xi_c)/c < X_c(t) \le c - y/c\} \\ &\quad = (1 + O(\xi_c^2))e^{y+\xi_c}\Psi(c) - (1 + O(\xi_c^2))e^y\Psi(c) \sim \xi_c e^y\Psi(c). \end{aligned}$$

By (**B**2), uniformly for $t \in [D]_\delta$ and $\gamma \le y \le y^*$,

$$\begin{aligned} &\mathbb{P}\left\{\sup_{u\in A_t} X_c(u) > c - \gamma/c, c - (y + \xi_c)/c < X_c(t) \le c - y/c\right\} \\ &\sim \mathbb{P}\left\{\sup_{0\le k_i<m_i,\forall i} W_t(\mathbf{a}k) > y - \gamma\right\}\mathbb{P}\{c - (y + \xi_c)/c < X_c(t) \le c - y/c\} \\ &\sim \mathbb{P}\left\{\sup_{0\le k_i<m_i,\forall i} W_t(\mathbf{a}k) > y - \gamma\right\}\xi_c e^y\Psi(c). \end{aligned} \tag{8.2.4}$$

Summing (8.2.4) over $y = j\xi_c + \gamma$ for $j = 0, 1, \cdots, \xi_c^{-1}(y^* - \gamma) - 1$, we obtain

$$\begin{aligned}
&\mathbb{P}\Big\{ \sup_{u \in A_t} X_c(u) > c - \gamma/c, c - y^*/c < X_c(t) \le c - \gamma/c \Big\} \\
&\quad \sim \Psi(c) \int_{\gamma}^{y^*-1} e^y \mathbb{P}\Big\{ \sup_{0 \le k_i < m_i, \forall i} W_t(\mathbf{a}k) > y - \gamma \Big\} dy \\
&\quad \sim \Psi(c - \gamma/c) \int_0^{y^*-\gamma-1} e^y \mathbb{P}\Big\{ \sup_{0 \le k_i < m_i, \forall i} W_t(\mathbf{a}k) > y \Big\} dy.
\end{aligned}$$

Plugging this in (8.2.2) and letting $y^* \to \infty$, we get (8.2.1).

Since $\int_0^\infty e^y \mathbb{P}\{W_t(\mathbf{a}k) > y\} dy < \infty$ for all $k$ and $A_t$ is a finite set, $H_{K,a}(t)$ is finite and its uniform continuity follows from (8.1.1) and (8.1.2), with the fact described in (**B**2) that the convergence in (8.1.2) is uniform over $t \in [D]_\delta$. Racall that $\sup_{t \in [D]_\delta, v \in \mathcal{S}} r_t(v) < \infty$, yielding the finiteness of $\sup_{t \in [D]_\delta} H_{K,a}(t)$. $\square$

**Theorem 8.2.2** *Let $K > 0$. Assume* (**C**) *and* (**B**1)-(**B**4). *Then as $c \to \infty$,*

$$\mathbb{P}\Big\{ \sup_{u \in I_{t,K\Delta_c}} X_c(u) > c \Big\} \sim \Psi(c)(1 + H_K(t))$$

*uniform over $t \in [D]_\delta$, where $H_K(t)$ is defined in (8.1.3) and is finite and uniformly continuous in $t \in [D]_\delta$.*

**Proof** Let $a > 0$. By (**B**4) and (8.2.1), we have for all large $c$,

$$
\begin{aligned}
0 &\le \left[\mathbb{P}\left\{\sup_{u\in I_{t,K\Delta_c}} X_c(u) > c\right\} - \mathbb{P}\left\{\sup_{u\in A_t} X_c(u) > c\right\}\right]\Big/\Psi(c) \\
&\le \left[\mathbb{P}\left\{c - \gamma_a/c < \sup_{u\in A_t} X_c(u) \le c\right\}\right. \\
&\quad \left. + \sum_{u\in A_t} \mathbb{P}\left\{\sup_{v\in I_{u,\mathbf{a}\Delta_c}} X_c(v) > c, X_c(u) \le c - \gamma_a/c\right\}\right]\Big/\Psi(c) \qquad (8.2.5) \\
&\le 2(\Psi(c-\gamma_a/c) - \Psi(c))(1 + H_{K,a}(t))/\Psi(c) + \prod_{i=1}^{d}(K/a^{1/p_i})N_a(\gamma_a) \\
&\le 3(e^{\gamma_a} - 1)(1 + H_{K,a}(t)) + (K^d/a^{\sum_{i=1}^d 1/p_i})N_a(\gamma_a).
\end{aligned}
$$

By (**B**4), for any $\varepsilon > 0$, we can choose $a^*$ small enough such that $N_a(\gamma_a)/a^{\sum_{i=1}^d 1/p_i} < \varepsilon/K^d$ and $3(e^{\gamma_a} - 1) < \varepsilon$ for all $0 < a \le a^*$. Therefore, by (8.2.1) and (8.2.5),

$$
(1-\varepsilon)(1 + H_{K,a}(t)) \le \mathbb{P}\left\{\sup_{u\in I_{t,K\Delta_c}} X_c(u) > c\right\}\Big/\Psi(c) \le (1+2\varepsilon)(1 + H_{K,a}(t)) + \varepsilon,
$$

for all large $c$ and all $t \in [D]_\delta$ and $0 < a \le a^*$. By uniform continuity of $W_t(u)$, $H_{K,a}(t) \to H_K(t)$ as $a \downarrow 0$. Therefore,

$$
1 + H_{K,a}(t) \le 1 + H_K(t) \le (1+\varepsilon)(1 + H_{K,a^*}(t)), \qquad (8.2.6)
$$

for all $t \in [D]_\delta$ and $0 < a \le a^*$.

First note that $M \triangleq 1 + \sup_{t\in[D]_\delta} H_K(t) < \infty$ in view of (8.2.6) and Lemma 8.2.1, therefore

$$
|H_K(t) - H_{K,a^*}(t)| \le M\varepsilon \qquad (8.2.7)
$$

for all $t \in [D]_\delta$, by (8.2.6) with $a = a^*$. Because $H_{K,a^*}$ is uniformly continuous by Lemma 8.2.1,

$$|H_{K,a^*}(t) - H_{K,a^*}(u)| \le \varepsilon, \qquad \forall \|t-u\| < \delta^*, t, u \in [D]_\delta, \tag{8.2.8}$$

for some $\delta^* > 0$. It follows from (8.2.7) and (8.2.8) that, if $\|t-u\| < \delta^*$,

$$\begin{aligned}
&|H_K(t) - H_K(u)| \\
&\quad \le |H_K(t) - H_{K,a^*}(t)| + |H_K(u) - H_{K,a^*}(u)| + |H_{K,a^*}(t) - H_{K,a^*}(u)| \\
&\quad \le 2M\varepsilon + \varepsilon.
\end{aligned}$$

Hence $H_K(t)$ is uniformly continuous in $t \in [D]_\delta$.

Combining (8.2.7) and the definition of $M$ yields that for all large $c$ and $t \in [D]_\delta$,

$$\begin{aligned}
-\varepsilon M - \varepsilon(1-\varepsilon)M &\le \mathbb{P}\Big\{\sup_{u \in I_{t,K\Delta_c}} X_c(u) > c\Big\} \Big/ \Psi(c) - (1 + H_K(t)) \\
&\le 2\varepsilon M + \varepsilon(1+2\varepsilon)M.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, this proves the theorem. □

**Lemma 8.2.3** *Under* (**C**) *and* (**B**1)-(**B**4), $\sup_{t\in[D]_\delta, K\ge 1} K^{-d}H_K(t) < \infty$ *and* $\{K^{-d}H_K : K \ge 1\}$ *is uniformly equicontinuous on* $[D]_\delta$*, that is,*

$$\sup_{K\ge 1, t,s\in[D]_\delta, \|t-s\|\le\varepsilon} |K^{-d}H_K(t) - K^{-d}H_K(s)| \to 0, \qquad \text{as } \varepsilon \to 0.$$

**Proof** Let

$$\mathbf{a} = (a^{1/p_1}, \cdots, a^{1/p_d}), \quad m_i = \lfloor K/a^{1/p_i} \rfloor,$$
$$N(K,a) = \lfloor (\prod_{i=1}^{d} m_i)/(\prod_{i=1}^{d} \lfloor a^{-1/p_i} \rfloor) \rfloor.$$

Note that the integrand of $H_{K,a}(t)$ involves the set $\{\mathbf{a}k : 0 \le k_i < m_i\}$, which can be partitioned into $N(K,a)+1$ disjoint subsets $L_j$ such that $\mathrm{card}(L_j) = \prod_{i=1}^{d} \lfloor a^{-1/p_i} \rfloor$ for $1 \le j \le N(K,a)$ and $\mathrm{card}(L_{N(K,a)+1}) = \prod_{i=1}^{d} m_i - N(K,a) \prod_{i=1}^{d} \lfloor a^{-1/p_i} \rfloor$. It is possible that $\mathrm{card}(L_{N(K,a)+1}) = 0$, in this case $L_{N(K,a)+1}$ is regarded as an empty set. We can therefore use the arguments at the end of the proof of Lemma 8.2.1 to bound

$$K^{-d} \sum_{j=1}^{N(K,a)+1} \left| \mathbb{P}\left\{ \sup_{k \in L_j} W_t(\mathbf{a}k) > y \right\} - \mathbb{P}\left\{ \sup_{k \in L_j} W_s(\mathbf{a}k) > y \right\} \right|$$

and thereby establish the uniform equicontinuity and boundedness of $\{K^{-d} H_{K,a} : K \ge 1\}$ on $[D]_\delta$. Moreover, by partitioning the cube $[0,K)^d$ similarly into $K^d$ cubes, it can be shown that $\sup_{K \ge 1, t \in [D]_\delta} |K^{-d} H_K(t) - K^{-d} H_{K,a}(t)| \to 0$, as $a \to 0$. Hence we can proceed as in (8.2.7) and (8.2.8) but with $H_{K,a}$ and $H_K$ replaced by $K^{-d} H_{K,a}$ and $K^{-d} H_K$ respectively to prove the uniform equicontinuity and boundedness of $\{K^{-d} H_K : K \ge 1\}$. $\square$

**Lemma 8.2.4** *Under* (**C**) *and* (**B**1)-(**B**5), *there exist constants $s_K \to 0$ as $K \to \infty$ such that*

$$\mathbb{P}\left\{ \sup_{u \in I_{t,K\Delta_c}} X_c(u) > c, \sup_{v \in B \setminus I_{t,K\Delta_c}} X_c(v) > c \right\} \le s_K K^d \Psi(c) \tag{8.2.9}$$

*for $c$ large enough, uniformly over $t \in [D]_\delta$ and over subsets $B$ of $[D]_\delta$.*

**Proof** Let $a > 0$ and $0 < q < q'$. Then by the property of $f$ described in (**B**5),

$$G_a \triangleq \sum_{w \in \mathbf{a}\mathbb{Z}^d} \exp\left(\sum_{i=1}^d |w_i|^{p_i}\right) f\left(\sum_{i=1}^d |w_i|^{p_i}\right) < \infty.$$

Let $n$ be positive integers that are large enough such that

$$\sum_{w \in \mathbf{a}\mathbb{Z}^d, \sum_{i=1}^d |w_i|^{p_i} \geq n^{p_1} a} \exp\left(\sum_{i=1}^d |w_i|^{p_i}\right) f\left(\sum_{i=1}^d |w_i|^{p_i}\right) < \varepsilon a^{\sum_{i=1}^d 1/p_i};$$

and $K > 0$ be large enough such that

$$\left[1 - \prod_{i=1}^d (1 - 2n/m_i)\right] G_a < \varepsilon a^{\sum_{i=1}^d 1/p_i},$$

where $m_i = \lfloor K/a^{1/p_i} \rfloor$. Let $F_{1,t} = \{t + \mathbf{a}k\Delta_c : n \leq k_i < m_i - n, k \in \mathbb{Z}^d\}$, $F_{2,t} = A_t \backslash F_{1,t}$, $B_t = \{t + \mathbf{a}k\Delta_c \in B \backslash I_{t,K\Delta_c}, k \in \mathbb{Z}^d\}$, $g_{uv} = \min\{c - \gamma_a, (\sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i})^q\}$. Then by (**B**5),

$$\begin{aligned} &\mathbb{P}\left\{X_c(u) > c - (\gamma_a + g_{uv})/c, X_c(v) > c - (\gamma_a + g_{uv})/c\right\} \\ &\leq \Psi(c - (\gamma_a + g_{uv})/c) f\left(\sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i}\right) \\ &\leq 2e^{g_{uv}} \Psi(c - \gamma_a/c) f\left(\sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i}\right), \end{aligned} \tag{8.2.10}$$

for all large $c$. For $u \in F_{1,t}$ and $v \in B_t$, $\sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i} \geq n^{p_1} a$ and $g_{uv} \leq (\sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i})^q$. Noting that $\mathrm{card}(F_{1,t}) \leq \prod_{i=1}^d m_i$, $\mathrm{card}(F_{2,t}) \leq \prod_{i=1}^d m_i - \prod_{i=1}^d (m_i - 2n) = \prod_{i=1}^d m_i [1 - \prod_{i=1}^d (1 - 2n/m_i)]$, and that $\sum_{u \in A_t} = \sum_{u \in F_{1,t}} + \sum_{u \in F_{2,t}}$, we obtain from (8.2.10) that for all large $c$,

$$\begin{aligned}
&\sum_{u\in A_t}\sum_{v\in B_t}\mathbb{P}\Big\{X_c(u)>c-(\gamma_a+g_{uv})/c, X_c(v)>c-(\gamma_a+g_{uv})/c\Big\}\\
&\le 2\Psi(c-\gamma_a/c)\Big(\prod_{i=1}^d m_i\Big)\Big\{\sum_{w\in\mathbf{a}\mathbb{Z}^d,\sum_{i=1}^d|w_i|^{p_i}\ge n^{p_1}a}\exp\Big(\sum_{i=1}^d|w_i|^{p_i}\Big)f\Big(\sum_{i=1}^d|w_i|^{p_i}\Big)\\
&\quad+\Big[1-\prod_{i=1}^d(1-2n/m_i)\Big]G_a\Big\}\\
&\le 4\varepsilon K^d\Psi(c-\gamma_a/c).
\end{aligned}\tag{8.2.11}$$

Define $\lambda_w=\min_{u\in A_t}g_{uw}$ if $w\in B_t$, and $\lambda_w=0$ if $w\in A_t$. Then

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{u\in I_{t,K\Delta_c}}X_c(u)>c,\sup_{v\in B\setminus I_{t,K\Delta_c}}X_c(v)>c\Big\}\\
&\le\sum_{u\in A_t}\sum_{v\in B_t}\mathbb{P}\Big\{X_c(u)>c-(\gamma_a+g_{uv})/c, X_c(v)>c-(\gamma_a+g_{uv})/c\Big\} \qquad (8.2.12)\\
&\quad+\sum_{w\in A_t\cup B_t}\mathbb{P}\Big\{\sup_{z\in I_{w,\mathbf{a}\Delta_c}}X_c(z)>c, X_c(v)\le c-(\gamma_a+\lambda_w)/c\Big\}.
\end{aligned}$$

On the right-hand side of (8.2.12), the first sum can be bounded by (8.2.11) and the second sum by

$$
\begin{aligned}
&\sum_{u\in A_t}\mathbb{P}\Big\{\sup_{z\in I_{u,\mathbf{a}\Delta_c}} X_c(z)>c, X_c(u)\le c-\gamma_a/c\Big\}\\
&\quad+\sum_{v\in B_t}\mathbb{P}\Big\{\sup_{v\in I_{v,\mathbf{a}\Delta_c}} X_c(z)>c, X_c(v)\le c-(\gamma_a+\lambda_v)/c\Big\}\\
&\quad\le\Big(\prod_{i=1}^d m_i\Big)N_a(\gamma_a)\Psi(c)+\sum_{v\in B_t}N_a(\gamma_a+\lambda_v)\Psi(c)\\
&\quad\le K^d a^{-\sum_{i=1}^d 1/p_i}N_a(\gamma_a)\Psi(c)+\sum_{v\in B_t}N_a(\gamma_a+\lambda_v)\Psi(c),
\end{aligned}
\tag{8.2.13}
$$

in view of (**B**4) and that $\text{card}(A_t)=\prod_{i=1}^d m_i$. To bound the last sum $\sum_{v\in B_t}$ in (8.2.13), first consider the case $d=1$. Since $\lambda_v\ge\min\{c-\gamma_a,(ak^p)^q\}$ if $ak^p\le\inf_{u\in A_t}(|v-u|/\Delta_c)^p<a(k+1)^p$, and since $N_a$ is nonincreasing, it follows that

$$
\begin{aligned}
&\sum_{v\in B_t}N_a(\gamma_a+\lambda_v)\\
&\qquad\le 2\Big\{\sum_{k=1}^\infty N_a(\gamma_a+(ak^p)^q)+N_a(c)\mu(B)/(a\Delta_c)\Big\}\\
&\qquad=2\Big\{\sum_{k=1}^\infty N_a(\gamma_a+(a^{1/p}k)^{pq})+N_a(c)\mu(B)/(a\Delta_c)\Big\}\\
&\qquad\le 2\Big\{a^{-1/p}\int_0^\infty N_a(\gamma_a+y^{pq})dy+\mu(B)N_a(c)/(a\Delta_c)\Big\}.
\end{aligned}
\tag{8.2.14}
$$

Making change of variable by $y^{pq}$ replaced by $\omega$, and noting that $N_a(\gamma_a)+\int_1^\infty\omega^s N_a(\gamma_a+\omega)d\omega=o(a)$ for all $s\ge0$ as described in (**B**4), yield that $a^{-1/p}\int_0^\infty N_a(\gamma_a+y^q)dy=o(1)$ as $a\to0$. Moreover, in view of (8.1.4), $N_a(c)/\Delta_c=O(\int_{c/2-\gamma_a}^{c-\gamma_a}\omega^s N_a(\gamma_a+\omega)d\omega)=o(a^{1/p})$ as $a\to0$ and $c\to\infty$, for $s>2/(\alpha p)$. Therefore, $\sum_{v\in B_t}N_a(\gamma_a+\lambda_v)\le\varepsilon$ for all large $c$ and small $a$.

In general, for $d > 1$, note that $\lambda_v \geq \min\{c - \gamma_a, (a\sum_{i=1}^d j^{p_i})^q\}$ if $a\sum_{i=1}^d j^{p_i} \leq \sum_{i=1}^d |(u_i - v_i)/\Delta_{c,i}|^{p_i} < a\sum_{i=1}^d (j+1)^{p_i}$. Let $U = \{\sigma \subset \{1, \cdots, d\} : \mathrm{card}(\sigma) = d-1\}$, since $N_a$ is nonincreasing, it follows that

$$\begin{aligned}
&\sum_{v\in B_t} N_a(\gamma_a + \lambda_v) \\
&\quad \leq 2\Big\{2^d \sum_{j=1}^{\infty}\Big(\sum_{\sigma\in U}\prod_{l\in\sigma}(a^{-1/p_l}K + 2j)\Big) N_a\Big(\gamma_a + \Big(a\sum_{i=1}^d j^{p_i}\Big)^q\Big) \\
&\qquad + \frac{N_a(c)\mu(B)}{\prod_{i=1}^d (a^{1/p_i}\Delta_{c,i})}\Big\} \\
&\quad \leq 2\Big\{2^d \sum_{j=1}^{\infty}\Big(\sum_{\sigma\in U}\prod_{l\in\sigma}(a^{-1/p_l}K + 2j)\Big) N_a(\gamma_a + d^q (a^{1/p_{i_0}} j)^{p_{i_0} q}) \\
&\qquad + \frac{N_a(c)\mu(B)}{\prod_{i=1}^d (a^{1/p_i}\Delta_{c,i})}\Big\} \\
&\quad \leq 2\Big\{2^d a^{-1/p_{i_0}} \int_0^{\infty}\Big(\sum_{\sigma\in U}\prod_{l\in\sigma}(a^{-1/p_l}K + 2a^{-1/p_{i_0}} y)\Big) N_a(\gamma_a + d^q y^{p_{i_0} q}) \\
&\qquad + \frac{N_a(c)\mu(B)}{\prod_{i=1}^d (a^{1/p_i}\Delta_{c,i})}\Big\} \\
&\quad \leq \varepsilon K^{d-1},
\end{aligned} \tag{8.2.15}$$

for all large $c$ and small $a$, as can be shown by arguments similar to those in the case $d = 1$. Combining (8.2.11)-(8.2.15) yields the desired conclusion. □

**Theorem 8.2.5** *Assume* (**C**) *and* (**B**1)-(**B**5). *Let $\ell_c \to \infty$ and $\ell_c = o(\Delta_{c,i})$ for all $i$ and hence $\ell_c\Delta_c = o(1)$ as $c \to \infty$. Then*

$$\mathbb{P}\Big\{\sup_{u\in I_{t,\ell_c\Delta_c}} X_c(u) > c\Big\} \sim H(t)\Psi(c)\ell_c^d, \tag{8.2.16}$$

$$\mathbb{P}\Big\{\sup_{u\in I_{t,\ell_c\Delta_c}} X_c(u) > c, \sup_{v\in B\setminus I_{t,\ell_c\Delta_c}} X_c(v) > c\Big\} = o(\Psi(c)\ell_c^d), \tag{8.2.17}$$

*as $c \to \infty$, uniformly over $t \in [D]_\delta$ and over subsets $B$ of $[D]_\delta$, where $H(t)$ is defined in (8.1.3) and is uniformly continuous and bounded below on $D$.*

**Proof** Let $\varepsilon > 0$. There exists $K^*$ such that $s_K \le \varepsilon/3$ for all $K \ge K^*$. For fixed $t \in D$ and $K \ge K^*$, define

$$\begin{aligned}
&\underline{\Lambda} = \{u \in K\Delta_c\mathbb{Z}^d : I_{u,K\Delta_c} \subset I_{t,\ell_c\Delta_c}\},\\
&\overline{\Lambda} = \{u \in K\Delta_c\mathbb{Z}^d : I_{u,K\Delta_c} \cap I_{t,\ell_c\Delta_c} \ne \emptyset\}, \qquad J_u = I_{u,K\Delta_c}.
\end{aligned} \tag{8.2.18}$$

Covering $I_{t,\ell_c\Delta_c}$ by rectangles with edges length $K\Delta_{c,i}$, $1 \le i \le d$. and letting $B$ be a subset of $[D]_\delta$ containing $I_{t,\ell_c\Delta_c}$, we have

$$\begin{aligned}
&\sum_{u\in\underline{\Lambda}}\Big(\mathbb{P}\Big\{\sup_{v\in J_u} X_c(v) > c\Big\} - \mathbb{P}\Big\{\sup_{v\in J_u} X_c(v) > c, \sup_{w\in B\setminus J_u} X_c(w) > c\Big\}\Big)\\
&\quad\le \mathbb{P}\Big\{\sup_{u\in I_{t,\ell_c\Delta_c}} X_c(u) > c\Big\} \le \sum_{u\in\overline{\Lambda}} \mathbb{P}\Big\{\sup_{v\in J_u} X_c(v) > c\Big\}.
\end{aligned} \tag{8.2.19}$$

By Theorem 8.2.2 and Lemma 8.2.4, as $c \to \infty$,

$$\begin{aligned}
&(1+o(1))\Psi(c)\sum_{u\in\underline{\Lambda}}(1 + H_K(u) - s_K K^d)\\
&\quad\le \mathbb{P}\Big\{\sup_{u\in I_{t,\ell_c\Delta_c}} X_c(u) > c\Big\} \le (1+o(1))\Psi(c)\sum_{u\in\overline{\Lambda}}(1 + H_K(u)),
\end{aligned} \tag{8.2.20}$$

uniformly over $t \in [D]_\delta$. In view of $\ell_c\Delta_c \to 0$ and the uniform equicontinuity in Lemma 8.2.3, we can choose $c^*$ large enough so that $|K^{-d}H_K(u) - K^{-d}H_K(t)| \le \varepsilon/3$ for all $c \ge c^*$, $\sqrt{\ell_c} \ge K \ge K^*$, $t \in D$ and $u \in \overline{\Lambda}$. Putting this and the bound $s_K \le \varepsilon/3$ in (8.2.20) and dividing (8.2.20) by $\Psi(c)\ell_c^d$, we obtain that for all $c \geq c^*$, $\sqrt{\ell_c} \geq K \geq K^*$ and $t \in D$,

$$\begin{aligned}(1-\varepsilon)(K^{-d}H_K(t) - 2\varepsilon/3) &\leq \mathbb{P}\left\{\sup_{u \in I_{t,\ell_c\Delta_c}} X_c(u) > c\right\} \Big/ (\Psi(c)\ell_c^d) \\ &\leq (1+\varepsilon)(K^{-d}H_K(t) + \varepsilon/3),\end{aligned} \tag{8.2.21}$$

since $\mathrm{card}(\overline{\Lambda}) \sim \mathrm{card}(\underline{\Lambda}) \sim K^{-d}\ell_c^d$. By Lemma 8.2.3, $M \triangleq \sup_{t \in [D]_\delta, K \geq 1} K^{-d}H_K(t) < \infty$. Therefore, it follows from (8.2.21) that

$$\sup_{t \in D}\left|\mathbb{P}\left\{\sup_{u \in I_{t,\ell_c\Delta_c}} X_c(u) > c\right\} \Big/ (\Psi(c)\ell_c^d) - K^{-d}H_K(t)\right| \leq \varepsilon M + 2\varepsilon/3, \tag{8.2.22}$$

for all $c \geq c^*$ and $\sqrt{\ell_c} \geq K \geq K^*$. Letting $c \to \infty$ in (8.2.22) yields

$$\sup_{t \in D}|K^{-d}H_K(t) - \tilde{K}^{-d}H_{\tilde{K}}(t)| \leq 2\varepsilon M + 4\varepsilon/3,$$

if $K, \tilde{K} \geq K^*$, establishing that $\{K^{-d}H_K\}$ is uniformly Cauchy. Hence $K^{-d}H_K(t)$ converges uniformly in $t \in D$ to $H(t)$, which is also bounded by $M$. We can therefore proceed as in the second paragraph of the proof of Theorem 8.2.2 to show that $H(t)$ is uniformly continuous in $t \in D$. Moreover, taking $K$ large enough such that $\sup_{t \in D}|K^{-d}H_K(t) - H(t)| \leq \varepsilon/3$, it follows from (8.2.22) that

$$\sup_{t \in D}\left|\mathbb{P}\left\{\sup_{u \in I_{t,\ell_c\Delta_c}} X_c(u) > c\right\} \Big/ \left(\Psi(c)\ell_c^d\right) - H(t)\right| \leq \varepsilon(M+1)$$

for all $c \geq c^*$, proving (8.2.16).

We next show that $\inf_{t \in D} H(t) > 0$. For the function $f$ in (**B**5), we can choose $a > 0$ large enough so that $\sum_{k \neq 0} f\left(a\sum_{i=1}^d |k_i|^{p_i}\right) \leq 1/2$. Let $K$ large enough and $m_i = \lfloor K/a^{1/p_i}\rfloor$,

$1 \leq i \leq d$, and $A_t = A_t(K, a, c)$ as defined before so that $\text{card}(A_t) = \prod_{i=1}^{d} m_i$. Then by (**B**1) and (**B**5), as $c \to \infty$,

$$\begin{aligned}
&\mathbb{P}\Big\{ \sup_{u \in A_t} X_c(u) > c \Big\} \\
&\geq \sum_{u \in A_t} \Big( \mathbb{P}(X_c(u) > c) - \sum_{v \in A_t, v \neq u} \mathbb{P}(X_c(u) > c, X_c(v) > c) \Big) \\
&\geq \sum_{u \in A_t} (1 + o(1)) \Psi(c)/2 \\
&= (1 + o(1)) \Big( \prod_{i=1}^{d} m_i \Big) \Psi(c)/2,
\end{aligned} \tag{8.2.23}$$

uniformly in $t \in D$ and $m_i \geq 2$. Combining (8.2.23) with Theorem 8.2.2 yields

$$\begin{aligned}
1 + H_K(t) &= \lim_{c \to \infty} \mathbb{P}\Big\{ \sup_{u \in I_{t, K\Delta_c}} X_c(u) > c \Big\} \Big/ \Psi(c) \\
&\geq \limsup_{c \to \infty} \mathbb{P}\Big\{ \sup_{u \in A_t} X_c(u) > c \Big\} \Big/ \Psi(c) \geq \Big( \prod_{i=1}^{d} m_i \Big)/2
\end{aligned}$$

uniformly in $t \in D$ and $m_i \geq 2$. Since $\lim_{K \to \infty} K^{-d} H_K(t) = H(t)$, it then follows that $H(t) \geq (\prod_{i=1}^{d} a^{-1/p_i})/2$ for all $t \in D$.

Finally, to prove (8.2.17), apply Lemma 8.2.4 to obtain that for all $t \in D$ and large $c$,

$$\begin{aligned}
&\mathbb{P}\Big\{ \sup_{u \in I_{t, \ell_c \Delta_c}} X_c(u) > c, \sup_{v \in B \setminus I_{t, \ell_c \Delta_c}} X_c(v) > c \Big\} \\
&\leq \sum_{u \in \overline{\Lambda}} \mathbb{P}\Big\{ \sup_{v \in J_u} X_c(v) > c, \sup_{v \in B \setminus J_u} X_c(v) > c \Big\} \\
&\leq \text{card}(\overline{\Lambda}) s_K K^d \Psi(c).
\end{aligned}$$

Since $s_K \to 0$ as $K \to \infty$ and $\text{card}(\overline{\Lambda}) \sim K^{-d} \ell_c^d$ as $\ell_c / K \to \infty$, (8.2.17) follows. $\square$

**Theorem 8.2.6** *Assume* (**C**) *and* (**B**1)-(**B**5). *Then as* $c \to \infty$,

$$\mathbb{P}\left\{\sup_{t\in D} X_c(t) > c\right\} \sim \Psi(c)\left(\prod_{i=1}^{d} \Delta_{c,i}^{-1}\right)\int_D H(t)dt.$$

**Proof** A basic idea of the proof is to cover the set $D$ by rectangles with edges length $\ell_c\Delta_{c,i}$, $1 \le i \le d$, and also $\ell_c \to \infty$ and $\ell_c\Delta_{c,i} \to 0$ for all $i$ and hence $\ell_c\Delta_c \to 0$ as $c \to \infty$. Define $\underline{\Lambda}, \overline{\Lambda}$ and $J_u$ as in (8.2.18) but with $K\Delta_c\mathbb{Z}^d$ replaced by $\ell_c\Delta_c\mathbb{Z}^d$, $I_{u,K\Delta_c}$ by $I_{u,\ell_c\Delta_c}$, and $I_{t,\ell_c\Delta_c}$ by $D$. Then (8.2.19) still holds with these new definitions of $\underline{\Lambda}, \overline{\Lambda}$ and $J_u$ and also with $B$ replaced by $[D]_\delta$. Labeling it as (8.2.19′), the upper and lower bounds in (8.2.19′) are both asymptotically equivalent to $(\ell_c^d \prod_{i=1}^d \Delta_{c,i})^{-1}(\ell_c^d)\Psi(c)\int_D H(t)dt = \Psi(c)(\prod_{i=1}^d \Delta_{c,i}^{-1})\int_D H(t)dt$, since $\ell_c\Delta_c \to 0$ and $H(t)$ is continuous. This finishes the proof. $\square$

## 8.3 Proof of Theorem 8.1.1

In view of Theorem 8.2.6, we only need to show that (**B**1)-(**B**5) holds for such Gaussian fields. (**B**1) is obvious. To show (**B**2), it follows from (8.1.5) that as $c \to \infty$,

$$\begin{aligned}
&\mathbb{E}\{c[X(t+u\Delta_c) - X(t)]|X(t) = c - y/c\} \\
&= -c[1-\rho(t, t+u\Delta_c)](c - y/c) \\
&\to -\left(\sum_{i=1}^{d} |u_i|^{p_i}\right)^{\alpha} r_t\left(\frac{|u_1|^{p_1}}{\sum_{i=1}^d |u_i|^{p_i}}, \cdots, \frac{|u_d|^{p_d}}{\sum_{i=1}^d |u_i|^{p_i}}\right),
\end{aligned} \qquad (8.3.1)$$

$$
\begin{aligned}
&\operatorname{Cov}\{c[X(t+u\Delta_c)-X(t)], c[X(t+v\Delta_c)-X(t)]|X(t)=c-y/c\} \\
&\quad = c^2[\rho(t+u\Delta_c, t+v\Delta_c)-\rho(t,t+u\Delta_c)\rho(t,t+v\Delta_c)] \\
&\quad \to \Big(\sum_{i=1}^d u_i^{p_i}\Big)^{\alpha} r_t\Big(\frac{u_1^{p_1}}{\sum_{i=1}^d u_i^{p_i}}, \cdots, \frac{u_d^{p_d}}{\sum_{i=1}^d u_i^{p_i}}\Big) \\
&\qquad + \Big(\sum_{i=1}^d v_i^{p_i}\Big)^{\alpha} r_t\Big(\frac{v_1^{p_1}}{\sum_{i=1}^d v_i^{p_i}}, \cdots, \frac{v_d^{p_d}}{\sum_{i=1}^d v_i^{p_i}}\Big) \\
&\qquad - \Big(\sum_{i=1}^d |u_i-v_i|^{p_i}\Big)^{\alpha} r_t\Big(\frac{|u_1-v_1|^{p_1}}{\sum_{i=1}^d |u_i-v_i|^{p_i}}, \cdots, \frac{|u_d-v_d|^{p_d}}{\sum_{i=1}^d |u_i-v_i|^{p_i}}\Big).
\end{aligned}
\tag{8.3.2}
$$

Since $\{c[X(t+\mathbf{a}k\Delta_c)-X(t)] : 0 \le k_i < m_i\}$ is multivariate normal, (**B**2) then follows.

Let $\gamma > 0$, $\phi$ be the density of standard normal. Since $\Psi(c-z/c) \sim e^z\Psi(c)$ for all $z \ge 0$ and there exist constants $B > 0$, $B' > 0$ such that $\mathbb{P}\{W_t(u) > z-\gamma\} \le B\exp(-B'z^2)$, it follows from (8.3.1) and (8.3.2) that as $c \to \infty$,

$$
\begin{aligned}
&\mathbb{P}\{X(t+u\Delta_c) > c-\gamma/c, X(t) \le c-y/c\} \\
&\quad = \mathbb{P}\{X(t) \le c-y/c\}\int_{-\infty}^{c-y/c} \mathbb{P}\{X(t+u\Delta_c) > c-\gamma/c|X(t)=x\}\phi(x)dx \\
&\quad = \mathbb{P}\{X(t) \le c-y/c\}\int_{y}^{\infty} \mathbb{P}\{X(t+u\Delta_c) > c-\gamma/c|X(t)=c-z/c\}c^{-1}\phi(c-z/c)dz \\
&\quad \le (1+o(1))\Psi(c)\int_y^{\infty} e^z\mathbb{P}\{W_t(u) > z-\gamma\}dz \\
&\quad \le h(y)\Psi(c),
\end{aligned}
$$

where $h(y) \to 0$ as $y \to \infty$, establishing (**B**3).

To show ($\mathbf{B}5$) holds, note that

$$
\begin{aligned}
&\mathbb{P}\{X(t) > c, X(t+u\Delta_c) > c\} \\
&\quad \le \mathbb{P}\{X(t) + X(t+u\Delta_c) > 2c\} \\
&\quad \sim \Psi\bigg(\bigg[\frac{2c^2}{1+\rho(t,t+u\Delta_c)}\bigg]^{1/2}\bigg) \\
&\quad = \Psi(c)\bigg(\frac{1+\rho(t,t+u\Delta_c)}{2}\bigg)^{1/2}\exp\bigg[-\frac{c^2}{1+\rho(t,t+u\Delta_c)}+\frac{c^2}{2}\bigg] \\
&\quad \le \Psi(c)\exp\bigg[-\frac{c^2}{2}\bigg(\frac{1-\rho(t,t+u\Delta_c)}{2}\bigg)\bigg].
\end{aligned}
$$

By (8.1.5), there exists $\eta > 0$ such that $c^2[1-\rho(t,t+u\Delta_c)] \ge \eta(\sum_{i=1}^d |u_i|^{p_i})^\alpha L(\sum_{i=1}^d |u_i|^{p_i})$ for all $t, t+u\Delta_c \in [D]_\delta$. Hence ($\mathbf{B}5$) holds with $f(y) = B_\lambda \exp(-y^\lambda)$ with $0 < \lambda < \alpha$, for some $B_\lambda > 0$.

Finally we turn to ($\mathbf{B}4$). Let $a > 0$, $\mathbf{a} = \{a^{1/p_1}, \cdots, a^{1/p_d}\}$, $0 < \zeta < \alpha$, $1 < \xi < 2^{p_{i_0}\zeta/2}$, $\kappa = \sum_{r=0}^\infty \xi^{-r}$ and $w_r = \xi^{-r}/(2\kappa)$. Define

$$
\begin{aligned}
B_r &= \{t + 2^{-r}\mathbf{a}k\Delta_c : 0 \le k_i < 2^r, k \in \mathbb{Z}^d\}, \\
F &= \Big\{\sup_{u \in I_{t,\mathbf{a}\Delta_c}} X(u) > c\Big\}, \\
E_{-1} &= \{X(t) \le c - \gamma/c\}, \\
E_r &= \Big\{\sup_{v \in B_r} X(v) \le c - \gamma(1 - w_0 - \cdots - w_r)/c)\Big\} \qquad \text{for } r \ge 0,
\end{aligned}
$$

recalling that $\sum_{r=0}^\infty w_r = 1/2$. Note that $B_r \subset B_{r+1} \subset I_{t,\mathbf{a}\Delta_c}$ and that by the continuity of

$X$, $\mathbb{P}(F \cap E_{-1}) \le \sum_{r=0}^{\infty} \mathbb{P}(E_{r-1} \cap E_r^c)$. Moreover,

$$
\begin{aligned}
&\mathbb{P}(E_{r-1} \cap E_r^c) \\
&\quad \le 2^{r+d} \sup_{v \in I_{t,\mathbf{a}\Delta_c}, \varepsilon \in \{0,1\}^d \setminus \{\mathbf{0}\}} \mathbb{P}\{X(v) \le c - \gamma(1 - w_0 - \cdots - w_{r-1})/c), \\
&\qquad\qquad X(v + \varepsilon 2^{-r}\mathbf{a}\Delta_c) > c - \gamma(1 - w_0 - \cdots - w_r)/c\}.
\end{aligned} \tag{8.3.3}
$$

Given $X(v) = c - y/c$, the conditional distribution of $c[X(v + \varepsilon 2^{-r}\mathbf{a}\Delta_c) - X(v)]$ is normal with mean $-c(c - y/c)[1 - \rho(v, v + \varepsilon 2^{-r}\mathbf{a}\Delta_c)] < 0$ and variance $c^2[1 - \rho^2(v, v + \varepsilon 2^{-r}\mathbf{a}\Delta_c)]$, which is bounded by $B(a\sum_{i=1}^{d} 2^{-rp_i})^{\zeta}$ for some $B > 0$, in view of (8.1.5). Hence

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{\varepsilon \in \{0,1\}^d} c[X(v + \varepsilon 2^{-r}\mathbf{a}\Delta_c) - X(v)] > w_r y | X(v) = c - y/c\Big\} \\
&\quad \le 2^d \exp\Big[-C(w_r y)^2 \Big/ \Big(a\sum_{i=1}^{d} 2^{-rp_i}\Big)^{\zeta}\Big] \\
&\quad \le 2^d \exp\Big[-C(w_r y)^2 \Big/ \Big(da2^{-rp_{i_0}}\Big)^{\zeta}\Big]
\end{aligned} \tag{8.3.4}
$$

for some $C > 0$.

Let $\eta \triangleq 2^{p_{i_0}\zeta}/\xi^2 > 1$. Combining (8.3.3) and (8.3.4) with fact $\mathbb{P}\{X(v) \in c - y/c\} \sim \Psi(c)e^y dy$ then yields

$$
\begin{aligned}
&\mathbb{P}(F \cap E_{-1}) \\
&\quad \le (1 + o(1))\Psi(c) \sum_{r=0}^{\infty} 2^{-r} \int_{\gamma/2}^{\infty} \exp[y - C\eta^r y^2/(4a^{\zeta}d^{\zeta}\kappa^2) + C'r]dy
\end{aligned} \tag{8.3.5}
$$

for some $C' > 0$. Let $\gamma_a = a^{\zeta/3}$. Then for large $c$ and $\gamma_a \le \gamma \le c$, (8.3.5) is bounded above by $\Psi(c)N_a(\gamma)$, where

$$N_a(\gamma) = 2\sum_{r=0}^{\infty} 2^{-r} \int_{\gamma/2}^{\infty} \exp[y - C\eta^r y^2/(4a^\zeta d^\zeta \kappa^2) + C'r]dy$$

satisfies $N_a(\gamma_a) + \int_1^\infty \omega^s N_a(\gamma_a + \omega)d\omega = o(a^l)$ for all $s > 0$ and $l > 0$.

## 8.4 Example: Standardized Fractional Brownian Sheet

For a given vector $H = (H_1, \cdots, H_d) \in (0,1)^d$, a $d$-fractional Brownian sheet $B^H = \{B^H(t) : t \in \mathbb{R}^d\}$ with Hurst index $H$ is a real-valued, centered Gaussian field with covariance function given by

$$\mathbb{E}(B^H(t)B^H(s)) = \prod_{i=1}^{d} \frac{1}{2}\Big(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}\Big), \quad t, s \in \mathbb{R}^d.$$

Let $D \subset \mathbb{R}^d$ such that $\bar{D}$ having no intersection with any coordinate, define the standardized field

$$X(t) = \frac{B^H(t)}{\sqrt{\mathrm{Var}(B^H(t))}}, \quad t \in D.$$

It follows that

$$\mathbb{E}(X(t)X(s)) = \prod_{i=1}^{d} \frac{|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}}{2|t_i s_i|^{H_i}},$$

and hence

$$\mathbb{E}(X(t)X(t+u)) = 1 - (1 + o(1))\frac{1}{2}\bigg(\sum_{i=1}^{d} \left|\frac{u_i}{t_i}\right|^{2H_i}\bigg), \tag{8.4.1}$$

as $\|u\| \to 0$, uniformly over $t \in [D]_\delta$. Hence (8.1.5) is satisfied with $p_i = 2H_i$ for $1 \le i \le d$, $\alpha = 1, L(\sum_{i=1}^d |u_i|^{2H_i}) \equiv 1/2$ and

$$r_t\Big(\frac{|u_1|^{2H_1}}{\sum_{i=1}^d |u_i|^{2H_i}}, \cdots, \frac{|u_d|^{2H_d}}{\sum_{i=1}^d |u_i|^{2H_i}}\Big) = \Big(\sum_{i=1}^d |u_i|^{2H_i}\Big)^{-1} \sum_{i=1}^d \left|\frac{u_i}{t_i}\right|^{2H_i}.$$

In other word,

$$r_t(v) = \sum_{i=1}^d \frac{v_i}{|t_i|^{2H_i}}, \quad t \in D, v \in \mathcal{S}.$$

Applying Theorem 8.1.1, we obtain

$$\mathbb{P}\Big\{\sup_{t\in D} X(t) > c\Big\} \sim \Psi(c)\Big(\frac{c^2}{2}\Big)^{\sum_{i=1}^d \frac{1}{2H_i}} \int_D H(t)dt, \tag{8.4.2}$$

where

$$H(t) = \lim_{K\to\infty} K^{-d} \int_0^\infty e^y \mathbb{P}\Big\{\sup_{0\le u_i \le K, \forall i} W_t(u) > y\Big\}dy,$$

and $\{W_t(u) : u \in [0,\infty)^d\}$ is a Gaussian random field such that $W_t(0) = 0$ and

$$\mathbb{E}(W_t(u)) = -\sum_{i=1}^d \left|\frac{u_i}{t_i}\right|^{2H_i},$$

$$\mathrm{Cov}(W_t(u), W_t(v)) = \sum_{i=1}^d \frac{u_i^{2H_i} + u_i^{2H_i} - |u_i - v_i|^{2H_i}}{|t_i|^{2H_i}}.$$

By similar discussions in Lemma 7.2.3, we obtain further that

$$\mathbb{P}\Big\{\sup_{t\in D} X(t) > c\Big\} \sim \Psi(c)\Big(\frac{c^2}{2}\Big)^{\sum_{i=1}^d \frac{1}{2H_i}} \widetilde{H} \int_D \Big(\prod_{i=1}^d \frac{1}{|t_i|^{2H_i}}\Big)dt, \tag{8.4.3}$$

where $\widetilde{H}$ is the Pickands' constant defined by

$$\widetilde{H} = \lim_{K\to\infty} K^{-d} \int_0^\infty e^y \mathbb{P}\left\{ \sup_{0\le u_i\le K, \forall i} \widetilde{W}(u) > y \right\} dy,$$

and $\{\widetilde{W}(u) : u \in [0,\infty)^d\}$ is a Gaussian random field such that $\widetilde{W}(0) = 0$ and

$$\mathbb{E}(\widetilde{W}(u)) = -\sum_{i=1}^d u_i^{2H_i},$$
$$\mathrm{Cov}(\widetilde{W}(u), \widetilde{W}(v)) = \sum_{i=1}^d \left( u_i^{2H_i} + u_i^{2H_i} - |u_i - v_i|^{2H_i} \right).$$

Especially, when $H = (1/2, \cdots, 1/2)$, then $\widetilde{H} = 1$ and thus

$$\mathbb{P}\left\{ \sup_{t\in D} X(t) > c \right\} \sim \Psi(c) 2^{-d} c^{2d} \int_D \left( \prod_{i=1}^d |t_i| \right)^{-1} dt. \qquad (8.4.4)$$

This result is very similar to Example 2.2 in Chan and Lai (2006).

It is worth mentioning here that we may also apply Piterbarg's result to get the approximation. Due to the covariance structure (8.4.1), applying Theorem 7.1 on Page 108 in Piterbarg (1996a), we obtain

$$\mathbb{P}\left\{ \sup_{t\in D} X(t) > c \right\} \sim \Psi(c) c^{\sum_{i=1}^d \frac{1}{H_i}} \widetilde{H} \int_D \left( \prod_{i=1}^d 2^{\frac{1}{2H_i}} |t_i|^{2H_i} \right)^{-1} dt,$$

which is the same as (8.4.3).

**Remark 8.4.1** In certain sense, Theorem 8.1.1 generalize Theorem 7.1 on Page 108 in

Piterbarg (1996a), since the latter one is the case that

$$r_t(v) = C_t v, \quad t \in D, v \in \mathcal{S},$$

where $C_t$ is some nondegenerate $d \times d$ matrix.

## 8.5 Example: Standardized Random String Processes

We study an anisotropic Gaussian field which is the solution to a stochastic partial differential equation in Mueller and Tribe (2002). We write the original process $\{U_t(x) : t \geq 0, x \in \mathbb{R}\}$ in Mueller and Tribe (2002) as $\{U(t) : t_1 \geq 0, t_2 \in \mathbb{R}\}$. Then it is a centered Gaussian field with stationary increments and $U(0) = 0$. It has the following covariance structure: for $t_2, s_2 \in \mathbb{R}, t_1 = s_1 \geq 0$,

$$\mathbb{E}\{(U(t) - U(s))^2)\} = |t_2 - s_2|,$$

and for $t_2, s_2 \in \mathbb{R}, t_1 > s_1 \geq 0$,

$$\mathbb{E}\{(U(t) - U(s))^2)\} = \sqrt{t_1 - s_1} F\left(\frac{|t_2 - s_2|}{\sqrt{t_1 - s_1}}\right),$$

where

$$\begin{aligned} F(a) &= (2\pi)^{-1/2} + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} G(a-z)G(a-z')(|z| + |z'| - |z-z'|)dzdz' \\ &= -(2\pi)^{-1/2} + \int_{\mathbb{R}} G(a-z)|z|dz \\ &= -(2\pi)^{-1/2} + 4G(a) + 2a \int_0^a G(z)dz, \end{aligned}$$

and $G(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$. Now we define the standardized field

$$X(t) = \frac{U(t)}{\sqrt{\mathrm{Var}(U(t))}}, \quad t \in \mathbb{R}_+ \times \mathbb{R}. \tag{8.5.1}$$

Note that

$$\begin{aligned} \mathrm{Var}(U(t)) &= \sqrt{t_1} F\left(\frac{|t_2|}{\sqrt{t_1}}\right) \\ &= \sqrt{t_1}\left(-\frac{1}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} e^{-\frac{t_2^2}{4t_1}}\right) + |t_2| \int_0^{\frac{|t_2|}{\sqrt{t_1}}} \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}} dz, \end{aligned}$$

and

$$\mathbb{E}(X(t)X(s)) = \frac{\mathrm{Var}(U(t)) + \mathrm{Var}(U(s)) - \mathbb{E}\{(U(t) - U(s))^2)\}}{2\sqrt{\mathrm{Var}(U(t))\mathrm{Var}(U(s))}}.$$

Thus we obtain that as $\|u\| \to 0$,

$$\mathbb{E}(X(t)X(t+u)) = 1 - (1 + o(1))\left[r_{t,1}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right)\sqrt{|u_1|} + r_{t,2}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right)|u_2|\right],$$

where

$$\begin{aligned} r_{t,1}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right) &= \frac{1}{2\mathrm{Var}(U(t))}\left(-\frac{1}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} e^{-\frac{u_2^2}{4|u_1|}}\right), \\ r_{t,2}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right) &= \frac{1}{2\mathrm{Var}(U(t))} \int_0^{\frac{|u_2|}{\sqrt{|u_1|}}} \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}} dz. \end{aligned}$$

Hence (8.1.5) is satisfied with $p_1 = 1/2$, $p_2 = 1$, $\alpha = 1$, $L(\sqrt{|u_1|} + |u_2|) \equiv 1$ and

$$\begin{aligned} & r_t\left(\frac{\sqrt{|u_1|}}{\sqrt{|u_1|} + |u_2|}, \frac{|u_2|}{\sqrt{|u_1|} + |u_2|}\right) \\ &= \frac{1}{\sqrt{|u_1|} + |u_2|}\left[r_{t,1}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right)\sqrt{|u_1|} + r_{t,2}\left(\frac{|u_2|}{\sqrt{|u_1|}}\right)|u_2|\right] \end{aligned}$$

In other words,

$$r_t(v) = r_{t,1}\left(\frac{v_2}{v_1}\right)v_1 + r_{t,2}\left(\frac{v_2}{v_1}\right)v_2, \quad t \in D, v \in \mathcal{S}.$$

Hence we can apply Theorem 8.1.1 to get the approximation to the excursion probability.

However, Piterbarg's result is not applicable for such case.

# Chapter 9

# Vector-valued Smooth Gaussian Random Fields

## 9.1 Joint Excursion Probability

Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field, where $T$ and $S$ are rectangles in $\mathbb{R}^N$. Let

$$\rho(t, s) = \mathbb{E}\{X(t)Y(s)\}, \quad \rho(T, S) = \sup_{t \in T, s \in S} \mathbb{E}\{X(t)Y(s)\}.$$

We will make use of the following conditions.

(**C**1). $X, Y \in C^2(\mathbb{R}^N)$ almost surely and their second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants $L, \eta > 0$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(t'))^2 \le L\|t - t'\|^{2\eta}, \quad \forall t, t' \in T, \ i, j = 1, \ldots, N,$$
$$\mathbb{E}(Y_{ij}(s) - Y_{ij}(s'))^2 \le L\|s - s'\|^{2\eta}, \quad \forall s, s' \in S, \ i, j = 1, \ldots, N.$$

(**C**2). For every $(t, t', s) \in T^2 \times S$ with $t \ne t'$, the Gaussian vector

$$(X(t), \nabla X(t), X_{ij}(t), X(t'), \nabla X(t'), X_{ij}(t'), Y(s), \nabla Y(s), Y_{ij}(s), 1 \le i \le j \le N)$$

is non-degenerate; and for every $(s, s', t) \in S^2 \times T$ with $s \neq s'$, the Gaussian vector

$$(Y(s), \nabla Y(s), Y_{ij}(s), Y(s'), \nabla Y(s'), Y_{ij}(s'), X(t), \nabla X(t), X_{ij}(t), 1 \leq i \leq j \leq N)$$

is non-degenerate.

(**C**3). For all $(t, s) \in T \times S$ such that $\rho(t, s) = \rho(T, S)$,

$$(\mathbb{E}\{X_{ij}(t)Y(s)\})_{i,j\in\zeta(t,s)}, \quad (\mathbb{E}\{X(t)Y_{i'j'}(s)\})_{i',j'\in\zeta'(t,s)}$$

are both negative semidefinite, where

$$\begin{aligned}\zeta(t, s) &= \{n : \mathbb{E}\{X_n(t)Y(s)\} = 0, 1 \leq n \leq N\},\\ \zeta'(t, s) &= \{n : \mathbb{E}\{X(t)Y_n(s)\} = 0, 1 \leq n \leq N\}.\end{aligned}$$

**Remark 9.1.1** Note that

$$\begin{aligned}\frac{\partial \rho}{\partial t_i}(t, s) &= \mathbb{E}\{X_i(t)Y(s)\}, \quad \frac{\partial^2 \rho}{\partial t_i \partial t_j}(t, s) = \mathbb{E}\{X_{ij}(t)Y(s)\},\\ \frac{\partial \rho}{\partial s_i}(t, s) &= \mathbb{E}\{X(t)Y_i(s)\}, \quad \frac{\partial^2 \rho}{\partial s_i \partial s_j}(t, s) = \mathbb{E}\{X(t)Y_{ij}(s)\}.\end{aligned}$$

Therefore, similarly to Remark 3.1.2, in order to verify (**C**3), it suffices to consider those points $(t, s) \in T \times S$ such that $t \in \partial_k T$ with $0 \leq k \leq N-2$ or $s \in \partial_{k'} S$ with $0 \leq k' \leq N-2$.

We decompose $T$ and $S$ into several faces as

$$T = \bigcup_{k=0}^{N} \partial_k T = \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} J, \quad S = \bigcup_{l=0}^{N} \partial_l S = \bigcup_{l=0}^{N} \bigcup_{L\in\partial_l S} L.$$

For each $J \in \partial_k T$ and $L \in \partial_l S$, define the number of extended outward maxima above level $u$ as

$$M_u^E(X, J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = k,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},$$
$$M_u^E(Y, L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \mathrm{index}(\nabla^2 Y_{|L}(s)) = l,$$
$$\varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \notin \sigma(L)\};$$

and define the number of maxima above level $u$ as

$$M_u(X, J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = k\},$$
$$M_u(Y, L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \mathrm{index}(\nabla^2 Y_{|L}(s)) = l\}.$$

Similarly to Lemma 2.3.1, we have the following result.

**Lemma 9.1.2** *Under* $(\mathbf{C}1)$ *and* $(\mathbf{C}2)$*, the following relation holds for each* $u > 0$*:*

$$\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} = \bigcup_{k,l=0}^{N} \bigcup_{J\in\partial_k T, L\in\partial_l S} \{M_u^E(X, J) \geq 1, M_u^E(Y, L) \geq 1\} \text{ a.s.}$$

It follows from Lemma 9.1.2 that

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} &\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{P}\{M_u^E(X, J) \geq 1, M_u^E(Y, L) \geq 1\} \\ &\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X, J) M_u^E(Y, L)\}.\end{aligned} \tag{9.1.1}$$

On the other hand, by the Bonferroni inequality, Lemma 9.1.2 implies

$$
\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \geq & \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \\
& - \sum_{k,l=0}^{N} \sum_{\substack{J,J'\in\partial_k T, J\neq J' \\ L\in\partial_l S}} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(J',X) \geq 1, M_u^E(Y,L) \geq 1\} \\
& - \sum_{k,l=0}^{N} \sum_{\substack{J\in\partial_k T \\ L,L'\in\partial_l S, L\neq L'}} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1, M_u^E(L',Y) \geq 1\} \\
& - \sum_{k,l=0}^{N} \sum_{\substack{J,J'\in\partial_k T, J\neq J' \\ L,L'\in\partial_l S, L\neq L'}} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(J',X) \geq 1, M_u^E(Y,L) \geq 1, M_u^E(L',Y) \geq 1\}.
\end{aligned}
$$

Let $p_{ij} = \mathbb{P}\{M_u^E(X,J) = i, M_u^E(Y,L) = j\}$, then $\mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} = \sum_{i,j=1}^{\infty} p_{ij}$ and $\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} = \sum_{i,j=1}^{\infty} ijp_{ij}$, and hence

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} - \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \\
&\quad = \sum_{i,j=1}^{\infty} (ij-1)p_{ij} \leq \sum_{i,j=1}^{\infty} [i(i-1)j + j(j-1)i]p_{ij} \\
&\quad = \mathbb{E}\{M_u^E(X,J)[M_u^E(X,J)-1]M_u^E(Y,L)\} + \mathbb{E}\{M_u^E(Y,L)[M_u^E(Y,L)-1]M_u^E(X,J)\}.
\end{aligned}
$$

We therefore obtain the following lower bound for the excursion probability,

$$
\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \geq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \Big\{&\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} \\
- \mathbb{E}\{M_u^E(X,J)[M_u^E(X,J)-1]M_u^E(Y,L)\} - \mathbb{E}\{M_u^E(Y,L)[M_u^E(Y,L)-1]M_u^E(X,J)\}\Big\}& \\
- \sum_{k,l=0}^{N} \sum_{\substack{J,J'\in\partial_k T, J\neq J' \\ L\in\partial_l S}} \mathbb{E}\{M_u^E(X,J)M_u^E(X,J')M_u^E(Y,L)\}& \\
- 2\sum_{k,l=0}^{N} \sum_{\substack{J\in\partial_k T \\ L,L'\in\partial_l S, L\neq L'}} \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)M_u^E(Y,L')\}.&
\end{aligned}
\tag{9.1.2}
$$

We will show that the upper bound in (9.1.1) makes the major contribution and the other terms in the lower bound in (9.1.2) are super-exponentially small.

**Lemma 9.1.3** *Let $D_i$ be compact sets in $\mathbb{R}^N$, $i = 1, 2, 3$. Let*

$$\{(\xi_1(x_1), \xi_2(x_2), \xi_3(x_3)) : (x_1, x_2, x_3) \in D_1 \times D_2 \times D_3\}$$

*be an $\mathbb{R}^3$-valued, $C^2$, centered, unit-variance, non-degenerate Gaussian random field and let*

$$
\begin{aligned}
\rho_{12}(x_1,x_2) &= \mathbb{E}\xi_1(x_1)\xi_2(x_2), \quad \rho_{12} = \sup_{x_1\in D_1, x_2\in D_2} \rho_{12}(x_1,x_2), \\
\rho_{13}(x_1,x_3) &= \mathbb{E}\xi_1(x_1)\xi_3(x_3), \quad \rho_{13} = \sup_{x_1\in D_1, x_3\in D_3} \rho_{13}(x_1,x_3), \\
\rho_{23}(x_2,x_3) &= \mathbb{E}\xi_2(x_2)\xi_3(x_3), \quad \rho_{23} = \sup_{x_2\in D_2, x_3\in D_3} \rho_{23}(x_2,x_3).
\end{aligned}
$$

*If $\rho_{12} \geq \rho_{13} \vee \rho_{23}$, then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$
\begin{aligned}
&\sup_{x_1 \in D_1, x_2 \in D_2, x_3 \in D_3} \mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)\xi_3(x_3)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u, \xi_3(x_3)\geq u\}}\} \\
&\quad = o\bigg( \exp\bigg\{ -\alpha u^2 - \frac{u^2}{1+\rho_{12}} \bigg\}\bigg),
\end{aligned} \tag{9.1.3}
$$

*where $m$ is a fixed positive number.*

**Proof** Let $\bar{\xi}(x_1, x_2, x_3) = [\xi_1(x_1) + \xi_2(x_2) + \xi_3(x_3)]/3$, then there exists a positive number $m'$ such that for all $(x_1, x_2, x_3) \in D_1 \times D_2 \times D_3$ and $u$ large enough,

$$
\begin{aligned}
&\mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)\xi_3(x_3)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u, \xi_3(x_3)\geq u\}}\} \\
&\quad \leq \mathbb{E}\{(\xi_1(x_1) + \xi_2(x_2) + \xi_3(x_3))^{m'} \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u, \xi_3(x_3)\geq u\}}\} \\
&\quad \leq \mathbb{E}\{(\xi_1(x_1) + \xi_2(x_2) + \xi_3(x_3))^{m'} \mathbb{1}_{\{[\xi_1(x_1)+\xi_2(x_2)+\xi_3(x_3)]/3\geq u\}}\} \\
&\quad = \mathbb{E}\{(3\bar{\xi}(x_1, x_2, x_3))^{m'} \mathbb{1}_{\{\bar{\xi}(x_1,x_2,x_3)\geq u\}}\}.
\end{aligned} \tag{9.1.4}
$$

It follows from the assumption $\rho_{12} \geq \rho_{13} \vee \rho_{23}$ that

$$
\begin{aligned}
&\sup_{x_1 \in D_1, x_2 \in D_2, x_3 \in D_3} \mathrm{Var}(\bar{\xi}(x_1, x_2, x_3)) \\
&= \sup_{x_1 \in D_1, x_2 \in D_2, x_3 \in D_3} \frac{3 + 2[\rho_{12}(x_1, x_2)) + \rho_{13}(x_1, x_3) + \rho_{23}(x_2, x_3)]}{9} \\
&\leq \frac{3 + 6\rho_{12}}{9} = \frac{1 + 2\rho_{12}}{3},
\end{aligned}
$$

and hence $\rho_{12} \in (-1/2, 1)$. Combining this with (9.1.4), we see that for any $\varepsilon > 0$, as $u \to \infty$, the first line in (9.1.3) is $o(\exp\{\varepsilon u^2 - \frac{3u^2}{2(1+2\rho_{12})}\})$. Now the result follows by taking

$\alpha$ to be a positive number less than

$$\frac{3}{2(1+2\rho_{12})} - \frac{1}{1+\rho_{12}} = \frac{1-\rho_{12}}{2(1+\rho_{12})(1+2\rho_{12})}.$$

$\square$

**Lemma 9.1.4** *Let $D_1, \ldots, D_n$ be compact sets in $\mathbb{R}^N$, where $n \geq 3$, and let*

$$(\xi_1(x_1), \xi_2(x_2), \xi_3(x_3), \ldots, \xi_n(x_n) : x_i \in D_i, i = 1 \ldots, n)$$

*be an $\mathbb{R}^n$-valued, $C^2$, centered, unit-variance, non-degenerate Gaussian random vector. Let $m$ be a fixed positive number and*

$$\rho_{12}(x_1, x_2) = \mathbb{E}\{\xi_1(x_1)\xi_2(x_2)\}, \quad \rho_{12} = \sup_{x_1 \in D_1, x_2 \in D_2} \rho_{12}(x_1, x_2).$$

*Then*

$$\begin{aligned} &\lim_{u\to\infty} u^{-2} \log \mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\} \\ &\leq -\frac{1}{1+\rho_{12}(x_1, x_2)}. \end{aligned}$$

*If*

$$\begin{aligned} &\{(x_1, \ldots, x_n) \in D_1 \times \cdots \times D_n : \\ &\rho_{12}(x_1, x_2) = \rho_{12}, \mathbb{E}\{(\xi_1(x_1) + \xi_2(x_2))\xi_i(x_i)\} = 0, \forall i = 3, \ldots, n\} = \emptyset, \end{aligned} \tag{9.1.5}$$

*then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\sup_{x_i \in D_i, i=1,\ldots,n} \mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\}$$
$$= o\bigg( \exp\bigg\{ -\alpha u^2 - \frac{u^2}{1+\rho_{12}} \bigg\} \bigg).$$

**Proof** Let $\bar{\xi}(x_1, x_2) = [\xi_1(x_1) + \xi_2(x_2)]/2$, then there exists a positive number $m'$ such that for all $x_i \in D_i, i = 1, \ldots, n$ and $u$ large enough,

$$\begin{aligned}
&\mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\} \\
&\quad \leq \mathbb{E}\{[(\xi_1(x_1) + \xi_2(x_2))/2]^{m'} \mathbb{1}_{\{[\xi_1(x_1)+\xi_2(x_2)]/2\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\} \\
&\quad = \mathbb{E}\{(\bar{\xi}(x_1, x_2))^{m'} \mathbb{1}_{\{\bar{\xi}(x_1,x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\}.
\end{aligned}$$

Note that

$$\mathrm{Var}(\bar{\xi}(x_1, x_2)|\xi_3(x_3) = \cdots = \xi_n(x_n) = 0) \leq \mathrm{Var}(\bar{\xi}(x_1, x_2)) = \frac{1 + \rho_{12}(x_1, x_2)}{2},$$

where the equality holds if and only if $\bar{\xi}(x_1, x_2)$ is independent of $(\xi_3(x_3), \ldots, \xi_n(x_n))$. Now our result follows from the continuity of the conditional expectation and the compactness of $D_i$, $i = 1, \ldots, n$. □

The following result is similar to Lemma 3 in Piterbarg (1996b).

**Lemma 9.1.5** *Let $(X, Y) = \{(X(t), Y(s)) : t \in K \subset \mathbb{R}^N, s \in D \subset \mathbb{R}^N\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field satisfying* (**C**1) *and* (**C**2)*. Then for any $\varepsilon > 0$,* *there exists $\delta > 0$ such that for $K$ with* $\operatorname{diam}(K) \le \delta$ *and $u$ large enough,*

$$\mathbb{E}\{M_u(X,K)[M_u(X,K)-1]M_u(Y,D)\} \le \operatorname{Vol}(K)\exp\Big\{-\frac{u^2}{2\beta_X(K,D)}+\varepsilon u^2\Big\},$$

*where*

$$\beta_X(K,D) = \sup_{t\in K, s\in D, e\in \mathbb{S}^{N-1}} \operatorname{Var}\Big(\frac{X(t)+Y(s)}{2}\Big|_{\nabla^2 X(t)e=0}^{\nabla X(t)=\nabla Y(s)=0,}\Big).$$

*Similarly, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $D$ with* $\operatorname{diam}(D) \le \delta$ *and $u$ large enough,*

$$\mathbb{E}\{M_u(X,K)M_u(Y,D)[M_u(Y,D)-1]\} \le \operatorname{Vol}(D)\exp\Big\{-\frac{u^2}{2\beta_Y(K,D)}+\varepsilon u^2\Big\},$$

*where*

$$\beta_Y(K,D) = \sup_{t\in K, s\in D, e\in \mathbb{S}^{N-1}} \operatorname{Var}\Big(\frac{X(t)+Y(s)}{2}\Big|_{\nabla^2 Y(s)e=0}^{\nabla X(t)=\nabla Y(s)=0,}\Big).$$

**Proof** The proof will be similar to the original proof of Lemma 3 in Piterbarg (1996b). The only difference is that the integral here involves both $X$ and $Y$ exceeding $u$. But we may apply the arguments for proving Lemma 9.1.3 and Lemma 9.1.4 to handle the double integral, to make it bounded above by the integral of $(X+Y)/2$ exceeding $u$, and then the desired result follows. □

**Lemma 9.1.6** *Let $(X,Y) = \{(X(t),Y(s)) : t \in J \subset \mathbb{R}^N, s \in L \subset \mathbb{R}^N\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field satisfying* (**C**1), (**C**2) *and* (**C**3). *Then there* *exists* $\alpha > 0$ *such that as* $u \to \infty$*,*

$$\begin{aligned} \mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\} &= o\Big(\exp\Big\{-\frac{u^2}{1+\rho(J,L)}-\alpha u^2\Big\}\Big), \\ \mathbb{E}\{M_u(X,J)M_u(Y,L)[M_u(Y,L)-1]\} &= o\Big(\exp\Big\{-\frac{u^2}{1+\rho(J,L)}-\alpha u^2\Big\}\Big), \end{aligned} \tag{9.1.6}$$

*where* $\rho(J,L) = \sup_{t\in J, s\in L} \rho(t,s)$*.*

**Proof** We only prove the first line in (9.1.6), since the proof for the second line is the same. The set $J$ may be covered by congruent cubes $K_i$ with disjoint interiors, edges parallel to coordinate axes and sizes so small that the conditions of Lemma 9.1.5 are satisfied for each union of two neighboring cubes $K_i$ and $K_j$. Then

$$\begin{aligned} &\mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\} \\ &\le \mathbb{E}\Big\{\Big(\sum_i M_u(X,K_i)\sum_j [M_u(X,K_j)-1]\Big)M_u(Y,L)\Big\} \\ &= \mathbb{E}\Big\{\Big(\sum_i M_u(X,K_i)\sum_j M_u(X,K_j) - \sum_i M_u(X,K_i)\Big)M_u(Y,L)\Big\} \\ &= \sum_i \mathbb{E}\{M_u(X,K_i)^2 M_u(Y,L)\} + \sum_{i\neq j}\mathbb{E}\{M_u(X,K_i)M_u(X,K_j)M_u(Y,L)\} - \\ &\quad - \sum_i \mathbb{E}\{M_u(X,K_i)M_u(Y,L)\} \\ &= \sum_i \mathbb{E}\{M_u(X,K_i)[M_u(X,K_i)-1]M_u(Y,L)\} + \sum_{i\neq j}\mathbb{E}\{M_u(X,K_i)M_u(X,K_j)M_u(Y,L)\}. \end{aligned} \tag{9.1.7}$$

Then by the Kac-Rice formula and Lemma 9.1.3, there exists $\alpha' > 0$ such that for $u$ large enough,

$$\sum_{|i-j|\geq 2} \mathbb{E}\{M_u(X,K_i)M_u(X,K_j)M_u(Y,L)\} \leq \exp\Big\{-\frac{u^2}{1+\rho(J,L)} - \alpha' u^2\Big\}. \tag{9.1.8}$$

If $K_i$ and $K_j$ are neighboring, say $j = i+1$, we have

$$\begin{aligned}
&\mathbb{E}\{M_u(X,K_i\cup K_{i+1})[M_u(X,K_i\cup K_{i+1})-1]M_u(Y,L)\}\\
&\quad = \mathbb{E}\{[M_u(X,K_i)+M_u(X,K_{i+1})][M_u(X,K_i)+M_u(X,K_{i+1})-1]M_u(Y,L)\}\\
&\quad = 2\mathbb{E}\{M_u(X,K_i)M_u(X,K_{i+1})M_u(Y,L)\} + \mathbb{E}\{M_u(X,K_i)[M_u(X,K_i)-1]M_u(Y,L)\}\\
&\qquad + \mathbb{E}\{M_u(X,K_{i+1})[M_u(X,K_{i+1})-1]M_u(Y,L)\}.
\end{aligned}$$

Applying Lemma 9.1.5, we see that for $u$ large enough,

$$\begin{aligned}
&\sum_i \mathbb{E}\{M_u(X,K_i)[M_u(X,K_i)-1]M_u(Y,L)\} + \sum_{|i-j|=1}\mathbb{E}\{M_u(X,K_i)M_u(X,K_j)M_u(Y,L)\}\\
&\quad \leq \exp\Big\{-\frac{u^2}{2\beta_X(J,L)} + \varepsilon u^2\Big\}.
\end{aligned} \tag{9.1.9}$$

It is obvious that $\beta_X(J,L) \leq \frac{1+\rho(J,L)}{2}$, and we will show

$$\beta_X(J,L) < \frac{1+\rho(J,L)}{2}. \tag{9.1.10}$$

By the definition of $\beta_X(J,L)$ in Lemma 9.1.5, if $\beta_X(J,L) = \frac{1+\rho(J,L)}{2}$, then due to the continuity, there are some $(t,s) \in \bar{J} \times \bar{L}$ and $e \in \mathbb{S}^{N-1}$ such that

$$\mathrm{Var}\Big(\frac{X(t)+Y(s)}{2}\Big|_{\substack{\nabla X(t)=\nabla Y(s)=0,\\ \nabla^2 X(t)e=0}}\Big) = \frac{1+\rho(J,L)}{2}. \tag{9.1.11}$$

This implies

$$\rho(t,s) = \rho(J,L), \quad \mathbb{E}\{X(t)\nabla Y(s)\} = \mathbb{E}\{Y(s)\nabla X(t)\} = 0.$$

By (**C**3), $\mathbb{E}\{Y(s)\nabla^2 X(t)\}$ becomes negative semidefinite. But $\mathbb{E}\{X(t)\nabla^2 X(t)\}$ is always negative definite due to the constant variance, so that

$$\big(\mathbb{E}\{Y(s)\nabla^2 X(t)\} + \mathbb{E}\{X(t)\nabla^2 X(t)\}\big)e \neq 0, \quad \forall e \in \mathbb{S}^{N-1}.$$

This contradicts (9.1.11) and hence (9.1.10) holds. Plugging (9.1.8) and (9.1.9) into (9.1.7), we finish the proof. $\square$

**Lemma 9.1.7** *Let $\{(X(t),Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field satisfying* (**C**1), (**C**2) *and* (**C**3). *Then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}\mathbb{E}\{M_u^E(X,J)M_u^E(X,J')M_u^E(Y,L)\} &= o\Big(\exp\Big\{-\frac{u^2}{1+\rho(J,L)} - \alpha u^2\Big\}\Big),\\ \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)M_u^E(Y,L')\} &= o\Big(\exp\Big\{-\frac{u^2}{1+\rho(J,L)} - \alpha u^2\Big\}\Big),\end{aligned} \tag{9.1.12}$$

*where $J$ and $J'$ are different faces of $T$, $L$ and $L'$ are different faces of $S$.*

**Proof** We only prove the first line in (9.1.12), since the proof for the second line is the same. If the two faces $J$ and $J'$ are not neighboring, by similar arguments in Lemma 2.3.6 and Lemma 9.1.3, it is straightforward to verify that the high moment in (9.1.12) is super-exponentially small. Thus we turn to considering the case when $J$ and $J'$ are neighboring, i.e., $I := \bar{J} \cap \bar{J}' \neq \emptyset$. Without loss of generality, assume

$$
\begin{aligned}
\sigma(J) &= \{1, \ldots, m, m+1, \ldots, k\}, \\
\sigma(J') &= \{1, \ldots, m, k+1, \ldots, k+k'-m\}, \\
\sigma(L) &= \{1, \ldots, l\},
\end{aligned}
$$

where $0 \le m \le k \le k' \le N$ and $k' \ge 1$. If $k = 0$, we consider $\sigma(J) = \emptyset$ by convention. Under such assumption, $J \in \partial_k T$, $J' \in \partial_{k'} T$, $\dim(I) = m$ and $L \in \partial_l S$. We assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1.

We first consider the case $k \ge 1$. By the Kac-Rice metatheorem,

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(X, J) M_u^E(X, J') M_u^E(Y, L)\} \\
&\quad \le \int_J dt \int_{J'} dt' \int_L ds \int_u^\infty dx \int_u^\infty dx' \int_u^\infty dy \\
&\qquad \int_0^\infty dz_{k+1} \cdots \int_0^\infty dz_{k+k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k \\
&\qquad \mathbb{E}\{|\mathrm{det} \nabla^2 X_{|J}(t)||\mathrm{det} \nabla^2 X_{|J'}(t')||\mathrm{det} \nabla^2 Y_{|L}(s)||X(t) = x, X(t') = x', Y(s) = y, \\
&\qquad\quad \nabla X_{|J}(t) = 0, X_{k+1}(t) = z_{k+1}, \ldots, X_{k+k'-m}(t) = z_{k+k'-m}, \\
&\qquad\quad \nabla X_{|J'}(t') = 0, X_{m+1}(t') = w_{m+1}, \ldots, X_k(t') = w_k, \nabla Y_{|L}(s) = 0\} \\
&\qquad \times p_{t,t',s}(x, x', y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k, 0) \\
&\quad := \int\int\int_{J \times J' \times L} A(t, t', s)\, dt dt' ds,
\end{aligned} \tag{9.1.13}
$$

where $p_{t,t',s}(x, x', y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k, 0)$ is the density of

$$
\begin{aligned}
&(X(t), X(t'), Y(s), \nabla X_{|J}(t), X_{k+1}(t), \ldots, X_{k+k'-m}(t),\\
&\nabla X_{|J'}(s), X_{m+1}(s), \ldots, X_k(s), \nabla Y_{|L}(s))
\end{aligned}
$$

evaluated at $(x, x', y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k, 0)$.

Similarly to Lemma 3.1.7, by Lemma 9.1.4 and continuity, if

$$
\begin{aligned}
I_0 := \{(t, s) \in I \times \bar{S} :& \rho(t, s) = \rho(T, S), \mathbb{E}\{X_i(t)Y(s)\} = \mathbb{E}\{X(t)Y_j(s)\} = 0,\\
&\forall i = 1, \ldots, k + k' - m, j = 1, \ldots, l\} = \emptyset,
\end{aligned}
$$

then $\mathbb{E}\{M_u^E(X, J)M_u^E(X, J')M_u^E(Y, L)\}$ is super-exponentially small. Therefore, similarly to the proof in Lemma 3.1.8, we only need to consider the alternative case, which is $I_0 \neq \emptyset$. Define

$$
B(I_0, \delta) := \{(t, t', s) \in J \times J' \times S : d((t, s), I_0) \vee d((t', s), I_0) < \delta\},
$$

where $\delta$ is a small positive positive number to be specified. Then the difference between $\int\int\int_{J\times J'\times L} A(t, t', s)dtdt'ds$ and $\int_{B(I_0,\delta)} A(t, t', s)dtdt'ds$ is super-exponentially small. Hence we turn to estimating $\int_{B(I_0,\delta)} A(t, t', s)\, dtdt'ds$.

Due to (**C**3), we may choose $\delta$ small enough such that for all $(t, t', s) \in B(I_0, \delta)$,

$$
\Lambda_{J\cup J'}(t, s) = -(\mathbb{E}\{[X(t) + Y(s)]\nabla^2 X(t)\})_{i,j=1,\ldots,k+k'-m}
$$

are positive definite.

Let $\{e_1, e_2, \ldots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For $t \in J$ and $s \in J'$, let $e_{t,t'} = (t'-t)^T/\|t'-t\|$ and let $\alpha_i(t,t',s) = \langle e_i, \Lambda_{J\cup J'}(t,s)e_{t,t'}\rangle$, then

$$\Lambda_{J\cup J'}(t,s)e_{t,t'} = \sum_{i=1}^{N} \langle e_i, \Lambda_{J\cup J'}(t,s)e_{t,t'}\rangle e_i = \sum_{i=1}^{N} \alpha_i(t,t',s)e_i. \tag{9.1.14}$$

There exists some $\alpha_0 > 0$ such that

$$\langle e_{t,t'}, \Lambda_{J\cup J'}(t,s)e_{t,t'}\rangle \geq \alpha_0 \tag{9.1.15}$$

for all $t$ and $t'$. Since all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, we have the following representation,

$$\begin{aligned} t &= (t_1, \ldots, t_m, t_{m+1}, \ldots, t_k, b_{k+1}, \ldots, b_{k+k'-m}, 0, \ldots, 0), \\ t' &= (t'_1, \ldots, t'_m, b_{m+1}, \ldots, b_k, t'_{k+1}, \ldots, t'_{k+k'-m}, 0, \ldots, 0), \end{aligned}$$

where $t_i \in (a_i, b_i)$ for all $i \in \sigma(J)$ and $t'_j \in (a_j, b_j)$ for all $j \in \sigma(J')$. Therefore,

$$\begin{aligned} &\langle e_i, e_{t,t'}\rangle \geq 0, \quad \forall\ m+1 \leq i \leq k, \\ &\langle e_i, e_{t,t'}\rangle \leq 0, \quad \forall\ k+1 \leq i \leq k+k'-m, \\ &\langle e_i, e_{t,t'}\rangle = 0, \quad \forall\ k+k'-m < i \leq N. \end{aligned} \tag{9.1.16}$$

Let

$$\begin{aligned} D_i &= \{(t,t',s) \in B(I_0,\delta) : \alpha_i(t,t',s) \geq \beta_i\}, \quad \text{if } m+1 \leq i \leq k, \\ D_i &= \{(t,t',s) \in B(I_0,\delta) : \alpha_i(t,t',s) \leq -\beta_i\}, \quad \text{if } k+1 \leq i \leq k+k'-m, \\ D_0 &= \left\{(t,t',s) \in B(I_0,\delta) : \sum_{i=1}^{m} \alpha_i(t,t',s)\langle e_i, e_{t,t'}\rangle \geq \beta_0\right\}, \end{aligned} \tag{9.1.17}$$

where $\beta_0, \beta_1, \ldots, \beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0$. It follows from (9.1.16) and (9.1.17) that, if $(t,s)$ does not belong to any of $D_0, D_m, \ldots, D_{k+k'-m}$, then by (9.1.14),

$$\langle \Lambda_{J\cup J'}(t,s) e_{t,t'}, e_{t,t'} \rangle = \sum_{i=1}^{N} \alpha_i(t,t',s) \langle e_i, e_{t,t'} \rangle \le \beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0,$$

which contradicts (9.1.15). Thus $D_0 \cup \cup_{i=m+1}^{k+k'-m} D_i$ is a covering of $B(I_0, \delta)$, by (9.1.13),

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,J) M_u^E(X,J') M_u^E(Y,L)\} \\
&\quad \le \int_{D_0} A(t,t',s)\, dt dt' ds + \sum_{i=m+1}^{k+k'-m} \int_{D_i} A(t,t',s)\, dt dt' ds.
\end{aligned}$$

We first show that $\int_{D_0} A(t,t',s)\, dt dt' ds$ is super-exponentially small.

$$\begin{aligned}
&\int_{D_0} A(t,t',s)\, dt dt' ds \\
&\quad \le \int_{D_0} dt dt' ds \int_u^\infty dx \int_u^\infty dy\, p_{\nabla X_{|J}(t), \nabla X_{|J'}(t'), \nabla Y_{|L}(s)}(0,0,0) \\
&\quad \times p_{X(t),Y(s)}(x,y | \nabla X_{|J}(t) = 0, \nabla X_{|J'}(t') = 0, \nabla Y_{|L}(s) = 0) \qquad (9.1.18) \\
&\quad \times \mathbb{E}\{|\mathrm{det} \nabla^2 X_{|J}(t)| |\mathrm{det} \nabla^2 X_{|J'}(t')| |\mathrm{det} \nabla^2 Y_{|L}(s)| | X(t) = x, Y(s) = y, \\
&\qquad \nabla X_{|J}(t) = \nabla X_{|J'}(t') = \nabla Y_{|L}(s) = 0\}.
\end{aligned}$$

Note that

$$
\begin{aligned}
&\operatorname{Var}(X(t)+Y(s)|\nabla X_{|J}(t), \nabla X_{|J'}(t'), \nabla Y_{|L}(s)) \\
&\le \operatorname{Var}(X(t)+Y(s)|X_1(t),\dots,X_m(t),X_1(t'),\dots,X_m(t')) \\
&= \operatorname{Var}(X(t)+Y(s)|X_1(t),\dots,X_m(t),X_1(t)+\langle\nabla X_1(t),t'-t\rangle+\|t'-t\|^{1+\eta}Y^1_{t,t'},\dots, \\
&\qquad\qquad X_m(t)+\langle\nabla X_m(t),t'-t\rangle+\|t'-t\|^{1+\eta}Y^m_{t,t'}) \\
&= \operatorname{Var}(X(t)+Y(s)|X_1(t),\dots,X_m(t),\langle\nabla X_1(t),e_{t,t'}\rangle+\|t'-t\|^{\eta}Y^1_{t,t'},\dots, \\
&\qquad\qquad \langle\nabla X_m(t),e_{t,t'}\rangle+\|t'-t\|^{\eta}Y^m_{t,t'}) \\
&\le \operatorname{Var}(X(t)+Y(s)|\langle\nabla X_1(t),e_{t,t'}\rangle+\|t'-t\|^{\eta}Y^1_{t,t'},\dots,\langle\nabla X_m(t),e_{t,t'}\rangle+\|t'-t\|^{\eta}Y^m_{t,t'}) \\
&= \operatorname{Var}(X(t)+Y(s)|\langle\nabla X_1(t),e_{t,t'}\rangle,\dots,\langle\nabla X_m(t),e_{t,t'}\rangle)+o(1).
\end{aligned}
\tag{9.1.19}
$$

Hence there exist constants $C_2>0$ and $\varepsilon_0>0$ such that for $\|t'-t\|$ sufficiently small,

$$
\begin{aligned}
&\operatorname{Var}((X(t)+Y(s))/2|\nabla X_{|J}(t), \nabla X_{|J'}(t'), \nabla Y_{|L}(s)) \\
&\qquad \le \frac{\rho(T,S)+1}{2} - C_2\sum_{i=1}^m \alpha_i^2(t,t',s)+o(1) < \frac{\rho(T,S)+1}{2}-\varepsilon_0,
\end{aligned}
\tag{9.1.20}
$$

where the last inequality is due to the fact that $(t,t',s)\in D_0$ implies

$$
\sum_{i=1}^m \alpha_i^2(t,t',s) \ge \sum_{i=1}^m \alpha_i^2(t,t',s)|\langle e_i, e_{t,t'}\rangle|^2 \ge \frac{1}{m}\bigg(\sum_{i=1}^m \alpha_i(t,t',s)\langle e_{t,t'},e_i\rangle\bigg)^2 \ge \frac{\beta_0^2}{m}.
$$

Similarly, we can use the techniques in the proof of Theorem 2.3.8 to show that for $i=m+1,\dots,k+k'-m$, $\int_{D_i} A(t,t',s)\,dtdt'ds$ are super-exponentially small, . $\square$

Now, the approximation obtained still contains the absolute values of determinants, which are hard to be computed. However, we will show that removing these absolute values only causes exponentially small difference, and then we will get the approximation based on the mean Euler characteristic of the excursion set.

**Proposition 9.1.8** *Let* $\{(X(t), Y(s)) : t \in T, s \in S\}$ *be an* $\mathbb{R}^2$*-valued, centered, unit-variance Gaussian random field satisfying* $(\mathbf{C}1)$, $(\mathbf{C}2)$ *and* $(\mathbf{C}3)$. *Then there exists* $\alpha > 0$ *such that for any* $J \in \partial_k T$, $L \in \partial_l S$, *as* $u \to \infty$,

$$\begin{aligned} &\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} \\ &= (-1)^{k+l}\int_J\int_L \mathbb{E}\{\det\nabla^2 X(t)\det\nabla^2 Y(s)\mathbb{1}_{\{X(t)\geq u,\ \varepsilon_j^* X_j(t)\geq 0 \text{ for all } j\notin\sigma(J)\}} \\ &\quad\times \mathbb{1}_{\{Y(s)\geq u,\ \varepsilon_j^* Y_j(s)\geq 0 \text{ for all } j\notin\sigma(L)\}} | \nabla X_{|J}(t) = \nabla Y_{|L}(s) = 0\} \\ &\quad\times p_{\nabla X_{|J}(t),\nabla Y_{|L}(s)}(0,0)dtds + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T,S)} - \alpha u^2\Big\}\Big), \end{aligned} \tag{9.1.21}$$

*where* $\rho(T,S) = \sup_{t\in T, s\in S}\rho(t,s)$.

**Proof** To simplify the proof, let us consider the case when $k = l = N$, and the proof for general cases is similar. By the Kac-Rice formula,

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} \\
&= \int_J\int_L \mathbb{E}\{|\mathrm{det}\nabla^2 X(t)||\mathrm{det}\nabla^2 Y(s)|\mathbb{1}_{\{X(t)\geq u,\ \mathrm{index}(\nabla^2 X(t))=N\}} \\
&\quad \times \mathbb{1}_{\{Y(s)\geq u,\ \mathrm{index}(\nabla^2 Y(s))=N\}}|\nabla X(t) = \nabla Y(s) = 0\} p_{\nabla X(t),\nabla Y(s)}(0,0)dtds \\
&= (-1)^{N+N}\int_J\int_L p_{\nabla X(t),\nabla Y(s)}(0,0)dtds\int_u^\infty\int_u^\infty \mathbb{E}\{\mathrm{det}\nabla^2 X(t)\mathrm{det}\nabla^2 Y(s) \\
&\quad \times \mathbb{1}_{\{\mathrm{index}(\nabla^2 X(t))=N\}}\mathbb{1}_{\{\mathrm{index}(\nabla^2 Y(s))=N\}}|X(t)=x, Y(s)=y, \nabla X(t) = \nabla Y(s) = 0\} \\
&\quad \times p_{X(t),Y(s)}(x,y|\nabla X(t) = \nabla Y(s) = 0)dxdy \\
&:= \int_J\int_L p_{\nabla X(t),\nabla Y(s)}(0,0)dtds\int_u^\infty\int_u^\infty A(x,y,t,s)dxdy.
\end{aligned}
$$

Similarly to the proof in Lemma 3.1.5, define

$$
\begin{aligned}
&O(J,L) = \{(t,s) \in \bar{J}\times\bar{L} : \rho(t,s) = \rho(T,S), \mathbb{E}\{X(t)\nabla Y(s)\} = \mathbb{E}\{Y(s)\nabla X(t)\} = 0\}, \\
&U_\delta(J,L) = \{(t,s) \in J\times L : d((t,s),O(J,L)) < \delta\},
\end{aligned}
$$

where $\delta$ is a small positive number to be specified. Then, similarly, we only need to estimate

$$
\int_{U_\delta(J,L)} p_{\nabla X(t),\nabla Y(s)}(0,0)dtds\int_u^\infty\int_u^\infty A(x,y,t,s)dxdy.
$$

Due to (**C**3), we may choose $\delta$ small enough such that $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 Y(s)\}$ and $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 X(t)\}$ are negative definite for all $(t,s) \in U_\delta(J,L)$. Also note that as

$\delta \to 0$, both $\mathbb{E}\{X(t)\nabla Y(s)\}$ and $\mathbb{E}\{Y(s)\nabla X(t)\}$ tend to 0, therefore,

$$
\begin{aligned}
&\mathbb{E}\{X_{ij}(t)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\} \\
&= (1+o(1))(\mathbb{E}\{X_{ij}(t)X(t)\}, \mathbb{E}\{X_{ij}(t)Y(s)\}) \begin{pmatrix} 1 & \rho(T,S) \\ \rho(T,S) & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \frac{1}{1-\rho(T,S)^2}(\mathbb{E}\{X_{ij}(t)X(t)\}, \mathbb{E}\{X_{ij}(t)Y(s)\}) \begin{pmatrix} x-\rho(T,S)y \\ y-\rho(T,S)x \end{pmatrix},
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\mathbb{E}\{Y_{ij}(s)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\} \\
&= \frac{1}{1-\rho(T,S)^2}(\mathbb{E}\{Y_{ij}(s)X(t)\}, \mathbb{E}\{Y_{ij}(s)Y(s)\}) \begin{pmatrix} x-\rho(T,S)y \\ y-\rho(T,S)x \end{pmatrix}.
\end{aligned}
$$

Note that $\mathbb{E}\{X(t)\nabla^2 X(t)\}$ and $\mathbb{E}\{Y(s)\nabla^2 Y(s)\}$ are both negative definite, $\mathbb{E}\{X(t)\nabla^2 Y(s)\}$ and $\mathbb{E}\{Y(s)\nabla^2 X(t)\}$ both negative semidefinite. There exists $\varepsilon_0 \in (0, 1-\rho(T,S))$ such that for $\delta$ small enough and all $(x,y) \in [u,\infty)^2$ with $(\varepsilon_0+\rho(T,S))y < x < (\varepsilon_0+\rho(T,S))^{-1}y$,

$$
\begin{aligned}
\Sigma_1(x,y) &:= \mathbb{E}\{\nabla^2 X(t)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\}, \\
\Sigma_2(x,y) &:= \mathbb{E}\{\nabla^2 Y(s)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\},
\end{aligned}
$$

are both negative definite. Define

$$
\Delta_1(x,y) = \nabla^2 X(t) - \Sigma_1(x,y), \quad \Delta_2(x,y) = \nabla^2 Y(s) - \Sigma_2(x,y).
$$

Then, conditioning on $(X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0)$, $\Delta_1(x,y)$ and $\Delta_2(x,y)$ are both centered Gaussian random matrices. We write

$$\int_u^\infty \int_u^\infty A(x,y,t,s)dxdy = \int_u^\infty dy \int_{(\varepsilon_0+\rho(T,S))y}^{(\varepsilon_0+\rho(T,S))^{-1}y} A(x,y,t,s)dx+ \\ + \int_u^\infty dy \int_{(\varepsilon_0+\rho(T,S))^{-1}y}^\infty A(x,y,t,s)dx + \int_u^\infty dx \int_{(\varepsilon_0+\rho(T,S))^{-1}x}^\infty A(x,y,t,s)dy.$$

Since $(\varepsilon_0+\rho(T,S))^{-1} > 1$, the last two integrals above are super-exponentially small. Meanwhile,

$$\begin{aligned}
&\int_u^\infty dy \int_{(\varepsilon_0+\rho(T,S))y}^{(\varepsilon_0+\rho(T,S))^{-1}y} A(x,y,t,s)dx \\
&= \int_u^\infty dy \int_{(\varepsilon_0+\rho(T,S))y}^{(\varepsilon_0+\rho(T,S))^{-1}y} \mathbb{E}\{\det(\Delta_1(x,y)+\Sigma_1(x,y))\det(\Delta_2(x,y)+\Sigma_2(x,y)) \\
&\quad \times \mathbb{1}_{\{\text{index}(\Delta_1(x,y)+\Sigma_1(x,y))=N\}} \mathbb{1}_{\{\text{index}(\Delta_2(x,y)+\Sigma_2(x,y)))=N\}} | \\
&\quad |X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\} p_{X(t),Y(s)}(x,y|\nabla X(t)=\nabla Y(s)=0)dx.
\end{aligned}$$

Using the same arguments in the proof of Lemma 2.3.2, we see that removing the two indicator functions above only causes a super-exponentially small difference. Therefore, there exists $\alpha>0$ such that for $u$ large enough,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} \\
&= \int_J \int_L p_{\nabla X(t),\nabla Y(s)}(0,0)dtds \int_u^\infty \int_u^\infty p_{X(t),Y(s)}(x,y|\nabla X(t)=\nabla Y(s)=0) \\
&\quad \times \mathbb{E}\{\det \nabla^2 X(t) \det \nabla^2 Y(s) | X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\} dxdy \\
&\quad + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T,S)} - \alpha u^2\Big\}\Big).
\end{aligned}$$

$\square$

Define the excursion set

$$A_u(X,T) = \{t \in T : X(t) \geq u\},$$

$$A_u(Y,S) = \{s \in S : Y(s) \geq u\},$$

$$A_u(X,T) \times A_u(Y,S) = \{(t,s) \in T \times S : X(t) \geq u, Y(s) \geq u\}.$$

Let

$$\mu_i(X,J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \mathrm{index}(\nabla^2 X_{|J}(t)) = i,$$

$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},$$

$$\mu_i(Y,L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \mathrm{index}(\nabla^2 Y_{|L}(s)) = i,$$

$$\varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \notin \sigma(L)\},$$

where $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues. Then, it follows from the Morse theorem that the Euler characteristic of the excursion set can be represented as

$$\varphi(A_u(X,T)) = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(X,J),$$

$$\varphi(A_u(Y,S)) = \sum_{l=0}^{N} \sum_{L \in \partial_l S} (-1)^l \sum_{i=0}^{l} (-1)^i \mu_i(Y,L).$$

Since for two sets $D_1$ and $D_2$, $\varphi(D_1 \times D_2) = \varphi(D_1)\varphi(D_2)$, we have

$$\begin{aligned} &\varphi(A_u(X,T) \times A_u(Y,S)) = \varphi(A_u(X,T)) \times \varphi(A_u(Y,S)) \\ &= \sum_{k,l=0}^{N} \sum_{J \in \partial_k T, L \in \partial_l S} (-1)^{k+l} \left( \sum_{i=0}^{k} (-1)^i \mu_i(J) \right) \left( \sum_{j=0}^{l} (-1)^j \mu_j(L) \right). \end{aligned} \tag{9.1.22}$$

Now we can state our result as follows.

**Theorem 9.1.9** *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian random field satisfying* (**C**1)*,* (**C**2) *and* (**C**3)*. Then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \\
&= \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X,J) M_u^E(Y,L)\} + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T,S)} - \alpha u^2\Big\}\Big) \\
&= \mathbb{E}\{\varphi(A_u(X,T) \times A_u(Y,S))\} + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T,S)} - \alpha u^2\Big\}\Big),
\end{aligned} \tag{9.1.23}$$

*where $\rho(T,S) = \sup_{t\in T, s\in S} \rho(t,s)$.*

**Proof** The first equality in (9.1.23) follows from the combination of (9.1.1), (9.1.2), Lemma 9.1.6 and Lemma 9.1.7. The second equality in (9.1.23) follows from applying Proposition 9.1.8 and (9.1.22). □

**Example 9.1.10** Let $T = S = [0, 1]$, then

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} = \mathbb{P}\{X(0) \geq u, Y(0) \geq u, X'(0) < 0, Y'(0) < 0\} \\
&\quad + \mathbb{P}\{X(0) \geq u, Y(1) \geq u, X'(0) < 0, Y'(1) > 0\} \\
&\quad + \mathbb{P}\{X(1) \geq u, Y(0) \geq u, X'(1) > 0, Y'(0) < 0\} \\
&\quad + \mathbb{P}\{X(1) \geq u, Y(1) \geq u, X'(1) > 0, Y'(1) > 0\} \\
&\quad + (-1)\int_0^1 p_{X'(t)}(0)dt \int_u^\infty \int_u^\infty \int_{-\infty}^0 p_{X(t),Y(0),Y'(0)}(x, y, z|X'(t) = 0) \\
&\qquad \times \mathbb{E}\{X''(t)|X(t) = x, Y(0) = y, Y'(0) = z, X'(t) = 0\}dxdydz \\
&\quad + (-1)\int_0^1 p_{X'(t)}(0)dt \int_u^\infty \int_u^\infty \int_0^\infty p_{X(t),Y(1),Y'(1)}(x, y, z|X'(t) = 0) \\
&\qquad \times \mathbb{E}\{X''(t)|X(t) = x, Y(1) = y, Y'(1) = z, X'(t) = 0\}dxdydz \\
&\quad + (-1)\int_0^1 p_{Y'(s)}(0)ds \int_u^\infty \int_u^\infty \int_{-\infty}^0 p_{X(0),Y(s),X'(0)}(x, y, z|Y'(s) = 0) \\
&\qquad \times \mathbb{E}\{Y''(s)|X(0) = x, Y(s) = y, X'(0) = z, Y'(s) = 0\}dxdydz \\
&\quad + (-1)\int_0^1 p_{Y'(s)}(0)ds \int_u^\infty \int_u^\infty \int_0^\infty p_{X(1),Y(s),X'(1)}(x, y, z|Y'(s) = 0) \\
&\qquad \times \mathbb{E}\{Y''(s)|X(1) = x, Y(s) = y, X'(1) = z, Y'(s) = 0\}dxdydz \\
&\quad + \int_0^1 \int_0^1 p_{X'(t),Y'(s)}(0, 0)dtds \int_u^\infty \int_u^\infty p_{X(t),Y(s)}(x, y|X'(t) = Y'(s) = 0) \\
&\qquad \times \mathbb{E}\{X''(t)Y''(s)|X(t) = x, Y(s) = y, X'(t) = Y'(s) = 0\}dxdy \\
&\quad + o\Big(\exp\Big\{-\frac{u^2}{1 + \rho(T, S)} - \alpha u^2\Big\}\Big).
\end{aligned}
$$

## 9.2 Vector-valued Gaussian Processes

Let $\{(X(t), Y(t)) : t \in T\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian process, where $T = [a, b]$ is a finite interval in $\mathbb{R}$. We want to estimate the following probability

$$\mathbb{P}\{\exists t \in T \text{ such that } X(t) \geq u, Y(t) \geq u\}.$$

Let

$$\rho(t) = \mathbb{E}\{X(t)Y(t)\}, \quad \rho(T) = \sup_{t \in T} \rho(t).$$

We will make use of the following conditions.

($\mathbf{D}1$). $X, Y \in C^2(\mathbb{R}^N)$ almost surely, and for each pair $(t, s) \in T^2$ with $t \neq s$,

$$(X(t), X'(t), X''(t), Y(t), Y'(t), Y''(t), X(s), X'(s), Y(s), Y'(s))$$

is non-degenerate.

($\mathbf{D}2$). For all $t \in T$ such that $\rho(t) = \rho(T)$ (hence $\mathbb{E}\{X'(t)Y(t)\} + \mathbb{E}\{X'(t)Y(t)\} = 0$), $\mathbb{E}\{X'(t)Y(t)\} = -\mathbb{E}\{X(t)Y'(t)\} \neq 0$.

Define

$$\begin{aligned}
\mu(X, \mathring{T}) &= \#\{t \in \mathring{T} : Y(t) > X(t) \geq u, X'(t) = 0, X''(t) < 0\}, \\
\mu(Y, \mathring{T}) &= \#\{t \in \mathring{T} : X(t) > Y(t) \geq u, Y'(t) = 0, Y''(t) < 0\}, \\
\mu(X = Y, \mathring{T}) &= \#\{t \in \mathring{T} : X(t) = Y(t) \geq u, X'(t)Y'(t) < 0\}, \\
\mu(X, a) &= \mathbb{1}_{\{Y(a) > X(a) \geq u, X'(a) < 0\}}, \\
\mu(Y, a) &= \mathbb{1}_{\{X(a) > Y(a) \geq u, Y'(a) < 0\}}, \\
\mu(X, b) &= \mathbb{1}_{\{Y(b) > X(b) \geq u, X'(b) > 0\}}, \\
\mu(Y, b) &= \mathbb{1}_{\{X(b) > Y(b) \geq u, Y'(b) > 0\}}.
\end{aligned} \tag{9.2.1}$$

**Lemma 9.2.1** *Under* (**D**1)*, for each $u > 0$, we have*

$$\begin{aligned}
&\{\exists t \in T \textit{ such that } X(t) \geq u, Y(t) \geq u\} \\
&= \{\mu(X, \mathring{T}) \geq 1\} \cup \{\mu(Y, \mathring{T}) \geq 1\} \cup \{\mu(X = Y, \mathring{T}) \geq 1\} \\
&\quad \cup \{\mu(X, a) \geq 1\} \cup \{\mu(Y, a) \geq 1\} \cup \{\mu(X, b) \geq 1\} \cup \{\mu(Y, b) \geq 1\} \text{ a.s.}
\end{aligned}$$

**Proof** Note that

$$\begin{aligned}
&\{\exists t \in T \text{ such that } X(t) \geq u, Y(t) \geq u\} \\
&\quad = \{\exists t \in T \text{ such that } (X \wedge Y)(t) \geq u\}.
\end{aligned}$$

Then the result follows similarly to Lemma 2.3.1. □

**Lemma 9.2.2** *Under* (**D**1) *and* (**D**2), *there exists some constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\mathbb{E}\mu(X, \overset{\circ}{T}) = o\Big( \exp\Big\{ -\frac{u^2}{1+\rho} - \alpha u^2 \Big\}\Big),$$
$$\mathbb{E}\mu(Y, \overset{\circ}{T}) = o\Big( \exp\Big\{ -\frac{u^2}{1+\rho} - \alpha u^2 \Big\}\Big).$$

**Proof** We only show the proof for $\mathbb{E}\mu(X, \overset{\circ}{T})$, since the proof for $\mathbb{E}\mu(Y, \overset{\circ}{T})$ will be the same. By the Kac-Rice formula,

$$\begin{aligned}\mathbb{E}\mu(X, \overset{\circ}{T}) = &\int_a^b p_{X'(t)}(0) \int_u^\infty dx \int_x^\infty dy\, p_{X(t),Y(t)}(x, y | X'(t) = 0) \\ &\times \mathbb{E}\{|X''(t)| \mathbb{1}_{\{X''(t)<0\}} | X(t) = x, Y(t) = y, X'(t) = 0\}.\end{aligned}$$

We only need to show that $\mathbb{P}\{Y(t) > X(t) \geq u | X'(t) = 0\}$ is super-exponentially small. But

$$\mathbb{P}\{Y(t) > X(t) \geq u | X'(t) = 0\} \leq \mathbb{P}\{(X(t) + Y(t))/2 \geq u | X'(t) = 0\},$$

and due to (**D**2), for each $t \in T$ such that $\rho(t) = \rho(T)$,

$$\mathrm{Var}((X(t) + Y(t))/2 | X'(t) = 0) < \mathrm{Var}((X(t) + Y(t))/2) = (1 + \rho(T))/2.$$

By continuity, we obtain

$$\sup_{t \in T} \mathrm{Var}((X(t) + Y(t))/2 | X'(t) = 0) < (1 + \rho(T))/2,$$

completing the proof. $\square$

**Lemma 9.2.3** *Under* $(\mathbf{D}1)$ *and* $(\mathbf{D}2)$, *there exists some constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})[\mu(X = Y, \overset{\circ}{T}) - 1]\} = o\Big(\exp\Big\{ - \frac{u^2}{1+\rho(T)} - \alpha u^2\Big\}\Big).$$

**Proof** By the Kac-Rice formula,

$$\begin{aligned}
&\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})[\mu(X = Y, \overset{\circ}{T}) - 1]\} \\
&= \int_a^b \int_a^b p_{X(t)-Y(t),X(s)-Y(s)}(0,0)dtds \\
&\quad \times \int_u^\infty \int_u^\infty dxdy\, p_{X(t),X(s)}(x, y|X(t) - Y(t) = 0, X(s) - Y(s) = 0) \\
&\quad \times \mathbb{E}\{|X'(t) - Y'(t)||X'(s) - Y'(s)|\mathbb{1}_{\{X'(t)Y'(t)<0\}}\mathbb{1}_{\{X'(s)Y'(s)<0\}}| \\
&\qquad\qquad |X(t) = x, X(s) = y, X(t) - Y(t) = 0, X(s) - Y(s) = 0\}.
\end{aligned}$$

Similarly to the proof of Lemma 3 in Piterbarg (1996b), we will write $T$ as the union of several small intervals, and then it suffices to prove that there exists $\alpha' > 0$ such that

$$\mathrm{Var}(X(t)|X(t) - Y(t), X(s) - Y(s)) < \frac{1+\rho(T)}{2} - \alpha', \quad \forall |t-s| < \delta \tag{9.2.2}$$

and

$$\begin{aligned}
&\lim_{u\to\infty} u^{-2} \log \mathbb{P}\{X(t) \ge u, Y(s) \ge u|X(t) - Y(t) = 0, X(s) - Y(s) = 0\} \\
&\quad < -\frac{1}{1+\rho(T)} - \alpha', \quad \forall |t-s| \ge \delta,
\end{aligned} \tag{9.2.3}$$

where $\delta$ is a small positive number to be specified.

Note that for all $t \in T$ ,

$$\operatorname{Var}(X(t)|X(t)-Y(t)) = 1 - \frac{(1-\rho(t))^2}{2(1-\rho(t))} = \frac{1+\rho(t)}{2} \le \frac{1+\rho(T)}{2}.$$

But for those $t$ such that $\rho(t) = \rho(T)$, $\mathbb{E}\{X(t)(X'(t) - Y'(t))\} = -\mathbb{E}\{X(t)Y'(t)\} \neq 0$ by (**D**2), it then follows from continuity that

$$\sup_{t \in T} \operatorname{Var}(X(t)|X(t)-Y(t), X'(t)-Y'(t)) < \frac{1+\rho(T)}{2}. \tag{9.2.4}$$

Therefore (9.2.2) follows by noting that as $|t-s| \to 0$,

$$\begin{aligned} &\operatorname{Var}(X(t)|X(t)-Y(t), X(s)-Y(s)) \\ &\quad = (1+o(1))\operatorname{Var}(X(t)|X(t)-Y(t), X'(t)-Y'(t)). \end{aligned}$$

Now we turn to proving (9.2.3). By continuity, it suffices to show that there is not any pair $(t, s)$ with $|t-s| \ge \delta$ such that

$$\begin{aligned} &\lim_{u\to\infty} u^{-2} \log \mathbb{P}\{X(t) \ge u, Y(s) \ge u | X(t)-Y(t) = 0, X(s)-Y(s) = 0\} \\ &\quad = -\frac{1}{1+\rho(T)}. \end{aligned} \tag{9.2.5}$$

By (9.2.4) and the following evident inequality

$$
\begin{aligned}
&\mathbb{P}\{X(t) \geq u, Y(s) \geq u | X(t) - Y(t) = 0, X(s) - Y(s) = 0\} \\
&\quad \leq \min \big\{ \mathbb{P}\{X(t) \geq u | X(t) - Y(t) = 0, X(s) - Y(s) = 0\}, \\
&\qquad\qquad \mathbb{P}\{Y(s) \geq u | X(t) - Y(t) = 0, X(s) - Y(s) = 0\}, \\
&\qquad\qquad \mathbb{P}\{[X(t) + Y(s)]/2 \geq u | X(t) - Y(t) = 0, X(s) - Y(s) = 0\} \big\},
\end{aligned}
$$

if (9.2.5) holds, then we have

$$
\begin{aligned}
&\rho(t) = \rho(s) = \rho(T), \\
&\mathbb{E}\{X(t)[X(s) - Y(s)]\} = \mathbb{E}\{Y(t)[X(s) - Y(s)]\} = 0, \\
&\mathbb{E}\{X(s)[X(t) - Y(t)]\} = \mathbb{E}\{Y(s)[X(t) - Y(t)]\} = 0,
\end{aligned} \tag{9.2.6}
$$

and

$$
\mathrm{Var}([X(t) + Y(s)]/2 | X(t) - Y(t) = 0, X(s) - Y(s) = 0) = \frac{1 + \rho(T)}{2}. \tag{9.2.7}
$$

But by the conditional formula for Gaussian variables, (9.2.6) implies

$$
\begin{aligned}
&\mathrm{Var}([X(t) + Y(s)]/2 | X(t) - Y(t) = 0, X(s) - Y(s) = 0) \\
&\quad = \frac{1 + \mathbb{E}\{X(t)X(s)\}}{2} - (1 - \rho(T)) = \frac{2\rho(T) + \mathbb{E}\{X(t)X(s)\} - 1}{2} \\
&\quad < \frac{1 + \rho(T)}{2},
\end{aligned}
$$

which contradicts (9.2.7). Thus there is no pair $(t, s)$ with $|t - s| \geq \delta$ such that (9.2.5) holds, and hence (9.2.3) is true, completing the proof. $\square$

**Lemma 9.2.4** *Under* (**D**1) *and* (**D**2), *there exists some constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\begin{aligned}
&\max\big\{\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(X, a)\}, \mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(Y, a)\},\\
&\qquad \mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(X, b)\}, \mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(Y, b)\}\big\}\\
&= o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T)} - \alpha u^2\Big\}\Big),
\end{aligned}$$

**Proof** We only show the proof for $\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(X, b)\}$, since the other terms can be proved similarly. By the Kac-Rice metatheorem,

$$\begin{aligned}
\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\mu(X, b)\} = \int_a^b &\mathbb{E}\{|X'(t) - Y'(t)|\mathbb{1}_{\{X(t)\ge u, X'(t)Y'(t)<0\}}\\
&\times \mathbb{1}_{\{Y(b)>X(b)\ge u, X'(b)>0\}}|X(t) - Y(t) = 0\}dt.
\end{aligned}$$

We only need to show that there exists $\alpha' > 0$ such that for $\delta$ small enough,

$$\begin{aligned}
&\lim_{u\to\infty} u^{-2}\log\mathbb{P}\{X(t) \ge u, X(b) - Y(b) < 0, X'(b) > 0|X(t) - Y(t) = 0\}\\
&\quad < -\frac{1}{1+\rho(T)} - \alpha', \quad \forall |t-b| < \delta,
\end{aligned} \tag{9.2.8}$$

and

$$\begin{aligned}
&\lim_{u\to\infty} u^{-2}\log\mathbb{P}\{X(t) \ge u, X(b) \ge u, Y(b) \ge u|X(t) - Y(t) = 0\}\\
&\quad < -\frac{1}{1+\rho(T)} - \alpha', \quad \forall |t-b| \ge \delta.
\end{aligned} \tag{9.2.9}$$

We show (9.2.8) first. Note that we only need to consider the case when $\rho(b) = \rho(T)$. Under this situation, by (**D**2), either $\mathbb{E}\{X'(b)Y(b)\} < 0$ or $\mathbb{E}\{Y'(b)X(b)\} < 0$. If $\mathbb{E}\{X'(b)Y(b)\} <$

0, then as $|t-b| \to 0$,

$$
\begin{aligned}
&\mathbb{E}\{X(t)X'(b)|X(t)-Y(t)=0\} \\
&= \mathbb{E}\{X(t)X'(b)\} - \frac{(1-\rho(t))\mathbb{E}\{X'(b)[X(t)-Y(t)]\}}{2(1-\rho(t))} \\
&= (1+o(1))\frac{\mathbb{E}\{X'(b)Y(b)\}}{2}.
\end{aligned}
$$

It then follows from Lemma 2.3.10 that

$$
\begin{aligned}
&\lim_{u\to\infty} u^{-2} \log \mathbb{P}\{X(t) \geq u, X'(b) > 0 | X(t)-Y(t)=0\} \\
&< -\frac{1}{1+\rho(T)} - \alpha', \quad \forall |t-s| < \delta.
\end{aligned} \tag{9.2.10}
$$

For the alternative case, $\mathbb{E}\{Y'(t)X(t)\} < 0$, consider Taylor's expansion,

$$
X(b)-Y(b) = X(t)-Y(t) + (b-t)(X'(t)-Y'(t)) + (b-t)^{1+\eta} Z_{t,b},
$$

where $\eta > 0$ and $Z_{t,b}$ is a Gaussian random field with uniformly finite variance. Then as $|t-b| \to 0$,

$$
\begin{aligned}
&\mathbb{E}\{X(t)[X(b)-Y(b)]|X(t)-Y(t)=0\} \\
&= \mathbb{E}\{X(t)[X(b)-Y(b)]\} - \frac{(1-\rho(t))\mathbb{E}\{[X(t)-Y(t)][X(b)-Y(b)]\}}{2(1-\rho(t))} \\
&= -(1+o(1))(b-t)\mathbb{E}\{X(b)Y'(b)\},
\end{aligned}
$$

and also $\mathrm{Var}(X(b)-Y(b)|X(t)-Y(t)) \le C_1(t-b)^2$ for some positive constant $C_1$. Therefore,

$$\frac{\mathbb{E}\{X(t)[X(b)-Y(b)]|X(t)-Y(t)=0\}}{\sqrt{\mathrm{Var}(X(t)|X(t)-Y(t))\mathrm{Var}(X(b)-Y(b)|X(t)-Y(t))}}$$
$$\ge -C_2\mathbb{E}\{X(b)Y'(b)\} > 0,$$

for some positive constant $C_2$. It then follows from Lemma 2.3.10 that

$$\begin{aligned}&\lim_{u\to\infty} u^{-2}\log\mathbb{P}\{X(t)\ge u, X(b)-Y(b)<0|X(t)-Y(t)=0\}\\ &< -\frac{1}{1+\rho(T)}-\alpha', \quad \forall |t-s|<\delta.\end{aligned} \tag{9.2.11}$$

Now, (9.2.10) and (9.2.11) imply (9.2.8).

We turn to proving (9.2.9). Note that

$$\mathbb{P}\{X(t)\ge u, X(b)\ge u, Y(b)\ge u|X(t)-Y(t)=0\}$$
$$\le \mathbb{P}\{X(t)\ge u, [X(b)+Y(b)]/2\ge u|X(t)-Y(t)=0\},$$

and

$$\begin{aligned}\mathrm{Var}(X(t)|X(t)-Y(t)) &\le \frac{1+\rho(T)}{2},\\ \mathrm{Var}([X(b)+Y(b)]/2|X(t)-Y(t)) &\le \frac{1+\rho(T)}{2}.\end{aligned}$$

Due to the regularity condition (**D**1), we obtain

$$\begin{aligned}&\lim_{u\to\infty} u^{-2}\log\mathbb{P}\{X(t)\ge u, [X(b)+Y(b)]/2\ge u|X(t)-Y(t)=0\}\\ &< -\frac{1}{1+\rho(T)}-\alpha', \quad \forall |t-b|\ge\delta,\end{aligned}$$

and then (9.2.9) follows. $\square$

Define the excursion set $A_u(T, X \wedge Y) = \{t \in T : (X \wedge Y)(t) \geq u\}$. Then Morse theorem gives

$$\begin{aligned}\varphi(A_u(T, X \wedge Y)) = \mu(X, \overset{\circ}{T}) - \mu'(X, \overset{\circ}{T}) + \mu(Y, \overset{\circ}{T}) - \mu'(Y, \overset{\circ}{T}) \\ + \mu(X = Y, \overset{\circ}{T}) + \mu(X, a) + \mu(Y, a) + \mu(X, b) + \mu(Y, b),\end{aligned} \tag{9.2.12}$$

where

$$\begin{aligned}\mu'(X, \overset{\circ}{T}) &= \#\{t \in \overset{\circ}{T} : Y(t) > X(t) \geq u, X'(t) = 0, X''(t) > 0\}, \\ \mu'(Y, \overset{\circ}{T}) &= \#\{t \in \overset{\circ}{T} : X(t) > Y(t) \geq u, Y'(t) = 0, Y''(t) > 0\},\end{aligned}$$

and the rest terms on the right hand side of (9.2.12) are defined in (9.2.1).

Now we obtain the following mean Euler characteristic approximation.

**Theorem 9.2.5** *Let $\{(X(t), Y(t)) : t \in \mathbb{R}\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian process satisfying* (**D**1) *and* (**D**2)*, and let $T = [a, b]$ be a closed finite interval in $\mathbb{R}$. Then there exists some constant $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}&\mathbb{P}\{\exists t \in T \text{ such that } X(t) \geq u, Y(t) \geq u\} \\ &\quad = \mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\} + \mathbb{E}\{\mu(X, a)\} + \mathbb{E}\{\mu(Y, a)\} + \mathbb{E}\{\mu(X, b)\} + \mathbb{E}\{\mu(Y, b)\} \\ &\qquad + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T)} - \alpha u^2\Big\}\Big) \\ &\quad = \mathbb{E}\{\varphi(A_u(T, X \wedge Y))\} + o\Big(\exp\Big\{-\frac{u^2}{1+\rho(T)} - \alpha u^2\Big\}\Big),\end{aligned} \tag{9.2.13}$$

*where $\rho(T) = \sup_{t \in T} \rho(t)$.*

**Proof** By Lemma 9.2.1, we can find the upper bound and lower bound, similarly to (2.3.1) and (2.3.2) respectively, for the excursion probability. Applying Lemma 9.2.2, Lemma 9.2.3

and Lemma 9.2.4 yields the first equality in (9.2.13). The last line of (9.2.13) follows from (9.2.12). $\square$

**Remark 9.2.6** Based on the proof of Lemma 9.2.3 and Lemma 9.2.4, the term $\mathbb{E}\{\mu(X = Y, \overset{\circ}{T})\}$ in the approximation (9.2.13) can be replaced by a simpler one $\mathbb{E}\{\mu'(X = Y, \overset{\circ}{T})\}$, where

$$\mu'(X = Y, \overset{\circ}{T}) = \#\{t \in \overset{\circ}{T} : X(t) = Y(t) \geq u\}.$$

It follows from the Kac-Rice metatheorem that

$$\mathbb{E}\{\mu'(X = Y, \overset{\circ}{T})\} = \int_a^b \mathbb{E}\{|X'(t) - Y'(t)|\mathbb{1}_{\{X(t)\geq u\}}|X(t) - Y(t) = 0\}dt.$$

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