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DEVELOPMENT OF A CONCEPTUAL FRAMEWORK FOR
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TEXT + BOOK = TEXTBOOK?
DEVELOPMENT OF A CONCEPTUAL FRAMEWORK FOR NON-TEXTUAL ELEMENTS IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

By

Rae Young Kim

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ABSTRACT

TEXT + BOOK = TEXTBOOK? DEVELOPMENT OF A CONCEPTUAL FRAMEWORK FOR NON-TEXTUAL ELEMENTS IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

By

Rae Young Kim

This study is an initial analytic attempt to iteratively develop a conceptual framework informed by both theoretical and practical perspectives that may be used to analyze non-textual elements in mathematics textbooks. Despite the importance of visual representations in teaching and learning, little effort has been made to specify in any systematic way the characteristics and roles of non-textual elements in mathematics textbooks. A non-textual element in this study denotes a visual representation that is comprised of components that are not purely verbal, numerical, or mathematical symbolic representations. Non-textual elements thus include, for example, photos, pictorial illustrations, graphs, and mathematical abstract figures.

Based on the results from the iterative investigations including pilot studies (a survey of students followed by a preliminary textbook analysis), and analyses of data from interviews with teachers (11 South Korean and 10 U.S. teachers) and curriculum developers (four South Korean and three U.S. textbook authors), this study has developed the Final Framework (F-Framework) for non-textual elements, in which *accuracy, connectivity, contextuality, simplicity*, and *aesthetics* are the central aspects. In particular, this F-Framework suggests that *accuracy* and *connectivity* are the fundamental aspects that every non-textual element should possess; *contextuality*, *simplicity*, and *aesthetics* are useful aspects that may facilitate the teaching and learning of mathematics. The

validity and reliability of the F-Framework were tested by examining lessons about angle, slope, and prime factorization in four South Korean and three U.S. mathematics textbooks.

Through the iterative process to develop the framework, this study found theoretical and practical issues around non-textual elements that may be useful for future research. Although all the five aspects of non-textual elements in the F-Framework were considered important, curriculum developers and teachers had different interpretations and emphases on each aspect in different contexts, which were also related to their theoretical and practical concerns. Continuous dialogue between teachers and curriculum developers is needed to explore ways in which the effects of non-textual elements on student learning can be improved. The results from the textbook analysis suggest that textbooks may provide different opportunities to learn different mathematical ideas, which may affect students' understanding of mathematical concepts.

This study provides useful implications on curriculum development, teacher education, and educational policy. Further investigations about how non-textual elements are used in other topics of mathematics textbooks as well as how to improve non-textual elements should be followed.

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Finally, I would like to thank my parents, Nak-Jin Kim and Hye-Kyung Kim, and my brother, Tae Young Kim, for their unconditional support and warm wishes.

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[1] Images in this dissertation are presented in color.

CHAPTER 1

INTRODUCTION AND OVERVIEW

More pictures and illustrations have been used in mathematics textbooks than in the past (Evans, Watson, & Willows, 1987). Woodward (1993) claims that it is due to the assumptions that such pictures may attract students' attention and generate educational effects. However, such assumptions have not been tested in mathematics education. In particular, unlike other subject areas, pictures and illustrations including visual mathematical figures in mathematics textbooks can serve not only as informative agents but also as "tools for thinking" (Cuoco, 2001). However, previous studies have generally failed to capture the analytical significance of understanding the roles of such pictures and illustrations in mathematics textbooks, called non-textual elements in this study, for teaching and learning. Little has been known about how such pictures and illustrations support students' mathematical learning and how they are interpreted and used in teaching and learning of mathematics. Further, no efforts have been devoted to develop a framework to examine the roles of pictures and illustrations in mathematics textbooks.

This study aims to develop a conceptual framework that may be used for analyzing non-textual elements in mathematics textbooks informed by both theoretical and practical perspectives. As a set of "design experiments" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), it has iteratively developed a conceptual framework that has emerged not only from theoretical ideas and scholarly debates but also from practitioners' (teachers and curriculum developers) points of view on non-textual elements. Also, this study tests the validity and practical application of the framework by examining non-textual elements in middle school mathematics textbooks from South Korea and the United States.

In this chapter, I first define and give some examples of non-textual elements. Then, I describe the significance of non-textual elements in teaching and learning mathematics based on a literature review, and introduce the research questions and design of this study with a brief description of the results of pilot studies.

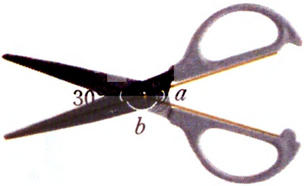
1.1 Definition of Non-Textual Elements

A non-textual element in this study denotes a visual representation that is comprised of components that are not purely verbal, numerical, or mathematical symbolic representations. For example, the equation $a^2 + b^2 = c^2$ used in the Pythagorean Theorem is not a non-textual element because it consists solely of symbolic notations. However, if it is explained by a picture of a right triangle where necessary symbols and marks are used, that picture is a non-textual element. Even though the picture has some symbols and marks, it is not a purely symbolic representation. Thus, it can be regarded as a non-textual element. Non-textual elements thus include, for example, graphic representations including photos and pictorial illustrations/pictures which portray objects in real world and mathematical visual representations including graphs and abstract mathematical figures.

My categorization is different from Sewell's (1994) classification despite some similarities. He classified "pictorial symbols" including 3D models, photos, illustrations/rendered pictures and "graphic symbols" including image-related graphics, concept-related graphics, and arbitrary graphics. However, since my study investigates mathematics textbooks which are designed to help students learn mathematical concepts, even photos and illustrations which Sewell categorized as pictorial symbols could have some connections with related mathematical concepts. Thus, I divide each type of graphic representations into two subcategories: photos *with* mathematical connection, and photos *without* mathematical connection; and illustrations *with* mathematical connection, and illustrations *without* mathematical connection. Figures 1 and 2 show illustrative examples of different types of non-textual elements.

Figure 1. Examples of Graphic Representations

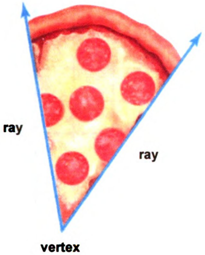
(a) Photo with mathematical connection



(b) Photo without mathematical connection

Figure 1. (cont'd)

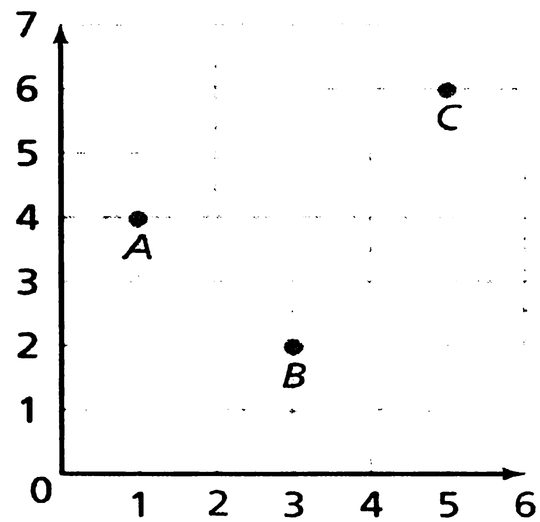
(c) Illustration/picture with mathematical connection



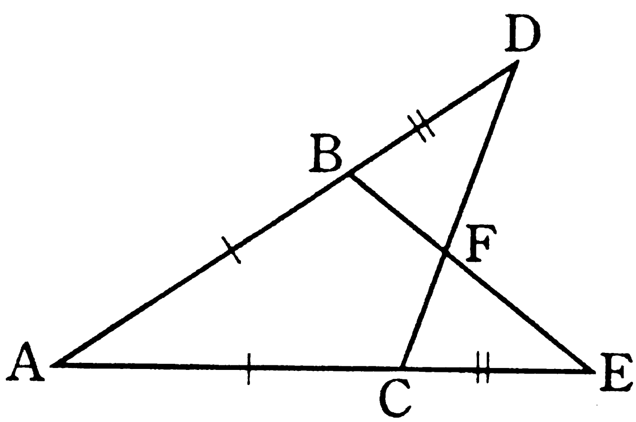
(d) Illustration/picture without mathematical connection

Figure 2. Examples of Mathematical Representations

(a) Graph



(b) Mathematical figure (other than graphs)



The pictures in Figure 1 show some examples of graphic representations used in mathematics textbooks. Figures 1 (*a*) and (*c*) show a photo and an illustration with mathematical connection, respectively. They explicitly show mathematical ideas within realistic contexts using mathematical terms and notations. In contrast, it is not easy to see what the pictures in Figures 1 (*b*) and (*d*) are used for in mathematics textbooks. In fact, Figure 1 (*b*) is used to introduce a story of the famous aviator Amelia Earhart who died in a flight crash because she was off course in order to show how to measure the angle. However, this picture does not explicitly provide any mathematical idea. Similarly, Figure 1 (*d*) is also used in a textbook for a mathematical purpose to find a slope as rate of change of population in Cactusville. Yet students may not find any mathematical idea in this picture to solve the problem. These pictures play decorative roles in textbooks and often no explicit mathematical meaning. Although Figures 1 (*b*) and (*d*) provide contextual information for the problems, they are counted as photos/illustrations *without* mathematical connection because they do not explicitly represent the mathematical concepts.

1.2 Research on Non-Textual Elements in Mathematics Textbooks

Existing research has emphasized the importance of mathematical representations in teaching and learning, including verbal, visual, and symbolic representations (Abrams, 2001; Arcavi, 2003; Brenner et al., 1997; Duval, 2006; Herman, 2007; Pape & Tchoshanov, 2001). The National Council of Teachers of Mathematics (NCTM) proposes representations as an integral component of its standards that all students are expected to meet in school mathematics (National Council of Teachers of Mathematics, 2000). Since representations are related to both content and process in learning mathematics, it is widely believed that representations, as "tools for thinking" (Cuoco, 2001), play important roles in one's creation of "concept image"[1] (Tall & Vinner, 1981), communication with others, and mathematical reasoning.

Visual representations, in particular, allow us to see what is not easily seen only from texts, such as patterns, visualized objects and images that can be manipulated and experimented with, and tendencies of data which allow us to predict the future (Arcavi, 2003). Since visual representations provide students with concrete and concise images of related concepts, they help improve students' understanding of the contents (Levin & Mayer, 1993). On the one hand, if a visual representation is strongly connected with the text, the representation contributes to students' learning from the text (Carney & Levin, 2002; Levin & Mayer, 1993). On the other hand, if representations are inappropriately or incorrectly used, they may lead to students' misunderstandings and confusions because they may convey incorrect information or images. Since pictorial representations can be memorized more easily than verbal representations (McDaniel & Pressley, 1987), incorrect visual representations may hinder students not only from their current learning of the concepts but also from their further learning of related contents. It is thus important for students to experience various appropriate representations.

Many studies on visual literacy have also emphasized the importance of visual representations in students' learning. In the process of internalizing and externalizing visual representations, visual literacy becomes more important (Burmark, 2002). Visual literacy is generally defined as "the ability to understand and use images, including the ability to think, learn, and express oneself in terms of images" (Braden, 1996, p13).

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[1] Concept image means "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 151). Students often create their own concept image, think about mathematical concepts and solve problems, and communicate with others through representations.

Vergnaud (1998) argues that representations should be considered a crucial subject, for everybody encounters representations and should learn how to interpret and use them because they do not merely present "reality" but "entities" such as "objects, properties, relationships, processes, actions, and constructs" (p. 167) on which people generally do not have a single common agreement. In fact, as each student may not have the same level of textual literacy as that of visual literacy, students who have difficulty in reading and understanding texts may get more information from, and better understand, visual representations. Winn (1987) asserts that graphical representations have, often but not always, more influence to improve performance of students with low-ability than those with high-ability. This suggests that visual representations also can provide some students with opportunities to learn mathematics which they cannot understand only from textual representations. Carney and Levin (2002) in their follow-up studies reaffirm that visual representations, especially pictorial illustrations, can improve students' understanding and learning.

Despite the importance of visual representations in the teaching and learning of mathematics, little effort has been made to specify in any systematic way the characteristics and roles of non-textual elements in mathematics textbooks. Most studies on visual representations in mathematics have mainly focused on their cognitive and psychological effects on students' mathematical learning, such as how students generate their own concept images into their mind (i.e. internal representations), how students express their mental images to communicate with others (i.e. external representations), and how students interpret "presented" representations (Goldin & Shteingold, 2001). However, such studies have not been conducted on mathematics textbooks. They have mainly analyzed the artifacts students created or interaction with others in classrooms through representations.

In particular, little attention has been given to the roles of non-textual elements in mathematics textbooks. Although there are many studies which investigate the impact of non-textual elements in textbooks on student learning and comprehension, such research has been done mainly in science and reading areas. Levin (1993) shows the importance of non-textual elements in understanding texts by describing five functions of pictures: *decorational, representational, organizational, interpretational,* and *transformational.* He defines decorational pictures as those without relation to texts. Representational pictures have to do with some parts of texts. Organizational pictures are used to provide a conceptual structure to understand the texts. Interpretational pictures help clarify the texts. Finally, transformational pictures help students remember the fact in texts. Although his findings are useful to see the roles of non-textual elements in reading and science textbooks, they are not enough to understand non-textual elements in mathematics textbooks. Unlike typical non-textual elements in reading and science, non-textual elements in mathematics are used not only as informative agents but also as integral "tools for thinking" (Cuoco, 2001) with which students manipulate and experiment in their minds and hands. Lack of research on non-textual elements in mathematics textbooks thus calls for the development of a conceptual framework to understand non-textual elements in mathematics textbooks.

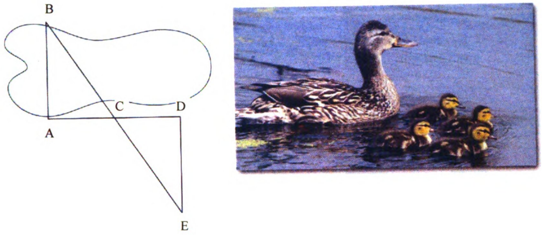
1.3 Pilot Studies

Preliminary framework (P-Framework)

In order to examine and analyze non-textual elements in mathematics textbooks, I undertook an iterative series of investigations to develop a non-textual element framework. The first draft of a framework was developed in 2006 to investigate how non-textual elements were used to teach geometry in Korean and U.S. textbooks. Based on a review of the literature (some of which is summarized in section 1.1 with other studies), I developed a preliminary framework (P-Framework) whose constructs were *validity, connectivity, authenticity,* and *variety.*

Validity refers to relevancy and clarity of non-textual elements in textbooks to explain a geometric concept or problem. It refers to how non-textual elements are appropriately used and placed, and how they represent concepts and ideas obviously. It is related to both the feature and content of a non-textual element. For example, Figure 3 shows pictures used in a problem to measure the length of the river using the concept of similarity. However, these pictures in Figure 3 have low level of validity for two reasons. First, even though the picture of the triangles in the left provides information about the problem, it is smaller than the picture of ducks on the right, which does not give any useful information to solve the problem. Also, the picture in the right has more colors than that in the left. Since bigger pictures and colors can attract students' attention more easily (Dwyer & Lamberski, 1983; Evans et al., 1987), these pictures may have a low level of validity. Second, even though the picture in the left shows the two triangles that may be used to solve the problem, it does not show all the conditions that students need to solve the problem. The picture in the right has nothing to do with the mathematics of the problem. These pictures in Figure 3 have a low level of validity because they do not explicitly represent the mathematical ideas.

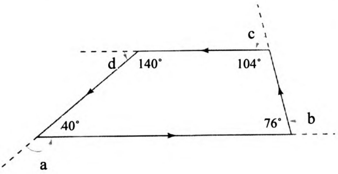
Figure 3. Examples of Validity



Connectivity means how non-textual elements are closely related to textual elements. Dwyer (1988) discovered a symbiotic connection between verbal and visual literacy when the two were united to help student achievement. That is, closeness between non-textual elements and texts can affect student learning. Figure 4 shows two different examples of connectivity. The picture on the left shows the shape of a quadrilateral using mathematical notations and marks. This picture is closely related to the texts. However, the picture of two children on the right is associated with contexts, not with the problem. Although the two pictures are used in the same problem, their connections to the texts are different. The picture on the left is associated with the mathematical content of the problem while the picture on the right is associated with objects (in-line skaters and a quadrilateral) or contexts (skating around a park) of the problem. Since their roles are different, what kinds of connectivity are between a non-textual element and texts should be considered.

Figure 4. Examples of Connectivity

Suppose an in-line skater skates around a park that has the shape of a quadrilateral. Suppose he skates once around the quadrilateral, turning each corner exactly once. What is the sum of the angles through which he turns?



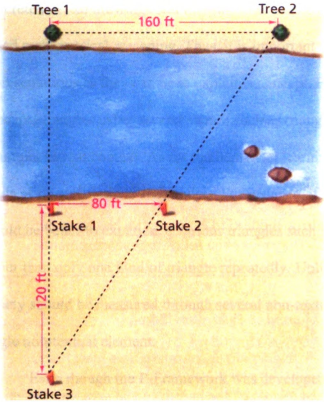
Authenticity implies the non-textual element being fully trustworthy and factual in relation to the content. This idea stems from Freudenthal's Realistic Mathematics Education (RME) (Freudenthal, 1991). Freudenthal argued that students could better understand mathematics if they learned it within a realistic context because they could see the values of mathematics in their lives. Figure 5 shows examples of authenticity. Figures 5 (1) and (2) each represent the similar mathematical problems to find the length indicated using two similar triangles. Whereas Figure 5 (1) shows the problem within realistic contexts, Figure 5 (2) shows the problem in abstract mathematical ways. Figure

5 (1) looks like a real situation where students may encounter such a problem. Students could see the application of the concept in the context rather easily. In contrast, Figure 5 (2) is an abstract mathematical figure to show the two similar triangles. Since this picture does not have any realistic situation or contexts, students may hardly think that this picture looks actual or real. Rather, students may see how the problem can be represented in abstract mathematical ways using mathematical marks and notations.

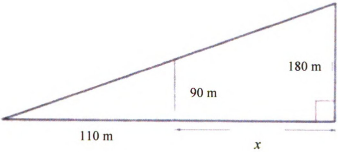
Thus, I made two categories for authenticity: instrumental authenticity and mathematical authenticity. A non-textual element such as Figure 5 (1) has instrumental authenticity because it looks authentic in realistic contexts. However, a non-textual element such as Figure 5 (2) has mathematical authenticity because it is being trustworthy in an abstract mathematical way.

Figure 5. Examples of Authenticity

(1) Find the distance between Stake 1 and Tree 1.



(2) What is the value for x?



Variety is measured by how many diverse types of non-textual elements are used to explain a certain concept or problem. Presmeg (1997) claimed that the "one-case concreteness" of drawings and images is the source of many difficulties in visualization-based mathematical reasoning. Thus, it is important to demonstrate as many possible representations as they can be to explain a concept. For example, we can teach the concept of angles using several shapes such as triangles, quadrilaterals, and other examples in real context. As for mathematical learning, it is important to present several kinds of representations and models. For instance, if we talk about angles in a triangle, it would be better to experience various triangles such as acute, obtuse, and right triangles rather than only one kind of triangle repeatedly. Unlike other aspects in the P-Framework, variety should be measured through several non-textual elements used in a lesson, not a single non-textual element.

Even though the P-Framework was developed based on a literature review, it is relatively humble in which each aspect should be articulated and tested. Since this study is an iterative exploration about non-textual elements, I tested the P-Framework through surveys and textbook analysis.

Survey as a pilot study

In order to test the validity of the P-Framework and see how people understand non-textual elements, I developed a survey including demographic information and 14 tasks about non-textual elements used in geometry, such as angles, parallel lines, similarity, and congruence. The tasks were composed of 10 multiple choice questions, with sub-questions to ask their reasons, and four open-ended questions. Twenty nine middle school students (12 U.S. students and 17 Korean students) and seven graduate students (two U.S. students and five Korean students) with teaching experience in K-12 schools who were studying mathematics education in a doctoral program volunteered to participate in the pilot study by responding to the survey. The data from the survey were analyzed using both quantitative and qualitative research methods.

The results from the students' responses show: (1) authentic non-textual elements which included realistic contexts played an important role in motivating students' interests, (2) younger students understood non-textual elements more easily when the elements were highly related to realistic situation, whereas older students tended to be comfortable with the elements presented in abstract mathematical ways, (2) inaccurate non-textual elements hindered their understanding of given concepts, (3) students could not understand what a picture meant when it had less connection to the content, (4) U.S. students preferred pictures with realistic contexts, whereas Korean students preferred mathematical models and figures in understanding concepts and solving problems, and (5) students felt attracted to colorful pictures.

The results from graduate students' responses were similar to middle school students' with regard to authenticity and validity. The graduate students thought that colors and realistic contexts in pictures were attractive. They thought that incorrect pictures were confusing even though they could point out why they were incorrect. Since they were knowledgeable in mathematics, all the graduate students could point out which non-textual elements were incorrect or unconnected to the content. In addition, even though they thought that pictures with realistic contexts would be more interesting to students, they preferred mathematical models and figures that were more familiar because they were accustomed to using such mathematical pictures in problem solving. Most of them also said that mathematical models and figures showed related concepts more clearly and explicitly with mathematical symbols and terms than pictures with realistic contexts because pictures with realistic contexts did not automatically explain the related concepts.

Textbook analysis as a pilot study

Based on the results from the pilot study, the P-Framework was revised and used to analyze selections from textbooks. The results from the survey with both secondary students and graduate students show that colors were attractive and motivational to students. This, however, did not fall into any category in the P-Framework I had initially developed. Drawing on many studies on visual literacy and Sinclair's (2004) work, I then included *aesthetics*. According to Sinclair (2004), *aesthetics* as an important factor in learning mathematics denotes how the non-textual elements are visually attractive enough to motivate students' learning.

Using a modified P-Framework that included *aesthetics*, I analyzed non-textual elements used to teach angles, parallel lines, similarity, and congruence in five U.S. standards-based textbooks[2] and 13 Korean[3] secondary mathematics textbooks that are currently used in each context. Each non-textual element was coded according to the modified P-Framework with *aesthetics* that I had developed.

[2] Although the number of U.S. textbooks in this study is small, the ones studied are used widely in U.S. secondary schools among standards-based textbooks (Senk & Thompson, 2003).
[3] The Korean textbooks in this study have been approved by the Ministry of Education, Science, and Technology (formerly, the Ministry of Education and Human Resources Development until 2007) following the Korean seventh national curriculum for middle school students. All kinds of the textbooks in Korea are examined in this study.

The findings from the pilot study show that non-textual elements in mathematics textbooks provide different opportunities to learn mathematics in different contexts according to their characteristics and roles in terms of *validity, connectivity, authenticity, aesthetics,* and *variety.* The results of the study show that even a single mathematical concept can be described and expressed in various ways through non-textual elements. Such variety may provide students with different experiences to conceptualize the mathematical concept. Contextual authentic representations (instrumental authenticity) may motivate students to think about the related mathematical problems within the real contexts whereas mathematical authentic representations (mathematical authenticity) may show how to represent a concept precisely in abstract mathematical ways. In addition, I found that connectivity and validity of non-textual elements help students learn because ill-connected or improper non-textual elements may impact students' misconception (Mishra, 1999). Such different characteristics of non-textual elements in mathematics textbooks in different contexts may provide different opportunities to learn; this may influence students' learning of mathematics as well as their beliefs about what mathematics is.

Learning from the pilot studies

The results from survey and textbook analysis provided me with useful implications for my dissertation study. First, *validity, connectivity, authenticity, aesthetics,* and *variety* appear to be important aspects of non-textual elements in teaching and learning mathematics.

Second, I found, however, that the terms of *authenticity* and *validity* should be better articulated because the terms as used initially are too broad to convey clearly their meaning. Defining *authenticity* as "being fully trustworthy and factual" leads to the question of what trustworthy and factual mean. The previous pilot studies show that both students and graduate students consider which contexts the content is located in rather than what looks actual or real. I thus replaced "*authenticity*" with "*contextuality.*" For *validity*, the students and graduate students used mathematical accuracy as a criterion to judge which non-textual element would be valid. Unlike my assumption that students may be affected by the features of a non-textual element, their responses in the survey did not show much evidence about that. Rather, both the students and graduate students considered mathematical accuracy when they were asked to judge validity. Thus, I redefined *validity* as *accuracy* in the R-Framework.

Based on the results from the pilot studies, the R-Framework consisted of five aspects of non-textual elements: *accuracy, connectivity, contextuality, aesthetics,* and *variety.* The R-Framework was the basis for the interviews with teachers and curriculum developers. Table 1 shows the descriptions of each aspect in the R-Framework.

Table 1. Descriptions of Each Aspect in the Revised Framework (R-Framework)

Aspect	Description
Accuracy	How mathematically clear and rigorous a non-textual element is to explain a concept
Connectivity	How non-textual elements are closely related to textual elements
Contextuality	What kinds of context(s) the content in a non-textual element is located in

Table 1. (cont'd)

Aspect	Description
Aesthetics	How the non-textual elements are visually attractive enough to motivate students' learning
Variety	How many diverse types of non-textual elements are used to explain a certain concept or problem

Third, the results from pilot survey and textbook analysis lead to further questions about how non-textual elements are understood and used in other topics. Both the survey and textbook analyses used only selected geometric topics. Geometry uses many visual representations in nature, but we do not know how non-textual elements are used in other topics, such as algebra, which sometimes use only verbal and symbolic representations.

Fourth, a fuller, more useful framework also needs to take into account or be responsive to how teachers and curriculum developers understand the characteristics and role of non-textual elements in mathematics textbooks as well as in teaching and learning. The results from pilot survey and textbook analysis suggest that students and graduate students have different interpretations and orientations toward non-textual elements depending on their knowledge and educational experiences. This implies that it is important to hear voices from various people who are involved with design and use of non-textual elements in mathematics textbooks. It is worthwhile to examine what criteria teachers, as active implementers of textbooks in classrooms, have to judge which non-textual elements are important and useful in student learning. It is also useful to inquire into curriculum developers' intentions and interpretations of non-textual elements in their textbooks. Such information can provide various perspectives on educationally significant aspects of non-textual elements and help develop a more robust conceptual framework to analyze non-textual elements in mathematics textbooks.

Fifth, the results from the pilot studies show some differences in the use of non-textual elements in South Korea and the United States. This suggests that there may be some socio-culturally different notions of which non-textual elements are useful for teaching and learning mathematics. Even though mathematics is considered a universal language, the way of learning mathematics through non-textual elements could vary because how to teach and learn mathematics is a cultural practice (Fernandez & Yoshida, 2004; Ma, 1999; Stigler & Hiebert, 1999). In the dissertation, I analyze non-textual elements in different contexts by finding patterns in the results from interviews with teachers and curriculum developers as well as textbook analysis.

Last, the results from pilot studies indicate that aesthetics as a motivational role is an important aspect of non-textual elements. In particular, realistic contexts and colors in non-textual elements are very attractive to both students and graduate students. Aesthetics, as defined as the motivational dimension of non-textual elements, was thus added to the R-Framework.

1.4 Significance of the Study

This study develops a conceptual framework through the iterative process between theory and practice similar to what "design experiments" (Cobb et al., 2003) do. Design experiments are implemented to develop theories through iterative processes where conjectures are tested and developed in designed contexts. Cobb et al. (2003) described design experiments as "extended (iterative), interventionist (innovative and design-based), and theory-oriented enterprises whose 'theories' do real work in practical educational contexts" (p. 13).

Even though this study does not focus on actual learning process in a certain context (e.g., classrooms, professional communities, and school districts) as many scholars did for design experiments, the rationales and approaches of this study are similar to design experiments. A preliminary framework (P-Framework) on non-textual elements was generated and tested through pilot studies and a literature review, and new ideas were developed through reflections on the findings from the pilot studies. These ideas reflected on a revised framework (R-Framework). The R-Framework was used to develop the interview protocols for teachers and curriculum developers about how non-textual elements in textbooks can be selected and used for teaching and learning mathematics. A final framework (F-Framework) was developed based on the results from the interviews, and tested by analyzing existing middle school mathematics textbooks. Through the iterative design process, the F-Framework evolved not only from theoretical perspectives and scholarly debates on important aspects of non-textual elements but also from empirical reflections on how to use and interpret non-textual elements for effective teaching and learning. Also, the F-Framework was tested using existing mathematics textbooks to show practical applications and implications for further research. Thus, this study was iteratively conducted. Each framework was generated based on reflections on the findings from the previous stage and tested with additional empirical data. Through such iterative and reflective processes, the conceptual framework was elaborated not only from theoretical perspectives about important aspects of non-textual elements but also from practical issues and challenges around teaching and learning of mathematics through non-textual elements.

The conceptual framework that emerges from this study will contribute to research and practice in many ways. It may be used to compare the characteristics and roles of non-textual elements in different textbooks, to examine how teachers use non-textual elements in their classrooms, and to design and assess the use of non-textual elements in textbooks and other materials as a tool. By doing so, it will contribute to the increased utility of non-textual elements in mathematics textbooks, which improves students' understanding of mathematical ideas.

In addition to the usefulness of the conceptual framework in analyzing non-textual elements in textbooks, instructional materials, and classrooms, this study provides some exploratory insights about teachers' and curriculum developers' understanding of non-textual elements in different contexts. Most studies on visual representations in textbooks have focused only on the relationship between representations and students' learning. However, we should not overlook the important roles of teachers and curriculum developers as important agents mediating the relationship.

It is significant and necessary to investigate teachers' cognition of the roles of non-textual elements in mathematics textbooks for their instruction. First, although pedagogical content knowledge has been emphasized as an important knowledge component for teachers, little attention has been paid to how teachers understand and use non-textual elements for their teaching and student learning. Shulman (1986) categorizes pedagogical content knowledge as a set of knowledge about "the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (p. 9). Since there is no single perfect representation for a concept, teachers must make their own decisions about which non-textual element is most appropriate for their students to learn the concept. Knowing about non-textual elements is part of knowing the appropriate ways of presenting the content to students (Ball, Thames, & Phelps, 2008). It thus becomes important to know teachers' understanding of non-textual elements as their pedagogical content knowledge because "teachers' orientations to content influence the ways in which they teach that content" (Ball et al., 2008, p. 393). In other words, teachers' orientations to non-textual elements can be closely related to the ways they use them in their classrooms to provide students with opportunities to learn mathematics. However, little has been explored about how teachers understand the nature of non-textual elements in mathematics textbooks and what criteria they use to select or modify non-textual elements in mathematics textbooks for their instruction. In addition, we don't know about how they judge and use appropriate representations for each topic in mathematics.

Second, even when the same textbooks are provided as written curriculum, teachers can use them in their classrooms differently by *offloading*, *adapting,* and *improvising* (Brown, 2002). In other words, teachers sometimes follow the given curriculum as provided (offloading), or revise the curriculum to meet their needs (adapting), or create their own curriculum without considering the given curriculum suggestions (improvising). In the process, depending on teachers' understanding of non-textual elements, they select and use non-textual elements which may affect opportunities to learn provided in classrooms. Although many studies have explored how teachers use textbooks in their classrooms (e.g., Remillard, 2005; Stein, Remillard, & Smith, 2007), there is little evidence about how teachers understand and use non-textual elements in mathematics textbooks for their classes.

Further, little is known about what intentions curriculum developers have in designing their textbooks in terms of non-textual elements. Also, whether or not teachers understand and use non-textual elements in the ways curriculum developers intended remains unanswered. Although many scholars have emphasized the importance of visual representations in teaching and learning, little has been known about how curriculum developers understand and use them in designing their textbooks. There are tensions around designing textbooks. Textbooks are designed to serve instructional purposes: teachers can use them as instructional materials for their lessons, and students can use them as references and resources. Thus, textbooks should include meaningful and useful non-textual elements to help students learn mathematics. At the same time, textbooks are influenced by the textbook market because publishers give priority to the marketability of their textbooks. Textbooks thus include many colorful and appealing pictures and illustrations to draw teachers' and students' attention. In the process of designing textbooks, textbook authors should take into consideration both instructional purposes and marketability of textbooks which are often in conflict with each other because little is known about how such increasing popularity of decorative illustrations and pictures in textbooks affect students' learning (Woodward, 1993). In addition, although many studies have discovered the efficacy of illustrations in education, such research has little impact on publishers' decisions for illustrations (Woodward, 1993). As Houghton and Willows (1987) describe, "at present, it would appear that a great deal of instructional text design is guided by intuition, prior practice, trial and error approaches, and marketability consideration" (p. iii). In this situation, we need to examine how textbook authors actually deal with this tension between educational and economic benefits from textbooks and how they make decisions about the use of non-textual elements in their mathematics textbooks.

In addition, we need to see whether or not teachers share assumptions and ideas about the roles of non-textual elements with curriculum developers. Yet often lacking is research that uncovers whether or not teachers and curriculum developers have different understandings of the roles of non-textual elements in each topic of mathematics. Although many studies on representations have explored what kinds of representations are used in class to teach a mathematical concept (e.g., Abrams, 2001; Brenner et al., 1997; Zazkis & Liljedahl, 2004), there is little effort to investigate how these representations are used in different mathematical topics and whether or not teachers or curriculum developers have different assumptions about the use of non-textual elements for each topic. Such information would contribute not only to a better understanding of non-textual elements but also to stronger coherence between curriculum and instruction.

Analyzing and comparing teachers' and curriculum developers' understanding of non-textual elements in South Korean and U.S. mathematics textbooks, this study provides patterns of non-textual elements applied in different contexts as well as alternative ways of using non-textual elements to improve the quality of non-textual elements in current mathematics textbooks.

1.5 Research Questions and Design of the Study

This study focuses on four issues: (1) how teachers understand non-textual elements in mathematics textbooks, (2) what intentions and perspectives curriculum developers have for non-textual elements in mathematics textbooks, (3) how these two analyses inform a framework of important roles of non-textual elements in mathematics textbooks, and (4) how non-textual elements are actually used in middle school mathematics textbooks (grades 6 – 8). The research questions guiding this study are as follows:

1. How do teachers think about non-textual elements in mathematics textbooks?
   (1) How do teachers understand non-textual elements in terms of *accuracy, connectivity, contextuality, aesthetics,* and *variety* as well as other aspects of non-textual elements?
   (2) How do teachers understand the roles of non-textual elements used for different topics?

2. What intentions and perspectives do curriculum developers have for non-textual elements in mathematics textbooks?
   (1) How do curriculum developers understand non-textual elements in terms of *accuracy, connectivity, contextuality, aesthetics,* and *variety* as well as other aspects of non-textual elements?
   (2) How do curriculum developers understand the roles of non-textual elements used for different topics?

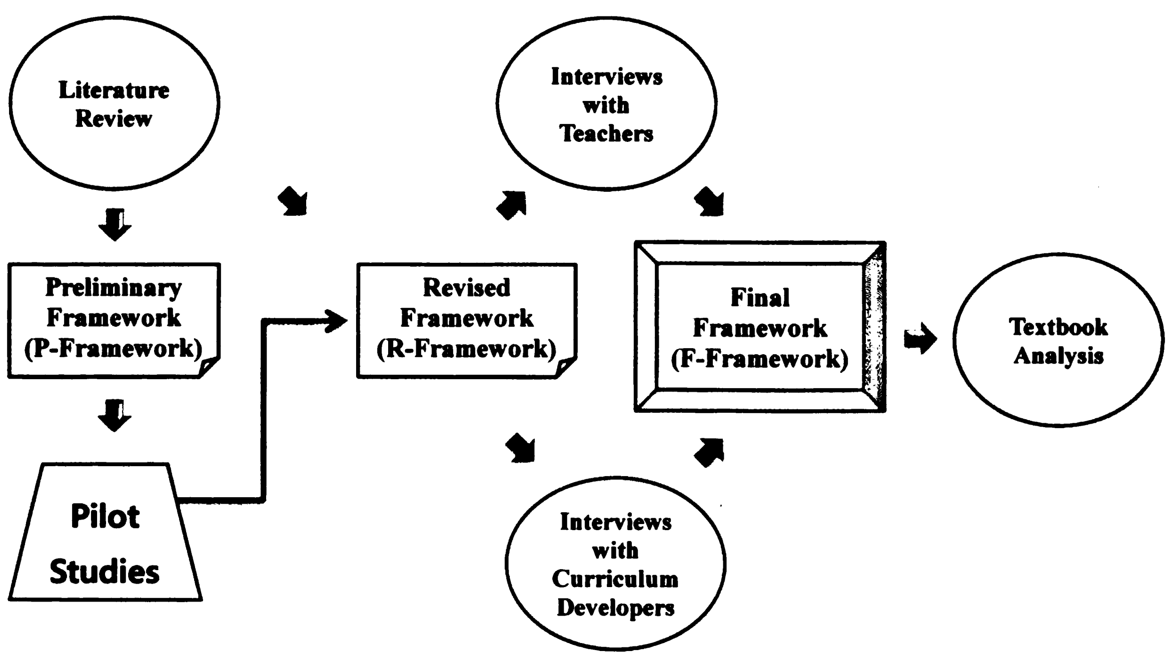
3. How do these two analyses inform a framework for the non-textual elements in mathematics textbooks?
   (1) What kinds of assumptions do teachers and curriculum developers have in common with regards to the roles of non-textual elements in mathematics textbooks?
   (2) To what extent do their assumptions display conflicting perspectives between teachers and curriculum developers?
   (3) How do these analyses help refine the non-textual element framework?

4. How are non-textual elements actually used in middle school mathematics textbooks in terms of the components of the refined framework?
   (1) What kinds of variation or commonality are found in the roles of the non-textual elements in current mathematics textbooks both within and across countries (e.g., South Korea and the United States)?
   (2) How are the non-textual elements in current mathematics textbooks used to support students' learning about specific mathematics and topics (angle, slope, and prime factorization)?

Figure 6 summarizes the design of this study. First, I developed a hypothetical conceptual framework called the Preliminary Framework (P-Framework) to analyze non-textual elements in mathematics textbooks throughout a literature review and pilot studies. A revised framework called the R-Framework resulted. This work was described in section 1.3 of this chapter. Then, I devised interview protocols based on the R-

Framework, and I interviewed teachers and curriculum developers from South Korea and the United States. The analyses of teachers' and curriculum developers' thinking about non-textual elements provide ideas about what intentions curriculum developers have when selecting non-textual elements in their textbooks and how non-textual elements are interpreted and used by teachers for teaching and learning of mathematics. Details of the methodology and results of analyses are reported in Chapters 3 (for teachers) and 4 (for curriculum developers). After analyzing and synthesizing the results from the interviews, I then developed an elaborated final framework named the F-Framework which is described in Chapter 5. Finally, to test the validity of the F-Framework, I used it to examine how non-textual elements are actually used in middle school mathematics textbooks from South Korea and the United States. This work is reported in Chapter 6. Conclusions and implications of this study are discussed in Chapter 7.

Figure 6. Research Design



This study uses mixed methods including qualitative analyses of interview data to generate the F-Framework and quantitative data from analyses of secondary mathematics textbooks. The analyses of interviews employ grounded theory, which is "a general methodology for developing theory that is grounded in data systematically gathered and analyzed" (Strauss & Corbin, 1994, p. 273). For the validation of the F-Framework, this study employs a comparative method to explore the roles of non-textual elements in different contexts – South Korea and the United States. This study does not attempt to evaluate the current use of non-textual elements in each county. Rather, this inquiry allows us to see various ways of understanding non-textual elements in mathematics textbooks across contexts as well as to verify the validity and reliability of the conceptual framework in different contexts.

CHAPTER 2

UNDERSTANDING NON-TEXTUAL ELEMENTS IN RESEARCH

Since the 1980s, many studies on the visual component of texts as a part of document design study have been conducted in diverse disciplines, including semiotics, rhetoric, linguistics, visual design, instructional design and technology, and cognitive psychology (Shriver, 1990). Document design is "the field concerned with creating texts, that is books, pamphlets, posters, and others that integrate words and pictures in ways that help people to achieve their specific goals for using texts at home, school, or work" (Shriver, 1997, p. 13). Since document design focuses on both verbal and visual features which are associated with human activities of generating and interpreting visual and verbal texts, many studies on document design have been conducted in various areas dealing with social, historical, psychological, and cultural issues. For example, many studies in rhetoric have investigated how people generate meaning when they read and write. Cognitive psychologists have explored how people think and learn in reading and writing. Researchers on instructional design and technology have focused on the structure of contents in texts themselves rather than on how people understand them. Theory, research, and practice in education have charged such interdisciplinary approaches in document design studies to find ways of improving educational quality and effects.

However, these studies have not explored pictorial representations in mathematics textbooks. This may be because they have been considered as a part of mathematical contents. At the same time, many studies in mathematics education have not examined pictorial representations because they are considered as a part of visual design of a textbook. Further, although pictorial representations and mathematical representations coexist in a textbook, no research has been done to reveal their relationship, roles, and effects on student learning.

This chapter reviews the literature regarding non-textual elements (including pictorial representations and mathematical representations) in multiple disciplines around education in order to elaborate a conceptual framework to analyze non-textual elements in mathematics textbooks. In addition, since this study focuses on the use of non-textual elements in the lessons of angle, slope, and prime factorization, I review the literature on teaching and learning of mathematics in these topics.

2.1 Research on Pictorial Representations

Since the late 1970s, many researchers have explored the instructional roles and effects of pictorial representations. Such studies have been mainly conducted in reading comprehension. Levin and Mayer (1993) argue that a picture is worth a thousand words because it is concentrated/focused, compact/concise, concrete, coherent, comprehensible, correspondent, codable, and collective. They argue that pictures are useful to draw students' attention by showing the most important information in texts. Since pictures provide concrete images of the content and structured information of the text, students can remember them much longer. Compared to texts, pictures make the content much easier to understand by linking between new concepts and what students already know. They argue that such advantages of pictures would be taken only when the pictures are accompanied with texts. Further, Willows (1978) asserts that if pictures are unrelated to the texts, they even can be distracters. This implies that conceptual proximity between texts and pictures is important to be effective in student learning.

On the other hand, even though giving contexts related to the content is helpful for students to understand the story, Smith and Watkins (1972) pointed out that if the contexts are not realistic, this could be distracting. In their experiments, they gave elementary students pictures of plants in a pond which was colored yellow. Only about one third of the students, mainly older students, could recognize it. Younger children could not. If color is essential to the concept being presented and if students concentrate on it, color can facilitate learning (Dwyer & Lamberski, 1983). How relevant the color of non-textual elements is used in textbooks may influence students' learning (Burmark, 2002). However, their experiments show that colors and unrealistic contexts can be distracting if they are inappropriate.

In addition to the effects of pictures on student learning, many studies on instructional design (e.g., Duchastel & Waller, 1979; Levie & Lentz, 1982) have examined reading textbooks to see how pictures are used. They found from their experiments that even though pictures could be used to introduce a new idea, to draw students' attention, and to provide concrete examples of concepts, very few textbooks use pictures like these. In fact, Levin and Mayer (1993) pointed out that even though many studies have examined pictorial representations, further investigations are necessary to figure out the relationship among illustrations, texts, and learners.

2.2 Research on Visual Mathematical Representations

Representations in mathematics education mean "packages that assign objects and their transformations to other objects and their transformations" (Cuoco, 2001, p. x). Many studies on representations in mathematics education have mainly focused on their cognitive and psychological effects on student learning, for instance, how students perceive images (internal representations) and how students generate images (external representations) to communicate with others. The interaction between internal representations and external representations is crucial in teaching and learning mathematics.

Teachers' use and interpretation of representations can be a basis for the development of students' internal representations (Goldin & Shteingold, 2001). Thus, it is important to see what kinds of representations teachers select and use for their classes and how teachers interpret the representations in relation to the concept. Also, students often use representations to develop and express their ideas through problem solving (Pape & Tchoshanov, 2001). Thus, students' external representations can be a good way to see how they understand the mathematical concepts. Thus, representations in mathematics education are not merely informative objects in reading and science but tools for thinking which allow students to manipulate objects and to generate mathematical ideas. Therefore, representations are very important tools for effective teaching and learning. Well-established representations may help students better understand mathematical concepts as well as allow teachers to facilitate decent mathematical discussions through representations more effectively. But what could be a good representation?

Many researchers have explored how students understand and use visual mathematical representations in their learning (e.g., Brenner et al., 1997; Fennell & Rowan, 2001; Goldin & Shteingold, 2001; Herman, 2007). In particular, by observing student-invented representations, researchers have attempted to uncover students' understanding of mathematics as well as difficulties and "hidden meaning" that students have (Zazkis & Gadowsky, 2001). For example, Coulombe and Berenson (2001) investigated how students generated and interpreted different kinds of representations, such as graphs and tables for linear functions. They found variations among students' understanding of linear functions through representations. Some students had difficulty in reading representations and interpreting them in realistic contexts. In particular, they had a hard time to interpret a negative slope of a linear function in relation to realistic contexts. Diexmann and English (2001) reported that students had difficulties in generating and understanding diagrams as well as reasoning with diagrams. It was not easy for students to create a diagram to fit the structure of problems. Students sometimes could not explain what they did with representations.

Such difficulties in students' use of representations lead to discussions for instruction using representations. Scheuermann and van Garderen (2008) found that students often have difficulties in understanding representations or problems because of lack of literacy skills: what the representation mean in the context of a problem and what is the important ideas in a problem. They argued that teachers should teach not only the meaning of representation and key ideas of a problem but also how to represent the ideas of a problem using representations. These two components should not be separately taught. Students need to be able to know key ideas of a problem in representations and understand representations with regard to key ideas of a problem, for representations are both "process" and "product" (National Council of Teachers of Mathematics, 2000). Representations are not only an outcome from reasoning but also part of the process of reasoning. However, representations in school mathematics are often considered only products (National Council of Teachers of Mathematics, 2000). This is one of the reasons that students may have difficulty in learning mathematics through representations.

For the further discussion about this issue, the literature about representations for the concepts of angle, slope, and prime factorization that I focused on for this study will be reviewed in the next section.

2.3 Research on Teaching and Learning Angle, Slope, and Prime Factorization

Angle, slope, and prime factorization are the fundamental concepts that elementary and middle school students should learn in school mathematics. I selected them as topics for focused analyses of non-textual elements in this dissertation.

Angle is one of the basic ideas in geometry to understand geometric shapes and structures as well as to analyze their relationships. Since geometry is "a complex interconnected network of concepts, ways of reasoning, and representation systems to conceptualize and analyze physical and imagined spatial environments" (Battista, 2007), it is natural that geometric ideas such as angle are often used to represent and analyze objects in mathematics and the real world. In the process of reasoning, it is evident that representations in geometry are crucial to develop geometric ideas.

As shown in the previous section, students often struggle with visual representations and drawings because they are asked to reason about abstract geometric ideas through concrete visual representations (Laborde, 1993). Since a diagram often represents a case of a geometric concept, it could be confusing and complicated for students to understand the concept (Clements & Battista, 1992). For example, parallel lines mean that two lines never meet or intersect. A line is theoretically considered to be infinite long. However, representations for parallel lines in textbooks such as | | or = are finite. Diagrams and drawings sometimes can be a distractor for student learning (Mesquita, 1998).

When angle as a geometric idea is applied to realistic contexts, similar problems occur. Angles can be found easily in the contexts of real life. However, students may think that angles from real life are different from angles in geometry that they have learned theoretically. Angle can be defined in three different ways: "an amount of turning about a point between two lines," "a pair of rays with a common end-point," and "the region formed by the intersection of two half-planes" (Mitchelmore & White, 2000). Indicating students' understanding of the concept, Mitchelmore and White (2000) described three stages of abstraction in terms of angle: "situated angle concepts," "contextual angle concepts," and "abstract angle concepts." These stages mean that students may understand a single concept of angle in a situation (situated angle concepts), find similarities between different situated angle concepts (contextual angle concepts), and then find similarities between different contextual angle concepts (abstract angle concepts). Through this process, they argue, students can generalize the concept of angle across contexts with connection among the different angle concepts as defined above. However, many studies have reported that students often have difficulties in understanding the concept of angle (e.g., Battista, 2007; Kieran, 1986; Mitchelmore &

White, 2000; Park, Kim, & Jeong, 2005). Mitchelmore and White (2000) gave one of the reasons that the angle concept applied to various contexts would be closely related to how an object looks and the structure of angle concept. For example, students may not find an angle between horizontal lines on the ground and the hill. Students may have a hard time to understand the relationship between angles of inclination and trigonometry.

Despite student difficulties with visual representations, Battista (2007) pointed out that researchers and teachers should consider strengths of visual representations that enable students to expand their mathematical ideas from particular figures and to apply abstract geometric ideas to a specific figure and object. From this perspective, it is important for students to experience various contexts to construct a geometric idea, not only from mathematical abstract figures but also realistic objects. This implies that students need to experience both abstract mathematical representations and visual representations with realistic contexts to develop their concept of angle. Also, this suggests that it is important to make connection between mathematical ideas and contexts.

Slope is one of the important ideas in both algebra and geometry. The concept of slope can be found in formulas and equations in algebra, in graphs in geometry, and as the tangent of an angle in trigonometry (Stump, 2001). Similar to the concept of angle, the concept of slope can be taught using abstract mathematical representations, mostly graphs, and visual representations within realistic contexts such as hills and stairs. Realistic representations in textbooks sometimes introduce the concept of slope in two different ways: physical situations like ski slopes and functional situations like the rate of growth. The former is related to slope as a measure of steepness and the latter is related to slope as a measure of rate of change (Stump, 2001).

Many studies have reported that students may have misconceptions about and difficulties in learning the concept of slope because they often cannot understand slope in relation to linear functions or graphs (Bell & Janvier, 1981; Schoenfeld, Smith, & Arcavi, 1993), and sometimes they have a hard time to consider slope as a ratio due to lack of proportional reasoning (Barr, 1981; Lamon, 1995). Even pre-service teachers could not understand the two different approaches, steepness and rate of change, as the concept of slope with connection to physical and realistic situation (Simon & Blume, 1994). While the pre-service teachers understood steepness of ski ramps with concrete models and found the idea of rate of change when they talked about velocity of bicycles, they could not make a connection between the two approaches. Stump (2001) argued that teachers should provide students with a lot of opportunities to discover the concept of slope through various activities that involve both physical and functional situations as well as to communicate their understanding of mathematics with regard to their language use. For doing this, she pointed out that it is important for students to understand the relationship between the concept of slope and visual representations. Even if students intuitively recognize a slope of a ski ramp, it does not mean that they understand the slope mathematically. It is important to make the connection between the concept and visual representations for student learning.

The Fundamental Theorem of Arithmetic, one of the most important in mathematics, states that any composite number can be presented as a unique product of prime numbers. Prime factorization is very important to understand "numbers and multiplicative relationships among numbers… the representation of numbers" (Zazkis & Liljedahl, 2004). However, although this theorem has been explored by many mathematicians, there are only few studies about how this topic is taught in schools and how teachers and students understand the concept. In the study of pre-service teachers' understanding of prime factorization through representations, Zazkis and Liljedahl (2004) concluded that lack of visible and clear representation for prime factorization could hinder students from constructing the concept. They found three different ways of finding prime numbers: factoring numbers, observing examples, and excluding composite numbers. Some of the pre-service teachers in the study factored numbers by checking if the numbers could be divisible by prime numbers. However, they could not fully make connections among the concept of factors, multiples, and divisibility. Some pre-service teachers observed examples of prime numbers to find out properties of prime numbers and test what they found with other numbers. In this case, according to examples with small numbers, they reached a wrong conclusion that every prime number is odd. Other pre-service teachers failed to represent prime factorization when they excluded composite numbers because they could not know how to represent it mathematically. From the observation for all the three cases, they found the important roles of visual representations that help students construct mental images and objects for the concept of prime factorization. Since it is not easy to objectify mathematical concepts (Sfard, 1991), unclear representations could be an impediment to constructing concept images (Zazkis & Liljedahl, 2004). This suggests that clear visual representations play an important role in learning mathematics as tools for thinking.

2.4 Learning from the Literature Review

As shown above, although pictorial representations and mathematical representations are parts of mathematics textbooks, research on these different types of representations in mathematics textbooks has not been conducted. In particular, there is no research on how these representations are interpreted and used in mathematics textbooks and classrooms as well as how these two kinds of representations support student learning together and what kinds of roles each representation play in explaining a concept. Research on pictorial representations gives some clue about what features of pictorial representations might affect student learning. For example, when pictorial representations are clearly connected with the texts, their effects on student learning increase. Realistic pictures could be helpful for students to understand the texts. Research on mathematical representations also provides some useful information about how students understand and use representations for their learning. However, unlike reading and science, since representations are tools for thinking in mathematics, both pictorial and mathematical representations in mathematics textbooks should be understood together with regard to their roles in teaching and learning.

In terms of content area, although angle, slope, and prime factorization are important and fundamental topics in mathematics, many studies have reported that students have various difficulties in understanding the concepts in relation to contexts and situations. Because the ideas of Freudenthal's Realistic Mathematics Education (RME) are influential in mathematics education, mathematics education researchers have examined mathematical learning in realistic contexts such as how students develop a mathematical model from given contexts, how students transform mathematical situations into symbolic representations, and how students develop representations to solve problems in real life (e.g., Meyer, 2001; Woleck, 2001). This work argues that realistic contexts themselves cannot be mathematics. They should be used as representations with which students could construct mathematical ideas. Only when students can "mathematize" realistic contexts and situation, can it be meaningful (Freudenthal, 1991). In addition, the literature review on teaching and learning of the three concepts show that students have a hard time applying mathematical ideas to realistic contexts as well as constructing mathematical ideas from realistic situated problems. This implies that it is important for students to experience both mathematical representations and contextual representations in which mathematical concepts are located in realistic contexts. Various experiences with different kinds of representations may help students construct mathematical concepts properly. This idea can be related to both contextuality and variety in this study. It is important not only to understand mathematical ideas in realistic contexts but also to experience various ways of presenting mathematical ideas. Although some teachers know that there are several definitions of each concept and multiple representations of mathematical concepts have more power to support student learning, they tend to use one representation to explain one of the definitions and textbooks generally show one representation (Jones & Swan, 2006). However, the literature shows how such "one-concreteness" (Presmeg, 1997) could be dangerous.

Another issue I learned is that accurate representations are important to construct concept images. Since students develop mathematical ideas through representations, it is very important to have accurate and clear representations. Unclear and inaccurate representations may impede students' understanding of the concept.

CHAPTER 3

TEACHERS' PERSPECTIVES ON NON-TEXTUAL ELEMENTS

Non-textual elements can be effective in student learning only when teachers explicitly give clues or instructions (Peeck, 1993). Teachers' understanding and interpretation of non-textual elements in mathematics textbooks can affect students' opportunities to learn mathematics through non-textual elements. It is thus important to see not only how teachers understand and interpret non-textual elements in mathematics textbooks but also what kinds of criteria teachers use to choose non-textual elements for their instruction. Examining the data collected through structured interviews with 10 Korean and 11 U.S. in-service secondary mathematics teachers (See Appendix A. for the interview protocol)[4], this chapter explores how teachers consider these five characteristics of non-textual elements in mathematics textbooks as well as what other characteristics teachers think important for teaching and learning.

3.1 Data Collection and Analysis

Participants

In order to investigate how teachers understand the roles of non-textual elements in mathematics textbooks, I interviewed 11 U.S. teachers and 10 South Korean secondary mathematics teachers using a structured interview and probes. The teachers were teaching mathematics in grades 6-9 (mostly middle school students)[5] at the time of the interviews. They were recruited at regional conferences in South Korea and the United States. Their

[4] All teachers' names are pseudonyms.
[5] Since South Korea and the United States have different school system, the range of students who the teachers teach is determined wider than each country's middle school ages for this study.

attendance in regional conferences implies that they have interests in professional development for teaching mathematics and their commitment to teaching is relatively high. Since the regional conferences were held for teachers to improve their teaching methods and mathematical knowledge in both countries, it may be that they have special interests in pedagogical content knowledge and new ideas of teaching methods. The teachers I selected from conferences are relevant for this study because there is a high possibility that they consider how to use representations and which representations are appropriate for their instruction as part of pedagogical content knowledge. All the teachers volunteered to participate in this study by responding to some questions in the interview protocol.

The demographic backgrounds of South Korean and U.S. teachers are quite similar. All the teachers have taught mathematics in secondary schools, and all have a bachelor's degree in education field. Three South Korean teachers and four U.S. teachers also have master's degrees. Most of the teachers have plenty of teaching experience. Seven out of the 11 U.S. teachers have taught mathematics over ten years. Similarly, seven out of the 10 South Korean teachers have taught mathematics over ten years. In terms of their majors, there are some differences. While all the South Korean teachers majored in mathematics education, U.S. teachers' majors varied, including mathematics, elementary education, secondary education, and others. However, considering the difference in teacher education system in the two countries, we may understand the differences. Whereas one must major in mathematics or mathematics education to become a teacher in Korea, there is variation across states in the United States according to their policy and teacher education system.

Interview protocols

Interviews allowed me to see teachers' rationales and criteria of good non-textual elements for mathematics learning as well as to get a sense of how they use the textbooks' non-textual elements in classes. The interview protocols for this study were developed on the basis of findings from preliminary work described above, including some of the non-textual elements used in the survey where significant differences among respondents' interpretations were found. The interview protocol for this study includes non-textual elements selected from lessons about angle, slope, and prime factorization. I chose slope because unlike topics in geometry which use many visual representations in nature, slopes are sometimes explained with only verbal representations. Since the concept of slope is related to that of angle, it is interesting to see how teachers and curriculum developers understand the roles of non-textual elements in each topic. Further, it is noteworthy to compare how teachers and curriculum developers understand non-textual elements in explaining prime factorization, which is rarely explained by visual representations to the understanding of teachers and curriculum developers in other two topics.

The protocol for teachers (See Appendix A) consists of two parts: (1) questions about non-textual elements, and (2) demographic information. The first part of the protocol includes seven tasks with sub-questions about non-textual elements in mathematics textbooks. The non-textual elements in the protocol were selected from both U.S. and Korean textbooks. I list possible components from the R-Framework for each question in Table 2. However, they simply reflected my assumptions about what kinds of aspects the teachers might consider for each non-textual element. I expected that the teachers might raise some other important aspects of non-textual elements that were not included in the R-Framework. Although I had the R-Framework which was the basis for the interview protocol, I wanted to listen to teachers' own thoughts about non-textual elements without any influence from my interpretation.

I first asked open-ended questions about their ideas of non-textual elements in teaching and learning mathematics such as mathematical meaningfulness and pedagogical usefulness. For example, when I asked which representation is more useful for students to understand the concept of angle, a teacher pointed out the picture in Task 1 (1) because it represented the definition of the concept accurately. Then, I asked sub-questions to know what it meant and why the teacher interpreted it as accurate. From such conversation with the teacher, I found that the teacher considered accuracy as an important aspect of non-textual elements for learning mathematics. I also asked which representation is more effective to teach the concept to support student learning. The teacher answered the picture in Task 1 (2) because it has realistic contexts which attracted students' attention. I also asked sub-questions about what this meant. Through the conversation, I could see what the teacher considered important for both teaching and learning mathematics.

I also asked the teachers to compare non-textual elements in the protocol. Since each task showed different types of non-textual elements used to explain a concept or solve a problem, this question allowed me to see how the teachers understand different types of non-textual elements. The question about modification also allowed me to see important aspects that a good non-textual element should possess for teaching and learning the concept. The following table shows the details of the tasks.

Table 2. Summary of Interview Protocol for Teachers

Task	Topic	Possible framework aspect	Subject of questions
1	Angle	1) Accuracy, connectivity 2) Aesthetics, contextuality, connectivity 3) Aesthetics, connectivity, variety	a. Pedagogical usefulness b. Mathematical meaningfulness c. Similarities or differences among them d. Possible modification
2	Slope	1) Accuracy, aesthetics, connectivity 2) Accuracy, aesthetics, contextuality, connectivity	a. Pedagogical usefulness b. Mathematical meaningfulness c. Similarities or differences among them d. Possible modification e. Roles of mathematically disconnected non-textual elements
3	Prime factoriza-tion	1) Accuracy, aesthetics, contextuality, connectivity 2) Accuracy, aesthetics, connectivity 3) Accuracy, aesthetics, contextuality, connectivity, variety	a. Pedagogical usefulness b. Mathematical meaningfulness c. Similarities or differences among them d. Possible modification
4	Roles of non-textual elements	Accuracy, aesthetics, contextuality, connectivity, variety	a. Roles of non-textual elements across topics b. Use of non-textual elements in class
5	Problem solving	1) Accuracy, aesthetics, contextuality 2) Accuracy, aesthetics	a. Usefulness of non-textual elements in problem solving b. Similarities or differences among them
6	Problem solving	1) Accuracy, aesthetics, connectivity, variety 2) Accuracy, aesthetics, connectivity, variety	a. Mathematical meaningfulness b. Pedagogical usefulness c. Similarities or differences among them d. Application of non-textual elements to contexts e. Possible modification

Table 2. (cont'd)

Task	Topics	Possible component	Questions
7	Use of non-textual elements in teaching	Accuracy, aesthetics, contextuality, connectivity, variety	a. Participants' use of non-textual elements b. Usefulness of visual representations c. Opinions on the use of non-textual elements in current textbooks

The demographic information part in the protocol includes questions about participants' major, teaching experience, materials they have used for their class, and other resources for their class. Although this study does not include systematic analysis of the relationship between participants' demographic background and their interpretation of non-textual elements, I assume that knowing their demographic information might allow for some understanding of their rationale behind their interpretation as well as allow me to raise some questions for further study.

Procedures

Interviewees signed a consent form to agree to participate in this study. They were asked to respond to some tasks and corresponding open-ended questions in the structured interview protocols for in-depth analysis and development of a conceptual framework. Although the tasks were designed on the basis of the non-textual elements framework, the interviews also examined which factor teachers consider in their judgment of which non-textual elements are useful and meaningful for teaching and learning. In order to see how the teachers understood non-textual elements in the textbooks that they have used for their classes and how they interpreted non-textual elements in relation to texts (connectivity), I asked some additional questions with both South Korean and U.S. textbooks at the end of each interview. Since U.S. teachers could not read texts in South Korean textbooks and South Korean teachers could not read U.S. textbooks, I translated excerpts in English or Korean to help them understand the texts. Since the interview protocol consisted of selected non-textual elements in South Korean and U.S. textbooks, I intended to show the teachers both sets of textbooks in order to see how they interpreted them with texts. This also allowed me to see how teachers understood non-textual elements that they were relatively unfamiliar with. Since the South Korean and U.S. textbooks look different in terms of their feature and structure, I could observe how teachers understood non-textual elements represented in different textbooks that they have never used before. In addition to the textbooks that I brought to the interviews, the teachers also showed me their textbooks to explain how they used non-textual elements from their textbooks for their classes. Each interview took between 30 and 45 minutes. The task-based interviews were audio-taped and transcribed for the analysis.

Data analysis

The interviews were analyzed by finding patterns across their responses to each question and coding key terms and sentences that they often used. I used emergent coding for each aspect. The patterns and key ideas which emerged from the data were categorized to elaborate the R-Framework. In order to find similarities and differences of the roles of non-textual elements perceived by the teachers from South Korea and the United States, the general patterns and key ideas were also analyzed by each group: South Korean teachers and U.S. teachers. The finding from the interviews is used to elaborate and expand a conceptual framework (the F-Framework) to examine non-textual elements in textbooks.

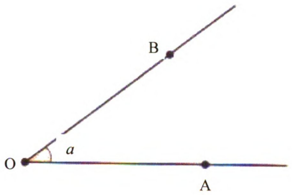
3.2 Accuracy, Connectivity, Contextuality, Aesthetics, and Variety of Non-Textual Elements

Accuracy

When I asked teachers about their criteria to select non-textual elements for their classroom teaching, all the teachers from South Korea and the United States emphasized the importance of accuracy in non-textual elements. In particular, accuracy of non-textual elements becomes more important when introducing a new concept because inaccurate non-textual elements can be coupled with the concept which may produce students' misconception and affect their future learning. Thus, the teachers stated that it is important to show the definition of the concept accurately in a non-textual element, i.e., how the non-textual element accurately represents the definition of the concept. For instance, when I asked which non-textual element would help students to better understand the concept of angle in mathematical ways, 10 teachers from both South Korea and the United States considered whether or not a given non-textual element accurately represents the definition of angle. Ms. Paek said, "When explaining a new concept, I always want to make sure that students understand the fact that every mathematical concept begins from its definition. So it is very important that a representation accurately shows the definition of the concept." She seemed to be concerned with the cumulative nature of mathematics that has every concept build upon its definition. If a non-textual element fails to represent the definition of a concept, it may lead to students' misconception on the concept. Mr. Evans also considered the definition of angle, saying that "I guess for [Figure 7] number one it depends on what definition you're looking for, but if the definition says ray, you'd certainly want to have arrows at the ends here." The data from the interviews show that teachers seek to ensure accuracy of non-textual elements by confirming that the non-textual elements accurately represent the definition of the concept.

Figure 7. Angles in Task 1

(1)



(2)

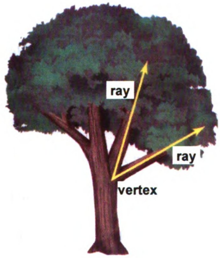
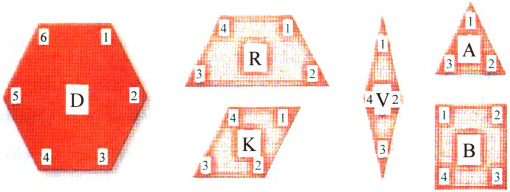


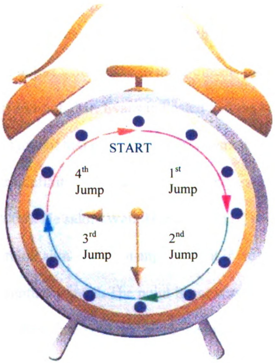
Figure 7. (cont'd)

(3)



Although all the teachers acknowledged that realistic contexts play an important role in both promoting student motivation and demonstrating meaningfulness of mathematics, it is notable that they give priority to accuracy over realistic contexts in non-textual elements. Even though a non-textual element with realistic contexts is intended to help students to understand a given concept, the role of the contexts would be questionable if it is inaccurate or unclear. When I talked about a picture of a clock explaining prime factorization (See Figure 8), Mr. Carter said, "It doesn't seem like the picture is . . . help[ful] that much. You know, cuz here, I mean you still have to already know what the prime factors are." This implies that although the picture in Figure 8 has a realistic object, students who don't know about the concept of prime factors may not be able to learn from this picture at all. Ms. Jung also said, "Since the picture is realistic but unclear, students who don't have any prior knowledge about factors will be hard to figure out what the picture really means. Even some students who already have prior knowledge may have a hard time to figure out what the picture is about and how this picture is related to factorization. So this picture is useless, especially when the concept is introduced for the first time." This shows that the teachers tended to think that unclear non-textual elements may not have much educational effectiveness and significance in student learning because they do not explicitly show what they represent and how they are associated with the mathematical concepts. It is hard to use them as tools for thinking because students may get confused by the pictures. In fact, some of the teachers I interviewed also asked me what this picture meant. Even though I showed Figure 8 in textbooks with texts, they said that they would not use it for their classes because it was still difficult to understand the ideas in the written text with the picture. It means that even teachers who have prior knowledge about the concept could struggle with interpretation of the picture. Thus, such an unclear image can be useless not only for students to learn the concept but also for teachers to use as a tool for their instruction.

Figure 8. Clock in Task 3



In terms of non-textual elements with realistic contexts, some teachers pointed out that when a realistic object in three-dimensional space is projected onto a two-dimensional plane, it is possible for students to get a misconception about the concept of angle. Mr. Lewis noted when he talked about a tree (See Figure 7, (2)):

Mr. Lewis: I would be careful because it's not flat. There's more three dimensional. One could be going – when they think of a real tree it could be going out. It's more of a three dimensional representation. A lot of students might have a misconception as to what angle means.

He argued that the most important criterion for a non-textual element is whether or not the non-textual element represents the concept in a mathematically correct way. In this respect, he added, if a three-dimensional object is projected onto a two-dimensional plane, it could be unclear which angle the picture intended to indicate among multiple angles the object would have in three-dimensional space.

In addition, Mr. Evans indicated that the branches of the tree were not straight lines. He thought that even though they came from a realistic context, they did not clearly present the definition of angle, "a pair of rays with a common end-point" (Mitchelmore & White, 2000). He said, "Well, [Figure 7 (2) is] probably not [helpful]. I mean, I think that once we think about what an angle is, that if we're trying to think of this as a ray, the branch eventually stops. The point is not very clearly defined, where the branches come together." He pointed out that since the branches of the tree are not rays, this picture is not appropriate for the definition of angle, which is formed by two rays sharing a common endpoint, the vertex of the angle. By definition, a ray is part of a straight line which is finite in one direction, but infinite in the other. However, he stressed that the branches in the picture were not really rays because "the branch eventually stops." A branch of a tree is not a straight line in most cases as in this picture. Students may have misconceptions from this picture because they might think that any type of lines including curvy ones can constitute an angle whenever they appear to share a common endpoint. In addition, he also mentioned that the angle in the picture does not satisfy the definition of angle because "the [end]point is not very clearly defined, where the branches come together." The endpoint or vertex of an angle is constituted when two rays meet. However, the branches of the tree in this picture do not really meet at a point from a geometric point of view. In Euclidean geometry, a point does not have space, but the branches of the tree in the picture share a certain amount of space and volume, and therefore, they cannot be seen as meeting at a point.

Realistic Mathematics Education (RME) suggests that the mathematical concepts existed in realistic contexts can be a good starting point for students to formulate the concepts in their mind (Freudenthal, 1991). In particular, students who have never learned the concept before may use such contextual situated mathematical concepts as a model in their learning (Widjaja & Heck, 2003). Thus, it is important to ensure that a realistic context or object is appropriate to present the definition of a given concept in terms of mathematical accuracy.

Patterns of *accuracy* among the different teachers

Interestingly, although both the South Korean and U.S. teachers I interviewed gave a strong emphasis on the importance of accuracy in non-textual elements, their attention to and rationale for accuracy of non-textual elements were different. While the South Korean teachers focused on accuracy of non-textual elements in relation to advanced mathematics which their students would learn in the future, the U.S. teachers paid attention to accuracy of non-textual elements when the non-textual elements presented mathematical concepts in realistic contexts.

Many South Korean teachers mentioned that non-textual elements should be accurate enough to be used in advanced mathematics. When we talked about Figure 7, Ms. Cho said, "The angles in (3) have only the degree of angles, not including the direction of the angles. So it is not easy to explain the direction of angles using this picture. Students may get confused when they learn negative angles or other way to represent an angle." Ms. Kang also said that if we always introduce an acute angle from x axis as shown in Figure 7 (1), students may have difficulty in expanding the concept to the concept of radian in trigonometry. She said that it is important for students to experience a variety of angles. If not, students may have difficulty in applying the idea to the concept of angle in advanced mathematics.

Unlike the South Korean teachers, the U.S. teachers focused on accuracy of non-textual elements which included realistic contexts. None of the U.S. teachers pointed out the extension of mathematical concepts presented in a non-textual element to advanced mathematics, considering students' future learning of mathematics. Rather, the U.S. teachers focused on each contextual non-textual element and attempted to examine if it was appropriate to explain a mathematical concept. As Mr. Lewis and Mr. Evans pointed out, the U.S. teachers were concerned with errors in non-textual elements when non-textual elements presented mathematical concepts in realistic contexts. Such concerns with realistic contexts in non-textual elements were also found in some of the South Korean teachers' interviews, but this was the only concern the U.S. teachers noticed. Even though teachers agreed upon the importance of accuracy in non-textual elements, their rationales and approaches can vary. Some teachers are likely to understand accuracy of non-textual elements with regards to mathematical structure and cumulative nature whereas other teachers tend to understand it with regards to mathematical application in realistic contexts.

Overall, although both South Korean teachers and U.S. teachers emphasized the importance of accuracy, only three out of 11 U.S. teachers could explain strengths and weaknesses of a particular non-textual element with some evidence whereas nine out of 10 South Korean teachers could. Knowing the importance of accuracy is different from knowing which non-textual elements are accurate or inaccurate. Even though all the teachers who participated in this study emphasized the importance of accuracy, there is variation among the teachers in terms of their perception of accuracy of a certain non-textual element. This implies that teachers should have opportunities not only to understand the importance of accuracy in non-textual elements but also to develop their critical eyes on specific non-textual elements in order to select the most appropriate non-textual elements for their instruction.

Connectivity

As shown in Chapter 1, many studies have called attention to the importance of connectivity between non-textual elements and texts (e.g., Levin & Mayer, 1993; Pape & Tchoshanov, 2001). Connectivity was defined as how non-textual elements are closely related to textual elements. All the teachers I interviewed also argued that connectivity of non-textual elements is one of the most important factors that make the elements more meaningful in learning mathematics. From the interviews with teachers, closeness between a non-textual element and texts were often interpreted as close association between a non-textual element and the content. Unlike reading, since non-textual elements in mathematics provide not only related information but also a tool for thinking that could be manipulated, it is necessary to investigate what teachers consider connectivity between non-textual elements and content. This became more obvious when we talked about the picture of a female skier falling down on the hill as shown in Figure 9 (2).

Figure 9. Slopes in Task 2

(1)

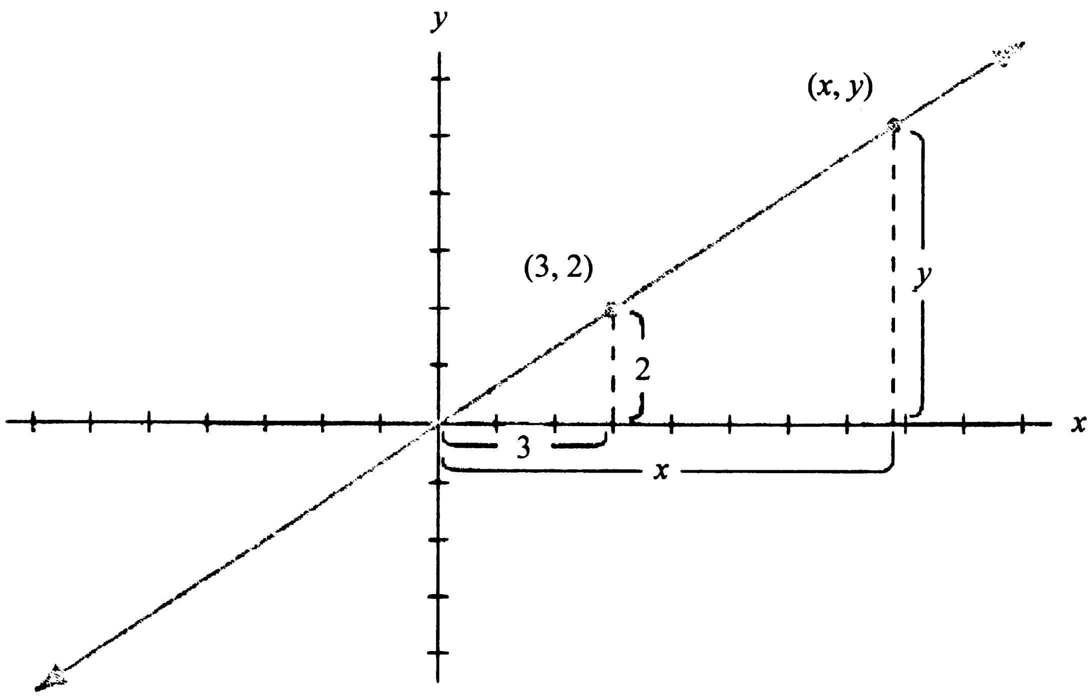
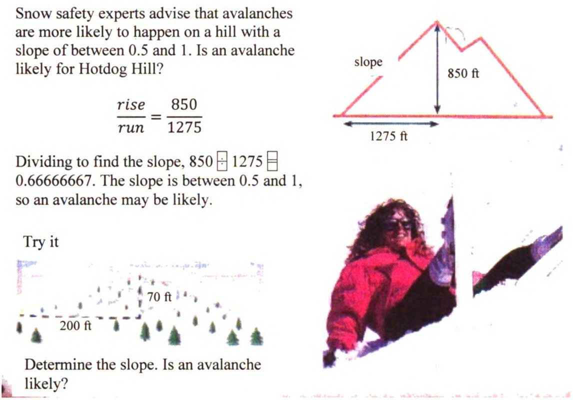


Figure 9. (cont'd)

(2)



Snow safety experts advise that avalanches are more likely to happen on a hill with a slope of between 0.5 and 1. Is an avalanche likely for Hotdog Hill?

$$\frac{rise}{run} = \frac{850}{1275}$$

Dividing to find the slope, $850 \boxed{\div} 1275 \boxed{=}$ 0.66666667. The slope is between 0.5 and 1, so an avalanche may be likely.

Try it

Determine the slope. Is an avalanche likely?

When I showed two non-textual elements in Task 2 (See Figure 9) and asked which picture would be more helpful for students to understand the concept of slope, none of the teachers mentioned the picture of the female skier shown in Figure 9 (2). All the teachers agreed that since Figure 9 (1) showed what a slope meant in mathematics and how to calculate the slope of the line in the graph, this kind of graphs were more helpful for students to understand the concept mathematically. In addition, some teachers mentioned that since the picture of mountain in red (the picture located in the right above the female skier) in Figure 9 (2) allowed students to find the slope of the mountain with an image of the slope in the real world, such a picture could be helpful for students to better understand the concept. This shows that even if a non-textual element presents realistic objects or contexts, it could be helpful if it has mathematical connectivity. In the case in Figure 9 (2), even though the picture presents a mountain which is a realistic object, the picture is useful because it shows what is the slope of the mountain and how it can be measured. Such mathematical connectivity in a non-textual element is an important criterion that teachers use to judge its usefulness.

In order to see how teachers understand the roles of non-textual elements which do not have mathematical connectivity with texts, I asked how they thought about the role of the picture of the female skier in Figure 9 (2). Although I wondered if they could see connectivity between the picture and texts, most of the teachers considered connection between the picture and the content (slope). Table 3 shows the teachers' opinions about the picture.

Table 3. Teachers' Opinions about the Role of the Picture of the Female Skier

The picture is...	Number of teachers	Reasons
Distracting	11	Too distracting to concentrate on the content No connection with the concept Attention to the woman instead of the problem Lacks mathematical significance/relevance
Helpful	4	Gives contextual information Longer retention Gives a clue by relating slope with steepness
Neither	6	Unnecessary/meaningless picture Depends on students' developmental/cognitive level

As shown in Table 3, more than half of the teachers thought the picture distracting rather than helpful because of the lack of connection with the mathematical concept or the problem. Since such a picture which does not have mathematical connectivity with the content is usually colorful and proportionally bigger than other components in a given problem, it easily attracts students' attention. Ms. Robinson said that "Well I think it would get their attention. It's gonna – their eyes are gonna be drawn to her, first. If they see that page or that problem, first thing – their eyes are gonna go there [indicating the picture of the female skier]." In addition, such a picture does not give students any help to solve the problem. Mr. Martin pointed out that "It has nothing to do with solving the problem, it's totally unrelated to the problem of slope, but in terms of maybe engaging some people, or getting their interest, because maybe they are skiers themselves, maybe it'll pull some more kids in that would, but in terms of the actual problem, insignificant." Although they might attract students' attention, such decorative pictures thus are not only helpless for students to solve the problem but also can distract students' understanding.

Ms. Morris: Yeah, I think she's just an additive. Maybe almost a distracter for some kids, because a lot of times, textbooks want to give you more information that you don't have to use, they just throw those numbers in and stuff, and that might be one of those things to some kids. That why is she falling down? Ask a lot of questions about why she's there rather than on their own actual.

The teachers commonly pointed out that the female skier in the picture was not helpful for students to solve the given problem because the skier was not relevant to the concept the problem intended to ask students to utilize in solving the problem. The given problem was about the concept of slope, but the skier did not give any information on what a slope meant in mathematics and how to measure or calculate a slope.

Further, Ms. Jung made an interesting point that such a picture could mislead students even though it provides contextual information. She said,

> Ms. Jung: When I looked at the picture at a glance, the picture attracted my attention. At the same time, the image of the picture was steepness, which was scary to me from my experience in skiing. So I immediately guess that this problem may ask, "What degree of slope of the hill may be scary to skiers?" But when I read the problem, it was totally different from what I thought. Wow, it's interesting. I think this kind of picture would be helpful for kids to see the context where the concept can be found. But this could mislead students as I did. I think this picture does not have to do with this problem at all. This picture is not helpful to solve this problem.

She warned the possibility of giving students some misleading information on the given problem. Students might simply guess what the problem asks without reading carefully the question as the picture would attract students' attention before they read the question. As a result, students might misinterpret the problem, just as Ms. Jung "immediately guess[ed]" from the picture what the problem would be, but she found her guess was not correct and was surprised by the gap between her guess and the actual problem.

In contrast, some teachers thought that the picture was helpful because it would provide contextual information or some clue about the relationship between slope and steepness. Mr. Flores said that "I guess it's kind of helpful. Again it provides more context, I teach a lot of students who have a tough time reading and so to see that picture of the girl on skis that would give them an idea, okay, I understand that my words are going to be based on somebody skiing down a mountain probably and so I think that it would be helpful for 'em." He considered different learning styles of individual students. For example, students who have some difficulty in reading would benefit from this picture because the picture might serve as a context that provides some cues for what the problem would be. In addition, this picture can help students to understand the concept of slope in relation to steepness in daily life. Ms. Paek said that "when students look at the picture of a female skier, they may get some sense of steepness from real world. They may figure out that 'Aha, slope is related to steepness. That's why the woman fell down on the hill.' So I think it is helpful for kids to understand it."

Although these two groups of teachers had opposite opinions about the usefulness of the picture of the female skier, it is notable that both considered connections with the content. Such commonality can be found in the third group who thought that it did not matter. These teachers were not sure if the picture was distracting or helpful. However, they said that such a picture was useless and unnecessary because it was not associated with the content. They said that they did not pay attention to such decorative pictures in class at all. Mr. Carter talked about the inefficiency of such kinds of pictures in textbooks.

Mr. Carter: So, a lot of the pictures, I mean they're nice and they're colorful, but . . . once again we're talking variability, and then a picture of two kittens, kind of thing. You know, it's just a lot of times the illustrations don't really seem to have that much relevance, at least in the urban settings. . . . I mean, if you really dig into it, you can start to make the connections. . . . But I don't think the kids necessarily see a lot of that stuff, to see any connections. It's almost like a lot of times the pictures don't seem to serve any purpose to them. . . . more paying attention [to] the text and things like that.

He said that a picture, even if it is a decorative one, should have meaningful connections to mathematics and/or students' daily life. Teachers do not use random pictures in their classrooms. They choose pictures that have clear relevance to the contents they are teaching and/or to some contexts where the contents can be situated. Also, students' everyday life is an important element to determine relevancy because the picture will provide students some familiar contexts that facilitate students' engagement in the content to which the picture is related. Including many decorative pictures that lack meaningful connections to mathematics and students' life will increase the volume of the textbooks without bringing about educational effects. Teachers will not use such pictures because "it's just a lot of times the illustrations don't really seem to have that much relevance." Overall, the teachers' responses suggest that connectivity is a very important criterion when they make decisions about how each picture will be useful in their instruction.

Even though each group of teachers had different opinions, they all agreed that connectivity is one of the most important factors when considering the role of non-textual elements in textbooks for teaching and learning mathematics. The difference between the groups was their opinions about what kinds of connectivity a non-textual element should have. They focused on different ways in connection to mathematical content, context, or expected behavior of students in an activity to a solve problem.

All the teachers agreed that non-textual elements should have mathematical connections with texts. For example, when the concept of angle is introduced, it will be more useful if some pictures of angles in triangles and rectangles are shown. Such a non-textual element can provide some ideas of what an angle is and how it can be found in several different shapes. However, teachers have some disagreement on the usefulness of connectivity with contexts or students' activities. Connectivity with contexts means that a non-textual element has to do only with contexts in the content or problem. For example, when a textbook introduces how to calculate a slope of the Hotdog Hill, it includes the picture that a female skier falls down on the hill (See Figure 9). The picture of the woman does not tell anything about how to find the slope of the hill mathematically. The picture is associated only with the context in the problem. Some teachers thought that it would be useful for students to understand the concept because it would allow students to understand the concept of slope within a realistic context. However, other teachers thought that it would be distracting or even useless because it had nothing to do with the concept and it would attract students' attention to the picture rather than the problem.

Even though I did not include any picture of students' behaviors or activities in the interview protocol, some teachers also noticed the role of pictures of students' behaviors or activities in textbooks. Mr. Carter showed me a picture from his textbook in which two students were drawing several triangles on a paper. He said,

> Mr. Carter: It was something that you could tie right in, you had a hands-on kind of an activity, you know, it tied right in to the lesson, it was a visual type thing. But that's something that just, from the very first time I saw this, really stuck with me. Whereas a lot of the others, it's just like, 'Yeah, that's a nice picture, but what's the point?

Ms. Jung also asserted that such kinds of pictures should be removed from textbooks. She said,

> Ms. Jung: For example, when problems about logic and statement are presented, the textbook I currently use has a picture of students who are talking with each other as a small group. Maybe it implies students' group activities that create statements and discuss about the logic behind it. But why do

> we have such a picture in the problem? It is simply unnecessary. I don't want to have such pictures in textbooks.

Both teachers saw some connection between the pictures and the texts. However, they thought that such pictures were unnecessary in textbooks because they had little to do with the substantive contents that they were teaching.

Contextuality

The P-Framework had two types of contextuality: instrumental contextuality and mathematical contextuality. *Instrumental contextuality* refers to how a mathematical concept is expressed by a non-textual element within a real context. Similarly, *mathematical contextuality* in this study denotes how a mathematical concept is expressed by a non-textual element within a mathematical context. A real context means a situation in which people live and experience. A mathematical context denotes a situation in which a mathematical element(s) serves as a context for explaining a concept.

My data from the interviews with teachers suggest that there are three different roles of non-textual elements in relation to contextuality: two of them are related to instrumental contextuality, and one is related to mathematical contextuality. The former includes motivation and application of mathematics in real life. The latter is mathematical abstraction which allows students to learn a concept in abstract mathematical ways through mathematical representations. In fact, while teachers used adjectives like "boring," "plain," "wishy washy," "scary," and "dry" for mathematical non-textual elements, they used adjectives like "interesting," "meaningful," "easier," and "attractive" for contextual non-textual elements. This shows how teachers differently understand the roles of mathematical contextuality and instrumental contextuality.

Motivation. When I asked about which non-textual element could be more interesting to students, most of the South Korean and U.S. teachers answered that realistic contexts would be interesting to students. In particular, many teachers asserted from their teaching experiences that realistic contexts motivated students to learn because such contexts were very effective to draw students' attention when they introduced a new concept. Ms. Vasquez said that at the beginning of a lesson, realistic contexts as "a grab" could attract students' attention. Since realistic contexts show "how it applies to their lives or how it applies to their world", students could get interested in the content more easily. Ms. Kang gave me a different perspective to explain this. She said, "Atypical and various shapes in real life can be more interesting to students because many pictures and illustrations in mathematics textbooks are more typical than real life. So, realistic contexts are helpful for students to have more interest in the topic." Mr. Carter said, "In this particular case, at least in terms of initially introducing it, I would tend to think once again this kind of draws their attention, and then you could get into the plotting points and determining slope and things like that and how they did it." He pointed out that such attraction from realistic contexts could be a useful pedagogical strategy to guide students to concentrate on mathematical tasks with understanding. It is notable that such a motivational role is also played by a non-textual element that is visually attractive, a point which will be discussed later in this chapter.

Application of mathematics in real life. Many teachers thought non-textual elements with realistic contexts useful because non-textual elements showed where mathematical concepts could be applied. From students' perspectives, mathematical concepts are given practical meaning when they are applied to some real-life situations. If non-textual elements are situated within realistic contexts, such non-textual elements can provide students with concrete examples showing how mathematical concepts are widely applied to various immediate surroundings. In this regard, when we talked about Task 2:

Ms. Morris: I like that idea of being able to show them that it can be used somewhere else.

Interviewer: In a mountain.

Ms. Morris: Right, a mountain, figure out what the slope is of the mountain, what the grade is, how long it's going to take you to walk up that type of thing. Where as this one, it's harder for them I think to visualize what the point that they're trying to make here is that you can measure this part.

Interviewer: I see. Do you think these kind of visual representations are helpful for learning and teaching and learning this kind of concept?

Ms. Morris: I think so, because I think this, I mean this could be a job they could do later in life with No. 2. Having to determine if an avalanche is likely and how much snow has to be removed so they have to know how, what the slope is, so that way they can figure it out.

Ms. Kang emphasized that it would be very helpful for students to find application of mathematics in realistic contexts as much as they can. She said that from such experiences, students could find some patterns across the contexts and formulate a mathematical concept from the experiences as well as apply mathematical knowledge to solve problems in various contexts.

This idea is consistent with Freudenthal's Realistic Mathematics Education (RME) which is a theory of teaching and learning in mathematics education. It emphasizes that it is important for students to learn mathematics in connection to realistic contexts and their everyday life experiences. Further, RME points out that "context problems and real-life situations are used both to constitute and to apply mathematical concepts" (Van den Heuvel-Panhuizen, 2001, p. 4). As the teachers pointed out, students could understand mathematics in realistic situations as well as apply mathematical ideas to realistic contexts.

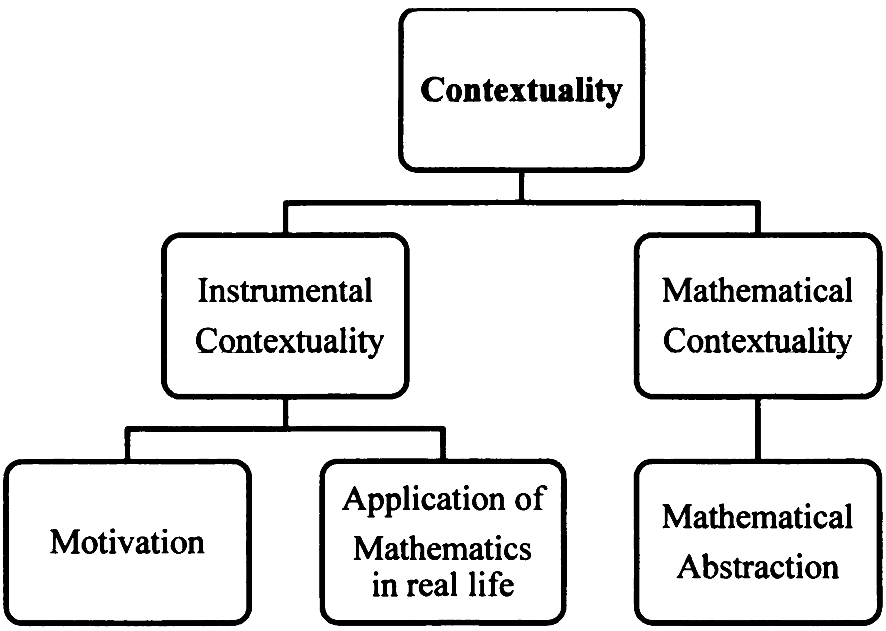
In addition, many teachers stated that students can better understand a mathematical concept if they understand it within a realistic situation because of its familiarity and usefulness of mathematics. If students feel that a mathematical concept can be found from anywhere in the world, they can see the usefulness of mathematics and feel more familiarity with the concept. When we talked about Task 2, Ms. Kwak said that the realistic shape of the hill and the picture in (2) would be more helpful for students to understand the concept of slope rather than showing only a graph like (1) because they feel more comfortable and friendly. In Task 1, Mr. Flores talked about the picture in (2) and said, "It gives a real world example of where you would find an angle like that, that it's not just an arbitrary picture on a page of two lines. That they could go out and they could see trees in their backyard like that and they could start realizing that there are angles all around them." He pointed out that students could better understand mathematical concepts when they could make connections with their lives.

Mathematical abstraction. From the interviews, I found that for these teachers, mathematical contextuality was always associated with mathematical accuracy. In other words, when the teachers talked about a non-textual element situated within a mathematical context, they said that such non-textual elements are more accurate and rigorous than contextual non-textual elements. Ms. Jung said, "As mathematical accuracy is important, I primarily use abstract mathematical figures in my classes. Realistic examples are not enough to show mathematical accuracy and rigor. Specifically, it is not easy to find the appropriate realistic examples for algebra which includes the concept of variable." Since realistic objects and contexts are more complicated and atypical than mathematical representations used in class, teachers seemed to think that it was not easy to find appropriate realistic examples that showed mathematical concepts more accurately. In addition, Ms. Cho said that "realistic contexts would be helpful for students to make sense of a concept, but they are not helpful for students to express the idea accurately in mathematical ways."

The data show that teachers distinguish the role of non-textual elements with instrumental contextuality from that with mathematical contextuality. When we talked about non-textual elements with instrumental contextuality, teachers mentioned "motivation", and "application of mathematics in real life." Teachers said that they used non-textual elements with instrumental contextuality as a trigger to call students' attention to a concept and as examples to show the usefulness of mathematics in students' life and other fields. In contrast, when considering non-textual elements with mathematical contextuality, teachers understood and used them for mathematical accuracy and rigor. Since mathematics has its own language and symbols to express a mathematical concept, teachers said that students should understand and express a mathematical idea in abstract mathematical ways.

The data from the interviews allow me to clarify the roles of non-textual elements in terms of contextuality. Figure 10 shows the summary of what I found in terms of contextuality.

Figure 10. Contextuality



Patterns of *contextuality* among the different teachers

Both South Korean and U.S. teachers emphasized the importance of contextuality of non-textual elements in student learning. However, their emphasis between instrumental and mathematical contextuality is different between the two groups. Overall, the South Korean teachers I interviewed are likely to put more emphasis on mathematical contextuality than instrumental contextuality whereas the U.S. teachers I interviewed are likely to emphasize the importance of instrumental contextuality in teaching and learning mathematics. Further, this is closely associated with their pedagogical strategies to teach mathematics in their classrooms.

All the South Korean teachers I interviewed seem to consider contextual non-textual elements a secondary tool to help students better understand a mathematical concept. They said that they first used contextual non-textual elements as a trigger to attract students' attention. However, when explaining a mathematical concept, they mainly used abstract mathematical non-textual elements to teach mathematical concepts because they thought that students need to know what mathematical symbols and abstract representations mean as well as to solve problems in mathematical ways. They said that once students understood a concept mathematically, they could apply the knowledge to realistic contexts. They thus use contextual non-textual elements when they want to show mathematical application at the end of lesson or chapter. This is a common pattern of their strategies to use non-textual elements in their classroom. This seems to be influenced by the structure of textbooks which cover national curriculum because they said that "most of textbooks are organized like this." This reminds us how textbooks influence teachers' instruction.

In contrast, many of the U.S. teachers slightly put more emphasis on instrumental contextuality in teaching and learning mathematics because it allows students to see the usefulness and application of mathematics in students' daily lives. They use "more realistic ones which tie into [students'] worlds instead of throwing it out as a concept."

Eight out of 10 U.S. teachers asserted that more contextual non-textual elements with mathematical connection were necessary in textbooks. Despite the importance of contextual non-textual elements in teaching and learning mathematics, they pointed out that there were not many contextual non-textual elements with mathematical connectivity in current textbooks. Although they want to use more contextual non-textual elements in their classrooms, they said that it was hard to find good ones for their instruction. Thus, some of the teachers use contextual non-textual elements at the beginning of the lesson to attract students' attention and then they use mathematical non-textual elements that are familiar to them. Others said that they use simple mathematical non-textual elements to introduce a concept and then they provide contextual non-textual elements which allow students to experience mathematics in various situations. The teachers mentioned that even though they use many mathematical non-textual elements in their classes, they want to have more useful contextual non-textual elements with mathematical connection in textbooks. This seems that these U.S. teachers appreciate the value of contextual non-textual elements in teaching and learning mathematics.

Aesthetics

Aesthetics in the framework denotes how the non-textual elements are visually attractive enough to motivate students' learning. Sinclair (2004) establishes the criteria of the motivational role of the aesthetic including "connectedness, fruitfulness, visual appeal, apparent simplicity, and surprise" (p. 47). When I asked teachers which non-textual element would be more attractive to motivate student learning, I initially expected three possible answers from teachers: connection with realistic contexts, the effects of color as visual appeal, and mathematical beauty as fruitfulness.

All the teachers talked about the effects of realistic contexts on students' motivation. As I mentioned in the section for contextuality, teachers recognized that realistic contexts played an important role in motivating students because students would be familiar with the contexts and could find the usefulness of mathematics in the contexts.

In addition, some of the teachers mentioned the effect of color to draw students' attention to the topic. If a non-textual element is colorful, it can catch students' eyes to the picture as well as related concepts. When we talked about Task 6 as shown in Figure 11, Mr. Carter told me, "In terms of the illustration, . . . color might draw it, . . . might draw their attention. . . . To tell you the truth, I didn't even look at the numbers, I just saw it as a whole thing, and this kind of gave that abstract, mathematical, like they might be used to seeing, but then a practical application and the color kind of brought it out too. This has some color." Ms. King also said, "Different colors and stuff can catch their attention." In short, colors can catch students' eyes even before they actually look at the content. When students open their textbooks, colorful pictures can grasp students' eyes quickly and then students can pay attention to the content.

However, Mr. Flores warned that colors were not requisite. He said, "I mean a math book can have all the pretty pictures in the world, but if it doesn't help them learn math it's not worth it." It implies that even though aesthetics is one of the important aspects of non-textual elements, this is not a fundamental aspect of non-textual elements which decides their usefulness and effectiveness on student learning.

Figure 11. Non-Textual Elements in Task 6

(1) Match each side and angle of the first shape in Exercises 1-4 with its congruent partner in the second shape.

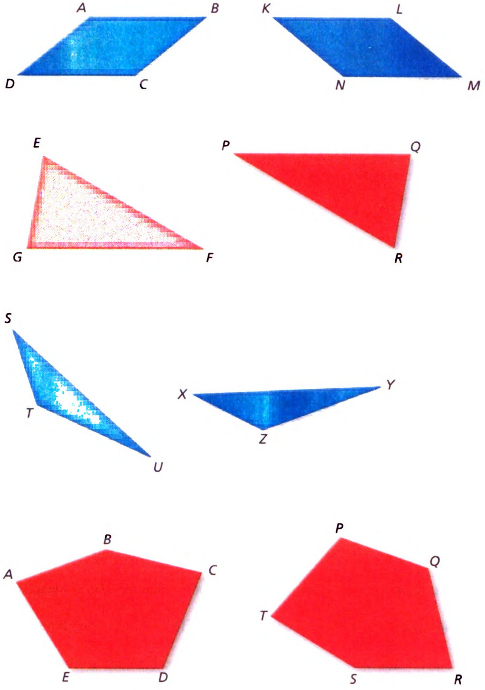
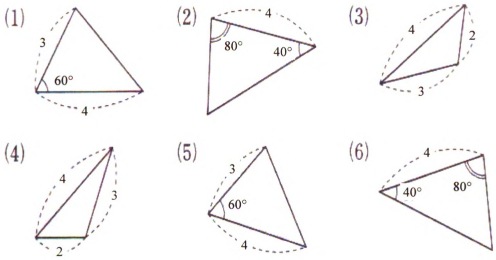


Figure 11. (cont'd)

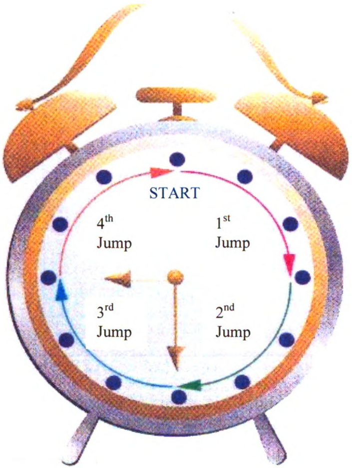
(2) Find congruent triangles in (1) ~ (6) and show why they are congruent.



Unlike what I expected, most of the teachers did not talk about the effect of mathematical beauty which comes from the nature of mathematics on motivating students' interests. While students solve a problem, they may find connections among mathematical concepts that they have never known before or have an a-ha moment in the process of solving the problem such as a pattern emerged in a sequence of numbers (Sinclair, 2006). However, when I talked about Task 3 (See Figure 12) with Ms. Koo, she mentioned that the picture in (3) could motivate students to engage with the activity and the concept because students could see the process of how to factor the number. She said that even though Figure 12 (1) has realistic contexts and has pretty colors, Figure 12 (3) would be more attractive because it clearly showed the process.

Figure 12. Non-Textual Elements in Task 3

(1)



(2)

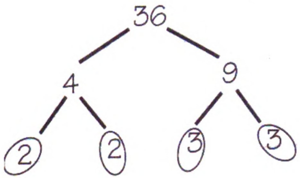
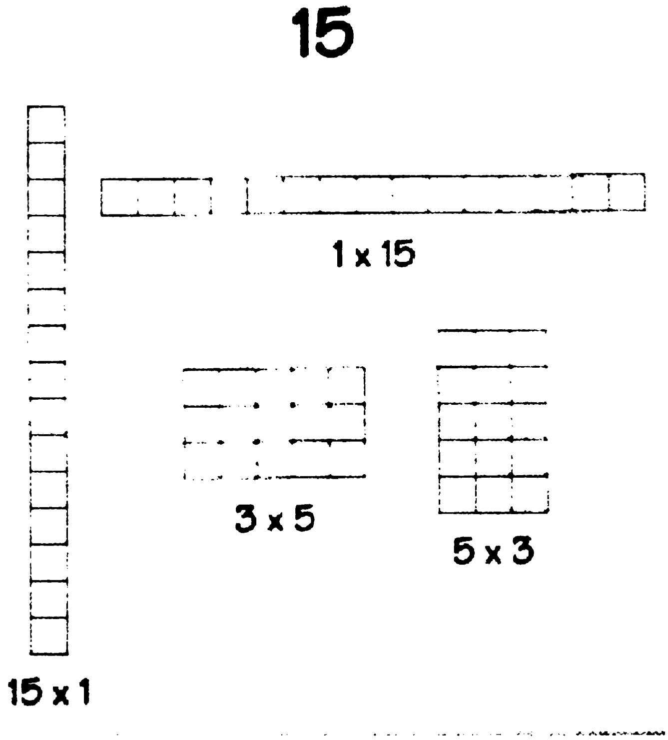


Figure 12. (cont'd)

(3)



Patterns of *aesthetics* among the different teachers

Both the South Korean and U.S. teachers regarded realistic contexts and colors as factors to attract students' attention. As shown in the section of contextuality, teachers have different emphasis and use of contextuality in their classrooms. As for colors, while eight out of 10 U.S. teachers mentioned the effects of colors on motivation, only three out of 11 South Korean teachers mentioned color effects on motivation. In fact, U.S. textbooks are printed in all kinds of natural colors while South Korean textbooks have only a small number of colors. Thus such different recognition may be derived from their experiences with their textbooks. Since U.S. teachers have more experiences with colorful pictures and illustrations than South Korean teachers, U.S. teachers perhaps could recognize the roles of colors more easily than South Korean teachers.

All the U.S. teachers agreed upon the effects of colors on motivation. However, some of the teachers said that colors may distract students' attention because it could "catch their eyes [rather] than the content." In other words, colors could be a double edged sword. Colors are very effective to motivate students to learn mathematics. But at the same time, colors can distract students by catching students' eyes before students look at the content. They argued that it would be very important to consider how to use colors effectively.

On the other hand, the three South Korean teachers focused more on the strengths of colors. They assumed that if there were more colors in textbooks, they could make students get more interested in mathematics. And they said unless colors in a non-textual element impede mathematical accuracy and clarity, colors could help students to better understand mathematical concepts by giving clear images which would make a concept or problem much easier.

Variety

People have different modes of representations to understand a concept (Khoury & Behr, 1982). Although one mathematical concept is explained, it is important to have various ways to approach the concept rather than one way to express it. This is a part of pedagogical content knowledge teachers should have (Shulman, 2004). When I asked the teachers about variety in general, Mr. Flores said, "Kids learn in so many different ways. So I think having different representations is definitely important."

However, when I asked if these various shapes in Figure 13 (3) could be more helpful to understand the concept of angle, the teachers had different ideas about variety. Ms. Robinson said, "To me, no it wouldn't be [helpful], cuz it – I mean if you just wanted to say that these shapes had interior angles, here's one, here's another one, that would be about the only use that I would see for that." She would rather recognize them as several angles in each shape rather than as different types of angles in various shapes. I intended to provide the picture (3) in Figure 13 to see how teachers think about various kinds of angles and their application into various shapes. But she did not seem to pay attention to various shapes where many different angles could be found. She seemed not to care about how various angles students need to experience for their learning of the concept of angle.

Further, some teachers regarded Figure 13 (3) as too complicated a picture for students because it has too many concepts at a time. Ms. Vasquez said, "I'd have problems with Number Three (3) at my kids' level because there are just so many and so many different ones unless they were named more." She thought that giving such various angles at a time would be more confusing without indicating each angle by naming. Mr. Martin also pointed out the complexity of Figure 13 (3) for student learning.

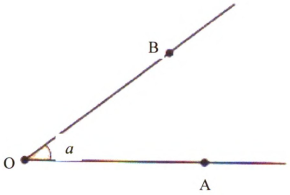
Mr. Martin: If you're just talking about an angle, I think there's too much going on, because you've got polygons themselves, you've got hexagon, trapezoid, looks like a rhombus, another rhombus, equilateral triangle and square. It just – be concerned that with some students who are struggling a little bit, you're throwing too many variable at them; there's too

> many concepts. Again, what's the angle? Is it the numbers, is it the part of the figure? So in that mind, that would be – and again, not wrong, but maybe just a little more information than they need when you're just introducing an angle.

Unlike my assumption that teachers would consider variety of non-textual elements in explaining a concept, teachers preferred simpler non-textual elements in order to give a clear explanation about the concept. Although many studies have emphasized the importance of various representations for students' understanding (Ainsworth, Bibby, & Wood, 2002), the teachers said that even one simple non-textual element could be better than various non-textual elements to explain a concept. When showing Figure 13 (3) to the teachers, there was no teacher who recognized it as giving various opportunities to learn the concept of angle. Rather, they thought that it was too complicated or "overwhelming." Ms. Robinson said, "This could mess kids up." They thus wanted to have one simple non-textual element which explicitly shows only the concept without any other distraction. When Ms. King talked about the picture in (1) for Task 2 (See Figure 13), she said, "Because I think that one [the picture in (1)] is probably the more simpler basic, but if you can get their interest first, then you're gonna be able to bring in the math." Ms. Perez pointed out that simpler non-textual elements allowed students to focus on the concept without distracting from other details.

Figure 13. Non-Textual Elements in Task 1

(1)



(2)

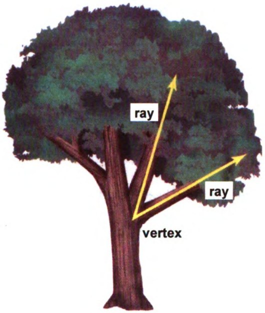
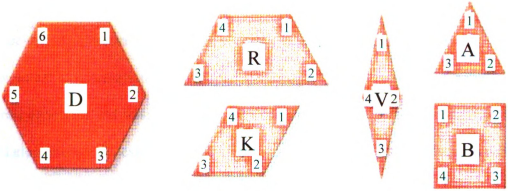


Figure 13. (cont'd)

(3)



With regard to types of non-textual elements, the teachers did not pay attention to the variety of different kind of non-textual elements: photos, illustrations, graphs, or mathematical figures. Teachers mainly focused on if a non-textual element was helpful for students to understand a concept. They did not care about if photos or illustrations were used to explain a concept.

3.3 Non-Textual Elements across Different Topics

None of the teachers recognized any difference in the use of non-textual elements across topics in textbooks as well as in their classrooms. They mentioned that any kind of non-textual elements could be used in lessons on angle, slope, and prime factorization if the non-textual elements are mathematically accurate and helpful for students to understand the concepts. For example, the teachers said that even in algebra, mathematical connected realistic contextual non-textual elements should be used.

However, the teachers said that they used more abstract mathematical non-textual elements in algebraic topics. I found two reasons from the interviews with teachers.

One is related to what algebra is. Some of the teachers argued that even though contextual non-textual elements could be helpful for students to understand an algebraic concept, students also need to know how to express the ideas in mathematical ways. They said that students need to know how to set up an equation using symbols and mathematical representations, how to read and interpret graphs, and how to solve algebraic equations and functions. Thus, they said that even though contextual non-textual elements are necessary, abstract mathematical non-textual elements mainly should be used.

The second one is a practical issue. The teachers complained that they have "never seen that kind of thing." They said that it was hard to find the most appropriate contextual non-textual elements for algebraic topics in textbooks. In fact, the results from the textbook analysis in this study show that only about 12 percent of the total non-textual elements in both textbooks are used in prime factorization. Among them, only about 20 percent have realistic contexts. This is a relatively small proportion compared to non-textual elements in angle, where about 40 percent have realistic contexts. The teachers thus wanted to have more useful contextual non-textual elements in algebraic topics in mathematics textbooks.

3.4 Teachers' Use of Non-Textual Elements in Classrooms

The results from the interviews with teachers show that the roles of non-textual elements are closely related to curriculum and instruction in classrooms. Teachers completely agreed that accuracy and connectivity are the most fundamental aspects that non-textual elements should always possess for effective teaching and learning. However, teachers make some distinctions about the roles of contextuality and simplicity in their instruction.

First, contextual non-textual elements play a role in teaching and learning of mathematics in a way different from abstract mathematical non-textual elements. Teachers distinguished the roles of contextual non-textual elements from those of abstract mathematical non-textual elements. Mr. Evans said that realistic contexts provide information about where mathematics can be applied, "Okay, here's some place where you may see angle, or you may have seen angles in the past." According to his teaching experience, students can easily understand the mathematical concept within a context. However, when it comes to talking about its definition or mathematical logic and rigor, he argued that abstract mathematics figures are more useful because they allow students to have accurate and clear images of the concept as well as to communicate in a mathematical way with others. In fact, there is some variation among the teachers how to use such different types of non-textual elements in their classrooms. Some teachers may use contextual non-textual elements first to motivate students, and then use abstract mathematical ones to develop the ideas. Other teachers may use abstract mathematical ones to introduce the definition of a concept then use contextual ones to show its mathematical application and problem solving. However, all the teachers agreed upon the differences between the roles of each type of non-textual elements.

Second, instructional decisions about non-textual elements depend on curricular phases and students' grade levels. Even though I have found important aspects of non-textual elements, their roles and uses are not static and equal. In other words, teachers put different emphases on them according to their instructional decisions which are based on their curricular structures and students' needs. For example, when we talked about Figure 7 which is about angles, Mr. Martin said that a simple picture like Figure 7 (1) could give "the best idea of what an angle is." He said, "I also think for some younger or less mature students, seeing a picture with an angle imposed on it, I think helps." However, he added, "[number] three, once they understand the concept of an angle, and as you're moving into polygons, then you can start to show them polygons. But they're not going to be confused." This shows that he considered and chose appropriate non-textual elements to meet students' needs and prior knowledge. Even though he thought that simple non-textual elements could give "the best idea of what an angle is" to his students, he would use angles in various polygons if his students already know the concept of angle. This idea is also related to his curricular phases. When he introduces the concept of angle for the first time, he would use the simple picture to show what an angle is. However, when he thinks that "they know what an angle is", he "can add figures with more than one angle." This shows that the roles and uses of aspects of non-textual elements are not static but flexible according to curricular phases and students' readiness and needs.

In sum, the interviews reveal how some teachers understand important aspects of non-textual elements in teaching and learning of mathematics. They thought that accuracy and connectivity are the fundamental aspects that a non-textual element should possess for effective teaching and learning. Contextuality and aesthetics were considered as important to motivate students and show mathematical usefulness and application. One interesting thing is that teachers prefer simplicity to variety in their instruction because simple non-textual elements could give students clear ideas of the concept. Even though these aspects of non-textual elements are all important, teachers put different emphases on them and use them differently according to their instructional decisions.

CHAPTER 4

VIEWS OF CURRICULUM DEVELOPERS

In addition to listening to teachers talking about their understanding of non-textual elements for their teaching, it is important to hear from curriculum developers about their views on the roles of non-textual elements in textbooks and their criteria for deciding on these elements for their textbooks. Even though curriculum developers have intentionally chosen non-textual elements in their textbooks, little has been known about how they chose and what criteria they had in their selection.

This chapter examines qualitative data collected through structured interviews with four South Korean and three U.S. curriculum developers who have written mathematics textbooks that are currently used in each country (See Appendix B for the interview protocol). In this chapter, I describe first my data collection and analysis and then explain the findings about: (1) who made decisions about the non-textual elements that would be included in their textbooks and the kinds of criteria they had when selecting non-textual elements for their textbooks, (2) how curriculum developers understood each aspect of non-textual elements within the framework that I have developed, (3) what other aspects of non-textual elements were considered by curriculum developers, (4) how curriculum developers understood the roles of non-textual elements across different topics, and (5) what kinds of suggestions they had for future research and textbook design.

4.1 Data Collection and Analysis

Participants

Curriculum developers were individually recruited and interviewed in 2008 and 2009. Three U.S. curriculum developers and four South Korean curriculum developers volunteered to participate in this study through a structured interview. They were purposely selected in terms of their various experiences in K-12 mathematics curriculum development. All the curriculum developers from both countries have written at least two series of secondary mathematics textbooks in each country. All the curriculum developers are professors in each country except one Korean curriculum developer, who is a mathematics teacher in a high school.

Interview protocols

The interviews with curriculum developers allowed me to see what rationales and criteria of good non-textual elements for mathematics learning they had as well as to get a sense of how they use non-textual elements in their mathematics textbooks. The interview protocol for this study (See Appendix B) was developed on the basis of the R-Framework. The interview protocol for curriculum developers includes not only some questions used for teachers but also additional questions to investigate what aspects of non-textual elements they consider in designing their textbooks, how they understand the roles of non-textual elements in mathematics textbooks, and how much they control selections and uses of non-textual elements in their textbooks.

Procedures

The procedures of the interviews with curriculum developers were the same as those with teachers. Most interviews took less than one hour although one of the interviews took five hours because he gave me a lot of information about textbook development as well as his view on mathematics which reflected on his decisions about non-textual elements for his textbook. The task-based interviews were audio-taped and transcribed for the analysis.

Data analysis

As I did for the interviews with teachers, the interviews with curriculum developers were analyzed by finding patterns across their responses to each question and coding key terms and sentences that they often used. In order to find similarities and differences of the roles of non-textual elements perceived by the curriculum developers from South Korea and the United States, the general patterns and key ideas were also analyzed by each group: South Korean curriculum developers and U.S. curriculum developers. The finding from the interviews is used to elaborate and expand a conceptual framework to examine non-textual elements in textbooks.

4.2 Process of and Criteria for Selecting Non-Textual Elements for Textbooks

As I will show below, although South Korea and the U.S. have different systems to publish secondary mathematics textbooks, the ways of selecting non-textual elements for mathematics textbooks are similar, especially with respect to selecting pictorial representations.

Since South Korea has a national mathematics curriculum, a mathematics textbook can get approval from the textbook examination committee only when it follows the detailed approval plan and guidelines created and announced by the Ministry of Education, Science, and Technology (MEST) (Pang, 2008). Professors and teachers may volunteer to write a secondary mathematics textbook.[6] Recently, an increasing number of teachers have participated in writing textbooks. Professors and teachers bring their knowledge from research and practice into writing their textbooks through regular meetings and conversations for one or two years. Authors have full authority to select non-textual elements including pictures and mathematical figures for their textbooks. All the South Korean authors I interviewed said that each author initially took a part in which he had expertise. Each person shared their drafts with other authors and got feedback from one another at weekly meetings for a year. During their meetings, they also discussed issues around which pictures and mathematical figures would be more helpful for students to understand mathematical concepts.

Once authors and their publishing company submit the draft of their textbook to the Korean Institute of Curriculum and Evaluation (KICE), the approval committee, the textbook is examined through one preliminary and two major examinations. If it passes all these three examinations, the authors should submit a draft of teachers' instructional guidebook that accompanies the textbook. If the guidebook is not approved, the original textbook also cannot get approval. Both the textbook and teachers' guidebook are examined in the final evaluation. If it passes the final evaluation, it can be published and used in secondary schools.

[6] Elementary mathematics textbooks have been written in different way. MEST assigns elementary textbook development to a university or research institute.

The following are the evaluation criteria for secondary mathematics textbooks (Pang, 2008, pp. 113-114).

1. Does a textbook sufficiently reflect the nature, objectives, contents, teaching and learning methods, and assessment of the national curriculum?
2. Is a textbook organized in a way to consider students' individual differences so that students learn mathematics at their own pace and ability?
3. Does a textbook provide contents in a way for students to attain their learning goals?
4. Does a textbook consider mathematical connections so that stepwise or level-based learning is fostered effectively?
5. Are contents adequate for the amount of learning in one class period?
6. Are there any errors of content or inaccurate theories?
7. Does a textbook provide students with reading materials that may help them recognize the usefulness of mathematics?
8. Does a textbook select contents related to real-life contexts and consider connections with other related subject areas?
9. As needed, does a textbook reflect learning areas applicable to all subject matters such as citizenship, humanity, environment, economy, energy, consumer, career counseling, information ethics, and gender equity?
10. Does a textbook provide various and effective teaching and learning methods to foster logical thinking, inquiry, problem solving, creativity, reasoning, and applications based on mathematical concepts, principles, and rules?

11. Does a textbook properly present data for learning activities, and provide methods of data collection, analysis, and application?
12. Does a textbook effectively use various manipulative materials and instructional technology including calculators, computers, and the internet?
13. Does a textbook introduce assessment methods and tasks that are compatible with the objectives, contents, and methods of mathematics education?
14. Does a textbook abide by the orthographic rules of Korean and foreign languages as well as the norms of standard language?
15. Does a textbook correctly use mathematical terms and notations as presented in the national curriculum without omission or repetition?
16. Is a textbook edited innovatively and does it use space effectively?
17. Are photos and illustrations clear and compatible with contents?
18. Does appearance of the textbook including format, length, and chromaticity adhere to the guidelines?
19. Does a textbook take advantage of novel ideas and organize contents in an innovative way?
20. Area teaching and learning processes as well as assessment methods creative?

Although these questions are general criteria for evaluating a textbook, it is notable that many of them pertain to the aspects of non-textual elements elaborated in my framework. In particular, question 17 explicitly asks if non-textual elements in a textbook are not only clear (i.e., accuracy) but also connected to the given contents (i.e., connectivity). Questions 4 and 9 are also related to connectivity and questions 6 and 15 to accuracy. Questions 7 and 8 are closely linked to contextuality in my framework. It appears that accuracy, connectivity, and contextuality are all considered important aspects of mathematics textbooks in South Korea.

Unlike the case of South Korea, the U.S. government does not regulate textbook development and revision. In other words, there is no governmental approval or accreditation system for mathematics textbooks in the U.S. because textbooks are published in demands of the market (Anderson, 1993). Although there are NCTM curriculum focal points and state-level policies that suggest curricular content and process for each grade level, they are not mandatory requirements for publishing textbooks. Publishers are not always attentive to the standards and policies because they are more concerned with increasing the marketability of their textbooks rather than with strictly following such standards and policy guidelines (Woodward, 1993). Thus, there is greater variation across U.S. textbooks than across South Korean ones in terms of content, structure, length, and volume (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002).

Such great variation across U.S. textbooks is also due to the process of developing textbooks. The U.S. authors I interviewed for this study said that they first drafted out their textbooks which were generated as what they intended with regard to theories and the results from empirical studies. Then, they tested the drafts in a small number of sampled schools while arranging several seminars to get feedback from teachers, students, and parents in the schools as well as other researchers. Finally, they revised their drafts based on the feedback. This entire process was repeated several times until they finalized writing up their textbooks. Such process of publishing a textbook in the United States clearly appears different from the South Korean textbook development that strictly follows highly prescribed national guidelines that are printed and given to the curriculum authors.

Although the systems to publish secondary mathematics textbooks are different between the two countries, the criteria they use to select non-textual elements for the textbooks are similar in the sense that their criteria are not very specific. All the authors from South Korea and the U.S. I interviewed mentioned that there were no specific criteria for non-textual elements they had when they wrote their textbooks. While the authors focused on mathematical contents and problems in their textbooks, they did not pay close attention to non-textual elements. Dr. Choi said that even though national guidelines for textbook authors require authors to include non-textual elements in textbooks, he thought that not many authors really cared about them. Although he has written textbooks over ten years, he has never thought about non-textual elements until he recently had a chance to observe and analyze how a teacher taught mathematics in her classroom. He realized from this experience the important roles non-textual elements played in teaching and learning mathematics in classrooms. He said that he therefore paid more attention to non-textual elements when he wrote a new textbook for the 8th national curriculum starting from 2010.

Among non-textual elements, the authors seemed to distinguish between abstract mathematical representations and pictorial representations. In particular, most authors I met seemed to regard pictorial representations as a part of the artistic or graphic design of their textbooks, not as a tool to convey educational and mathematical meanings. As Shriver (1990) pointed out, document design is often misunderstood as design such as "graphic design, industrial design, [and] fashion design" (p. 4). Thus, while the authors emphasized the importance of mathematical representations in students' learning, they did not care about the role of pictorial representations in teaching and learning of mathematics. In my interview with him, Dr. Zimmerman asserted that we should distinguish "technical art" from "photographs." According to his definitions, technical art includes abstract mathematical representations such as triangles and geometric figures. Since these pictures are directly linked to content, they are different from photographs, especially gratuitous pictures which "do not represent anything" and "have nothing to do with the content." Gratuitous pictures mostly have no pedagogical significance, only "attracting students' eyes to look at the pictures, not the texts." He said that even such gratuitous pictures were selected by visual design specialists, not by textbook authors in the first edition of his textbook series. Although he has been involved with selection of such pictures from the second edition of his books, it was a selection of some of the pictures that visual design specialists or publishers gave them, not that textbook authors designed or carefully chose considering their educational significance.

For South Korean textbooks, Dr. Park also pointed out that even though an increasing number of pictures is included in mathematics textbooks according to the national guidelines for textbook authors, most textbook authors do not pay much attention to pictorial representations in textbooks. Although textbook authors think of mathematical representations as part of the content, they do not think of pictorial representations as part of it. All the authors in this study said that although they paid attention to mathematical representations including technical art, they did not care about pictures that were not explicitly associated with the contents.

For mathematical representations, all the authors emphasized their importance in student learning and said that they took them into thoughtful consideration in relation to the content. Dr. Zimmerman said that mathematical representations are actually mathematics, which means that they are tools for showing how people understand mathematics as well as tools for solving problems. Thus, he said that every representation is selected and examined carefully by the authors. Dr. Lewis and her colleagues who wrote mathematics textbooks also thought about which representations would be most helpful for teaching and learning mathematical concepts.

Dr. Lewis: I sort of think of it as their suitcase of ways of thinking about mathematics. So we had a lot of arguments among the five of us about what the set of representations that we would use in the development of the fraction to rational numbers [and] proportional reasoning strand because that's sort of the heart of middle school mathematics. We were very selective. We wanted visual representations that were going to be useful in multiple kinds of settings. Something that not just would invite a kid to think in a particular way about a problem, but something that would be easy for kids to do themselves. So for example, the visual representations that we use in the development of the whole fraction strand starts with strips. We don't use circles at all. We start with strips and we start with kids subdividing because what we have in mind is what are the fundamental underlying things that children need to understand to understand rational numbers.

Further, Dr. Choi said that since creating a representation is part of "doing mathematics," Dr. Choi and his co-authors gave serious thought to whether or not to add a non-textual element for a given problem included in their textbook. They thought that some problems may be given without a non-textual element to let students create their own representations for the problem. Overall, all the South Korean and U.S. authors whom I interviewed chose mathematical representations for their textbooks carefully despite their lack of attention given to pictorial representations.

However, as for non-textual elements that were not directly linked to contents, the U.S. authors left publishers in full charge of selecting such elements or selected ones from the pictures provided by publishers or visual designers, who majored in neither mathematics nor mathematics education. The South Korean authors who were in charge of all the contents in their textbooks also did not consider which non-textual elements would be more helpful for students when selecting such non-textual elements. Dr. Choi said that when they chose a decorative picture which was not explicitly linked to the content, they just looked for a tiny connection to the given problem without thinking about its pedagogical relevance. For example, when there was a problem to find the probability of winning a soccer game, they simply put a picture of a soccer player. Although that was not helpful to solve the problem, they put that picture along with the problem for decorative purposes. Dr. Lewis said, "The pictures that are in just for entertainment, like this, those are done by the publisher. We reject some, but the diagrams and pictures that are germane to the intent of the problems themselves, those we made decisions about."

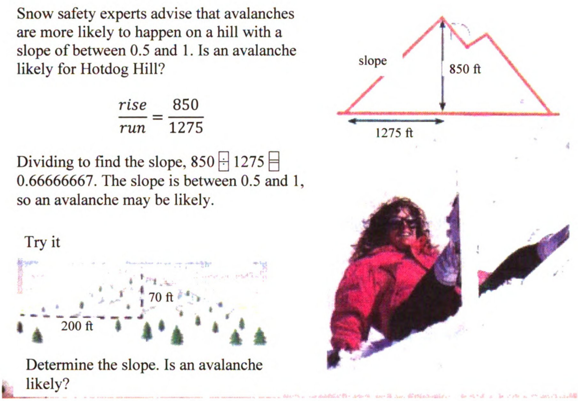
All the authors were selective when they chose mathematical representations which were closely associated with the content. In this case, they seemed to understand the role of non-textual elements as part of mathematics. They saw non-textual elements as a tool for thinking. However, although pictorial representations are part of a textbook, the authors did not pay as much attention to them. Further, most of the authors from both South Korea and the United States asserted that such pictures should be removed from textbooks because they were useless and distracting for student learning. Recently, many appealing and colorful non-textual elements are used in mathematics textbooks because publishers believe that such non-textual elements are effective to attract "customers" to purchase their textbooks. Consequently, it is inevitable that there be "a constant fight" between publishers and authors over which non-textual elements should be included in their textbooks.

4.3 Accuracy, Connectivity, Contextuality, Aesthetics, and Variety of Non-Textual Elements

Accuracy

The National Mathematics Advisory Panel reported that many errors were found in many secondary mathematics textbooks, and they suggested that publishers should confirm accuracy of their textbooks (National Mathematics Advisory Panel, 2008). All the authors I interviewed seriously considered accuracy of non-textual elements in mathematics textbooks. When I asked about the picture of slope in Figure 14, Dr. Robinson pointed out inaccuracy of the picture.

Figure 14. Female Skier in Task 2



Snow safety experts advise that avalanches are more likely to happen on a hill with a slope of between 0.5 and 1. Is an avalanche likely for Hotdog Hill?

$$\frac{rise}{run} = \frac{850}{1275}$$

Dividing to find the slope, 850 $\boxed{\div}$ 1275 $\boxed{=}$ 0.66666667. The slope is between 0.5 and 1, so an avalanche may be likely.

Try it

Determine the slope. Is an avalanche likely?

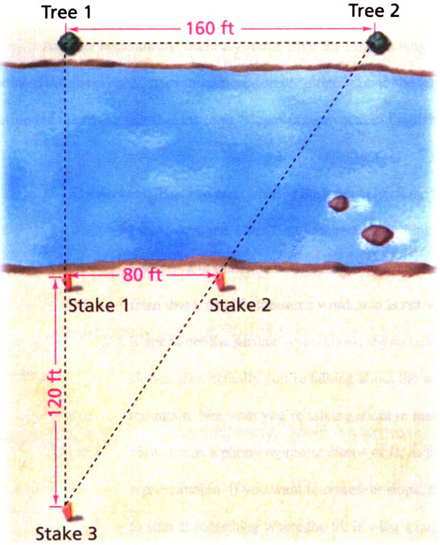
Dr. Robinson: The mountain is a very nice metaphor and lots of people use it. In particular you can use it for a slope of zero and a slope that's undefined because zero would mean flat and students would understand that it's not interesting to ski but it's possible, whereas undefined it's impossible, totally unsafe, vertical line. One thing that you have to also help them with is that we're always going from left to right, which on the

> mountain goes both ways. Depends on which way you're hiking. Actually when I looked at Number 2, one of the things that struck me was that the scale on the two dimensions is not the same and so the picture is sort of deceiving because it looks like the slope is a slope of 1 or a 45 degree angle, but according to the numbers it's something else. So that's something where the graphic artist and the author or the editor really have to be careful to make diagrams that have the accuracy that you want.

She indicated not only benefits that students might get from the picture but also the inaccuracy of the picture which could lead to students' misconceptions.

Such concerns about inaccurate non-textual elements were often observed in the interviews with South Korean textbook authors. For example, when we talked about Figure 15, Dr. Choi said that although these two triangles were supposed to be right triangles, this picture did not give any information like that. There is no clue that the line between the two trees is parallel to the line between the two stakes. He pointed out that such required information should be provided and indicated accurately in the picture.

Figure 15. An Example of Missing Information



Interestingly, Dr. Park mentioned that accuracy was considered not only by students but also mathematics teachers. Indicating Figure 13 (3), which is used in lessons on angle, he said that mathematics teachers would complain about such picture which did not indicate each angle using symbols and marks precisely. He actually had such experiences with mathematics teachers who complained about missing marks for an angle in his textbooks. He said that mathematical accuracy would be the most important attribute that a non-textual element should have.

Another issue around accuracy stems from the English language. Dr. Zimmerman pointed out the possibility of misconception of "slope" due to the term itself. He reminded me of the fact that the term "slope" is equivocal in English.

> *Dr. Zimmerman:* That is a case where a designer said, "Oh, 'Slope.' When I think of slope, I think mountains. And I think ski slopes. So, a skier." But in mathematics, slope has a – although it comes from there, and it's the same word, that is not what it means. Slope is not the surface – you know, if you talk about ski slopes, then actually you're talking about the surface of a mountain. Not what you're talking about in mathematics. I view this as a phony representation – or is, in fact a false representation. If you want to represent slope, then you have to start at something where the tilt is what's important. You know, how tilted is it. That's what slope is – slope is measurement of tilt. Slope is very much related, in that sense, to angle. And you could argue that slope – you could easily argue that slope is a numerical representation of angle. Let's say angle from the horizontal. The measure of an angle from the horizontal.

By pointing out the possible misconception of "slope" because of the picture, he emphasized how important it is to ensure accurate and appropriate non-textual elements to explain a concept in mathematical way. In South Korea, the mathematical term for "slope" in Korean (giulgi) means the degree of tilt which is different from the term for surface of a hill (bital) in Korean. This distinction is not very clear in the term "slope" in English. The picture of a female skier (Figure 14) thus may lead to a misconception by simply associating the mathematical concept of slope with the surface of a hill for skiing. His point suggested that cultural and linguistic factors also should be considered in evaluating accuracy of non-textual elements. Further, this implied that it could be dangerous if we understand non-textual elements only as design because they could produce unintended misconceptions.

Connectivity

Connectivity is one of the aspects of non-textual elements that all the authors considered most important. This is also the reason why the authors thought decorative pictures are useless. Dr. Choi said that if a non-textual element did not have connectivity, it could not give any support to students and sometimes even hinder students' understanding of a given concept or problem. He suspected that it was why teachers or students did not often use such pictures in their classrooms. He also said that non-textual elements which lack connectivity represent a waste of the space in textbooks where more useful information and pictures could have been included to support student learning.

Interviewer: You said that if you write your textbook again maybe you want to put a thinner textbook and more robust representations there. So, do you think that decorative representations are not necessary in textbooks?

Dr. Robinson: Right. Right. It's so American. It's just superficial. It should be integrated. Like our lesson on slope uses the roof, our initial lesson, and the roof really is used in that storyline. So it's not like it's just superficial. It's quite integrated.

Dr. Robinson pointed out that even a picture which had a realistic object or situation should have a mathematical connection to make it more meaningful in the textbook. This is more clearly explained by Dr. Zimmerman.

Dr. Zimmerman: Freudenthal and other people made the point: angle is not a mathematical concept to begin with. Angle's a real concept – a real thing. Angle came from the world, this is not -- so, you know, you're saying "Do you represent it?" No, mathematics is representing the real world; the real world isn't representing the mathematics. So I, for instance, take a door, and I have a door and I open the door and I think of the angle between the door and the wall. I have an angle between the door and the wall. That's a real thing. That's not mathematics yet. So the mathematics is really representing the real thing, and it's in fact picturing the mathematics.

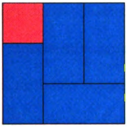
Well sure, that's one way of saying it. But the other way of saying it is the mathematics is abstracting the picture.

Dr. Zimmerman posited that a realistic picture itself cannot be mathematics. Rather, "mathematics is abstracting the picture." If this idea is applied to non-textual elements in textbooks, only giving a picture would not give mathematical meanings. Such a picture can be useful in learning mathematics only when it has a mathematical connection and meanings. "The door was there, whether the mathematics is there or not," he explicated.

Dr. Robinson was concerned that teachers did not seem to care about this important idea. She said, "When I work with the middle school teachers now I realize that they see them as a hook, as a motivator for kids, but they don't see them as just a natural part of the whole concept of modeling." She exemplified how they tried to make a connection between realistic contexts and mathematical concepts that students need to learn.

Dr. Robinson: In the books that we did for [the publisher's name] we tried to make the photographs more integral with the activities we were asking the students to do. We didn't have photographs like that that just basically elaborate. We tried to make more of a substantive connection. I mean, here's one I'm looking at on Page XX where it's a Japanese woman with a very pretty screen but the story is about the Tatami mats and how they just fit into the floor. So it's really about a space and a covering problem. This is just typically American to make things look interesting and nice and to sugar coat it. I would rather spend the money and the effort doing something that's more helpful in terms of the mathematical concepts per se.

Figure 16. Tatami Mats



1 blue mat = 2 red mats

She showed how to transform realistic objects and situations to mathematical ones by making a connection between them. Students may not only be interested in what these are but also be motivated to solve the problems in mathematical ways. This idea is also related to contextuality.

Although all the authors regarded connectivity as an important aspect of non-textual elements, they had some different opinions about what kinds of connectivity a non-textual element should have. One is that textbooks should have only essential pictures which are necessary to understand a concept or solve a problem in mathematical ways. The other is that even pictures that are only loosely linked to the context of a problem may provide students with some information that may be helpful to solve the problem. For example, if the picture in the right in Figure 16 is only given to show tatami mats, this may provide contextual information about how tatami mats look like. Even though the picture does not give any idea about how this picture can be used to solve the mathematical problem, this may be helpful for students to see what tatami mats are.

Since this is quite a small set of data from the interviews with curriculum developers, it is impossible to generalize any pattern. However, it was interesting that all four of the Korean authors had the former opinion whereas two of the three U.S. authors had the latter one. Dr. Lee said that there were too many pictures in textbooks which might distract students' learning. He said that this kind of pictures were not helpful for students to solve problems, making it hard for students to focus on them. Dr. Choi said that textbooks should have concise and essential pictures which could help students to create their own strategies in mathematical ways without any distraction. In contrast, although the U.S. authors also thought that such pictures with no direct mathematical connection were of no use, they said that if such pictures provided some contextual information, they also could be helpful for students, especially those who had low reading ability.

> *Dr. Zimmerman:* They have no pedagogical significance – well, they're always pedagogical, is they might attract some student to look at the page. In fact, they usually attract the student to look at the picture, and not look at the text. But they might. Now, on the other hand, you might imagine that if we drew a lot where we had, let's say some populations of some cities,

> so you have the population of Istanbul, let's say. Istanbul, Turkey. So you have – and then you have a picture of Istanbul. Is that a false representation? And I would say, no, that's not bad, because you're actually – for many students, they don't even have any idea what Istanbul is? And you are giving them context for the problem that they did not have. So you can distinguish that from just some sort of picture that has nothing to do with.

He argued that even though such pictures did not provide mathematical information, they could help students to understand related contexts and to see what they were solving within the contexts. Dr. Lewis also said, "Even if it isn't essential to solve the problem, . . . but if it helps to understand the problem, then we're happy to have that." While Dr. Lewis looked at Figure 14 (female skier in Task 2) as an example, she said, "The kids found it interesting to see what the woman looked like, but this is not essential. Something like this is very helpful. It raises all sorts of interesting mathematical questions."

This shows how the textbook authors saw the relationship between non-textual elements and textual elements in textbooks. Some curriculum developers regarded only mathematical connection between them as valuable. Other curriculum developers regarded both mathematical and contextual connection between non-textual elements and textual elements as meaningful. That is, connectivity can be helpful if a non-textual element explicitly shows the mathematical concept or provide only contextual information. Such differences may be derived from their perspectives to see what mathematics is and how mathematics is taught. The bottom line of both perspectives is to pursue the meaningfulness of mathematics. Even though contextual information could provide some useful information to understand the mathematical concept, it could be helpful for students to understand it within contexts. It is important to see if a non-textual element provides mathematical connection and information which is helpful to solve the problem.

Contextuality

Contextuality in the R-framework was defined as what kinds of context(s) the content in a non-textual element is located in. As shown above, some authors talked about the roles of realistic non-textual elements in mathematics textbooks when they considered connectivity between non-textual elements and the content. The authors thought that realistic pictures could be helpful for students to understand the related situation to the problem. However, they warned that if the pictures were merely given to students, they might not be as helpful as the authors intended because the pictures were not mathematics themselves. Dr. Robinson said, "this issue of students' confusion – well, they look at a representation in a geometry book and sometimes it's intended to be a specific case and sometimes it's intended to be a general case. And we don't always make it clear to the students which is which and they don't always think about it in the way we had intended." Thus, what she attempted to do was "choosing a context where it's more familiar to the students or they see the rationale for why would we bother to measure, that might be a little more helpful." Otherwise, there is a cost for students to understand the context.

Further, if students could understand mathematics in contexts, they may apply the mathematics to other situations and academic areas. Dr. Lewis said, "Kids are able to see into these representations see what the mathematics is in the situation and they are then able to in other situations where those might be useful, they are able to bring them into those situations." The authors seemed to pay much attention to benefits from realistic situations to learn mathematical concepts, not merely to attractiveness of realistic contexts and objects.

Aesthetics

While many of the teachers I interviewed emphasized color effects on student learning as "a grab," all the authors focused more on the beauty of mathematics that students could find through non-textual elements in textbooks than on color effects. For example, Dr. Kim said that "a colorful picture is very attractive to students. But it is superficial attraction. You know, such beauty is only skin deep. Rather, we should support students to be motivated by the beauty of mathematics, inner beauty." In fact, the authors thought that although colors could catch students' eyes, it could not create students' intrinsic motivation to learn mathematics. The authors pointed out that publishers overvalued colorful and appealing pictures in textbooks.

Dr. Lewis: We had three years of trials in the school. The trial materials didn't have any of these. The trial materials didn't have color.

> The kids loved them. I think that the publishing companies overestimate the appeal of all of this part of the materials to kids. The people who gravitate to this stuff and to all of the color that goes on in these materials are the people who are selecting the textbooks, the teacher. I've spent time in publisher booths at conferences just listening. Teachers will come in and say, 'Look how beautiful this is. Now look, this looks really interesting.' So their focus is not so much on the mathematics as it is on the color and jazziness of the materials, but in that trial work for five years we had materials out in schools; three iterations of every single year's materials. Five years we had stuff out in schools. Not a streak of color anywhere. No pictures whatsoever that were not relevant to the problems themselves and we never got a single complaint from a single kid. I think they're irrelevant. I would like not to have them. They're very distracting.

Dr. Lewis indicated that although teachers assumed that students could be motivated by colors, it was teachers, not students, who were attracted by colors when they purchased and used textbooks. She said, "I think for the students the color is irrelevant. I think for teachers to get them to look at the textbook they sometimes choose by – they don't necessarily choose the final version by the book, but to even get it in their hands the front cover needs to be interesting."

Variety

Because multiple representations have been emphasized in many studies, my assumption about the variety of non-textual elements was that teachers and curriculum developers would consider it important how various types of non-textual elements were used to explain a concept in a textbook. For example, they might want to have various shapes and situations to explain parallel lines in a textbook. However, the results from the interviews were different from my expectation. Rather, the curriculum developers seemed to think that one good simple representation could be better and more powerful than various representations. Dr. Robinson said, "It may be better to use one context for a variety of lessons or for a series of lessons than to keep shifting to a new context all the time" because students may not have to spend much time and efforts to understand what each context and representation means. Rather than using various types of non-textual elements to explain a concept, she said that just one useful non-textual element could be even more effective for students to understand the concept. However, she mentioned, "I don't think textbook publishers think about that one much." In addition, Dr. Lewis said that one powerful representation could be used across similar topics to show how these concepts are related to each other or expand one idea to the other idea. She said, "If you look at, in the algebra work in the [title of her book], the development of the distributive property you'll see some of the same representations that we've used in fractions. So we're trying to choose a very small number that are powerful and have uses over time."

Further, Dr. Zimmerman pointed out that this could be related to the reality in classroom. He said, "There are a lot of ways of representing mathematics and you have to choose between them. Teachers don't have so much time, so they try to pick the one that they're most comfortable with. But in all the cases, you try to use the representation as an avenue straight to the mathematics." He expected that teachers might want to have the best picture that presented the concept they were teaching because they could not use all the pictures and representations in textbooks and other instructional materials within restricted time and space. In other words, teachers pursue efficiency of non-textual elements to teach a concept by choosing the most efficient and productive picture to teach the concept within limited time. This is consistent with what teachers talked about the variety of non-textual elements through the interviews.

4.4 Non-Textual Elements across Different Topics

Unlike most of the teachers, the textbook authors I interviewed distinguished the roles of non-textual elements across different topics. Dr. Robinson mentioned, "Geometry is so visual that that by itself could be a lot. I mean in number, in measurement, in different domains you have different kinds of representation and they may have a different role. I think it depends on what the problem is. I think you're not going to do angles without pictures and you're not going to do slope without pictures. But you could do prime factorization without pictures." She distinguished topics in geometry from algebraic topics because she thought that algebraic topics usually did not depend on visual aids. Dr. Zimmerman also had a similar opinion, but his reason was different from Dr. Robinson's. He said, "Geometry really represents the arithmetic, I think. A whole number can be factored if I've got a rectangle, or let's say I've got an array of chocolate candies. And I want to know is 46 prime or not? And so I take 46 and I put it into – I say, 'Oh I can take 23 rows of 2. Oh, that tells me I put it in rows there's none left over, because it's not prime, it's a composite.' Well I represented the number 46 geometrically. It's a rectangular array. And there's no question which comes first – the number came first there." He did not think that geometric and algebraic topics needed different representations as Dr. Robinson said. Rather, he thought that geometric topics could be represented in an algebraic way and vice versa.

Interestingly, whereas the U.S. curriculum developers interestingly mentioned differences of non-textual elements in each topic, the South Korean curriculum developers I interviewed did not talk about this issue. I was able to get some clue about the reason from the interviews with the curriculum developers. I think that it was related to the writing process of textbooks. Each South Korean curriculum developer wrote only one chapter in which he or she had expertise. A curriculum developer who majored in geometry or had an expertise in geometry, for example, wrote a chapter of geometry in a textbook. Although other contributors to the textbook gave him or her feedback on the chapter, he or she was the only person who had both responsibility and autonomy for the chapter. Thus, there would be few chances to consider differences of non-textual elements across topics.

4.5 Additional Suggestions from Curriculum Developers

The authors raised some important issues and provided some specific ideas about non-textual elements in mathematics textbooks. First, textbooks should provide students with opportunities to explore mathematical concepts and problems by themselves. Dr. Lewis said, "We believe that giving kids manipulatives that are related to things they can draw themselves that gives them an opportunity to look inside a figure like this and see triangles and know why buildings use triangles. It's a stable figure." A non-textual element should not be a static or fixed object but a flexible and manipulative one as a "kinetic action of a concept" (Dr. Robinson).

Dr. Robinson: Once you go to the textbook the students learn in a different way. They learn by referring back rather than by reconstructing for themselves. I mean, in the community college and in the high school I've seen a lot of this where they want to find the box that shows you how to do the – that models what it is I'm doing. And if they can't find the box then they feel stuck. Rather than saying to themselves, "Well, what do I know? What can I use? How can I make a diagram that's helpful to me to understand this?" We're really preempted their opportunity to think by giving them recipes to follow instead. So I think that this research you're doing is very important because it makes people stop and think about what do we really want books to do?"

In order to do this, some authors suggested that textbook authors should understand student-generated representations in order to develop better non-textual elements in textbooks. Dr. Robinson argued that we need to look more carefully at how students generate their own representations and how teachers think about the students' work. Such studies would give us better ideas of what kinds of representations would be more helpful for students to understand mathematics and how non-textual elements in mathematics textbooks could be improved to support student learning. Dr. Lewis said, "It's the mathematics in the representation and the way in which that kind of representation can become a thinking tool for the kids. Those are very important criteria for us." She said, "We should be looking at the mathematical substance. What are the opportunities for kids?"

Second, useless pictures and representations should be removed from textbooks. Such pictures are not only useless but also distracting. Dr. Lewis mentioned, "It's a waste of – the pictures waste – makes the materials longer. They are distracting." Dr. Robinson said, "If our economy is falling apart, why should we buy and manufacture these huge expensive glossy heavy books when we could have manufactured these very simple ones that are much more well thought out." Further, Dr. Choi asserted that if necessary, not giving a picture would be better for students to learn than providing a pre-established picture. He said that since creating an appropriate representation is part of doing mathematics, textbook authors should deeply deliberate if a representation should be given to students.

As a result, even though the writing process and policy with regard to publishing a textbook is quite different between South Korea and the United States, it appears in the findings that both the South Korean and U.S. curriculum developers do not consider much about the roles of pictorial representations in their textbooks. Although both groups of curriculum developers considered abstract mathematical representations as a part of the content, they did not pay much attention to pictorial representations.

All the curriculum developers in the two countries considered the five aspects of non-textual elements important for teaching and learning. Similar to the teachers I interviewed, the curriculum developers thought mathematical accuracy and connection as important for teaching and learning. They also pointed out the roles of contextuality in motivation and mathematical usefulness. They also preferred simplicity to variety of non-textual elements. Further, curriculum developers raised a linguistic issue with regard to non-textual elements and emphasized mathematization through non-textual elements.

Although this is too small an amount of data to generalize any pattern in each country, it is interesting that South Korean and U.S. curriculum developers had different opinions about connectivity. While the South Korean authors focused only on mathematical connectivity, the U.S. authors appreciate both mathematical connectivity and contextual connectivity. In addition, whereas the South Korean authors I interviewed did not recognize differences of non-textual elements across topics, the U.S. authors have some notion of how non-textual elements are used differently in each topic.

CHAPTER 5

REFINEMENTS LEADING TO THE FINAL FRAMEWORK

5.1 Synthesis of the Interview Analysis

As shown in Chapter 3 and 4, the results from interviews with teachers and curriculum developers show that they had many mutual agreements on which aspects of non-textual elements were important for student learning. They said that non-textual elements play an important role for student learning as tools for thinking and motivators. They also asserted that rather than including decorative pictures in textbooks, they wanted to have simple and powerful non-textual elements which were accurate, closely connected with the content, meaningful in students' lives, and mathematically attractive.

In particular, both teachers and curriculum developers thought that mathematical accuracy and connectivity with the content were the most important aspects of non-textual elements when evaluating them. However, it is notable that both groups preferred simplicity of non-textual elements to variety. Both groups thought that giving a "simple and powerful" non-textual element would be more effective for students to understand a mathematical concept more clearly than showing various non-textual elements as examples. Table 4 shows commonalities between teachers and curriculum developers in terms of non-textual elements.

Table 4. Commonalities in Teachers' and Curriculum Developers' Perceptions of Non-Textual Elements in Textbooks

Aspects of NTE	Common Feature
Accuracy	• This is the most important aspect of non-textual elements. • Accurate visual representations are important to show conceptual definitions as well as to communicate in a mathematical way.
Aesthetics	• Visual appeal can draw students' attention. • Realistic contexts with which students feel more familiar motivate them to learn mathematics.
Connectivity	• Mathematical connectivity to content is important. • Contextual connectivity can provide contextual information that is helpful to solve problems. However, it is controversial. • Non-textual elements which lack connectivity to content are useless and distracting.
Contextuality	• This is useful to motivate students. • This allows students to make connections to their lives and to see the usefulness of mathematics. • This is related to application of mathematics. • Mathematical contextuality should be accompanied with mathematical accuracy.
Variety	• Simplicity, rather than variety, is considered useful and effective for teaching and learning. • Difference in types of non-textual elements was not given any attention.

There are some differences between teachers' and curriculum developers' perceptions of non-textual elements in mathematics textbooks. While teachers thought that colorful and realistic pictures were effective in attracting students' attention, curriculum developers argued that such pictures were not enough to motivate students to learn mathematics because they could catch students' eyes to the pictures, but not to the texts and mathematical content. Rather, curriculum developers said that the beauty of mathematics in a non-textual element could be a motivator and attractor. For example, let's assume that there is a picture of a beautiful carpet on the floor. As Dr. Zimmerman said, even though it shows beautiful patterns on the carpet which could be visually attractive, it does not mean that it is mathematics. When students can find interesting patterns and how they make sense in mathematical way from the picture, they may feel more attractiveness from this picture. Dr. Robinson said, "When I worked with the middle school teachers now I realize that they see them as a hook, as a motivator for kids, but they don't see them as just a natural part of the whole concept of modeling."

Although there are differences between teachers and curriculum developers, it does not mean that they have different criteria of non-textual elements. In terms of aesthetics, whereas teachers put more emphasis on visual attractiveness by colors and beautiful realistic pictures, curriculum developers focused more on the beauty of mathematics, which seems to be defined as mathematical attractiveness. However, both the teachers and the curriculum developers agreed that attractive non-textual elements played an important role in motivating students' interests. Such agreement between the two groups implies that the five aspects of non-textual elements in the framework are reasonable and important in mathematics textbooks.

5.2 Implications from the Interview Analysis on Refining the Framework

The results from the interviews with teachers and curriculum developers allow me to articulate the definitions of each aspect of non-textual elements in the framework. When the teachers and curriculum developers considered *accuracy*, they considered the definition of a concept in a non-textual element using appropriate symbols and marks. In particular, accuracy becomes a more important issue in a non-textual element with realistic contexts. Since realistic contexts are more complicated, just giving a realistic context could be confusing to students who learn a mathematical concept for the first time. Many textbooks use non-textual elements with realistic contexts by simplifying the contexts. In the process, some realistic contexts are oversimplified or distorted to fit in the mathematical concept. Both teachers and curriculum developers pointed out this problem in current textbooks. Thus, it is very important to see if a non-textual element presents a concept accurately with regard to the definition of the concept.

For *connectivity*, I initially wanted to see how a non-textual element is associated with textual elements in mathematics textbooks. I define textual elements as the content in the non-textual elements. When I talked with teachers and curriculum developers, I asked questions about copies of non-textual elements I showed them in the interview protocol as well as about non-textual elements in their own textbooks. At that time, many teachers and curriculum developers showed their textbooks and talked about the use of non-textual elements in their textbooks. When they explained connections between non-textual elements and textual elements, they did not care about the text per se but the mathematical content in the text.

In addition, I found two different kinds of connectivity were considered by both teachers and curriculum developers: mathematical connectivity and contextual connectivity. Although all the teachers and curriculum developers emphasized mathematical connectivity, they did not agree upon the importance of contextual connectivity because of its implicit or indirect connection to the mathematical content.

Thus, I redefined connectivity as the relationship between non-textual elements and the mathematical content.

Compared to accuracy and connectivity as the essential aspects, *contextuality* is often considered as the most important facilitator in teaching and learning of mathematics. As mentioned above, contextuality is considered with mathematical accuracy and connectivity by both teachers and curriculum developers because they argued that there were many realistic contextual pictures which were inaccurate and mathematically meaningless. The initial ideas of contextuality (or authenticity in the P-Framework) include instrumental contextuality and mathematical contextuality. However, I found from the interviews that instrumental contextuality is also closely associated with mathematical accuracy and connectivity. Since mathematical contextuality is considered distinct from instrumental contextuality and highly connected with mathematical accuracy and connectivity, I decided to consider only instrumental contextuality in this category. Thus, I redefined contextuality as contextualization of mathematical ideas in realistic context(s) with mathematical connection. An explanation about contextuality with an example is given later in this chapter.

Although I initially focused on the motivational roles of *aesthetics,* I found from the interviews that motivational roles of a non-textual element were explained by contexuality because all the teachers and curriculum developers emphasized the motivational roles of mathematics in realistic contexts. Thus, I redefined aesthetics as visual attractiveness to motivate students to learn. In this regard, the results from the interviews show that colors can be considered as aesthetics.

Neither the teachers nor the curriculum developers considered variety of non-textual elements to be important in mathematics textbooks. Rather, they suggested that simple non-textual elements could be more helpful for students to understand concepts or solve problems. “Simple” can be construed in two different ways: one is consistency of a non-textual element across the contents, and the other is clarity and ease of a non-textual element. The former was mentioned by two curriculum developers I interviewed. They said that one simple non-textual element across the related contents could be more effective than various non-textual elements in each topic because it allowed students to see how the contents were connected to each other. This could be much helpful for students to understand the relationship among topics. However, this idea was not considered by any other teachers and curriculum developers. In addition, this is contradictory to the idea of variety. Considering the fact that the “one-case concreteness” of drawings and images is the source of many difficulties in visualization-based mathematical reasoning (Presmeg, 1997), this idea should be further examined.

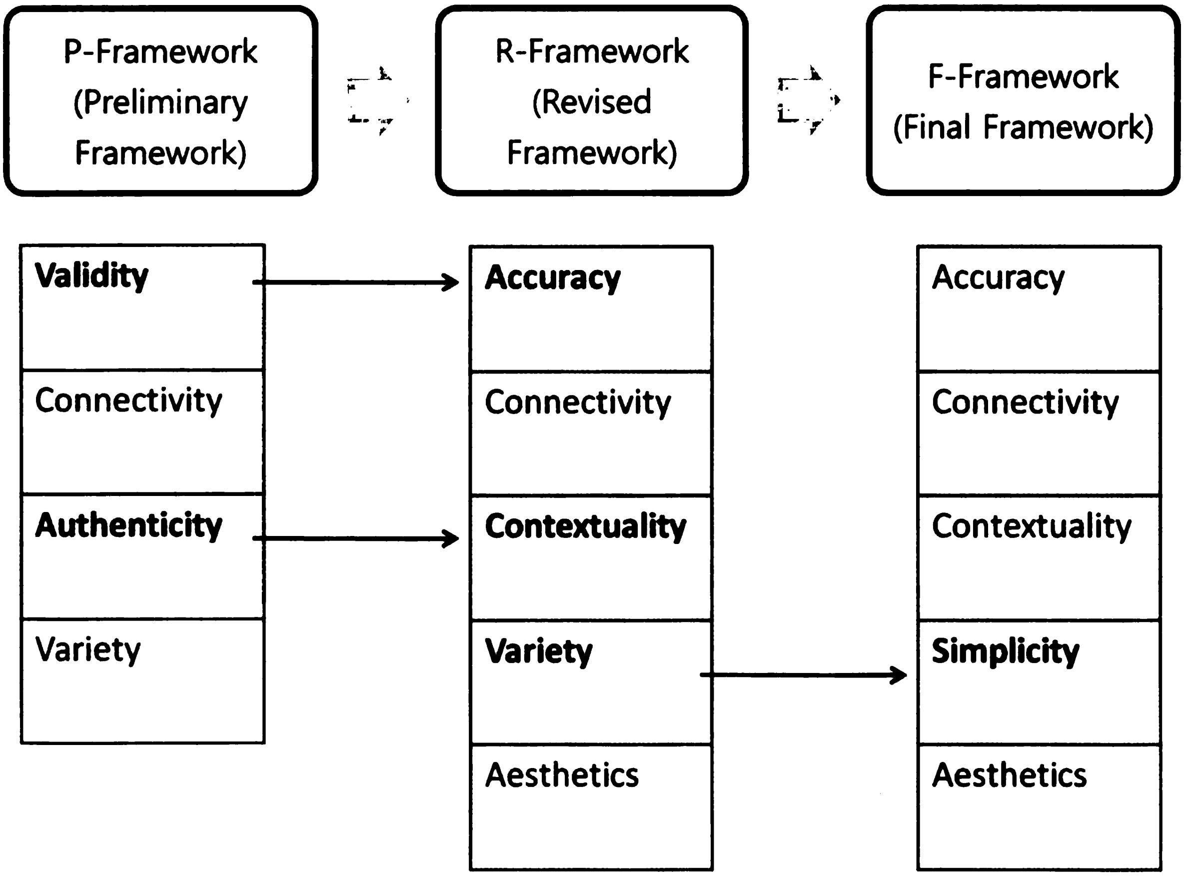
Considering “simple” to mean clear and easy to understand, both teachers and curriculum developers pointed out that it would be more effective if a non-textual element clearly explained only one concept without any distraction, especially in the introduction of a new concept. However, they did not care about variety, or how many different types of non-textual elements need to be provided to explain a concept. They focused more on each non-textual element to see if it was simple enough to understand a concept easily. Thus, instead of variety as an element in the framework, I added “simplicity,” which denotes conciseness of a non-textual element.

The definition of each aspect of non-textual elements in the F-Framework is explained and examples of each aspect of non-textual elements is provided in next section.

5.3 The Final Framework (F-Framework)

The final framework (F-Framework) has emerged from a series of studies described in Chapters 1 through 4. Figure 17 shows each phase of the framework development through an iterative series of investigations. The P-Framework was used for pilot studies, and the R-Framework was used to develop interview protocols for the interviews with teachers and curriculum developers. The components of the F-Framework are: *accuracy, connectivity, contextuality, aesthetics,* and *simplicity*. Each conceptual component in the framework is grounded in the basis of a diverse body of research on mathematics education, semiotics, metaphor theory, visual rhetoric, and information design. The concepts of *accuracy*, *connectivity*, and *contextuality* mainly draw on the NCTM standards (2000) that emphasize the importance of using appropriate representations (*accuracy*) with interconnection among mathematical ideas (*connectivity*) applied to various contexts (*contextuality*). In addition, *aesthetics* as an important factor in learning mathematics (Sinclair, 2004) denotes how the non-textual elements are visually attractive enough to motivate students' learning. Further, I added *simplicity* to the F-Framework from the interview analysis while *variety* was omitted. The R-Framework initially included *variety* as an aspect of non-textual element. However, the results from the interview analysis show that variety of non-textual elements was not counted as an important aspect of non-textual elements by teachers or curriculum developers. Rather, they argued that even a simple and powerful non-textual element could be more effective for student learning than various complicated non-textual elements to explain a concept. Thus, *simplicity* was included in the F-Framework.

Figure 17. Development of the Final Framework (F-Framework)



As the F-Framework was developed through a series of investigations, Table 5 shows how each aspect of non-textual elements has been articulated and changed in the process. The definitions of each aspect of non-textual elements in the F-Framework were used to develop a coding scheme for textbook analysis, which is described in Chapter 6.

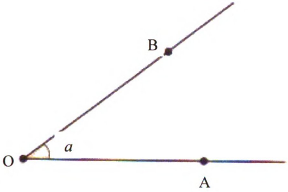
Table 5. Changes of Definitions of Each Aspect of Non-Textual Elements

P-Framework		R-Framework		F-Framework	
Validity	Relevancy and clarity of non-textual elements	Accuracy	How mathematically clear and rigorous a non-textual element is to explain a concept	Accuracy	Mathematical clarity and rigor of non-textual elements by definition of the concept
Connectivity	How non-textual elements are closely related to the written textual elements	Connectivity	How non-textual elements are closely related to the written textual elements	Connectivity	Close linkage between non-textual elements and the mathematical content.
Authenticity	Being fully trustworthy and factual to show contents	Contextuality	What kinds of context(s) the content in a non-textual element is located in	Contextuality	Contextualization of mathematical ideas in realistic context(s)
Variety	How many diverse types of non-textual elements are used to explain a certain concept or problem	Variety	How many diverse types of non-textual elements are used to explain a certain concept or problem	Simplicity	Mathematical conciseness of a non-textual element
		Aesthetics	How the non-textual elements are visually attractive enough to motivate students' learning	Aesthetics	Visual attractiveness of non-textual elements to motivate students

Accuracy indicates mathematical clarity and rigor of non-textual elements by definition of a concept in mathematics textbooks. In other words, it measures how non-textual elements represent concepts and ideas obviously and correctly in mathematical ways. How obviously the concepts and contents are represented is important to communicate without misunderstanding. If an illustration offers unclear or wrong information, a student may have more confusion and difficulty in solving the related problem. For example, in the survey from a pilot study, students were asked to interpret the picture in Figure 18 (2). Although this picture was used to introduce the concept of parallel lines in a textbook, the lines seem to meet somewhere at horizontal line near the forest. Although most of the older students who knew the concept of parallel lines could interpret this picture correctly, younger students who did not know much about parallel lines were confused by this picture. Some students focused on each cabbage in the field as a dot so that they could not see lines as well as how the lines could be parallel. Some other students thought that these were slight curves, not lines. They thus drew a pair of curves as parallel lines. Such an inaccurate picture can mislead students to conclude that any two curvy lines can be parallel if they do not meet or parallel lines can meet somewhere if they are extended further. This picture thus cannot satisfy the definitions of parallel lines. In contrast, Figure 18 (1) explicitly shows the definition of angle that "a pair of rays with a common end-point" (Mitchelmore & White, 2000) with appropriate symbols and notations. This picture may help students understand what an angle means and how it can be represented.

Figure 18. Examples of Accuracy

(1)



(2)

Connectivity means how non-textual elements are closely related to the mathematical content. Braden (1983) coins the terms "visual-verbal symbiosis" and "visual-verbal discontinuity" for describing the relationship between visual and verbal elements. Visual-verbal symbiosis means well connectedness and supportiveness to each other between visual and verbal elements, while visual-verbal discontinuity indicates disconnection between visual and verbal representations. Many studies have found that a symbiotic connection between verbal and visual literacy helps improve student achievement when the two are united (Braden, 1983; Dwyer, 1988; Herbel-Eisenmann, 2002). Since visual representations can serve as concrete models to show what students cannot see in texts and symbols (e.g., patterns and concrete images) and as tools to solve problems (Arcavi, 2003), it is very important to have close connection between verbal and visual representations to support students' learning.

The results from the interviews show that teachers and curriculum developers considered mathematical content in texts when they considered connection between the written text and non-textual elements. When I interviewed teachers and curriculum developers, they focused on mathematical content in texts when I showed the picture of female skier in Figure 14 with the written text. In fact, since the problem given in the text is to find the slope of Hotdog hill regarding the relationship between avalanche and slope, the picture of the female skier has nothing to do with the text. However, all the teachers except one concentrated on whether or not the picture of female skier can help students understand the concept of slope, not the problem as written. Such a phenomenon was similarly observed when I showed non-textual elements with surrounding text in textbooks. In fact, Levin and Mayer (1993) claim that all kinds of text-connected pictorial representations improve students' understanding from texts whereas disconnected pictorial representations do not. They argue that decorative pictures or illustrations cannot help students understand the content. However, from the interview analysis with teachers and curriculum developers, I noticed that some of them mentioned the effects of decorative non-textual elements on student learning in their classes by providing contextual or behavioral information. Thus, I found that connectivity between non-textual elements and texts can be often interpreted as connectivity between non-textual elements and the mathematical content in written text. Thus, I modified the definition of connectivity to be connection between mathematical content and non-textual elements.

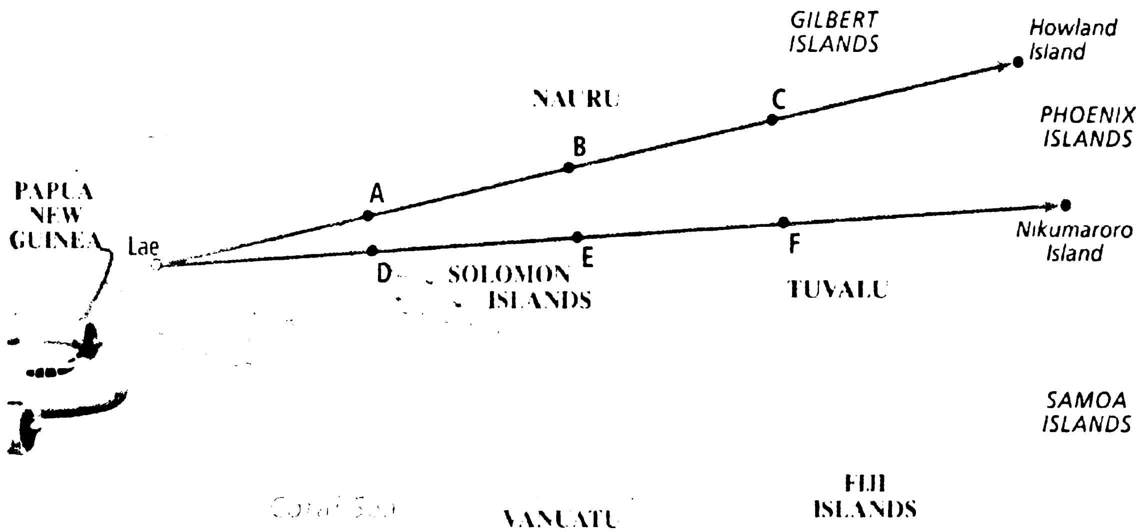
Figure 19 shows examples of this new definition of connectivity. Both pictures in Figure 19 are used in a textbook to introduce a story of Amelia Earhart, a woman who died in a plane crash, as an example of using the concept of angle to find her wrong plane trajectory. Even though the two pictures are used to explain the same story, their roles are different in terms of connectivity. Figure 19 (1) does not contribute to understanding the concept of angle. Figure 19 (2), in contrast, has a mathematical connection and makes visual what happened so that students can see the application of the concept of angle and start thinking about how to measure the degrees she was off course. Although both pictures in Figure 19 have realistic contexts, Figure 19 (1) does not have any mathematical connectivity while Figure 19 (2) has mathematical connectivity. Thus, I defined connectivity in the framework as mathematical connectivity between non-textual elements and mathematical content.

Figure 19. Examples of Connectivity

(1)

Figure 19. (cont'd)

(2)



Contextuality denotes contextualization of mathematical ideas in realistic context(s). Freudenthal's (1991) realistic mathematics advocates that a real-world situation or authentic context should serve as the starting point for learning mathematics. NCTM standards (2000) emphasize the importance of students' mathematical experiences in a context because "using mathematics in applied situations leads to deeper understanding" (p 93). Wiggins (1993) asserts the compartmentalization of knowledge and the decontextualization of knowing are problematic because competency requires both context and reasoning. This claim is consistent with the results from Ferratti and Okolo's research (1996) that students' thinking skills and attitudes are enhanced when they collaborate in the solution of authentic problems. I think contextuality is not only a matter of texts and problems. Contextual non-textual elements may influence students' understanding and learning with connections between mathematics and real life. Such experiences in the textbooks may give students an opportunity to think about mathematics in contexts and deepen their understanding (National Council of Teachers of Mathematics, 2000).

Figure 20 provides examples of contextuality. Even though realistic contexts are helpful for students to understand mathematics, simply giving realistic contexts does not makes it mathematics (Freudenthal, 1991). Both pictures in Figure 20 have realistic objects which are scissors. However, as both teachers and curriculum developers pointed out, it is important to mathematize the realistic objects. While students may have difficulties to find mathematical ideas from Figure 20 (1), they may easily see the usefulness of mathematics in Figure 20 (2) because it shows how the realistic object can be used to understand the concept of angle. In other words, Figure 20 (1) gives just realistic objects while Figure 20 (2) gives ideas about how to use realistic objects in mathematical ways. Thus, I defined contextualization as setting mathematical ideas in realistic contexts. Figure 20 (3) shows an example of non-textual elements without contextuality. In other words, it does not have any realistic contexts to explain the concept. Although Figure 20 (3) has high level of mathematical connectivity, it does not have contextuality because it is not represented in realistic contexts.

Figure 20. Examples of Contextuality

(1)

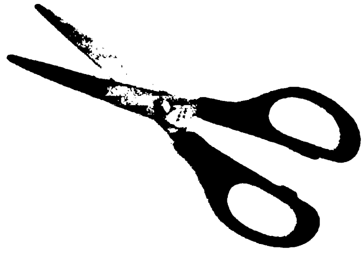
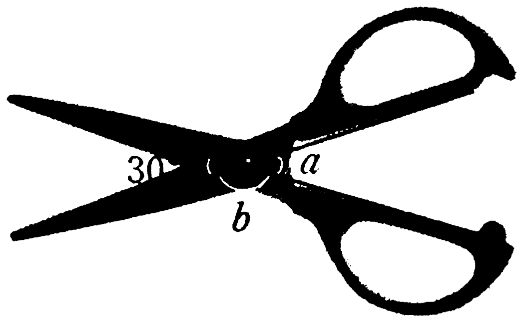
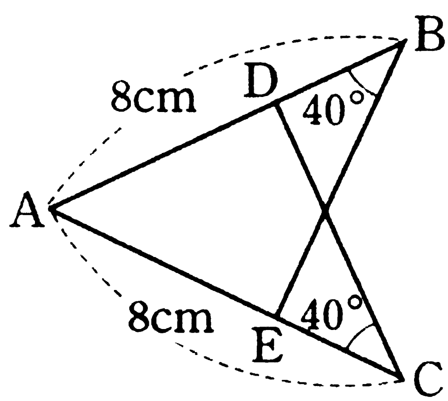


Figure 20. (cont'd)

(2)



(3)



Aesthetics as used in this study denotes how the non-textual elements are visually attractive enough to motivate students' learning. Many scholars have questioned about how to define and measure aesthetic values in mathematics because it seems to be subjective and interpretive. Given the importance of aesthetics, Sinclair (2006) claims that the aesthetic should be considered as an essential lens to see mathematical learning because the aesthetic values are closely related to how students understand and learn mathematics. Aesthetics is important not only to facilitate students to discover the beauty in mathematics but also to influence students' affective domain such as emotions, beliefs, and attitudes toward mathematics (Goldin, 2000). In particular, the motivational role of aesthetics concerns the value of stimulating students to engage with a particular problem. Sinclair (2004) establishes the criteria of the motivational role of the aesthetic including "connectedness, fruitfulness, visual appeal, apparent simplicity, and surprise" (p. 47). The results from the interview analysis suggest that both the connectedness between mathematics and students' lives and colors as visual appeal were counted as motivational factors by teachers and curriculum developers. That is, the "connectedness" Sinclair mentioned means connectedness between mathematics and realistic contexts, which falls into the category of "contextuality" in the F-Framework. In this study, I therefore restrict aesthetics to mean colors.

Simplicity implies mathematical conciseness of a non-textual element. Simplicity suggests how a non-textual element is concise and neat in presenting a concept or problem without any redundant, unnecessary or distracting factor. This idea stems from the results of the interview analysis. Both teachers and curriculum developers mentioned that one simple non-textual element for a given concept could be more effective than various complicated non-textual elements. This becomes more important when a new concept is introduced because a simple non-textual element can convey the meaning and idea more clearly and effectively. If a non-textual element includes many factors or components which are not necessary to explain a given concept, there can be ambiguity about what the non-textual element tries to explain. Since mathematics is a remarkably precise subject, such ambiguity may hinder students' understanding of the concept (Goldin & Shteingold, 2001). Some teachers also mentioned that such simple non-textual elements might allow students to better understand the core ideas of the concept as well as find connections among the related concepts if reiterated through related ideas. Figure 21 shows examples of simplicity. Figure 21 (1) shows the definition of angle which is "an amount of turning about a point between two lines" (Mitchelmore & White, 2000) without any other distracting factors. It is easy to find the concept in the picture. However, in Figure 21 (2), it is complicated to find the mathematical ideas. Even though the data are given to solve the problem of finding the slope of the equation from the data, there is other information such as weather and days of the week. Thus, students may easily get confused by the other factors such as weather and days of the week. Such unnecessary components of the picture may not be helpful for students to see how this picture can be used to solve the problem. This shows how simplicity is somewhat related to effectiveness of non-textual elements on learning.

Figure 21. Examples of Simplicity

(1)

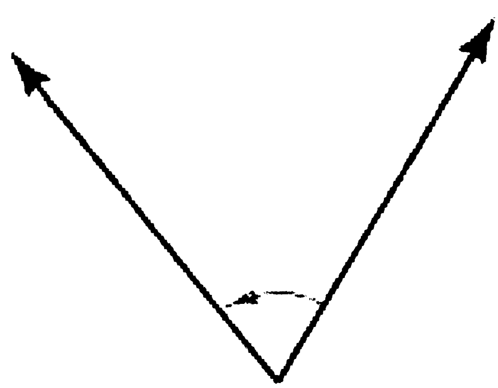
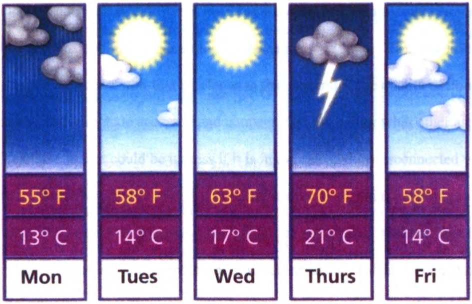


Figure 21. (cont'd)

(2)



In sum, Table 6 shows the definitions of each aspect of non-textual elements in the F-Framework.

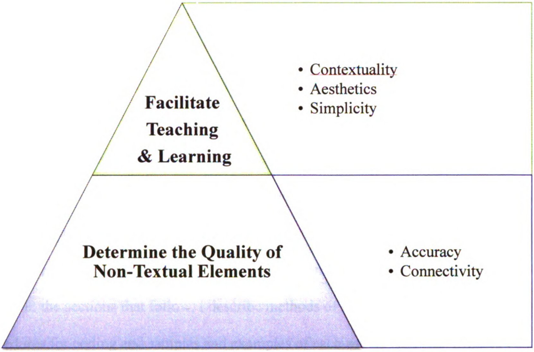
Table 6. Definition of Each Aspect in the F-Framework

Aspect of NTE	Definition
Accuracy	Mathematical clarity and rigor of non-textual elements by definition of the concept
Connectivity	Close linkage between non-textual elements and the mathematical content.
Contextuality	Contextualization of mathematical ideas in realistic context(s)
Simplicity	Mathematical conciseness of a non-textual element
Aesthetics	Visual attractiveness of non-textual elements to motivate students

5.4 Structure of the F-Framework

According to the results from my interviews with teachers and curriculum developers, there are two different levels of the aspects of non-textual elements in terms of their roles in teaching and learning mathematics (See Figure 22). The first level denotes the aspects of non-textual elements that determine the quality of non-textual elements. These include accuracy and connectivity. No matter what other aspects it has, a non-textual element could be useless if it is inaccurate and/or disconnected with the content. These two aspects of non-textual elements are essential factors to support student learning. The second level includes the aspects of non-textual elements that facilitate teaching and learning. These are simplicity, contextuality, and aesthetics. Although these aspects of non-textual elements are not requisites to support student learning, these aspects of non-textual elements facilitate student learning by providing concise images for a given concept, making connection between mathematics and realistic contexts, and motivating students to learn mathematics. Teachers also take advantage of such aspects of non-textual elements to make mathematics more meaningful and interesting for students.

Figure 22. Structure of the F-Framework



CHAPTER 6

NON-TEXTUAL ELEMENTS IN MATHEMATICS TEXTBOOKS

In order to test the validity of the F-Framework and find the patterns of non-textual elements used in current mathematics textbooks in different contexts, this study analyzes non-textual elements used in the lessons of angle, slope, and prime factorization in four South Korean textbooks and three U.S. textbooks. According to the F-Framework, each non-textual element was coded and analyzed using quantitative methods such as descriptive statistics, t-tests, ANOVA, and correlations.

In the sections that follow, I describe methods of textbook analysis including procedures for coding and samples and examples of each aspect for coding. Then, I show the results from the textbook analysis by topic, textbook, and country. Since the textbook analysis is purposely implemented to test the validity of the F-Framework for analyzing non-textual elements, I concluded with explanation of how the F-Framework can be used to analyze non-textual elements and to find patterns of non-textual elements in different textbooks across contexts.

6.1 Methods for the Textbook Analysis

In this section, I first describe the selection procedure for the textbook sample. I then outline procedures for coding with examples of non-textual elements. Then discuss inter-rater reliability.

Samples

In order to test the validity of the F-Framework and find the patterns of non-textual elements used in current mathematics textbooks in different contexts, I analyzed non-textual elements in the lessons of angle, slope, and prime factorization in three U.S. secondary mathematics textbooks and three South Korean ones. Because one of the three Korean textbooks did not include any lessons about slope, I also analyzed the lessons on slope from the fourth South Korean textbook. Although the number of textbooks reported in this chapter enable to investigate the current use of non-textual elements in some depth as well as to test the validity of the F-Framework. These textbooks were selected based on the criteria and process described as follows.

First, I considered textbooks which were either written by the curriculum developers or used by the teachers whom I interviewed for this study. Then, I considered different types of textbooks. All South Korean mathematics textbooks are written based on the national curriculum and are approved by the government. Thus, the features of all South Korean textbooks are quite similar to each other. However, U.S. textbooks include a range that reflects the great difference between traditional and standards-based mathematics approaches to mathematics education. U.S. standards-based textbooks have many features different from conventional textbooks because they aim to challenge traditional beliefs on both what is important mathematics and what is the most effective way to teach and learn mathematics (Senk & Thompson, 2003). For example, standards-based textbooks possess more mathematical problems embedded in real contexts and fewer questions requiring only memorization or simple computation. Thus, I attempted to select one traditional textbook and one standards-based textbook for the U.S. textbooks.

Despite the small number of textbooks, these textbooks which are used in many U.S. schools are enough to study in depth as well as to provide some ideas about the use of non-textual elements in terms of different types of textbooks.

Drawing from the set of textbooks that met these criteria in each country, I selected two textbooks that were used by most of the teachers I interviewed and one textbook that was written by one of the curriculum developers I met in each country. Finally, I considered if the textbooks include all the three topics, i.e., angle, slope, and prime factorization. However, as stated earlier, one of the South Korean textbooks that met the criteria did not include lessons of slope. Thus, for the lessons of slope, I added one additional textbook which is written by one of the South Korean curriculum developers who I interviewed. Table 7 shows the information about the selected textbooks.

Table 7. Information about the Selected Textbooks

Country	Textbook Code	Nature	Reason for Selection
South Korea	SK1	Approved by the Government, Coded only for the lessons of angle and prime factorization	Used by five of the Korean teachers whom I interviewed
	SK2	Approved by the Government, Coded for all the three topics	Used by two of the Korean teachers whom I interviewed
	SK3	Approved by the Government, Coded for all the three topics	Written by one of the curriculum developers whom I interviewed

Table 7. (cont'd)

Country	Textbooks	Nature	Reason
South Korea	SK4	Approved by the Government, Coded only for the lessons of slope	Written by one of the curriculum developers whom I interviewed
The United States	US1	Standards-based textbooks, Coded for all the three topics	Written by one of the curriculum developers whom I interviewed
	US2	Conventional textbooks, Coded for all the three topics	Used by four of the U.S. teachers whom I interviewed
	US3	Standards-based textbooks, Coded for all the three topics	Used by three of the U.S. teachers whom I interviewed

Procedures for Coding

A coding scheme for textbook analysis was developed according to the F-Framework. I as an investigator tested and revised the coding scheme using a small random sample of non-textual elements used in lessons of angle, slope, and prime factorization in mathematics textbooks from each country. The revised coding scheme used to code non-textual elements in four South Korean textbooks and three U.S. textbooks is described in Table 8.

Table 8. Coding Scheme for Textbook Analysis

Aspects of Non-textual elements	Score	Description
Accuracy (For mathematical representations)	2	An NTE correctly shows the definition of a concept, or it is accurate to show a concept based on its definition.
	1	An NTE makes sense in terms of the definition or meaning of a concept. But it does not show every required mathematical condition (e.g., some requisite notations are missing or misleading).
	0	An NTE is inaccurate in terms of the definition of a concept (e.g., it has an obvious error).
Accuracy (For contextual representations)	2	The realistic object or context is appropriate to explain a concept in terms of its definition.
	1	An NTE conveys what it means. But some attributes of the realistic object or context are not appropriate to explain a concept.
	0	An inappropriate realistic object is used to present a concept. There is a major error or concern to use the realistic object or context for a concept. Or there is no mathematical concept in the NTE.
Connectivity	2	An NTE is explicitly and fully associated with the mathematical content in the text. It directly shows a concept or problem.
	1	An NTE is partly related to the mathematical content in the text. There is some missing information in an NTE. It shows the content but it does not explicitly show how it is connected with the content.
	0	An NTE has nothing to do with the content mathematically. It can give some clue about contexts in texts (e.g., river when the problem is about length of river).
Simplicity	2	An NTE is straightforward to show a concept or problem without any distracting or other factor.
	1	An NTE is straightforward to show a concept or problem with some other factors that might be helpful for the concept.

Table 8. (cont'd)

Aspects of Non-textual elements	Score	Description
Simplicity	0	An NTE has distracting or other factors that are useless in addition to factors needed to show a concept or problem.
Contextuality	2	A realistic object or context is used in an NTE with mathematical connection.
	1	No mathematical connection exists, but there is some realistic contextual information (that is used to solve the problem or to facilitate related activities).
	0	Neither realistic object nor realistic context is included in an NTE.

Since aesthetics is defined as color effects in the F-Framework, aesthetics is not coded in the textbook analysis because the same number of colors is used on every page for each textbook. Thus, all the textbooks are coded and analyzed by accuracy, connectivity, contextuality, and simplicity in this chapter.

Examples of Non-Textual Elements for Coding

Each non-textual element in the selected textbooks was coded and analyzed to find the patterns of aspects of non-textual elements used in textbooks in terms of country, textbooks, and topics. Since aspects of non-textual elements cannot be evaluated separately from the content, each non-textual element was coded in relation to the adjacent content on the same page. If there is a group of non-textual elements to serve the content on the page, the unit of analysis is determined in terms of their roles. If each non-textual element in a group could separately serve the content or be used for the different purpose, each non-textual element in the group was coded separately. If each non-textual element in a group cannot fully explain the content or concept separately, the group of non-textual elements was coded as one unit. Figure 23 shows an example of each case.

Figure 23. Examples of unit of analysis

(1)

1. Use this information to write an equation for the relationship between degrees Fahrenheit and degrees Celsius.
2. How did you find the y-intercept? What does the y -intercept tell you about this situation?

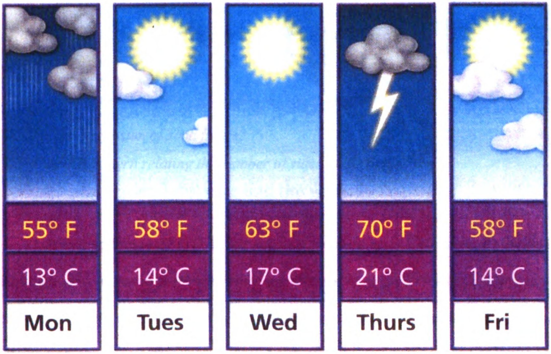
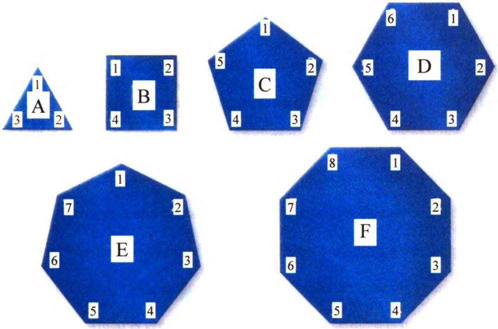


Figure 23. (cont'd)

(2)



What is the angle sum of each figure?
Do you see a pattern relating the number of sides to the angle sum?

Although Figure 23 (1) has five separate pictures, no one picture can provide full information to solve the problems, which are to write an equation for the relationship between degrees Fahrenheit and degrees Celsius and to find y-intercept of the equation. Since one picture presents only one degree Fahrenheit and one degree Celsius, it is not enough to find an equation which represents the relationship between them. Each picture cannot fully serve the content. I thus counted the group of pictures as one unit of analysis for the reason. In contrast, although Figure 23 (2) also has several separate pictures, each picture can be used independently for the problem. Since the problem is to find angle sum of each figure, each picture can serve the problem separately and provide various opportunities to explore the concept of angle sum in different polygons. In this case, I counted each picture as a unit of analysis. Specific examples of non-textual elements for each aspect of non-textual elements in the F-Framework and details about how to code each non-textual element are provided in the next section.

Accuracy is the extent to which a non-textual element correctly represents the concept or problem with regard to its mathematical definition. Figure 24 (1) shows an example of accurate non-textual elements. This picture is well drawn to show clearly the definition of angle which is "the figure formed by two rays sharing a common endpoint" (Sidorov, 2001) with appropriate symbols and marks. On the other hand, Figure 24 (2) is used to introduce the concept of acute angle. However, there are two angles around the endpoint according to the definition of angle. One is an acute angle and the other is another angle which is greater than 270 degrees. Since there is no mark to indicate the acute angle, it could be confusing to students who have never learned this concept before even though this picture shows the definition of angle well. One point was given to this picture because it did not clearly indicate the acute angle. There were no mathematical figures which are completely incorrect in the textbooks that I investigated.

For non-textual elements with realistic contexts, I found different levels of accuracy in such non-textual elements. Figures 24 (3), (4), and (5) are non-textual elements with realistic contexts. Since Figure 24 (3) accurately represents angles that are measured to find the zenith from horizons, two points were given to this figure for accuracy. Figure 24 (4) shows how to measure the length of the river using the concept of angle and similarity. Although this figure shows two triangles (one triangle which is made by the two trees and stake 3, and the other triangle which is made by the three stakes) to solve the problem, it is hard to see if the two triangles are similar. For example, if the two triangles are similar, corresponding angles should be the same. But there is no clue that the line between the trees is parallel to that between the stakes 1 and 2. Thus, this figure does not represent all the conditions that are necessary to solve the problem. This figure was graded as one point by the coders. Figure 24 (5) is used to explain the concept of parallel lines. However, this figure is not appropriate to explain the concept because an array of the cabbages is not a straight line mathematically. Further, the lines in the picture look like they meet each other somewhere along the horizontal line between the forest and the field. Thus, students may be confused about what exactly parallel lines mean and may develop wrong images of parallel lines either consciously or unconsciously through these pictures. In fact, when I surveyed 30 students as a pilot study, 34% of the students, particularly younger children and low achievers, had hard time to understand the concept of parallel lines using this picture. This figure is an example of inaccurate non-textual elements.

Figure 24. Examples of Accuracy for Coding

(1) 2 points (mathematical representation)

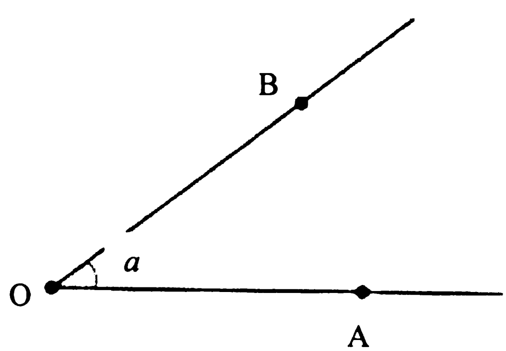
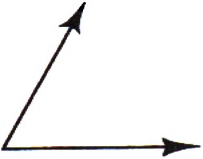


Figure 24. (cont'd)

(2) 1 point (mathematical representation)



(3) 2 points (contextual representation)

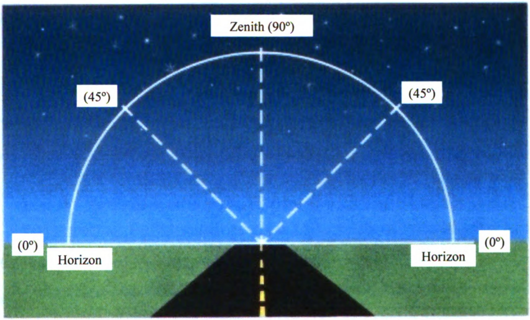
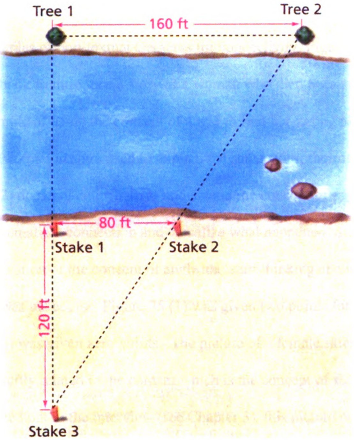


Figure 24. (cont'd)

(4) 1 point (contextual representation)

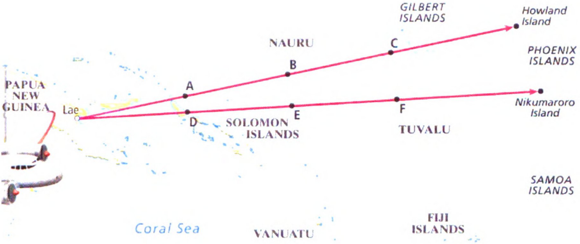


(5) 0 points (contextual representation)

In this study, *connectivity* means mathematical connection between a non-textual element and the content on the page. It is measured by how a non-textual element is explicitly associated with the mathematical ideas in the content. Figure 25 shows some examples of non-textual elements for connectivity. Figures 25 (1) and (2) are used to in a textbook to introduce a story of a woman who died because of a flight crash as an example of using the concept of angle to find her wrong plane trajectory. Even though the two pictures are used to explain the same story, their roles are different. While Figure 25 (2) does not contribute to understand the concept of angle, Figure 25 (1) has mathematical connection and visualize what happened. Students can thus see the application of the concept of angle and start thinking about how to measure the degrees she was off course. Figure 25 (1) was given two points for connectivity whereas Figure 25 (2) was given zero points. The picture of a female skier in Figure 25 (3) is not explicitly related to the content which is the concept of slope. However, as some teachers pointed out in the interview (see Chapter 3), this picture provides some clue that steepness of the hill is related to the concept of slope in mathematics. Thus, even though this picture does not explicitly show the relationship, students may better understand the concept of slope in relation to that of steepness in real life. This kind of picture can be distinguished from pictures like Figure 25 (2) because Figure 25 (2) does not provide any clue about the concept of angle. Thus, one point was given to the pictures which implicitly provide related ideas to the concept or content like Figure 25 (3).

Figure 25. Examples of Connectivity for Coding

(1) 2 points

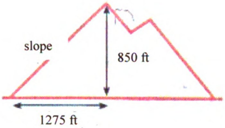


(2) 0 points

Figure 25. (cont'd)

(3) 1 point

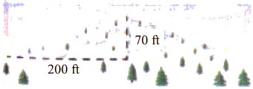
Snow safety experts advise that avalanches are more likely to happen on a hill with a slope of between 0.5 and 1. Is an avalanche likely for Hotdog Hill?



$$\frac{rise}{run} = \frac{850}{1275}$$

Dividing to find the slope, $850 \boxed{\div} 1275 \boxed{=}$ 0.66666667. The slope is between 0.5 and 1, so an avalanche may be likely.

Try it



Determine the slope. Is an avalanche likely?

Simplicity is measured by the degree of conciseness of a non-textual element. This idea is derived from the interviews with teachers and curriculum developers. Figure 26 (1) is used to show an angle without any other concept or component. If a teacher uses this picture to explain the concept of angle, students may focus on the concept without any distraction and complexity. The score of this figure for simplicity is two points. Figure 26 (2) is used to explain the concept of right angle. This picture drawn by Mondrian clearly shows many right angles in various rectangles in different colors. Although this picture is more complicated than a picture which has only one example of right angle, this allows students to find right angles in various rectangles. In addition, since there are different colors in rectangles, they are helpful for students to find right angles in a shape more easily. One point was given to pictures like Figure 26 (2) which show the concept with some other factors like different colors and shapes that might be helpful for the concept. Figure 26 (3) is used in a problem to find an equation and its slope which show the relationship between degrees Fahrenheit and degrees Celsius. The picture provides not only degrees Fahrenheit and degrees Celsius but also days of the week and weather for each day. The information about weather for each day is not needed to solve this problem. In addition, since the size of the picture for the weather is bigger than the degrees which are necessary to solve this problem, this could distract students' attention to the data. The information about the weather could mislead students because students may think that this problem is about the relationship between weather and temperature. Zero points were thus given to pictures like Figure 26 (3) which provide useless or distracting information to solve the given problem.

Figure 26. Examples of Simplicity for Coding

(1) 2 points

(2) 1 point

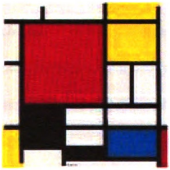
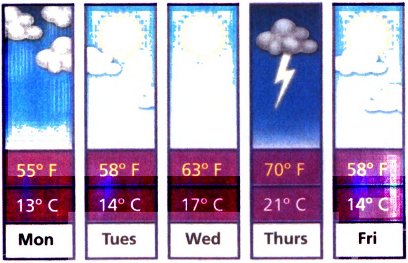


Figure 26. (cont'd)

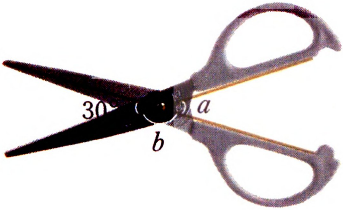
(3) 0 points



Contextuality is measured by the degree of realistic contexts in a non-textual element. Both Figures 27 (1) and (2) include realistic objects. These realistic objects play different roles in mathematical learning. While the scissors in Figure 27 (1) serve as a tool for solving the problem which asks to find angles *a* and *b*, the picture of an old man in Figure 27 (2) does not contribute to the problem, "Mr. Rawlings has 60 cookies. He wants to give each of his 16 grandchildren the same number of cookies for a snack. What is the greatest number of cookies he can give each child?" at all. It is important to see if realistic contexts or objects in a non-textual element themselves are used as contexts in which mathematical concepts or problems are embedded because their educational effects could be different. However, Figure 27 (2) provides some contextual information which might help students see the contexts to which the problem is related. Figure 27 (2) were given one point for contextuality whereas pictures like Figure 27 (1) were given two points. Abstract mathematical figures like Figure 27 (3) were given zero points for contextuality because they do not include any realistic contexts or objects.

Figure 27. Examples of Contextuality for Coding

(1) 2 points



(2) 1 point

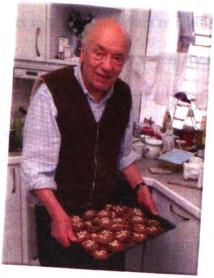
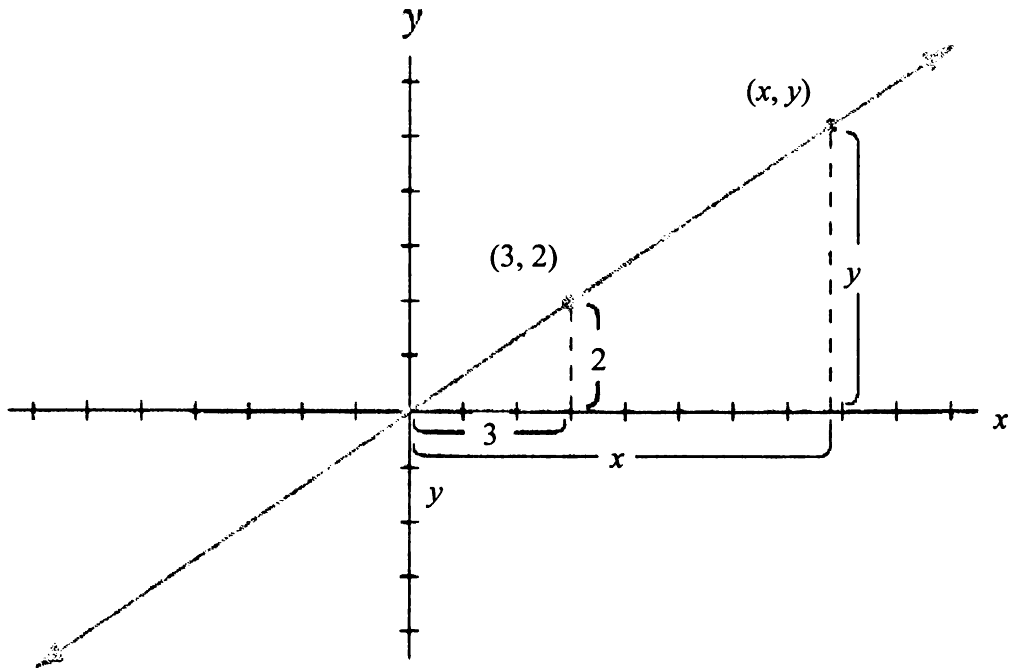


Figure 27. (cont'd)

(3)



Reliability of Coding

In order to check inter-rater reliability and take into account cultural differences, three coders who had different national backgrounds and experiences with textbooks participated in coding the textbooks. Since I have experience with both South Korean and U.S. mathematics textbooks, I (rater A) invited two coders with different backgrounds. One coder (rater B) was an American doctoral student in mathematics and science education who had no experience with South Korean textbooks and schooling. The other coder (rater C) was a Korean doctoral student in education who minored in mathematics at the undergraduate level and, at the time of the coding, had no experience with U.S. textbooks and K-12 schooling. Coding non-textual elements by three coders who have different backgrounds was helpful for me not only to check inter-rater reliability among the coders but also to consider patterns of their interpretation in relation to their backgrounds.

Before coding non-textual elements, the coders read the coding scheme and were allowed to ask questions to clarify what each sentence in the coding scheme meant. I did not train the coders using specific examples because my interpretation of a non-textual element could affect their decisions when they coded. Thus, before they coded, I explained what each aspect meant and how each score could be distinguished without any specific example. Understanding the meaning of each category in the coding scheme, each coder coded all the non-textual elements in both South Korean and U.S. textbooks independently. Since I needed to translate texts in textbooks to each coder as I did for teachers to code connectivity, I sat with coders when they were coding. However, I did not do anything but translate into Korean or English. This could be a limitation of this study because they coded each non-textual element with my oral translation for connectivity, which may be different from reading written texts by themselves in the textbooks.

I collected the codes for each non-textual element from the three coders and analyzed the data using quantitative research methods. When the coders disagreed, the median of the three coders' code for a non-textual element was used as its final code. Overall, 76 percent of non-textual elements earned the same score from all the three coders, and another 24 percent of non-textual elements earned the same score from two out of the three coders.

An inter-rater reliability analysis using the Kappa statistic between a pair of coders was performed to determine consistency among the raters. The inter-rater reliability for the raters A (researcher) and B (American) was found to be Kappa = .63 ($p < .001$), with 95% CI (.537, .728). The inter-rater reliability for the raters B (American) and C (Korean) was also Kappa = .63 ($p < .001$), with 95% CI (.537, .726). The inter-rater reliability for the rater A and C was Kappa = .88 ($p < .001$), with 95% CI (.813, .937). According to a rule of thumb, Kappa coefficients between .40 and .59 indicate a moderate agreement between raters, coefficients between .60 and .79 a substantial agreement, and coefficients .80 or above an outstanding agreement (Landis & Koch, 1977). Thus, the overall Kappa coefficients found in this study can be interpreted as indicating a substantial or outstanding level of inter-rater reliability. They show that the scores of aspects of non-textual elements are substantially consistent among the three coders and the F-Framework is sufficiently reliable to analyze non-textual elements in mathematics textbooks.

Unlike my assumption that inter-rater reliability for the raters B and C would be the least one among the three reliabilities, the inter-rater reliability for raters A and B is similar to that for raters B and C. In addition, although only three coders are not enough to see cultural differences in their interpretation, I compared three coders' scores for each non-textual element and found the patterns of their interpretations by examining non-textual elements on which they disagreed.

In fact, there was no non-textual element upon which all the three coders completely disagreed in scoring. In order to see the patterns of the agreement between the three coders, I calculated the percentages of full agreements (i.e. the case that all the three coders gave the same score for a non-textual element) by topic, country, and aspect of non-textual elements as shown in Table 9.

Table 9. Percentages of Full Agreements among Three Coders

		Percent
Topics	Angle	78.26
	Slope	76.56
	Prime factorization	66.67
Country	South Korea	84.09
	U.S.	68.27
Aspects	Accuracy	66.67
	Connectivity	72.92
	Simplicity	68.75
	Contextuality	93.75

My initial assumption on the difference of coders' interpretation was that their different backgrounds might affect their perception of contextuality because realistic contexts that people experience and interpret could be different in each country according to its social, historical, and cultural contexts. However, contextuality is the aspect of non-textual elements on which the highest level of consensus (93.75%) was observed among the three coders.

Rather, accuracy is the aspect of non-textual elements on which the lowest level of consensus was observed among them (33% of non-textual elements were rated with two different scores: two out of the three coders gave the same score and the other coder gave different one). Examining each non-textual element that was rated with two different scores, I found that about 90% of the non-textual elements have contextuality. While the three coders had strong consensus on mathematical visual non-textual elements which were presented without real context, the three coders have somewhat different interpretation on accuracy of non-textual elements with realistic contexts no matter which textbooks they examined. In particular, raters B (American) and C (Korean) who have different national background rated about 80% out of the non-textual elements with realistic contexts differently. While rater B gave full points to most of the non-textual elements with realistic contexts in terms of accuracy, rater C gave them partial points. This tendency could be found across South Korean and U.S. textbooks that they examined. In other words, rater B gave full points to most of contextual non-textual elements in both South Korean and U.S. textbooks while rater C gave them partial points across South Korean and U.S. textbooks. Implications of discrepancies in coding are discussed in chapter 7.

6.2 Results from the Textbook Analysis

The textbook data were analyzed using quantitative research methods such as descriptive statistics, t-tests, ANOVA, and correlations. In order to find patterns of non-textual elements and variation among the coders' interpretation, the data were analyzed and compared in terms of topics, textbook, country, as well as coders for each aspect of non-textual elements.

Based on the coding scheme for each aspect of non-textual elements as explained above, three coders coded each non-textual element in four South Korean and three U.S. mathematics textbooks. Overall, the total number of non-textual elements coded in this study was 266 from 98 pages in both South Korean and U.S. textbooks. To compare South Korean and U.S. textbooks, I calculated the average number of non-textual elements per page used in each topic as shown in Table 10. General patterns in the use of non-textual elements in each topic are similar between South Korea and the United States. The greatest number of non-textual elements is used in lessons about angle, and the least number of non-textual elements is used in lessons about prime factorization in both South Korean and U.S. textbooks. Over 50% of the total number of non-textual elements is used in angle in both South Korean and U.S. textbooks. In comparison between the two countries, South Korean textbooks have more non-textual elements per page than their counterpart for each topic.

Table 10. Number of Non-Textual Elements Used in Each Topic and Average Number, Per Page

	Angle		Slope		Prime factorization		Overall	
	n	mean	*n*	mean	*n*	mean	*n*	mean
Korea	52	5.2	56	2.7	17	1.4	125	2.9
U.S.	82	3.9	43	2.0	16	1.2	141	2.6

Patterns of Non-Textual Elements by Topic

Table 11 presents means and the results for the four aspects of non-textual elements used in mathematics textbooks by topic. Accuracy, connectivity, and simplicity of non-textual elements are not significantly different across topics such as angle, slope, and prime factorization even though the total number of non-textual elements is quite different across topics. Accuracy and connectivity are the fundamental aspects of non-textual elements which decide the quality of non-textual elements in mathematics textbooks. This result shows that these important aspects of non-textual elements do not depend on topics. In fact, the mean scores of accuracy, connectivity, and simplicity are higher than 1.5 points across topics. Considering that a code of two points means the full score for each aspect, it is good to have such high scores for these fundamental aspects of NTEs. In particular, considering simplicity that teachers and curriculum developers in the interviews thought as an effective way to help students better understand related concepts, this result implies that simplicity of non-textual elements is considered regardless of topics.

Table 11. Descriptive Statistics of the Four Aspects of Non-Textual Elements in Textbooks, by Topic

		F-test				Scheffe's test		
		n	Mean	*SD*	*F*	A-B	B-C	C-A
Accuracy	Angle (A)	134	1.522	.584	.070			
	Slope (B)	99	1.556	.759				
	Prime factorization (C)	33	1.545	.794				
Connectivity	Angle (A)	134	1.836	.428	1.948			
	Slope (B)	99	1.737	.564				
	Prime factorization (C)	33	1.667	.645				
Simplicity	Angle (A)	134	1.679	.608	.915			
	Slope (B)	99	1.596	.755				
	Prime factorization (C)	33	1.515	.795				
Contextuality	Angle (A)	134	.813	.967	4.460 *	*		
	Slope (B)	99	.465	.812				
	Prime factorization (C)	33	.576	.83				

*p < .05.

However, the mean scores for contextuality are lower than those for other aspects. All the mean scores are less than one point. It implies that non-textual elements in mathematics textbooks are likely not to use realistic contexts to explain concepts across topics. Further, contextuality of non-textual elements appears differently across topics.

The mean score for contextuality in the lessons about angle is higher than those in other topics. Although both teachers and curriculum developers in the interviews did not report any difference of non-textual elements across topics, the results from the textbook analysis show that non-textual elements with realistic contexts are used less in algebraic topics such as slope and prime factorization while angle as a geometric topic is more likely to be presented within realistic contexts than in other algebraic topics. In fact, about 40 percents of non-textual elements in lessons of angle used realistic contexts to explain the concept whereas only 20 percents of those used in lessons of slope and 21 percents of those in lessons of prime factorization.

Considering correlations among the four aspects of non-textual elements used for different topics, I found Spearman's correlations among them as shown in Table 12. The data analyzed in Table 12 suggest that accuracy, connectivity, and simplicity are significantly and positively related to each other in all the topics whereas contextuality is negatively related to them across topics. This implies that contextual non-textual elements in the textbooks are likely to be inaccurate, disconnected with the content, and more complicated across topics. This tendency may be associated with the nature of realistic contexts in which there are generally many irregular or untypical objects and complicated phenomenon.

Table 12. Spearman's Correlations among the Four Aspects of Non-Textual Elements Used for Different Topics

		Connectivity		Simplicity		Contextuality	
Angle (n=134)	Accuracy	.44	**	.35	**	-.26	**
	Connectivity			.67	**	-.40	**
	Simplicity					-.55	**
Slope (n=99)	Accuracy	.86	**	.86	**	-.77	**
	Connectivity			.87	**	-.82	**
	Simplicity					-.79	**
Prime factorization (n=33)	Accuracy	.83	**	.86	**	-.53	**
	Connectivity			.79	**	-.48	**
	Simplicity					-.60	**

**$p < .01$.

Patterns across the Sampled U.S. and South Korean Textbooks

Table 13 presents descriptive statistics for the scores of the four aspects of non-textual elements between South Korea and the United States. The mean scores of all the four aspects of non-textual elements in the selected South Korean textbooks are higher than the counterpart. However, the standard deviations for the U.S. textbooks are greater than those of the Korean textbooks for those of the four aspects. The results from the *t*-tests show that the mean differences in the scores of all the aspects except contextuality between the two countries are significant. This means that non-textual elements in the South Korean textbooks tend to be more accurate and more mathematically connected with the contents tan the U.S. textbooks studied. But the U.S. books show more variation.

Table 13. Descriptive Statistics of the Four Aspects of Non-Textual Elements between the Selected South Korean and U.S Textbooks

	South Korea (*n*=125)		United States (*n*=141)			*p*-value
	Mean	*SD*	Mean	*SD*	*t*	(2-tailed)
Accuracy	1.73	.56	1.37	.73	4.53	<.001
Connectivity	1.87	.38	1.70	.60	2.92	<.01
Simplicity	1.76	.59	1.51	.75	3.03	<.01
Contextuality	.74	.96	.57	.86	1.51	.131

In order to see if there is any difference of aspects of non-textual elements across topics in different contexts, I calculated mean differences in the four aspects of non-textual elements in the selected South Korean textbooks and U.S. textbooks by topic as shown in Tables 14 and 15. While accuracy, connectivity, and simplicity in the South Korean textbooks are not different across topics, contextuality in the selected South Korean textbooks studied is significantly different, especially between angle and slope. This is consistent with the results shown in Table 11. This implies that the uses of non-textual elements with realistic contexts are different between angle and slope whereas there is no significant difference in other aspects of non-textual elements across topics. In other words, non-textual elements with realistic contexts are less often used in lessons on slope than they are in lessons on angle. This means that compared to the lessons of angle, the concept of slope is often explained with abstract visual non-textual elements rather than contextual non-textual elements. The concept of angle as a geometric topic is often explained in the context of real life. However, in the middle school mathematics curriculum, the concept of slope is most often explained as an algebraic topic as "rate of change" in lessons of linear function or as a geometric topic which is associated with "steepness" in real life (Stump, 2001). Since the results in Table 14 show that less realistic contextual representations are used in the lessons of slope compared to those of angle, this implies that the South Korean textbooks are more likely to introduce the concept of slope using abstract algebraic representations which is more related to "rate of change."

Table 14. Mean Differences in the Four Aspects of Non-Textual Elements in the Sampled South Korean Textbooks, by Topic

		F-test					Scheffe's test		
		n	Mean	*(SD)*	*F*		A-B	B-C	C-A
Accuracy	Angle (A)	52	1.73	(.45)	.053				
	Slope (B)	56	1.71	(.65)					
	Prime factorization (C)	17	1.76	(.56)					
Connectivity	Angle (A)	52	1.90	(.30)	.359				
	Slope (B)	56	1.86	(.40)					
	Prime factorization (C)	17	1.82	(.53)					
Simplicity	Angle (A)	52	1.75	(.59)	.013				
	Slope (B)	56	1.77	(.60)					
	Prime factorization (C)	17	1.76	(.56)					
Contextuality	Angle (A)	52	1.15	(1.00)	9.833	***	***		
	Slope (B)	56	.39	(.78)					
	Prime factorization (C)	17	.65	(.93)					

*** $p < .001$.

In the U.S. textbooks studied, however, there is no significant difference of contextuality across topics as well as accuracy, connectivity, and simplicity. Even though there are significant differences of connectivity and simplicity across the sampled U.S. textbooks (See Table 15), such differences are not observed in different topics in the U.S.

textbooks. Accuracy as the most basic aspect of non-textual elements does not differ regardless of topics and textbooks. Although connectivity and simplicity vary across the textbooks I studied, there are no significant differences in connectivity and simplicity across topics. This implies that such differences do not come from the nature of different topics.

The results in Table 15 show that unlike the South Korean textbooks, contextual non-textual elements are used to explain the concept of slope as often as they are to explain the concept of angle. This implies that the U.S. textbooks studied explain the concept of slope using realistic contextual non-textual elements as they did for the concept of angle.

Table 15. Mean Differences in the Four Aspects of Non-Textual Elements in the sampled U.S. Textbooks, by Topic

		F-test				Scheffe's test		
		n	Mean	*(SD)*	*F*	A-B	B-C	C-A
Accuracy	Angle (A)	82	1.39	(.62)	.097			
	Slope (B)	43	1.35	(.84)				
	Prime factorization (C)	16	1.31	(.95)				
Connectivity	Angle (A)	82	1.79	(.49)	2.801			
	Slope (B)	43	1.58	(.70)				
	Prime factorization (C)	16	1.50	(.73)				
Simplicity	Angle (A)	82	1.63	(.62)	2.869			
	Slope (B)	43	1.37	(.87)				
	Prime factorization (C)	16	1.25	(.93)				
Contextuality	Angle (A)	82	.60	(.89)	.097			
	Slope (B)	43	.56	(.85)				
	Prime factorization (C)	16	.50	(.73)				

The F-Framework can be used to find differences across textbooks in a particular context or between contexts. I used ANOVA to analyze variation in the use of non-textual elements across textbooks and explore the patterns of these elements in different type of textbooks. Table 16 shows the results according to topics because textbook SK1 was used only for angle and prime factorization and textbook SK4 was used only for the lessons of slope (See Table 7 for the details). Table 16 reports that there is no significant difference in the mean scores of accuracy, connectivity, and simplicity across the selected South Korean textbooks by topic. In contrast, those of contextuality among the South Korean textbooks are significantly different in the lessons of slope as shown Table 16 (b). This shows that accuracy, connectivity, and simplicity of non-textual elements do not differ in terms of topics. However, it is notable that the mean difference of contextuality for the lessons of slope is significant among the South Korean textbooks studied while that of contextuality for the lessons of angle and prime factorization is insignificant. This implies that the South Korean textbooks use various approaches to non-textual elements in lessons on slope. Unlike geometric and numeric topics such as angle and prime factorization, some textbooks focus on abstract visual representations for the lessons of slopes and other textbooks use non-textual elements with realistic contexts to present this content.

Table 16. (a) Descriptive Statistics of the Four Aspects of Non-Textual Elements in the Selected South Korean Textbooks for Angle and Prime Factorization, by Textbook

		F-test			
	Textbook	*n*	Mean	*(SD)*	*F*
Accuracy	SK1	25	1.80	(.41)	4.114
	SK2	17	1.94	(.24)	
	SK3	27	1.56	(.58)	
Connectivity	SK1	25	1.88	(.33)	.308
	SK2	17	1.94	(.24)	
	SK3	27	1.85	(.46)	
Simplicity	SK1	25	1.68	(.69)	.622
	SK2	17	1.88	(.33)	
	SK3	27	1.74	(.59)	
Contextuality	SK1	25	1.04	(1.02)	1.411
	SK2	17	.71	(.99)	
	SK3	27	1.22	(.97)	

(b) Descriptive Statistics of the Four Aspects of Non-Textual Elements in the Selected South Korean Textbooks for Slope, by Textbook

		F-test					Scheffe's test		
	Textbook	*n*	Mean	*(SD)*	*F*		B-C	C-D	D-B
Accuracy	SK2	14	1.57	(.76)	.789				
	SK3	22	1.68	(.72)					
	SK4	20	1.85	(.49)					
Connectivity	SK2	14	1.79	(.43)	.330				
	SK3	22	1.86	(.35)					
	SK4	20	1.90	(.45)					
Simplicity	SK2	14	1.57	(.76)	1.235				
	SK3	22	1.77	(.61)					
	SK4	20	1.90	(.45)					
Contextuality	SK2	14	1.00	(1.04)	7.286	**	*		**
	SK3	22	.27	(.70)					
	SK4	20	.10	(.31)					

*p < .05; **p < .01.

Compared to the South Korean textbooks, the U.S. textbooks studied have more variation in non-textual elements across textbooks. Table 17 shows that there are significant mean differences in connectivity, simplicity, and contextuality across textbooks. Unlike the South Korean textbooks, the sampled U.S. textbooks vary in terms of the four aspects of non-textual elements. For example, the degree of mathematical connectivity of non-textual elements differs across textbooks. In the lessons about angle, slope, and prime factorization, textbook US1 has more complicated non-textual elements while textbook US2 has simpler non-textual elements. Mean scores of simplicity between the two textbooks are also significantly different. Textbook US1, a standards-based textbook, is significantly different from other textbooks US2 and US3. In terms of contextuality, textbooks US2 is traditional and textbook US3 is considered to be a standards-based textbook. Even though standards-based textbooks are assumed to have more mathematical problems embedded in real contexts, the extent of use of non-textual elements in realistic contexts can vary across different textbooks. Overall, whereas the South Korean textbooks studied are not significantly different in terms of aspects of non-textual elements except contextuality, the U.S. textbooks in the sample have significant differences in terms of all the aspects of non-textual elements except accuracy.

Table 17. Descriptive Statistics of the Four Aspects of Non-Textual Elements in the Selected U.S. Textbooks, by Textbook

		F-test					Scheffe's test		
		n	Mean	*(SD)*	*F*		E-F	F-G	G-E
Accuracy	US1	70	1.29	(.78)	.910				
	US2	21	1.43	(.60)					
	US3	50	1.46	(.71)					
Connectivity	US1	70	1.56	(.65)	4.112	*			
	US2	21	1.90	(.30)					
	US3	50	1.80	(.57)					
Simplicity	US1	70	1.33	(.79)	4.680	*	*		
	US2	21	1.81	(.51)					
	US3	50	1.64	(.72)					
Contextuality	US1	70	.93	(.95)	14.250	***	**		***
	US2	21	.29	(.72)					
	US3	50	.20	(.49)					

*p < .05; **p < .01; ***p < .001.

CHAPTER 7

CONCLUSION AND DISCUSSION

As described in chapter 1, the purpose of this study is to develop a conceptual framework that may be used to analyze non-textual elements in mathematics textbooks. Four research questions guided this investigation. In order to streamline this section, major findings about the views of teachers and curriculum developers are summarized first, followed by a summary of how the analyses of interviews with teachers and curriculum developers informed revision of the conceptual framework, and a summary of the findings from the textbook analyses.

7.1 Summary of the Findings

Interviews with Teachers and Curriculum Developers

This study provides important ideas about what intentions curriculum developers had to select non-textual elements in mathematics textbooks and how teachers understand the roles and characteristics of non-textual elements in mathematics textbooks for teaching and learning. Although the five aspects of non-textual elements in the F-Framework were considered important, emphases on each aspect were different across teachers and curriculum developers in different countries according to their beliefs and prior experiences, their students' readiness, phases of the lesson, and administrative restriction. For example, at the beginning of a lesson, Mr. Evans intended to use realistic contextual non-textual elements to attract students' attention to the content as well as to pose a question about the concept with connection to their life experiences. Then, he developed the ideas with his students using abstract mathematical non-textual elements.

However, Mr. Martin believed that a simple abstract mathematical non-textual elements should come first to give clear ideas about what the concept means. Then, he could offer various contextual non-textual elements to apply the idea into realistic contexts as well as to show the usefulness of mathematics. It also shows how teachers recognize different roles of different types of non-textual elements to teach mathematics.

I also noted a difference between curriculum developers and teachers. While most of the curriculum developers paid little attention to non-mathematical pictures and illustrations in mathematics textbooks, many teachers considered their motivational roles. I noticed that although curriculum developers considered how teachers might use non-textual elements from their textbooks, they concentrated more on mathematical ideas and the structure of textbooks with regard to connections among mathematical ideas. As described above, teachers recognize different roles of each type of non-textual elements and use them in various ways. However, curriculum developers did not pay much attention to various pedagogical approaches in classrooms.

On the other hand, while curriculum developers considered accuracy of non-textual elements with relation the structure of mathematics and connections among mathematical ideas in several cognitive domains, teachers did not seem to see the meaning in their practice. When it comes to motivational roles of non-textual elements, curriculum developers provided more sophisticated insights that mathematical reasoning through representations itself can be a good motivator while visual attractiveness could be superficial and momentary motivation. The curriculum developers' point of view on motivation is consistent with the roles of representations NCTM standards suggest (National Council of Teachers of Mathematics, 2000). However, teachers as practitioners did not seem to consider this idea seriously. For example, angle can be understood in three different ways: edge, interior space, and turn (Mitchelmore & White, 2000). Although there are three different approaches to teach the concept of angle, many textbooks still have only a traditional approach to teach the concept of angle which is "two rays with a common endpoint" (Keiser, 2000). Considering development of the concept of angles through various approaches in relation to contexts (Mitchelmore & White, 2000), this could affect students' understanding of the concept as well as their future learning of other mathematical ideas.

Revisions to the Framework for Non-Textual Elements

Through the iterative process of exploring aspects of non-textual elements, each aspect of non-textual elements was developed and articulated. For example, the broader definitions of validity and contextuality were reduced as mathematical accuracy and realistic contextuality. The definition of connectivity was changed into the relationship between non-textual elements and mathematical ideas in texts. Aesthetics was added to the framework as a factor of attracting students' attention. Variety was replaced by simplicity from the results of the interview analysis. Although the F-Framework as an initial analytic tool should be articulated more in future research, this study shows possible ways of using the F-Framework to analyze non-textual elements in mathematics textbooks across contexts.

Textbook Analyses

The validity of the F-Framework was tested through the textbook analysis to look for patterns of non-textual elements by textbook, topic, and country. As shown in this chapter, the F-Framework was useful to find patterns of non-textual elements. In general, the results from the textbook analysis show that accuracy, connectivity, and simplicity are significantly positively associated with each other while contextuality is negatively related to those aspects of non-textual elements across topics. In particular, accuracy and connectivity are strongly related to each other regardless of topics, textbook, and country. However, the negative relationship between contextuality and other aspects implies that more accurate, connected, and simpler representations are likely not to have realistic contexts, especially in South Korean textbooks. Considering the importance of understanding of mathematics within contexts, this calls attention to the development of non-textual elements with contextuality which possesses accuracy, connectivity, and simplicity as well.

For the variation of non-textual elements across topics, the results show that even though there are no significant differences of accuracy, connectivity, and simplicity across topics and countries, the scores for contextuality in the lessons about angle in the selected South Korean textbooks are significantly higher than those in the lessons about slope. Angle as a geometric topic is presumably taught within realistic contexts where students can find the connection between mathematics and their lives. However, slope is taught by either mathematical visual non-textual elements as an important mathematical concept to understand linear functions or contextual non-textual elements as understanding of steepness in real life. The results imply that the concept of slope is likely to be taught using abstract mathematical representations in South Korea while it is taught using contextual non-textual elements in U.S as many as the concept of angle is. Such patterns of non-textual elements are not only a matter of type of representations but also reflect opportunities to learn mathematics.

7.2 Limitations of the Study

There were some limitations of the study with respect to conceptualization of the framework, design of the interview protocols, procedure for coding, and size of the samples.

Although the inter-rater reliability is quite substantial overall, it was not as high in the scores of accuracy and simplicity. Although the definitions of accuracy and simplicity were evolved from the interviews with teachers and curriculum developers, it implies that the definitions of each aspect of non-textual elements should be more articulated and elaborated. In particular, accuracy for a non-textual element with realistic contexts should be considered as shown in Chapter 6. Since I did not show any example when the coders coded non-textual elements, it could be related to their interpretation of what extent of accuracy and simplicity could be acceptable for non-textual elements. For this case, the coding schemes should be articulated as well.

One limitation is the lack of independence between connectivity and contextuality. According to the idea of Realistic Mathematics Education, realistic contexts, abstract mathematical concepts, and representations should be linked. The results from my interviews also show that it is important to have mathematical connectivity for contextual representations. However, because I changed the definition of connectivity in the F-

Framework into the relationship between a non-textual element and mathematical ideas in texts, the distinction between connectivity and contextuality becomes hazy for contextual representations. My initial idea about connectivity was the relationship between a non-textual element and the text around it. However, due to language barriers, i.e., U.S. teachers could not read Korean textbooks and Korean teachers could not read U.S. ones, I omitted the surrounding text from the interview protocol. Rather took both U.S. and Korean textbooks to the interviews and talked about non-textual elements in textbooks with my oral translation, not with surrounding texts as they are, this could have affected their interpretation of connectivity. Further investigations should be done to articulate the definitions of connectivity and contextuality as well as to explore how connectivity is interpreted when non-textual elements accompany written translated texts.

Last, although I collected the data from South Korea and the United States for the interviews and textbooks, it is hard to generalize the patterns observed in this study due to the small sample size. This attempt was made to show an example of how the F-Framework could be used to analyze non-textual elements in different contexts. However, I found some interesting results from the analyses. It could be interesting if future research could be done with large sized data.

7.3 Conclusion and Discussion

Despite the several limitations of this study, it is beneficial for several important reasons. First, this study provides an analytic tool for examining non-textual elements in mathematics textbooks and other educational materials. Recently, increasing number of non-textual elements is included in mathematics textbooks to draw students' and teachers' attention. However, there was no framework and tool to analyze non-textual elements in mathematics textbooks. In fact, the National Mathematics Advisory Panel (2008) reported the scarcity of research on mathematics textbooks and inaccuracy of some recently published mathematics textbooks. Thus, although the F-Framework has some limitations, the development of F-Framework as an initial analytic tool is still meaningful. In order to know and improve non-textual elements in mathematics textbooks, the F-Framework could be a useful tool to diagnose the current status of non-textual elements and to improve the quality of non-textual elements in mathematics textbooks.

Second, this study provides information about the important aspects of non-textual elements in teaching and learning of mathematics based on theoretical and empirical data from research and practice. Despite the popularity of visual representations in teaching and learning of mathematics, little research has been conducted to find why and how they are chosen and used in textbooks and classrooms. In particular, even though both pictorial and mathematical visual representations coexist in mathematics textbooks, there was no research on how these two different types of representations support student learning and what kinds of roles each type of representations play in teaching and learning.

I argue that we should not overlook the important roles of teachers and curriculum developers as important agents mediating the relationship. Despite the importance of teachers' and curriculum developers' cognition of non-textual elements in mathematics textbooks for teaching and learning, little has been known about this issue. Noticeably, non-textual elements that are not mathematical representations in mathematics textbooks have not yet been explored even though they are important parts of textbooks. The results provide intriguing insights into teachers' and curriculum developers' thinking.

Third, even though realistic contexts are useful for students to learn mathematics, this cannot be an excuse that inaccurate, disconnected, and more complicated realistic contexts can be used in mathematics textbooks. Van den Heuvel-Panhuizen (2001) warns that only giving realistic contexts which do not have any mathematical connection does not mean that students can do "mathematization" (Freudenthal, 1968) in which students recreate mathematics by solving problems in real life. Realistic contexts could be meaningful only when students can do mathematics in the realistic contexts. It calls attention to the quality of non-textual elements with realistic contexts in current mathematics textbooks. Despite the nature of realistic contexts, it is necessary to deliberate what kind of realistic contexts would be the most appropriate for a given concept with accuracy, connectivity, and simplicity. In particular, accuracy and connectivity are the essential factors that teachers and curriculum developers pointed out. In order to maximize the effectiveness of non-textual elements, non-textual elements with realistic contexts should have at least both accuracy and connectivity. From this perspective, we need to develop non-textual elements with realistic contexts which are accurate, simplistic, and connected with the content in mathematical ways.

Furthermore, with regard to mathematical conceptualization, we should introduce both contextual and mathematical non-textual elements in teaching and learning. By approaching differently, textbooks offer different opportunities to learn of mathematical ideas. For example, whereas the South Korean textbooks sampled are likely to use abstract mathematical non-textual elements to explain the concept of slope, the U.S.

textbooks are likely to use contextual non-textual elements to explain it as many as those to explain the concept of angle. Such different approaches may provide students with different opportunities to conceptualize the concept in each country. As shown in Chapter 2, angle, slope, and prime factorization have several definitions and meanings in mathematics. In particular, slope can be explained as an algebraic idea that is rate of change or as a geometric idea that is steepness (Battista, 2007; Mitchelmore & White, 2000). It could be hard for students to understand the connection between the two different approaches as well as connection between the concept of slope and its application into the world (Wagener, 2009). While steepness of a hill can be recognized intuitively from our lives, rate of change in a graph may be hard for students to understand. Thus, the different way of using non-textual elements means not merely different types of non-textual elements. It is more related to mathematical concepts. Considering that students should understand the two different ideas together to learn advanced algebra and geometry, it would be important to give opportunities to learn both perspectives. Thus, it is important for teachers and curriculum developers to make sure that they provided students with various experiences to understand a mathematical concept.

Fourth, international comparative studies may allow us to see students' opportunities to learn in different countries as well as to find alternative ways beyond our local discourse and practice (Blömeke & Paine, 2008, p. 2036). Many comparative studies have found substantial gaps between Korean and U.S. students' performance (e.g., Mayer, Sims, & Tajika, 1995; Schmidt et al., 2001; Stigler, 1990; Stigler, Lee, Lucker, & Stevenson, 1982) as well as similar gaps between Korean and American teachers' mathematical content knowledge (Mathematics Teaching in the 21st Century (MT21), 2007). Opportunities to learn mathematics through non-textual elements could be one more explanation for the gaps. Even though the coders rated all of the textbooks, the mean scores of all the aspects in the South Korean textbooks are higher than those in the U.S. textbooks. However, while the South Korean textbooks do not differ in terms of accuracy, connectivity, and simplicity, there are significant variations among the sampled U.S. textbooks in terms of all the aspects but accuracy. This could be also related to opportunities to learn mathematics provided in mathematics classrooms. Also, by looking at different approaches of non-textual elements in mathematics textbooks across contexts, we can find alternative ways to use and improve the quality of non-textual elements in mathematics textbooks. Further investigations remain to be done.

However, since this study is an initial analytic attempt to find important aspects of non-textual elements, each aspect of non-textual elements should be scrutinized and elaborated with regard to both theory and practice. This study also should be extended and developed with larger data in terms of textbooks, topics, grade levels, teachers, curriculum developers, students, and contexts. It will give clearer and deeper understanding of non-textual elements in teaching and learning mathematics.

Also, although this study did not analyze coherence between how they understood non-textual elements and how non-textual elements were actually used in their textbooks, this would be interesting to see the patterns of aspects of non-textual elements in their textbooks using the F-Framework on which their opinions were reflected.

On the other hand, the three coders with different national backgrounds evaluated each non-textual element in both South Korean and U.S. mathematics textbooks. The inter-rater reliability among the coders appeared to be fairly high. However, when evaluating mathematical accuracy of a non-textual element with realistic contexts, they sometimes had different interpretations. Considering that it is hard to generalize the interpretations from only the three coders, we need further investigation about such challenge with larger data. Further, the results from this study show a possibility that there may be a "visual culture" which is how people recognize representations and see the world through representations in the context (Latour, 1990). In other words, the difference of the roles of non-textual elements in mathematics textbooks may reflect their socio-cultural values on what mathematics is and how mathematics is taught (Presmeg, 2007; Seah & Bishop, 2000). Although this study as an initial analytic attempt investigates a small number of data from South Korea and the United States, the common patterns among South Korean textbooks and among U.S. textbooks imply that there may be "visual culture" to learn mathematics. Different interpretations between the two coders who have different national background also pose a question if the way of understanding mathematics is affected by socio-cultural values. Since this study used only a small set of textbooks, topics, and countries, it is too early to answer the question. Further investigations with larger data could be done.

One more thing that I want to emphasize is that it is necessary to have intensive and deliberate dialogue about non-textual elements in mathematics textbooks between curriculum developers and teachers in order to enhance the effectiveness of non-textual elements on student learning. Even though teachers and curriculum developers agreed upon the important aspects of non-textual elements for teaching and learning, it seems that curriculum developers and teachers need to share ideas and continuing discussions from both theory and practice.

7.4 Implications

This study makes a unique contribution to the literature on non-textual elements in mathematics education and provides implications for curriculum development, educational policy, and teacher education and professional development.

The Non-Textual Elements Framework can be useful for both research and practice in mathematics education. Recently, in order to attract students' interests to mathematics textbooks, many textbooks in both countries have included more non-textual elements than before. However, the impact of such movement on students learning remains unknown. This framework can be used for researchers and policymakers to compare opportunities to learn through non-textual elements in various mathematics textbooks and contexts as well as to examine teachers' curriculum use in terms of non-textual elements in their classrooms. By doing so, researchers can establish theoretical and practical foundation for analyzing and developing non-textual elements in a wider range of research and practice across contexts. Policymakers may find more effective non-textual elements for teaching and learning as well as give systematic support for teachers and curriculum developers at policy level.

Teachers can use this framework to judge which non-textual elements could be the most effective for student learning as well as to develop useful non-textual elements for their classes to meet their students' needs. The framework can provide useful criteria to make instructional decisions about non-textual elements for their classrooms.

Teacher education programs and professional development are also able to provide pre-service and in-service teachers with the framework as "pedagogical tools to support teacher learning" (Putnam & Borko, 2000) to allow them to better understand the roles and characteristics of non-textual elements in mathematical teaching and learning and to experience various ways of using non-textual elements for teaching in schools. Considering non-textual elements as tools for thinking in mathematics, pre-service and in-service teachers should know how to use non-textual elements effectively for their classes as a part of their pedagogical content knowledge (Izsák & Sherin, 2003; Shulman, 2004).

For curriculum developers, the framework can be a tool and a standard to design and evaluate non-textual elements in their textbooks. Publishers and textbook authors need to consider the potential educational effects of non-textual elements on students' learning and be careful to use them in a productive way. Thus, this framework as a tool and a standard may contribute to curriculum development for effective teaching and learning.

It is necessary to improve non-textual elements in current mathematics textbooks. Even though this study use only a few textbooks from South Korea and the United States, the results from the textbook analysis show that some of the textbooks have many inaccurate and disconnected non-textual elements. As I mentioned in Chapter 5, accuracy and connectivity are the sine qua non that every non-textual element should possess to support student learning. However, some of the textbooks lack accuracy and connectivity in terms of non-textual elements. National Mathematics Advisory Panel (2008) also pointed out that this is a critical problem with current mathematics textbooks. Therefore, considering the importance of non-textual elements in teaching and learning of mathematics, we should examine non-textual elements in current mathematics textbooks and put more efforts to improve the quality of non-textual elements in mathematics textbooks.

Even though contextuality within a non-textual element itself can be easily recognized, it may be that mathematical accuracy within a contextual non-textual element can be differently interpreted according to the interpreter's socio-cultural background. I also conjecture that there are two more possible reasons to explain this. First, the difference in rating may not be a matter of what is mathematics but a matter of how to apply mathematics to real life and how mathematics in real life can be interpreted as a mathematical object. In fact, the three coders have much agreement upon mathematical accuracy in abstract visual representations. However, when they evaluated mathematical accuracy of a non-textual element with realistic contexts, they had different interpretations of mathematical accuracy. This suggests that even though all of the coders have strong mathematical knowledge and background, they may have different understanding of mathematical application in realistic contexts. A final alternative explanation related to the fact, as shown in Table 11, the South Korean textbooks studied have higher scores for accuracy and connectivity compared to the sampled U.S. textbooks. Thus, it could be possible that the rater B (American) may have less experience with accurate and mathematically connected non-textual elements through his K-12 school experiences than the rater C (Korean) may have. In fact, in a pilot study which analyzed five U.S. textbooks and 13 South Korean textbooks, the results show that the South Korean textbooks have more contextual non-textual elements *with* mathematical connection than those *without* such connection whereas the U.S. textbooks use more contextual non-textual elements *without* mathematical connection than those *with* such connection. Thus, my findings raises a question about that even though it is hard to generalize this with only the two coders, this phenomenon may be related to opportunities to learn mathematics provided in K-12 school mathematics classrooms.

Further investigation should be done about such challenges. In this study, since all the coders evaluated every non-textual element in both U.S. and Korean textbooks, such differences may not threaten the effort to compare the scores of each non-textual element. However, this raises important issues around evaluation of non-textual elements. Further systematic analyses remain to be done with regard to such socio-cultural challenges.

Further discussions and policy debates on how to develop and use non-textual elements in appropriate ways in textbooks and other educational materials should be followed. In addition, further systematic analyses from practice remain to be done such as how non-textual elements have been used historically in mathematics textbooks, how non-textual elements in mathematics textbooks are actually used in classrooms, and how non-textual elements have been used in other topics and textbooks.

APPENDIX A. Interview Protocol for Teachers

1. Introduction

1) Thank you very much for your participation in my study. My name is Rae-Young. This study is to explore how teachers (or textbook authors) recognize and interpret the roles of non-textual elements in secondary mathematics textbooks. I will give some tasks and ask some questions about your thoughts and opinions.

2) It will take 30 ~ 40 minutes in total.

3) Do you mind if I record and transcribe this interview and take some notes from your responses during this interview?

4) Ask the respondent to sign the consent form if (s)he agrees to participate in this interview

5) (Check recorder and batteries and see if they are working well)

2. Demographic information

1) What was your major in undergraduate/graduate?

2) How long have you been teaching in middle schools? Do you have any other teaching experience?

3) What grade levels have you taught?

4) Which courses (or subjects) have you taught? (algebra, geometry, calculus, statistics, etc.)

5) What is your main material for your instruction?

→ If the teacher uses textbooks,

① What kinds of textbook have you used for your class?

Example: Cognitive Tutor ______

Connected Mathematics Project (CMP); Prentice Hall _____

Heath Mathematics Connections; DC Heath & Co. ______

Math Advantage; Harcourt Brace ______

Math Alive; Visual Mathematics ______

Mathematics; Applications & Connections; Glencoe/McGraw Hill ______

Mathematics in Context (MiC); Holt, Rinehart & Winston _____

Mathematics Plus; Harcourt Brace & Co._____

MathScape; Seeing and Thinking Mathematically; Glencoe/McGraw Hill ____

MathThematics (STEM); McDougal Littell _____

Middle Grades Math; Prentice Hall ______

Middle School Math; Scott Foresman/Adison Wesley _____

Middle School Mathematics Through Applications Project (MMAP); Unpublished _____

Passport Series; McDougal-Littell _____

Pre-Algebra: An Integrated Transition to Algebra and Geometry; Glencoe/McGraw Hill _____

Saxon Math ______

University of Chicago School Mathematics Project Integrated Mathematics, Grades 7-12; Prentice Hall _______

Other (please specify) ____

② How often do you use textbooks in your class?

③ Do you have any other materials you use in your class?

→ If the teacher does not use textbooks, ① What are the resources for your own materials?

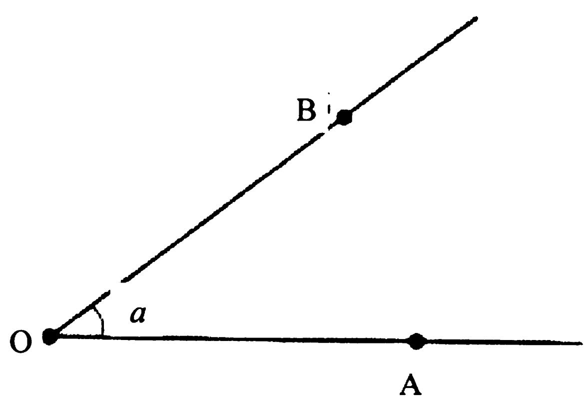
② Why don't you use textbooks?

♣ In tasks 1, 2, and 3, I'm going to show you several representations from mathematics textbooks for three different topics: angles, slopes and prime factorizations. Then, I will ask your opinions about these representations.

Task 1

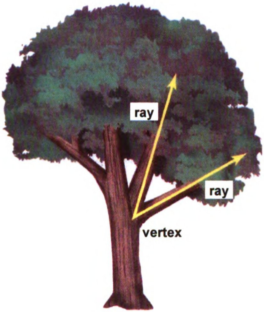
Following are three representations used in mathematics textbooks to introduce the concept of *angle*.

(1)

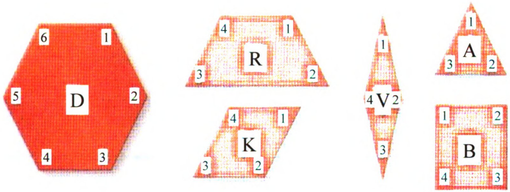


Task 1 (cont'd)

(2)



(3)



a. Which one do you think is more useful for students' understanding of the concept of angle? (Which one do you think is more effective to teach the concept?) Why?

b. Which one explains or illustrates more clearly the concept of angle in mathematical ways? Why?

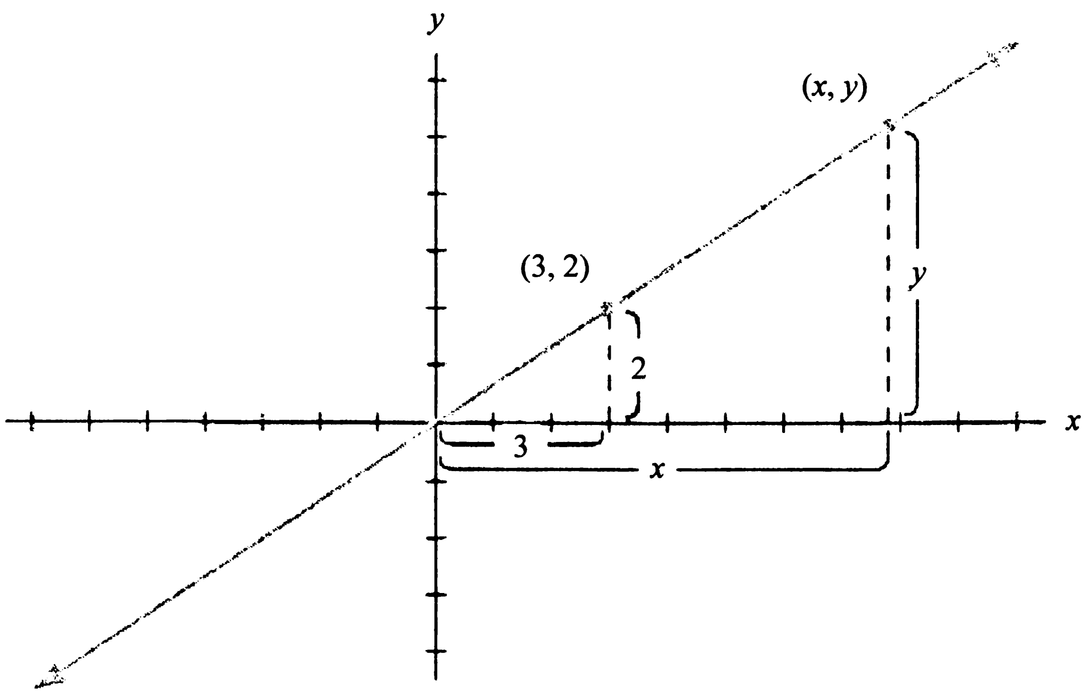
c. What difference or similarity do you see among the roles of representations between (1), (2), and (3)?

d. Do you want to modify any of them or use different representations to teach the concept of angle? If so, please give me some examples.

Task 2

These representations are used to explain the concept of *slope* in mathematics textbooks.

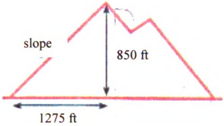
(1)



Task 2 (cont'd)

(2)

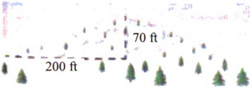
Snow safety experts advise that avalanches are more likely to happen on a hill with a slope of between 0.5 and 1. Is an avalanche likely for Hotdog Hill?



$$\frac{rise}{run} = \frac{850}{1275}$$

Dividing to find the slope, 850 $\boxed{\div}$ 1275 $\boxed{=}$ 0.66666667. The slope is between 0.5 and 1, so an avalanche may be likely.

Try it



Determine the slope. Is an avalanche likely?

a. Which one do you think is more useful for students' understanding of the concept of slope? (Which one do you think is more effective to teach the concept?) Why?

b. Which one explains or illustrates more clearly the concept of slope in mathematical ways? Why?

c. What difference or similarity do you see among the roles of representations between (1) and (2)?

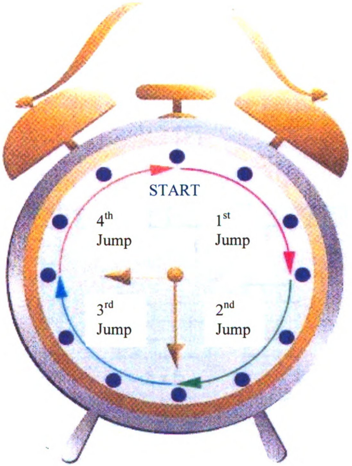
d. Do you want to modify any of them or use different representations to teach the concept of slope? If so, please give me some examples.

e. The picture at the bottom shows a skiing resort at the mountain. How do you think about this picture? Do you think it is useful to learn the concept of slopes or solve this problem? Why?

Task 3.

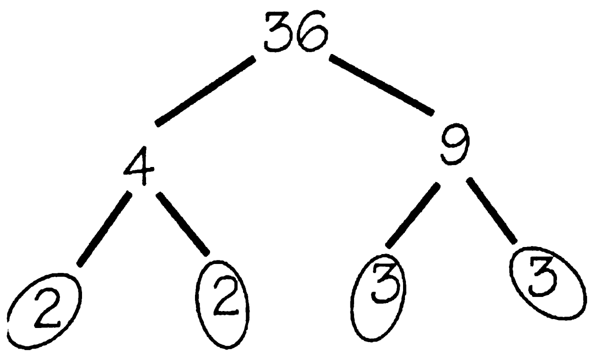
These representations are used to explain *prime factorization*.

(1)

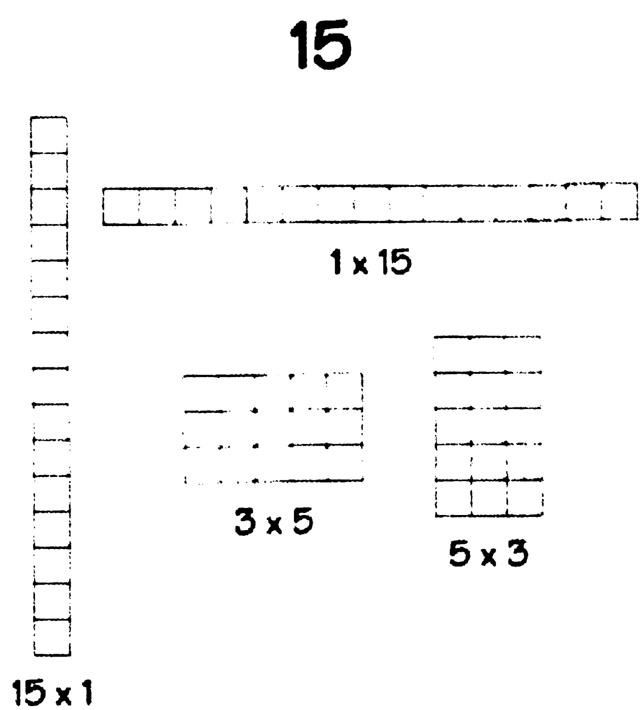


Task 3 (cont'd)

(2)



(3)



a. Which one do you think is most useful for students' understanding of the concept of prime factorization? (Which one do you think is more effective to teach the concept?) Why?

b. Which one explains or illustrates more clearly the concept of prime factorization in mathematical ways? Why?

c. What difference or similarity do you see among the roles of representations between (1), (2), and (3)?

d. Do you want to modify any of them or use different representations to teach the concept of prime factorization? If so, please give me some examples.

♣ In tasks 4, I will ask some questions about the three sets of representations in tasks 1, 2, and 3.

Task 4.

a. Let's look at the representations in Tasks 1 (angles), 2 (slopes), and 3 (prime factorizations) again. Can you describe any difference or similarity among these representations in terms of the roles of representations?

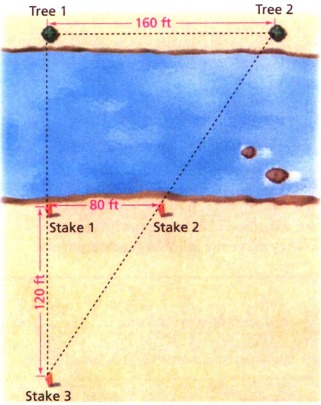
b. According to the topics, have you used visual representations differently in your class? If so, why? If not, why?

♣ In tasks 5 and 6, I'm going to show you several representations used in mathematics textbooks to solve problems. Then, I will ask your opinions about these representations.

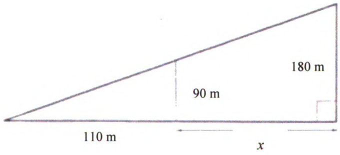
Task 5.

The following pictures are used to solve a problem using the concept of *similarity* in geometry textbooks.

(1) Find the distance between Stake 1 and Tree 1.



(2) What is the value for x?



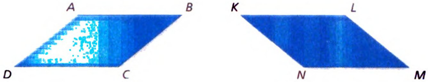
a. Which one do you think is more helpful for students to solve the problem stated in the box? Why?

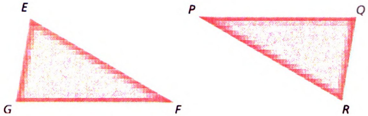
b. Do you see any difference or similarity among them in terms of the roles of representations?

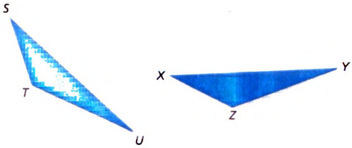
Task 6.

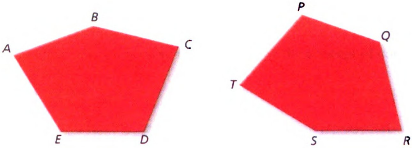
The following TWO pictures are used to solve problems about finding corresponding sides of congruent figures in geometry textbooks. Answer the following questions.

(1) Match each side and angle of the first shape in Exercises 1-4 with its congruent partner in the second shape.



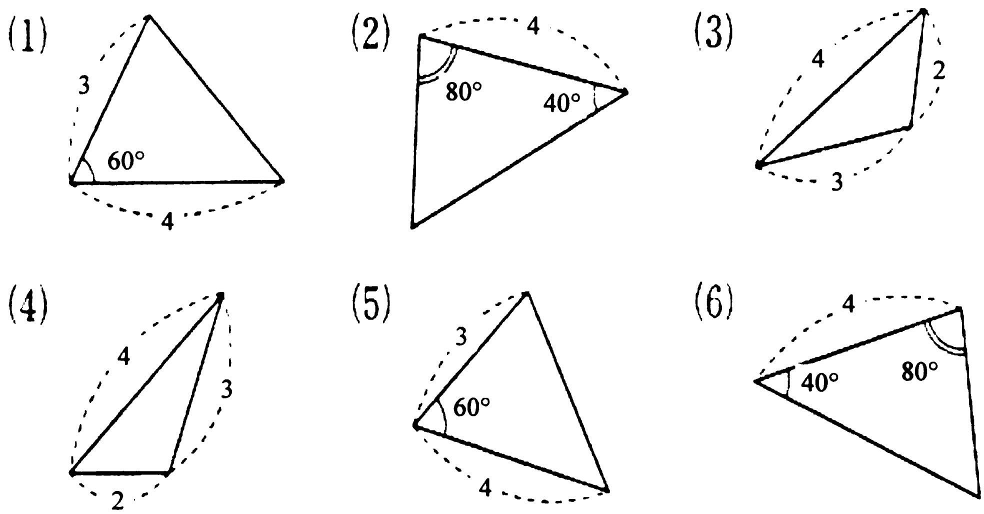






Task 6 (cont'd)

(2) Find congruent triangles in (1) ~ (6) and show why they are congruent.



a. Which representation is more helpful to understand the concept of congruence? (Which one do you think is more effective to teach the concept?) Why?

b. Which set of figures do you think provides students with more various mathematical experiences for their learning?

c. Do you see any difference or similarity between them in terms of the roles of representations?

d. Which representation is more helpful to apply the concept of congruence in various contexts?

e. Do you want to modify any of them or use different representations to teach the concept? Please give me some examples.

Task 7. I'm going to ask your opinions about the use of visual representations in teaching.

a. How often do you use visual representations in your classroom? How do you usually use visual representations?

b. Could you tell me your opinions about the usefulness of visual representations compared to symbols and textual information in mathematics textbooks?

c. Do you think we need more visual representations in textbooks than we have now? Why?

APPENDIX B. Interview Protocol for Curriculum Developers

1. Introduction

1) My name is Rae-Young Kim, and my study explores how textbook authors consider the roles of non-textual elements in secondary mathematics textbooks. I would like to listen to your opinions and experiences about the use of non-textual elements in your textbooks.
2) It will take about 30 minutes in total.
3) Do you mind if I record and transcribe this interview and take some notes from your responses during this interview?
4) Please sign the consent form if you agree to participate in this interview.
5) (Check recorder and batteries and see if they are working well)

2. Curriculum development

1) First, I would like to ask about how the content in your textbook was selected and who made decisions.
   - Who made decisions of which content should be in the textbooks?
   - Who made decisions on selecting non-textual elements (photos, illustrations, or mathematical figures) in textbooks? Who decides the number and type of non-textual elements in textbooks?
   - What is textbook authors' and publishers' responsibility? How did you collaborate to publish your textbooks?
   - What were you mostly concerned about when you designed your textbooks?

- Do you have any theoretical framework about how students learn mathematics thus the framework informs the design in your textbooks?
- What do you think distinguishes your textbook from others? Did that inform you decisions about what would be in the textbook and how it would be designed or is this something you have noticed once the book was completed?

2) Non-textual Elements (Visual representations)
   - What were you mostly concerned about when you thought about non-textual elements in your textbooks?
   - What kinds of roles do you think non-textual elements in textbooks play for teaching and learning?
   - What is your expectation for teachers and students with regard to their use of non-textual elements in textbooks?
   - How did you select non-textual elements for your textbooks?
   - What criteria did you have to select them? What kinds of resources did you use?
   - In selecting non-textual elements, what are the roles or responsibilities of publishers?
   - Recently, more pictures and figures are included in mathematics textbooks. How do you think about this trend?
   - Could you tell me your opinions about the usefulness of visual representations compared to symbols and textual information in

mathematics textbooks? Could you give me an example from your textbook or another text?

3. In your textbooks…

1) I would like to talk about non-textual elements in your textbook. What aspects of visual representations in that can be helpful to attract students' attention to the content in textbooks?
2) What aspects of visual representations should be considered to support students' mathematical learning in textbooks?
3) How do you think about the roles of decorative representations in textbooks? What about in your book? Do you think such representations are helpful for students to understand the concept or solve the problem? Why?
4) How about representations that used to give instruction for students' behaviors? For example, when you introduce "making a table" in one of your textbooks, you may put the picture of two students who are comparing their sheets. I think that it seems to encourage students to compare their work with their friends. Do you think such representation is helpful to support students' learning?

(Showing some representations from my protocol)

Here are some different types of visual representations used in mathematics textbooks to introduce the concept of *angle*.

Tasks 1, 2, and 3 from the interview protocol for teachers are used in the same way. In addition, I will ask, "Which one is the best example from your textbook to illustrate or address the concept?"

4. Future works

1) What kinds of non-textual elements do you think should be added to future mathematics textbooks? How do you want to modify current non-textual elements in future textbooks? (instructional effectiveness) Why?
2) I'm working on developing a conceptual framework to understand non-textual elements in mathematics textbooks. Do you have something you want me to consider?
3) Do you have any comments or suggestions for my study?

5. Demographic information

6) What was your major at undergraduate and graduate levels?
7) What kinds of jobs have you had regarding curriculum development?
8) Do you have any teaching experience in K-12 schools?

--- If so, how long did you teach?

What grade levels have you taught?

Which courses (or subjects) did you teach? (algebra, geometry, calculus, statistics, etc.)

What material(s) did you use for your class?

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