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PARTICLE TRACKING USING STOCHASTIC
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PARTICLE TRACKING USING STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY PURE JUMP LÉVY PROCESSES

By

Paramita Chakraborty

A DISSERTATION

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## ABSTRACT

## PARTICLE TRACKING USING STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY PURE JUMP LÉVY PROCESSES

**By**

**Paramita Chakraborty**

Stochastic diffusion driven by a pure jump Lévy process is an important core concept for particle tracking methods used in stochastic hydrology and for tempered anomalous diffusion models used in (Geo) Physics. In this work we discuss the jump Lévy diffusion in terms of stochastic differential equations (SDEs). We examine the existence and uniqueness of solutions of stochastic differential equations of the form

$$dY_t = a(Y_t)dt + b(Y_t)dX_t$$

where $\{X_t\}$ is a pure jump Lévy process. Further, we rigorously derive the infinitesimal generator and the backward equation. It can be shown that the infinitesimal generator is a pseudo differential operator. Using this form with the backward equation, we derive the forward equation by an involution type technique. The forward equation associated with the transition density of the solution process is analogous to the governing advection-dispersion equation used in particle tracking of heavy tailed flows and tempered anomalous diffusion models.

# DEDICATION

**To:** Shova Chakraborty, my Mother.

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I wish to express my deepest gratitude to my advisor Dr. Mandrekar. His guidance, patience and thoughtfulness made this work possible. It was a privilege to work with him.

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## Notations

1. $\mathbb{R}$ : Set of real numbers.

2. $\mathbb{R}_0$ : $\{x : x \in \mathbb{R}; x \neq 0\}$.

3. $\mathbb{N} := \{1, 2, 3, ...\}$

4. $\mathbb{N}_0 : =\{0, 1, 2, 3, ...\}$

5. $\mathbb{R}_+ : = \{x \in \mathbb{R} : x \geq 0\}$

6. $C_0^2(\mathbb{R})$ : All functions $f$ defined on $\mathbb{R}$ with compact support and bounded second order derivatives.

7. $C^m(\mathbb{R})$ : m-times continuously differentiable functions on $\mathbb{R}$.

8. $C^\infty(\mathbb{R}) : = \bigcap_{m \in \mathbb{N}} C^m(\mathbb{R})$

9. $C_\infty(\mathbb{R}) :=$ class of continuous functions on $\mathbb{R}$ vanishing at $\infty$.

10. $(\Omega, \mathcal{F}, P)$: Probability space.

11. $L^0(\Omega)$ : Set of all real valued random variables defined on $\Omega$.

12. $S_\alpha(\sigma, \beta, \mu)$ : Stable distribution with index of stability $\alpha$, the skewness parameter $\beta$, the scale parameter $\sigma$ and the shift parameter $\mu$.
For $0 < \alpha \leq 2$, $\sigma \geq 0$,$-1 \leq \beta \leq 1$ and $\mu$ real.

13. N(ds,du): Poisson random measure defined on $\mathcal{B}([0, \infty) \times \mathbb{R})$ with mean function $\nu(ds, du)$.

14. q(ds,du):Compensated Poisson random measure defined as :
$q(ds, du) = N(ds, du) - \nu(ds, du)$.

# Chapter 1

# Introduction

Stochastic diffusion is a very useful tool in hydrology. It is widely used to describe water flow through a porous medium. The main hypothesis is that the path of a water (quasi) molecule is governed by a Markov process. This idea has been discussed by Bhattacharya, Gupta and Sposito [6, 7, 15]. They argued that the trajectory of a quasi molecule is a Markov process with continuous sample path. With the drift coefficient chosen to be the drift speed of a quasi molecule and the diffusion matrix providing a measure for the random variation of the increment of the sample path, the Markov process can be shown to be the unique solution of Itô's stochastic differential equation (SDE) driven by a Brownian motion. Further, the conditional probability density function of the Markov process solves a forward equation that is analogous to the advection dispersion equation (ADE) that governs the flow. This mechanical advection dispersion equation is derived from the Buckingham-Darcy equation [4] that forms the basis for most current hydrological models of contaminant transport

[4, 14].

Recently, it has been observed that this advection dispersion equation may not be adequate for modeling the heavy tailed contaminant transport in saturated porous media. For such flows an alternative model has been advocated that describes the dispersive flux by a fractional space derivative (the space-fractional advection-dispersion equation or fADE [5]). A stochastic differential equation driven by a stable-Lévy process can be associated with this fADE model. If there exists a Markov process that solves an SDE driven by a stable-Lévy noise then it can be shown that the forward equation of this Markov process is analogous to the governing equation in the fADE model. This theory has been used by hydrologists as the basis of random walk particle tracking methods used to solve the fADE to model heavy tailed ground water contamination transport (Zhang et al [25],[26],[27]). The primary purpose of this work is to lay out a rigorous mathematical foundation for this theory that has been successfully used in hydrology.

Another example of an SDE driven by a jump Lévy process is the tempered anomalous diffusion model [10, 20], which has useful applications in (geo)-Physics. Stochastic diffusion driven by a general jump Lévy process can be used to describe all these cases. With this motivation we start by deriving the governing forward equation of an SDE driven by a general jump Lévy process. Next, as a special case, we derive the governing forward equation of an SDE driven by a stable-Lévy process that can be used for particle tracking of heavy tailed flows.

In Chapter 2 we give preliminary definitions and results required for the main part of the thesis. In Chapter 3 we discuss an SDE driven by a general jump Lévy process and derive the infinitesimal generator and forward equation. Using concepts developed in Chapter 3, we derive the governing forward equation of an SDE driven by a stable-Lévy process in Chapter 4.

More detailed discussion and application of the theory presented here can be found in [11] and [12].

# Chapter 2

# Preliminaries

The purpose of this chapter is to provide background material for the subsequent chapters.

## 2.1 Lévy Processes and Lévy Itô Decomposition

We shall be studying a stochastic differential equation driven by a *Lévy Process.* This involves defining stochastic integration (in the Itô sense) with respect to a Lévy process. We begin by defining the Lévy process . Let $\{\Omega, \mathcal{F}, P\}$ be the underlying probability space.

**Definition 2.1.1.** *A stochastic process* $\{X_t : \ t \geq 0\}$ *is a* Lévy process *if the following conditions are satisfied:*

1. *For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 < ... < t_n$*, random variables* $X_{t_0}$, $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent (independent increment*

*property).*

2. $X_0 = 0$ *a.s.*

3. *The distribution of* $X_{s+t} - X_s$ *does not depend on s (stationary increment property).*

4. *It is stochastically continuous, i.e* $P(|X_t - X_s| > \epsilon) \to 0$ *as* $s \to t$ *for every* $\epsilon > 0$.

5. *We assume that* $\{X_t, t \in \mathbb{R}_+\}$ *is cádalàg without loss of generality. (Sato [23]).*

   *(There is* $\Omega_0 \in \mathcal{F}$ *with* $P[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$, $X_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has left limit in* $t > 0$.*)*

<u>Remark:</u>

**(i)** $\{X_t\}$ is called a *Lévy process in law,* if it satisfies (1), (2), (3) and (4).

**(ii)** A stochastic process satisfying (1), (2), (4) and (5) is called an *additive process.*

**(iii)** An *additive process in law* is a stochastic process satisfying (1),(2) and (4).

**Definition 2.1.2.** *A probability measure* $\mu$ *on* $\mathbb{R}$ *is* infinitely divisible *if, for any positive integer n, there is a probability measure* $\mu_n$ *on* $\mathbb{R}$ *such that* $\mu = \mu_n^n$.

<u>Remark:</u>

**a)** Here $\mu_n^n$ gives the $n$-fold convolution of $\mu_n$, i.e, $\mu_n^n = \underbrace{\mu * \mu * \ldots * \mu}_{n}$.

**b)** We can write $\mu_n = \mu^{1/n}$ and it is uniquely determined. Using this we can define $\mu^{r_n}$ for any rational number $r_n$.

**c)** For any non-integer $t \in (0, \infty)$ we can choose a sequence of rational numbers $\{r_n\}$ such that $r_n \to t$ and define the $t$-fold convolution of $\mu$ as $\mu^t = \lim_{n\to\infty} \mu^{r_n}$. For detailed construction and proof of existence of such limit see Sato ([23], page 35).

Observe that by (1),(2) and (3) in definition 2.1.1 above and the fact that for each $t > 0$, $X_t = X_t - X_{\frac{(n-1)t}{n}} + \cdots + X_{\frac{2t}{n}} - X_{\frac{t}{n}} + X_{\frac{t}{n}} - X_0$ for every n, we can say for each $t$, $X_t$ = sum of independent identically distributed random variables. Let $\mu_{X_t}$ be the measure associated with $\{X_t\}$ and $\mu_n$ be the measure associated with $\{X_{\frac{t}{n}} - X_0\}$, then $\mu_{X_t} = \mu_n * \mu_n \cdots * \mu_n = \mu_n^n$. Thus $\mu_{X_t}$ is infinitely divisible for each t. The next theorem precisely gives the relation between a Lévy process and an infinitely divisible process.

**Theorem 2.1.1.** *(Theorem 7.10: [23], page 35)*

**(i)** *If $\{X_t \;:\; t \geq 0\}$ is a Lévy process in law on $\mathbb{R}$, then, for any $t \geq 0$, $P_{X_t}$ is infinitely divisible and, letting $P_{X_1} = \mu$, we have $P_{X_t} = \mu^t$.*

**(ii)** *Conversely, if $\mu$ is an infinitely divisible distribution on $\mathbb{R}$, then there is a Lévy process in law $\{X_t \;:\; t \geq 0\}$ such that $P_{X_1} = \mu$.*

An infinitely divisible process can be specified by its characteristic function, which is given by the *Lévy-Khintchine representation* as follows:

**Theorem 2.1.2.** *(Theorem 8.1: [23], page 37)* Lévy-Khintchine representation

**(i)** *If $\mu$ is an infinitely divisible distribution on $\mathbb{R}^d$, then its characteristic function has following representation:*

$$\begin{aligned}\hat{\mu}(z) &= \int e^{izx}\mu(dx) \\ &= \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbb{R}^d}\Big(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_D(x)\Big)\nu(dx)\Big] \quad (2.1)\end{aligned}$$

*for $z \in \mathbb{R}^d$, where $D = \{x : |x| \leq 1\}$, $A$ is a symmetric nonnegative definite $d \times d$ matrix, $\nu$ is a measure on $\mathbb{R}^d$ satisfying,*

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$$

*and $\gamma \in \mathbb{R}^d$. The representation of $\hat{\mu}$ in (i) by $A$, $\nu$, and $\gamma$ is unique.*

**(ii)** *Conversely, if $A$ is a symmetric nonnegative-definite $d \times d$ matrix, $\nu$ is a measure as above, and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution $\mu$ whose characteristic function is given by* (2.1).

*Remark: We call $(A, \nu, \gamma)$ in the above theorem the* generating triplet *of $\mu$.*

In this work we express the stochastic differentiation w.r.t a jump Lévy process as the stochastic differentiation w.r.t a compensated Poisson random measure. To understand this representation we need to know the decomposition of general Lévy processes in terms of integration w.r.t Poisson random measures. This is called the Lévy-Itô decomposition.

We start with the definition of Poisson random measures. Let

$$\bar{\mathbb{Z}}_{+} = \{0,1,2,3,...\} \cup \{+\infty\}$$

**Definition 2.1.3.** *Let $(\Theta, \mathcal{B}, \rho)$ be a $\sigma$-finite measure space. A family of $\bar{\mathbb{Z}}_{+}$-valued random variables $\{N(B) : B \in \mathcal{B}\}$ is called a Poisson random measure on $\Theta$ with intensity measure or mean function $\rho$, if the following hold:*

**(1)** *for every B, N(B) has Poisson distribution with mean $\rho(B)$;*

**(2)** *if $B_1$, ..., $B_n$ are disjoint, then $N(B_1)$, ..., $N(B_n)$ are independent;*

**(3)** *for every $\omega$, $N(.,\omega)$ is a measure on $\Theta$.*

*The random measure q defined by $q(B) = N(B) - \rho(B)$ is called the compensated Poisson random measure.*

The next theorem gives a constructive decomposition of a Lévy process as a sum of a *jump part* and a *continuous part*. In this theorem we shall use stochastic integration w.r.t a Poisson random measure (see Section 2.2 for precise definitions). Let $H = (0,\infty) \times (\mathbb{R}^d \setminus \{0\})$. The Borel $\sigma$-algebra of $H$ is denoted by $\mathcal{B}(H)$.The basic decomposition theorem is given by,

**Theorem 2.1.3.** *(Theorem 19.2 : [23] page 120)*

*Let $\{X_t : t \geq 0\}$ be an additive process on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, P)$*

*with system of generating triplets* $\{(A_t, \nu_t, \gamma_t)\}$ *and define the measure* $\tilde{\nu}$ *on* $H$ *by* $\tilde{\nu}((0,t] \times D) = \nu_t(D)$ *for* $D \in \mathcal{B}(\mathbb{R}^d)$. *Using* $\Omega_0$ *from definition(1.1) of an additive process, define for* $B \in \mathcal{B}(H)$,

$$J(B,\omega) = \begin{cases} \#\{s : \big(s, X_s(\omega) - X_{s-}(\omega)\big) \in B\} & ,for \quad \omega \in \Omega_0, \\ 0 & ,for \quad \omega \notin \Omega_0, \end{cases}$$

*Then the following holds:*

**(i)** $\{J(B) : B \in \mathcal{B}(H)\}$ *is a Poisson random measure on* $H$ *with intensity measure* $\tilde{\nu}$.

**(ii)** *There is* $\Omega_1 \in \mathcal{F}$ *with* $P[\Omega_1] = 1$ *such that, for any* $\omega \in \Omega_1$,

$$X_t^{(1)}(\omega) = \lim_{\epsilon \downarrow 0} \int_{(0,t]\times\{\epsilon<|x|<1\}} \{xJ(d(s,x),\omega) - x\tilde{\nu}(d(s,x))\} + \int_{(0,t]\times\{|x|\geq 1\}} xJ(d(s,x),\omega)$$

*is defined for all* $t \in [0,\infty)$ *and the convergence is uniform in* $t$ *on any bounded interval. The process* $\{X_t^{(1)}\}$ *is an additive process on* $\mathbb{R}^d$ *with* $\{(0, \nu_t, 0)\}$ *as the system of generating triplet.*

**(iii)** *Define,* $X_t^{(2)}(\omega) = X_t(\omega) - X_t^{(1)}(\omega)$ *for* $\omega \in \Omega_1$.

*There is* $\Omega_2 \in \mathcal{F}$ *with* $P[\Omega_2] = 1$ *such that, for any* $\omega \in \Omega_2$, $X_t^{(2)}(\omega)$ *is continuous in* $t$. *The process* $\{X_t^{(2)}\}$ *is an additive process on* $\mathbb{R}^d$ *with* $\{(A_t, 0, \gamma_t)\}$ *as the system of generating triplet.*

**(iv)** *The two process* $\{X_t^{(1)}\}$ *and* $\{X_t^{(2)}\}$ *are independent.* $\{X_t^{(1)}\}$ *is called the jump part and* $\{X_t^{(2)}\}$ *is called the continuous part of the process* $\{X_t\}$.

Finally the *Lévy-Itô decomposition* of a Lévy process is given as follows:

**Theorem 2.1.4.** *(Theorem 2.4.16:[23],page 108) (The Lévy-Itô decomposition) If $X$ is a Lévy process in $\mathbb{R}^d$ with generating triplet $(A, \nu, \gamma)$, then there exists a Brownian motion $B_A$ with covariance matrix* A *and an independent Poisson random measure* N *on $\mathbb{R} \times (\mathbb{R}^d \setminus \{0\})$ with intensity measure $\tilde{\nu}((0,t] \times B) = t \times \nu(B)$ for $B \in \mathcal{B}(\mathbb{R}_0)$, such that, for each $t \geq 0$,*

$$X(t) = \gamma t + B_A(t) + \int_{|x|<1} xq(t, dx) + \int_{|x|\geq 1} xN(t, dx)$$

*where $q$ is the compensated Poisson random measure associated with $N$ as in Definition 2.1.3. The Brownian motion part gives the continuous part of the Lévy process whereas the integration w.r.t Poisson and compensated Poisson random measure contribute to the jump part. In the next section we shall formally define integration with respect to a Poisson random measure and a compensated Poisson random measure.*

In this work we shall consider a pure jump Lévy process with $A = 0$ and $\gamma = 0$, i.e a Lévy processes of the form:

$$X(t) = \int_{|x|<1} xq(t, dx) + \int_{|x|\geq 1} xN(t, dx)$$

## 2.2 Stochastic Calculus w.r.t Lévy Processes

In the previous section the Lévy-Itô decomposition shows a Lévy processes can be expressed in terms of a Brownian motion and a Poisson integral. However, here we shall be considering only pure jump Lévy processes. Therefore for the main results we shall ignore the Brownian motion part and concentrate solely on stochastic integration w.r.t a Poisson (compensated) random measure. We shall follow the concepts discussed by A.V Skorokhod in [24] to define stochastic integration w.r.t a compensated Poisson random measure.

Let us consider $0 \leq t_0 < T < \infty$ and let $\nu$ be a measure defined on $[t_0, T] \times \mathbb{R}$. Let $\mathcal{B}$ be the ring of all Borel sets $A$ in $[t_0, T] \times \mathbb{R}$ for which $\nu(A) < \infty$.
Let $N$ be a Poisson random measure defined on $\mathcal{B}$ with intensity measure $\nu$. For every $A \in \mathcal{B}$ we shall denote by *q(A)* the random measure defined by the relation:

$$q(A) \;=\; N(A) - EN(A) \;= N(A) - \nu(A)$$

We suppose that for every $t \in [t_0, T]$, a $\sigma$-algebra $\mathcal{F}_t$ of events $A$ is defined, such that $A \subset [t_0, T] \times \mathbb{R}$ (that is, $(s, u) \in A$ only if $s \in [t_0, t]$). Also we assume that $N(A)$ is measurable w.r.t $\mathcal{F}_t$. Also, for all sets $A_1, A_2, \cdots, A_k$ in $[t, T] \times \mathbb{R}$ the quantities $N(A_i)$ are mutually independent of any event in $\mathcal{F}_t$.

We shall use the following function spaces in our definition:

$$\bar{M}(\mathcal{F}_t) \;=\; \{f(t,u) : f(t,u) \text{ is a random function measurable w.r.t } \mathcal{F}_t\}.$$

We shall call the function *f(t,u)* a step function if there exists $t_0 < t_1 < \cdots < t_n = T$ and Borel sets $B_1, B_2, \cdots, B_n$ in $\mathbb{R}$ such that *f(t,u)* is constant on every set $[t_k, t_{k+1}] \times B_j$, $k = 0, 1 \cdots (n-1)$, $j = 0, 1 \cdots n$, and $\bigcup B_j = \mathbb{R}$. We set the following:

$$\bar{M}_0(\mathcal{F}_t) = \{\text{step functions } f(t,u) \text{ in } \bar{M}(\mathcal{F}_t) : \exists\, \epsilon > 0, f(t,u) = 0 \text{ for } |u| \leq \epsilon\}.$$

$$\begin{aligned}
\bar{M}_p^{(1)} &= \{f(t,u) : \int_{t_0}^{T}\int_{\mathbb{R}} E|f(t,u)|\nu(dt,du) < \infty\}. \\
\bar{M}_p^{(2)} &= \{f(t,u) : P\Big[\int_{t_0}^{T}\int_{\mathbb{R}} |f(t,u)|\nu(dt,du) < \infty\Big] = 1\}. \\
\bar{M}_q^{(1)} &= \{f(t,u) : \int_{t_0}^{T}\int_{\mathbb{R}} E|f(t,u)|^2\nu(dt,du) < \infty\}. \\
\bar{M}_q^{(2)} &= \{f(t,u) : P\Big[\int_{t_0}^{T}\int_{\mathbb{R}} |f(t,u)|^2\nu(dt,du) < \infty\Big] = 1\}.
\end{aligned}$$

**Definition 2.2.1.** *Stochastic Integral w.r.t* ***compensated*** *Poisson random measure (See [24], page 35-37):*

**(a)** *Assume $f(t,u) \in \bar{M}_0(\mathcal{F}_t)$. Taking $u_j \in B_j$, we define*

$$\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du) = \sum_{\substack{0\leq k\leq (n-1)\\ 1\leq j\leq n}} f(t_k, u_j)q([t_k, t_{k+1}] \times B_j).$$

*Remark :* Note that in this case it can be shown that the following Itô isometry holds:

$$E\left|\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du)\right|^2 = \int_{t_0}^{T}\int_{\mathbb{R}} E|f(t,u)|^2\nu(dt,du)$$

**(b)** *It can be shown that For every $f(t,u) \in \bar{M}_q^{(1)}(\mathcal{F}_t)$, there exists a sequence of functions $f_n(t,u) \in \bar{M}_0(\mathcal{F}_t) \bigcap \bar{M}_q^{(1)}(\mathcal{F}_t)$ such that*

$$\lim_{n\to\infty}\int_{t_0}^{T}\int_{\mathbb{R}} E|f_n(t,u) - f(t,u)|^2\nu(dt,du) = 0$$

*Hence the sequence of random variables $\int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)q(dt,du)$ will converge in probability to some particular random variable which we shall denote by: $\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du)$, whenever $f(t,u) \in \bar{M}_q^{(1)}(\mathcal{F}_t)$.*

**(c)** *Now let $f(t,u) \in \bar{M}_q^{(2)}(\mathcal{F}_t)$, define $g_n(x)=1$ for $|x| \leq n$ and $g_n(x)=0$ for $|x| > n$. Then for all $n \in \mathbb{N}$,*

$$f_n(t,u) = f(t,u)g_n\left(\int_{t_0}^{T}\int_{\mathbb{R}} |f(t,u)|^2 q(dt,du)\right)$$

*belongs to $\bar{M}_q^{(1)}(\mathcal{F}_t)$, and consequently the expression $\int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)q(dt,du)$ is* *meaningful from (b). Also it can be shown for $n' > n$,*

$$P\left[\left|\int_{t_0}^{T}\int_{\mathbb{R}} f_{n'}(t,u)q(dt,du) - \int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)q(dt,du)\right| > 0\right]$$
$$\leq P\left[\int_{t_0}^{T}\int_{\mathbb{R}} |f(t,u)|^2\nu(dt,du) > n\right]$$

*Since the probability on righthand side approaches to zero as $n \to \infty$, the sequence of random variables $\int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)q(dt,du)$ will converge in probability to some particular random variable which we shall denote by $\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du)$, whenever $f(t,u) \in \bar{M}_q^{(2)}(\mathcal{F}_t)$.*

Next, we give the definition of a stochastic integral with respect to a Poisson random measure $N$.

**Definition 2.2.2.** *Stochastic Integral w.r.t Poisson random measure (See [24], page 38-41):*

**(a)** *If $f(t,u) \in \bar{M}_q^{(2)}(\mathcal{F}_t) \bigcap \bar{M}_p^{(2)}(\mathcal{F}_t)$, then $\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du)$ and $\int_{t_0}^{T} f(t,u)\nu(dt,du)$ both are finite. We set for $f(t,u) \in \bar{M}_q^{(2)}(\mathcal{F}_t) \bigcap \bar{M}_p^{(2)}(\mathcal{F}_t)$*

$$\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)N(dt,du) = \int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)q(dt,du) + \int_{t_0}^{T} f(t,u)\nu(dt,du)$$

**(b)** *For every function $f(t,u) \in \bar{M}_p^{(1)}(\mathcal{F}_t)$, it is possible to construct a sequence*

$f_n(t,u) \in \bar{M}_p^{(1)}(\mathcal{F}_t) \bigcap \bar{M}_p^{(2)}(\mathcal{F}_t)$ *such that*

$$\lim_{n\to\infty} \int_{t_0}^{T}\int_{\mathbb{R}} E|f_n(t,u) - f(t,u)|\nu(dt,du) = 0$$

*and therefore*

$$\lim_{\substack{n\to\infty \\ l\to\infty}} \int_{t_0}^{T}\int_{\mathbb{R}} E|f_n(t,u) - f_l(t,u)|\nu(dt,du) = 0$$

*Hence the sequence of variables* $\int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)N(dt,du)$ *will converge in probability to some limit, which we shall denote by*

$$\int_{t_0}^{T}\int_{\mathbb{R}} f(t,u)N(dt,du)$$

**(c)** *For* $f(t,u) \in \bar{M}_p^{(2)}(\mathcal{F}_t)$. *Then for every* $n \in \mathbb{N}$,

$$f_n(t,u) \;=\; f(t,u)g_n\left(\int_{t_0}^{T} |f(t,u)|^2\nu(dt,du)\right)$$

*will belong to* $\bar{M}_p^{(1)}(\mathcal{F}_t)$, *and for* $n' > n$,

$$P\left[\left|\int_{t_0}^{T}\int_{\mathbb{R}} f_{n'}(t,u)N(dt,du) \;-\; \int_{t_0}^{T}\int_{\mathbb{R}} f_n(t,u)N(dt,du)\right| > 0\right]$$

$$\leq P\left[\int_{t_0}^{T}\int_{\mathbb{R}} |f(t,u)|\nu(dt,du) \ > n\right].$$

*Consequently* $\lim_{n\to\infty} \int_{t_0}^{T} f(t,u)N(dt,du)$ *exists in the sense of convergence of probability. We shall denote this limit by,*

$$\int_{t_0}^{T} f(t,u)N(dt,du) \ \ for \ \ f(t,u) \in \bar{M}_p^{(2)}(\mathcal{F}_t).$$

Now we can proceed to define stochastic integration w.r.t jump Lévy processes.

**Definition 2.2.3.** *(Stochastic Integral w.r.t Lévy processes):*

*Let* $\{X_t\}$ *be a pure jump Lévy process taking values on* $\mathbb{R}$ *with generating triplet* $(0,\nu,\gamma)$. *We want to define stochastic integration of the form* $Y_t = Y_0 + \int_0^t L(s)dX_s$. *The Lévy-Itô decomposition shows that* $\{X_t\}$ *can be expressed in terms of a Poisson integral. Using that form we can define:*

$$\begin{aligned} Y_t &= Y_0 + \int_0^t L(s)dX_s \\ &= Y_0 + \int_0^t \gamma ds + \int_0^t \int_{|x|<1} xL(s)q(ds,dx) + \int_0^t \int_{|x|\geq 1} xL(s)N(ds,dx) \end{aligned}$$

*where we assume* $t_0 = 0$ *and* $L(t) \in \bar{M}_q^{(2)}(\mathcal{F}_t)|_{u=0}$.

In general a stochastic process taking values in $\mathbb{R}$ is called *a jump Lévy-type* *stochastic integral* if it can be written in the following form:

$$Y_t = Y_0 + \int_0^t G(s)ds + \int_0^t \int_{|x|<1} H(s,x)q(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)N(ds,dx)$$

where $|G|^{1/2} \in \bar{M}_q^{(2)}(\mathcal{F}_t)|_{u=0}$ ; $H(t,\cdot) \in \bar{M}_q^{(2)}(\mathcal{F}_t)$ and $K$ predictable. Here $N$ is a Poisson random measure on $(\mathbb{R}^+ \times \mathbb{R}_0)$ and $q$ is the corresponding compensated Poisson random measure.

## 2.3 Stochastic Diffusion Driven by a Jump Lévy Process & Existence and Uniqueness of the Solution

Let $\{X_t\}$ be a pure jump Lévy process with generating triplet $(0,\nu,0)$ of following form

$$X(t) = \int_{0<|x|<1} xq(t,dx) + \int_{|x|\geq 1} xN(t,dx) \tag{2.2}$$

where $N$ is a Poisson random measure with intensity measure $\tilde{\nu}((0,t] \times D) = t\nu(D)$ for $D \in \mathcal{B}(\mathbb{R}_0)$.

Let us consider the drift coefficient functions $a$ and the dispersion coefficient function $b$, where $a,\ b : \mathbb{R} \to \mathbb{R}$. We are interested in the diffusion equation driven by pure jump Lévy process $\{X_t\}$, i.e, SDE of the form:

$$dY_t = a(Y_t)dt + b(Y_t)dX_t \tag{2.3}$$

Using the Lévy-Itô decomposition of $X_t$ we can rewrite (2.3) as :

$$dY_t = a(Y_t)dt + \int\limits_{0<|x|<1} b(Y_t)xq(dt,dx) + \int\limits_{|x|\geq 1} b(Y_t)xN(dt,dx) \tag{2.4}$$

Thus, we get a special case of the *(jump)* Lévy-*type stochastic differential equation.*

Under certain conditions on the coefficient functions $a$ and $b$ of the diffusion process, we can show that there exists a unique solution process to the diffusion equation (2.3). The general stochastic differential equation w.r.t a Lévy process with the existence and uniqueness of the solution is discussed by D. Applebaum ([1] : section 6.2). Therefore for the following theorem we give just a sketch of the proof and refer to [1] for details.

**Theorem 2.3.1.** *Let coefficient functions $a$ and $b$ of diffusion equation* (2.3) *satisfy the following growth condition and Lipschitz condition*

**(A)** *Growth condition: there exists constant $C > 0$ such that $\forall$ $y$ in $\mathbb{R}$*

$$|a(y)|^2 + |b(y)|^2 \leq C\left(1 + |y|^2\right)$$

**(B)** *Lipschitz condition: there exists constant* $C' > 0$ *such that* $\forall\ y_1\ \ y_2$ *in* $\mathbb{R}$

$$|a(y_1) - a(y_2)|^2 + |b(y_1) - b(y_2)|^2 \ \leq C' \left( |y_1 - y_2|^2 \right)$$

*Then, there exists a unique solution process* $Y_t$ *for the equation* (2.3)*; also the solution process is continuous w.r.t the initial value.*

*Proof.* To prove the theorem use the alternative form of the diffusion process as in (2.4). The proof is done in two steps. In the first step consider equation (2.4) up to small jumps only, i.e consider

$$dZ_t = a(Z_t)dt + \int\limits_{0<|x|<1} b(Z_t)xq(dt, dx)$$

Such a process $\{Z_t\}$ can be constructed using *Picard's* method (Skorokhod [24]) and using conditions (A), (B). Further it can be shown the solution $\{Z_t\}$ is unique.
In the second step the large jumps can be added to $\{Z_t\}$ using an interlacing technique (Ikeda and Watanabe [17]) and hence the solution process $\{Y_t\}$ for the main equation (2.4) is constructed. Uniqueness follows from uniqueness of $\{Z_t\}$ and the interlacing structure.

For detailed constructions and proof of uniqueness see [1] (Theorem 6.2.3, page 304 & Theorem 6.2.9, page 311). For the proof of continuity w.r.t initial value and Markov property see [24]. $\square$

# Chapter 3

# Derivation of the Infinitesimal Generator, Backward and Forward Equations

In the previous chapter we observed that, under certain restrictions on the drift and the diffusion coefficients, a unique solution process $\{Y_t\}$ of the diffusion equation in (2.3) exists. Further, the solution is a Markov process. In this chapter we derive the infinitesimal generator and associated backward equation of the Markov process $\{Y_t\}$. Using Fourier analysis concepts we can show that the infinitesimal generator is a pseudo-differential operator. Then we derive the governing forward equation from the backward equation using an involution type technique.

For simplification of calculations we restrict the study to one dimension and also to the case where drift and diffusion coefficients are only space dependent. The theory can be easily extended to an SDE with time homogeneous coefficients defined on $\mathbb{R}^d$.

## 3.1 Properties of the Solution Process

### 3.1.1 Time Homogeneity of the Solution Process

Let us consider the SDE in (2.3) with a little modification. Consider

$$dY_t \;=\; a(Y_t)dt \;+\; b(Y_t)dX_t \tag{3.1}$$

for $t \geq s$, given $Y_s = y$ and $X_s$ is a jump Lévy process.

Also, we assume coefficient functions $a$ and $b$ satisfy growth condition and Lipschitz condition from Theorem 2.3.1.

Denote the unique solution of (3.1) by $Y_t = Y_t^{y,s}$ . Then,

$$\begin{aligned} Y_{t+h}^{y,t} &= y \;+\; \int_t^{t+h} a\left(Y_u^{y,t}\right) du \;+\; \int_t^{t+h} b\left(Y_u^{y,t}\right) dX_u \\ &= y \;+\; \int_0^h a\left(Y_{t+v}^{y,t}\right) dv \;+\; \int_0^h b\left(Y_{t+v}^{y,t}\right) d\tilde{X}_v \end{aligned}$$

with $u = t + v$ and $\tilde{X}_v \;=\; X_{t+v} - X_t$. On the other hand,

$$Y_h^{y,0} \;=\; y \;+\; \int_0^h a\left(Y_v^{y,0}\right) dv \;+\; \int_0^t b\left(Y_v^{y,0}\right) dX_v$$

Now from the definition of Lévy processes, $\{\tilde{X_v}\}$ and $\{X_v\}$ have same $X_0$-distribution. Then it follows from the uniqueness of the solution for SDE in (3.1) that $\{Y_{t+h}^{y,t}\}_{h\geq 0}$ and $\{Y_h^{y,0}\}$, $h \geq 0$ have same $Y_0$-distribution; i.e, $\{Y_t\}_{t\geq 0}$ is time homogeneous.

### 3.1.2 Markov Property of the Solution Process

Let $\{\mathcal{F}_t\}$ be the $\sigma$-field generated by $\{X_s : s \leq t\}$. Let $\{Y_t\}$ be the solution process of equation (2.3). From the equation we know that $Y_t$ is $\mathcal{F}_t$ measurable.

**Theorem 3.1.1.** *If there exists a unique solution of SDE* (2.3), *then the solution process is a Markov process.*

*Proof.* By construction we can re-write $Y_{t+s}$ as follows:

$$Y_{t+s} = Y_s + M(s, t+s)$$

where

$$M(s, t+s) = \int_s^{t+s} a(Y_v)dv + \int_s^{t+s} b(Y_v)dX_v$$

Following the Lévy-Itô decomposition $M(s, t+s)$ is $\sigma\{N(v, A) - N(u, A), s \leq u < v \leq t, A \in \mathcal{B}(\mathbb{R}_0)\}$ measurable ( $\mathcal{B}(\mathbb{R}_0)$ is the Borel sigma field of $\mathbb{R}_0$). Clearly $M(s, t+s)$ is independent of $\mathcal{F}_s \subseteq \sigma\{N(u, A), u \leq s, A \in \mathcal{B}(\mathbb{R}_0\}$ and that leads to the Markov property. See [24] (page 75, Theorem 1) for a more detailed proof. □

### 3.1.3 Feller Property of the Solution Process

Let $\{Y_t\}$ be the solution process of equation (2.3). Let $\{Y_t^y\}$ be the solution process with initial condition $y$. We want to show that the solution process is continuous in terms of initial value. Let functions $a$ and $b$ be the drift and diffusion coefficients as in (2.3). We assume the coefficients satisfy the growth condition in Theorem 2.3.1.

Define

$$Z_t^y = y + \int_0^t a(Z_s^y)ds + \int_0^t \int_{0<|u|<1} b(Z_s^y)uq(ds, du)$$

where compensated random variable $q$ is defined as in (2.4). We want to show that if $y_n \to y$, then $Z_t^{y_n} \xrightarrow{L^2} Z_t^y$ for $0 < t < T$. Now

$$\begin{aligned} |Z_t^{y_n} - Z_t^y|^2 \leq \; & 3|y_n - y|^2 + 3\left|\int_0^t \left[a(Z_s^{y_n}) - a(Z_s^y)\right]ds\right|^2 \\ & + 3\left|\int_0^t \int_{0<|u|<1} \left[b(Z_s^{y_n}) - b(Z_s^y)\right]uq(ds, du)\right|^2 \end{aligned}$$

If we write $\tilde{K}_\nu = \int_{0<|u|<1} u^2\nu(du)$, then using Cauchy-Schwarz inequality and Doob's martingale inequality in the right hand expression above, we get:

$$\begin{aligned} E\Big(\sup_{0<t<T} |Z_t^{y_n} - Z_t^y|^2\Big) \leq \; & 3|y_n - y|^2 + 3T\int_0^t E\Big(\sup_{0<t<T} |a(Z_s^{y_n}) - a(Z_s^y)|^2\Big)ds \\ & + 12\tilde{K}_\nu \int_0^t E\Big(\sup_{0<t<T} |b(Z_s^{y_n}) - b(Z_s^y)|^2\Big)ds \end{aligned}$$

By the growth condition

$$E\Big(\sup_{0<t<T} |Z_t^{y_n} - Z_t^y|^2\Big) \leq 3|y_n - y|^2 + C_T \int_0^t E\Big(\sup_{0<t<T} E\,|Z_s^{y_n} - Z_s^y|^2\Big) ds$$

where $C_T$ is a constant that depends on $T$ and constant $C'$ in growth condition given in theorem 2.3.1 only.

Hence by Gronwall's lemma we can say, if $y_n \to y$ then $E\Big(\sup_{0<t<T} |Z_t^{y_n} - Z_t^y|^2\Big) \to 0$.

Adding the large jump part to $\{Z_t\}$ by interlacing we can get $\{Y_t\}$ and thus the process $\{Y_t\}$ is continuous in terms of initial value.

## 3.2 Infinitesimal Generator and Backward Equation

Let $P(t, x, y)$ be the transition probability for the Markov process $Y_t$. That is, for any $A \in \mathcal{B}(\mathbb{R})$, ($\mathcal{B}(\mathbb{R})$ is Borel sigma field of $\mathbb{R}_0$), we have $P(t, x, A) = P_x\{Y_t \in A\}$. Here, $P_x(\cdot)$ should be interpreted as conditional probability with given condition $Y_0 = x$. The transition probabilities of a Markov process defines a transition semigroup; and the semigroup uniquely defines the infinitesimal generator. Let us start with the definition of a semigroup.

**Definition 3.2.1.** *A family $\{T_t; t \geq 0\}$ of bounded linear operators on a Banach space $B$ is called a strongly continuous semigroup if*

**(a)** $T_t T_s \;=\; T_{t+s}$ *for* $t, s \in [0, \infty)$

**(c)** $\lim_{t \downarrow 0} T_t f \;=\; f$ *for any* $f \in B$.

**Definition 3.2.2.** *The infinitesimal generator of a semigroup* $\{T_t\}$ *is defined by the formula*

$$Af = \lim_{h \downarrow 0} \frac{T_h f - f}{h} \;.$$

*Its domain* $\mathcal{D}_A$ *consist of all* $f \in B$ *for which the above limit exists.*

Let $\mathcal{C}$ be the space of bounded continuous functions on $\mathbb{R}$. For $f \in \mathcal{C}$ define the following operator:

$$T_t f(x) = \int f(y) P(t, x, dy) = E^x[f(Y_t)]$$

This infinitesimal generator is also called the infinitesimal generator of the Markov process $\{Y_t\}$.

In section 3.1 we already observed that the solution process $\{Y_t\}$ of the SDE in (3.1) is a Markov process. In this section we shall derive the infinitesimal generator of the solution process. We shall use the Itô formula for stochastic integrals with respect to Poisson random measure. The Itô formula for general Lévy type integrals is given by Ikeda and Watanabe [17]. Using their theorem we can state the Itô formula for our case as follows:

**Theorem 3.2.1.** *([17] Theorem 4.1, page 66): Let* $\{Y_t\}$ *is a pure jump Lévy type* *integral, i.e,*

$$Y(t). = Y(0) + \int_0^t G(s)ds + \int_0^t \int_{0<|x|<1} H(s,x)q(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)N(ds,dx)$$

*where $N$ is a random Poisson process on $\big([0,\infty)\times\mathbb{R}_0\big)$ with intensity measure $\tilde{\nu}$, with $H(t,\cdot) \in \bar{M}_q^{(2)}(\mathcal{F}_t)$ and $K$ predictable. Let $C^2(\mathbb{R})$ be the set of twice differentiable functions on $\mathbb{R}$. Then for any function $F \in C^2(\mathbb{R})$ following holds:*

$$\begin{aligned}
&F\Big(Y(t)\Big) - F\Big(Y(0)\Big) \\
&= \int_0^t F'\Big(Y(s)\Big)G(s)ds + \int_0^t \int_{|x|\geq 1} \Big\{F\Big(Y(s-) + K(s,x)\Big) - F\Big(Y(s-)\Big)\Big\}N(ds,dx) \\
&\quad + \int_0^t \int_{0<|x|<1} \Big\{F\Big(Y(s-) + H(s,x)\Big) - F\Big(Y(s-)\Big)\Big\}q(ds,dx) \\
&+ \int_0^t \int_{0<|x|<1} \Big\{F\Big(Y(s-) + H(s,x)\Big) - F\Big(Y(s-)\Big) - H(s,x)F'\Big(Y(s-)\Big)\Big\}\tilde{\nu}(ds,dx)
\end{aligned}$$

Let us assume the coefficient functions $a$ and $b$ satisfy growth condition and Lipschitz condition as in Theorem 2.3.1. Then a unique solution process $\{Y_t\}$ for the SDE in (3.1) exists and it is a Markov process. For $f \in C$ if we define $(T_t f)(x) = E^x f(Y_t)$, then the infinitesimal generator of $T_t$ exists.

Note that equation (3.1) or (2.3) has alternative form as a pure jump Lévy type

SDE as in equation (2.4), so we can express $Y_t$ as :

$$Y_t = Y_0 + \int_0^t a(Y_s)ds + \int_0^t \int_{|x|<1} b(Y_s)xq(ds,dx) + \int_0^t \int_{|x|\geq 1} b(Y_s)xN(ds,dx) \quad (3.2)$$

where $N$ is Poisson random measure with intensity measure $\tilde{\nu}(dt,dx) = dt\nu(dx)$. Equation (3.2) with Theorem 3.2.1 gives us the infinitesimal generator as follows:

**Theorem 3.2.2.** ***(Infinitesimal Generator)***: *Let $\{Y_t\}$ be the solution process of the stochastic differential equation: $dY_t = a(Y_t)dt + b(Y_t)dX_t$, where $\{X_t\}$ is a real valued pure jump Lévy process with generating triplet $(0,\nu,0)$. Also, suppose that the coefficient functions $a$ and $b$ satisfy the growth condition (A) and Lipschitz condition (B) as in Theorem 2.3.1.*

*If $C_\nu = \int_{|x|\geq 1} x\nu(dx) < \infty$, then for any function $f \in C_0^2(\mathbb{R})$ the infinitesimal generator $A$ of $\{Y_t\}$ is given by:*

$$Af(y) = f'(y)\Big(a(y) + C_\nu b(y)\Big) + \int_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x \Big\} \nu(dx) \quad (3.3a)$$

*alternatively, if $K_\nu = \int_{0<|x|<1} x\nu(dx) < \infty$, then for any function $f \in C_0^2(\mathbb{R})$ the infinitesimal generator $A$ of $\{Y_t\}$ is given by:*

$$Af(y) = f'(y)\Big(a(y) - K_\nu b(y)\Big) + \int_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) \Big\} \nu(dx) \quad (3.3b)$$

*Proof.* Let us assume $Y_0 = y$. Using Theorem 3.2.1 for any $f \in C_0^2(\mathbb{R})$,

$$
\begin{aligned}
& f(Y_t) - f(y) \\
&= \int_0^t f'(Y_s)a(Y_s)ds + \int_0^t \int_{|x|\geq 1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) \right\} N(ds, dx) \\
&+ \int_0^t \int_{0<|x|<1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) \right\} q(ds, dx) \\
&+ \int_0^t \int_{0<|x|<1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) - b(Y_s)xf'\Big(Y(s-)\Big) \right\} \tilde{\nu}(ds, dx)
\end{aligned}
$$

By definition of infinitesimal generator we have:

$$
\begin{aligned}
Af(y) &= \lim_{t \downarrow 0} \frac{E^y\left[f(Y_t)\right] - f(y)}{t} \\
&= \lim_{t \downarrow 0} \frac{1}{t} E^y \Bigg[ \int_0^t f'(Y_s)a(Y_s)ds \\
&\quad + \int_0^t \int_{|x|\geq 1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) \right\} N(ds, dx) \\
&\quad + \int_0^t \int_{0<|x|<1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) \right\} q(ds, dx) \\
&+ \int_0^t \int_{0<|x|<1} \left\{ f\Big(Y(s-) + b(Y_s)x\Big) - f\Big(Y(s-)\Big) - b(Y_s)xf'\Big(Y(s-)\Big) \right\} \tilde{\nu}(ds, dx) \Bigg] \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int_0^t E^y \Big[ f'(Y_s)a(Y_s) \Big] ds
\end{aligned}
$$

$$+\lim_{t\downarrow 0}\frac{1}{t}\int_0^t\int_{|x|\geq 1} E^y\Big[f\Big(Y(s-)+b(Y_s)x\Big)-f\Big(Y(s-)\Big)\Big]ds\nu(dx)$$

$$+\lim_{t\downarrow 0}\frac{1}{t}\int_0^t\int_{0<|x|<1} E^y\Big[f\Big(Y(s-)+b(Y_s)x\Big)-f\Big(Y(s-)\Big)-b(Y_s)xf'\Big(Y(s-)\Big)\Big]ds\nu(dx)$$

$$\begin{aligned} &= f'(y)a(y)+\int_{|x|\geq 1}\Big\{f\Big(y+b(y)x\Big)-f(y)\Big\}\nu(dx) \\ &\qquad +\int_{0<|x|<1}\Big\{f\Big(y+b(y)x\Big)-f(y)-b(y)xf'(y)\Big\}\nu(dx) \qquad (3.4)\end{aligned}$$

Now let us write

$$\begin{aligned} K_\nu &= \int_{0<|x|<1} x\nu(dx) \\ C_\nu &= \int_{|x|\geq 1} x\nu(dx).\end{aligned}$$

Then

$$Af(y)=\begin{cases} f'(y)\Big(a(y)-K_\nu b(y)\Big)+\int_{\mathbb{R}_0}\Big\{f\Big(y+b(y)x\Big)-f(y)\Big\}\nu(dx), \\ \hfill if\ K_\nu<\infty; \\ f'(y)\Big(a(y)+C_\nu b(y)\Big)+\int_{\mathbb{R}_0}\Big\{f\Big(y+b(y)x\Big)-f(y)-f'(y)b(y)x\Big\}\nu(dx), \\ \hfill if\ C_\nu<\infty.\end{cases}$$

This completes the proof. □

Remark:

**i)** In case $C_\nu$ and $K_\nu$ both are finite we can use either form because both forms will be equivalent.

**ii)** In case neither of $C_\nu$ and $K_\nu$ are finite we can use equation (3.4) form of the infinitesimal generator.

We can show, for the infinitesimal generator of any Markov process, in particular for the solution process of the SDE driven by pure jump Lévy process the following backward equation holds.

**Theorem 3.2.3.** *(**The backward equation**) : Let $A$ be the infinitesimal generator as in Theorem 3.2.2 for the solution process $\{Y_t\}$ of the SDE* (3.1). *Let $f \in C_0^2(\mathbb{R})$. Define, $u(y,t) = E^y\left[f(Y_t)\right]$. Then, $\frac{\partial u}{\partial t}$ exists and*

$$\frac{\partial u}{\partial t} = A(u) \tag{3.5}$$

*where the R.H.S is to be interpreted as $A$ applied to the function $y \mapsto u(y,t)$.*

*Proof.* Let $g(x) = u(x,t)$. Then, using Markov Property,

$$\begin{aligned}
\frac{E^y\left[g(Y_r)\right] - g(y)}{r} &= \frac{1}{r}\left\{E^y\left[E^{Y_r}\left(g(Y_t)\right)\right] - E^y\left[g(Y_t)\right]\right\} \\
&= \frac{1}{r}\left\{E^y\left[E^y\left(g(Y_{t+r})|\mathcal{F}_r\right)\right] - E^y\left[g(Y_t)\right]\right\} \\
&= \frac{1}{r}E^y\left[g(Y_{t+r}) - g(Y_t)\right] \\
&= \frac{u(y,t+r) - u(y,t)}{r} \to \frac{\partial u}{\partial t} \text{ as } r \downarrow 0.
\end{aligned}$$

Therefore, $A(u) = \lim_{r\downarrow 0} \frac{E^y[g(Y_r)] - g(y)}{r}$ exists, and $\frac{\partial u}{\partial t} = A(u)$. Hence the backward equation. $\square$

## 3.3 Pseudo Differential Operator Form of the Infinitesimal Generator and the Forward Equation

Using Fourier analysis we can show that the infinitesimal generator in (3.3) is a pseudo-differential operator (in sense of Jacob [18], definition 3.3.3) defined on the anistropic Sobolev space $H^{\xi^2,2}(\mathbb{R})$. Here we show that the transition probability density function of the solution process satisfies a deterministic differential equation *viz.* the *forward equation.* The forward equation can be derived from the backward equation using the infinitesimal generator in its pseudo differential operator form. This forward equation gives the governing equation of diffusive flows and thus validates the key role of jump Lévy SDE in stochastic modeling of anomalous diffusion. To derive the forward equation we assume the density function of the solution process belongs to the anistropic Sobolev space $H^{\xi^2,2}(\mathbb{R})$. For this section we are going to use the Fourier transform of a real function $f$ as follows:

$$\hat{f}(\xi) = F\big(f(\xi)\big) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx$$

We shall discuss other required concepts and definitions as we proceed.

**Lemma 3.3.1.** *Let $\{Y_t\}$ be the solution process of the SDE in* (3.1). *Let $A$ be the infinitesimal generator for $\{Y_t\}$, given as in Theorem 3.2.2. Let us make a change* *of variable* $b(y)x = -v$ *in* (3.3). *Let* $J(v)$ *be the Jacobian of the transformation. We define* $\nu_1(y,dv) = J(v)\nu(\frac{-dv}{b(y)})$. *Then* $A$ *can be expressed as follows:*

$$Af(y) = B(y)f'(y) + \int_{\mathbb{R}_0} \left[f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2}\right]\nu_1(y,dv) \qquad (3.6)$$

*where* $B(y) = \begin{cases} \left[a(y) + C_\nu b(y) + \int_{\mathbb{R}_0} v(\frac{|v|^2}{1+|v|^2})\nu_1(y,dv)\right], & if \quad C_\nu < \infty; \\ \left[a(y) - K_\nu b(y) - \int_{\mathbb{R}_0} (\frac{v}{1+|v|^2})\nu_1(y,dv)\right], & if \quad K_\nu < \infty. \end{cases}$

*Proof.* From Theorem 3.2.2, if we consider form (3.3a), the infinitesimal generator can be written as:

$$Af(y) = f'(y)\Big(a(y) + C_\nu b(y)\Big) + \int_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x\Big\}\nu(dx)$$

Let us consider the change of variable $b(y)x = -v$. Letting $J(v)$ be the Jacobian of this transformation we define : $\nu_1(y,dv) = J(v)\nu\left(\frac{-dv}{b(y)}\right)$. Then

$$\begin{aligned} Af(y) &= f'(y)\Big(a(y) + C_\nu b(y)\Big) + \int_{\mathbb{R}_0} \Big\{f(y-v) - f(y) + f'(y)v\Big\}\nu_1(y,dv) \\ &= f'(y)\Big[a(y) + C_\nu b(y) + \int_{\mathbb{R}_0} v\Big(\frac{|v|^2}{1+|v|^2}\Big)\nu_1(y,dv)\Big] \\ &\quad + \int_{\mathbb{R}_0} \Big\{f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2}\Big\}\nu_1(y,dv) \\ &= B(y)f'(y) + \int_{\mathbb{R}_0} \left[f(y-v) - f(y) + f'(y)\frac{v}{1+|v|^2}\right]\nu_1(y,dv) \end{aligned}$$

where $B(y) = \left[a(y) + C_\nu b(y) + \int\limits_{\mathbb{R}_0} v\left(\frac{|v|^2}{1+|v|^2}\right)\nu_1(y, dv)\right]$.

From Theorem 3.2.2, if we consider form (3.3b), the infinitesimal generator can be written as

$$Af(y) = f'(y)\Big(a(y) - K_\nu b(y)\Big) + \int\limits_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y)\Big\}\nu(dx)$$

Again change of variable $b(y)x = -v$ gives,

$$\begin{aligned} Af(y) &= f'(y)\Big(a(y) - K_\nu b(y)\Big) + \int\limits_{\mathbb{R}_0} \Big\{f(y - v) - f(y)\Big\}\nu_1(y, dv) \\ &= B(y)f'(y) + \int\limits_{\mathbb{R}_0} \left[f(y - v) - f(y) + f'(y)\frac{v}{1 + |v|^2}\right]\nu_1(y, dv) \end{aligned}$$

where $B(y) = \left[a(y) - K_\nu b(y) - \int\limits_{\mathbb{R}_0} \left(\frac{v}{1+|v|^2}\right)\nu_1(y, dv)\right]$.

Hence the lemma is proved. □

**Proposition 3.3.1.** *(Lévy Khinchin representation, see [18])*

*We say $\psi : \mathbb{R} \to \mathbb{C}$ is a continuous negative definite function, if $\psi$ has the following representation:*

$$\psi(\xi) \;=\; c + di\xi + \tilde{q}(\xi) + \int\limits_{\mathbb{R}_0} \left(1 - e^{-ix\xi} - \frac{ix\xi}{1 + |x|^2}\right)\nu(dx) \tag{3.7}$$

*with $c > 0$, $d \in \mathbb{R}$, $\tilde{q}$ is symmetric positive semidefinite quadratic form on $\mathbb{R}$ and $\nu$*

*is the Lévy measure associated with $\psi$ such that,*

$$\int_{\mathbb{R}_0} (|x|^2 \wedge 1)\nu(dx) < \infty$$

*and $\psi$ is uniquely determined by $(c, d, \tilde{q}, \nu)$.*

**Definition 3.3.1.** *We call a function $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ a continuous negative definite symbol if $Q$ is locally bounded and for each $x \in \mathbb{R}$ the function $Q(x, .) : \mathbb{R} \to \mathbb{C}$ is continuous negative definite.*

**Definition 3.3.2.** *We define the Schwartz space $\mathcal{S}(\mathbb{R})$ as all functions $u \in C^{\infty}(\mathbb{R})$ such that for all $m_1,\ m_2\ \in \mathbb{N}_0$*

$$p_{m_1,m_2}(u) := \sup_{x \in \mathbb{R}} \left[ (1 + |x|^2)^{\frac{m_1}{2}} \sum_{k \leq m_2} |\partial^k u(x)| \right] \ < \ \infty$$

The pseudo-differential operator associated with the symbol $Q(x, \xi)$ are defined as follows:

**Definition 3.3.3.** *For a continuous negative definite symbol $Q(x, \xi)$, we define the pseudo-differential operator $Q(x, D)$ by :*

$$Q(x, D)u(x) \ := \ (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} Q(x, \xi)\hat{u}(\xi)d\xi \tag{3.8}$$

*for $u \in \mathcal{S}(\mathbb{R})$.*

The next theorem gives the pseudo-differential operator representation of the infinitesimal generator. Consider an SDE of form (3.1). For the coefficient functions $a$ and $b$ let us define

$$M(x) := \max\{|a(x)|, |b(x)|^2\} \tag{3.9}$$

**Theorem 3.3.1.** *Let us use the measure $\nu_1$ and coefficient function $B$ from Lemma 3.3.1 to define following continuous negative definite symbol:*

$$Q(x,\xi) \;=\; \int_{\mathbb{R}_0} \left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi \tag{3.10}$$

*Let* A *be the infinitesimal generator defined in section 3.2. Then the infinitesimal generator, restricted in $\mathcal{S}(\mathbb{R})$ is a pseudo-differential operator with negative definite symbol $Q$ as above. That is :*

$$Af(x) = -Q(x,D)f(x) \;=\; -(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} Q(x,\xi)\hat{f}(\xi)d\xi \tag{3.11}$$

*where $\hat{f}$ is the Fourier transformation of $f \in \mathcal{S}(\mathbb{R})$.*

*Proof.* To prove this theorem we need the following bound of symbol (3.10).

**Lemma 3.3.2.** *Let the function $M(\cdot)$ be defined as* (3.9)*. For the continuous negative definite symbol $Q(x,\xi)$ given in* (3.10)*, then for some constant $c$*

$$|Q(x,\xi)| \le cM(x)(1+\xi^2). \tag{3.12}$$

*Proof.* Consider $Q(x,\xi)$ given in (3.10).

First, consider the form $B(x) = a(x) + C_\nu b(x) + \int\limits_{\mathbb{R}_0} v\frac{|v|^2}{1+|v|^2}\nu_1(y,dv)$

$$Q(x,\xi) = \int\limits_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi$$
$$= \int\limits_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - i\frac{v}{1+|v|^2}\xi - iv\frac{|v|^2}{1+|v|^2}\xi\right)\nu_1(x,dv) - i\Big[a(x) + C_\nu b(x)\Big]\xi$$
$$= \int\limits_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - iv\xi\right)\nu_1(x,dv) - i\Big[a(x) + C_\nu b(x)\Big]\xi$$

Using reverse transformation $b(x)y = -v$

$$Q(x,\xi) = \int\limits_{\mathbb{R}_0}\left(1 - e^{ib(x)y\xi} + ib(x)y\xi\right)\nu(dy) - i\Big[a(x) + C_\nu b(x)\Big]\xi$$
$$= \int\limits_{\mathbb{R}_0}\left(1 - e^{ib(x)y\xi} + ib(x)y\xi\right)\nu(dy) - i\xi\Big[a(x) + b(x)\int\limits_{|y|\geq 1} y\nu(dy)\Big]$$
$$= \int\limits_{0<|y|<1}\left(1 - e^{ib(x)y\xi} + ib(x)y\xi\right)\nu(dy) + \int\limits_{|y|\geq 1}\left(1 - e^{ib(x)y\xi}\right)\nu(dy) - i\xi a(x)$$

Next, consider the form $B(x) = a(x) - K_\nu b(x) - \int\limits_{\mathbb{R}_0} \frac{v}{1+|v|^2}\nu_1(y,dv)$.

$$Q(x,\xi) = \int\limits_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi$$
$$= \int\limits_{\mathbb{R}_0}\left(1 - e^{-iv\xi}\right)\nu_1(x,dv) - i\Big[a(x) - K_\nu b(x)\Big]\xi$$

Using reverse transformation $-b(x)y = v$ as the previous case,

$$Q(x,\xi) = \int_{\mathbb{R}_0} \left(1 - e^{ib(x)y\xi}\right)\nu(dy) - i\left[a(x) - K_\nu b(x)\right]\xi$$

$$= \int_{0<|y|<1} \left(1 - e^{ib(x)y\xi} + ib(x)y\xi\right)\nu(dy) + \int_{|y|\geq 1} \left(1 - e^{ib(x)y\xi}\right)\nu(dy) - i\xi a(x)$$

Thus, for both forms of $B(\cdot)$ we can write:

$$Q(x,\xi) = \int_{0<|y|<1} \left(1 - e^{ib(x)y\xi} + ib(x)y\xi\right)\nu(dy) + \int_{|y|\geq 1} \left(1 - e^{ib(x)y\xi}\right)\nu(dy) - i\xi a(x) \tag{3.13}$$

Further, note that we can get the following bounds :

$$\left|1 - e^{ib(x)y\xi} + ib(x)y\xi\right| \leq |b(x)\xi y|^2 \textit{ and } |1 - e^{ib(x)y\xi}| \leq 2$$

also, $|\xi| \leq (1 + |\xi|^2)$. Recall that $\nu$, the Lévy measure for $Q(x, D)$ in (3.10), satisfies $\int_{0<|y|<1} |y|^2\nu(dy) < \infty$. Therefore, since $\int_{|y|\geq 1} \nu(dy) < \infty$ we have the following:

$$\begin{aligned} |Q(x,\xi)| &\leq |\xi|^2|b(x)|^2 \int_{0<|y|<1} |y|^2\nu(dy) + \int_{|y|\geq 1} 2\nu(dy) + |a(x)||\xi| \\ &\leq cM(x)(1 + |\xi|^2) \end{aligned}$$

where $M(x) = \max\{|b(x)|^2, a(x)\}$ and $c$ is a constant that depends only on $\nu$. $\square$

**Proof of Theorem 3.3.1 continued**

We shall use the notation $\hat{f} = F(f)$ to denote Fourier transform as defined in the beginning of this section. Also recall that, the inverse Fourier transform of a function $g$ is given by $F^{-1}g(\eta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix\eta} g(x)dx$ and that $F(g^{(n)})(k) = (ik)^n \hat{g}$ where $g^{(n)}$ is the $n^{th}$-derivative of $g$, $n \in \mathbb{N}$ .

Using Lemma 3.3.2 we can say for $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}\left|\int e^{ix\xi} Q(x,\xi)\hat{f}(\xi)d\xi\right| &\leq cM(x)\int (1+\xi^2)|\hat{f}(\xi)|d\xi \\ &\leq cM(x)\int |f(z)| + |f''(z)|dz\end{aligned}$$

Since the functions in $\mathcal{S}(\mathbb{R})$ are rapidly decreasing, it is easy to see that $f,\ f'' \in L^1(\mathbb{R})$ (see the norm used in Definition 3.3.2). This justifies the use of the Fubini theorem the next steps of the proof. By Lemma 3.3.1,

$$\begin{aligned}
-Q(x,D)f &= -(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} Q(x,\xi)\hat{f}(\xi)d\xi \\
&= -(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} \left[\int_{\mathbb{R}_0} \left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi\right]\hat{f}(\xi)d\xi \\
&= -\int_{\mathbb{R}_0} \left[(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} F\left(f(\xi) - f(\xi - v) - \frac{v}{1+|v|^2} f'(\xi)\right)d\xi\right]\nu_1(x,dv) \\
&\qquad + B(x)\left[(2\pi)^{-\frac{1}{2}} \int e^{ix\xi} F\Big(f'(\xi)\Big)d\xi\right] \\
&= -\int_{\mathbb{R}_0} \left[f(x) - f(x-v) - f'(x)\frac{v}{1+|v|^2}\right]\nu_1(x,dv) \ + \ B(x)f'(x) \\
&= Af(x)
\end{aligned}$$

Hence, Theorem 3.3.1 is proved. □

Next,the we shall discuss spaces associated with the pseudo differential operators. For a real valued continuous negative definite function $\psi(\xi)$ and $s \geq 0$ let us define a norm

$$\|u\|_{\psi,s}^2 := \int_{\mathbb{R}} [1+\psi(\xi)]^s |\hat{u}(\xi)|^2 d\xi$$

An anistropic Sobolev space with a negative definite function $\psi$ is given by

$$H^{\psi,s}(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : \|u\|_{\psi,s}^2 < \infty\}$$

These are Hilbert spaces under norm $\|u\|_{\psi,s}^2$ and arise naturally in the discussion of the pseudo-differential operators (See [16, 19]). Jacob and Schilling [19] showed that with appropriate choice of $\psi$, a pseudo-differential $Q(x, D)$ operator associated with the generator of a Lévy type processes maps the space $H^{\psi,s+2}$ to $H^{\psi,s}$ and hence using Sobolev's embedding theorem, $Q(x, D)$ can be extended to $C_\infty(\mathbb{R})$. Thus we can say the pseudo differential operator representation of infinitesimal generator $A$ can be extended to $C_\infty(\mathbb{R})$.

In our case we shall use the Sobolev space $H^{\xi^2,2}$. In most cases, the density of a Lévy type processes belongs to this class, for example use the stable characteristic function to see that the stable-Lévy density belongs to $H^{\xi^2,2}$.

**Theorem 3.3.2.** ***(The forward equation)***

*Let $\{Y_t\}$ be the solution process for the SDE in* (2.3). *Let us further assume that* *there exists a transition probability density for $Y_t$ and that $p_x(t,y)$ is the density of $Y_t$, given $Y_0 = x$. We assume $p_x(t,\cdot)$ is in $H^{\xi^2,2}$ for all $x \in \mathbb{R}$.*

*Let us make a change of variable $b(y)x = -v$ in (3.3). Let $J(v)$ be the Jacobian of the transformation. We define $\nu_1(y,dv) = J(v)\nu(\frac{-dv}{b(y)})$. We consider the case when the measure $\nu_1$ is of the form $\nu_1(x,dy) = h(x)\mu(dy)$, where $h$ is a measurable function on $\mathbb{R}$ and $\mu$ is a measure on $\mathbb{R}$.*

*For coefficient functions $a$ and $b$ in the SDE (2.3) define $M(x) := \max\{|a(x)|, |b(x)|^2\}$. Let us assume the coefficient functions are such that*

$$\int_{\mathbb{R}} M^2(x)dx < \infty \tag{3.14}$$

*Then the transition probability density function satisfies the following equation: if $C_\nu < \infty$,*

$$\begin{aligned}\frac{\partial}{\partial s}p_x(s,y) &= \int\limits_{\mathbb{R}_0} \Big[(p_x\cdot h)(s,y-r) - (p_x\cdot h)(s,y) + r(p_x\cdot h)'(s,y)\Big]\mu(d(-r)) \\ &\quad - \frac{\partial}{\partial y}(p_x\cdot G)(s,y) \end{aligned}\tag{3.15a}$$

*where $(p_x\cdot h)(s,y) = h(y)p_x(s,y)$ and $G(y) = a(y) + C_\nu b(y)$.*

*Alternatively, if $K_\nu < \infty$,*

$$\frac{\partial}{\partial s}p_x(s,y) = \int\limits_{\mathbb{R}_0} \Big[(p_x\cdot h)(s,y-r) - (p_x\cdot h)(s,y)\Big]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x\cdot H)(s,y) \tag{3.15b}$$

*where* $(p_x \cdot h)(s,y) = h(y)p_x(s,y)$ *and* $H(y) = a(y) - K_\nu b(y)$.

*Proof.* Let $p_x(t,y)$ be the transition probability density of $Y_t$ starting at $Y_0 = x$. Let $u_0 \in L^2(\mathbb{R})$ be a twice differentiable bounded function. Let us write

$$u(x,t) = E^x[u_0(Y_t)] = \int_{\mathbb{R}} u_0(y)p_x(t,y)dy.$$

Then $Au(x,t) = \frac{\partial}{\partial t}u(x,t)$ is defined. Since $u_0 \in L^2(\mathbb{R})$ we can have a constant $c'$ so that following holds:

$$\begin{aligned}
u(x,t) &= Eu_0(Y_t + x) = \int_{\mathbb{R}} u_0(y+x)p_0(t,y)dy \\
&= \int_{\mathbb{R}} u_0(z)p_0(t,z-x)dz \\
\Rightarrow \hat{u}(\xi,t) &= \int_{\mathbb{R}} u_0(z)\hat{p}_0(t,z-\xi)dz \\
&= \int_{\mathbb{R}} u_0(z)e^{-iz\xi}\hat{p}_0(t,-\xi)dz \\
\Rightarrow |\hat{u}(\xi,t)|^2 &\leq \int_{\mathbb{R}} |u_0(z)|^2|\hat{p}_0(t,-\xi)|^2dz \\
\Rightarrow |\hat{u}(\xi,t)|^2 &\leq c'|\hat{p_0}(t,\xi)|^2 \qquad (3.16)
\end{aligned}$$

If we assume the transition density function vanishes at $t = \infty$, then integration by parts gives

$$\int_{-\infty}^{\infty}\int_0^{\infty} \frac{\partial}{\partial t}u(y,t)p_x(t,y)dtdy = -\int_{-\infty}^{\infty}\int_0^{\infty} u(y,t)\frac{\partial}{\partial t}p_x(t,y)dtdy \qquad (3.17)$$

Now substituting the backward equation in the left hand side we get,

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} p_x(t,y)A\Big(u(y,t)\Big)dtdy \;=\; -\int_{-\infty}^{\infty}\int_{0}^{\infty} u(y,t)\frac{\partial}{\partial t}p_x(t,y)dtdy \qquad (3.18)$$

The operator $A$ acts on *u(y,t)* as $u$ being a functions of $y$. Consider the integration part with respect to $y$ in the left hand side of (3.18) and to simplify the notations we ignore the other variables in the term for next steps of computation and write:

$$\int_{-\infty}^{\infty} p_x(t,y)A\Big(u(y,t)\Big)dy = \int p(y)Au(y)dy$$

Using the Cauchy-Schwarz inequality,

$$\left|\int p(y)Au(y)dy\right|^2 \le \int |p(y)|^2dy \int |Au(y)|^2dy$$

The Parseval identity gives

$$\int |p(y)|^2dy \;=\; \int |\hat{p}(\xi)|^2d\xi \;\le\; \int (1+\xi^2)|\hat{p}(\xi)|^2d\xi \;=\; \|p\|^2_{\xi^2,2}$$

Then the pseudo-differential operator form of the generator, equation (3.16) and Lemma 3.3.2 gives

$$\begin{aligned}\int |Au(y)|^2dy \;&=\; (2\pi)^{-1}\int\left|\int e^{iy\xi}Q(y,\xi)\hat{u}(\xi)d\xi\right|^2 dy\\ &\le\; (2\pi)^{-1}\int\int |Q(y,\xi)|^2|\hat{u}(\xi)|^2d\xi dy\end{aligned}$$

$$\begin{aligned}
&\le \int M^2(y)\int (1+\xi^2)|\hat{u}(\xi)|^2 d\xi dy \\
&\le c'\|p\|^2_{\xi^2,2}\int M^2(y)dy
\end{aligned}$$

Now by hypothesis of the theorem $\int M^2(y)dy < \infty$, therefore since $p_x \in H^{\xi^2,2}$ $\forall x \in \mathbb{R}$, we have $\left|\int p(y)Au(y)dy\right|^2 < \infty$. Thus we can apply the Fubini theorem for to (3.18).

Also for the negative definite symbol $Q$ given in (3.10) the following holds:

$$Q(x,\xi) = \int_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - \frac{iv\xi}{1+|v|^2}\right)\nu_1(x,dv) - iB(x)\xi$$

If we use $B(x) = a(x) + C_\nu b(x) + \int_{\mathbb{R}_0} v\frac{v^2}{1+v^2}\nu_1(x,dv)$ then

$$Q(x,\xi) = \int_{\mathbb{R}_0}(1 - e^{-iv\xi} - iv\xi)\nu_1(x,dv) - i\xi\Big[a(x) + C_\nu b(x)\Big]$$

we write

$$Q(x,\xi) = \int_{\mathbb{R}_0}\left(1 - e^{-iv\xi} - iv\xi\right)\nu_1(x,dv) - iG(x)\xi \qquad (3.19a)$$

where $G(x) = \Big[a(x) + C_\nu b(x)\Big]$.

If we use $B(x) = a(x) - K_\nu b(x) - \int_{\mathbb{R}_0} \frac{v}{1+v^2}\nu_1(x,dv)$ then

$$Q(x,\xi) = \int_{\mathbb{R}_0} (1 - e^{-iv\xi})\nu_1(x, dv) - i\xi\Big[a(x) - K_\nu b(x)\Big]$$

we write

$$Q(x,\xi) \;=\; \int_{\mathbb{R}_0} \big(1 - e^{-iv\xi}\big)\nu_1(x, dv) - iH(x)\xi \qquad (3.19b)$$

where $H(x) = \Big[a(x) - K_\nu b(x)\Big]$.

In order to get a closed form of the forward equation, we need to assume $\nu_1$ has the form $\nu_1(x, dy) = h(x)\mu(dy)$, where $h$ is real valued function and $\mu$ is a measure on $\mathbb{R}$. Using the pseudo differential representation of $A$ and Fubini theorem we get:

$$\begin{aligned}
\int_{-\infty}^{\infty} p(x)Au(x)dx &= -(2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi} Q(x,\xi)\hat{u}(\xi)p(x)d\xi dx \\
&= -(2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi} Q(x,\xi)u(x')p(x)d\xi dx dx' \\
&= \int_{-\infty}^{\infty} \left[-(2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi} Q(x,\xi)p(x)d\xi dx\right] u(x')dx' \\
&= \int_{-\infty}^{\infty} Ip(x')u(x')dx' \qquad (3.20)
\end{aligned}$$

$$\text{where } Ip(x') = \left[-(2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi} Q(x,\xi)p(x)d\xi dx\right] \qquad (3.21)$$

Using $Q(x,\xi)$ from (3.19a)

$$
\begin{aligned}
&Ip(x') \\
&= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}\left[\int_{\mathbb{R}_0}\left(1-e^{-iv\xi}-iv\xi\right)\nu_1(x,dv)-iG(x)\xi\right]p(x)d\xi dx \\
&= I_1 + I_2
\end{aligned}
$$

$$
\begin{aligned}
\text{where } I_1 &= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}\left(1-e^{-iv\xi}-iv\xi\right)p(x)\nu_1(x,dv)d\xi dx \\
I_2 &= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ix'\xi}iG(x)\xi p(x)d\xi.
\end{aligned}
$$

For the first part we have,

$$
\begin{aligned}
I_1 &= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}\left(1-e^{-iy\xi}-iy\xi\right)p(x)\nu_1(x,dy)d\xi dx \\
&= -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}\left(1-e^{-iy\xi}-iy\xi\right)p(x)h(x)\mu(dy)d\xi dx \\
&= -(2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{-ix'\xi}\left(1-e^{-iy\xi}-iy\xi\right)F\Big((p\cdot h)(-\xi)\Big)\mu(dy)d\xi \\
&= -(2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix'\xi}\left(1-e^{iy\xi}+iy\xi\right)F\Big((p\cdot h)(\xi)\Big)\mu(dy)d\xi \\
&= -(2\pi)^{-1/2}\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} e^{ix'\xi}F\Big[(p\cdot h)(\xi)-(p\cdot h)(\xi+y)+y(p\cdot h)'(\xi)\Big]d\xi\mu(dy)
\end{aligned}
$$

$$= -\int_{\mathbb{R}_0} F^{-1}\circ F\Big[(p\cdot h)(x') - (p\cdot h)(x'+y) + y(p\cdot h)'(x')\Big]\mu(dy)$$
$$= \int_{\mathbb{R}_0} \Big[(p\cdot h)(x'+y) - (p\cdot h)(x') - y(p\cdot h)'(x')\Big]\mu(dy)$$
$$= -\int_{\mathbb{R}_0} \Big[(p\cdot h)(x'-y) - (p\cdot h)(x') + y(p\cdot h)'(x')\Big]\mu(d(-y))$$

For the other part we have,

$$\begin{aligned}
I_2 &= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix'\xi}e^{ix\xi}i\xi G(x)p(x)d\xi dx \\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ix'\xi}i\xi F\Big[(p\cdot G)(-\xi)\Big]d\xi \\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ix'\xi'}F\Big[(p\cdot G)'(\xi')\Big]d\xi' \\
&= F^{-1}\circ F(p\cdot G)'(x') \;=\; (p\cdot G)'(x')
\end{aligned}$$

Thus, in case we use the negative definite symbol from (3.19a), we can write

$$Ip(x') = -\left[\int_{\mathbb{R}_0}\Big[(p\cdot h)(x'-y) - (p\cdot h)(x') + y(p\cdot h)'(x')\Big] - (G\cdot p)'(x')\right] \tag{3.22a}$$

Using $Q(x,\xi)$ of the from (3.19b) in (3.22) $Ip(x') = I_3 + I_4$. Where

$$I_3 = -(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{R}_0} e^{ix\xi}e^{-ix'\xi}(1-e^{-iv\xi})p(x)\nu_1(x,dv)d\xi dx$$

$$I_4 = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{-ix'\xi} iH(x)\xi p(x) d\xi dx.$$

Here for the first part we have,

$$\begin{aligned}
I_3 &= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{-ix'\xi}(1 - e^{-iy\xi}) F\Big((p \cdot h)(-\xi)\Big) \mu(dy) d\xi \\
&= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{R}_0} e^{ix'\xi}(1 - e^{iy\xi}) F\Big((p \cdot h)(\xi)\Big) \mu(dy) d\xi \\
&= -(2\pi)^{-1/2} \int_{\mathbb{R}_0} \int_{-\infty}^{\infty} e^{ix'\xi} F\Big[(p \cdot h)(\xi) - (p \cdot h)(\xi + y)\Big] d\xi \mu(dy) \\
&= -\int_{\mathbb{R}_0} F^{-1} \circ F\Big[(p \cdot h)(x') - (p \cdot h)(x' + y)\Big] \mu(dy) \\
&= \int_{\mathbb{R}_0} \Big[(p \cdot h)(x' + y) - (p \cdot h)(x')\Big] \mu(dy) \\
&= -\int_{\mathbb{R}_0} \Big[(p \cdot h)(x' - y) - (p \cdot h)(x')\Big] \mu(d(-y))
\end{aligned}$$

For the second integration term,

$$\begin{aligned}
I_4 &= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix'\xi} e^{ix\xi} i\xi H(x) p(x) d\xi dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix'\xi} i\xi F\Big[(p \cdot H)(-\xi)\Big] d\xi \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix'\xi'} F\Big[(p \cdot H)'(\xi')\Big] d\xi' \\
&= F^{-1} \circ F(p \cdot H)'(x') \;=\; (p \cdot H)'(x')
\end{aligned}$$

Therefore, in case we use the negative definite symbol from (3.19b), we can write:

$$Ip(x') = -\left[\int_{\mathbb{R}_0} \Big[(p\cdot h)(x'-y) - (p\cdot h)(x')\Big]\mu(d(-y)) - (H\cdot p)'(x')\right] \qquad (3.22\text{b})$$

Then from (3.20) and (3.18) we have,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{0}^{\infty} u(y,t) Ip_x(t,y)\,dt\,dy \\
= \;& -\int_{-\infty}^{\infty}\int_{0}^{\infty} u(y,t)\frac{\partial}{\partial t}p_x(t,y)\,dt\,dy \\
\Rightarrow \;& \int_{-\infty}^{\infty}\int_{0}^{\infty}\left[Ip_x(t,y) + \frac{\partial}{\partial t}p_x(t,y)\right]u(y,t)\,dt\,dy \\
= \;& 0
\end{aligned}$$

Note that, this is true for any arbitrary choice of twice differentiable bounded $u_0(\cdot) \in L^2(\mathbb{R})$. Hence we must have $[Ip_x(t,y) + \frac{\partial}{\partial t}p_x(t,y)] = 0$, thus $\frac{\partial}{\partial t}p_x(t,y) = -Ip_x(t,y)$. Then, combining (3.22a) and (3.22b), the forward equation is given as follows:

$$\frac{\partial}{\partial s}p_x(s,y) = \int_{\mathbb{R}_0}\Big[(p_x\cdot h)(s,y-r) - (p_x\cdot h)(s,y) + r(p_x\cdot h)'(s,y)\Big]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x\cdot G)(s,y) \qquad (3.23\text{a})$$

where $(p_x\cdot h)(s,y) = h(y)p_x(s,y)$ and $G(y) = a(y) + C_\nu b(y)$ ; in case $C_\nu < \infty$.

Alternatively,

$$\frac{\partial}{\partial s}p_x(s,y) = \int\limits_{\mathbb{R}_0} \Big[(p_x \cdot h)(s,y-r) - (p_x \cdot h)(s,y)\Big]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x \cdot H)(s,y) \tag{3.23b}$$

where $(p_x \cdot h)(s,y) = h(y)p_x(s,y)$ and $H(y) = a(y) - K_\nu b(y)$ ; in case $K_\nu < \infty$.

That concludes the theorem. $\square$

**Remark:** The forward equation theorem can be used to solve an interesting analytical problem. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distribution which is the dual space of $\mathcal{S}(\mathbb{R})$. If we assume $u(y,t)$ in $\mathcal{S}(\mathbb{R})$, then $Au(y,t) = -Q(y,D)u$ belongs to $\mathcal{S}(\mathbb{R})$. The solution of the forward equation actually produces an element in $\mathcal{S}'(\mathbb{R})$, which is the solution of the adjoint operator (forward operator) given by the transition function.

# Chapter 4

# Application to a Special Case : Diffusion Driven by an $\alpha$-stable Lévy Process

In this section we shall discuss a special case of SDE driven by a pure jump Lévy process, *viz.* an SDE driven by a stable Lévy process. Generally, the fractional advection dispersion or fADE is used to model a variety of anomalous diffusion processes, where observation shows that the plume spreads away from its center of mass faster than $(t^{1/2})$ scaling implied by the Brownian motion model. This is called super diffusion. A diffusion equation driven by an $\alpha$-Stable Lévy noise can be applied to these situations. Here we show that the transition density of the solution process of a stochastic differential equation driven by an $\alpha$-Stable Lévy process solves the fADE equation.

**Definition 4.0.4.** *A random variable $X$ is said to have a stable distribution with index of stability $\alpha$, scale parameter $\sigma$, skewness parameter $\beta$ and the shift parameter $\mu$ if its characteristic function has the following form:*

$$E[\exp i\theta X] = \begin{cases} \exp\Big\{-\sigma^{\alpha}|\theta|^{\alpha}\big(1 - i\beta(sign\ \theta)\tan(\frac{\pi\alpha}{2})\big) + i\mu\theta\Big\}, & if \quad \alpha \neq 1 \\ \exp\Big\{-\sigma|\theta|\big(1 + i\beta\frac{2}{\pi}(sign\ \theta)\ln|\theta|\big) + i\mu\theta\Big\}, & if \quad \alpha = 1 \end{cases}$$

*where* $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, $\mu \in \mathbb{R}$ *and*

$$sign\ \theta = \begin{cases} 1 & ,if \quad , \theta > 0, \\ 0 & ,if \quad , \theta = 0, \\ -1 & ,if \quad , \theta < 0. \end{cases}$$

*We write* $X \sim S_{\alpha}(\sigma, \beta, \mu)$.

**Definition 4.0.5.** $\{X_t\}$ *is $\alpha$-Stable Lévy Process if*

1. *X(0)= 0 a.s.*

2. *X has independent increments.*

3. *X(t)-X(s)*$\sim S_{\alpha}\big((t-s)^{1/\alpha}, \beta, 0\big)$ *; for* $0 \leq s < t < \infty$ *and* $-1 \leq \beta \leq 1$.

## 4.1 Existence and Uniqueness

Consider the SDE

$$dY_t = a(Y_t)dt + b(Y_t)dX_t \tag{4.1}$$

here $X_t$ is a standard, centered stable Lévy process with the index of stability $\alpha$, $(0 < \alpha < 2)$ and the skewness parameter $\beta$, $(-1 \leq \beta \leq 1)$. That is, $\{X_t\}$ is a Lévy process with $(X_t - X_s) \sim S_\alpha\Big((t-s)^{1/\alpha}, \beta, 0\Big)$.

Since $X_t$ is a pure jump Lévy process, from Theorem 2.3.1 and from section 3.1 we get the following:

**Proposition 4.1.1.** *Suppose the coefficient functions a and b satisfy the growth condition and Lipschitz condition as in section 3, i.e*

**(A)** *Growth condition: there exists constant $C > 0$ such that $\forall$ $y$ in $\mathbb{R}$*

$$|a(y)|^2 + |b(y)|^2 \leq C\left(1 + |y|^2\right)$$

**(B)** *Lipschitz condition: there exists constant $C' > 0$ such that $\forall$ $y_1$, $y_2$ in $\mathbb{R}$,*

$$|a(y_1) - a(y_2)|^2 + |b(y_1) - b(y_2)|^2 \leq C'\left(|y_1 - y_2|^2\right)$$

*Then there exists a unique stochastic process $\{Y_t\}$ that satisfies the stochastic differential equation (4.1). Also, $\{Y_t\}$ is a time homogeneous Markov process.*

## 4.2 Infinitesimal Generator, Backward and Forward Equations

Since $\{Y_t\}$ is a Markov process we can get the corresponding infinitesimal generator and associated forward and backward equation. For an SDE driven by an $\alpha$-stable Lévy process, the infinitesimal generator of the solution process can be expressed in terms of fractional derivatives of order $\alpha$. This forward equation given in terms of fractional derivatives of order $\alpha$ is used in hydrology to model ground water flows. First let us define the fractional derivative of order $\alpha$.

**Definition 4.2.1.** *The fractional derivative of order $\alpha$ for a function $f$ is derived by solving inverse Fourier transform. Let $g(x) = \frac{\partial^\alpha}{\partial(\pm x)^\alpha} f(x)$, then $\hat{g}(\xi) = (\pm i\xi)^\alpha \hat{f}(\xi)$. Hence, $g(x) := F^{-1}\left[(\pm i\xi)^\alpha \hat{f}(\xi)\right]$.*

**Proposition 4.2.1.** *(see [3] for details.)*

*The fractional derivative of order $\alpha$ for a function $f$ can be expressed as follows:*

*(a) For $0 < \alpha < 1$,*

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x-v) - f(x)\Big) \frac{dv}{|v|^{1+\alpha}} \tag{4.2}$$

$$\frac{\partial^\alpha}{\partial(-x)^\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^0 \Big(f(x-v) - f(x)\Big) \frac{dv}{|v|^{1+\alpha}} \tag{4.3}$$

*(b) For* $1 < \alpha < 2$,

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \Big( f(x-v) - f(x) + v f'(x) \Big) \frac{dv}{|v|^{1+\alpha}} \tag{4.4}$$

$$\frac{\partial^\alpha}{\partial (-x)^\alpha} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_{-\infty}^0 \Big( f(x-v) - f(x) + v f'(x) \Big) \frac{dv}{|v|^{1+\alpha}} \tag{4.5}$$

Now we return to SDE driven by $\alpha$-Stable Lévy process. We shall use the following theorem to get the Lévy type stochastic integration representation of $\{Y_t\}$.

**Theorem 4.2.1.** *(Theorem 3.12.2: [22]):*

*Let $N$ be a Poisson random measure defined on $[0, \infty) \times \mathbb{R}_0$ with intensity measure*

$$\begin{aligned} n(ds, du) = E[N(ds, du)] &= \begin{cases} \Big(1+\beta\Big) ds \frac{du}{|u|^{1+\alpha}}, & if \quad u > 0, \\ \Big(1-\beta\Big) ds \frac{du}{|u|^{1+\alpha}}, & if \quad u < 0. \end{cases} \\ &= I_\beta(u) ds \frac{du}{|u|^{1+\alpha}} \end{aligned}$$

$$Where \ \ I_\beta(u) = \begin{cases} (1+\beta), & if \quad u > 0, \\ (1-\beta), & if \quad u < 0. \end{cases}$$

*Now if we set $\beta$ to be the skewness parameter of an $\alpha$-Stable Lévy process $\{X_t\}$; then, for a random function $f : [0, \infty) \times \Omega \to \mathbb{R}$,* *(i) For* $0 < \alpha < 1$*:*

$$\int_0^t f dX_s \stackrel{d}{=} C_\alpha \int_0^t \int_{\mathbb{R}_0} f(s) u N(ds, du)$$

*(ii) For* $1 < \alpha < 2$*:*

$$\int_0^t f dX_s \stackrel{d}{=} C_\alpha \times \lim_{\delta \downarrow 0} \Big( \int_0^t \int_{(-\delta,\delta)^c} f(s) u \; q(ds, du) \Big)$$

*where constant* $C_\alpha$ *is defined as follows:*

$$(C_\alpha)^\alpha = \begin{cases} \left(2\alpha^{-1}(\Gamma(1-\alpha)) \cos \frac{\pi\alpha}{2}\right)^{-1}, & if \;\; 0 < \alpha < 1, \\ \left(2\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}(-\cos \frac{\pi\alpha}{2})\right)^{-1}, & if \;\; 1 < \alpha < 2. \end{cases}$$

Using the above two theorems we can derive the precise infinitesimal generator of the solution process $\{Y_t\}$ of the SDE (4.1). For rest of this chapter we shall use the following notation:

$$b^\alpha(\cdot) = \begin{cases} |b(\cdot)|^\alpha & , \; if \quad b(\cdot) > 0, \\ -|b(\cdot)|^\alpha & , \; if \quad b(\cdot) < 0. \end{cases}$$

**Theorem 4.2.2.** *Consider a stochastic differential equation driven by an* $\alpha$*-Stable Lévy process as in* (4.1)*. Suppose the solution process* $\{Y_t\}$ *exists. Then the infinites-*

*imal generator of* $\{Y_t\}$ *is given by: if* $0 < \alpha < 1$,

$$\begin{aligned} Af(y) &= a(y)f'(y) + \Bigg[(1-\beta)\Big(2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}f(y) \\ &\quad + (1+\beta)\Big(2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}f(y)\Bigg] \qquad (4.6a) \end{aligned}$$

*if* $1 < \alpha < 2$,

$$\begin{aligned} Af(y) &= a(y)f'(y) + \Bigg[(1-\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}f(y) \\ &\quad + (1+\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}f(y)\Bigg] \qquad (4.6b) \end{aligned}$$

*Proof.* Let us define a Poisson random measure $N$ as in Theorem 4.2.1. Then,

<u>for $0 < \alpha < 1$</u>

$$\begin{aligned} Y_t &= Y_0 + \int_0^t a(Y_s)ds + \int_0^t b(Y_s)dX_s \\ &= Y_0 + \int_0^t a(Y_s)ds + C_\alpha \int_0^t \int_{\mathbb{R}_0} b(Y_s)uN(ds,du) \end{aligned}$$

Let us define a Poisson random measure $N_\alpha(\cdot,\cdot)$ as $N_\alpha(s,u) = N(s, C_\alpha u)$. Let $\tilde{\nu}$ be the intensity measure of $N_\alpha$. Then a change of variable gives

$$\tilde{\nu}(ds,du) = n\Big(ds, d(C_\alpha u)\Big) = I_\beta(u)ds(C_\alpha)^\alpha \frac{du}{|u|^{1+\alpha}} = ds\nu_\alpha(du), \text{ (say)}$$

where $\nu_\alpha(du) = I_\beta(u)(C_\alpha)^\alpha \frac{du}{|u|^{1+\alpha}}$ ; $I_\beta(\cdot)$ defined as in Theorem 4.2.1. Further, we define the compensated Poisson random measure $q_\alpha$ by $q_\alpha(\cdot,\cdot) = N_\alpha(\cdot,\cdot) - \tilde{\nu}(\cdot,\cdot)$. Thus a change of variable gives:

$$
\begin{aligned}
Y_t &= Y_0 + \int_0^t a(Y_s)ds + \int_0^t \int_{\mathbb{R}_0} b(Y_s)uN_\alpha(ds,du) \\
&= Y_0 + \int_0^t a(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) \\
&\qquad + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) + \int_0^t \int_{0<|u|<1} b(Y_s)u\tilde{\nu}(ds,du) \\
&= Y_0 + \int_0^t a(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uds\nu_\alpha(du) \\
&\qquad + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du)
\end{aligned}
$$

Let us denote by $\tilde{a}$, the function: $\tilde{a}(y) = a(y) + b(y) \int_{0<|u|<1} u\nu_\alpha(du)$ . Then we have:

$$
Y_t = Y_0 + \int_0^t \tilde{a}(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) \quad (4.7)
$$

This is of the same form as the equation given in (3.2). Also, we have function $\tilde{a}$ in place of $a$ and the Poisson random measure $N_\alpha$. From Theorem 3.2.2 using the infinitesimal generator as in (3.3b), we can derive the infinitesimal generator in case of an $\alpha$-stable Lévy process as follows:

$$Af(y) = f'(y)\Big(\tilde{a}(y) - K_{\nu_\alpha} b(y)\Big) + \int\limits_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) \Big\} \nu_\alpha(dx)$$

Here, $K_{\nu_\alpha} = \int\limits_{0<|x|<1} x\nu_\alpha(dx)$. Thus,

$$\begin{aligned}
Af(y) &= f'(y)\Big[a(y) + b(y) \int\limits_{0<|u|<1} u\nu_\alpha(du) - b(y) \int\limits_{0<|x|<1} x\nu_\alpha(dx)\Big] \\
&\qquad + \int\limits_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) \Big\} \nu_\alpha(dx) \\
&= a(y)f'(y) + \int\limits_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) \Big\} \nu_\alpha(dx) \\
&= a(y)f'(y) + (C_\alpha)^\alpha \int\limits_{\mathbb{R}_0} \Big\{ f\Big(y + b(y)x\Big) - f(y) \Big\} I_\beta(x) \frac{dx}{|x|^{1+\alpha}} \\
&= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y) \int\limits_{\mathbb{R}_0} \Big\{ f\Big(y - v\Big) - f(y) \Big\} I_\beta(-v) \frac{dv}{|v|^{1+\alpha}} \\
&= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta) \int\limits_0^\infty \Big\{ f\Big(y - v\Big) - f(y) \Big\} \frac{dv}{|v|^{1+\alpha}} \\
&\qquad +(C_\alpha)^\alpha b^\alpha(y)(1+\beta) \int\limits_{-\infty}^0 \Big\{ f\Big(y - v\Big) - f(y) \Big\} \frac{dv}{|v|^{1+\alpha}} \\
&= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\Big[\frac{\Gamma(\alpha-1)}{\alpha} \frac{d^\alpha}{dy^\alpha} f(y)\Big] \\
&\qquad +(C_\alpha)^\alpha b^\alpha(y)(1+\beta)\Big[\frac{\Gamma(\alpha-1)}{\alpha} \frac{d^\alpha}{d(-y)^\alpha} f(y)\Big]
\end{aligned}$$

Hence, for $0 < \alpha < 1$, the infinitesimal generator of the solution process for SDE in

(4.1) can be written as:

$$Af(y) = a(y)f'(y) + \Big[(1-\beta)\Big(2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}f(y) + (1+\beta)\Big(2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}f(y)\Big] \quad (4.8)$$

<u>for $1 < \alpha < 2$</u>: Again from Theorem 4.2.1

$$\begin{aligned} Y_t &= Y_0 + \int_0^t a(Y_s)ds + \int_0^t b(Y_s)dX_s \\ &= Y_0 + \int_0^t a(Y_s)ds + C_\alpha \lim_{\delta\downarrow 0}\int_0^t \int_{(-\delta,\delta)^c} b(Y_s)uq(ds,du) \end{aligned}$$

Using a similar change of variable as in previous case,

$$\begin{aligned} Y_t &= Y_0 + \int_0^t a(Y_s)ds + \lim_{\delta\downarrow 0}\int_0^t \int_{(-\delta,\delta)^c} b(Y_s)uq_\alpha(ds,du) \\ &= Y_0 + \int_0^t a(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) \\ &\quad + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) - \int_0^t \int_{|u|\geq 1} b(Y_s)u\tilde{\nu}(ds,du) \\ &= Y_0 + \int_0^t a(Y_s)ds - \int_0^t \int_{|u|\geq 1} b(Y_s)u\tilde{\nu}(ds,du) \\ &\quad + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) \end{aligned}$$

Let us denote by $\tilde{a}$, the function, $\tilde{a}(y) = a(y) - b(y)\int_{|u|\geq 1} u\nu_\alpha(du)$. Then, we have:

$$Y_t = Y_0 + \int_0^t \tilde{a}(Y_s)ds + \int_0^t \int_{0<|u|<1} b(Y_s)uq_\alpha(ds,du) + \int_0^t \int_{|u|\geq 1} b(Y_s)uN_\alpha(ds,du) \tag{4.9}$$

which is of the same form as the equation given in (3.2). Also we have the function $\tilde{a}$ in place of $a$ and the Poisson random measure $N_\alpha$. From Theorem 3.2.2 using the infinitesimal generator as in (3.3a), we can derive the infinitesimal generator in case of $\alpha$-Stable Lévy process as follows:

$$Af(y) = f'(y)\Big(\tilde{a}(y) + C_{\nu_\alpha}b(y)\Big) + \int_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x\Big\}\nu_\alpha(dx)$$

Here, $C_{\nu_\alpha} = \int_{|x|\geq 1} x\nu_\alpha(dx)$. Therefore,

$$\begin{aligned}
Af(y) &= f'(y)\Bigg(a(y) - b(y)\int_{|u|\geq 1} u\nu_\alpha(du) + C_{\nu_\alpha}b(y)\Bigg) \\
&\qquad + \int_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x\Big\}\nu_\alpha(dx) \\
&= f'(y)a(y) + \int_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x\Big\}\nu_\alpha(dx) \\
&= a(y)f'(y) + (C_\alpha)^\alpha \int_{\mathbb{R}_0} \Big\{f\Big(y + b(y)x\Big) - f(y) - f'(y)b(y)x\Big\}I_\beta(x)\frac{dx}{|x|^{1+\alpha}} \\
&= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y) \int_{\mathbb{R}_0} \Big\{f\Big(y - v\Big) - f(y) + f'(y)v\Big\}I_\beta(-v)\frac{dv}{|v|^{1+\alpha}}
\end{aligned}$$

$$= a(y)f'(y) + (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\int_0^\infty \Big\{f\Big(y-v\Big) - f(y) + f'(y)v\Big\}\frac{dv}{|v|^{1+\alpha}}$$

$$+ (C_\alpha)^\alpha b^\alpha(y)(1+\beta)\int_{-\infty}^0 \Big\{f\Big(y-v\Big) - f(y) + f'(y)v\Big\}\frac{dv}{|v|^{1+\alpha}}$$

$$= a(y)f'(y) \; + \; (C_\alpha)^\alpha b^\alpha(y)(1-\beta)\Big[\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\frac{d^\alpha}{dy^\alpha}f(y)\Big]$$

$$+(C_\alpha)^\alpha b^\alpha(y)(1+\beta)\Big[\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\frac{d^\alpha}{d(-y)^\alpha}f(y)\Big]$$

Hence, in case we have $1 < \alpha < 2$, the infinitesimal generator for SDE in (4.1) can be written as:

$$Af(y) = a(y)f'(y) + \Bigg[(1-\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^\alpha(y)\frac{d^\alpha}{dy^\alpha}f(y)$$

$$+(1+\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1} b^\alpha(y)\frac{d^\alpha}{d(-y)^\alpha}f(y)\Bigg] \quad (4.10)$$

Combining Equation (4.8) and Equation (4.10) we get the infinitesimal generator in (4.6), and the theorem is proved. □

**The Backward equation** :

The backward equation can be obtained using Theorem 3.2.3. Let $A$ be the infinitesimal generator for the solution process $\{Y_t\}$ of SDE driven by $\alpha$-Stable Lévy process as in (4.1). Let $f \in C_0^2(\mathbb{R})$.

Define, $u(y,t) = E^y\left[f(Y_t)\right]$ . Then, $\frac{\partial u}{\partial t} = A(u)$ . That is:

if $0 < \alpha < 1$

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \left[(1-\beta)\Big(2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) \right.$$
$$\left. +(1+\beta)\Big(2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\right] \quad (4.11a)$$

if $1 < \alpha < 2$

$$\frac{\partial u(y,t)}{\partial t} = a(y)u'(y,t) + \left[(1-\beta)\Big(-2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{dy^{\alpha}}u(y,t) \right.$$
$$\left. +(1+\beta)\Big(-2\cos(\frac{\pi\alpha}{2})\Big)^{-1} b^{\alpha}(y)\frac{d^{\alpha}}{d(-y)^{\alpha}}u(y,t)\right] \quad (4.11b)$$

**The Forward equation** :

Theorem 3.3.2 can be used to obtain the forward equation. Let $\{Y_t\}$ be the solution process for the SDE in (4.1). Let us further assume that there exists a transition probability density for $Y_t$ and let $p_x(t,y)$ be the transition p.d.f of $Y_t$, given $Y_0 = x$. We have already stated that the Lévy measure for the $\alpha$-Stable Lévy process $\{X_t\}$ is given by $\nu_\alpha(dy) = I_\beta(y)(C_\alpha)^\alpha \frac{dy}{|y|^{1+\alpha}}$.

Note that, the measure $\nu_1(x,u)$ in Theorem 3.3.2 is the measure derived from the Lévy measure $\nu_\alpha(dy)$ by change of variable $-u = b(x)y$. So in this case the change of variable leads to $\nu_1(x,u) = I_\beta(u)(C_\alpha)^\alpha (b(x))^\alpha \frac{du}{|u|^{1+\alpha}}$.

Now we see $\nu_1$ is of the form $\nu_1(x,dy) = h(x)\mu(dy)$, where $h(x) = b^\alpha(x)$ and $\mu \equiv \nu_\alpha$. Thus, if condition (3.14) holds, then the transition probability density function of

$Y_t$ satisfies the forward equation given in Theorem 3.3.2. This leads us to the next theorem.

**Theorem 4.2.3.** ***(The forward equation for an $\alpha$-stable Lévy diffusion)***

*Consider a stochastic differential equation driven by an $\alpha$-stable Lévy process ($\alpha \neq 1$, $0 < \alpha < 2$) as in* (4.1), *such that the solution process $Y_t$ exists and is unique. Then if the coefficient functions* a *and* b *satisfy assumption* (3.14), *and if there exists a transition probability density function $p_x(s,y)$ of the solution process $\{Y_s\}$ given $Y_0 = x$, then the following forward equation holds:*

*if* $0 < \alpha < 1$,

$$\begin{aligned}\frac{\partial}{\partial s}p_x(s,y) &= \left[(1+\beta)\Big(2\cos\left(\frac{\pi\alpha}{2}\right)\Big)^{-1}\frac{d^\alpha}{dy^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\right.\\ &\left.+(1-\beta)\Big(2\cos\left(\frac{\pi\alpha}{2}\right)\Big)^{-1}\frac{d^\alpha}{d(-y)^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\right] - \frac{\partial}{\partial y}\Big[a(y)p_x(s,y)\Big] \quad (4.12a)\end{aligned}$$

*if* $1 < \alpha < 2$,

$$\begin{aligned}\frac{\partial}{\partial s}p_x(s,y) &= \left[(1+\beta)\Big(-2\cos\left(\frac{\pi\alpha}{2}\right)\Big)^{-1}\frac{d^\alpha}{dy^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\right.\\ &\left.+(1-\beta)\Big(-2\cos\left(\frac{\pi\alpha}{2}\right)\Big)^{-1}\frac{d^\alpha}{d(-y)^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\right] - \frac{\partial}{\partial y}\Big[a(y)p_x(s,y)\Big] \quad (4.12b)\end{aligned}$$

*Proof.* <u>**Case I :** $0 < \alpha < 1$</u>: In this case

$$K_{\nu_\alpha} = (C_\alpha)^\alpha \int_{0<|x|\le 1} xI_\beta(x)\frac{dx}{|x|^{1+\alpha}} = 2(C_\alpha)^\alpha \frac{\beta}{1-\alpha} < \infty.$$

Therefore we shall use the forward equation from (3.3b), i.e,

$$\frac{\partial}{\partial s}p_x(s,y) = \int\limits_{\mathbb{R}_0} \Big[(p_x \cdot h)(s, y-r) - (p_x \cdot h)(s,y)\Big]\mu(d(-r)) - \frac{\partial}{\partial y}(p_x \cdot H)(s,y)$$

where $(p_x \cdot h)(s,y) = h(y)p_x(s,y)$ and $H(y) = \tilde{a}(y) - K_{\nu_\alpha} b(y)$.

Note, in this case $H(y) = \tilde{a}(y) - K_{\nu_\alpha} b(y) = a(y)$. Thus, the forward equation:

$$\begin{aligned}
\frac{\partial}{\partial s}p_x(s,y) &= \int\limits_{\mathbb{R}_0} \Big[(p_x \cdot b^\alpha)(s, y-r) - (p_x \cdot b^\alpha)(s,y)\Big]\nu_\alpha(d(-r)) - \frac{\partial}{\partial y}(p_x \cdot a)(s,y) \\
&= (C_\alpha)^\alpha \int\limits_{\mathbb{R}_0} \Big[(p_x \cdot b^\alpha)(s, y-r) - (p_x \cdot b^\alpha)(s,y)\Big] I_\beta(-r)\frac{dr}{|r|^{1+\alpha}} - \frac{\partial}{\partial y}(p_x \cdot a)(s,y) \\
&= (1+\beta)(C_\alpha)^\alpha \int\limits_0^\infty \Big[(p_x \cdot b^\alpha)(s, y-r) - (p_x \cdot b^\alpha)(s,y)\Big]\frac{dr}{|r|^{1+\alpha}} \\
&\quad +(1-\beta)(C_\alpha)^\alpha \int\limits_{-\infty}^0 \Big[(p_x \cdot b^\alpha)(s, y-r) - (p_x \cdot b^\alpha)(s,y)\Big]\frac{dr}{|r|^{1+\alpha}} \\
&\qquad\qquad\qquad - \frac{\partial}{\partial y}(p_x \cdot a)(s,y) \\
&= \Bigg[(1+\beta)\Big(2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1}\frac{d^\alpha}{dy^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big) \\
&\quad +(1-\beta)\Big(2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1}\frac{d^\alpha}{d(-y)^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\Bigg] - \frac{\partial}{\partial y}\Big[a(y)p_x(s,y)\Big]
\end{aligned}$$

<u>**Case II :** $1 < \alpha < 2$:</u> In this case

$$C_{\nu_\alpha} = \int\limits_{|x|\geq 1} x\nu_\alpha(dx) = 2(C_\alpha)^\alpha \frac{\beta}{\alpha - 1} < \infty.$$

Therefore we shall use the forward equation from (3.3a) here, i.e,

$$\frac{\partial}{\partial s}p_x(s,y) = \int\limits_{\mathbb{R}_0} \Big[(p_x\cdot h)(y-r,s)-(p_x\cdot h)(s,y)+r(p_x\cdot h)'(y,s)\Big]\mu(d(-r))-\frac{\partial}{\partial y}(p_x\cdot G)(y,s)$$

where $(p_x \cdot h)(s,y) = h(y)p_x(s,y)$ and $G(y) = \tilde{a}(y) + C_{\nu_\alpha}b(y)$.

In this case $G(y) = \tilde{a}(y) + C_{\nu_\alpha}b(y) = a(y)$ . Thus, the forward equation is given by:

$$\begin{aligned}
\frac{\partial}{\partial s}p_x(s,y) &= \int\limits_{\mathbb{R}_0} \Big[(p_x\cdot b^\alpha)(y-r,s)-(p_x\cdot b^\alpha)(y,s)+r(p_x\cdot b^\alpha)'(y,s)\Big]\nu_\alpha(d(-r)) \\
&\qquad\qquad\qquad\qquad -\frac{\partial}{\partial y}(p_x\cdot a)(y,s) \\
&= (C_\alpha)^\alpha \int\limits_{\mathbb{R}_0} \Big[(p_x\cdot b^\alpha)(s,y-r)-(p_x\cdot b^\alpha)(s,y)+r(p_x\cdot b^\alpha)'(s,y)\Big]I_\beta(-r)\frac{dr}{|r|^{1+\alpha}} \\
&\qquad\qquad\qquad\qquad -\frac{\partial}{\partial y}(p_x\cdot a)(s,y) \\
&= (1+\beta)(C_\alpha)^\alpha \int\limits_0^\infty \Big[(p_x\cdot b^\alpha)(s,y-r)-(p_x\cdot b^\alpha)(s,y)+r(p_x\cdot b^\alpha)'(s,y)\Big]\frac{dr}{|r|^{1+\alpha}} \\
&\quad +(1-\beta)(C_\alpha)^\alpha \int\limits_{-\infty}^0 \Big[(p_x\cdot b^\alpha)(s,y-r)-(p_x\cdot b^\alpha)(s,y)+r(p_x\cdot b^\alpha)'(s,y)\Big]\frac{dr}{|r|^{1+\alpha}} \\
&\qquad\qquad\qquad\qquad -\frac{\partial}{\partial y}(p_x\cdot a)(s,y) \\
&= \Bigg[(1+\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1}\frac{d^\alpha}{dy^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big) \\
&\qquad +(1-\beta)\Big(-2\cos\Big(\frac{\pi\alpha}{2}\Big)\Big)^{-1}\frac{d^\alpha}{d(-y)^\alpha}\Big(b^\alpha(y)p_x(s,y)\Big)\Bigg] - \frac{\partial}{\partial y}\Big[a(y)p_x(s,y)\Big]
\end{aligned}$$

We combine the two cases to get the forward equation of the form (4.12). This concludes the proof. $\square$

Remark : The above fractional derivative form representation of the forward equation is just a special representation in this case. This form agrees with the space-fractional advection-dispersion equation used in hydrology. The main forward equation form derived as in (3.3a) or (3.3b) can be used for analytical purposes.

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