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EFFECTS OF VIRTUAL MANIPULATIVES WITH OPEN-ENDED VERSUS STRUCTURED QUESTIONS ON STUDENTS' KNOWLEDGE OF SLOPE

By

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ABSTRACT

EFFECTS OF VIRTUAL MANIPULATIVES WITH OPEN-ENDED VERSUS STRUCTURED QUESTIONS ON STUDENTS' KNOWLEDGE OF SLOPE

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Virtual Manipulatives (VMs) are computer-based, dynamic, visual representations of mathematics concepts, which have become widely used in mathematics instruction in recent years. Despite their wide use, there is little empirical research about the use of VMs in mathematics learning and teaching. To fill this gap, this study examined the effects of using VMs with two different instructional approaches on students' learning of slope. Approximately 50 students who were taking a remedial mathematics course at Michigan State University completed all sessions of the study. They were randomly assigned into two groups: OVM and SVM groups. After completing a pretest assessing their initial knowledge of slope, participants were randomly assigned into two groups to complete four 30- to 45-minute intervention sessions. Students in the OVM group worked with VMs to respond to open-ended exploratory questions; students in the SVM group used the same VMs to respond to structured mathematics questions. Then, students completed a posttest consisting of the same set of questions as the pretest. OVM students showed considerably higher pre- to posttest gain scores than SVM students on items requiring conceptual knowledge, whereas SVM students obtained notably higher gains on items requiring procedural and a combination of conceptual and procedural knowledge. These results suggest that VMs can be used with various instructional approaches to improve different types of students' mathematics knowledge.

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CHAPTER 1

INTRODUCTION

In the last two decades, computer and the Internet technology have shown breathtaking advances, and people have already experienced the effects of these technologies in various areas of their life. Education is one of the areas where the impact of technology can be clearly observed. Because various applications of computer technology have already been widely used in many areas of everyday life (e.g., communication, business, medicine), it is impossible to neglect the developments of computer technology in the area of education. Today, students are more often using computer technology and the Internet to improve their learning of school subjects than students in the past. School mathematics is one of the areas that today's students use computer technology to enhance their learning.

Many researchers and leading mathematics education organizations (e.g., NCTM, 2000; Steen et al., 2006; Wenglinsky, 1998) have argued that using technology can advance students' mathematics learning. For example, Wenglinsky (1998) examined the impact of using simulation and higher-order thinking technologies on students' performance in mathematics. He analyzed a U.S. national sample of 6,227 fourth and 7,146 eighth-grade students' mathematics scores on the National Assessment of Educational Progress (NAEP) and found that students who worked with the technologies improved their mathematics achievement. The National Council of Teachers of Mathematics (NCTM) is one of the professional mathematics education organizations in the United States. NCTM's *Principles and Standards for School Mathematics* (2000) declared that use of technology is one of the six major principles of a high-quality mathematics instruction. The technology principle states that "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." (NCTM, 2000, p. 11). With the aid of technology, students effectively focus on decision making, reflection, and problems solving; thus students can develop deep understanding of mathematics concepts by using computer technology (NCTM, 2000). Many types of computer technologies have been used to support learning and teaching of mathematics. For example, there are a variety of educational mathematics software such as Geometer's Sketchpad (Jackiw, 2001), Cognitive Tutor Algebra Software (Koedinger et al., 1997), SimCalc Mathworlds (Roschelle & Kaput, 1996); computer simulations such as Java applets; and Virtual Manipulatives (VMs). Among these technologies, VMs are a recent player in the mathematics education scene.

Recent advances in computer and Internet technology have made it possible for students to manipulate objects on the computer screen as easily as they can manipulate physical manipulatives. These VMs are dynamic visual representations of physical manipulatives, have additional features, and provide various opportunities to improve learning and teaching. VMs provide interactive learning environments where students can engage with various cognitive activities (e.g., generalizing, planning, testing) to learn mathematical ideas. Since VMs present interactive learning activities, they go beyond the other media that only present or deliver information. Therefore, Clark's (1983) famous controversial claim that media only delivers the information without influencing learning does not hold for using VMs in mathematics instruction. However, Clark's suggestion about exploring the impact of different instructional methods used with media on learning rather than only studying the impact of one specific media on learning is crucial. With such knowledge, mathematics teachers can better choose the most appropriate instructional methods to use with VMs after looking at the results of the research studies exploring the effects of different instructional approaches used with VMs on students' mathematics learning. However, there is a need for research that will study the impact of various instructional approaches with VMs on students' learning of mathematics concepts. Therefore, this study examined the effects of VMs used with two different instructional approaches on students' knowledge of slope, an important mathematical concept that is often problematic for students.

With the help of current computer technology, a wide variety of VMs have become available and practical for both students and teachers. This availability increased the need for research studies analyzing the affordances and constraints of VMs in learning and teaching mathematics. However, few studies have examined the impact of using VMs on students' understanding of mathematical concepts. Many researchers (e.g., Reimer & Moyer, 2005; Steen et al., 2006; Suh & Moyer, 2007) have pointed out the lack of research on using VMs in mathematics learning.

This study aimed to fill the gap in research literature and inform current practices with VMs in mathematics instruction by exploring the impact of VMs used with two different instructional approaches on students' learning of slope. Slope is one of the concepts that students often have difficulties in their courses. Because slope is a widely-used concept in various areas of school mathematics, students' knowledge of slope influences their learning of other mathematics subjects (Thompson, 1994). Thus, several researchers examined students' difficulties in learning slope and suggested various ways for developing instructional activities to improve students' learning.

CHAPTER 2

REVIEW OF THE LITERATURE

In this chapter, first, the relations between media and learning will be discussed through presenting a brief summary of different views about media and its effects on learning. Second, theoretical assumptions of various learning perspectives will be examined to identify possible effects of using VMs on students' mathematics learning. Then individual research studies analyzing the effects of using VMs on learning and teaching mathematics will be presented. Finally, research about students' learning of slope will be reviewed.

Media and Learning

Many researchers (e.g., Kaput, 1992; Kulik et al., 1983) have examined the impact of various types of media on students' learning of particular subject areas. Studies analyzed the relationships between media and learning unveiled conflicting views among researchers about the effects of media on learning. Researchers have been discussing the effects of media on learning for a long time. Clark (1983) stimulated the discussion about media and its effects on learning by asserting that choice of media has no influence on learning. After reviewing media comparison studies that scrutinized the effects of media on learning, Clark concluded that media do not influence learning under any condition. He claimed that finding larger positive effect sizes for one media than another can be attributed to differences in instructional method, novelty of the media, and editorial bias.

Clark (1983) viewed media based instructional materials as "mere vehicles that deliver instruction but do not influence student achievement any more than the truck that delivers our groceries causes changes in our nutrition" (p. 445). He suggested that researchers should avoid conducting additional studies that explore the relationship between media and learning. In response to Clark claims about media and learning, Kozma (1994) emphasized that as technologies develop and their use is systematically examined, the effects of media on learning will eventually be identified and utilized.

As Clark (1983) reviewed the media comparison studies that were conducted prior to 1983, his view of media is quite limited in the light of various types of current media. Clark considered media as presentations of information, or the tools that deliver information. Many types of current media, however, do more than presenting and delivering information. Since computer and the Internet technologies developed dramatically in the last two decades, there are various technologies that have been used in instruction, and have advanced capabilities (e.g., interactivity) and obviously they do more than only presenting and delivering information. One of these technologies that have been recently used in educational settings is VMs.

VMs and Learning

VMs can be considered as visual representations of dynamic objects that support students' understanding of mathematics concepts. They are a different type of computer technology that has potential to meet the shortcomings of physical manipulatives in learning and teaching (Dorward, 2002; Sarama et al. 1996). VMs provide interactive learning environments where students can engage with various cognitive activities (e.g., generalizing, planning, testing) to understand mathematics concepts. Because VMs enable students to interact with mathematics concepts, they differ from the media that simply present or deliver information. Therefore, Clark's claim about the media and learning is not valid for using virtual manipulatives in learning and teaching of mathematics. However, his suggestion about exploring the effects of different instructional methods within various media on learning rather than only studying the impact of one specific media on learning is important. Thus, research studies need to examine the effects of various instructional methods with VMs on students' learning.

Theoretical Assumptions

According to assumptions of various learning perspectives, VMs have potential to support learning and teaching mathematics in the following ways.

Constructivist perspective on learning. VMs provide interactive learning environments where students can instantly observe the effects of their actions, monitor their own learning, and form and test hypotheses in order to construct or modify their knowledge of mathematical concepts. The constructivist perspective (Anderson et al., 2000; Brown et al., 1989; Greeno et al., 1996) on learning assumes that learners construct their own knowledge by interacting with their physical and social environment. Therefore, students' interactions (e.g., receiving immediate and specific feedback) with VMs may enable them to understand mathematical concepts.

Researchers (Clements & McMillen, 1996; Kaput, 1995; Reimer & Moyer 2005; Suh et al., 2005; Suh & Moyer, 2007) have emphasized that VMs can support students' understanding of mathematical concepts by providing interactive learning environments. For example, Kaput (1995) used the term *cybernetic manipulatives* to refer the manipulatives presented on the computers. He emphasized that cybernetic manipulatives can enable students to make connections between their actions in different notation systems. While using physical materials, students exhaust all their cognitive resources to perform actions in one or the other notation systems, and therefore they cannot track relations between their actions. However, cybernetic manipulatives can provide immediate feedback to students' actions, and this enables students to identify relationships between their actions in different notation systems.

Exemplifications. Another perspective on learning that might be used to support the contributions of VMs on students' understanding of mathematical ideas is Nesher's (1989) *Learning System (LS)* model. Nesher developed a LS model based on two components: a knowledge component and an exemplification component. Knowledge component refers to the unit of knowledge to be instructed, based on expert knowledge. The exemplification component includes an illustrative domain that corresponds to the knowledge component and is purposely chosen to serve as an exemplification. In this LS model, learners make connections between the objects, relations, and operations in exemplification and knowledge components to understand abstract concepts.

VMs may help students exemplify mathematical concepts by providing concrete representations of formal mathematical concepts. The exemplification components include illustrations that are familiar to learners' experiences, thus learners can match the objects, operations, and relations in exemplification components with those in knowledge components in order to understand abstract concepts. VMs can be thought as an example of Nesher's LS model. For example, some VMs use balance scales to support students' learning of solving linear equations. While working with these VMs, learners can use balance scale as an exemplification component to comprehend mathematical properties of solving linear equations.

Multiple representations. The multiple representations view of learning can also be employed to support the educational benefits of using VMs in learning and teaching mathematics. The multiple representations perspective of learning assumes that people can improve their understanding of mathematical ideas through working with multiple representations of the same concepts. Researchers (Goldenberg, 1995; Heid & Edwards, 2001; Kaput, 1998) have also pointed out that using multiple representations (e.g., graphical, symbolic) and making connections among these representations is one of the important factors in modeling and understanding mathematical concepts.

Students can move from less abstract representations of mathematics concepts to more abstract ones while working with VMs. This enables them to begin with familiar concepts then continue with less familiar ones in their learning processes. Kaput (1995) suggested that computer software should enable students to extend their knowledge moving from familiar, concrete contexts to less familiar, abstract contexts. He pointed out that students can make this movement by using the links (provided by the computer software) between more familiar representations and less familiar ones. Kaput maintained that in software-based learning environments students should engage with a set of representations, beginning at the concrete level and *ramping upward* in abstractness to more abstract representations. VMs can help students initially work with less abstract representations of mathematics concepts and then engage with more abstract ones.

As VMs provide various representations of mathematical concepts, first, they may present representations which are concrete to students, and then gradually increase the level of abstractness of representations in order to support students' mathematics learning. For example, some virtual manipulatives offer various forms (e.g., verbal, tabular, graphical, and algebraic) forms of linear functions. These VMs can be designed in a way that enables students to work with less abstract representations in the beginning and then work with more abstract forms of linear functions. For example, while working with VMs, students can initially engage with verbal forms of linear functions then they can study tabular and graphical representations.

Dual coding theory. One of the perspectives that can be used to support the effectiveness of using VMs on learning and teaching is Dual Coding theory. The theory assumes that individuals can enhance their understanding of concepts when they receive both verbal and visual forms of the same information (Clark & Paivio, 1991). VMs may improve students' learning of mathematical concepts by providing visual and verbal forms of the same concepts in a coordinated way. They can present verbal information (either as narrative or written format) while students manipulate objects on the computer screen. Thus, students may recognize and focus on mathematical ideas and concepts behind their actions with VMs by having verbal information coordinated with their manipulations.

Having verbal forms of information coordinated with individuals' actions on VMs also supports teachers' use of VMs in their courses. For example, Crawford and Brown (2003) designed a survey to identify teachers' evaluations of VMs and their views about integration of VMs into classroom settings. In the study, after reviewing a number of VMs on the web, many classroom teachers emphasized that because the VMs (that they examined) provided neither clear instructions about their use nor feedback while students working with the VMs, they rated these manipulatives as unsatisfactory for their instructional practices. Therefore, as dual coding theory suggested, it is important to provide verbal information coordinated with students' actions while they work with VMs in order to improve their learning.

Researchers (e.g., Suh & Moyer, 2007) have already used some of these learning perspectives (multiple representations, constructivism) to explore the effects of VMs on students' understanding of mathematical concepts. However, dual coding theory and exemplifications approaches to learning have not been used in the research on VMs in mathematics education. It is important to use these learning perspectives or their various combinations (e.g., dual coding theory and multiple representations) to increase the quality of research studies that examine the impact of VMs on mathematics learning.

VMs and Learning Mathematics

In this section, individual research studies exploring the effects of using VMs on learning and teaching mathematics will be presented. Although there are few studies on VMs and mathematics learning, this review aims to present research evidence regarding the affordances and constraints of VMs on mathematics learning and teaching.

One of the few studies examining the features of VMs that support students' learning of mathematics concepts was conducted by Suh et al. (2005), who designed a qualitative study that included interviews with students, observation notes, and videotapes of classroom sessions to identify the characteristics of the VMs that supported students' learning of equivalence and fraction addition. Forty-six fifth-grade students were identified as low-, average-, and high-level achievers based on standardized test results used at their school. These three groups of students independently worked with VMs after the same elementary teacher introduced a unit on fractions and presented mathematical tasks. The researchers found that a virtual fraction applet enabled students to make conjectures, connections between symbolic and iconic forms of fractions, and discover various properties of fractions. Furthermore, the study indicated that playing a fraction track game supported students' communications of mathematical ideas and their ability to apply previously learned ideas into a different mathematical context. The findings of the study implied that as students learn mathematics concepts, they may have benefits of some aspects of VMs tutorials such as having combined visual and symbolic information in a linked format, and being able to test their hypotheses in a secure environment.

Steen et al. (2006) examined the impact of using VMs on students' academic achievement, attitudes, and interactions during a geometry unit. In the study, 31 Grade 1 students in an urban elementary school were randomly assigned to the treatment or control group for a geometry unit. Students were given a pretest that included two tests, one is Grade 1 and the other is Grade 2 level (provided by the publishers of the textbooks) to assess each student's prior knowledge of geometry concepts such as patterns, symmetry, and shape identification. Students in the control group used their regular textbooks, and worked with physical manipulatives for practice. Treatment group students interacted with VMs, but used the same textbooks. The corresponding forms of Grade 1 and Grade 2 tests were given as a posttest to both groups. All students completed the pre- and posttest on the paper-pencil format.

Steen et al. (2006) found that the use of VMs as an instructional tool was helpful for the treatment group. The researchers conducted a *t*-test to analyze the changes in both control and treatment group students' scores over the pre-and posttest. On the Grade 2 test, students in the treatment group had a mean change of 7.25 ($SD = 4.74$) and control group students had a mean change of 3.33 ($SD = 3.20$). Steen et al. reported that the *t*-test indicated a significant difference between groups on the Grade 2 test ($p < 0.05$).

Similarly, *t*-test showed a significant difference between control and treatment groups on the Grade 1 test ($p < 0.05$). In addition to the pretest and posttest data, the researchers examined the treatment teacher's daily journal to identify the teacher's observations about students' attitudes and interactions while working with VMs. The analysis of teacher's daily journal indicated that VMs enabled students to make changes on their learning, do more number of practices than they do in traditional classroom settings, and have benefits of the same high quality lessons and activities.

Reimer and Moyer (2005) also examined the effects of using VMs on students' mathematics learning. They analyzed the impact of using VMs on 19 third-grade students' conceptual and procedural knowledge of fractions in a classroom setting. These students worked with several VMs in a computer laboratory during a two-week unit on fractions. Reimer and Moyer designed pre- and posttest to identify students' conceptual and procedural knowledge of fractions at the beginning and end of the study. They also conducted interviews to identify the changes in students' knowledge of fractions, and used student attitudes survey. The analysis of the pre-and posttest data indicated that students considerably improved their conceptual knowledge of fractions. Specifically, researchers found that students displayed higher performance on the posttest ($M = 11.0$, $SD = 3.61$) than they did on the pretest ($M = 9.58$, $SD = 4.53$). Furthermore, the analysis of student interviews and attitude surveys also revealed that VMs improved students' learning of fractions by providing immediate and specific feedback. Reimer and Moyer found that VMs helped students enhance their mathematics learning. However, since the researchers used neither control nor comparison group in their study, it is difficult to identify various effects of using VMs on students' conceptual knowledge of fractions.

Suh and Moyer (2007) studied students' learning while working with physical manipulatives and VMs in order to identify the features of these manipulatives that enhanced students' learning. In the study, 36 third-grade students worked with physical and virtual balance scales in their regular mathematics course sessions. They were assigned into two groups; one group worked with the Virtual Balance Scale applet from the National Library of Virtual Manipulatives (NLVM) to solve linear equations. The other group worked with a physical manipulative called Hands-On Equations® (Borenson, 1997) to respond questions. Suh and Moyer used observational field notes and interviews to identify the effects of using physical manipulatives and VMs on students' solving linear equations. They found that the virtual balance applet improved students' mathematical thinking by providing explicit links of symbolic and visual models, step-by-step support in algorithmic processes, immediate feedback and self-checking system. However, the physical balance manipulative provided opportunities for invented strategies, mental mathematics, and tactile features to support students' learning. The study also indicated that although physical manipulatives and VMs have distinctive aspects to support students' learning, and both were helpful in improving students' knowledge and algebraic reasoning.

Studies of VMs in mathematics learning and teaching have reported that VMs supported students' learning by providing immediate feedback, multiple representations of mathematics concepts and interactive learning environment. However, these studies examined the effects of VMs while students studying mathematical ideas according to traditional instructional methods (e.g., using standard textbook). They had no attention to the impact of VMs with different instructional approaches on students' learning of mathematical concepts. Therefore, there is a need for research that will examine the effects of using VMs with different instructional approaches on students' mathematics learning.

Learning about Slope

Many researchers have emphasized the importance of the students' understanding of slope concept in mathematics learning (e.g., Leinhardt et al., 1990; Thompson, 1994; Lobato et al., 2003). Slope can be thought of as a powerful concept that can help students make connections between a line and its algebraic equation (Leinhardt et al., 1990). Students' knowledge of slope can affect their understanding of advanced mathematics concepts (Thompson, 1994). After analyzing college seniors' and graduate students' understanding of the fundamental theorem of calculus, Thompson found that students' impoverished knowledge about slope and rate of change caused difficulties in their learning of the theorem. Students' familiarity with point-slope formula for a line may enable them to improve their comprehension of the fundamental theorem of calculus (Macula, 1995). Furthermore, Macula mathematically showed how the point-slope formula of a line leads to the fundamental theorem of calculus.

Although several researchers have mentioned the importance of deep understanding of slope concept for students' mathematics learning, few studies have specifically focused on students' understanding of slope concept. Rather, researchers have often examined the concept of slope as a part of students' understanding of linear functions. As Lobato and Siebert (2002) pointed out, since most researchers and reform documents have focused on the ability to make connections among multiple representations of [linear] functions, students' knowledge of particular concepts such as slope has not been adequately examined to develop a type of instruction to support students' learning of slope. Few researchers have examined particular aspects of students' knowledge of slope. For example, Lobato et al. (2003) underlined two major parts of conceptual understanding of slope in their study. These are considering slope as a rate of the change in one quantity as related to the change in another quantity, and viewing slope as forming ratios for measures of particular attributes such as constant speed or the steepness of a ramp.

Research exploring students' knowledge of slope has focused on various aspects of slope knowledge, such as identifying the value of slope and interpreting its meaning from various representations (e.g., algebraic, geometric, physical, functional), making connections among these representations of slope (Leinhardt et al., 1990; Knuth, 2000), and considering slope as a measure to identify steepness or rate of change in functional situations where two quantities co-vary (Stump, 2001; Lobato et al., 2003). For example, Stump (2001) examined high school students' understanding of slope as a measure in two different contexts: physical contexts where slope can be used to measure the steepness, and functional contexts in which slope can be considered as a measure of rate of change. In another study, Stump (1997) identified six different representations of slope concept (geometric, algebraic, physical, trigonometric, functional, and ratio) and studied secondary mathematics teachers' knowledge of slope by analyzing the types of representations that teachers mostly used in their responses to the survey questions about slope.

Research has also revealed students' various difficulties in learning slope. The most common difficulties include (a) slope versus height confusion (McDermott et al., 1987), (b) making connections from an algebraic formula of slope to its graphical (Leinhardt et al., 1990, Knuth 2000) and functional (Lobato et al. 2003) representation, (c) identifying the distinction between the slope of a line and the slope of a function while finding slope value on a non-homogenous coordinate system that has different scales on each axis (Zaslavsky et al., 2002; Rasslan & Vinner, 1995), and (d) interpreting slope as a measure of rate of change (Stump, 2001). In addition to these common difficulties, researchers have also underscored students' other difficulties in learning slope such as considering m in y = b + mx as a difference rather than a ratio (Lobato et al., 2003); identifying the relations between the concept of slope and the angle that a line forms with the x-axis (Rasslan & Vinner, 1995, Stump 1997).

Slope-height confusion is one problem commonly reported for students having difficulties in identifying slope values while working on graphs. In their review of studies on the learning and teaching of functions and graphs, Leinhardt et al. (1990) underlined slope-height confusion as one of the three major categories of students' difficulties regarding analysis of graphs. McDermott et al. (1987) reported the slope-height confusion of students who were trying to compare the velocity of two moving objects by analyzing their position versus time graph. In that study, researchers presented students (who were enrolled at introductory physics course at college) position versus time graphs of two moving objects, and then asked them to compare the velocity of two objects at a particular time. Most students provided incorrect answers to the question, and their responses implied that instead of looking at the difference between slopes of two lines on the position versus time graph of two moving objects, students focused on the heights of the lines in order to compare the velocity of the objects at a particular time.

Another area where students often struggle while studying the slope concept is making connections between the algebraic formula of slope and its graphical and functional representations. When students are asked to find the slope of a line, they often apply the algebraic formula of $m = \frac{y_2 - y_1}{x_2 - x_1}$ and do the computation (Crawford & Scott, 2000). Finding the numeric value of slope from the formula, however, does not show that students have mentally constructed slope as the ratio of the change in one variable to the change in the other related variable, or slope as the ratio of the length of perpendicular line segment ($y_2 - y_1$) to the length of horizontal line segment ($x_2 - x_1$) on the Cartesian plane (Lobato et al., 2003, Schoenfeld et al., 1993). For example, Schoenfeld et al. examined one student's understanding of linear functions in a computer setting where the student was playing a game and working with a tutor. They found that the student's initial knowledge of slope included no connection between slope formula and its graphical representation on the Cartesian plane. In his study with high-school students who were enrolled in first-year algebra through calculus, Knuth (2000) found that the majority of the students were successful in finding slope values by using algebraic formula, but their responses to the questions about slope showed no connection between graphical and algebraic representations of slope.

Research focused on identifying slope of a line on a non-homogenous coordinate system (e.g., Rasslan & Vinner, 1995; Zaslavsky et al., 2002) indicated that students often experience difficulties in finding slope value. Zaslavsky et al. asked their subjects with various levels of experiences in learning and teaching mathematics (from high-school students to mathematics educators and mathematicians) to find the slope of a function whose graph was presented in two different coordinate systems: homogenous versus non-homogenous systems. Both students and inservice mathematics teachers experienced confusion while trying to find the slope of a linear function on a non-homogenous system, and they could not identify the distinction between the rate of change of the function and the slope of its line. In particular, most participants in the study could not recognize that the rate of the change of a function (or the slope of a function) is the slope of the line representing the function in a homogenous system; however, there is no relation between the rate of change (or slope) of the function and the slope of its line on a non-homogenous system.

Leinhardt et al. (1990), Crawford and Scott (2000), and Stump (2001) emphasized that students often have difficulties in considering slope as a measure for the rate of change where two quantities covary. Only 3 out of 22 high school students in Stump's (2001) study used the slope of a line to identify the rate of change in functional contexts where two quantities changed relatively. Similarly, after examining preservice and inservice secondary mathematics teachers' mathematical understanding of slope in various representations, Stump (1997) found that most teachers did not consider slope as a rate of change while analyzing the changes in two quantities. As Crawford and Scott (2000) pointed out, reform based projects and documents (e.g., NCTM, 2000) placed considerable emphasis on studying variability and change, thus it has become essential for students understanding slope as a rate of change.

After identifying students' difficulties in understanding slope concept, researchers have recommended various instructional strategies to help students overcome their struggles and improve their learning of slope concept. Their most common suggestions for teaching slope can be summarized in the following way. First, introduce slope concept as a rate of change by using real life examples, such as distance versus time relation for traveling cars in order to help students develop some understanding of slope (Crawford & Scott, 2000; Leinhardt et al, 1990). Second, revisit students' knowledge of Cartesian coordinate system and help students refresh and improve their knowledge about the Cartesian system in order to enable them to understand slope as a ratio by making connections between the algebraic formula of slope and its graphical representation (Leinhardt et al, 1990; Schoenfeld et al., 1993; Stump, 2001). Finally, help students realize the distinction between the slope of a linear function and the slope of a line by allowing them to work on both homogenous and non-homogenous coordinate systems while studying slope concept (Rasslan & Vinner, 1995; Zaslavsky et al., 2002).

Research Focus

VMs have the potential to meet the shortcomings of physical manipulatives in learning and teaching (Dorward, 2002; Sarama et al. 1996) by providing multiple linked representations (e.g., algebraic, graphic, and tabular) of linear functions to help students identify slope value in various forms of linear functions. Furthermore, VMs may enable students to recognize the relations between making changes on the slope value in one representation of linear functions and the effects of these changes on the other forms of linear functions. With some VMs, for example, once students make changes on the slope value in a particular algebraic form of a linear function, they can instantly observe the effects of their changes on the graph of the linear function.

An analysis of several VMs on the web revealed that various instructional approaches can be used with VMs to improve students' learning of mathematics. Among these approaches, two instructional approaches were chosen for this study because both of them were commonly used in mathematics teaching and they also underlined different aspects of mathematics learning. In one of these two instructional approaches, one group of participants (OVM) used VMs to answer open-ended exploratory questions asking them to observe the activities on the VMs, make reflection on their interactions with the VMs, and identify the relations among various mathematical ideas based on their observations. In the other instructional approach, the other group of participants (SVM) worked with the same set of VMs that OVM group used to answer structured mathematics questions requiring them to mostly focus on numeric expressions, quantitative relations, calculations and mathematical procedures.

This study examined the effects of using VMs with these two different instructional approaches on students' knowledge of slope. In the following sections, the knowledge of slope that served as a focus for the study will be described and the research questions that guided the inquiry will be presented.

Students' Knowledge of Slope

Review of research literature about students' learning of slope indicated that there are major areas of slope knowledge that need particular attention to improve students' understanding of slope. This study focused on the following aspects of students' knowledge of slope.

Knowledge of making connections between slope concept and real life settings:

- Considering slope as a rate of change while exploring graphs that compare two quantities from real life contexts (e.g., distance-time vs. velocity-time graphs), and being able to identify slope values from these graphs.
- Forming a graph from real life data, then finding slope value on the graph, and identifying its contextual meaning (e.g., slope as the rate of increase in the perimeter of circle).
- Identifying slope as the steepness of physical objects (e.g., ramps) while solving problems given in real life settings.

Knowledge of making connections between algebraic and graphical forms of linear functions, and identifying slope value on different coordinate systems:

- Forming lines that have the same or different slope values on the Cartesian coordinate system and identifying how making changes on an algebraic form of a linear function affect its graph on the Cartesian plane, and vice versa.

- Identifying the relations between different numeric values of slope and the steepness of a line.

Knowledge of identifying slope value from tabular and various algebraic forms (e.g., slope-intercept form) of linear functions:

- Finding the slope of a linear function from its tabular representation, and identifying slope value from various algebraic representations of linear functions (e.g., slope intercept form).
- Forming an algebraic equation by using the slope value and a pair of the points that the equation satisfies.

The study also examined students' knowledge of slope through three types of questions: The questions mainly concerning use of *conceptual knowledge of slope* constitute the first type of questions. The second type of the questions mostly requires use of *procedural knowledge of slope*. The other type of questions primarily entails use of co*mbination of conceptual and procedural knowledge of slope*. Hiebert and Lefevre's (1986) definitions of procedural and conceptual knowledge were extended to describe conceptual, procedural, and combination of conceptual and procedural knowledge of slope. Hiebert and Lefevre define conceptual knowledge as the knowledge that has rich relationships, and consider procedural knowledge as rules or procedures to solve mathematics problems. Based on these definitions of conceptual and procedural knowledge in mathematics, conceptual, procedural, and combination of conceptual and procedural knowledge of slope can be described in the following way:

- Conceptual knowledge of slope includes identifying mathematical relations among various representations (e.g., algebraic, graphic) of slope concept, defining slope concept based on its different representations, and interpreting the meaning of slope values in real life settings.

- Procedural knowledge of slope consists of finding slope values in algebraic, tabular, and graphic representations of linear functions by using different formulas or procedures.
- Combined conceptual and procedural knowledge of slope contains various amounts of procedural and conceptual knowledge of slope. For example, finding the value of slope in a graphical representation, and identifying the relationships between algebraic and graphical representations of slope.

Research Questions

The purpose of this study is to examine the effects of using VMs with two different instructional approaches on students' knowledge of slope through the following overall question:

What are the effects of VMs used with open-ended exploratory questions versus structured mathematics questions on students' knowledge of slope?

This overall question gives rise to two more specific research questions:

1. What are the effects of using VMs with open-ended exploratory questions versus structured mathematics questions on the pre- to posttest gains?
2. What are the effects of using VMs with open-ended exploratory questions versus structured mathematics questions on the pre- to posttest gains on the procedural, conceptual, and combined conceptual and procedural questions?

CHAPTER 3

METHOD

College students taking a remedial mathematics course completed a pretest that assesses their knowledge of slope, randomly assigned into two groups, individually worked with a set of VMs to answer open-ended exploratory versus structured mathematics questions on their laptops throughout four 30-to-45 minute intervention sessions, and then completed a posttest having the same set of questions with the pretest.

Participants

Sixty-five students taking Intermediate Algebra (MTH 1825) at Michigan State University in the Fall 2008 Semester participated in the study, with 48 of the students completing all of the experimental tasks to be included in the analysis (See Table 1). MTH 1825 is a three-credit remedial mathematics course. According to the university's regulations, students who score 7 or below on the Mathematics Placement Service (MPS) exam, or have a subscore below 12 on the Elementary Algebra section of the ACT exam (a national college admission examination) are required to take MTH 1825. The course covers basic mathematics topics such as properties of real numbers, factoring, roots, radicals, first and second degree equations, and linear inequalities.

Table 1

Number of Students Participated to the Sessions

Sessions	**Number of Participants**	
	OVM	SVM
Session 1	32	33
Session 2	24	25
Session 3	24	25
Session 4	24	24
Session 5	24	24
Session 6	24	24

Students volunteered to participate in the study through their MTH 1825 instructors' announcement about the study, flyers around their classrooms, and a recruiter who was hired by the researcher. Students' participation to the study was encouraged in two ways. First, the researcher offered each student $40 to participate in all six sessions of the study. Students received $5 for their participation to each session of the study, and they received $10 as bonus in addition to $30 after participating in all six sessions. Second, the researcher informed students that through their participation in the study, they could learn mathematics concepts pertaining to the future topics of their current MTH 1825 course. The details about the recruitment of students can be found in Appendix A.

Setting

The study was conducted in the rooms that have wireless connection. All participants responded to sets of questions by using the VMs and typing their answers on the computer.

Materials

Each participant in the study completed a background questionnaire, pretest, posttest, and used the VMs to answer the questions on the computer.

Background questionnaire. The background questionnaire (see Appendix B) consisted of open-ended questions that aim to identify students' experiences in learning mathematics, and using computer and the Internet technology in (or outside of) their courses.

Pretest and posttest. All participants answered to the questions about slope on a pretest and posttest (see Appendix C). Both tests have the same set of questions, presented in different order. The pre- and posttest measured students' knowledge of slope by including questions in three major areas (see Table 2):

- Making connections between slope concept and real life experiences.
- Identifying relations between algebraic and graphical forms of linear functions, and finding slope of the functions on different coordinate systems.
- Finding slope value from tabular and various algebraic forms (e.g., slope-intercept form) of linear functions.

Table 2

Categories of Slope Knowledge and Related Pretest (Posttest) and Session Questions

Categories of Slope Knowledge	Examples	Related Pretest (Posttest) Questions	Related Intervention Sessions
Connections between slope concept and real life experiences	considering slope as a rate of change while exploring graphs that compare two quantities from real life contexts and identifying slope values from these graphs building a graph from a real life data, then finding slope value on the graph, and identify its contextual meaning identifying slope as the steepness of physical objects (e.g., ramps) while solving problems given in real life settings	Pre (Post) Question 5 (7), 8 (5), and 9 (3)	Session 1
Identifying relations between algebraic and graphical forms of linear functions	constructing lines that have the same or different slope values how making changes on an algebraic form of linear function affects its graph identifying the relations between the numeric value of slope and the steepness of a line finding slope of a function when its graphs is drawn on a Cartesian plane or a non-homogenous coordinate system	Pre (Post) Question 1(1), 4 (6), 7 (9), and 10 (10)	Session 2 and 3
Finding slope value from tabular and various algebraic forms of linear functions	finding the slope of a linear function from its tabular representation identifying slope value from various algebraic representations of linear functions (e.g., slope-intercept form) forming an algebraic equation by using the slope value and a pair of the points that the equation satisfies	Pre (Post) Question 2 (4), 3 (2), 10 (10)	Session 3 and 4

As shown in Table 3, pre- and posttest also consisted of three types of questions assessing students' conceptual, procedural, and combined conceptual and procedural knowledge of slope.

Table 3

Categories of Questions throughout the Pre-and Posttest

Pretest	**Posttest**	**Type of knowledge needed to answer the question**
Q1	Q1	Conceptual
Q2	Q4	Procedural
Q3	Q2	Procedural
Q4	Q6	Combined conceptual and procedural
Q5	Q7	Combined conceptual and procedural
Q7	Q9	Procedural
Q8	Q5	Conceptual
Q9	Q3	Conceptual
Q10	Q10	Combined conceptual and procedural

Sources of VMs. In the study, students used the VMs from the National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu), SeeingMath (http://seeingmath.concord.org), and National Council of Teachers of Mathematics (NCTM) Illuminations (http://illuminations.nctm.org) web sites. All of the VMs were incorporated into a standard computer interface, which presented each VM along with exploratory open-ended or structured mathematics questions.

Procedures

Each student participated in six sessions, one session to complete the background questionnaire and pretest, four sessions for the intervention, and one session to complete the posttest (see Table 4). In the first session, participants completed the background questionnaire and the pretest on the paper-pencil format, with 45 minutes to complete the pretest. Then, students were randomly assigned into two groups: VMs with open-ended exploratory questions (OVM) group and VMs with structured mathematics questions (SVM) group.

Throughout the intervention sessions, OVM students used VMs to answer open-ended exploratory questions asking them to observe the activities on the VMs, think about their interactions with VMs, and identify the relations among different mathematical ideas based on their observations. While working with VMs, OVM students used their own initial values based on their choices to answer open-ended exploratory questions that rarely required using numeric expressions, algebraic relations, calculations, and mathematical procedures. In contrast to OVM group, SVM students worked with VMs to answer the structured mathematics questions that mostly entailed using formal mathematics language such as numeric values, expressions, calculations and mathematical procedures during the intervention sessions. Structured mathematics questions mostly have one correct answer and required students completing particular steps or executing specific formulas. Moreover, these questions posed particular initial values that SVM students had to use while working with VMs. Therefore, SVM students had considerably less freedom than OVM group in their activities with VMs to answer the intervention questions.

Students participated in four 30-to-45 minute intervention sessions, they completed first two sessions with 10-15 minutes break between the sessions in one day, and at least one day at most one week later they completed the other two sessions with 10-15 minutes break between the sessions. In the intervention sessions, students worked with VMs individually in a computer laboratory, with OVM students working on open-ended exploratory questions, and SVM students working on structured mathematics questions.

Students in the SVM group were presented with structured mathematics questions on the right side of the computer screen, next to the VM. After working with VMs to answer the intervention session questions, SVM students typed their answers on the spaces provided on the computer. Students in the OVM group worked with the same VMs, but used them to answer open-ended exploratory questions. They, too, typed their responses on the computer. Throughout the intervention sessions, students' written answers were automatically saved by the computer.

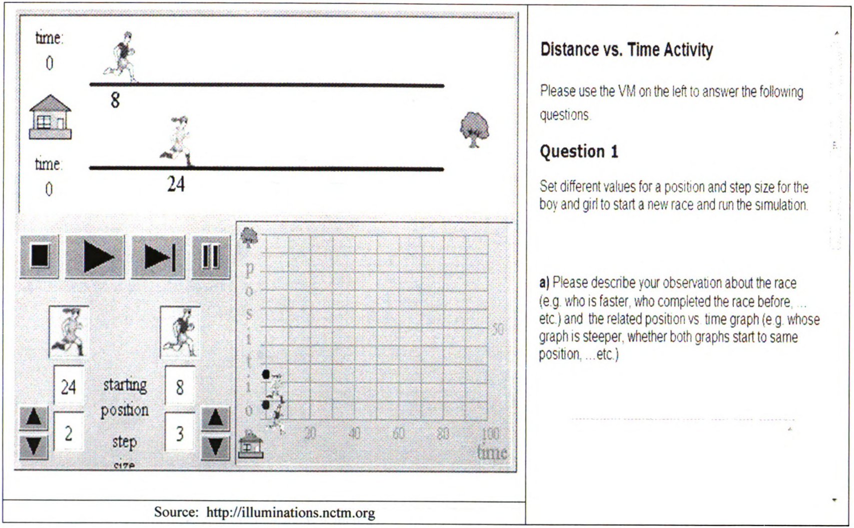
Table 4

Design of the Study for the VMs Used with Exploratory versus Structured Mathematics Questions

Sessions		**VMs with Exploratory Q.**	**VMs with Structured Q.**	**Order of Sessions**
Pretest		OVM	SVM	Session 1
Intervention Sessions	**Distance vs. Time Activity**	OVM	SVM	Session 2
	Geoboard Activity	OVM	SVM	Session 3
	Graphs Activity	OVM	SVM	Session 4
	Linear Trans. Activity	OVM	SVM	Session 5
Posttest		OVM	SVM	Session 6

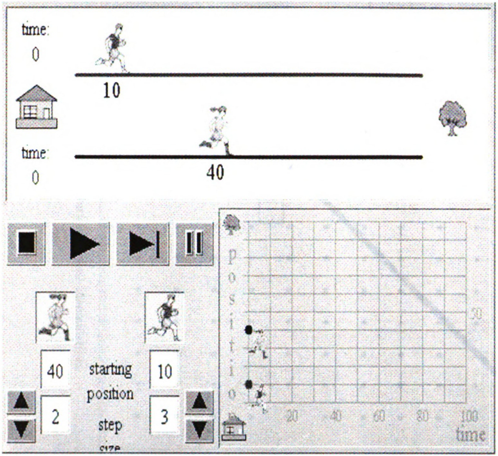
Intervention Session 1 (Session 2 - Distance vs. Time Activity). This session aimed to help students understand the meaning of slope in a real-life setting by providing a VM that provides the simulation of two runners' actions along a track and presents position-time graphs of these two runners. Students in the OVM group worked with the VM (see Figure 1), guided by open-ended exploratory questions (see Appendix D) about the rate of change and its relation to slope. Students in the SVM group used the same VM (see Figure 2) to answer structured mathematics questions (see Appendix D) focusing on the relations between slope and rate of change.

Figure 1. A *screenshot of the VM with exploratory open-ended questions.*



Source: http://illuminations.nctm.org

Figure 2. *A screenshot of the VM with structured mathematics questions.*



Source: http://illuminations.nctm.org

Distance vs. Time Activity

Please use the VM on the left to solve the following questions.

Question 1

Please set the starting position and step size for both runners on the VM in the following way

Boy

Starting position: 10 Step size: 3

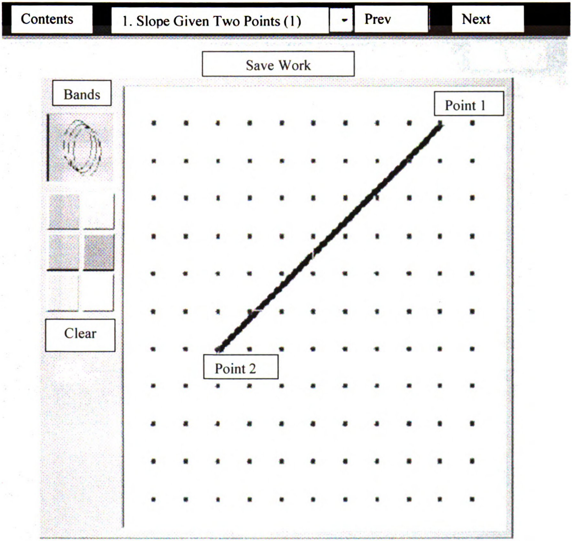
Girl

Starting position: 40 Step size: 2

a) Then, run the simulation. Now, write a story that describes the trip. For example, "*The girl is going really fast. She catches up to and passes the boy, who is going slow.*" or "*The girl started way behind the boy, who was already halfway to the tree by the time she got going. She went really fast and caught up to him more and more. Finally, at 75 she passed him and kept going really fast and got to the tree first.*"

Intervention Session 2 (Session 3 - Geoboard Activity). In this session, students were expected to improve their ability to identify the slope of a linear function when they had two different points that satisfy the function. Students used the VM (see Figure 3) that enabled them to draw lines between different points on a Geoboard to respond the open-ended exploratory or structured mathematics questions (see Appendix D).

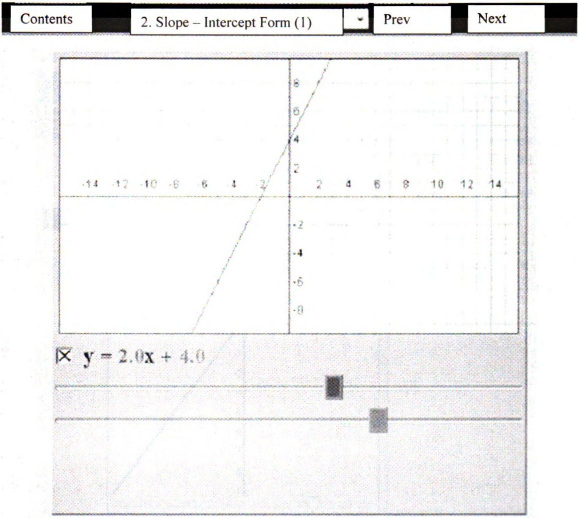
Figure 3. *A screenshot of the VM that students used in Intervention Session 2.*



Source: National Library of Virtual Manipulatives

Intervention Session 3 (Session 4 - Graphs Activity). This session aimed to help students develop their ability to find slope value from algebraic and graphical representations of linear functions and identify various meanings of slope value through these representations. Students worked with the VM (see Figure 4) that provides both algebraic and graphical representations of linear functions to respond open-ended exploratory or structured mathematics questions (see Appendix D).

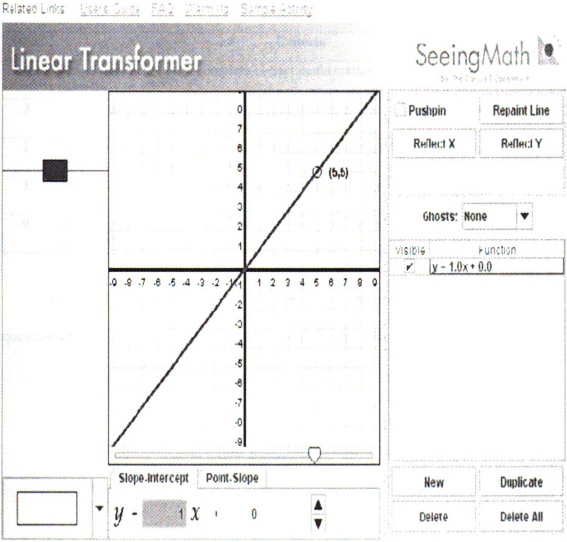
Figure 4. *A screenshot of the VM that students used in Intervention Session 3.*



Source: National Library of Virtual Manipulatives

Intervention Session 4 (Session 5 - Linear Transformer Activity). In this session, students were expected to improve their ability to make connections between slope-intercept and point-slope forms of linear functions by using slope concept and its graphical representation. In the session, students used the VM (see Figure 5) that provides graphical representations correspondent with slope-intercept and point-slope forms of linear functions to answer open-ended exploratory or structured questions (see Appendix D).

Figure 5. *A screenshot of the VM that students used in Intervention Session 4.*



Source: http://seeingmath.concord.org

Posttest Session. After completing the four intervention sessions, students received the posttest (see Appendix C) which has the same set of questions with the pretest in a different order, and had 45 minutes to complete the test.

Pre- and Posttest Scoring

Responses on the pre- and posttest were scored using the rubric presented in Table 4. The researcher developed this rubric using the approach taken in his work with colleagues on MSU's Teachers for a New Era (TNE) project.

Table 5

General Rubric for Pre- and Posttest Scoring

Score	Criteria
4	Responses that present complete correct mathematical explanations with correct answers.
3	Responses that lack clarity in their explanations and/or include minor calculation errors.
2	Responses that show a chain of mathematical reasoning but include major conceptual errors or incomplete solutions.
1	Responses that involve at least one correct and relevant mathematical statement.
0	Responses consisting only of mathematically incorrect, irrelevant, or blank statements.

Based on this general rubric, the researcher developed specific rubrics for each question on the pre-and posttest (See Table 6 for a particular rubric).

Table 6

Rubric Used Scoring Students' Answers on the Pre (Post) Question 8 (5)

Score	Criteria
4	Responses that show the steepness of current ramp with correct mathematical explanations, and include complete correct explanations about how to change the dimensions of the ramp to obtain a new ramp that has the same steepness with the current ramp.
3	Responses that provide correct mathematical explanations to find the steepness of current ramp, and to determine the new dimensions of a new ramp with the same steepness, but include some calculation errors and therefore provide incorrect results for the steepness of the ramp, and the dimensions of the new ramp or Responses that correctly show the dimensions for a new ramp which has the same steepness by using proportions but include incomplete or unclear explanations about how he found the dimensions of new ramp.
2	Responses that correctly show the dimensions of a new ramp (that has the same steepness with current ramp), but did not provide any explanation about how he found the dimensions of the new ramp or Responses that show a chain of reasoning (that focuses on the relation between height and length of the ramp) in their explanations about the dimensions of a new ramp but involve incomplete explanations and/or provide incorrect values about the dimensions of the new ramp.
1	Responses that include at least one correct and relevant mathematical statement.
0	Responses that present mathematically incorrect, irrelevant, or blank statements.

To determine the Interrater reliability of the scoring, a second rater who is a mathematics education doctoral student scored the pre-and posttest responses of 10 randomly selected participants. The Interrater reliability is defined as the Pearson correlation between two scores from two different raters. The Pearson correlation between two raters' scores was calculated to identify the Interater reliability of scoring. As seen from Table 7, there is a high level of agreement between two scorers.

Table 7

Interrater Reliability of Scoring for Overall and Subsections of the Pre-and Posttest

Sections	**Interrater Reliability**	
	Pretest	**Posttest**
Overall	0.93	0.95
Conceptual	0.92	0.95
Procedural	0.94	0.96
Combined Con. and Proc.	0.94	0.95

The scoring process revealed problems with one item: Pretest Question 6 (Posttest Question 8). The purpose of this item was to evaluate students' understanding of slope on a non-homogenous system whose axes had different scales. Because of unclear wording of the item, students' performance could not be validly assessed on that item. Therefore, the item was dropped from the data analysis.

CHAPTER 4

RESULTS

Research Question 1 concentrated on students' overall performance throughout the pre- and posttest. Therefore, all students' gain scores were found in the following way. First, students' overall scores on the pretest and posttest were calculated by summing their scores on the nine questions in the pretest and posttest. Then, students' overall gain scores were computed by subtracting their overall pretest scores from their overall posttest scores. Table 8 presents overall mean pretest, posttest, and gain scores for both OVM and SVM groups.

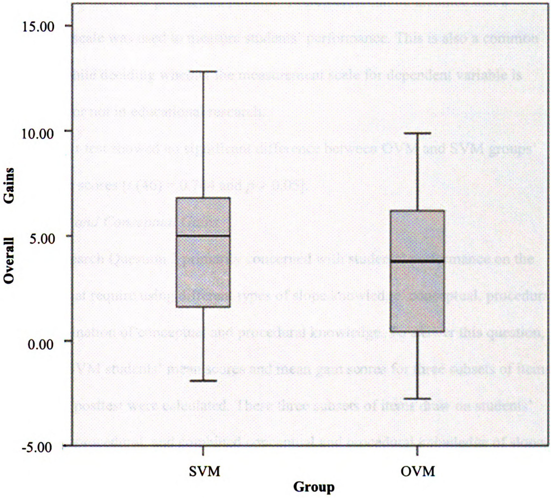
Table 8

Means for OVM and SVM Groups' Overall and Gain Scores

Groups	**Pretest**		**Posttest**		**Gain**	
	M	*SD*	*M*	*SD*	*M*	*SD*
OVM	7.78	4.16	11.48	5.96	3.70	3.59
SVM	8.74	4.46	13.23	5.27	4.49	3.68
Overall	8.26	4.29	12.36	5.63	4.1	3.62

OVM and SVM group students had similar overall mean scores on the pretest and posttest, and revealing no large differences in their gain scores (see Figure 6).

Figure 6. *Comparison of OVM and SVM groups' overall gain scores.*



An independent *t*-test was conducted to test whether there is significant difference between OVM and SVM students' overall gains. The data met the following assumptions of the independent *t*-test. First, both OVM and SVM group's overall gain scores have a roughly normal distribution. Second, the distribution of scores in each group was homogenous (in other words, both groups had similar variance). Third, the observations in two groups were independent since students were randomly assigned into the groups. Finally, *t*-tests are appropriate for continuous scale data. Although ordinal scale (from 0 to 4) was used to assess students' knowledge in each question, their scores for nine questions on the pretest and posttest were summed. That means each student can have scores from 0-36 on the pretest and posttest. Therefore, it can be assumed that a continuous scale was used to measure students' performance. This is also a common approach while deciding whether the measurement scale for dependent variable is continuous or not in educational research.

The *t*-test showed no significant difference between OVM and SVM groups' overall gain scores [$t\ (46) = 0.744$ and $p > 0.05$].

Procedural and Conceptual Gains

Research Question 2 primarily concerned with students' performance on the questions that require using different types of slope knowledge: conceptual, procedural, and a combination of conceptual and procedural knowledge. To answer this question, OVM and SVM students' mean scores and mean gain scores for three subsets of items on the pre-and posttest were calculated. These three subsets of items draw on students' conceptual, procedural, and combined conceptual and procedural knowledge of slope, respectively (see Table 3).

Procedural items. As shown in Table 9, SVM students attained considerably higher gain scores than OVM group in the subset of procedural knowledge questions.

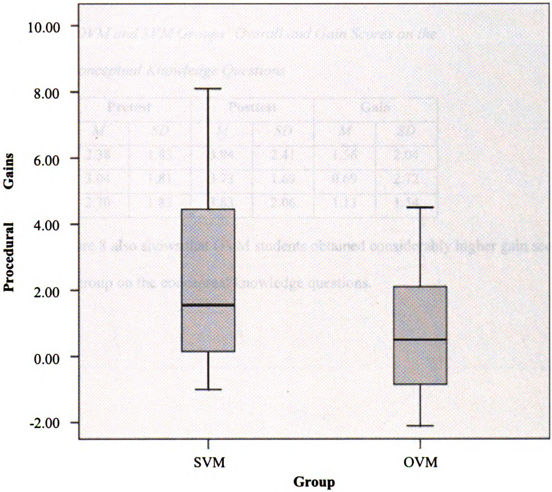
Table 9

Means for OVM and SVM Groups' Overall and Gain Scores on the Subset of Procedural Knowledge Questions

Groups	**Pretest**		**Posttest**		**Gain**	
	M	*SD*	*M*	*SD*	*M*	*SD*
OVM	2.84	2.26	3.73	2.96	0.89	2.04
SVM	3.3	2.43	5.51	3.04	2.21	2.72
Overall	3.07	2.33	4.62	3.1	1.55	2.47

As seen from Figure 7, there is a substantial difference between OVM and SVM students' scores on the subset of procedural knowledge questions.

Figure 7. *Comparison of OVM and SVM groups' gain scores on procedural knowledge questions.*



Conceptual items. Although students in OVM group showed lower performance than SVM students on the procedural knowledge questions, they had higher gain scores than SVM students for conceptual knowledge questions. As seen in Table 10, OVM students' mean gain scores is 1.56 and the mean of SVM students' gain scores is 0.69.

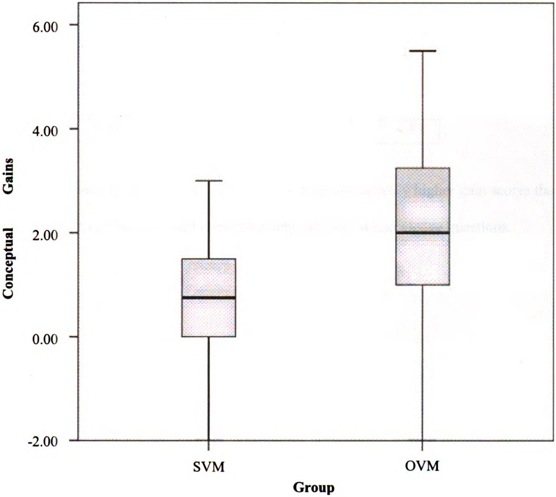
Table 10

Means for OVM and SVM Groups' Overall and Gain Scores on the Subset of Conceptual Knowledge Questions

Groups	**Pretest**		**Posttest**		**Gain**	
	M	*SD*	*M*	*SD*	*M*	*SD*
OVM	2.38	1.83	3.94	2.41	1.56	2.04
SVM	3.04	1.81	3.73	1.69	0.69	2.72
Overall	2.70	1.83	3.83	2.06	1.13	1.54

Figure 8 also shows that OVM students obtained considerably higher gain scores than SVM group on the conceptual knowledge questions.

Figure 8. *Comparison of OVM and SVM groups' gain scores on conceptual knowledge questions.*



Combined conceptual and procedural items. SVM group students also obtained higher gain scores than OVM students on the subset of questions that combined conceptual and procedural knowledge of slope. As presented in Table 11, SVM students' mean gain score is 1.6 and the mean of OVM students' gain scores is 0.66.

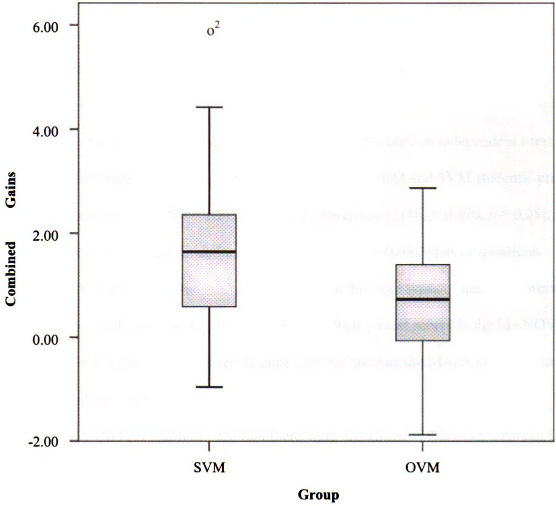
Table 11

Means for OVM and SVM Groups' Overall and Gain Scores on the Subset of Combined Conceptual and Procedural Knowledge Questions

Groups	**Pretest**		**Posttest**		**Gain**	
	Mean	SD	*Mean*	*SD*	*Mean*	*SD*
OVM	2.66	1.55	3.32	1.81	0.66	1.09
SVM	2.62	1.52	4.22	1.73	1.6	1.6
Overall	2.64	1.52	3.77	1.81	1.13	1.43

As seen from Figure 9, SVM students had considerably higher gain scores than OVM group on the combined conceptual and procedural knowledge questions.

Figure 9. *Comparison of OVM and SVM groups' gain scores on the combined conceptual and procedural knowledge questions.*



In addition to these descriptive statistics, a Multivariate Analysis of Variance (MANOVA) test was conducted for OVM and SVM students' gain scores on conceptual, procedural, and combined conceptual and procedural knowledge of slope questions.

OVM and SVM students' gain scores in three types of questions satisfied the following assumptions of the MANOVA test: normal distribution of students' gain scores in each type of knowledge, equality of variance between groups in three types of knowledge, independence of observations between OVM and SVM groups in the study.

Moreover, there are two main reasons for using the MANOVA test to determine the significance among three outcomes (students' gain scores in three types of knowledge). First, there is a significant correlation between OVM and SVM students' gain scores on procedural, conceptual and combined conceptual and procedural knowledge questions. This means that there is some relation (or overlap) between students' gain scores. Therefore, using a MANOVA test is more appropriate than an ANOVA test that requires all outcome variables have no relation with each other. Second, an independent t-test indicated that there is no significant difference between OVM and SVM students' pretest scores on conceptual [$t\ (46) = 1.272, p > 0.05$], procedural [$t\ (46) = 0.670, p > 0.05$], and combined conceptual and procedural [$t\ (46) = -\ 0.093, p > 0.05$] types of questions. Because OVM and SVM students' pretest scores on the three types of questions were not statistically significant, there is no need to control their pretest scores in the MANOVA test. Thus, using the MANOVA test is more appropriate than the MANCOVA test to analyze students' gain scores.

A MANOVA test indicated that there were significant differences between OVM and SVM students' gain scores in three types of questions. The results of the MANOVA test can be summarized in the following way. First, SVM students had significantly higher gains than OVM students in the subset of procedural questions [$F\ (1, 46) = 4.409$, $p < 0.05$, partial $\eta^2 = 0.087$]. Second, SVM group had significantly higher gains than OVM students in the questions that require using combined conceptual and procedural knowledge of slope [$F\ (1, 46) = 5.688$, $p < 0.05$, partial $\eta^2 = 0.110$]. Finally, OVM students attained significantly higher gains than SVM students in the conceptual questions [$F\ (1, 46) = 13.562, p < 0.05$, partial $\eta^2 = 0.228$].

Item by Item Analysis

To identify the learning activities in the intervention sessions that helped students obtain the most gains on the pre-posttest items, OVM and SVM students' mean gain scores on each test item were examined. This analysis also shed light on how OVM and SVM students' activities with VMs in the intervention sessions enabled them to show considerable differences between their gain scores on three types of questions throughout the pre- and posttest.

Mean gain scores were computed for the OVM and SVM groups on each pre-posttest item. The OVM group attained largest mean gain score on conceptual question 3, and the SVM group had the highest mean gain score on procedural question 2. In addition, these two questions are the items on which both OVM and SVM group showed the largest differences between their mean gain scores. As seen in Table 12, the difference between OVM and SVM groups' mean gain scores are 0.94 and 0.72 on conceptual question 3 and procedural question 2 respectively.

Table 12

Mean Gain Scores Item by Item

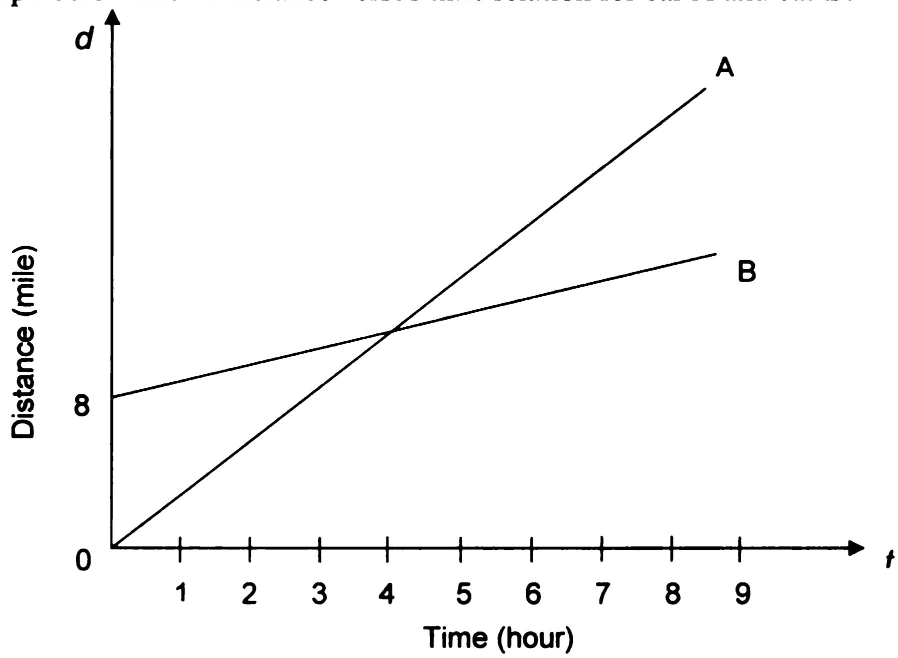
Question Type	Pretest (Posttest) Q. No.	Groups	
		OVM	SVM
Con. Q. 1	Pre Q. 1 (Post Q. 1)	0.79	0.42
Con Q. 2	Pre Q. 8 (Post Q. 5)	0.5	0.21
Con Q. 3	Pre Q. 9 (Post Q. 3)	0.90	-0.04
Proc. Q. 1	Pre Q. 2 (Post Q. 4)	0.13	0.88
Proc. Q. 2	Pre Q. 3 (Post Q. 2)	0.34	1.06
Proc. Q. 3	Pre Q. 7 (Post Q. 9)	0.40	0.38
Com. Q. 1	Pre Q. 4 (Post Q. 6)	0.19	0.31
Com. Q. 2	Pre Q. 5 Post Q. 7)	0.13	0.71
Com. Q. 3	Pre Q. 10 (Post Q. 10)	0.34	0.58

Conceptual question 3 (see Figure 10) that OVM group obtained the highest mean gain score aimed to measure students' conceptual knowledge of slope by asking them to compare the speed of two cars based on their distance vs. time graphs.

Procedural question 2 (see Figure 11) that SVM group had the largest mean gain score entailed students finding the slope values of linear functions presented in different algebraic forms in order to assess their procedural knowledge of slope.

Figure 10. *Conceptual question 3.*

The graph below shows distance versus time relation for car A and car B.



a) When t = 2 hours, is the speed of car A greater than, less than, or equal to the speed of car B? Please explain your answer with reasons.

b) Do car A and car B ever have the same speed? If so, at what times? Please explain your answer with reasons.

Figure 11. *Procedural question 2.*

Please find the slope of each of the following equations and provide your explanations.

a) $f(x) = \frac{1}{2} - 4x$

b) $f(x) = 2x + 1 - 2x$

c) $f(x) = -x - \frac{1}{4} + \frac{1}{3}x + 1$

d) $y + 0.25 = \frac{1}{2}(x-2) - 2(x-2)$

e) $3y - 2 = \frac{1}{2}(x - 1)$

These two items with large differences in mean gain scores may help explain how students' learning activities on the intervention sessions enabled them to have higher gains on different types of slope knowledge. Because students showed the largest increase in their gain scores on these two items, analysis of how they responded to the questions in the intervention sessions that were most related to these items may provide considerable evidence about the effects of students' different learning activities on their gain scores. Throughout all the intervention sessions, OVM and SVM students engaged with activities pertaining to pre-and posttest items. Intervention Session 1 and 4 included questions that were particularly related to conceptual questions 3 and procedural question 2 respectively.

Students' responses on these two intervention sessions were examined. The analysis indicated that students' answers can be categorized in two ways. One way of categorization is assigning students' responses into different groups based on the type of expressions they included. Throughout the intervention session 1 and 4, students' responses mostly included one of the following four types of expressions.

Responses including only statements (Stat.). This type of response contains only statements that present no explanations or descriptions of activities with VMs. For example:

"As the position increase so does the time."

"The equation is $y = 0.5x + 3.9$."

"The runner's speed is dependent on the time he or she is allotted."

Responses with the descriptions of activities on the VMs (Desc. of acts. on VMs). On this type of responses, students described the events that occurred on the VMs while interacting with them. For example:

"y=3.0+2.0 and point slope i get y-11.0=3.0(x-3.0) i just plugged in numbers until i got what you were asking for."

"The girl started ahead of the boy. She really does go fast until boy catched [caught] up! They were both very close at the end and the girl won by an inch."

"I found slope intercept form. I used the points (-2, 3) and plotted it on the graph then moved the line until 0.5 became my x."

Responses consisting of explanations with reasons (Exp.). This type of responses involves at least one explanation with reasons. These responses often display cause-effect relations or provide evidences to support their conclusions. For example:

"The boy will have a different distance vs. time because he now has a different feet [foot] per second and the graph will be different because they are at different positions."

"Slope-intercept form is extremely easier to deal with because it has a simple graphing demonstration compared to the point-slope form it is more difficult because it has a distributing involved. The similarities are when there changes in the slope and *y*-intercept."

"The more steps per second, the faster the person will finish the race. This is so, because the more steps an individual does, their rate becomes faster."

Responses with explanations and descriptions of activities (Exp. & Desc.). Some of the explanation responses also included descriptions of learning activities with VMs. Responses in this category often provide descriptions of events occurring on VMs as evidence to support their conclusions. For example:

"The boy completed the race before because his step size was at 5 the girl at 3. The boy and girl both started at the same spot but the boy still finished within 15 seconds and the girl in a little over 20 seconds."

"The speed is slower, but because the step size is larger it helps them to win the race. Example- the girls step size was 3 and she finished 60 (time) before the guy whose step size was 1."

"When I observe the graph of the linear equation I can see that the changes I create affect the line in both the slope-intercept and the point-slope. When I change the slope in the slope-intercept form it rises [raises] the line on the *y*-axis and can also lower on the y-axis. Looking at the point-slope form I can see that the change I make in the equation moves the line down the *y*-axis because I am increasing the number. All the changes I make reflect on the *x* and *y*-axes dramatically."

The percentages for these four types of responses in intervention Session 1 and Session 4 were presented on Table 13.

Table 13

Categories of Students' Responses in Session 1 and 4 (by %)

Sessions	Group	Exp.	Desc. with VMs	Exp. & Desc.	Stat.
Session 1	OVM	69	39	24	16
	SVM	43	48	20	30
Session 4	OVM	63	20	6	24
	SVM	39	12	0	56

Throughout Sessions 1 and 4, OVM students more often provided explanations in their responses than SVM group. As shown in Table 13, 69% and 63% of OVM students explained their answers on Session 1 and 4 respectively. In contrast to OVM students, only 43% and 39% of SVM students made explanations on these two sessions. Similarly, SVM students' answers more often included statements than OVM group over the two sessions. As presented in the table, 30% and 56% of SVM students provided statements in their responses on Session 1 and 4 respectively, however only 16% and 24% of OVM students included statements in their answers on the two sessions respectively.

Both OVM and SVM students also differed in the percentage of their responses between Session 1 and 4. For example, more number of SVM students provided descriptions of their activities with VMs than OVM students in Session 1. However, this difference occurred in the opposite way in Session 4. As seen in Table 13, in Session 1, 48% of SVM students included descriptions of their activities with VMs whereas 39% of OVM students provided descriptions in their responses. However, in Session 4, only 12% of SVM students contained descriptions in contrast to 20% of OVM students. In addition to these differences, both groups also showed similar changes in their responses between Session 1 and 4. For example, both OVM and SVM groups had less percentage of responses including descriptions of activities with VMs on Session 4 than they did on Session 1. As shown in Table 13, the percentage of OVM and SVM students whose responses involved descriptions of activities with VMs decreased from 39% to 20% and from 48% to 12% throughout the Session 1 and 4 respectively.

The other way of categorizing students' responses is based on whether their responses include quantitative or qualitative relations. On the intervention session 1 and 4, students' answers often contained one of the following three types of expressions.

Responses with numeric relations or algebraic expressions (Num.). This type of responses contained numeric relations among quantities or algebraic equations. For example:

"The girl step size was half the size of the boy and the boy reached the end of the race in slightly more then [than] half of the time it took the girl to finish the race. The boy's graph was steeper."

"The girls speed is 2 steps per second. The boys speed is 3 steps per second. So, the boy covered more distance."

"$y = 0.5x = 3$ and $y-3 = 0.5(x-3)$ would be the two equations for the line."

Responses including calculations (Calc.). These responses included various types of calculations. For example:

"75-20 = 55- boy's, 20+20 = 40- girl's"

"x = 40-0/20 = 20 20 = x (the girl), x = 100-10/20 = 4.5"

"Boy: 30/20 = 1.5, Girl: 70/20 = 3.5"

Responses involving qualitative relations (Qual.). This type of response included qualitative relations among variables such as relations between graphs and runners' speeds or point-slope forms of linear equations. For example:

"When both the boy and girl have the same starting position, the person with the larger step size is faster and has a steeper graph."

"The girl and boy both started at 10. The boy had step of 5 and the girl had 3. The boy wins this race. He is faster than her because he takes larger steps and covers more ground. The larger the steps the steeper the line."

"When I change the slope in point-slope form, it changes the steepness of the line about a point. The line tilts about a fixed point. If I keep increasing it, it slowly turns about the point till it is nearly a vertical straight line, crosses the Y-axis, and slopes negatively."

As in the first way of categorization, these three types of responses emerged from students' answers in Intervention Session 1 and 4. The percentages for the three types of responses on these two sessions are presented in Table 14.

Table 14

Percentage of Students' Responses by Category in Session 1 and 4 (by %)

Sessions	Group	Num.	Calc.	Qual.
Session 1	OVM	9	0	36
	SVM	23	11	8
Session 4	OVM	22	1	39
	SVM	43	1	0

More number of OVM students than SVM group used qualitative relations in their responses throughout the session 1 and 4. As shown in Table 13, 36% and 39% of OVM students provided qualitative relations in their answers in Session 1 and 4 respectively while 8% of and no SVM students included qualitative relations in their responses in Session 1 and 4 respectively.

While making these two ways of categorization, it has not been taken the correctness of students' responses on the intervention sessions into consideration. In other words, including mathematically correct or incorrect explanations and statements had no effect on the categorization of students' answers. There are two main reasons for why students' responses were not examined in terms of their mathematical correctness. One reason is that since some questions entailed students making observations and reflecting their ideas about what they did with VMs, students' answers to this type of open-ended questions cannot be evaluated as mathematically correct or incorrect. The other reason is that sometimes students chose the equations or numeric values themselves under certain conditions and worked with their own input to answer the questions, and in some cases students did not explicitly present their initial values in their responses, thus it is difficult to identify these students' responses as mathematically correct or incorrect.

CHAPTER 5

DISCUSSION

This study examined the effects of using VMs with two different instructional approaches on students' knowledge of slope. Analysis of students' responses on the pre- and posttest showed that both OVM and SVM groups increased their overall knowledge of slope from the pre- to posttest with similar amounts of improvement in their overall knowledge of slope. This suggests that VMs with two different instructions enabled both groups to improve their knowledge of slope.

Although students in both groups showed similar amount of improvement in their overall knowledge of slope, SVM students differed from OVM students in their gains on the subsets of items tapping three types of slope knowledge; conceptual, procedural, and combined conceptual and procedural knowledge of slope. In particular, the OVM group had higher gains on the conceptual items; SVM students showed higher gains on the procedural and combined conceptual and procedural items. It is important to identify the reasons why the groups differed in the amount of improvement in three types of slope knowledge.

Analysis of students' pre-and posttest data indicated that the OVM group students had significantly higher improvement than the SVM group in their conceptual knowledge of slope. OVM students' activities with open-ended exploratory questions on the intervention sessions might have contributed to this higher improvement in the conceptual knowledge of slope. Throughout the intervention sessions, OVM students worked on the questions similar to the conceptual knowledge questions on the pre-and posttest — questions that emphasized providing explanations with reasons and identifying relations among different mathematical ideas. In the intervention sessions, the OVM group answered open-ended exploratory questions that enabled them to think about their observations related to their interactions with VMs, identify relations between their observations and mathematical ideas, and explain their responses with reasons. These activities might have helped OVM students obtain higher gain scores than SVM students in the conceptual knowledge questions that require reflection, providing responses with reasons, and making connections among various representations of slope concept.

In contrast to the OVM group, SVM students worked with the questions that mostly required using mathematics procedures and numeric values throughout the intervention sessions. These questions rarely entailed students making reflections on their observations or identifying relations between their observations and ideas.

As OVM students worked on the questions requiring explanations and making connections among various mathematical ideas during the intervention sessions, they included more explanations and qualitative relations in their responses than SVM group in these sessions. For example, in Session 1, 69% of OVM students included explanations in their responses while 43% of SVM group explained their answers with reasons. Furthermore, 36% of OVM students presented qualitative relations in their responses whereas only 8% of SVM students included these relations in Session 1. Because more OVM than SVM students included explanations with reasons and qualitative relations in their responses over the intervention sessions, OVM students might have had more potential than SVM students to increase their scores on the conceptual questions.

In contrast to OVM students' greater performance on the conceptual knowledge questions, SVM students improved more than the OVM group on the procedural questions. Throughout the intervention sessions, SVM students worked on structured mathematics questions that led them to engage with more mathematical language, such as mathematical symbols, formulas, and procedures than the other group. These structured mathematics questions often presented particular numbers as input, and required students to find specific numeric values (e.g., what's the speed of the boy and girl runner in the first 30-second?). Like the procedural questions on the pre- and posttest, structured mathematics questions required using mathematics symbols, numbers, formulas and procedures. Also, the analysis of SVM students' responses in the intervention sessions indicated that, as expected, they included more numeric expressions and relations, calculations, and mathematical procedures than OVM students. For example, in Session 1, 23% and 11% of SVM students included numeric expressions, relations and calculations in their responses, respectively, whereas only 9 % of OVM students used numeric expressions and none of them included calculations in their responses. Because SVM students included more numeric expressions and relations, calculations, and mathematical procedures in their responses than the OVM group throughout the intervention sessions, they became more familiar with procedural questions than the other group. Therefore, SVM students' activities with structured mathematics questions during the intervention sessions might have helped them attain higher gain scores than the OVM group on the procedural questions.

In addition to their improvement on the procedural questions, SVM students also obtained higher gain scores on the combined conceptual and procedural items than the OVM group. However, this does not mean that instruction for the SVM group is also effective for improving students' conceptual knowledge of slope, given that SVM students had low gain scores on the conceptual items over the pre-and posttest. There might be several reasons for why SVM students did significantly better than the other group students in the questions that require using combined conceptual and procedural knowledge of slope. These questions might have focused on students' procedural knowledge of slope more than their conceptual knowledge. If these combined-type questions emphasized students' procedural knowledge of slope more than their conceptual knowledge of slope, SVM students might have obtained higher gain scores than the OVM group by using their procedural knowledge.

Another reason might be that combined conceptual and procedural questions required using first procedural knowledge, and then applying conceptual knowledge of slope. If students possessed the required procedural knowledge, they could move on the next step that required using their conceptual knowledge of slope. If OVM students had had insufficient procedural knowledge to solve combined-type questions, they might have been unable to use their conceptual knowledge successfully in these questions. Therefore, OVM students might have obtained lower gain scores than the other group in these questions.

Students' responses on the intervention sessions also provided some insights about the reasons of why SVM students had higher gain scores than OVM students on combined conceptual and procedural items. In session 1, many SVM students included both explanations and descriptions of their activities with VMs, and their percentage (20%) was close to the percentage of OVM students (24%) who provided both explanations and descriptions in their responses. SVM students often used their descriptions of activities with VMs as examples to explain their answers. This implies that since SVM students had already become familiar with explaining their answers with reasons throughout the intervention sessions, they might have increased their scores on the combined conceptual and procedural items over the posttest.

This study also examined students' responses throughout the intervention sessions to identify how students' learning activities with VMs enabled them to show considerable differences in their gain scores on the conceptual, procedural, and combined conceptual and procedural questions over the pre-and posttest. During the intervention sessions, OVM students' responses involved more explanations with reasons and descriptions of qualitative relations among variables than the SVM group. It is possible that OVM students' engagement with explanations and relations between variables helped them acquire higher scores than SVM group in the conceptual items that require providing explanations with reasons and making connections among various representations of mathematical ideas.

In contrast to the OVM group, SVM students' responses over the intervention sessions involved more numeric relations, expressions, and calculations. Because the SVM group mostly focused on mathematical relations, expressions, and procedures while working with VMs, they might have become more experienced than the OVM group in solving the procedural questions that mostly require using formal mathematics language (i.e., algebraic expressions and procedures). This experience with numeric expressions and mathematical procedures across the intervention sessions may explain, in part, the SVM group's increased scores on the procedural and combined conceptual and procedural items on the posttest.

Analysis of students' responses on the intervention sessions also indicated that intervention questions helped identify how students interact with the VMs. OVM and SVM students used the VMs for different purposes to respond the questions. OVM students often used VMs to explain their answers by describing the events on the VMs as examples for their claims. In contrast, SVM students mostly interacted with VMs to find particular algebraic equations, graphs or numeric values. Thus, questions in the intervention sessions directly influenced how students interacted with the VMs.

Throughout the intervention sessions, both groups provided the descriptions of their activities with the VMs in their responses. The amount of VM activity description, however, varied across sessions. For example, in Session 1, 39% of OVM and 48% of SVM students described their activities with VMs in their answers while in session 4, only 20% of OVM and 12% of SVM students talked about their activities with VMs. Although students were explicitly asked to use VMs to answer the questions at the beginning of each intervention session, in some sessions they rarely referred to their activities with VMs. However, it should be noted that if the students do not mention the events on the VMs or their activities with VMs in their responses, this does not mean that they did not use the VMs. It is possible that students might have used VMs to answer the questions even though they did not refer to their activities with VMs in their responses. However, intervention questions in Session 1 are more effective than the questions in Session 4 in encouraging students to use VMs in their answers. Because students could respond the questions in Session 4 without using VMs, they might have answered these questions with little or no interaction with VMs. Therefore, instructional tasks should be designed in a way that encourages and supports students' learning activities with VMs.

Once instructional activities effectively support students' interactions with VMs, students may realize that they need to use VMs to provide successful responses to given instructional questions. Only stating that students in a learning setting should use VMs to answer the questions might not be enough to initiate and foster students' effective interactions with the VMs.

Throughout the four intervention sessions, two different instructional approaches have been used with the same set of VMs. In one approach, the SVM group worked on structured mathematics questions. While studying these questions, SVM students were exposed to more mathematics symbols, expressions, and procedures than the other group. In this type of questions, SVM students often needed to begin with given numeric values and apply some mathematical procedures or formulas to answer the questions. In the other instructional approach, OVM students mostly focused on exploratory learning activities (e.g., interpreting the activities on the VMs, making connections among their observations, and supporting their answers with reasons). These two instructional approaches were chosen for the study for two main reasons. First, these approaches at some level reflect traditional versus reform-based approaches to mathematics instruction. Second, these two approaches can also be considered to be an extension of discussion about whether conceptual or procedural type of knowledge has the most importance in students' mathematics learning among mathematics education researchers.

Because the two instructional approaches used in this study have deep roots in both practical and theoretical areas of mathematics education, this study attempted to show how using the same technology with these two approaches affected students' mathematics learning. The two different approaches helped students improve different types of mathematics knowledge. They had also influences on students' interactions with VMs. This shows how technologies with different instructional approaches affect both our knowledge and our learning activities with them. This also confirms the claim that technologies do not only influence our knowledge but also our learning activities with them (Pea, 1987).

Clark (1983) recommended that researchers should explore the effects of different instructional methods within various media on learning instead of studying the impact of one specific media on learning. This study followed Clark's suggestion by exploring the effects of a single medium—a set of VMs—with two different instructions on students' knowledge of slope.

Strengths and Limitations

In the study, students' written responses were automatically recorded by the computer throughout the intervention sessions. Students never wasted time by writing their answers on the paper while working with VMs. In other words, students worked with VMs without having any distraction, i.e. using paper and pencil. All students completed their activities on the computer during the intervention sessions. This might have helped students deeply focus on their learning activities, and had more opportunities to improve their knowledge of slope by using VMs.

Forty-eight students completed all session of the study. The number of students taking MTH 1825 in Fall 2008 semester was about 800. Having 48 participants out of nearly 800 students in the study considerably increased the chance to identify the differences between OVM and SVM students' knowledge of slope.

The students who participated in the study were taking a remedial mathematics course. Because the effects of VMs on students' learning of slope were particularly examined, all participants completed the study before working on the slope concept in their remedial courses. Therefore, students' learning regarding slope concept from their remedial courses did have no influence on their knowledge about slope while they were participating to the study. However, it is always possible that students could learn about slope in their other mathematics or related courses. In these courses, instructors might have slightly mentioned about the slope concept to help students learn other relevant mathematics concepts or ideas. This study was designed to help students complete all sessions before they began learn slope concept in their remedial mathematics courses in order to reduce the effects of students' learning about slope outside of the study.

Like every study, this study had several limitations that need to be taken into consideration when interpreting its results and contributions. First, although each participant used additional time to become familiar with the features of VMs, some participants might still have not become comfortable in using the VMs, and therefore they might have not used the VMs effectively to answer the intervention questions. Furthermore, some participants might have misunderstood some tools of the VMs. This might have led them to incorrectly (or inadequately) use some tools of the VMs while responding to the questions. Therefore, they might have had some incorrect or incomplete knowledge about the slope concept.

Another limitation of the study is that students interacted with VMs for a small amount of time. Students worked with VMs for only four 30-45 minute sessions during the two weeks. They had no access to use VMs outside of the study sessions. Therefore, students might not have had adequate time to show considerable changes in their slope knowledge.

A third limitation of the study was the selection of the participants. Students were encouraged to participate in the study with the recommendation of their instructors and a small amount of money offered by the researcher. Students who accepted the offer participated in the study. Therefore, the participants of the study were not randomly selected from the actual population of MTH 1825 students. The lack of random selection of students limits the generalization of this study's findings to a larger population of students.

The other limitation of the study was regarding the data collection. During the intervention sessions, some participants provided short responses. They wrote one or two sentences, they did not elaborate their answers even though the researcher prompted them and encourage them to present their answers with details. Also, when some participants were using the VMs, the Internet was suddenly disconnected. This occurred when students completed their activity and were saving their answers. Because they lost the work, they had to return the activity and complete it again.

Conclusions

This study reported the effects of using VMs with two different instructional approaches on students' knowledge of slope. These two different instructions were (a) providing open-ended exploratory questions that need to be answered by using a set of VMs, and (b) asking students to solve structured mathematics questions while working with the VMs. The results of the study indicated that each type of instruction helped students improve different types of slope knowledge. In particular, the instruction that focused on mathematical procedures and numerical values resulted in greater improvement on tasks requiring procedural and combined conceptual and procedural knowledge of slope. However the instruction that emphasized exploratory learning activities such as observing, analyzing, and interpreting helped students develop their conceptual knowledge of slope.

These findings suggest that teachers can alternatively use VMs with different instructions to help their students improve different types of mathematics knowledge. For example, when students have difficulties in using mathematical procedures, teachers can use VMs with procedural based instructional activities. Teachers can also integrate VMs into their instructional activities that focus on exploratory learning activities to help students improve their conceptual knowledge. As Stein et al. (2000) pointed out, different instructional tasks can be used to improve students' learning of various types of mathematics knowledge. Therefore, VMs with different instructional approaches can help students increase their mathematics knowledge.

This study also revealed that instructional activities should support students' effective interactions with VMs to facilitate their learning of mathematics concepts. As pointed out earlier, only asking students to use the VM to answer a set of questions might not be enough to initiate and foster productive interactions with VMs in their learning. While engaging with instructional activities, students should use VMs to answer the questions. Therefore, teachers need to revise and redesign their instructional activities in a way that enables their students to effectively interact with VMs while learning mathematics concepts.

In the research literature, few studies focused on the impact of VMs on mathematics learning and teaching. This study examined the effects of using VMs with two different instructional approaches on students' learning of a particular mathematics concept to fill this gap in the literature. Based on the findings of this research study, future research can deeply examine the use of VMs with a variety of instructional approaches to improve students' learning of different mathematics concepts.

Appendix A

Information about Recruitment of Students

- The researcher arranged meetings with the instructors of MTH 1825 in the first week of the semester and provided information about the educational benefits of the study for the students.
- The instructors of MTH 1825 were given a paper that provides brief information about the study to make announcement for the study in their courses. The researcher also gave sign-up papers to the instructors.
- Instructors made the announcement for the study in their courses and distributed sign-up papers among their students. Studens who were interested in participate to the study wrote their contact information (name, phone number, e-mail address) and available times to participate the study on these sign-up papers. At the end of the course session, each instructor collected these sign-up papers and gave them to the researcher. Then the researcher contacted with the students in the sign-up papers via e-mail or phone and arranged a meeting time and place with students.
- The researcher hired a college student as a recruiter and informed the recruited about the design of the study. The researcher and recruiter put flyers of the study on announcement boards in classroom halls and dormitories. Flyers included brief information about the study; the rules of the participation, the amount of money students would receive at the end of their participation, and contact information for the researcher (e-mail address and phone number).

- The researcher also prepared small cards that include short information about the study. The recruiter distributed these cards among the students in classroom halls and dormitories. The recruiter also put an announcement for the study on a web site which is popular among college students in the campus.
- Students who had the information about the study from flyers, small cards, or web contacted with the researcher via e-mail or phone. Then the researcher arranged a meeting time and place with students. In each meeting, the researcher met at least 2 at most 5 students. The researcher reserved rooms having wireless connection as meeting places for the study.
- In the first meeting with students, the researcher thanked for their interest to the study and provided detailed information about the study and responded students' questions. Then each student was given a consent form and provided time to read the form. Students who accepted participate to the study signed the consent form and gave it to the researcher.
- Having assigned the consent form, students completed the pretest in the first session of the study, and scheduled a time with the researcher for the next session of the study.

Appendix B

Background Questionnaire

Gender: M F

Age:

1. How many hours per week do you study mathematics on average?

2. How many math courses did you take in the last year of your high school? Please write the name of courses.

3. How many math courses have you taken so far at college? Please write the name of courses.

4. Which specific topics in algebra are most difficult for you? Why are these topics difficult to understand for you?

5. Have you ever been used any technology (e.g. mathematics software, math simulations, applets, virtual (computer) manipulatives,...etc.) in your math courses or outside the school?

    a. If you have used any technology, please write the name of the type of technology.

    b. If you used technology in your courses, please write your grade level and the name of course(s) that you used technology.

    c. Please write 2-3 sentences about your experiences with each type of technology that you used in your courses or outside the school.

6. Do you believe that using educational mathematics software, applets, simulations, or virtual (computer) manipulatives (in school or outside the school) is helpful for students' learning algebra? Why? Please explain your answer with 2-3 sentences.

Appendix C

Pretest (Posttest)

Please show all your work clearly!

<u>**Note:**</u> Use a pen for all your work. If you make a mistake, you can lightly cross out your work. You have **45** minutes to complete this test.

Pretest Question 1(Posttest Question 1)

a) Please write a **definition** of slope with your own words.

b) If you choose any three points on the coordinate plane, can you find a straight line that passes all these three points? Please **explain** your answer with reasons.

c) What do you think about the conditions that make three points on the same straight line? Please **explain** your answer with reasons.

Pretest Question 2 (Posttest Question 4)

x	-4	-2	4
f(x)	3	2	-1

a) Please **find** the slope of the f(x), and **explain** your answer with reasons.

b) Please write an equation for f(x), and **explain** your answer.

Pretest Question 3 (Posttest Question 2)

Please find **the slope** of each of the following equations and **provide your explanations**.

f) $f(x) = \frac{1}{2} - 4x$

g) $f(x) = 2x + 1 - 2x$

h) $f(x) = -x - \frac{1}{4} + \frac{1}{3}x + 1$

i) $y + 0.25 = \frac{1}{2}(x-2) - 2(x-2)$

j) $3y - 2 = \frac{1}{2}(x-1)$

Pretest Question 4 (Posttest Question 6)

Note: This question was adapted from Moschkovich (1996) study.

If you change $f(x) = x$ to the $f(x) = \frac{1}{2} - 5x$, what can you tell about the changes in the graphs of $f(x)$?

a) Does the graph of new $f(x)$ steeper than the graph of previous $f(x)$? Why? Please **explain** your answer.

b) Is it always true that when the numeric value of slope is bigger, the graph of the line always becomes steeper? Please **explain** your answer with reasons.

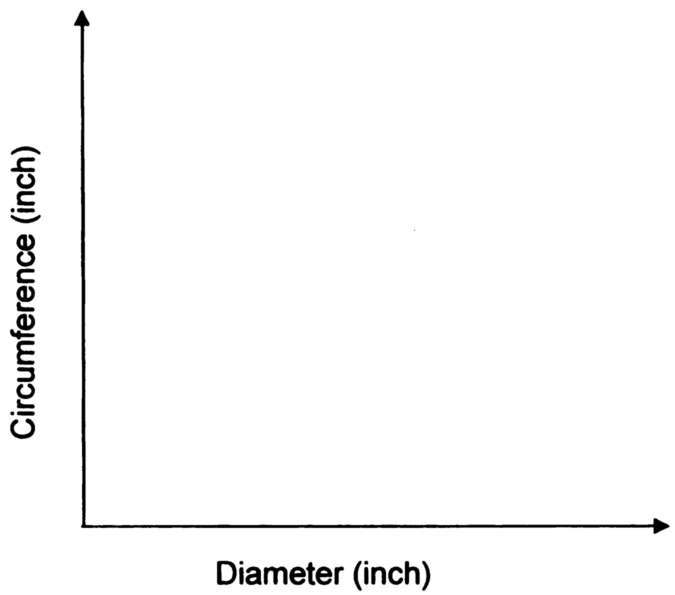
c) Is there any relation between the slope value and the steepness of the line? Please **explain** your answer with reasons.

Pretest Question 5 (Posttest Question 7)

Note: This question was adapted from Stump (1997) study.

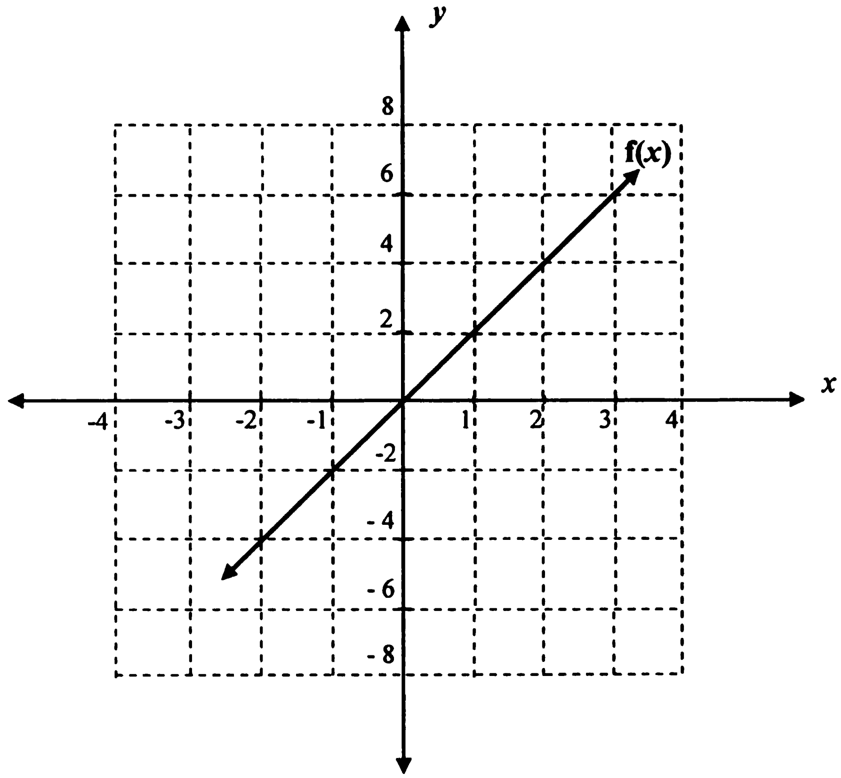
Suppose that you are measuring the diameters and the circumferences of several aluminum disks. If you plot a graph of your data with diameter on the x-axis (in inch), and circumference on the y-axis (in inch), what can you say about the graph you obtained? Can you find the slope of your graph? Describe the meaning of the slope in the context of the problem. Please **explain** your answer, and **show all your work.**

(Note: The Circumference of a circle whose diameter is R is equal to $R\pi$, and assume that $\pi = 3.14$)



Pretest Question 6 (Posttest Question 8)

Note: This question was adapted from Zaslavsky et al. (2002) study.

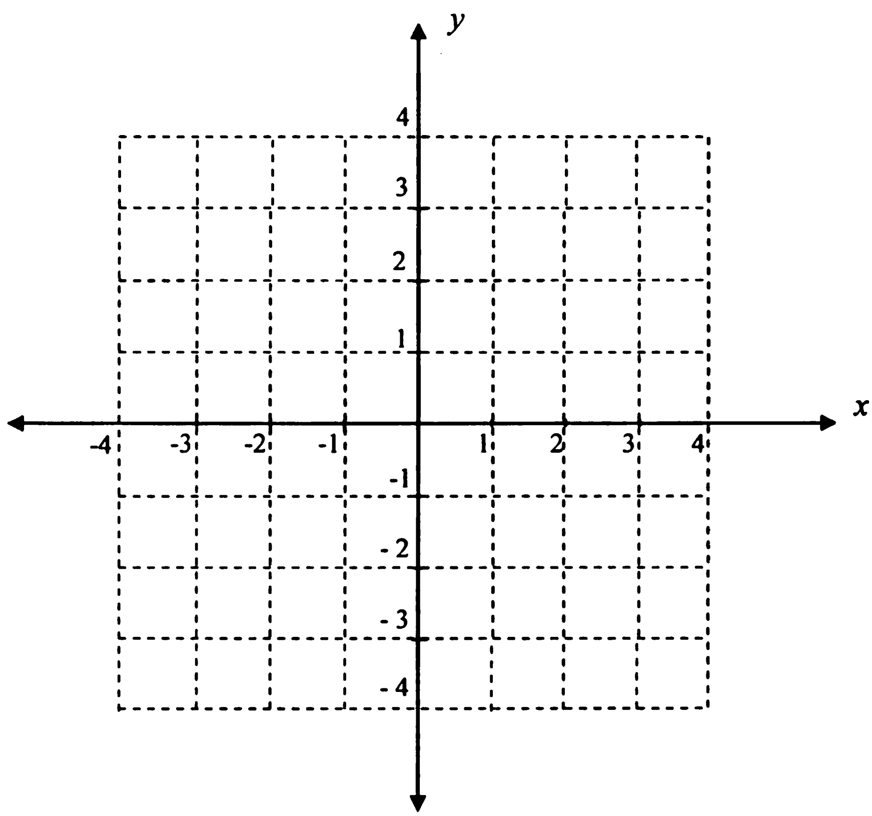


a) What is the slope of the function f(x)? How did you find it? Please **explain** your answer.

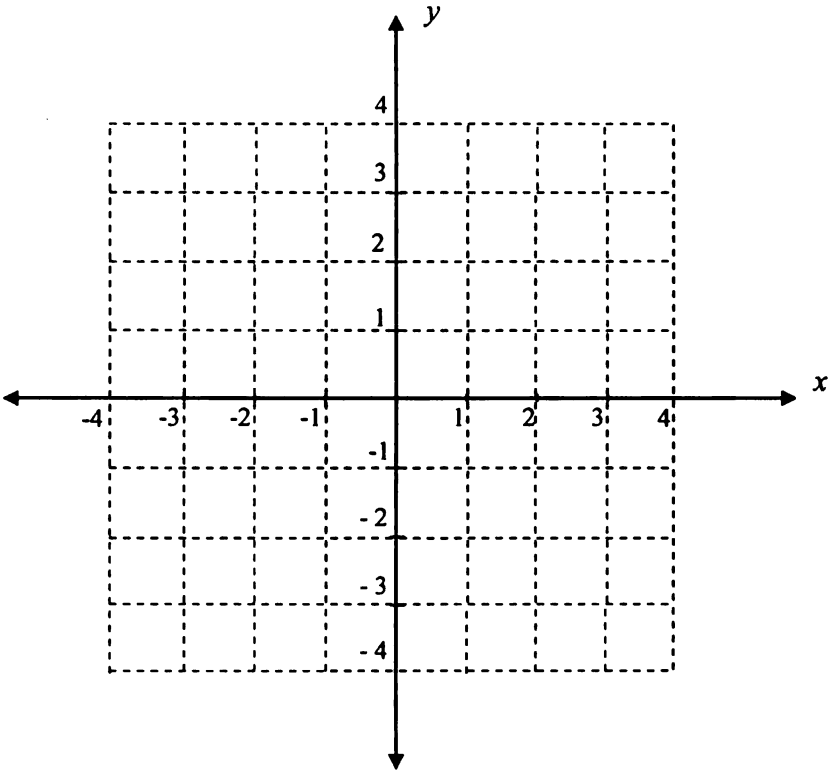
b) Please write an equation for the line on the graph above.

Pretest Question 7 (Posttest Question 9)

a) Please draw **three straight lines** whose slopes are **-2, 0, 1** respectively on the following Cartesian plane.



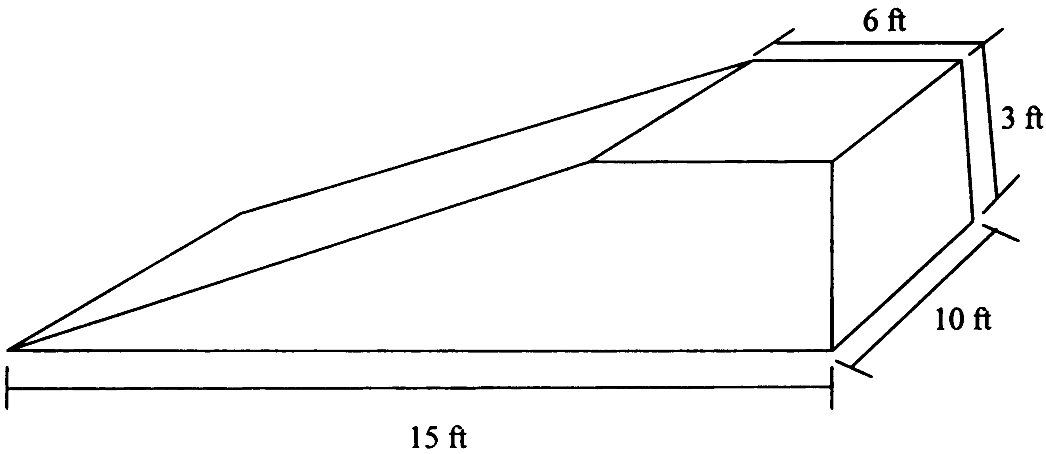
b) Please draw **two different straight lines** that have **the same slope** which is $-\frac{1}{2}$ on the following Cartesian plane.



Pretest Question 8 (Posttest Question 5)

Note: This question was adapted from Lobato & Siebert (2002) study.

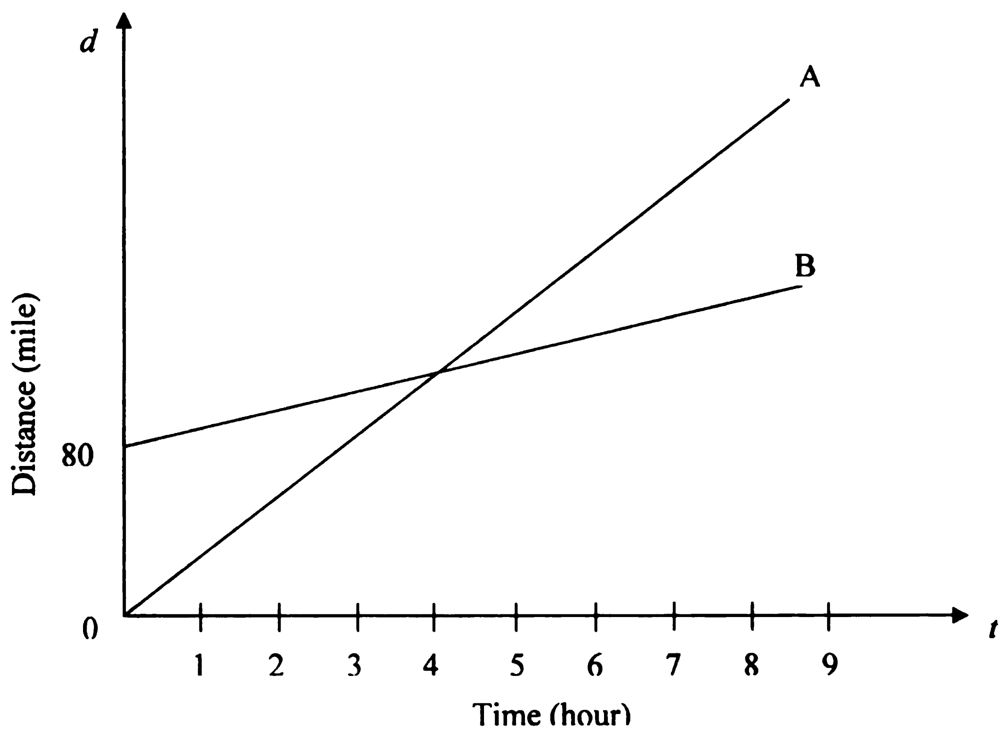
Suppose that a customer needs a wheelchair ramp that reaches to his doorstep, which is 4 ft high (point to ground and door). This particular ramp will not reach since it is only 3 ft high. How can you change the dimensions so that you have a new ramp that is the same steepness as this ramp but it reaches the door? **Please show your work with details.**



Pretest Question 9 (Posttest Question 3)

Note: This question was adapted from Stump (1997) study.

The graph below shows distance versus time relation for car A and car B.



c) When t = 2 hours, is the speed of car A greater than, less than, or equal to the speed of car B? Please **explain your answer with reasons.**

d) Do car A and car B ever have the same speed? If so, at what times? Please **explain your answer with reasons.**

Pretest Question 10 (Posttest Question 10)

Which of the following statements are true? Please **circle your choice** and **give your explanation with reasons**.

We can always consider **slope** of a linear function as a **measure** for the rate of change in the function.

True / False

Explanation: ....

The **x-intercept** of the graph of a linear function is **always dependent** on the **slope** of the linear function.

(**Note:** The *x*-intercept of a graph is the point at which the graph crosses the *x*-axis.)

True / False

Explanation: ....

If the **slope** of a linear function is **positive**, then the **y-intercept** of the graph of the linear function **must be positive**.

(**Note:** The *y*-intercept of a graph is the point at which the graph crosses the *y*-axis.)

True / False

Explanation: ....

If the y-intercept of a linear function is **negative**, then the slope of the linear function **must be negative**.

(**Note:** The *y*-intercept of a graph is the point at which the graph crosses the *y*-axis.)

True / False

Explanation: ....

If two lines are **intersecting each** other on the Cartesian plane, one line **always** has **positive slope** and the other line **always** has **negative slope**.

True / False

Explanation: ....

If the graph of a line gets **steeper**, the **absolute value of its slope** becomes bigger.

True / False

Explanation: ....

If we choose **any two points** on a straight line, we always find the **same slope value**.

True / False

Explanation: ....

The slope of a linear function **does not always** indicate a ratio of changes on y-values over changes on x-values.

True / False

Explanation: ....

Appendix D

Intervention Session Questions

Exploratory Questions for Distance vs. Time Activity (OVM Group)

Please use the VM on the left to answer the following questions.

Question 1

Note: This question was adapted from http://standards.nctm.org/.

Set different values for a position and step size for the boy and girl to start a new race and run the simulation.

a) Please describe your observation about the race (e.g. who is faster, who completed the race before,...etc.) and the related position vs. time graph (e.g. whose graph is steeper, whether both graphs start to same position,...etc.)

b) Now, make changes on your settings (e.g., change the value of the size of the step size for the boy, and change the girl's starting position).How will the change you made influence the race and the related position vs. time graph? Please explain your answer with details.

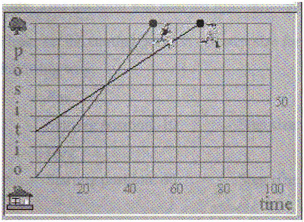
Question 2

a) Set different values for the step size for the boy and girl runner (with the same starting position) each time and run the simulation. Based on your observations on the race and the related position vs. time graph, please describe the relation between the step size of a runner and the speed of the runner. For example, if a runner is using a larger step size, how does this affect his/her speed in the race?

b) How do the different step sizes affect the position vs. time graphs of runners? Please explain the relation between the step size and the position vs. time graphs of runners.

Question 3

The following graph shows position vs. time graphs of the boy and girl runner. The blue line shows girl's graph, and red line shows boy's graph.



Based on your observations on this graph, please answer the following questions.

a) What can you tell about the step size of the boy and girl runner? Does the girl have bigger step size than the boy, or do they have the same step size? Please explain your answer.

b) What can you tell about the speed of the boy and girl runner? Do they have the same speed or different throughout the race? Please explain your answer.

c) As you see in the graph, two lines intersect. What can you tell about the speed of the both runners when their position vs. time graph crosses each other? Do they have the same speed? Please explain your answer.

Structured Mathematics Questions for Distance vs. Time Activity (SVM Group)

Please use the VM on the left to solve the following questions.

Question 1

Note: This question was adapted from http://standards.nctm.org/.

Please set the starting position and step size for both runners on the VM in the following way.

Boy

Starting position: 10 Step size: 3

Girl

Starting position: 40 Step size: 2

a) Then, run the simulation. Now, write a story that describes the trip. For example, *"The girl is going really fast. She catches up to and passes the boy, who is going slow." or "The girl started way behind the boy, who was already halfway to the tree by the time she got going. She went really fast and caught up to him more and more. Finally, at 75 she passed him and kept going really fast and got to the tree first."*

b) Based on your descriptions of the trip in your story at a), does the speed of the boy equal to the girl all the time in the race? Please explain your answer with reasons.

c) What's the speed of the boy and girl runner in the first 20 seconds of race? Please show your work.

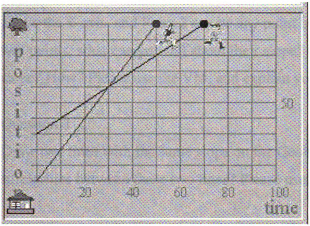
Question 2

Set starting position 10 for the boy and girl runner, and make step size 1 for the boy runner, and 3 for the girl runner.

a) What's the speed of the boy and girl runner in the first 20 seconds of the race? Please show your work.
b) What's the speed of the boy and girl runner throughout the race? Please show your work.
c) Does each of the runners have the constant speed throughout the race? Please explain your answer.
d) What is the relationship between the step size of a runner and the speed of the runner? Please explain your answer.

Question 3

The following graph shows position vs. time graphs of the boy and girl runner. The blue line shows girl's graph, and red line shows boy's graph.



Based on your observations on this graph, please answer the following questions.

a) Does the girl have higher speed than the boy in the first 30 seconds of the race? What's the speed of the boy and girl runner in the first 30 seconds? Please show your work.
b) As you see in the graph, the graphs of two runners intersect each other. Do both runners have the same speed at 30^{th} second? Please explain your answer.
c) What is the speed of the boy and girl runner over the race? Does each of them have constant speed throughout the race? Please explain your answers.
d) What is the relationship between the speed of a runner and the position vs. time graph of the runner? Please explain your answer.

Exploratory Questions for Geoboard Activity (OVM Group)

Please use the VM on the left to answer the following questions.

Question 1

Choose any two points on the Geoboard in the following way: Point 1 can be any point above the x-axis, and Point 2 can be any point below the x-axis. When you connect these two points, what can you tell about their slope? Can you find the value of the line's slope? How? Please use the VM, and explain your answer.

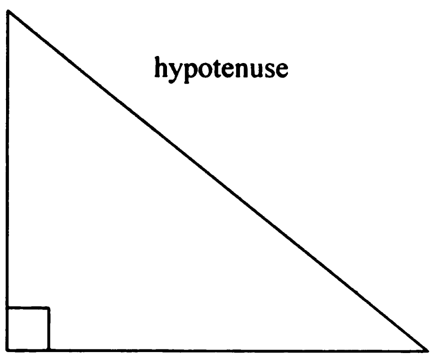
Question 2

Draw two lines with the same slope on the Geoboard. What are the differences and similarities between these two lines? Please use the VM to explain your answers.

Question 3

Please construct two different right triangles on the Geoboard. Then, answer the following questions by using the VM.

a) What do you notice about the lengths of the sides of right triangles? Please explain your answer.
b) Can you find slope of the hypotenuse (please see note below) in each of these two right triangles? Please explain your answer.



Note: Hypotenuse is the longest side on a right triangle.

c) Describe the relationship between the slope of hypotenuse and the lengths of other two sides in a right triangle. Please explain your answer.

Structured Mathematics Questions for Geoboard Activity (SVM Group)

Please use the VM to answer the following questions.

Question 1

By using the VM, please draw a line that connects the following two points (1, -1) and (-1, 3).

a) Does this line have a slope? Is the slope of the line positive or negative? Why? Please explain your answer.
b) Can you find the slope value of the line that connects (1,-1) and (-1, 3)? How can you find it? Please explain your answer.
c) Is it possible that you can find a different line that connects (1, -1) and (-1, 3)? Please explain your answer.

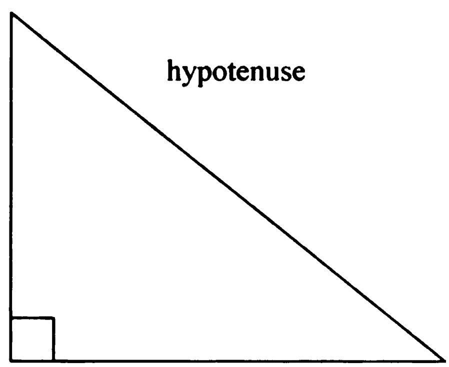
Question 2

Can you find two different lines that have a slope of **- 1/2**? After you draw these two lines on the Geoboard, what are the differences and similarities between these two lines? Please explain your answer.

Question 3

Please construct a right triangle by connecting **(-1, 3)**, **(-1, -3)**, and **(2, -3)** on the Geoboard. Then answer the following questions by using the VM.

a) What is the slope of the hypotenuse (please see the note below) in the triangle that you just constructed? Please explain your answer.



Note: Hypotenuse is the longest side on a right triangle.

b) What is the relationship between the slope of hypotenuse and the lengths of other two sides in your right triangle? Please explain your answer.

c) Please construct a right triangle to find the slope of a line that connects **(0, 3)** and **(-4, -3)**. Is the slope positive or negative? What's the slope of the line? Please explain your answer.

Exploratory Questions for Graphs Activity (OVM Group)

Please explore the graphs of various linear equations by using the VM on the left side of the computer screen. Click and drag the sliders under the graph to change the slope and y-intercept of the line shown.

Please use the VM on the left to answer to the following questions.

Question 1

What do you observe on the graph when you change the value of slope in the equation? Please use the VM to answer the question, and explain your answer.

Question 2

How does the graph of a linear function change when its slope value takes positive versus negative values? Please use the VM to answer the question, and explain your answer.

Question 3

How does changing the values of y-intercept affect the graph of a linear function? Please use the VM to answer the question, and explain your answer.

Question 4

Are there any relations between the values of y-intercept and slope on the linear equations? Please use the VM to answer the question, and explain your answer.

Structured Mathematics Questions for Graphs Activity (SVM Group)

On the VM, please adjust the green and brown sliders to graph the equations to answer the following questions.

Question 1

a) Please first draw the graph of **y = 1.0*x* + 1.0** and then draw the graph of **y = 2.0*x* + 1.0** by using the VM. What did you observe when you move from the graph of **y =1.0*x* + 1.0** to the graph of **y = 2.0*x* + 1.0**? Please explain your answer.

b) Please first draw the graph of **y = 2.0*x* + 1.0** and then draw the graph of **y = -2.0*x* + 1.0** by using the VM. What changes happened when you move from the graph of **y = 2.0x + 1.0** to the graph of **y = -2.0*x* + 1.0**? Please explain your answer.

c) What is the relationship between slope value of linear functions and their graphs? Please explain your answer.

Question 2

a) Please first draw the graph of **y = 2.0*x* + 1.0** and then draw the graph of **y = 2.0*x* + 3.0** by using the VM. What did you observe when you move from the graph of **y = 2.0*x* + 1.0** to the graph of **y = 2.0*x* + 3.0**? Please explain your answer.

b) Please first draw the graph of **y = 2.0*x* + 3.0** and then draw the graph of **y = 2.0*x* -1.0** by using the VM. What changes happened when you move from the graph of **y = 2.0x + 3.0** to the graph of **y = 2.0*x* - 1.0**? Please explain your answer.

c) What is the relationship between y-intercept value of linear functions and their graphs? Please explain your answer with reasons (**Note**: The *y*-intercept of a graph is the point at which the graph crosses the *y*-axis).

Question 3

Please draw the graph of y = **- 1.0*x* + 2.0** by using the VM.

a) Can you find another linear equation that has the same graph as the graph of ***y* = - 1.0x + 2.0**? Please explain your answer.

b) Is the graph of y = **2x** same as the graph of y **-6** = **2(x-3)**? Please explain your answer.

c) Is it always true that we can find two different forms of linear equations that have the same graph? Please explain your answer.

Exploratory Questions for Linear Transformation Activity (OVM Group)

The VM on the left side provides point-slope and slope-intercept forms of linear equations. You can make changes on the values of slope, x-intercept, and y-intercept by changing the numeric values on the boxes with the aid of arrows.

Please use the VM on the left to answer the questions below.

Question 1

What do you observe on the graph of a linear equation when you make changes on the slope values in the point-slope form of the linear equation? Please explain your answer with details.

Question 2

a) What are the differences and similarities between slope-intercept and point-slope forms of linear equations? Please explain your answer.

b) Does the value of slope or y-intercept change from moving slope-intercept form to point-slope form of a linear equation? Please explain your answer.

Question 3

a) Choose a line (such as, a line that passes from (0, 2), and whose slope is 3). Then, please find two different algebraic forms (slope-intercept and point-slope) for the line that you chose. Pease explain your answer.

b) Which form (slope-intercept or point-slope) of linear functions that you formed first to find the algebraic forms of the line in your answer at a)? Please explain your answer.

Structured Mathematics Questions for Linear Transformation Activity (SVM Group)

The VM on the left side provides point-slope and slope-intercept forms of linear equations. You can make changes on the values of slope, x-intercept, and y-intercept by changing the numeric values on the boxes with the aid of arrows.

Please use the VM on the left to answer the following questions.

Question 1

a) By using the VM, please find the equation of a line that passes through **(-2, 3)** and has a **slope of 0.5**?

b) Please find the **slope-intercept form** and **point-slope form** of the equation that you found at a).

c) In **your answer at a)**, which form of the equations (slope-intercept or point-slope) that **you found first**? How did you find your first equation? Please explain your answer.

Question 2

Please compare these two algebraic forms **(that you found in Question 1)** with each other. What changes or does not change while moving from slope-intercept formula to point-slope formula of a linear equation? What are the differences between these two algebraic forms? Please explain your answer.

Question 3

a) Please find **two different** linear equations for the line that passes through **(-1, 3)** and whose **slope is -2**. Show all your work.

b) Please find **two different** linear equations for the line that passes through **(-1, 0)** and **(1, 2)**. Show all your work.

c) Which form (slope-intercept or point-slope) of liner functions that you used first to find the algebraic form of the lines in your answer at a) and b)? Please explain your answer.

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