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Attaullah Y. Memon

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OUTPUT REGULATION OF NONLINEAR SYSTEMS USING CONDITIONAL SERVOCOMPENSATORS

By

Attaullah Y. Memon

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ABSTRACT

OUTPUT REGULATION OF NONLINEAR SYSTEMS USING CONDITIONAL SERVOCOMPENSATORS

By

Attaullah Y. Memon

The design of output feedback controllers that solve the output regulation problem for nonlinear systems is considered, with emphasis on improving their transient performance. We concentrate on the output regulation of nonlinear systems using *conditional servocompensators*, an idea that was introduced in the earlier work of Khalil and co-researchers. The conditional servocompensator acts as a traditional servocompensator in a neighborhood of the zero-error manifold while acting as a stable system otherwise, leading to improvement in the transient response while achieving zero steady-state regulation error. The idea was introduced in a sliding mode control framework. We extend the technique of conditional servocompensators to more general feedback controllers by using *Lyapunov redesign* and *saturated high-gain feedback*.

The striking feature of our approach is the flexibility of starting with any stabilizing state feedback controller and then including a conditional servocompensator to achieve zero steady-state regulation error without degrading the transient performnace. We give regional as well as semi-global analytical results for error convergence. The flexibility offered by our design approach allows us to consider as a special case, the output regulation problem for control-constrained linear systems. Such a design approach, however, assumes the availability of all state variables to meet the control objectives, and the output feedback control can only be implemented using a full-order observer. An extension of methodology to nonlinear systems would necessitate the development of a full-order nonlinear observer. We develop a full-order high-gain observer for nonlinear systems, extending earlier work on reduced-order high-gain observers for nonlinear systems and full-order observers for linear systems.

To my father... and his fond memories!

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Chapter 1

Introduction

This dissertation concentrates on the design of feedback controllers that solve the output regulation problem for a class of minimum-phase nonlinear systems, with emphasis on improving the transient performance. For this class of systems, robust continuous feedback control techniques like Lyapunov redesign (min-max control) or sliding mode control can be used to ensure convergence of the tracking error to a small ball around the origin of the closed-loop system, while rejecting bounded disturbances. However, rendering the tracking error arbitrarily small requires the use of high-gain feedback near the origin, which is traditionally achieved by using the classical idea of including a *servocompensator* with the *stabilizing compensator*.

We concentrate on the output regulation of nonlinear systems using *conditional servocompensators*, an idea that was introduced in the earlier work [52, 53, 54] of Khalil and co-researchers. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state stable system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators enables us to achieve zero steady-state tracking error without degrading the transient response of the system. The idea was introduced in [52] and [53] in a sliding mode control framework, where [52] dealt with the special case of conditional integrator for constant exogenous signals, while the more general case of time-varying signals was treated in [53]. We focus on extending the technique of *conditional servocompensators* [53] to more general feedback controllers by using Lyapunov redesign and saturated high-gain feedback.

The flexibility offered by the Lyapunov redesign framework of starting with any stabilizing state feedback controller and then including a conditional servocompensator, allows us to consider as a special case, the output regulation problem for linear systems subject to control constraints. The presence of saturation in the input channel imposes strong limitations to the achievable control objectives such as transient performance. Because of the constraint on the control, the mechanism of solving the stabilization problem through Algebraic Riccati Equation (ARE) necessitates the use of full-state feedback. The sliding mode control design of [53] uses partial state feedback, and therefore, can not be used to address this problem. We exploit the two-time-scale approach to the observer design described in [13] to design an output feedback control to achieve regulation in the presence of input constraints. Naturally, the first thing that comes to mind is to extending the methodology to nonlinear systems. However, such a design approach assumes the availability of all state variables in order to meet the control objectives. Consequently, the output feedback control will be implemented using a full-order nonlinear high-gain observer. This motivates us to develop a full-order high-gain observer for minimum-phase nonlinear systems.

In Section 1.1, we briefly review some of the main background elements of this dissertation. These include the problem of nonlinear output regulation, Lyapunov redesign, and output feedback using high-gain observers. In Sections 1.2 and 1.3, we provide the background and evolution of the problem of output regulation for nonlinear systems and the motivation of this work. Finally, Section 1.4 gives an overview of this dissertation.

1.1 Preliminaries

1.1.1 The Nonlinear Output Regulation Problem

One of the most fundamental problems in control theory is the output regulation problem, alternatively known as the servomechanism problem, which deals with the design of a feedback controller for a fixed plant that yields a prescribed steady-state response to every external command in a given family of functions. This includes the problem of rendering the output y of a fixed plant to asymptotically track a reference signal r in a certain class of functions, as well as the problem of asymptotically rejecting a disturbance signal d in a certain class of disturbances. The objective is to force the difference between the reference input r and the actual output y decay to zero as time tends to infinity, for every reference input r and every disturbance signal d over prespecified families of functions.

Consider a nonlinear system modeled by equations

$$\dot{x} = f(x, w, u) \tag{1.1}$$

$$e = h(x, w) \tag{1.2}$$

where $x \in R^n$ is the *state* of the plant, $u \in R^m$ is the *control input*, $w \in R^r$ represents the *exogenous input* variables which includes refrences to be tracked and/or disturbances to be rejected, and $e \in R^m$ denotes an *error* variable expressed as a function of the state x and of the exogenous input w. It is assumed that the functions $f(x, w, u)$ and $h(x, w)$ are sufficiently smooth, $f(0, 0, 0) = 0$, and $h(0, 0) = 0$. The exogenous input signals $w(t)$ are assumed to be the solutions of a homogenous differential equation

$$\dot{w} = Sw \tag{1.3}$$

where S has distinct eigenvalues on the imaginary axis. This mathematical model that generates all possible exogenous input functions is called the *exosystem.* Suppose there exists a *feedback controller* which can process the information available from the plant, and whose output u is a function of x and w, given by

$$u = \gamma(x, w) \tag{1.4}$$

The interconnection of (1.1)-(1.2) and (1.4) yields a closed-loop system characterized by the equations

$$\begin{aligned} \dot{w} &= Sw & (1.5) \\ \dot{x} &= f(x, w, \gamma(x, w)) & (1.6) \end{aligned}$$

The closed-loop system (1.5)-(1.6) is said to have the *property of output regulation* if the control u can be designed such that for every exogenous input w (in a prescribed family of signals) and for every initial state in some neighborhood of the origin, the output e decays to zero as time tends to infinity.

The structure of the feedback controller usually depends on the amount of information available for feedback. If all the components of the state x of the plant and the state w of the exosystem are available, then the control law takes the form of equation (1.4), which is a memoryless system. A more realistic situation is when only the error e is available for measurement. In this case, the control signal u is synthesized by means of a dynamical nonlinear system

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e) & (1.7) \\ u &= \theta(\xi) & (1.8) \end{aligned}$$

with internal state ξ. The interconnection of (1.1)-(1.2) and (1.7)-(1.8) yields a closed-loop system characterized by the equations

$$
\begin{aligned}
\dot{w} &= Sw && (1.9)\\
\dot{x} &= f(x, w, \theta(\xi)) && (1.10)\\
\dot{\xi} &= \eta(\xi, h(x, w)) && (1.11)
\end{aligned}
$$

The above formulation is known as *error feedback output regulation problem.* Together with the notion of *immersion*, it is shown in [24] that the error feedback output regulation problem is solvable if there exists a continuously differentiable mapping $x = \pi(w)$, with $\pi(0) = 0$, and a continuous mapping $\chi(w)$, with $\chi(0) = 0$, that solve the *nonlinear regulator equations*

$$
\begin{aligned}
\frac{\partial \pi(w)}{\partial w} Sw &= f(\pi(w), w, \chi(w)) && (1.12)\\
0 &= h(\pi(w), w) && (1.13)
\end{aligned}
$$

In other words, the above condition means that the error feedback output regulation problem is solvable if it is possible to find a mapping $\chi(w)$ that renders the identities of equations (1.12)-(1.13) satisfied for some $\pi(w)$, and is such that control input generated by the autonomous system

$$
\begin{aligned}
\dot{w} &= Sw && (1.14)\\
u &= \chi(w) && (1.15)
\end{aligned}
$$

is precisely the control u required to achieve zero steady-state error, in the presence of any exogenous input w.

1.1.2 Lyapunov Redesign [29]

The term Lyapunov redesign refers to a nonlinear control technique where a stabilizing state feedback control can be constructed with knowledge of the Lyapunov function of a nominal system, resulting in a control design which is robust to large matched uncertainties. Consider the system

$$\dot{x} = f(t,x) + G(t,x)[u + \delta(t,x,u)] \tag{1.16}$$

where $x \in R^n$ is the state and $u \in R^p$ is the control input. Assume that the functions f, G, and δ are piecewise continuous in t and locally Lipschitz in x and u. The functions f and G are known precisely, while the function δ is an unknown function which satisfies the matching condition and represents various uncertain terms pertaining to model simplification, parameter uncertainty etc. A nominal model of the system can be taken as

$$\dot{x} = f(t,x) + G(t,x)u \tag{1.17}$$

Suppose we can design a stabilizing state feedback control law $u = \psi(t,x)$ such that the origin of the nominal closed-loop system

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x) \tag{1.18}$$

is uniformly asymptotically stable. Suppose further that we know a continuously differentiable Lyapunov function $V(t,x)$ for (1.18) that satisfies the inequalities

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{1.19}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t,x) + G(t,x)\psi(t,x)] \leq \alpha_3(\|x\|) \tag{1.20}$$

where α_1, α_2, and α_3 are class $\mathcal{K}$ functions. Now assume that, with $u = \psi(t,x) + v$, the uncertain term δ satisfies the inequality

$$\|\delta(t,x,\psi(t,x)) + v\| \leq \rho(t,x) + \kappa_0\|v\|, \quad 0 \leq \kappa_0 < 1 \tag{1.21}$$

where ρ is a nonnegative continuous function, which represents a measure of the size of the uncertainty. It is shown in [29] that with the knowledge of the Lyapunov function $V(t,x)$, the function ρ, and the constant κ_0 in (1.21), the additional feedback control component v can be designed such that the overall control $u = \psi(t,x) + v$ stabilizes the actual system (1.16) in the presence of the uncertainty δ. The design of v is called *Lyapunov redesign.*

1.1.3 High-Gain Observers

High-Gain observers provide an important technique for the design of output feedback controllers for minimum-phase nonlinear systems. A high-gain observer is essentially an approximate differentiator that robustly estimates the derivatives of the output. The gain of a high-gain observer depends on a small parameter ϵ, which can be adjusted to guarantee that the estimation error decays to an $O(\epsilon)$ value arbitrarily fast. High-gain observers are applicable to a class of nonlinear systems that can be transformed into the *normal form*

$$\dot{z} = \psi(z,x) \tag{1.22}$$

$$\dot{x} = Ax + B\phi(z,x,u) \tag{1.23}$$

$$y = Cx \tag{1.24}$$

where $z \in R^l$ and $x \in R^r$ are the system states, $u \in R$ is the control input and $y \in R$ is the measured output. The $r \times r$ matrix A, the $r \times 1$ matrix B and the $1 \times r$ matrix

C, are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Over the past several years, many researchers have contributed toward the investigation of output feedback control for the class of systems of the form (5.1)-(5.3). Of significant relevance are the works [2, 3, 13, 29], which solve the problem of robust output feedback stabilization, in the large, of the input-output linearizable nonlinear dynamic systems by means of bounded partial-state feedback control (e.g. $u = \gamma(x)$) and high-gain observer, with subsequent substitution of the estimate of x, provided by the high-gain observer, in the feedback. The boundedness of the control protects the state of the plant from *peaking* when the high-gain observer estimates are used instead of the true states. As a result, the output feedback control recovers the performance under the state feedback control. One of the important consequences of this technique is the ability to separate the design of output feedback control for nonlinear systems into a state feedback design followed by the design of the high-gain observer. Teel and Praly [57] developed a generic separation principle, which showed that (semi)global stabilizability via state feedback plus uniform observability imply semiglobal stabilizability via output feedback. Atassi and Khalil [2] provided a more comprehensive separation principle and showed that the output feedback controller recovers the performance of the state feedback controller in the sense of recovering asymptotic stability of an equilibrium point, its region of attraction, and its trajectories. An exhaustive survey of the use of high-gain observers in nonlinear control can be found in [31, 34]. A review of the many approaches to design observers for stabilization of nonlinear dynamical systems appears in [16].

1.2 Background and Motivation

The output regulation problem was first studied for multivariable linear systems under various names, such as the robust servomechanism problem by Davison [11] and the structurally stable output regulation problem by Francis and Wonham [14]. The solvability conditions for the output regulation problem were worked out either in terms of the solvability of a system of two linear matrix equations called the *regulator equations* or in terms of the location of the transmission zeros of a composite system that incorporates the plant and the exosystem. A salient result of the theory is the observation that any structurally stable (robust) controller which solves the regulation problem can always be viewed as the interconnection of two components; a servocompensator and a stabilizing compensator. The servocompensator is a device that incorporates an *internal model* of the exosystem, a model capable of generating the reference and disturbance signals produced by the exosystem. The role of the stabilizing compensator is to stabilize the augmented system comprising of the plant and the servocompensator. A general setup is shown in Figure 1.1. The above mentioned property is known as the *internal model principle*, which reduces in the special case of constant references and disturbances to the classical integral control. The significance of the internal model principle is that it enables formulation of the output regulation problem into the well-known stabilization problem for an augmented linear system formed of the plant and the servocompensator.

Over the past two decades many researchers have contributed toward the investigation of the nonlinear output regulation problem, among whom we specifically mention Isidori and co-researchers [4, 5, 6, 7, 8, 25, 26, 45, 46, 49, 50, 51], Huang and

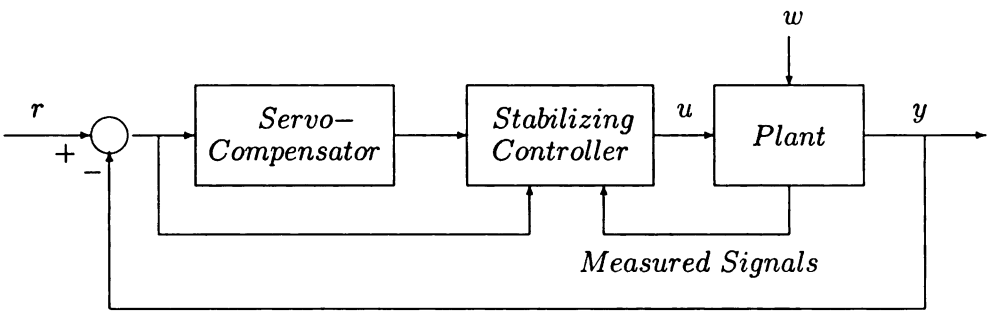


Figure 1.1: A general setup for the solution of output regulation problem

co-researchers [9, 10, 20, 21, 22, 23, 36] and Khalil and co-researchers [30, 32, 40, 53]. The pioneering work of Isidori and Byrnes [26] showed how the results of Francis and Wonham [14] could be extended to nonlinear plants and nonlinear neutrally stable exosystems with their formulation of the nonlinear regulator equations. The results in [26] were local and required smallness of both the exogenous signals and the initial states. Huang and Rugh [22] pursued a computational approach involving a power series expansion of the solution of the nonlinear regulator equations. Their method allowed for large exogenous signals, but was still local in terms of the initial states. Regional and semi-global results first appeared, for the case of fully linearizable systems, in the work of Khalil [30]. An important contribution of [30] was the observation that in the nonlinear case the internal model must be able to generate not only the trajectories of the exosystem, but also a number of their higher-order harmonics. This idea was also elaborated independantly by Huang and Lin [23] and Priscoli [44]. Together with the idea of *immersion*, Isidori [24] provided a complete set of necessary and sufficient conditions for the existence of a solution to the local output regulation problem. An interesting generalization of the results of [30], in terms of the structurally stable output regulation approach of [24] and the robust control approach of [30], was presented by Isidori [25]. A good account of the available results for the nonlinear output regulation problem can be found in [7, 20, 24, 27]. A simplification of the robust servomechanism design of [40] was presented in [32], where the only precise information that is required in the design of the controller is the relative degree of the plant, the sign of its high-frequency gain and the linear internal model. Some results that relax the assumption of input-to-state stability of the zero dynamics and allow for the frequencies of the exosystem to be unknown, thereby making use of an adaptive internal model, can be found in [50, 51]. A result that relaxes the assumption that the solution of the regulator equations be a polynomial in the exogenous signals can be found in [10]. Recent works have focused on the identification of design procedures yielding *nonlinear internal models*. In this respect, [9] focuses on an internal model that is constituted by a linear system having a nonlinear output map, whereas [5, 45, 46] focus on the design of nonlinear internal models having nonlinear observability forms. A more recent result [41] considers the practical nonlinear output regulation with respect to the dimension of the internal model and to the gain of the stabilizing compensator near the zero-error manifold.

1.3 From the *Conventional* to the *Conditional* Servocompensators

The classical idea of the internal model principle design has been used by Khalil and co-researchers in [30, 32, 33, 39, 40] to achieve asymptotic output regulation for a class of nonlinear systems. For the case of constant references and disturbances [33, 39], the servocompensator is simply an integrator driven by the tracking error and its inclusion creates an equilibrium point at which the tracking error is zero. For the more general case [30, 32, 40], a linear internal model is identified which generates the trajectories of the exosystem and, along with them, a number of higher-order harmonics generated by the system nonlinearities. This is then used to synthesize a servocompensator, the inclusion of which creates an invariant manifold on which the regulation error is zero. In order to achieve nonlocal stabilization of the disturbance-dependant equilibrium point or zero-error manifold, the stabilizing compensator is designed using robust continuous feedback control techniques like min-max control or sliding mode control (SMC). The error feedback controller is designed using the separation approach of Esfandiari and Khalil [13], where a state feedback control is designed first and then a saturated high-gain observer is used to recover the performance of the state feedback design.

While the above described designs achieve robust output regulation, the steady-state performance achieved often happens at the expense of degradation of the transient performance. This is due in part to the increase in system order as a result of servocompensator, and in part to the interaction of the servocompensator with the control saturation. To address the issue of transient performance degradation in the *conventional* integrator and servocompensator designs of [30, 32, 33, 39, 40], the idea of *conditional* integrators and servocompensators was introduced by Seshagiri and Khalil [52, 53], in a sliding mode control framework. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators makes it possible to achieve zero steady-state tracking error without degrading the transient response of the system. The special case of conditional integrator for constant exogenous signals was addressed in [52], while the more general case of time-varying signals was treated in [53]. To extend the design beyond the sliding mode control, [54] developed the conditional integrator using Lyapunov redesign and saturated high-gain feedback. Starting with any stabilizing state feedback controller, [54] shows how to include a conditional integrator by modifying the original controller.

1.4 An overview of the Dissertation

In this dissertation, we focus on the design of controllers that solve the output regulation problem for a class of minimum-phase nonlinear systems with emphasis on improving the transient performance. We extend the technique of conditional servocompensators [53] to more general feedback controllers by using Lyapunov redesign and saturated high-gain feedback.

In Chapter 2, we consider the problem of state feedback regulation of nonlinear systems using conditional servocompensators. We use the Lyapunov redesign and saturated high-gain feedback approach to design the stabilizing compensator, and include a conditional servocompensator by modifying the original controller that yields asymptotic error regulation without degrading the transient performance. We prove that the trajectories of the closed-loop system under saturated high-gain feedback control with a conditional servocompensator approach those of a closed-loop system under saturated high-gain feedback control without a servocompensator. We provide analytical results for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally.

In Chapter 3, we consider the output regulation problem for a class of minimum-phase input-output linearizable nonlinear systems. The state feedback control design of Chapter 2 is specialized to partial state feedback control design. This partial state feedback controller can be viewed as an intermediate step towards the output feedback controller of Chapter 3, which is implemented using a reduced-order high-gain observer. We also prove that the output feedback controller with conditional servocompensator recovers the performance of a state feedback controller that does not include any servocompensator.

In Chapter 4, we consider the output regulation problem of linear systems subject to control constraints. We apply the Lyapunov-redesign-servocompensator approach of Chapter 2 to the linear output regulation problem under input constraints. The output feedback control is implemented using a two-time-scale full-order observer design of [13] and the performance recovery is shown using the separation principle of [1, 3].

In Chapter 5, we study the problem of state estimation of a minimum-phase nonlinear system using a full-order high-gain observer. The observer comprises two components, a slow open-loop observer that estimates the state of the internal dynamics, and a fast observer that estimates the state of the external dynamics. The observer design approach is based on the two-time scale observer design of Esafandiari and Khalil [13], and the performance recovery is shown using the separation principle of Atassi and Khalil [2].

Chapter 2

Regulation of Minimum-Phase Nonlinear Systems Using Conditional Servocompensators

2.1 Introduction

We consider the problem of state feedback regulation of nonlinear systems using *conditional servocompensators*, an idea that was introduced in an earlier work [52, 53, 54] of Khalil and co-researchers. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state stable system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators enables us to achieve zero steady-state tracking error without degrading the transient response of the system. The idea was introduced in [52] and [53] in a sliding mode control framework, where [52] dealt with the special case of conditional integrator for constant exogenous signals, while the more general case of time-varying signals was treated in [53]. To extend the design beyond the sliding mode control, [54] developed the conditional integrator using Lyapunov redesign and saturated high-gain feedback. Starting with any stabilizing state feedback controller, [54] shows how to include a conditional integrator by modifying the original controller. We aim to extend the development of [54] by addressing the servomechanism problem for time-varying exogenous signals.

2.2 System Description and Assumptions

Consider the nonlinear system

$$\begin{aligned} \dot{\zeta} &= \tilde{f}(\zeta, w) + \tilde{G}(\zeta, w)u \\ e &= \tilde{h}(\zeta, w) \end{aligned} \tag{2.1}$$

where $\zeta \in R^n$ is the state and $u \in R^m$ is the control input. The plant is subjected to a set of *exogenous* input variables w that belong to a compact set $\mathcal{W} \in R^w$, which include unknown disturbances to be rejected and references to be tracked. The variable $e \in R^p$ denotes the regulation error. The functions $\tilde{f}$, $\tilde{G}$ and $\tilde{h}$ are sufficiently smooth in ζ on a domain $\Xi \subset R^n$ and are continuous in w for $w \in \mathcal{W}$. Our goal is to design a controller to asymptotically regulate e to zero.

Assumption 2.1. *$w(t)$ is generated by the known exosystem*

$$\dot{w} = S_0 w \tag{2.2}$$

where S_0 has distinct eigenvalues on the imaginary axis.

The requirement that S_0 has distinct eigenvalues on the imaginary axis implies that $w(t)$ is bounded and *persistent* in time, i.e. $w(t) \nrightarrow 0$ as $t \to \infty$. In the context of output regulation problem it is particularly meaningful when the exogenous signals are persistent in time, as it is in the case of any periodic (and bounded) function. In these cases, the system is expected to exhibit a steady-state response that is itself a persistent function of time, and whose characteristics depend on the specific input imposed on the system rather than the state in which the system was at the initial time. For example, the following choice of S_0 can generate bounded sinusoidal signals w_1 and w_2 of frequency ω and a constant signal w_3

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

We only require S_0 to be known which is equivalent to precisely knowing the frequencies of the sinusoidal signals generated by the exosystem (2.2). The amplitude and phase of the sinusoidal signals are allowed to be unknown since we do not require the initial conditions $w(0)$ to be known. For the case where the frequecies of the exosystem are unknown, an alternate design, that makes use of an adaptive internal model whose *natural frequencies* are automatically tuned to match those of the unknown exosystem, can be found in [51].

Assumption 2.2. *There exist a continuously differentiable mapping $\zeta = \pi(w)$, with $\pi(0) = 0$, and a continuous mapping $\chi(w)$ that solve the equations*

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} S_0 w &= \tilde{f}(\pi, w) + \tilde{G}(\pi, w)\chi(w) \\ 0 &= \tilde{h}(\pi, w) \end{aligned} \tag{2.3}$$

for all $w \in \mathcal{W}$.

The above assumption states a necessary and sufficient condition for the solution of the output regulation problem. It essentially means that $\zeta = \pi(w)$ is a zero-error invariant manifold and $\chi(w)$ is the steady-state control that maintains the motion on this manifold, in the presence of any exogenous input w.

Assumption 2.3. *There exists a set of real numbers $c_0, ..., c_{q-1}$ such that $\chi(w)$ satisfies the identity*

$$L_s^q \chi = c_0 \chi + c_1 L_s \chi + \cdots + c_{q-1} L_s^{q-1} \chi \tag{2.4}$$

for all $w \in \mathcal{W}$, where $L_s \chi = (\partial \chi / \partial w)\, S_0 w$ and the characteristic polynomial

$$p^q - c_{q-1} p^{q-1} - \cdots - c_0$$

has distinct roots on the imaginary axis.

Motivation for the above assumption comes from the nonlinear version of the internal model principle, which recognizes that in the nonlinear case, the controller must be able to reproduce not only the trajectories generated by the exosystem, but also a number of higher order nonlinear harmonics, an idea that was elaborated independantly by Khalil [30], Huang and Lin [23] and Priscoli [44]. Assumption 2.3, along with the notion of *immersion* [24], allows the construction of a finite-dimentional linear internal model as follows. Defining

$$S = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ c_0 & \cdots & \cdots & \cdots & c_{q-1} \end{bmatrix}, \qquad \tau = \begin{bmatrix} \chi \\ L_s \chi \\ \vdots \\ L_s^{q-2} \chi \\ L_s^{q-1} \chi \end{bmatrix}$$

and $\Gamma = [1 \ 0 \ \cdots \ 0]_{1 \times q}$, it can be shown that the steady-state control $\chi(w)$ is generated by the internal model

$$\frac{\partial \tau(w)}{\partial w} S_0 w = S \tau(w), \quad \chi(w) = \Gamma \tau(w) \tag{2.5}$$

The internal model (2.5) is valid only when $\chi(w)$ contains a finite number of harmonics, which will always be the case if $\chi(w)$ is a polynomial function of w.[1] The coefficients $c_0, c_1, \cdots, c_{q-1}$ in (2.4) are required to be known, even in cases when $\chi(w)$ is uncertain. To further elaborate on this, let $\chi(w) = aw_1 + bw_1^3$, where a and b are unknown constants, and $w_1 = sin(\omega t + \phi)$, in which ω is the frequency and ϕ is the phase. It can be shown that in this case Equation (2.4) can be satisfied with $L_s^4\chi(w) = -9\omega^4\chi(w) - 10\omega^2 L_s^2\chi(w)$. It is important to note that the coefficinets of the polynomial (in this case, $c_0 = -9\omega^4$, and $c_2 = -10\omega^2$) only depend on frequency ω, and not on the unknown constants a and b. Thus, by construction, the internal model (2.5) includes harmonics of the sinusoids generated by the exosystem (2.2).

With the change of variables $x = \zeta - \pi$, the system (2.1) can be represented by

$$\dot{x} = f(x, w) + G(x, w)[u - \chi(w)] \tag{2.6}$$

where $f(x, w) = \tilde{f}(x + \pi, w) - \tilde{f}(\pi, w) + [\tilde{G}(x + \pi, w) - \tilde{G}(\pi, w)]\chi(w)$ and $G(x, w) = \tilde{G}(x + \pi, w)$. The system (2.6) is in the form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating $\chi(w)$ as matched uncertainty. We assume that a stabilizing state feedback controller is available for the system

$$\dot{x} = f(x, w) + G(x, w)u$$

Assumption 2.4. *There exists a locally Lipschitz function $\psi(x, w)$, with $\psi(0, w) = 0$, and a continuously differentiable Lyapunov function $V(x, w)$, possibly unknown, such that*

$$\alpha_1(\|x\|) \leq V(x, w) \leq \alpha_2(\|x\|) \tag{2.7}$$

[1]An example where this assumption is not satisfied is when $\chi(w) = sin(w)$, where $w = sin(at)$, in which a is a constant.

$$\frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \leq -W(x) \tag{2.8}$$

$\forall x \in X \subset R^n$ and $\forall w \in \mathcal{W}$, where α_1 and α_2 are class $\mathcal{K}$ functions, $W(x)$ is a continuous positive definite function, and X is a given domain that contains the origin.

The system (2.6) can be written as

$$\begin{aligned} \dot{x} &= f(x,w) + G(x,w)\psi(x,w) \\ &\quad + G(x,w)u - G(x,w)[\chi(w) + \psi(x,w)] \end{aligned} \tag{2.9}$$

In what follows, we use Lyapunov redesign [29] to construct the robust stabilizing feedback control to deal with the uncertain term $\chi(w)$. Towards that end, let $\Omega = \{sup_{w \in \mathcal{W}} V(x,w) \leq c_1\} \subset X$, for some $c_1 > 0$, and $\delta(x)$ be a continuous function such that

$$\|\chi(w) + \psi(x,w)\| \leq \delta(x) \qquad \forall x \in \Omega, \quad \forall w \in \mathcal{W} \tag{2.10}$$

The derivative of $V(x,w)$ along the trajectories of (2.9) is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\ &\quad + \frac{\partial V}{\partial x}G(x,w)u - \frac{\partial V}{\partial x}G(x,w)[\chi(w) + \psi(x,w)] \\ &\leq -W(x) + \frac{\partial V}{\partial x}G(x,w)u - \frac{\partial V}{\partial x}G(x,w)[\chi(w) + \psi(x,w)] \end{aligned} \tag{2.11}$$

The first term on the right-hand side of (2.11) is due to the nominal closed-loop system. The second and third terms represent, respectively, the effect of the control u and the uncertain term $[\chi(w)+\psi(x,w)]$ on $\dot{V}(x,w)$. Due to the matching condition, this uncertain term appears on the right-hand side exactly at the same point where u appears. Consequently, it is possible to design u to cancel the (destabilizing) effect of $[\chi(w)+\psi(x,w)]$ on $\dot{V}(x,w)$. Before proceeding further, we introduce the following assumption.

Assumption 2.5. *$(\partial V/\partial x)G(x,w)$ can be expressed as*

$$(\partial V/\partial x)G(x,w) = v^T(x)H(x,w) \tag{2.12}$$

where $v(x)$ is a known, locally Lipschitz function, with $v(0) = 0$, and $H(x,w)$ is a, possibly unknown, function that satisfies

$$H^T(x,w) + H(x,w) \geq 2\lambda I_m, \;\; \|H(x,w)\| \leq k; \; k \geq \lambda > 0 \tag{2.13}$$

$\forall x \in \Omega$ and $\forall w \in \mathcal{W}$, where I_m is $m \times m$ identity matrix.

The inequality (2.13) means that the uncertainity in the term $(\partial V/\partial x)G(x,w)$ can be bounded by a known positive constant λ. The matrix H will simply be an identity matrix if there is no uncertainty in the term $(\partial V/\partial x)G(x,w)$. Taking

$$u = -\alpha(x)\phi\left(\frac{v}{\mu}\right) \tag{2.14}$$

where the continuous function $\alpha(x)$ is chosen such that

$$\alpha(x) \geq \frac{k}{\lambda}\delta(x) + \alpha_0, \quad \alpha_0 > 0 \tag{2.15}$$

the saturation function $\phi\left(\frac{v}{\mu}\right)$ is defined as

$$\phi\left(\frac{v}{\mu}\right) = \begin{cases} v/\|v\| & \text{if} \quad \|v\| \geq \mu \\ \\ v/\mu & \text{if} \quad \|v\| \leq \mu \end{cases} \tag{2.16}$$

and $\mu > 0$ is a small design parameter. With this control, we can see that when

$\|v\| \geq \mu$, the equation (2.11) yields

$$
\begin{aligned}
\dot{V} &\leq -W(x) - \alpha(x)\lambda \|v\| + k\delta(x) \|v\| \\
&\leq -W(x) + \mu\lambda\alpha_0
\end{aligned} \tag{2.17}
$$

The same result can be obtained when $\|v\| \leq \mu$. Hence, with control (2.14), the derivative of $V(x, w)$ along the trajectories of the closed-loop system (2.9) is negative over the domain of interest. In what follows, we will refer to this control law as *saturated high-gain feedback control.*

2.3 Control Design

The robust state feedback control designed in the previous section will achieve practical regulation in the sense that the regulation error will be $O(\mu)$, so that it can be made arbitrarily small by choosing μ small enough. However, a very small μ will induce *chattering*. Therefore, we can not rely on reducing μ as a mechanism to render the error arbitrarily small. We refer to the eralier work of Khalil and co-researchers [30, 40, 52, 53] where a *servocompensator* is used to ensure that the regulation error converges to zero, without requiring μ to be arbitrarily small so as to render the regulation error arbitrarily small. Motivated by the latest results of Seshagiri and Khalil [53], we introduce the *conditional servocompensator* via the saturated high-gain feedback controller

$$
u = -\alpha(x)\phi\left(\frac{s}{\mu}\right) \tag{2.18}
$$

where $s = v(x) + K_1\sigma$, and σ is output of the conditional servocompensator

$$
\dot{\sigma} = (S - JK_1)\sigma + \mu J\phi\left(\frac{s}{\mu}\right) \tag{2.19}
$$

in which $J = [0, \cdots, 0, 1]^T$, and K_1 is chosen such that $S - JK_1$ is Hurwitz, which is always possible since the pair (S, J) is controllable.

By appropriately choosing the initial condition $\sigma(0)$, the solution of (2.19) will be $O(\mu)$ for all $t \geq 0$. In particular, consider the Lyapunov function $V_0(\sigma) = \sigma^T P_0 \sigma$, where P_0 is the symmetric positive definite solution of the Lyapunov equation $P_0 A_\sigma + A_\sigma^T P_0 = -I$, in which $A_\sigma =: S - JK_1$. The derivative of V_0 satisfies the inequality

$$\dot{V}_0 \leq -\|\sigma\|^2 + 2\mu \|\sigma\| \|P_0 J\|$$

Therefore, $\dot{V}_0 \leq 0$ on the surface $V_0(\sigma) = \mu^2 c_2$ for the choice $c_2 = 4\|P_0 J\|^2 \lambda_{max}(P_0)$. Hence, the set $\left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ is positively invariant, and we require $\sigma(0)$ to belong to it.

The closed-loop system is given by

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)\phi\left(\frac{s}{\mu}\right) \\
&\quad - G(x,w)[\chi(w) + \psi(x,w)] \qquad (2.20) \\
\dot{\sigma} &= A_\sigma \sigma + \mu J \phi\left(\frac{s}{\mu}\right)
\end{aligned}$$

We will now show that, for sufficiently small μ, the set $\Psi = \Omega \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ is a subset of the region of attraction, and for all initial conditions in Ψ, every trajectory of the closed-loop system (2.20) asymptotically approaches an invariant manifold on which the error is zero. The forthcoming analysis shares many points in common with the ones in [54], apart from various technical differences due to the presence of the time-varying signal $w(t)$. We start by showing that the set Ψ is positively invariant and there is a class $\mathcal{K}$ function ρ such that every trajectory in Ψ enters the set $\Psi_\mu = \{\|x\| \leq \rho(\mu)\} \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ in finite time and stays thereafter. The derivative of V along the trajectories of the closed-loop system (2.20) satisfies

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\
&\quad - \frac{\partial V}{\partial x}G(x,w)\alpha(x)\phi\left(\frac{s}{\mu}\right) - \frac{\partial V}{\partial x}G(x,w)[\chi(w) + \psi(x,w)] \\
&= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\
&\quad - v^T(x)H(x,w)\alpha(x)\phi\left(\frac{s}{\mu}\right) - v^T(x)H(x,w)[\chi(w) + \psi(x,w)] \\
&\leq -W(x) - \alpha(x)[s - K_1\sigma]^T H(x,w)\phi\left(\frac{s}{\mu}\right) \\
&\quad - [s - K_1\sigma]^T H(x,w)[\chi(w) + \psi(x,w)] \\
&= -W(x) - \alpha(x)s^T H(x,w)\phi\left(\frac{s}{\mu}\right) + \alpha(x)(K_1\sigma)^T H(x,w)\phi\left(\frac{s}{\mu}\right) \\
&\quad - s^T H(x,w)[\chi(w) + \psi(x,w)] + (K_1\sigma)^T H(x,w)[\chi(w) + \psi(x,w)]
\end{aligned}$$

Inside Ψ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Using this along with (2.13), (2.16) and (2.15), it can be shown that when $\|s\| \geq \mu$ we have

$$\begin{aligned}
\dot{V} &\leq -W(x) - \lambda\alpha(x)\|s\| + k\delta(x)\|s\| \\
&\quad + \|K_1\|\,\|\sigma\|\,k\,[\alpha(x) + \delta(x)] \\
&\leq -W(x) + \mu\gamma_1
\end{aligned} \tag{2.21}$$

where $\gamma_1 = \max_{x\in\Omega} kk_0[\alpha(x) + \delta(x)]$ and $k_0 = \|K_1\|\sqrt{c_2/\lambda_{min}(P_0)}$. Similarly, when $\|s\| \leq \mu$ we have

$$\begin{aligned}
\dot{V} &\leq -W(x) - \lambda\alpha(x)\frac{\|s\|^2}{\mu} + k\delta(x)\|s\| \\
&\quad + \alpha(x)\|K_1\|\,\|\sigma\|\,k\frac{\|s\|}{\mu} + \delta(x)\|K_1\|\,\|\sigma\|\,k \\
&\leq -W(x) + \mu\gamma_2
\end{aligned} \tag{2.22}$$

where $\gamma_2 = \max_{x \in \Omega} \; kk_0 \left[\alpha(x) + \delta(x)(1 + 1/k_0)\right] \; \geq \gamma_1$. From (4.18) and (4.19),

$$\dot{V} \leq -W(x) + \mu\gamma_2, \quad \forall (x, \sigma) \in \Psi$$

Hence, from [29, Theorem 4.18], for sufficiently small μ, Ψ is positively invariant and all trajectories starting in Ψ enter Ψ_μ in finite time and stay thereafter.

Next, we use $V_1 = \frac{1}{2}s^T s$ and Assumption 2.6, below, to show that the trajectories reach the boundary layer $\{\|s\| \leq \mu\}$ in finite time.

Assumption 2.6. *$N(x, w) =: (\partial v/\partial x)G(x, w)$ satisfies*

$$N(x, w) + N^T(x, w) \geq 2\lambda_p I_m, \; \|N(x, w)\| \leq k_p \tag{2.23}$$

where $k_p \geq \lambda_p > 0$, $\forall x \in \{\|x\| \leq \rho(\mu)\}$ and $\forall w \in \mathcal{W}$. Moreover, $\alpha(0) \geq \frac{k_p}{\lambda_p}\delta(0) + \alpha_0, \; \alpha_0 > 0$.

Remark 2.1. *The above assumption is automatically satisfied in the special case when $V = x^T P x$ and $G = B$, where P and B are known constant matrices. In this case, from Assumption 2.5, we get $v = (\partial V/\partial x)B = 2B^T P x$, and $H(x, w) = I$. Applying Assumption 2.6 yields us $N = (\partial v/\partial x)B = 2B^T P B$, which is a positive definite matrix, independant of (x, w).*

For $(x, \sigma) \in \Psi_\mu$

$$\begin{aligned} s^T \dot{s} &= -\alpha(x) s^T N(x, w)\phi\left(\frac{s}{\mu}\right) + s^T \frac{\partial v}{\partial x}[f(x, w) + G(x, w)\psi(x, w)] \\ &\quad - s^T N(x, w)[\chi(w) + \psi(x, w)] + s^T K_1 A_\sigma \sigma + \mu s^T K_1 J \phi\left(\frac{s}{\mu}\right) \end{aligned}$$

When $\|s\| \geq \mu$ we have

$$
\begin{aligned}
s^T \dot{s} \;\; \leq \;\; & -\alpha(x)\lambda_p \|s\| + \|N(x,w)\| \, \|\chi(w) + \psi(x,w)\| \, \|s\| \\
& + \left\| \frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \right\| \|s\| \\
& + (\|\sigma\| \, \|K_1\| \, \|A_\sigma\| + \mu \, \|K_1\| \, \|J\|) \, \|s\| \qquad (2.24)
\end{aligned}
$$

Inside Ψ_μ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $\frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)]$ is continuous such that $\frac{\partial v}{\partial x}[f(0,w) + G(0,w)\psi(0,w)] = 0$. Therefore, the norm $\left\|\frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)]\right\|$ together with the norms $\|\sigma\| \, \|K_1\| \, \|A_\sigma\|$, $\mu \, \|K_1\| \, \|J\|$, $\|\alpha(x) - \alpha(0)\|$, and $\|\delta(x) - \delta(0)\|$ can be bounded by a class $\mathcal{K}$ function $\rho_1(\mu)$. Hence,

$$
\begin{aligned}
s^T \dot{s} \;\; & \leq \;\; -\alpha(0)\lambda_p \|s\| + k_p\delta(0) \|s\| + \rho_1(\mu) \|s\| \\
\Rightarrow \qquad \dot{V}_1 \;\; & \leq \;\; -\lambda_p \left[\alpha_0 - \frac{\rho_1(\mu)}{\lambda_p}\right] \|s\| \qquad (2.25)
\end{aligned}
$$

Thus, for sufficiently small μ, all trajectories inside Ψ_μ would reach the boundary layer $\{\| s \| \leq \mu\}$ in finite time.

Finally, we show that inside the boundary layer the trajectories of the closed-loop system asymptotically approach an invariant manifold on which the error is zero. In order to prove local convergence to this manifold, we shall make the following assumption.

Assumption 2.7. *There exist non-negative constants k_1 to k_6 such that*

$$
\begin{aligned}
\|\psi(x,w)\| \leq k_1 \|v(x)\| + k_2\sqrt{W(x)} \\
\left\|\tfrac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)]\right\| \leq k_3 \|v(x)\| + k_4\sqrt{W(x)} \\
\left|\tfrac{\alpha(x) - \alpha(0)}{\alpha(0)}\right| \leq k_5 \|v(x)\| + k_6\sqrt{W(x)}
\end{aligned}
$$

$\forall w \in \mathcal{W}$, in some neighborhood of $x = 0$.

Remark 2.2. *If $W(x) \geq k_i \|x\|^2$, $k_i > 0$, near the origin $x = 0$, then any locally Lipschitz function can be bounded by $k_j\sqrt{W(x)}$ for some $k_j > 0$. Furthermore, if the equilibrium point of the closed-loop system is exponentially stable, such that Equations (2.7) and (2.8) are satisfied with α_1, α_2 and W being quadratic in $\|x\|$, and all other functions are locally Lipschitz, then the above assumption is automatically satisfied.*

Inside the boundary layer, the closed-loop system (2.20) is given by

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)\,(s/\mu) \\
&\quad - G(x,w)[\chi(w) + \psi(x,w)] \\
\dot{\sigma} &= S\sigma + Jv(x)
\end{aligned} \tag{2.26}$$

From [53], there exists a unique matrix Λ such that

$$S\Lambda = \Lambda S \qquad \text{and} \qquad -K_1\Lambda = \Gamma$$

We define

$$\mathcal{M}_\mu = \{x = 0, \sigma = \bar{\sigma}(w)\}$$

where $\bar{\sigma}(w) = (\mu/\alpha(0))\Lambda\, \tau(w)$. It is easy to verify by direct substitution that $\mathcal{M}_\mu$ is an invariant manifold of (2.26) for all $w \in \mathcal{W}$.

Defining $\tilde{\sigma} = \sigma - \bar{\sigma}(w)$ and $\tilde{s} = v + K_1\tilde{\sigma}$, the closed-loop system inside the boundary layer can be written as

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)\frac{\tilde{s}}{\mu} \\
&\quad + G(x,w)\left(\frac{\alpha(x) - \alpha(0)}{\alpha(0)}\right)\chi(w) - G(x,w)\psi(x,w) \\
\dot{\tilde{\sigma}} &= A_\sigma\tilde{\sigma} + J\tilde{s} \;=\; S\tilde{\sigma} + Jv
\end{aligned} \tag{2.27}$$

Consider the Lyapunov function candidate

$$V_2 = V(x,w) + \frac{b}{\mu}\tilde{\sigma}^T P_0 \tilde{\sigma} + \frac{c}{2}\tilde{s}^T \tilde{s} \tag{2.28}$$

where b and c are positive constants to be chosen, and P_0 is the symmetric positive definite solution of the Lyapunov equation $P_0 A_\sigma + A_\sigma^T P_0 = -I$. Calculating $\dot{V}_2$ along the trajectories of the system (2.27), we obtain

$$\dot{V}_2 = \dot{V} + \frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] + c\tilde{s}^T \dot{\tilde{s}}$$

The first term of $\dot{V}_2$ can be written as

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial w} S_0 w + \frac{\partial V}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\
&\quad + \frac{\partial V}{\partial x} G(x,w)\left[-\alpha(x)\frac{\tilde{s}}{\mu} + \left(\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right)\chi(w)\right] \\
&\quad - \frac{\partial V}{\partial x} G(x,w)\psi(x,w) \\
&\le -W(x) - v^T H(x,w)\frac{\alpha(x)}{\mu}(v + K_1\tilde{\sigma}) \\
&\quad + v^T H(x,w)\left(\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right)\chi(w) - v^T H(x,w)\psi(x,w)
\end{aligned}$$

Using Assumptions 2.4 - 2.7 yields

$$\begin{aligned}
\dot{V} &\le -W(x) - \alpha_0(\lambda/\mu)\|v\|^2 + \|v\|\, k(\bar{\alpha}/\mu)\,\|K_1\|\,\|\tilde{\sigma}\| \\
&\quad + \|v\|\, k \left\|\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right\| \|\chi(w)\| + \|v\|\, k\,\|\psi(w)\| \\
&\le -W(x) - [\alpha_0(\lambda/\mu) - k_7]\,\|v\|^2 + k_8\,\|v\|\sqrt{W(x)} \\
&\quad + (k_9/\mu)\,\|v\|\,\|\tilde{\sigma}\|
\end{aligned} \tag{2.29}$$

where $\bar{\alpha}$ is an upper bound on $\alpha(x)$ and k_7 to k_9 are some positive constants. Similarly, the second term of $\dot{V}_2$ can be written as

$$\begin{aligned}
\frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] &= -\frac{b}{\mu}\|\tilde{\sigma}\|^2 + \frac{b}{\mu}[\tilde{\sigma}^T P_0 J \tilde{s} + \tilde{s}^T J^T P_0^T \tilde{\sigma}] \\
&\leq -\frac{b}{\mu}\|\tilde{\sigma}\|^2 + \frac{2bk_{10}}{\mu}\|\tilde{\sigma}\|\,\|\tilde{s}\| \qquad (2.30)
\end{aligned}$$

where $k_{10} = \lambda_{max}(P_0)$. Next, we have

$$\begin{aligned}
\dot{\tilde{s}} &= \frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\
&\quad + \frac{\partial v}{\partial x}G(x,w)\left[-\alpha(x)\frac{\tilde{s}}{\mu} + \left(\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right)\chi(w)\right] \\
&\quad - \frac{\partial v}{\partial x}G(x,w)\psi(x,w) + K_1(S\tilde{\sigma} + Jv)
\end{aligned}$$

or

$$\begin{aligned}
c\tilde{s}^T\dot{\tilde{s}} &= c\tilde{s}^T\frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \\
&\quad + c\tilde{s}^T N(x)\left[-\alpha(x)\frac{\tilde{s}}{\mu} + \left(\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right)\chi(w)\right] \\
&\quad - c\tilde{s}^T N(x)\psi(x,w) + c\tilde{s}^T N(x)K_1(S\tilde{\sigma} + Jv)
\end{aligned}$$

or

$$\begin{aligned}
c\tilde{s}^T\dot{\tilde{s}} &\leq -c\alpha_0(\lambda_p/\mu)\|\tilde{s}\|^2 + ck_{11}\|\tilde{s}\|\,\|\tilde{\sigma}\| \\
&\quad + ck_{12}\|\tilde{s}\|\,\|v\| + ck_{13}\|\tilde{s}\|\sqrt{W(x)} \qquad (2.31)
\end{aligned}$$

for some positive constants k_{11} to k_{13}. From (2.29), (2.30) and (2.31), we have

$$\begin{aligned}\dot{V}_2 \;\leq\; & -W(x) - [\alpha_0(\lambda/\mu) - k_7]\,\|v\|^2 - \frac{b}{\mu}\,\|\tilde{\sigma}\|^2 \\ & - c\alpha_0(\lambda_p/\mu)\,\|\tilde{s}\|^2 + k_8\,\|v\|\,\sqrt{W(x)} + ck_{13}\,\|\tilde{s}\|\,\sqrt{W(x)} \\ & + (k_9/\mu)\,\|v\|\,\|\tilde{\sigma}\| + (2bk_{10}/\mu + ck_{11})\,\|\tilde{\sigma}\|\,\|\tilde{s}\| + ck_{12}\,\|\tilde{s}\|\,\|v\| \qquad (2.32)\end{aligned}$$

The right-hand side of (2.32) can be arranged in the following quadratic form of $\Pi = [\sqrt{W}\;\; \|v\|\;\; \|\tilde{\sigma}\|\;\; \|\tilde{s}\|]^T$:

$$\dot{V}_2 \;\leq\; -\Pi^T \Delta \Pi \qquad (2.33)$$

where the symmetric matrix Δ is given by

$$\Delta = \begin{pmatrix} 1 & \frac{-k_8}{2} & 0 & \frac{-ck_{13}}{2} \\ \frac{-k_8}{2} & \left(\frac{\alpha_0\lambda}{\mu} - k_7\right) & -\frac{k_9}{2\mu} & \frac{-ck_{12}}{2} \\ 0 & -\frac{k_9}{2\mu} & \frac{b}{\mu} & \left(-\frac{bk_{10}}{\mu} - \frac{ck_{11}}{2}\right) \\ \frac{-ck_{13}}{2} & \frac{-ck_{12}}{2} & \left(-\frac{bk_{10}}{\mu} - \frac{ck_{11}}{2}\right) & \frac{c\alpha_0\lambda_p}{\mu} \end{pmatrix}$$

If the principal leading minors of Δ can be made positive by choosing the constants b and c appropriately, and by choosing μ sufficiently small, then $\dot{V}_2$ will be negative definite. This would imply that, inside the boundary layer, the trajectories of the closed-loop system will asymptotically approach $\mathcal{M}_\mu$ as $t \to \infty$. Towards that end, we partition the matrix Δ as

$$\Delta = \begin{pmatrix} 1 & -q_{12}^T \\ & \\ -q_{12} & \frac{1}{\mu}Q_{22} + \Delta_{22} \end{pmatrix} \tag{2.34}$$

where

$$q_{12} = \begin{pmatrix} \frac{k_8}{2} & 0 & \frac{ck_{13}}{2} \end{pmatrix}^T \tag{2.35}$$

$$Q_{22} = \begin{pmatrix} \alpha_0\lambda & -\frac{k_9}{2} & 0 \\ & & \\ -\frac{k_9}{2} & b & -bk_{10} \\ & & \\ 0 & -bk_{10} & c\alpha_0\lambda_p \end{pmatrix} \tag{2.36}$$

and

$$\Delta_{22} = \begin{pmatrix} -k_7 & 0 & \frac{-ck_{12}}{2} \\ & & \\ 0 & 0 & -\frac{ck_{11}}{2} \\ & & \\ \frac{-ck_{12}}{2} & -\frac{ck_{11}}{2} & 0 \end{pmatrix} \tag{2.37}$$

From (2.36), it is easy to see that by choosing b and c, we can successively make the principal leading minors of Q_{22} positive. First, b is chosen large enough to make the 2×2 minor positive, and then, c is chosen large enough to make the 3×3 minor positive. Finally, choosing μ small enough will render

$$det \begin{pmatrix} 1 & -q_{12}^T \\ \\ -q_{12} & \frac{1}{\mu}Q_{22} + \Delta_{22} \end{pmatrix} > 0 \tag{2.38}$$

Consequently, $\dot{V}_2$ will be negative definite. Therefore, the trajectories of the closed-loop system will asymptotically approach $\mathcal{M}_\mu$ as $t \to \infty$. Our conclusions are summarized in the following theorem.

Theorem 2.1. *Suppose Assumptions 2.1 - 2.7 are satisfied and consider the closed-loop system (2.20). Suppose* $w(0) \in \mathcal{W}$. *Then, there exists* $\mu^* > 0$ *such that* $\forall \mu \in (0, \mu^*]$, *the set* $\Psi = \Omega \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ *is a subset of the region of attraction, and for all initial conditions in* Ψ, *the state variables are bounded and* $\lim_{t\to\infty} e(t) = 0$.

Remark 2.3. *The analytical results are provided for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally. For example, if Assumptions 2.4 - 2.6 hold globally, then the compact set of the initial conditions* Ω *can be chosen arbitrarily large, and will yield semi-global results.*

2.4 Performance

In section 2.3, we introduced a saturated high-gain feedback controller that incorporated a conditional servocompensator, and subsequently showed that trajectories of the closed-loop system under this scheme are bounded and yield zero steady-state regulation error. In this section we show that the conditional servocompensator does not degrade the transient response of the system. The trajectories of the closed-loop system under saturated high-gain feedback control with servocompensator approach those of a closed-loop system under saturated high-gain feedback control without servocompensator. The latter control is given by

$$u = -\alpha(x)\phi\left(\frac{v(x)}{\mu}\right) \tag{2.40}$$

which is similar to (2.18) except that $v(x) + K_1\sigma$ is replaced by $v(x)$, and yields the closed-loop system

$$\begin{aligned} \dot{w} &= S_0 w \\ \dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)\phi\left(\frac{v(x)}{\mu}\right) \\ &\quad - G(x,w)[\chi(w) + \psi(x,w)] \end{aligned} \tag{2.41}$$

In both cases, the trajectories eventually enter a boundary layer, which is $\|v(x)\| \leq \mu$ for the closed-loop system (2.41). We will denote the solution of (2.41) by x^* to distinguish it from the solution of (2.20). There could be a period of time when trajectories enter and leave their boundary layers before eventually settling inside. The challenging part in showing the closeness of the trajectories of the two systems is to keep track of when trajectories enter and leave the boundary layers since the entry and exit times will be different for the two systems. For convenience, we impose conditions which ensure that once the trajectories enter the boundary layer they do not leave. Recall that in the proof of Theorem 1 we showed that trajectories in Ψ_μ enter the boundary layer in finite time. The new feature here is that we need to show this property for all trajectories in Ω and not just Ψ_μ. Towards that end, the choice of $\alpha(x)$ in the control laws (2.18) and (2.40) is modified to

$$\alpha(x) \geq max\left\{\frac{k}{\lambda}\delta(x),\ \frac{k_p}{\lambda_p}\delta(x) + \frac{1}{\lambda_p}\beta(x)\right\} + \alpha_0 \tag{2.42}$$

where $\beta(x)$ is a known continuous function such that

$$\left\| \frac{\partial v}{\partial x}[f(x,w) + G(x,w)\psi(x,w)] \right\| \leq \beta(x)$$

for all $x \in \Omega$. Suppose that the inequality (2.23) holds for all $x \in \Omega$. Then, outside the boundary layer $\{\|s\| \geq \mu\}$, the trajectories of (2.20) satisfy

$$\begin{aligned}
s^T \dot{s} &\leq -\alpha(x)\lambda_p \|s\| + \beta(x)| \|s\| + k_p \delta(x) \|s\| \\
&\quad + (\|\sigma\| \|K_1\| \|A_\sigma\| + \mu \|K_1\| \|J\|) \|s\| \\
&\leq -\lambda_p \left[\alpha(x) - \frac{k_p}{\lambda_p}\delta(x) - \frac{1}{\lambda_p}\beta(x) - \mu k_a \right] \|s\| \\
&\leq -\lambda_p [\alpha_0 - \mu k_a] \|s\| \qquad (2.43)
\end{aligned}$$

for some positive constant k_a. Thus, for sufficiently small μ, the trajectories starting inside Ω would reach the boundary layer in finite time and stay there for all future time. A similar result can be obtained for the system (2.41) using $V_1^* = \frac{1}{2}v(x^*)^T v(x^*)$. Inside the boundary layer the systems (2.20) and (2.41) exhibit slow and fast dynamics for small μ. To show closeness of their trajectories, we will compare the equations of the two systems represented in the singularly perturbed form.

Assumption 2.8. *There exists a mapping $T_w(x)$, continuously differentiable in (x,w), such that the Jacobian $\frac{\partial T_w(x)}{\partial x}$ is nonsingular, uniformly in w, and*

$$\frac{\partial T_w(x)}{\partial x} G(x,w) = \begin{bmatrix} 0 \\ I_m \end{bmatrix}; \quad \forall x \in \Omega \ ; \ \forall w \in \mathcal{W}$$

The change of variables $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = T_w(x)$ transforms the system (2.6) into the form

$$\begin{aligned}
\dot{\varphi}_1 &= \pi_1(\varphi_1, \varphi_2, w) \\
\dot{\varphi}_2 &= \pi_2(\varphi_1, \varphi_2, w) + [u - \chi(w)]
\end{aligned}$$

In the new variables, $v(x)\Big|_{x=T_w^{-1}(\varphi)} = v_1(\varphi_1, \varphi_2, w)$.

Assumption 2.9. *The equation* $v_1(\varphi_1, \varphi_2, w) + K_1\sigma = 0$ *has a unique solution* $\varphi_2 = \varpi(\varphi_1, \sigma, w)$ *for all* $x \in \Omega$ *and* $w \in \mathcal{W}$*, and the system*

$$\dot{\sigma} = A_\sigma \sigma \tag{2.44}$$

$$\dot{\varphi}_1 = \pi_1(\varphi_1, \varpi(\varphi_1, \sigma, w), w) \tag{2.45}$$

has an exponentially stable equilibrium point at $\varphi_1 = 0$.

Remark 2.4. *Exponential stability of (2.44) - (2.45) implies exponential stability of*

$$\dot{\varphi}_1 = \pi_1(\varphi_1, \varpi(\varphi_1, 0, w), w) \tag{2.46}$$

On the other hand, exponential stability of (2.46) together with continuous differentiability of $\pi_1(\varphi_1, \varpi(\varphi_1, \sigma, w), w)$ *implies exponential stability of (2.44) - (2.45).*

Outside their respective boundary layers, the two systems can be represented by

$$\dot{x} = f(x, w) - G(x, w)\chi(w) - \alpha(x)G(x, w)\left[\frac{v(x) + K_1\sigma}{\|v(x) + K_1\sigma\|}\right] \tag{2.47}$$

$$\dot{\sigma} = A_\sigma\sigma + \mu J\left[\frac{v(x) + K_1\sigma}{\|v(x) + K_1\sigma\|}\right] \tag{2.48}$$

and

$$\dot{x^*} = f(x^*, w) - G(x^*, w)\chi(w) - \alpha(x^*)G(x^*, w)\left[\frac{v(x^*)}{\|v(x^*)\|}\right] \tag{2.49}$$

Suppose t_0 is the time instant when one of the two systems reaches its boundary layer. Since $\|\sigma\| = O(\mu)$, the difference in the right-hand side of state equations for $\dot{x}$ and $\dot{x^*}$ is $O(\mu)$ $\forall t \in [0, t_0]$. From [29, Theorem 3.4], $\|x(t) - x^*(t)\| = O(\mu)$ $\forall t \in [0, t_0]$. Therefore, when the trajectory of one system reaches the boundary layer, then for sufficiently small μ, the trajectory of the other system will be in some $O(\mu)$ neighborhood of its boundary layer. Since $\dot{V_1}$ and $\dot{V_1^*}$ are strictly negative outside their boundary layers, uniformly in μ, it must be true that the trajectory of the other system also reaches its boundary layer in time $t_1 = t_0 + O(\mu)$. For all $t \geq t_1$, when the trajectories of the two systems are inside their boundary layers, the two systems can be represented by

$$\dot{x} = f(x,w) - G(x,w)\chi(w) - \alpha(x)G(x,w)\left(\frac{v(x) + K_1\sigma}{\mu}\right) \tag{2.50}$$

$$\dot{\sigma} = A_\sigma \sigma + J\left[v(x) + K_1\sigma\right] \tag{2.51}$$

and

$$\dot{x^*} = f(x^*,w) - G(x^*,w)\chi(w) - \alpha(x^*)G(x^*,w)\left(\frac{v(x^*)}{\mu}\right) \tag{2.52}$$

With the change of variables $\varphi = T_w(x)$ and $\varphi^* = T_w(x^*)$, the two systems can be written in the singularly perturbed forms

$$\dot{\sigma} = A_\sigma \sigma + J\left[v_1(\varphi_1, \varphi_2, w) + K_1\sigma\right] \tag{2.53}$$

$$\dot{\varphi_1} = \pi_1(\varphi_1, \varphi_2, w) \tag{2.54}$$

$$\mu\dot{\varphi_2} = \mu[\pi_2(\varphi_1, \varphi_2, w) - \chi(w)] - \alpha_1(\varphi_1, \varphi_2, w)\left[v_1(\varphi_1, \varphi_2, w) + K_1\sigma\right] \tag{2.55}$$

for (2.50)-(2.51) and

$$\dot{\varphi}_1^* = \pi_1(\varphi_1^*, \varphi_2^*, w) \tag{2.56}$$

$$\mu\dot{\varphi}_2^* = \mu[\pi_2(\varphi_1^*, \varphi_2^*, w) - \chi(w)] - \alpha_1(\varphi_1^*, \varphi_2^*, w)\,(\upsilon_1(\varphi_1^*, \varphi_2^*, w)) \tag{2.57}$$

for (2.52), where $\alpha_1(\varphi_1, \varphi_2, w) = \alpha(x)\Big|_{x=T_w^{-1}(\varphi)}$. For the singularly perturbed system (2.53)-(2.55), the slow model is

$$\dot{\sigma} = A_\sigma \sigma \tag{2.58}$$

$$\dot{\varphi}_1 = \pi_1(\varphi_1, \varpi(\varphi_1, \sigma, w), w) \tag{2.59}$$

and the fast model is

$$\frac{d\varphi_2}{d\tau} = -\alpha_1(\varphi_1, \varphi_2, w)\,[\upsilon_1(\varphi_1, \varphi_2, w) + K_1\sigma] \tag{2.60}$$

where $\tau = t/\mu$. Similarly, for the singularly perturbed system (2.56)-(2.57), the slow model is

$$\dot{\varphi}_1^* = \pi_1(\varphi_1^*, \varpi(\varphi_1^*, 0, w), w) \tag{2.61}$$

and the fast model is

$$\frac{d\varphi_2^*}{d\tau} = -\alpha_1(\varphi_1^*, \varphi_2^*, w)\upsilon_1(\varphi_1^*, \varphi_2^*, w) \tag{2.62}$$

The definitions of the slow and fast models follow the procedures in [35]. We require the two boundary-layer systems to be exponentially stable.

Assumption 2.10. *The two boundary-layer systems given by (2.60) and (2.62) have* *exponentially stable equilibrium points at* $(\varphi_1, \varphi_2) = (0,0)$, *and* $(\varphi_1^*, \varphi_2^*) = (0,0)$, *respectively.*

Because σ is $O(\mu)$, [29, Theorem 9.1] confirms that the solutions of (2.58)-(2.59) and (2.61) are $O(\mu)$ close to each other. Furthermore, [29, Theorem 11.2] shows the closeness of the solutions of the systems (2.53)-(2.55) and (2.56)-(2.57) to those of their respective slow and fast models given by (2.58)-(2.60) and (2.61)-(2.62). Therefore, $\| x(t) - x^*(t) \| = O(\mu) \;\; \forall t \geq 0$. The foregoing conclusions can be summarized in the following theorem, which confirms that the controller (2.18) recovers the performance of the controller (2.40).

Theorem 2.2. *Let* $(x(t), \sigma(t))$ *be the state of the closed-loop system (2.20), and* $x^*(t)$ *be the state of the closed-loop system without servocompensator (2.41). Let* $x(0) = x^*(0) \in \Omega$. *Suppose Assumptions 2.8 - 2.10 are satisfied and* $\alpha(x)$ *satisfies (2.42). Then, under the hypotheses of Theorem 2.1, there exists* $\mu^* > 0$ *such that* $\forall \mu \in (0, \mu^*]$, $\| x(t) - x^*(t) \| = O(\mu) \;\; \forall t \geq 0$.

Remark 2.5. *The performance recovery provided by Theorem 2.2 is with respect to a system controlled by the saturated high-gain feedback controller (2.40), rather than the original stabilizing controller* $\psi(x, w)$.

2.5 Example

Consider a second-order system modeled by the equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\theta_1(\tilde{x}_1 - \tilde{x}_1^3/6) + \theta_2 u \\ y &= \tilde{x}_1 \end{aligned} \tag{2.63}$$

where θ_1 and θ_2 are (possibly uncertain) constants. with the reference signal $r(t) = r_0 sin(\omega t)$, which is generated by the exosystem

$$\dot{w} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} w, \qquad w^T(0) = \begin{pmatrix} 0 \\ r_0 \end{pmatrix}, \quad r(t) = w_1$$

With the change of variables $x_1 = \tilde{x}_1 - r$, $x_2 = \tilde{x}_2 - \dot{r}$, we have

$$\begin{aligned} \dot{x} &= Ax + B[f(x,w) + G(x,w)(u - \chi(w))] \\ e &= Cx \end{aligned} \tag{2.64}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \;\; 0),$$

$$f(x,w) = -\theta_1 x_1 + \tfrac{\theta_1}{6}\left[(x_1 + w_1)^3 - w_1^3\right], \quad G(x,w) = \theta_2, \text{ and}$$

$\chi(w) = \frac{1}{\theta_2}\left[\theta_1 w_1 - \frac{1}{6}\theta_1 w_1^3 - \omega^2 w_1\right]$. It can be verified that $\chi(w)$ satisfies the identity $L_s^4\chi = -9\omega^4\chi - 10\omega^2 L_s^2\chi$.

We compare the performance of saturated high-gain feedback stabilizing controller without a servocompensator (Design 1), with two control designs that use servocompensators (Design 2 and Design 3). Design 2 uses the conditional servocompensator (3.13), while Design 3 uses a conventional servocompensator [40]. In Design 3, we augment a fourth-order conventional servo-compensator $\dot{\sigma} = S\sigma + Je$, with the system (2.64) to obtain an *augmented system* of the form

$$\begin{aligned} \dot{\xi} &= A_1\xi + B_1[f(x,w) + G(x,w)\psi(x,w)] \\ e &= C_1\xi \end{aligned} \tag{2.65}$$

where

$$A_1 = \begin{pmatrix} S & JC \\ 0 & A \end{pmatrix}, \quad B_1 = (0 \;\; B)^T, \quad C_1 = (0 \;\; C) \tag{2.66}$$

and $\xi = \begin{pmatrix} \sigma & x \end{pmatrix}^T$ is the state. A stabilizing feedback controller is designed via Lyapunov redesign [40], which yields

$$\psi(x,w) = \frac{1}{\theta_2}\left[\theta_1 x_1 - \frac{\theta_1}{6}\left((x_1+w_1)^3 - w_1^3\right) - \frac{1}{k}\phi(\varpi/\mu)\right] \tag{2.67}$$

where $\varpi = 2B^T P\xi$, and $P = P^T > 0$ is the solution of the Lyapunov equation

$$P(A_1 + B_1K_2) + (A_1 + B_1K_2)^T P = -I$$

in which K_2 is chosen such that $A_1+B_1K_2$ is Hurwitz. For the conditional servocompensator design (Design 2), Assumption 2.4 is satisfied with $V(x) = \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2)$ and $\psi(x,w) = \frac{1}{\theta_2}\left[\theta_1 x_1 - \frac{\theta_1}{6}[(x_1+w_1)^3 - w_1^3] - k_1x_1 - k_2x_2\right]$ where $k_1 = k_2 = 1$. Assumption 2.5 is satisfied with $v(x) = x_1 + 2x_2$ and $H(x,w) = 1$. We use the following numerical values in the simulation: $\theta_1 = 1$, $\theta_2 = 3$, $\omega = 0.5$ rad/s, $r_0 = 1$, $k = 1/5$, and $\mu = 0.1$. For the conventional servocompensator design (Design 3), K_2 is chosen so as to assign the eigenvalues of $A_1 + B_1K_2$ at $-0.5, -1, -1.5$, -2, -2.5 and -3. For the conditional servocompensator (Design 2), K_1 is chosen so as to assign the eigenvalues of $S - JK_1$ at $-0.5, -1, -1.5$ and -2. The control u is given by

$$u = -10 \; sat\left(\frac{x_1 + 2x_2 + K_1\sigma}{\mu}\right)$$

Figure 2.1 shows the tracking error during the transient period and Figure 2.2 shows the steady state tracking error for the three designs. The transient response of

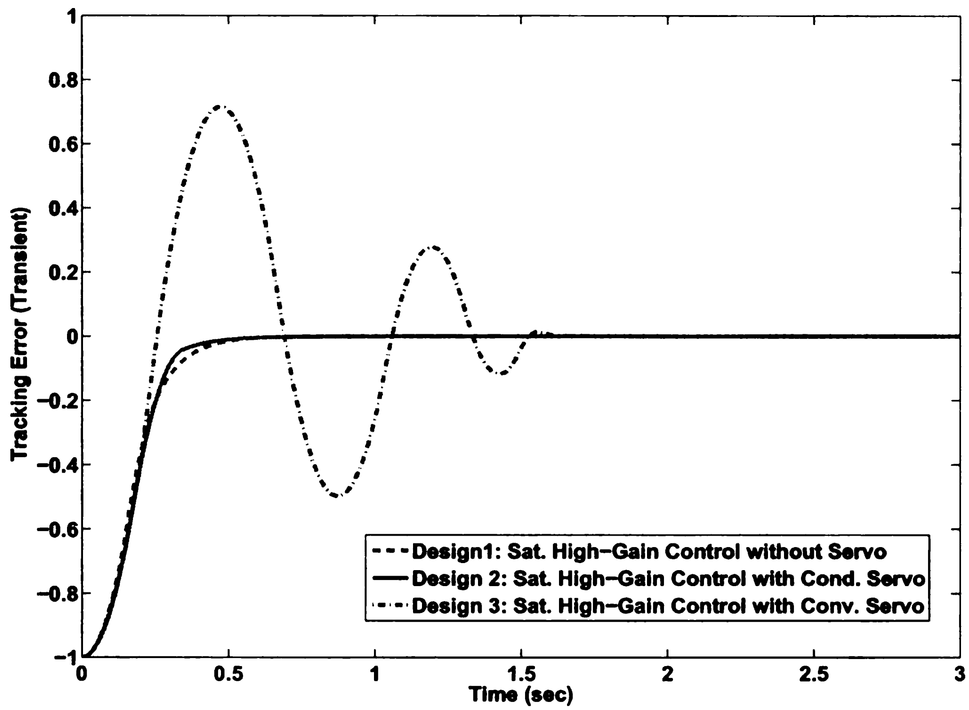


Figure 2.1: Tracking error during the transient period

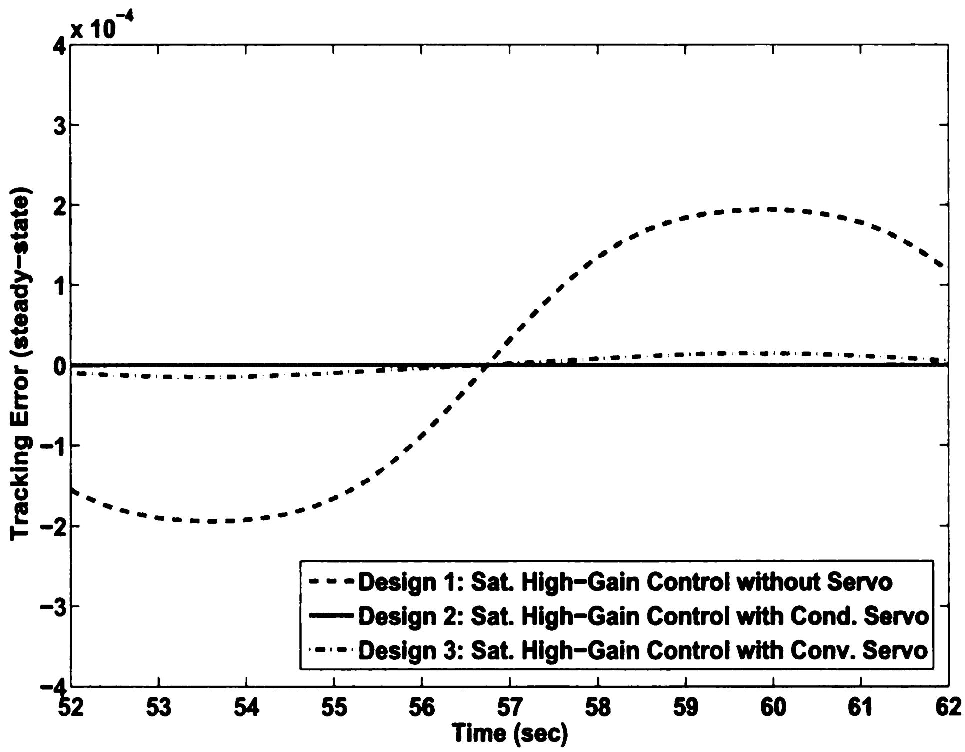


Figure 2.2: Steady-state Tracking error

the controller design without a servocompensator (Design 1) is close to the one with conditional servocompesator (Design 2), but it does not result in asymptotic error convergence. Asymptotic error convergence to zero is achieved with the conventional servocompensator design (Design 3), however, at the expense of a degraded transient performance.

2.6 Conclusions

We have extended the state feedback regulator of [54], which uses conditional integrators, Lyapunov redesign, and saturated high-gain feedback, to a more general case of time-varying signals by using conditional servocompensators. We showed that the use of conditional servocompensators enables us to achieve zero steady-state regulation error, in the presence of time-varying exogenous signals that are generated by a known exosystem. Analytical results are provided for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally.

Chapter 3

Output Feedback Regulation of Input-Output Linearizable Nonlinear Systems

3.1 Introduction

We consider the output feedback regulation problem for minimum-phase nonlinear systems that can be transformed into the normal form. We first design a partial-state feedback control that regulates the error to zero in the presence of time-varying refrence and disturbance signals. The output feedback control is then implemented using a high-gain observer. We show that the output feedback controller with conditional servocompensator recovers the performance of a state feedback controller that does not include any servocompensator.

3.2 System Description and Assumptions

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x,w) + G(x,w)[u - \chi(w)] \\ e &= h(x,w) \end{aligned} \tag{3.1}$$

where e is the measured output that we want to regulate to zero. This problem relates to a more realistic and practical situation when only the error variable e is available for measurement, rather than the complete state x as in Chapter 2. It is well known [24] that if the above system has a well-defined vector relative degree and the distribution span $\{g_1, \ldots, g_m\}$ is involutive, where $g_1, \ldots, g_m$ are the columns of G, then it can be transformed into the *normal form*

$$\begin{aligned} \dot{\xi} &= A\xi + B\{f_1(\xi, z, w) + G_1(\xi, z, w)[u - \chi(w)]\} \\ \dot{z} &= f_2(\xi, z, w) \\ e &= C\xi \end{aligned} \tag{3.2}$$

where ξ and z belong to the sets $X_\xi \subset R^{n-r}$ and $X_z \subset R^r$, respectively. The $r \times r$ matrix A, the $r \times m$ matrix B and the $m \times r$ matrix C, given by

$$A = \text{ block diag } [A_1, \ldots, A_m],$$

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{r_i \times r_i}$$

$$B = \text{ block diag } [B_1, \dots, B_m], \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r_i \times 1}$$

$$C = \text{ block diag } [C_1, \dots, C_m],$$

$$C_i = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{1 \times r_i}$$

where $1 \leq i \leq m$ and $r = r_1 + \ldots + r_m$, represent m chains of integrators. In the new coordinates, the zero-error manifold can be written as

$$\mathcal{M}_\mu = \{\xi = 0, z = 0, \sigma = \bar{\sigma}(w)\}$$

In what follow, we first state the assumptions required to address the output feedback regulation problem, and then we elaborate on how they are related to those in Chapter 2.

Assumption 3.1. *The following inequalities hold for all* $(\xi, z, w) \in X_\xi \times X_z \times \mathcal{W}$*:*

$$G_1(\xi, z, w) + G_1^T(\xi, z, w) \geq 2\lambda_1 I_m, \ \lambda_1 \geq \lambda > 0$$
$$B^T P_1 B G_1(0, 0, w) + G_1^T(0, 0, w) B^T P_1 B \geq 2\lambda_2 I_m, \ \lambda_2 \geq \lambda > 0$$

where $P_1 = P_1^T$ *is the solution of the Lyapunov equation* $P_1(A-BK)+(A-BK)^T P_1 = -I_n$ *and* K *is chosen such that* $A - BK$ *is Hurwitz.*

Assumption 3.2. *There exists a Lyapunov function* $V_z(z, w)$ *such that for all* (ξ, z, w) $\in X_\xi \times X_z \times \mathcal{W}$

$$k_{12} \|z\|^2 \leq V_z(z, w) \leq k_{13} \|z\|^2 \tag{3.3}$$

$$\frac{\partial V_z}{\partial w} S_0 w + \frac{\partial V_z}{\partial z} f_2(\xi, z, w) \leq -k_{14} \|z\|^2 + k_{15} \|z\| \|\xi\| \tag{3.4}$$

for some positive constants k_{12}*,* k_{13}*,* k_{14} *and* k_{15}*.*

Let $V(\xi, z, w) = V_z(z, w) + k_{16}[\xi^T P_1 \xi]$ for some positive constant k_{16}. In this case, Assumption 2.4 is satisfied with $\psi(\xi, z, w) = G_1^{-1}(\xi, z, w)[-f_1(\xi, z, w) - K\xi]$ and inequalities (2.7) and (2.8) are satisfied with α_1, α_2 and W which are quadratic in $\left\| [\begin{array}{cc} \xi & z \end{array}]^T \right\|$. Consequently, the set $\mathcal{M}_\mu$ is exponentially stable, uniformly in w. Assumption 2.5 is satisfied with $v^T(\xi) = 2k_{16}\xi^T P_1 B$ and $H(\xi, z, w) = G_1(\xi, z, w)$.

Remark 3.1. *For a SISO nonlinear system in the normal form, Assumptions 3.1 - 3.2 imply Assumptions 2.4 - 2.7, and boil down to the requirements that the system is minimum-phase and the control coefficient $\alpha(.)$ is positive and bounded away from zero.*

3.3 Control Design and Analysis

We first derive a partial state-feedback control which will then be implemented as an output feedback control using a high-gain observer. Towards that end, we re-write the inequality (2.10) as

$$\|\chi(w) + \psi(\xi, 0, w)\| \leq \delta(\xi) \qquad \forall (\xi, z, w) \in X_\xi \times X_z \times \mathcal{W} \tag{3.5}$$

Consequently, (2.15) can be modified as

$$\alpha(\xi) \geq \frac{k}{\lambda}\delta(\xi) + \alpha_0, \quad \alpha_0 > 0 \tag{3.6}$$

and from (2.18), a partial state feedback control can be taken as

$$u = -\alpha(\xi)\phi\left(\frac{s_1}{\mu}\right) \tag{3.7}$$

where $s_1 = v(\xi) + K_1\sigma$. Under the partial state feedback control, the closed-loop system is represented by

$$\dot{\xi} = A\xi + B\left\{f_1(\xi, z, w) + G_1(\xi, z, w)[-\alpha(\xi)\phi\left(\frac{s_1}{\mu}\right) - \chi(w)]\right\} \tag{3.8}$$

$$\dot{z} = f_2(\xi, z, w) \tag{3.9}$$

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\phi\left(\frac{s_1}{\mu}\right) \tag{3.10}$$

The control (3.7) can be implemented as an output feedback controller that uses the high-gain observer

$$\dot{\hat{\xi}} = A\hat{\xi} + L(e - C\hat{\xi}) \tag{3.11}$$

to estimate ξ, where the observer gain L is chosen as

$$L = \text{ block diag } [L_1, \ldots, L_m], \quad L_i = \begin{bmatrix} \frac{\alpha_1^i}{\epsilon} \\ \frac{\alpha_2^i}{\epsilon^2} \\ \vdots \\ \vdots \\ \frac{\alpha_{r_i}^i}{\epsilon^{r_i}} \end{bmatrix}_{r_i \times 1}$$

in which ϵ is a positive constant and the positive constants α_j^i are chosen such that the roots of

$$\lambda^{r_i} + \alpha_1^i \lambda^{r_i - 1} + \cdots + \alpha_{r_i - 1}^i \lambda + \alpha_{r_i}^i = 0$$

are in the open left-half plane, for all $i = 1, \ldots, m$. The output feedback control is given by

$$u = -\alpha(\hat{\xi})\phi\left(\frac{\hat{s}_1}{\mu}\right) \tag{3.12}$$

where $\hat{s}_1 = v(\hat{\xi}) + K_1\sigma$ and σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\phi\left(\frac{\hat{s}_1}{\mu}\right) \tag{3.13}$$

Next, we show that the output feedback control (3.12) recovers the performance of the state feedback control for sufficiently small ϵ. For the purpose of analysis, we replace the observer dynamics by the equivalent dynamics of the scaled estimation error $D\eta = \xi - \hat{\xi}$, where

$$\begin{aligned} D &= \text{ block diag } [D_1, \ldots, D_m], \\ D_i &= \left[\ \epsilon^{r_i-1}, \epsilon^{r_i-2}, \cdots, 1\ \right]_{r_i \times r_i} \end{aligned}$$

The closed-loop system can be represented by

$$\begin{aligned} \dot{\xi} &= A\xi + B\{f_1(\xi, z, w) + \\ &\quad G_1(\xi, z, w)[-\alpha(\xi - D\eta)\phi\left(\frac{v(\xi - D\eta) + K_1\sigma}{\mu}\right) - \chi(w)]\} && (3.14) \\ \dot{z} &= f_2(\xi - D\eta, z, w) && (3.15) \\ \dot{\sigma} &= A_\sigma\sigma + \mu J\phi\left(\frac{v(\xi - D\eta) + K_1\sigma}{\mu}\right) && (3.16) \\ \epsilon\dot{\eta} &= A_0\eta + \epsilon B\{f_1(\xi, z, w) + \\ &\quad G_1(\xi, z, w)[-\alpha(\xi - D\eta)\phi\left(\frac{v(\xi - D\eta) + K_1\sigma}{\mu}\right) - \chi(w)]\} && (3.17) \end{aligned}$$

where $A_0 = \epsilon\, D^{-1}(A - LC)D$ is an $r \times r$ Hurwitz matrix. The system (3.14) - (3.17) is a standard singularly perturbed system, and $\eta = 0$ is the unique solution of (3.17) when $\epsilon = 0$. The reduced system is the closed-loop system under state feedback control. For convenience, we write $\Theta^T = \begin{bmatrix} \xi^T & z^T \end{bmatrix}$. Let the initial states be $\Theta(0) = \Theta_0 \in \mathcal{B}$, $\sigma(0) = \sigma_0 \in \{V_0(\sigma) \le \mu^2 c_2\}$ and $\hat{\xi}(0) = \hat{\xi}_0 \in \mathcal{Q}$, where $\mathcal{B}$ is a compact set that contains $\mathcal{M}_\mu$, and $\mathcal{Q}$ is any compact subset of R^r. Let $(\Theta(t,\epsilon), \sigma(t,\epsilon), \eta(t,\epsilon))$ denote the trajectory of the system (3.14) - (3.17) starting from $(\Theta_0, \sigma_0, \hat{\xi}_0)$. The system (3.14) - (3.17) fits into the framework of Atassi & Khalil [3] so that the results provided therein can be conveniently applied. In particular, the system (3.14) - (3.17) satisfies Assumptions 1 and 2 of [3], which require the functions $f_1(\xi, z, w)$, $f_2(\xi, z, w)$ and $G_1(\xi, z, w)$ to be locally Lipschitz in their arguments uniformly in w over the domain of interest.

Let $(\Theta_r(t), \sigma_r(t))$ be the solution of the closed-loop system under state feedback, starting from (Θ_0, σ_0). Using [3, Theorems 1 & 3], it can be shown that, the trajectories of (3.14) - (3.17) are bounded, and $\Theta(t, \epsilon) - \Theta_r(t)$ and $\sigma(t, \epsilon) - \sigma_r(t)$ converge to zero, as $\epsilon \to 0$, uniformly in t, for all $t \geq 0$. Using [3, Theorems 2 & 5] , we can show that there is a neighborhood $\mathcal{N}$ of $\mathcal{M}_\mu \times \{\eta = 0\}$, independent of ϵ, and $\epsilon_2 > 0$ such that for every $0 < \epsilon \leq \epsilon_2$, the set $\mathcal{M}_\mu \times \{\eta = 0\}$ is exponentially stable and every trajectory in $\mathcal{N}$ converges to this set as $t \to \infty$. Furthermore, from [3, Theorems 1, 2 & 5], there is $\epsilon_3 > 0$ such that for every $0 < \epsilon \leq \epsilon_3$, the solutions starting in $\mathcal{B} \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\} \times \mathcal{Q}$ enter $\mathcal{N}$ in finite time. Hence, for every $0 < \epsilon \leq \epsilon_4 = min\{\epsilon_2, \epsilon_3\}$, the set $\mathcal{M}_\mu \times \{\eta = 0\}$ is exponentially stable and $\mathcal{B} \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\} \times \mathcal{Q}$ is a subset of the region of attraction. Thus, for sufficiently small ϵ, the closed-loop system (3.14) - (3.17), under the output feedback controller (3.12), is uniformly exponentially stable with respect to the set $\mathcal{M}_\mu \times \{\xi - \hat{\xi} = 0\}$, and $\lim_{t\to\infty} e(t) = 0$. The foregoing conclusions are summarized in the following theorem.

Theorem 3.1. *Suppose Assumptions 2.1 - 2.3 and 3.1 - 3.2 are satisfied. Then, given any $\gamma > 0$, there exists $\mu^* > 0$ such that $\forall \mu \in (0, \mu^*]$, there exists $\epsilon^*(\mu) > 0$ such that $\forall \epsilon \in (0, \epsilon^*]$, the closed-loop system under the output feedback (3.14) - (3.17)* *is uniformly exponentially stable with respect to the set* $\mathcal{M}_\mu \times \{\xi - \hat{\xi} = 0\}$, *and*

$$\left\| \begin{pmatrix} \xi(t,\epsilon) \\ z(t,\epsilon) \\ \sigma(t,\epsilon) \end{pmatrix} - \begin{pmatrix} \xi_r(t,\epsilon) \\ z_r(t,\epsilon) \\ \sigma_r(t,\epsilon) \end{pmatrix} \right\| \leq \gamma$$

for all $t \geq 0$ *and* $w \in \mathcal{W}$.

3.4 Examples

In this section, we present simulations for two examples. In the first example we consider the magnetic suspension system [29] to illustrate the performance of the saturated high gain feedback control using conditional servocompensator and compare it with the earlier results of Mahmoud & Khalil [40] , that involved the traditional approach of augmenting a servocompensator with the plant. In the second example, we revisit the simulation example treated in Serrani, Isidori & Marconi [50] , to compare the performance of our control design with that of [50].

Example 3.4.1. Magnetic suspension system

Consider the magnetic suspension system [29] given by

$$m\ddot{y} = -k\dot{y} + mg - \frac{L_0\, i^2}{2a(1 + y/a)^2} + f_d$$

where m is the mass of the ball, $y \geq 0$ is the downward vertical position of the ball measured from a reference point $y = 0$, k is a viscous friction coefficient, g is the acceleration due to gravity, L_0 is the inductance of the electromagnet, a is a positive constant and f_d is an external disturbance force. Let y^* be a nominal equilibrium point and i^* be the corresponding nominal value of the current i that maintains the equilibrium condition

$$\hat{m}g = \frac{a\ L_0\ i^*}{2(a+y^*)^2}$$

where $\hat{m}$ is the nominal value of the mass of the ball. Defining $\zeta_1 = \frac{y-y^*}{y^*}$, $\zeta_2 = \frac{\dot{y}}{y^*\omega_n}$, $u = \frac{i^{*2}-i^2}{i^{*2}}$ and $\tau = \omega_n t$, yields the normalized model

$$\dot{\zeta}_1 = \zeta_2 \tag{3.18}$$

$$\dot{\zeta}_2 = -b\zeta_2 + \Delta(\zeta_1) + g(\zeta_1)u + d(t) \tag{3.19}$$

where $\Delta(\zeta_1) = 1 - \frac{\hat{m}}{m}(\frac{a+y^*}{a+y^*+\zeta_1 y^*})^2$, $g(\zeta_1) = 1 - \Delta(\zeta_1)$, $b = \frac{k}{m\omega_n}$, $d = \frac{f_d}{my^*\omega_n^2}$, $\omega_n^2 = g/y^*$ and $(\dot{.})$ denotes the derivative with respect to τ. It is desired to balance the ball at a certain constant position r_0 in the presence of a sinusoidal disturbance signal $d(t) = d_0 sin(\omega t)$. These signals are generated by the exosystem

$$\dot{w} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} w, \quad w^T(0) = (d_0, 0, r_0), \quad d(t) = w_1, \quad r_0 = w_3$$

With the change of variables $x_1 = \zeta_1 - w_3$, $x_2 = \zeta_2$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 + \Delta(x_1 + w_3) + g(x_1 + w_3)u + w_1 \\ e &= x_1 \end{aligned} \tag{3.20}$$

The system (3.20) can also be written as

$$
\begin{aligned}
\dot{x} &= Ax + B[f_2(x,w) + g(x_1 + w_3)(u - \chi(w))] \\
e &= Cx
\end{aligned}
\tag{3.21}
$$

where

$$
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},
$$

$f_2(x,w) = -bx_2 + \Delta(x_1 + w_3) + w_1 + g(x_1 + w_3)\chi(w)$, and $\chi(w) = -\left(\frac{w_1 + \Delta(w_3)}{g(w_3)}\right)$. It can be verified that $\chi(w)$ satisfies the identity $L_s^3 \chi = -\omega^2 L_s \chi$. We compare the performance of a saturated high-gain feedback stabilizing controller without a servocompensator (Design 1), with two control designs that use servocompensators (Designs 2 and 3). Design 2 uses the conditional servocompensator (3.13). For this design, Assumption 2.4 is satisfied with $V(x) = \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2)$ and $\psi(x,w) = \frac{1}{g(x_1+w_3)}[bx_2 - \Delta(x_1 + w_3) - g(x_1 + w_3)\chi(w) - w_1 - k_1x_1 - k_2x_2]$ where $k_1 = k_2 = 1$. Assumption 2.5 is satisfied with $v(x) = x_1 + 2x_2$ and $H(x,w) = 1$. Assumption 2.7 is also satisfied since the system (3.21) is locally exponentially stable (*see* Remark 2.2). The control u for Design 2 is given by

$$
\begin{aligned}
\dot{\sigma} &= (S - JK_1)\sigma + \mu J sat\left(\frac{x_1 + 2\hat{x}_2 + K_1\sigma}{\mu}\right) \\
u &= -12\ sat\left(\frac{x_1 + 2\hat{x}_2 + K_1\sigma}{\mu}\right)
\end{aligned}
\tag{3.22}
$$

The state estimate $\hat{x}_2$ is provided by the high-gain observer

$$
\dot{\hat{x}}_1 = \hat{x}_2 + g_1(x_1 - \hat{x}_1)/\epsilon, \quad \dot{\hat{x}}_2 = g_2(x_1 - \hat{x}_1)/\epsilon^2 \tag{3.23}
$$

where g_1 and g_2 are chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. The control u for Design 1 is given by

$$u = -12\ sat\left(\frac{x_1 + 2\hat{x}_2}{\mu}\right)$$

which is similar to (3.22) except that it does not have a servocompensator included. In Design 3, we augment a third-order conventional servo-compensator $\dot{\sigma} = S\sigma + Je$, with the system (3.21) to obtain an *augmented system* of the form

$$\begin{aligned} \dot{\xi} &= A_1\xi + B_1[-bx_2 + \Delta(x_1 + w_3) + w_1 + g(x_1 + w_3)u] \\ e &= C_1\xi \end{aligned} \tag{3.24}$$

where

$$\xi^T = \begin{pmatrix} \sigma^T & x^T \end{pmatrix}, \quad A_1 = \begin{pmatrix} S & JC \\ 0 & A \end{pmatrix}, \quad B_1 = (0\ \ B)^T, \quad C_1 = (0\ \ C)$$

A stabilizing feedback controller, designed via Lyapunov redesign [40], is given by

$$u = \frac{1}{g(x_1 + w_3)}[bx_2 - \Delta(x_1 + w_3) - g(x_1 + w_3)\chi(w) - w_1 - \frac{1}{k_m}\phi(\varpi/\mu)]$$

where k_m is a positive constant, $\varpi = 2B^T P\xi$, the state estimate $\hat{x}_2$ is provided by the high-gain observer (3.33) and $P = P^T > 0$ is the solution of the Lyapunov equation

$$P(A_1 + B_1K_2) + (A_1 + B_1K_2)^T P = -I$$

in which K_2 is chosen such that $(A_1 + B_1K_2)$ is Hurwitz.

We use the following numerical values in the simulation: $\hat{m} = m = 0.2$ kg, $k = 0.001$, $a = 0.01$, $g = 9.81$ m/s, $L_0 = 1$, $\omega = 2$ rad/s, $d_0 = 1$, $y^* = 0.1$m,

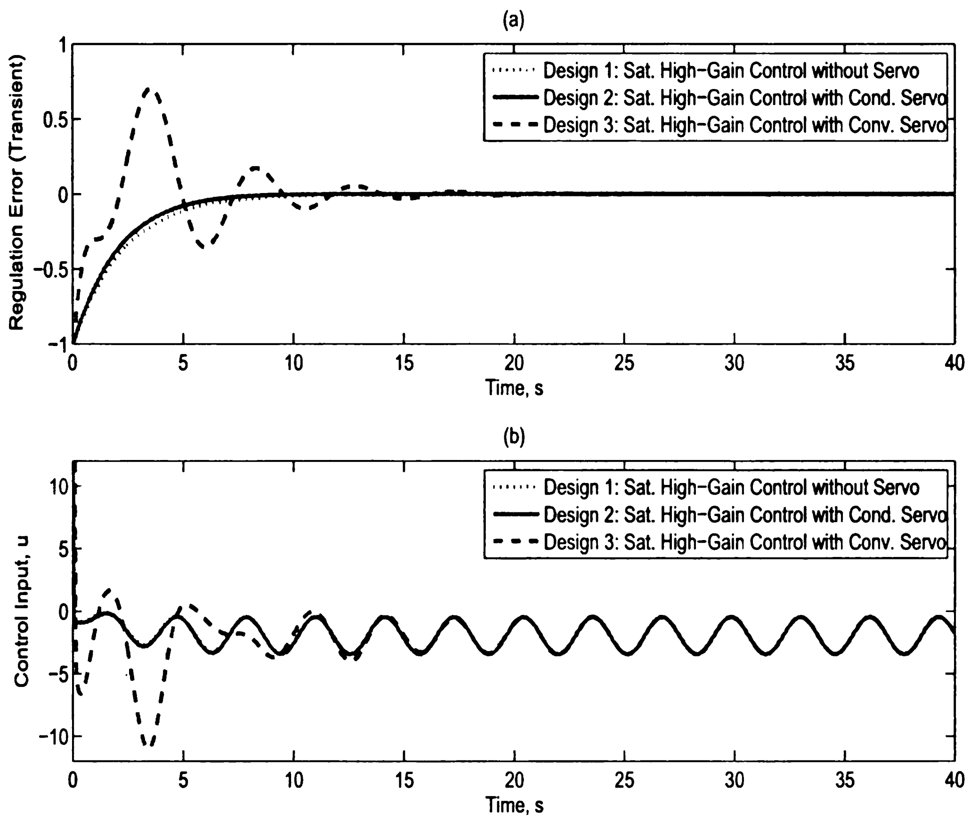


Figure 3.1: Regulation error during the transient period

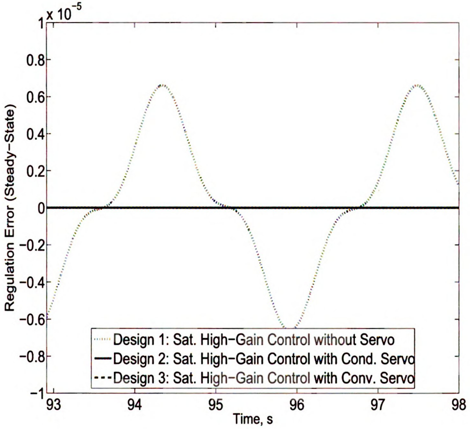


Figure 3.2: Steady-state regulation error

$k_m = 1/10$, $\mu = 0.1$, $g_1 = 2$, $g_2 = 1$ and $\epsilon = 0.01$. K_1 is chosen so as to assign the eigenvalues of $(S - JK_1)$ at $-0.5, -1, -1.5$ and -2. K_2 is chosen so as to assign the eigenvalues of $(A_1 + B_1K_2)$ at $-0.5, -1, -1.5$, -2 and -2.5.

Figure 3.1(a) shows the regulation error during the transient period and Figure 3.2 shows the steady-state regulation error for the three designs. The transient response of the controller design without a servocompensator (Design 1) is close to the one with conditional servocompesator (Design 2), but it does not result in asymptotic error convergence. Asymptotic error convergence to zero is achieved with the conventional servocompensator design (Design 3); however, at the expense of degraded transient performance.

Example 3.4.2.

Consider the single-input single-output nonlinear system [50]

$$\begin{aligned}
\dot{z} &= -z + x_1 + zx_1 - w_1 z \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= z + x_2^2 - x_1 w_2^2 + u \qquad\qquad (3.25) \\
y &= x_1
\end{aligned}$$

It is desired that the output y of the system (3.25) asymptotically tracks a desired constant set point w_1, while rejecting the sinusoidal disturbance w_2. These signals are generated by the exosystem

$$\dot{w} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix} w$$

With the change of variables $\xi_1 = x_1 - w_1$, $\xi_2 = x_2$, we have

$$\begin{aligned}
\dot{z} &= -z + (w_1 + 1 + z)\xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= z + \xi_2^2 - \xi_1 w_2^2 + (u - \chi(w)) \\
e &= \xi_1
\end{aligned} \tag{3.26}$$

where $\chi(w) = -w_1 + w_1 w_2^2$ is the steady-state control input. It can be verified that $\chi(w)$ satisfies the identity $L_s^3\chi = -\omega^2 L_s\chi$. It can be seen that the zero dynamics are globally exponentially stable. The system (3.26) can be written as

$$\begin{aligned}
\dot{z} &= f_1(z, \xi, w) \\
\dot{\xi} &= A\xi + B[f_2(z, \xi, w) + (u - \chi(w))] \\
e &= C\xi
\end{aligned} \tag{3.27}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{3.28}$$

$f_1(z, \xi, w) = -z + (w_1 + 1 + z)\xi_1$, and $f_2(z, \xi, w) = z + \xi_2^2 - \xi_1 w_2^2$.

We compare the performance of two output feedback control designs. Design 1 implements the dynamic output feedback control law of [50] with all design parameters chosen as presented therein, and is given by

$$\dot{\eta} = P\eta + Qe \tag{3.29}$$

$$\dot{\zeta} = \Phi\zeta + N sat(l, \hat{\theta}) \tag{3.30}$$

$$u = -K sat(l, \hat{\theta}) + \varepsilon K M \zeta \tag{3.31}$$

where $\hat{\theta} = \eta_r + k^{r-1}a_0\eta + \cdots + ka_{r-2}\eta_{r-1}$, in which r denotes the relative degree of the system, the saturation function $sat(l, \hat{\theta})$ is defined as

$$sat(l,\hat{\theta}) = \begin{cases} \hat{\theta} & \text{if} \quad \left|\hat{\theta}\right| \le l \\ \frac{\hat{\theta}}{\left|\hat{\theta}\right|} & \text{if} \quad \left|\hat{\theta}\right| > l \end{cases}$$

$$P = \begin{pmatrix} -400 & 1 \\ -(200)^2 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} -400 \\ -(200)^2 \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{pmatrix}, \ N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The following design parameters (as given in [50]) are used in the simulation: $k = 10$, $K = 25$, $\varepsilon = 2$, and $l = 50$. Design 2 uses the conditional servocompensator (3.13). Assumption 3.1 is satisfied since the system is minimum-phase and the zero dynamics are globally exponentially stable. Assumption 2.4 is satisfied with $V(\xi) = \frac{1}{2}(3\xi_1^2 + 2\xi_1\xi_2 + \xi_2^2)$ and $\psi(\xi, w) = -\xi_2^2 + \xi_1 w_2^2 - k_1\xi_1 - k_2\xi_2$ where $k_1 = k_2 = 1$. Assumption 2.5 is satisfied with $v(\xi) = \xi_1 + 2\xi_2$ and $H(x, w) = 1$. Since the closed-loop system is locally exponentially stable, Assumption 2.7 is also satisfied. The control u for Design 2 is given by

$$u = -12\ sat\left(\frac{\xi_1 + 2\hat{\xi}_2 + K_1\sigma}{\mu}\right) \tag{3.32}$$

The state estimate $\hat{\xi}_2$ is provided by the high-gain observer

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + g_1(\xi_1 - \hat{\xi}_1)/\epsilon, \quad \dot{\hat{\xi}}_2 = g_2(\xi_1 - \hat{\xi}_1)/\epsilon^2 \tag{3.33}$$

where g_1 and g_2 are chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. We use the following numerical values in the simulation: $\mu = 0.2$, $g_1 = 2$, $g_2 = 1$ and

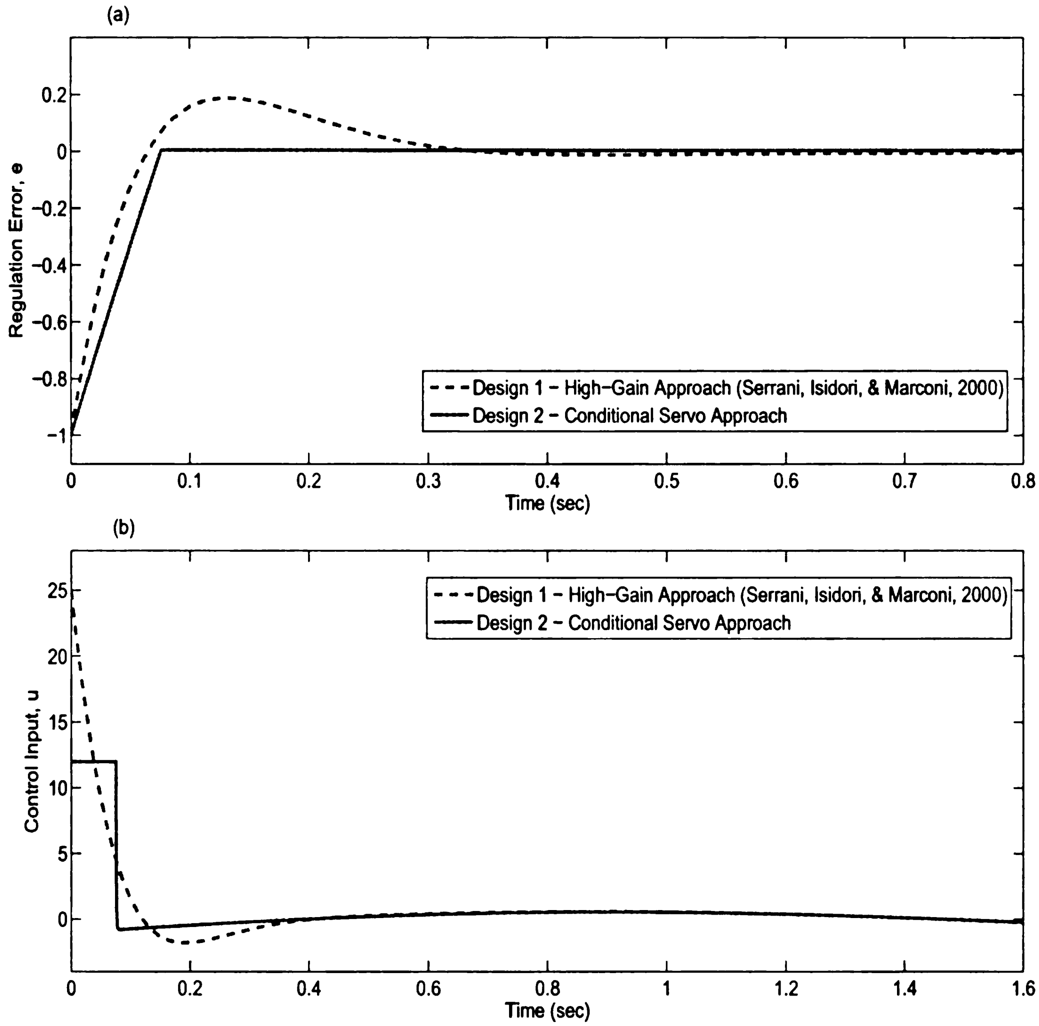


Figure 3.3: Performance of the two control designs (a) Regulation error 'e' during the transient period (b) Control input 'u'

$\epsilon = 0.01$. The gain vector K_1 is chosen so as to assign the eigenvalues of $(S - JK_1)$ at $-0.5, -1, -1.5$ and -2.

Figure 3.3(a) shows the regulation error during the transient period and Figure 3.3(b) shows the control input for the two designs. The transient response of the controller design using high-gain approach (Design 1) has a better rise-time but results in overshoot, which is not present in the case of the controller design using a conditional servocompensator (Design 2). Notice that, in case of Design 1, a large control effort is needed to regulate the error to zero, which in case of Design 2, is saturated to a pre-determined threshold. The use of high-gain feedback, by its nature, leads to a large spike in the control during initial period, and this can be seen in Figure 3(b). It turns out that the trajectories of the closed-loop systems of Design 1 and Design 2 are close to each other under high-gain feedback, so that for the same set of initial conditions or for the same given region of attraction (achieved by adjusting the saturation level), the performance of both designs are close to each other.

3.5 Conclusions

We considered output feedback regulation problem for a class of minimum-phase, input-output linearizable, nonlinear systems, where the states of the system are regulated to a disturbance-dependent invariant manifold on which the regulation error is zero. We showed that the use of conditional servocompensators enables us to achieve zero steady-state regulation error, in the presence of time-varying exogenous signals that are generated by a known exosystem. The output feedback control is implemented using a high-gain observer. The state feedback controller of Chapter 2 can be viewed as an intermediate step towards the output feedback controller of Chapter 3. Analytical results are provided for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally.

Chapter 4

Regulation of Linear Systems Subject to Input Constraints

4.1 Introduction

We consider the problem of output regulation for linear systems subject to input saturation. Stabilization of linear systems under input constraints has been extensively studied over the past decade, c.f., [15, 17, 38, 47, 55, 56]. When the open-loop eigenvalues are in the closed left-half plane, global or semi-global stabilization can be achieved by low-gain feedback or by a combination of low-gain and high-gain feedback. The techniques are extended to the servomechanism problem in [37]. Although more recent results [12, 19] have also dealt with cases with right-half plane eigenvalues; we consider here the case of left-half-plane eigenvalues. The presence of saturation in the input channel imposes strong limitations to the achievable control objectives such as transient performance. In order to achieve desired control objectives we cast the output regulation problem for linear systems subject to input constraints in the Lyapunov redesign framework as presented in Chapter 2. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state stable system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators enables us to achieve zero steady-state tracking error without degrading the transient response of the system. The goal of this work is to apply the Lyapunov-redesign-servocompensator approach of Chapter 2 to the linear regulation problem under input constraints and compare its performance with the approach presented in [37].

We extend the state feedback design to output feedback by using a full-order high-gain observer. This is different from Chapter 3 where a reduced-order high-gain observer was used because the state feedback control was a partial one. In the current problem, because of the control constraint, the mechanism of solving the stabilization problem through ARE necessitates the use of full-state feedback. We use the singular perturbation approach to the observer design described in [13].

4.2 Low-gain design for linear systems

In this section, we briefly review the approach presented in [37] for the semiglobal output regulation problem of linear systems subject to input saturation. Consider a single-input single-output linear system

$$\begin{aligned} \dot{\zeta} &= A\zeta + B\varrho(u) + Ew \\ e &= C\zeta + Fw \end{aligned} \tag{4.1}$$

where $\zeta \in R^n$ is the state, u is the control input, e is the regulation error and $w(t)$ is an *exogenous* input that belongs to a compact set $\mathcal{W} \in R^w$, and is generated by the internal model

$$\dot{w} = Sw \tag{4.2}$$

where S has distinct eigenvalues on the imaginary axis. The function ϱ is defined as

$$\varrho(y) = \begin{cases} y & \text{if} \quad |y| \leq 1 \\ -1 & \text{if} \quad y < -1 \\ 1 & \text{if} \quad y > 1 \end{cases} \tag{4.3}$$

Assumption 4.1. *The matrix A has all eigenvalues in the closed left-half plane, the pair (A, B) is stabilizable, and the pair (A, C) is detectable.*

Assumption 4.2. *There exist matrices Π, Γ such that*

$$\Pi S = A\Pi + B\Gamma + E \ , \quad 0 = C\Pi + F \tag{4.4}$$

where $|\Gamma w| \leq 1 - \delta$ for all $w \in \mathcal{W}$, for some $0 < \delta < 1$.

It is shown in [37] that if ζ and w were available for feedback, a stabilizing state feedback control law can be taken as

$$u = -K(\lambda)\zeta + [K(\lambda)\Pi + \Gamma]w \tag{4.5}$$

where $K(\lambda) = B^T P(\lambda)$, and $P(\lambda)$ is the positive definite solution of the Riccati equation

$$P(\lambda)A + A^T P(\lambda) - P(\lambda)BB^T P(\lambda) + Q(\lambda) = 0 \tag{4.6}$$

in which $Q(\lambda)$ is a positive definite matrix that satisfies $\lim_{\lambda \to 0} Q(\lambda) = 0$. The positive parameter λ is chosen small enough that the control does not saturate over the domain of interest. Assuming that

$$\left(\begin{bmatrix} C & F \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \right)$$

is detectable, the state feedback design is extended to an error feedback design by using the observer

$$\begin{bmatrix} \dot{\hat{\zeta}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varrho(u) + \begin{bmatrix} L_A \\ L_S \end{bmatrix} \left(e - \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \hat{w} \end{bmatrix} \right) \tag{4.7}$$

where the matrices L_A and L_S are chosen such that the matrix

$$\bar{A} \triangleq \begin{pmatrix} A - L_A C & E - L_A F \\ -L_S C & S - L_S F \end{pmatrix} \tag{4.8}$$

is Hurwitz. The error feedback control law is given by

$$u = -K(\lambda)\hat{\zeta} + [K(\lambda)\Pi + \Gamma]\hat{w} \tag{4.9}$$

With the change of variables, $\xi = \zeta - \Pi w$, $\tilde{\zeta} = \zeta - \hat{\zeta}$ and $\tilde{w} = w - \hat{w}$, the closed-loop system can be written as

$$\begin{aligned} \dot{\xi} &= A\xi + B\varrho[-K(\lambda)\xi + \Gamma(w - \tilde{w}) + K(\lambda)(\tilde{\zeta} - \Pi\tilde{w})] \\ &\quad + (A\Pi - \Pi S + E)w \\ \dot{\tilde{\zeta}} &= (A - L_A C)\tilde{\zeta} + (E - L_A F)\tilde{w} \\ \dot{\tilde{w}} &= -L_S C\tilde{\zeta} + (S - L_S F)\tilde{w} \end{aligned} \tag{4.10}$$

The key feature of the approach [37] is designing an observer with much faster dynamics than those of the original system. Since the last two equations of (4.10) are homogeneous, designing the observer dynamics arbitrarily fast yields rapid error convergence to zero. Hence, $\lim_{t\to\infty} e(t) = 0$.

4.3 Problem Statement and Control Design

We now cast the output regulation problem for linear systems subject to input constraints in the Lyapunov redesign framework as presented in Chapters 2 and 3. Our goal here is to design an output feedback controller for the system (4.1) to stabilize the system when $w = 0$ and to asymptotically regulate e to zero when $w \neq 0$. With the change of variables $x = \zeta - \pi$, the system (4.1) can be written as

$$\begin{aligned} \dot{x} &= Ax + B[\varrho(u) - \Gamma w] \\ e &= Cx \end{aligned} \tag{4.11}$$

The system (4.11) is in a form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating Γw as a matched uncertainty. We design a low-gain feedback control law to achieve stabilization and then introduce a conditional servocompensator through a saturated high-gain feedback. Towards that end, let $K(\lambda) = B^T P(\lambda)$ be the state feedback gain matrix. The derivative of the Lyapunov function $V(x) = x^T P(\lambda) x$ with respect to the nominal system $\dot{x} = [A - BK(\lambda)]x$ is

$$\dot{V}(x) = -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x \tag{4.12}$$

For convenience we write $(\partial V/\partial x)B = v(x)$, where $v(x) = 2B^T P(\lambda)x$. The system (4.11) can be written as

$$\dot{x} = (A - BK)x + B\varrho(u) + B[Kx - \Gamma w] \tag{4.13}$$

We design a saturated high-gain feedback controller for this system to deal with the uncertain term Γw. Let $\Omega = \{V(x) \leq c_1\} \subset \mathcal{X}$ be a compact set for some $c_1 > 0$ and

$$|K(\lambda)x - \Gamma w| \leq 1 - \delta_0, \quad \delta_0 > 0 \;\; \forall x \in \Omega, \forall w \in \mathcal{W} \tag{4.14}$$

We introduce the *conditional servocompensator* via the saturated high gain feedback controller

$$u = -\left(\frac{s}{\mu}\right) \tag{4.15}$$

where $s = v(x) + K_1\sigma$ and σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J \varrho\left(\frac{s}{\mu}\right) \tag{4.16}$$

where $\mu > 0$ is the width of the boundary layer, (S, J) is controllable and K_1 is chosen such that $S - JK_1$ is Hurwitz. Equation (4.16) is a perturbation of the exponentially stable system $\dot{\sigma} = (S - JK_1)\sigma$, with the norm of the perturbation bounded by μ. In order to show that σ is always $O(\mu)$, we define the Lyapunov function

$$V_0(\sigma) = \sigma^T P_0 \sigma$$

where the symmetric positive definite matrix P_0 is the solution of $P_0 A_\sigma + A_\sigma^T P_0 = -I$ and $A_\sigma \triangleq S - JK_1$. Consider the compact set $\{\sigma : V_0(\sigma) \leq \mu^2 c_2\}$, where c_2 is a positive constant. Let $\sigma(0)$ belong to this set. Using the inequality

$$\dot{V}_0(\sigma) \leq -\|\sigma\|^2 + 2\mu\,\|\sigma\|\,\|P_0 J\|$$

it is easy to show that $\dot{V}_0(\sigma) \leq 0$ on the boundary $V_0(\sigma) = \mu^2 c_2$ for the choice $c_2 = 4\,\|P_0 J\|^2\,\lambda_{max}(P_0)$. Hence, the set $\{\sigma : V_0(\sigma) \leq \mu^2 c_2\}$ is positively invariant.

4.4 Closed-Loop Analysis

In this section, we will show that, for sufficiently small μ, every trajectory of the closed-loop system asymptotically approaches an invariant manifold on which the error is zero. The forthcoming analysis follows the procedure of Section 2.3 and is presented mainly to delineate the more sharper results that can be obtained in case of the linear systems. Towards that end, the closed-loop system is given by

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= [A - BK]x - B\varrho\left(\frac{s}{\mu}\right) + B[Kx - \Gamma w] \\ \dot{\sigma} &= A_\sigma \sigma + \mu J \varrho\left(\frac{s}{\mu}\right) \end{aligned} \tag{4.17}$$

We start by showing that the set $\Psi = \Omega \times \left\{V_0(\sigma) \le \mu^2 c_2\right\}$ is positively invariant and every trajectory in Ψ reaches the positively invariant set $\Psi_\mu = \{V(x) \le \rho(\mu)\} \times \left\{V_0(\sigma) \le \mu^2 c_2\right\}$ in finite time, where ρ is a class $\mathcal{K}$ function.[1]

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}[A - BK]x - \frac{\partial V}{\partial x}B\varrho\left(\frac{s}{\mu}\right) + \frac{\partial V}{\partial x}B[Kx - \Gamma w] \\ &= -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x - (s - K_1\sigma)\varrho\left(\frac{s}{\mu}\right) \\ &\quad + (s - K_1\sigma)[Kx - \Gamma w] \\ &= -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x - s\varrho\left(\frac{s}{\mu}\right) \\ &\quad + K_1\sigma\varrho\left(\frac{s}{\mu}\right) + s[Kx - \Gamma w] - K_1\sigma[Kx - \Gamma w] \end{aligned}$$

[1] The set Ψ_μ is defined as a positively invariant set, unlike the definition used in Chapter 2 where $\Psi_\mu = \{\|x\| \le \rho(\mu)\} \times \left\{V_0(\sigma) \le \mu^2 c_2\right\}$. The sharper definition used here is due to the fact that $V(x)$ is independent of w.

Inside Ψ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. When $|s| \geq \mu$, from (4.3) and (4.14), we have

$$\begin{aligned} \dot{V} &\leq -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x - |s| + \|K_1\|\,\|\sigma\| + |s| + \|K_1\|\,\|\sigma\| \\ &\leq -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x + \mu\gamma_1 \end{aligned} \tag{4.18}$$

where $\gamma_1 = 2\,\|K_1\|\,\sqrt{c_2/\lambda_{min}(P_0)}$. Similarly, when $|s| \leq \mu$

$$\begin{aligned} \dot{V} &\leq -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x - \frac{|s|^2}{\mu} + \|K_1\|\,\|\sigma\|\,\frac{|s|}{\mu} + |s| + \|K_1\|\,\|\sigma\| \\ &\leq -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x + \mu\gamma_2 \end{aligned} \tag{4.19}$$

where $\gamma_2 = \gamma_1 + (1/4)$. From (4.18) and (4.19), we have

$$\dot{V} \leq -x^T[Q(\lambda) + P(\lambda)BB^TP(\lambda)]x + \mu\gamma_2, \quad \forall (x,\sigma) \in \Psi \tag{4.20}$$

Hence, from [29, Theorem 4.18], for sufficiently small μ, Ψ is positively invariant and all trajectories starting in Ψ enter the positively invariant set Ψ_μ in finite time.

Next, we use $V_1 = \frac{1}{2}s^2$ to show that the trajectories reach the boundary layer $\{|s| \leq \mu\}$ in finite time. Since $P(\lambda)$ is positive definite, $B^TP(\lambda)B > 0$. Let $k_p = 2B^TP(\lambda)B$. For $(x,\sigma) \in \Psi_\mu$, we have

$$\begin{aligned} s\dot{s} &= -2sB^TPB\varrho\left(\frac{s}{\mu}\right) + 2sB^TP[A - BK]x \\ &\quad + 2sB^TPB[Kx - \Gamma w] + sK_1A_\sigma + \mu sK_1J\varrho\left(\frac{s}{\mu}\right) \end{aligned}$$

Outside the boundary layer, i.e. when $|s| \geq \mu$, we have

$$\begin{aligned} s\dot{s} &\leq -k_p|s| + \|2B^TP[A - BK]x\|\,|s| + k_p\big|Kx - \Gamma w\big|\,|s| \\ &\quad + (\|\sigma\|\,\|K_1\|\,\|A_\sigma\| + \mu\,\|K_1\|\,\|J\|)\,|s| \end{aligned} \tag{4.21}$$

Inside Ψ_μ, $\|\sigma\| \le \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $[A - BK(\lambda)]x$ is continuous and vanishes at $x = 0$. Therefore, the norm $\|2B^T P(\lambda)[A - BK(\lambda)]x\,\|$ together with the norms $\|\sigma\|\,\|K_1\|\,\|A_\sigma\|$ and $\mu\,\|K_1\|\,\|J\|$ can be bounded by a class $\mathcal{K}$ function $\rho_1(\mu)$. Hence,

$$\begin{aligned} s\dot{s} &\le -k_p|s| + k_p(1-\delta_0)|s| + \rho_1(\mu)|s| \\ \Longrightarrow \qquad \dot{V}_1 &\le -k_p\left[\delta_0 - \frac{\rho_1(\mu)}{k_p}\right]|s| \end{aligned}$$

Thus, for sufficiently small μ, all trajectories inside Ψ_μ would reach the boundary layer $\{|s| \le \mu\}$ in finite time.

Finally, we show that inside the boundary layer the trajectories of the closed-loop system asymptotically approach an invariant manifold on which the error is zero. Inside the boundary layer, the closed-loop system (4.17) is given by

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= [A - BK]x - B\left(\frac{s}{\mu}\right) + B[Kx - \Gamma w] \\ \dot{\sigma} &= S\sigma + Jv(x) \end{aligned} \tag{4.22}$$

From [53], there exists a unique matrix Λ such that

$$S\Lambda = \Lambda S, \qquad -K_1\Lambda = \Gamma \tag{4.23}$$

We define $\mathcal{N}_\mu = \{x = 0, \sigma = \bar{\sigma}\}$, where $\bar{\sigma} = \mu\Lambda w$. It is easy to verify by direct substitution that $\mathcal{N}_\mu$ is an invariant manifold of (4.22) for all $w \in \mathcal{W}$. Defining $\tilde{\sigma} = \sigma - \bar{\sigma}$ and $\tilde{s} = v + K_1\tilde{\sigma}$, the closed-loop system inside the boundary layer can be written as

$$
\begin{aligned}
\dot{w} &= S_0 w \\
\dot{x} &= [A - BK]x - B\left(\frac{\tilde{s}}{\mu}\right) + BKx \\
\dot{\tilde{\sigma}} &= A_\sigma \tilde{\sigma} + J\tilde{s} \;=\; S\tilde{\sigma} + Jv
\end{aligned} \tag{4.24}
$$

Define the Lyapunov function

$$
V_2 = V(x) + \frac{b}{\mu}\tilde{\sigma}^T P_0 \tilde{\sigma} + \frac{c}{2}\tilde{s}^2 \tag{4.25}
$$

where b and c are positive constants to be chosen. Calculating $\dot{V}_2$ along the trajectories of the system (4.24), we obtain

$$
\dot{V}_2 = \dot{V} + \frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] + c\tilde{s}\dot{\tilde{s}} \tag{4.26}
$$

Calculating $\dot{V}$ along the trajectories of (4.24), we have

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}[A - BK(\lambda)]x - \frac{\partial V}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial V}{\partial x}BK(\lambda)x \\
&\leq -\lambda_{min}(Q)\|x\|^2 - \frac{|v|^2}{\mu} + \frac{k_a}{\mu}|v|\,\|\tilde{\sigma}\| + k_0|v|\|x\|
\end{aligned} \tag{4.27}
$$

where k_a and k_0 are the upper bounds on $\|K_1\|$ and $\|K\|$, respectively. The second term of $\dot{V}_2$ satisfies the inequality

$$
\begin{aligned}
\frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] \;\leq\; & -\frac{b}{\mu}\|\tilde{\sigma}\|^2 \\
& + \frac{2bk_1}{\mu}\|\tilde{\sigma}\|\,|\tilde{s}|\lambda_{max}(P_0)
\end{aligned} \tag{4.28}
$$

where k_1 is the upper bound on $\|J\|$. Next, we have

$$
\begin{aligned}
\dot{\tilde{s}} &= \frac{\partial v}{\partial x}[A - BK(\lambda)]x - \frac{\partial v}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial v}{\partial x}BK(\lambda)x \\
&\quad + K_1(S\tilde{\sigma} + Jv) \\
c\tilde{s}\dot{\tilde{s}} &\leq -c(k_p/\mu)|\tilde{s}|^2 + ck_2|\tilde{s}|\|x\| + ck_3|\tilde{s}|\,\|\tilde{\sigma}\| \\
&\quad + ck_4|\tilde{s}||v| \qquad (4.29)
\end{aligned}
$$

where k_2, k_3 and k_4 are some positive constants. From (4.27), (4.28) and (4.29), we have

$$
\begin{aligned}
\dot{V}_2 &\leq -\lambda_{min}(Q)\|x\|^2 - \frac{1}{\mu}|v|^2 - \frac{b}{\mu}\,\|\tilde{\sigma}\|^2 - c(k_p/\mu)|\tilde{s}|^2 \\
&\quad + k_0|v|\|x\| + \frac{k_a}{\mu}|v|\,\|\tilde{\sigma}\| + ck_4|\tilde{s}||v| + ck_2|\tilde{s}|\|x\| \\
&\quad + [(2bk_1/\mu)\lambda_{max}(P_0) + ck_3]\,\|\tilde{\sigma}\|\,|\tilde{s}| \qquad (4.30)
\end{aligned}
$$

The right-hand side of (4.30) can be arranged in the following quadratic form of $\Pi = [\ \|x\|\ |v|\ \|\tilde{\sigma}\|\ |\tilde{s}|\ ]^T$:

$$
\dot{V}_2 \leq -\Pi^T\Delta\Pi \qquad (4.31)
$$

where the symmetric matrix Δ is given by

$$
\Delta = \begin{pmatrix}
\lambda_{min}(Q) & -\frac{k_0}{2} & 0 & \frac{-ck_2}{2} \\
 & \frac{1}{\mu} & -\frac{k_a}{2\mu} & \frac{-ck_4}{2} \\
 & & \frac{b}{\mu} & -\frac{bk_1\lambda_{max}(P_0)}{\mu} - \frac{ck_3}{2} \\
 & & & \frac{ck_p}{\mu}
\end{pmatrix}
$$

Similar to Chapter 2, the leading principal minors of Δ can be made positive by first choosing b large enough, and then, choosing c large. Finally, by choosing μ

sufficiently small, $\dot{V}_2$ will be negative definite. Therefore, inside the boundary layer, the trajectories of the closed-loop system will exponentially approach the zero-error manifold $\mathcal{N}_\mu$ as $t \to \infty$. Our conclusions can be summarized in the following theorem.

Theorem 4.1. *Suppose Assumptions 4.1 - 4.2 are satisfied and consider the closed-loop system comprising of the system (4.11), the servocompensator (4.16) and the state feedback control (4.15). Then, there exists $\mu^* > 0$ such that $\forall \mu \in (0, \mu^*]$, the set the set $\Psi = \Omega \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ is a subset of the region of attraction, and for all initial conditions in Ψ, the state variables are bounded and $\lim_{t\to\infty} e(t) = 0$.*

4.5 Output Feedback Design

In this section, we extend the state feedback controller of the previous section to output feedback by using a full-order high-gain observer. This is different from Chapter 3 where a reduced-order high-gain observer was used because the state feedback control was a partial one. In the current problem, because of the control constraint, the mechanism of solving the stabilization problem through ARE necessitates the use of full-state feedback. We use the singular perturbation approach to the observer design described in [13].

The state feedback control (4.15) is implemented as an observer-based controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\varrho(u) + L(e - C\hat{x}) \\ u &= -\left(\frac{\hat{s}}{\mu}\right) \end{aligned} \tag{4.32}$$

where L is the vector of observer gains to be designed and $\hat{s} = 2B^T P(\lambda)\hat{x} + K_1\sigma$. The estimation error, $\eta = x - \hat{x}$, satisfies the equation

$$\dot{\eta} = (A - LC)\eta - B\Gamma w \tag{4.33}$$

The observer design starts by transforming the system into the normal form. There is a nonsingular matrix T such that $x = T \begin{pmatrix} x_a \\ x_f \end{pmatrix}$ transforms the system (4.11) into the form

$$\begin{aligned} \dot{x}_a &= A_{aa}x_a + A_{af}y && (4.34) \\ \dot{x}_f &= A_f x_f + B_f[E_a x_a + E_f x_f + \varrho(u) - \Gamma w] && (4.35) \\ y &= C_f x_f && (4.36) \end{aligned}$$

where (A_f, B_f, C_f) represents a chain of integrators. The eigenvalues of A_{aa} are the zeros of the triplet (A, B, C). The observer gain L is designed as [13]

$$L(\varepsilon) = T \begin{pmatrix} A_{af} \\ M(\varepsilon)L_f \end{pmatrix} \tag{4.37}$$

where L_f assigns the eigenvalues of $(A_f - L_f C_f)$ in the open left-half plane, $M(\varepsilon) = blkdiag[\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \cdots, \frac{1}{\varepsilon^q}]$, $q = dim(x_f)$, and ε is a small positive constant. The observer gain $L(\varepsilon)$ assigns the observer eigenvalues into two groups: $(n - q)$ eigenvalues are assigned at the open-loop invariant zeros and q eigenvalues are assigned at $O(1/\varepsilon)$ locations, approaching the eigenvalues of $(A_f - L_f C_f)/\varepsilon$ as $\varepsilon \to 0$. The state component x_a is estimated by the observer

$$\dot{\hat{x}}_a = A_{aa}\hat{x}_a + A_{af}y \tag{4.38}$$

and the state x_f is estimated using the high-gain observer

$$\dot{\hat{x}}_f = A_f\hat{x}_f + M(\varepsilon)L_f(y - C_f\hat{x}_f) + B_f\varrho(u) \tag{4.39}$$

The change of variables $\eta = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} \eta_s \\ \bar{\eta}_f \end{pmatrix}$ transforms the error equation (4.33) into

$$\begin{aligned} \dot{\eta}_s &= A_{aa}\eta_s && (4.40) \\ \dot{\bar{\eta}}_f &= [A_f - M(\varepsilon)L_fC_f]\bar{\eta}_f + B_f[E_a\eta_s + E_f\bar{\eta}_f - \Gamma w] && (4.41) \end{aligned}$$

where $\eta_s = x_a - \hat{x}_a$ and $\bar{\eta}_f = x_f - \hat{x}_f$. To bring the system (4.40)-(4.41) into the standard singularly perturbed form we need to scale $\bar{\eta}_f$ as

$$\eta_f = N^{-1}(\varepsilon)\bar{\eta}_f \tag{4.42}$$

where $N(\varepsilon) = blkdiag[\varepsilon^{q-1}, \cdots, \varepsilon, 1]$. With the special structure of the matrices $N(\varepsilon)$, A_f, B_f, C_f, $M(\varepsilon)$ and L_f, it is shown in [13] that

$$\begin{aligned} N^{-1}(\varepsilon)B_f &= B_f \\ N^{-1}(\varepsilon)[A_f - M(\varepsilon)L_fC_f]N(\varepsilon) &= \frac{1}{\varepsilon}[A_f - L_fC_f] \end{aligned}$$

where $[A_f - L_fC_f]$ is Hurwitz. The scaling (4.42) transforms (4.40)-(4.41) into the standard singularly perturbed system

$$\begin{aligned} \dot{\eta}_s &= A_{aa}\eta_s && (4.43) \\ \varepsilon\dot{\eta}_f &= [A_f - L_fC_f]\eta_f + \varepsilon B_f[E_a\eta_s + E_fN(\varepsilon)\eta_f - \Gamma w] && (4.44) \end{aligned}$$

We notice that the perturbation term Γw is multiplied by ε so that its effect diminishes asymptotically as $\varepsilon \longrightarrow 0$.

We now analyze the closed-loop system composed of (4.11), (4.16) and (4.32).

Using $(x, \sigma, \eta_s, \eta_f)$ as the state vector, the closed-loop system is given by

$$\dot{x} = [A - BK]x + B\varrho\Big[-\frac{1}{\mu}\Big(s - 2B^T P(\lambda)\begin{bmatrix} T_1 & T_2 N(\varepsilon) \end{bmatrix}\begin{bmatrix} \eta_s \\ \eta_f \end{bmatrix}\Big)\Big] + B[Kx - \Gamma w]$$

$$\dot{\sigma} = A_\sigma \sigma + \mu J \varrho\Big[\frac{1}{\mu}\Big(s - 2B^T P(\lambda)\begin{bmatrix} T_1 & T_2 N(\varepsilon) \end{bmatrix}\begin{bmatrix} \eta_s \\ \eta_f \end{bmatrix}\Big)\Big]$$

$$\dot{\eta}_s = A_{aa}\eta_s \tag{4.45}$$

$$\varepsilon\dot{\eta}_f = [A_f - L_f C_f]\eta_f + \varepsilon B_f[E_a \eta_s + E_f N(\varepsilon)\eta_f - \Gamma w]$$

The system (4.45) is a standard singularly perturbed system with (x, σ, η_s) as the slow variable and η_f as the fast variable. The slow model of (4.45) is obtained by setting $\varepsilon = 0$ in the last equation of (4.45). Since $[A_f - L_f C_f]$ is Hurwitz, hence non-singular, we obtain the unique root $\eta_f = 0$. Substitution of $\eta_f = 0$ in (4.45) results in the slow model

$$\dot{x} = [A - BK]x + B\varrho\Big[-\frac{1}{\mu}\Big(s - 2B^T P(\lambda) T_1 \eta_s\Big)\Big] + B[Kx - \Gamma w] \tag{4.46}$$

$$\dot{\sigma} = A_\sigma \sigma + \mu J \varrho\Big[\frac{1}{\mu}\Big(s - 2B^T P(\lambda) T_1 \eta_s\Big)\Big] \tag{4.47}$$

$$\dot{\eta}_s = A_{aa}\eta_s \tag{4.48}$$

which appears as the cascade connection of (4.48) and the closed-loop system under the state feedback (4.17). We will now show that, for sufficiently small μ, every trajectory of the system (4.46)-(4.48) exponentially approaches an invariant manifold on which the error is zero. Towards that end, we define the Lyapunov function

$$V_3 = V(x) + \alpha_s V_s(\eta_s) \tag{4.49}$$

where $\alpha_s > 0$ and $V_s = \eta_s^T P_s \eta_s$, in which P_s is the positive definite solution of the

Lyapunov equation

$$P_s A_{aa} + A_{aa}^T P_s = -I \tag{4.50}$$

The function $V_s = \eta_s^T P_s \eta_s$ satisfies $\dot{V}_s \leq -\|\eta_s\|^2$. Let $\Omega_0 = \{V_3 \leq c_{11}\} \subset \mathcal{X}$ be a compact set for some $c_{11} > 0$. We start by showing that the set $\Sigma = \Omega_0 \times \{V_0(\sigma) \leq \mu^2 c_2\}$ is positively invariant and every trajectory in Σ reaches the positively invariant set $\Sigma_\mu = \{V_3 \leq \rho_0(\mu)\} \times \{V_0(\sigma) \leq \mu^2 c_2\}$ in finite time, where ρ_0 is a class $\mathcal{K}$ function.

$$\begin{aligned}
\dot{V}_3 &= \frac{\partial V}{\partial x}[A - BK]x - \frac{\partial V}{\partial x}B\varrho\left(\frac{s}{\mu}\right) + \frac{\partial V}{\partial x}B[Kx - \Gamma w] \\
&\quad + \frac{\partial V}{\partial x}B\left[\varrho\left(\frac{s}{\mu}\right) - \varrho\left(\frac{s - 2B^T P(\lambda) T_1 \eta_s}{\mu}\right)\right] + \alpha_s \dot{V}_s(\eta_s) \\
&= -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x - (s - K_1\sigma)\varrho\left(\frac{s}{\mu}\right) + (s - K_1\sigma)[Kx - \Gamma w] \\
&\quad + (s - K_1\sigma)\left[\varrho\left(\frac{s}{\mu}\right) - \varrho\left(\frac{s - 2B^T P(\lambda) T_1 \eta_s}{\mu}\right)\right] + \alpha_s \dot{V}_s(\eta_s) \\
&= -x^T[Q + PBB^T P]x - s\varrho\left(\frac{s}{\mu}\right) + K_1\sigma\varrho\left(\frac{s}{\mu}\right) + s[Kx - \Gamma w] \\
&\quad - K_1\sigma[Kx - \Gamma w] + s\left[\varrho\left(\frac{s}{\mu}\right) - \varrho\left(\frac{s - 2B^T P(\lambda) T_1 \eta_s}{\mu}\right)\right] \\
&\quad - K_1\sigma\left[\varrho\left(\frac{s}{\mu}\right) - \varrho\left(\frac{s - 2B^T P(\lambda) T_1 \eta_s}{\mu}\right)\right] + \alpha_s \dot{V}_s(\eta_s)
\end{aligned} \tag{4.51}$$

Inside Σ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. When $|s| \geq \mu$, from (4.3) and (4.14), we have

$$\begin{aligned}
\dot{V}_3 &\leq -\lambda_{min}(Q)\|x\|^2 - |s| + \|K_1\|\,\|\sigma\| + |s| + \|K_1\|\,\|\sigma\| \\
&\qquad + \{\|v(x)\| + \|K_1\|\,\|\sigma\|\}\, L_1\,\|\eta_s\| + \|K_1\|\,\|\sigma\|\, L_1\,\|\eta_s\| - \alpha_s\,\|\eta_s\|^2 \\
&\leq -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)L_1\,\|x\|\,\|\eta_s\| - \alpha_s\,\|\eta_s\|^2 + \mu\gamma_3
\end{aligned} \tag{4.52}$$

where $\gamma_3 = 2\|K_1\|\sqrt{c_2/\lambda_{min}(P_0)}(1 + L_1\|\eta_s\|)$, in which L_1 is a positive Lipschitz constant that satisfies the inequality

$$\varrho\left(\frac{s - 2B^T P(\lambda)T_1\eta_s}{\mu}\right) - \varrho\left(\frac{s}{\mu}\right) \le L_1\|\eta_s\|$$

Similarly, when $|s| \le \mu$, we have

$$\begin{aligned}\dot{V}_3 \;&\le\; -\lambda_{min}(Q)\|x\|^2 - \frac{|s|^2}{\mu} + \|K_1\|\|\sigma\|\frac{|s|}{\mu} + |s| + \|K_1\|\|\sigma\| \\ &\quad + \{\|v(x)\| + \|K_1\|\|\sigma\|\}L_1\|\eta_s\| + \|K_1\|\|\sigma\|L_1\|\eta_s\| - \alpha_s\|\eta_s\|^2 \\ &\le\; -\lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)L_1\|x\|\|\eta_s\| - \alpha_s\|\eta_s\|^2 + \mu\gamma_4 \qquad (4.53)\end{aligned}$$

where $\gamma_4 = \gamma_3 + (1/4)$. From (4.52) and (4.53), we have

$$\dot{V}_3 \le \lambda_{min}(Q)\|x\|^2 + 2\lambda_{max}(P)L_1\|x\|\|\eta_s\| - \alpha_s\|\eta_s\|^2 + \mu\gamma_4,\ \forall(x,\sigma,\eta_s) \in \Sigma \quad (4.54)$$

The right-hand side of (4.54) can be arranged in the following quadratic form of $\Pi_1 = [\|x\|\ \ \|\eta_s\|]^T$:

$$\dot{V}_3 \;\le\; -\Pi_1^T\Delta_1\Pi_1 + \mu\gamma_4 \qquad (4.55)$$

where the symmetric matrix Δ_1 is given by

$$\Delta_1 = \begin{pmatrix} \lambda_{min}(Q) & -\lambda_{max}(P)L_1 \\ -\lambda_{max}(P)L_1 & \alpha_s \end{pmatrix}$$

By choosing α_s large enough the matrix Δ_1 can be made positive definite. Then, from [29, Theorem 4.18], for sufficiently small μ, Σ is positively invariant and all trajectories starting in Σ enter the positively invariant set Σ_μ in finite time. In the next step, we use $V_1 = \frac{1}{2}s^2$ to show that the trajectories reach the boundary layer

$\{|s| \leq \mu\}$ in finite time. For $(x, \sigma, \eta_s) \in \Sigma_\mu$, we have

$$\begin{aligned} s\dot{s} &= 2sB^T P[A - BK]x - 2sB^T PB\varrho\left(\frac{s}{\mu}\right) + 2sB^T PB[Kx - \Gamma w] \\ &+ 2sB^T PB\left[\varrho\left(\frac{s}{\mu}\right) - \varrho\left(\frac{s - 2B^T P(\lambda)T_1\eta_s}{\mu}\right)\right] \\ &+ sK_1 A_\sigma + \mu s K_1 J\varrho\left(\frac{s - 2B^T P(\lambda)T_1\eta_s}{\mu}\right) \end{aligned}$$

Outside the boundary layer, i.e. when $|s| \geq \mu$, we have

$$\begin{aligned} s\dot{s} &\leq \|2B^T P[A - BK]x\| |s| - k_p|s| + k_p|Kx - \Gamma w||s| \\ &+ k_p L_1 \|\eta_s\| |s| + (\|\sigma\| \|K_1\| \|A_\sigma\| + \mu \|K_1\| \|J\| \|\eta_s\|) |s| \end{aligned} \quad (4.56)$$

Inside Σ_μ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $[A - BK(\lambda)]x$ is continuous and vanishes at $x = 0$. Therefore, the norm $\|2B^T P(\lambda)[A - BK(\lambda)]x\|$ together with the norms $k_p L_1 \|\eta_s\|$, $\|\sigma\| \|K_1\| \|A_\sigma\|$ and $\mu \|K_1\| \|J\|$ can be bounded by a class $\mathcal{K}$ function $\rho_2(\mu)$. Hence,

$$\begin{aligned} \dot{V}_1 = s\dot{s} &\leq -k_p|s| + k_p(1 - \delta_0)|s| + \rho_2(\mu)|s| \\ &\leq -k_p\left[\delta_0 - \frac{\rho_2(\mu)}{k_p}\right]|s| \end{aligned}$$

Thus, for sufficiently small μ, all trajectories inside Σ_μ would reach the boundary layer $\{|s| \leq \mu\}$ in finite time.

Finally, we show that inside the boundary layer the trajectories of the closed-loop system (4.46)-(4.48) exponentially approach an invariant manifold on which the error is zero. Inside the boundary layer, the closed-loop system (4.46)-(4.48) is given by

$$\dot{x} = [A - BK]x - B\left(\frac{s}{\mu}\right) + B\left(\frac{2B^T P(\lambda) T_1 \eta_s}{\mu}\right) + B[Kx - \Gamma w] \quad (4.57)$$

$$\dot{\sigma} = S\sigma + Js - J\left(\frac{2B^T P(\lambda) T_1 \eta_s}{\mu}\right) \quad (4.58)$$

$$\dot{\eta}_s = A_{aa}\eta_s \quad (4.59)$$

Next, we define

$$Z_\mu = \{x = 0, \sigma = \bar{\sigma}, \eta_s = 0\}, \quad \bar{\sigma} = \mu\Lambda w \quad (4.60)$$

which is an invariant manifold of (4.57)-(4.59) for all $w \in \mathcal{W}$. Defining $\tilde{\sigma} = \sigma - \bar{\sigma}$ and $\tilde{s} = v + K_1\tilde{\sigma}$, the closed-loop system inside the boundary layer can be written as

$$\dot{x} = [A - BK]x - B\left(\frac{\tilde{s}}{\mu}\right) + B\left(\frac{2B^T P(\lambda) T_1 \eta_s}{\mu}\right) + BKx \quad (4.61)$$

$$\dot{\tilde{\sigma}} = A_\sigma\tilde{\sigma} + J\tilde{s} \quad (4.62)$$

$$\dot{\eta}_s = A_{aa}\eta_s \quad (4.63)$$

Define the Lyapunov function

$$V_4 = V(x) + \frac{\beta_s}{\mu}V_s(\eta_s) + \frac{p}{\mu}\tilde{\sigma}^T P_0 \tilde{\sigma} + \frac{q}{2}\tilde{s}^2 \quad (4.64)$$

where β_s, p and q are positive constants to be chosen. Calculating $\dot{V}_4$ along the trajectories of the system (4.61)-(4.63), we obtain

$$\dot{V}_4 = \dot{V} + \frac{\beta_s}{\mu}\dot{V}_s + \frac{p}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] + q\tilde{s}\dot{\tilde{s}} \quad (4.65)$$

Calculating $\dot{V} + \frac{\beta_s}{\mu}\dot{V}_s$ along the trajectories of (4.61)-(4.63), we have

$$\begin{aligned}
\dot{V} + \frac{\beta_s}{\mu}\dot{V}_s &= \frac{\partial V}{\partial x}[A - BK(\lambda)]x - \frac{\partial V}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial V}{\partial x}B\left(\frac{2B^T P(\lambda)T_1\eta_s}{\mu}\right) \\
&\quad + \frac{\partial V}{\partial x}BK(\lambda)x + \frac{\beta_s}{\mu}\dot{V}_s \\
&\leq -\lambda_{min}(Q)\|x\|^2 - \frac{|v|^2}{\mu} + \frac{k_a}{\mu}|v|\,\|\tilde{\sigma}\| + \frac{2\lambda_{max}(P)}{\mu}|v|\,\|\eta_s\| \\
&\quad + k_0|v|\|x\| - \frac{\beta_s}{\mu}\|\eta_s\|^2 \qquad (4.66)
\end{aligned}$$

where k_a and k_0 are the upper bounds on $\|K_1\|$ and $\|K\|$, respectively. The third term of $\dot{V}_4$ satisfies the inequality

$$\frac{p}{\mu}\left[\tilde{\sigma}^T P_0\dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0\tilde{\sigma}\right] \leq -\frac{p}{\mu}\|\tilde{\sigma}\|^2 + \frac{2pk_1}{\mu}\|\tilde{\sigma}\|\,|\tilde{s}|\lambda_{max}(P_0) \qquad (4.67)$$

where k_1 is the upper bound on $\|J\|$. Next, we have

$$\begin{aligned}
\dot{\tilde{s}} &= \frac{\partial v}{\partial x}Ax - \frac{\partial v}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial v}{\partial x}B\left(\frac{2B^T P(\lambda)T_1\eta_s}{\mu}\right) + K_1(S\tilde{\sigma} + Jv) \\
q\tilde{s}\dot{\tilde{s}} &\leq qk_5|\tilde{s}|\|x\| - q(k_p/\mu)|\tilde{s}|^2 + 2q(k_p/\mu)\lambda_{max}(P)\,|\tilde{s}|\,\|\eta_s\| \\
&\quad + qk_6|\tilde{s}|\,\|\tilde{\sigma}\| + qk_7|\tilde{s}||v| \qquad (4.68)
\end{aligned}$$

where k_5, k_6 and k_7 are some positive constants. From (4.66), (4.67) and (4.68), we have

$$\begin{aligned}
\dot{V}_4 &\leq -\lambda_{min}(Q)\|x\|^2 - \frac{|v|^2}{\mu} + \frac{k_a}{\mu}|v|\,\|\tilde{\sigma}\| + \frac{2\lambda_{max}(P)}{\mu}|v|\,\|\eta_s\| \\
&\quad + k_0|v|\|x\| - \frac{\beta_s}{\mu}\|\eta_s\|^2 - \frac{p}{\mu}\|\tilde{\sigma}\|^2 + \frac{2pk_1}{\mu}\|\tilde{\sigma}\|\,|\tilde{s}|\lambda_{max}(P_0) \\
&\quad + qk_5|\tilde{s}|\|x\| - q(k_p/\mu)|\tilde{s}|^2 + 2q(k_p/\mu)\lambda_{max}(P)\,|\tilde{s}|\,\|\eta_s\| \\
&\quad + qk_6|\tilde{s}|\,\|\tilde{\sigma}\| + qk_7|\tilde{s}||v| \qquad (4.69)
\end{aligned}$$

The right-hand side of (4.69) can be arranged in the following quadratic form of $\Pi_2 = [\ \|x\|\ \ \|\eta_s\|\ \ |v|\ \ \|\tilde{\sigma}\|\ \ |\tilde{s}|\ ]^T$:

$$\dot{V}_4 \leq -\Pi_2^T \Delta_2 \Pi_2 \tag{4.70}$$

where the symmetric matrix Δ_2 is given by

$$\Delta_2 = \begin{pmatrix} \lambda_{min}(Q) & 0 & -\frac{k_0}{2} & 0 & \frac{-qk_5}{2} \\ & \frac{\beta_s}{\mu} & -\frac{\lambda_{max}(P)}{\mu} & 0 & -\frac{qk_p\lambda_{max}(P)}{\mu} \\ & & \frac{1}{\mu} & -\frac{k_a}{2\mu} & \frac{-qk_7}{2} \\ & & & \frac{p}{\mu} & -\frac{pk_1\lambda_{max}(P_0)}{\mu} - \frac{qk_6}{2} \\ & & & & \frac{qk_p}{\mu} \end{pmatrix}$$

As before, if the principal leading minors of Δ_2 can be made positive by choosing the constants β_s, p and q appropriately, and by choosing μ sufficiently small, then $\dot{V}_4$ will be negative definite. This would imply that, inside the boundary layer, the trajectories of the closed-loop system will exponentially approach Z_μ as $t \to \infty$. Towards that end, we partition the matrix Δ_2 as

$$\Delta_2 = \begin{pmatrix} \lambda_{min}(Q) & -d_{12}^T \\ & \\ -d_{12} & \frac{1}{\mu}D_{22} + \Delta_{22} \end{pmatrix} \tag{4.71}$$

where

$$d_{12} = \begin{pmatrix} 0 & \frac{k_0}{2} & 0 & \frac{qk_5}{2} \end{pmatrix}^T \tag{4.72}$$

$$D_{22} = \begin{pmatrix} \beta_s & -\lambda_{max}(P) & 0 & -qk_p\lambda_{max}(P) \\ & 1 & -\frac{k_a}{2} & 0 \\ & & p & -pk_1\lambda_{max}(P_0) \\ & & & qk_p \end{pmatrix} \tag{4.73}$$

and

$$\Delta_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-qk_7}{2} \\ 0 & 0 & 0 & -\frac{qk_6}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.74}$$

From (4.73), it is easy to see that by choosing β_s, p and q, we can successively make the principal leading minors of D_{22} positive. First, β_s is chosen large enough to make the 2×2 minor positive, then, p is chosen large enough to make the 3×3 minor positive, and then, q is chosen large enough to make the 4×4 minor positive. Finally, choosing μ small enough will render

$$det \begin{pmatrix} \lambda_{min}(Q) & -d_{12}^T \\ & \\ -d_{12} & \frac{1}{\mu}D_{22} + \Delta_{22} \end{pmatrix} > 0 \tag{4.75}$$

Consequently, $\dot{V}_4$ will be negative definite. Therefore, inside the boundary layer, the trajectories of the closed-loop system (4.61)-(4.63) will exponentially approach the zero-error manifold Z_μ.

Let the initial states $(x(0), \sigma(0), \eta_s(0)) \in \mathcal{G}$ and $\eta_f(0) \in \mathcal{H}$, where $\mathcal{G}$ is a compact set which contains Z_μ. Using Theorems 2 and 5 of Atassi & Khalil [3], we can show that there is a neighborhood $\mathcal{N}$ of $Z_\mu \times \{\eta_f = 0\}$, independent of ε, and $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$, the set $Z_\mu \times \{\eta_f = 0\}$ is exponentially stable and every trajectory in $\mathcal{N}$ converges to this set as $t \to \infty$. Furthermore, from [3, Theorems 1, 2 & 5], there is $\varepsilon_2 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_2$, the solutions starting in $\mathcal{G} \times \mathcal{H}$ enter $\mathcal{N}$ in finite time. Hence, for every $0 < \varepsilon \leq \varepsilon_3 = min\{\varepsilon_1, \varepsilon_2\}$, the set $Z_\mu \times \{\eta_f = 0\}$ is exponentially stable and $\mathcal{G} \times \mathcal{H}$ is a subset of the region of attraction. Thus, for sufficiently small ε, the closed-loop system (4.45), under the output feedback controller (4.32), is uniformly exponentially stable with respect to the set $Z_\mu \times \{\eta_f = 0\}$. Hence, $\lim_{t\to\infty} e(t) = 0$. The foregoing conclusions are summarized in the following theorem:

Theorem 4.2. *Suppose Assumptions 4.1 - 4.2 are satisfied and consider the closed-loop system comprising of the system (4.11), the servocompensator (4.16) and the observer-based feedback control (4.32). Then, there exists $\varepsilon^* > 0$ such that $\forall \varepsilon \in (0, \varepsilon^*]$, the closed-loop system, is uniformly exponentially stable with respect to the set $Z_\mu \times \{\eta_f = 0\}$.*

4.6 Example

Consider a minimum-phase linear system, with the transfer function from u to e

$$T(s) = \frac{s+1}{s^2(s+5)}$$

that corresponds to (4.1) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

with the signal w generated by the exosystem

$$\dot{w} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} w, \quad w^T(0) = (0, w_0)$$

We show the performance of two designs: Design I incorporates the saturated high-gain feedback using a conditional servocompensator and the full-order observer (5.12)-(5.13). For this design, K_1 is chosen so as to assign the eigenvalues of $S - JK_1$ at -0.5 and -1, and the observer gain $L(\varepsilon)$ is designed such that the eigenvalue of (5.12) is assigned at the location of the invariant zero of the triplet (A, B, C), i.e. at -1, and $L_f = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is chosen such that the polynomial $\varpi^2 + g_1\varpi + g_2$ is Hurwitz. Design II is based on the linear observer-based error-feedback control approach [37], reviewed in Section III. For this design, a fifth-order linear observer of the form (4.7) is constructed where the matrices L_A and L_S are chosen such as to assign the eigenvalues of the matrix $\bar{A}$ at $[-22, -23, -24, -25, -26]$. We use the following numerical values in the simulation: $\omega = 1$ rad/s, $w_0 = 0.5$, $\mu = 0.1$, $g_1 = 2$, $g_2 = 1$, $\lambda = 0.05$ and $\varepsilon = 0.05$.

Figure 4.1(a) shows the regulation error during the transient period for the two designs and Figure 4.1(b) shows the corresponding control input. The regulation error goes to zero sharply in the case of Design I, where as in Design II, the same oscillates before eventually converging to zero. Note that due to the fact that a higher dimensional (fifth-order) observer is used in Design II, in order to achieve reasonable performance, the observer gains were required to be pushed very high e.g. $O(10^7)$, in contrast to $O(10^3)$ for those in Design I. Figure 4.2 shows the performance of the two control designs when the control coefficient is perturbed by 40% (from 1 to 1.4).

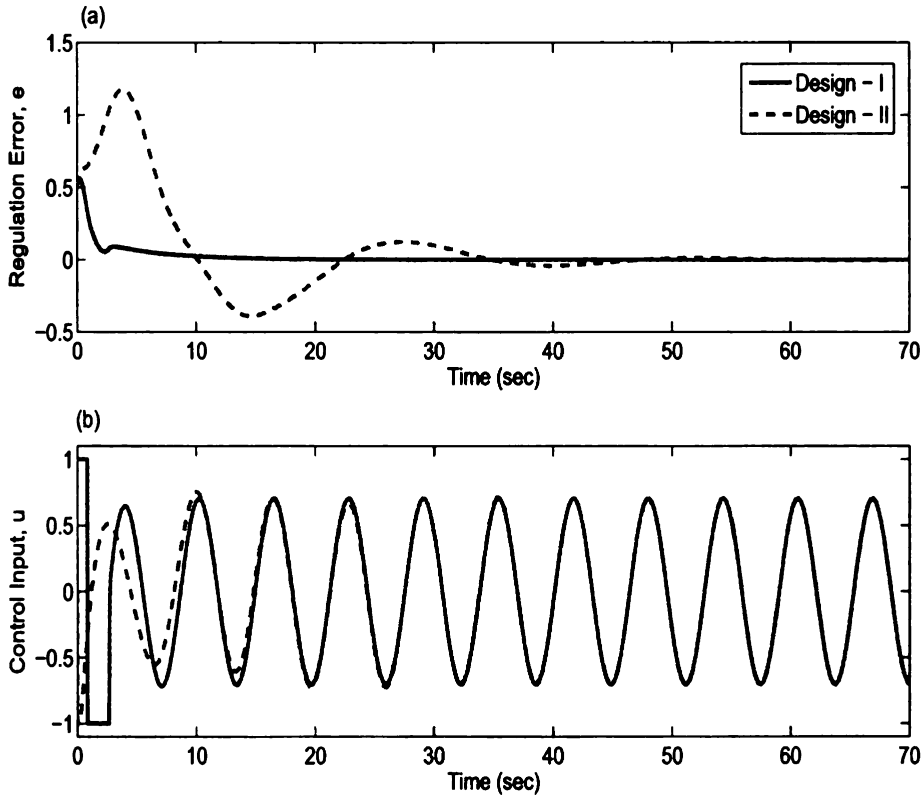


Figure 4.1: Performance comparison of the two control designs (a) Regulation error 'e' during the transient period (b) Corresponding control input, 'u'

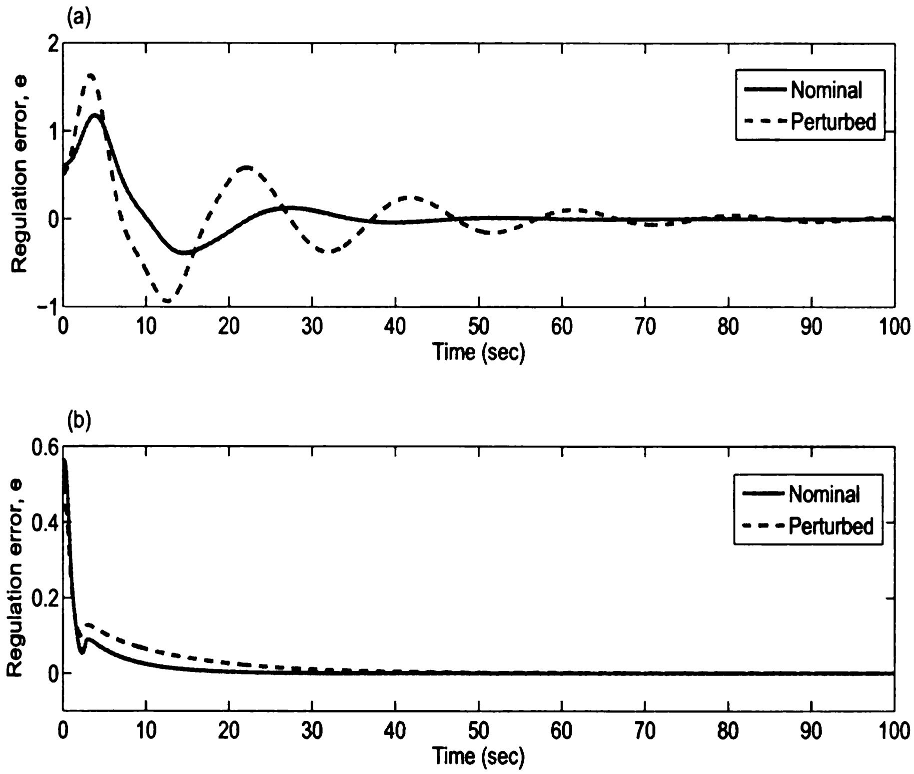


Figure 4.2: Transient performance of the two control designs when control coefficient is perturbed by 40 percent (a) Design II - Nominal vs Perturbed (b) Design I - Nominal vs Perturbed

4.7 Conclusions

In this chapter, we considered the output regulation problem of linear systems subject to input constraints. We presented a novel control design that includes a conditional servocompensator, introduced via Lyapunov redesign and saturated high-gain feedback. The use of a conditional servocompensator enables us to achieve zero steady-state regulation error, without degrading the transient response. The output feedback control is implemented using a two-time-scale observer design of [13] and the performance recovery is shown using the separation principle of [3, 1]. The performance of the control design is demonstrated by simulation.

Chapter 5

Full-Order High-Gain Observers for Minimum Phase Nonlinear Systems

5.1 Introduction

Most nonlinear control techniques assume availability of all state variables to achieve control objectives like stability or asymptotic tracking. Since, in many practical problems we cannot measure all state variables due to technical or economic reasons, a state observer is used to estimate the system states from the output measurements. An exhaustive review of the many approaches to design observers for stabilization of nonlinear dynamical systems appears in [16]. Among these, one popular approach is the *high-gain observer* which is attractive because of its ability to estimate the unmeasured states while rejecting the effect of disturbances. High-gain observers are applicable to a class of nonlinear systems that can be transformed into the *normal* *form*

$$\dot{z} = \psi(z, x) \tag{5.1}$$

$$\dot{x} = Ax + B\phi(z, x, u) \tag{5.2}$$

$$y = Cx \tag{5.3}$$

where $z \in R^l$ and $x \in R^r$ are the system states, $u \in R$ is the control input and $y \in R$ is the measured output. The $r \times r$ matrix A, the $r \times 1$ matrix B and the $1 \times r$ matrix C, are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Over the past several years, many researchers have contributed toward the investigation of output feedback control for the class of systems of the form (5.1)-(5.3). Of significant relevance are the works [2, 3, 13, 29], which solve the problem of robust output feedback stabilization, in the large, of the input-output linearizable nonlinear dynamic systems by means of bounded partial-state feedback control (e.g. $u = \gamma(x)$) and high-gain observer, with subsequent substitution of the estimate of x, provided by the high-gain observer, in the feedback. The boundedness of the control protects the state of the plant from *peaking* when the high-gain observer estimates are used instead of the true states.

In certain applications, feedback control synthesis requires availability of the full state vector (i.e. x and z), rather than just the partial state x. One such example is the output regulation of linear dynamic systems under input contraints treated in Chapter 4. Because of the control constraint, the mechanism of solving the stabilization problem through Algebraic Riccati Equation (ARE) necessitates the use of full-state feedback. In order to design an output feedback control to achieve the desired control objective in the presence of input constraints, the full-order observer design described in [13] is exploited. Extensions to nonlinear systems of the form (5.1)-(5.3) would necessitate the development of a full-order nonlinear high-gain observer. A rather similar situation arises in applications that incorporate optimal stabilizing controllers. The optimality of the stabilizing controller yields strong robustness properties by virtue of its design through the existence of a *Control Lyapunov Function* (CLF) [48]. However, the optimal control design approach assumes the availability of all state variables in order to meet the control objectives. Consequently, the output feedback control can only be implemented using a full-order observer.

A few researchers have considered the estimation of the full state of nonlinear systems using high-gain observers. Esfandiari and Khalil, in [13], use a pole placement/ singular perturbation approach to design a one-parameter observer gain, in order to recover the robustness properties of a state feedback controller designed to stabilize a fully linearizable system of the form

$$\dot{x} = Ax + B\phi(x)u \tag{5.4}$$

$$y = C(x) \tag{5.5}$$

The state feedback control $u = F(x)$ is implemented as an observer-based controller

$$\dot{\hat{x}} = A\hat{x} + B\phi(\hat{x})u + L(y - C\hat{x}) \tag{5.6}$$

$$u = F(\hat{x}) \tag{5.7}$$

where L is the observer gain. The observer design starts by transforming the system into the normal form. There is a nonsingular matrix T such that $x = T \begin{pmatrix} x_a \\ x_f \end{pmatrix}$ transforms the system (5.4)-(5.5) into the form

$$\dot{x}_a = A_{aa}x_a + A_{af}y \tag{5.8}$$

$$\dot{x}_f = A_f x_f + B_f[E_a x_a + E_f x_f + u] \tag{5.9}$$

$$y = C_f x_f \tag{5.10}$$

where (A_f, B_f, C_f) represents a chain of integrators. The eigenvalues of A_{aa} are the zeros of the triplet (A, B, C). The observer gain L is designed as

$$L(\varepsilon) = T \begin{pmatrix} A_{af} \\ M(\varepsilon)L_f \end{pmatrix} \tag{5.11}$$

where L_f assigns the eigenvalues of $(A_f - L_f C_f)$ in the open left-half plane, $M(\varepsilon) =$ block-diag $[\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \cdots, \frac{1}{\varepsilon^q}]$, $q = dim(x_f)$, and ε is a small positive constant. The observer gain $L(\varepsilon)$ assigns the observer eigenvalues into two groups: $(n-q)$ eigenvalues are assigned at the open-loop zeros and q eigenvalues are assigned at $O(1/\varepsilon)$ locations, approaching the eigenvalues of $(A_f - L_f C_f)/\varepsilon$ as $\varepsilon \to 0$. The state component x_a is estimated by the open-loop observer

$$\dot{\hat{x}}_a = A_{aa}\hat{x}_a + A_{af}y \tag{5.12}$$

and the state x_f is estimated using the high-gain observer

$$\dot{\hat{x}}_f = A_f\hat{x}_f + M(\varepsilon)L_f(y - C_f\hat{x}_f) + B_f\varrho(u) \tag{5.13}$$

This design results in a standard two-time scale singularly perturbed system. Subsequent analysis in [13] reveals the potential of using this approach towards development of full-order high-gain observers for the class of nonlinear systems under investigation here.

A recent result [28] designs a full-order observer for the class of nonlinear systems of the form (5.1)-(5.3). The observer is designed as

$$\dot{\hat{z}} = \psi(\hat{z}, \hat{x}) \tag{5.14}$$

$$\dot{\hat{x}} = A\hat{x} + B\phi(\hat{z}, \hat{x}, u) + H(y - C\hat{x}) \tag{5.15}$$

where the observer gain H is chosen as

$$H = \begin{bmatrix} \frac{\alpha_1}{\epsilon} \\ \frac{\alpha_2}{\epsilon^2} \\ \vdots \\ \vdots \\ \frac{\alpha_r}{\epsilon^r} \end{bmatrix}$$

in which $\epsilon > 0$ is a design parameter and the positive constants $\alpha_i, i = 1, \dots, r$ are chosen such that the roots of

$$\lambda^r + \alpha_1 \lambda^{r-1} + \cdots + \alpha_{r-1}\lambda + \alpha_r = 0$$

are in the open left-half plane. Assuming that the state component x can be rapidly estimated by the observer (5.14)-(5.15), reduces the problem to only that of ensuring that the observation error of the state component z (the internal dynamics) goes to zero. The analysis in [28] focuses on the observer design, and the performance of the closed-loop system under output feedback is not considered. The drawback of the above approach is that the observer design relies on the exact knowledge of the nonlinear functions $\psi(z,x)$ and $\phi(z,x,u)$, and does not take into consideration the effects of model uncertainties. Moreover, the effect of peaking is not investigated, since the functions $\psi(z,x)$ and $\phi(z,x,u)$ are required to be *globally* Lipschitz in x and z. This yields global results and, therefore, the problem of destabilization due to peaking does not arise, though peaking is present, and can lead to a degraded transient behavior of the closed-loop system under output feedback.

The goal of this chapter is to design a full-order observer for a class of nonlinear systems of the form (5.1)-(5.3). Our design approach is based on the earlier work of Esafandiari and Khalil [13]. In contrast to [28], we allow for model uncertainties of the functions $\psi(z,x)$ and $\phi(z,x,u)$, and do not require them to be *globally* Lipschitz in x and z. We establish the performance recovery properties of the output feedback design using the separation principle of [2].

The rest of the chapter is organized as follows. Section 5.2 states the problem formulation and motivation for the observer design which comes from the earlier work [13, 2] and the more recent result [28]. Section 5.3 recalls the separation results of [2] in order to establish the performance recovery properties of the output feedback design. Section 5.4 discusses the main results.

5.2 System Description and Problem Formulation

Consider the nonlinear system given by equations (5.1)-(5.3). The synthesis process of an output feedback controller for this system involves two steps. First, a state feedback controller that uses measurements of the states (z,x) is designed to asymptotically stabilize the origin. Then, a state observer is designed to estimate (z,x). The system (5.1)-(5.3) is required to satisfy the following assumption:

Assumption 5.1. • *The functions ψ and ϕ are locally Lipschitz in their arguments for $(z,x,u) \in D_z \times D_x \times R$, where $D_z \subset R^{n-r}$ and $D_x \subset R^n$ are domains* *that contain their respective origins.*

- $\psi(z,x) = (0,0)$, *and* $\phi(z,x,u) = (0,0,0)$.

The stabilizing state feedback controller takes the form

$$\dot{\vartheta}_a = \Gamma_a(\vartheta_a, z, x) \tag{5.16}$$

$$u = \gamma(\vartheta_a, z, x) \tag{5.17}$$

The state feedback design is required to satisfy the following properties:

Assumption 5.2. • *The functions* Γ_a *and* γ *are locally Lipschitz functions in their arguments over the domain of interest.*

- Γ_a *and* γ *are globally bounded functions of* x.

- $\Gamma_a(0,0,0) = 0$, *and* $\gamma(0,0,0) = 0$.

The closed-loop system under the state feedback controller (5.16)-(5.17) is given by

$$\dot{z} = \psi(z,x) \tag{5.18}$$

$$\dot{x} = Ax + B\phi(z,x,\gamma(\vartheta_a,z,x)) \tag{5.19}$$

$$\dot{\vartheta}_a = \Gamma(\vartheta_a, z, x) \tag{5.20}$$

Consider now the full-order observer, given by the equations

$$\dot{\hat{z}} = \psi_0(\hat{z}, \hat{x}) \tag{5.21}$$

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{z}, \hat{x}, u) + H(y - \hat{x}_1) \tag{5.22}$$

where $\psi_0(z,x)$ and $\phi_0(z,x,u)$ are, respectively, nominal models of the nonlinear functions $\psi(z,x)$ and $\phi(z,x,u)$. The functions $\psi_0(z,x)$ and $\phi_0(z,x,u)$ are required to satisfy the following assumption.

Assumption 5.3. • *The functions $\psi_0(z,x)$ and $\phi_0(z,x,\gamma(\vartheta,z,x))$ are locally Lipschitz in their arguments over the domain of interest.*

- *The functions $\psi_0(z,x)$ and $\phi_0(z,x,\gamma(\vartheta,z,x))$ are globally bounded in x.*

- $\psi_0(z,x) = (0,0)$, *and* $\phi_0(z,x,u) = (0,0,0)$.

Remark 5.1. *We require the functions $\psi_0(z,x)$ and $\phi_0(z,x,u)$ to be globally bounded in x to avoid the effect of peaking, which occurs in the observed states $\hat{x}$ and propagates to the state variables $(z,x,\hat{z})$ through the control law.*

The observer (5.21)-(5.22) comprises two subsystems, each with different dynamics. Equation (5.21) is the observer for the internal dynamics and, by design, is an open-loop observer, yielding in its slow dynamics. Equation (5.22) is a high-gain observer, with fast dynamics. We proceed with the analysis of the closed-loop system in two steps. In the first step, we consider a situation when x is available for feedback, and all we need is the estimate of z, which is provided by the observer

$$\dot{\hat{z}} = \psi_0(\hat{z},x) \tag{5.23}$$

The state feedback control (5.16)-(5.17) is modified to include the observer (5.23) as

$$\begin{aligned} \dot{\vartheta}_a &= \Gamma_a(\vartheta_a,\hat{z},x) && (5.24)\\ \dot{\hat{z}} &= \psi_0(\hat{z},x) && (5.25)\\ u &= \gamma(\vartheta_a,\hat{z},x) && (5.26) \end{aligned}$$

We define

$$\vartheta =: \begin{bmatrix} \vartheta_a \\ \hat{z} \end{bmatrix}, \qquad \Gamma =: \begin{bmatrix} \Gamma_a \\ \psi_0 \end{bmatrix} \tag{5.27}$$

Using (5.27), the *partial-state* feedback control (5.24)-(5.26) becomes

$$\dot{\vartheta} = \Gamma(\vartheta, x) \tag{5.28}$$

$$u = \gamma(\vartheta, x) \tag{5.29}$$

Then, the closed-loop system under the partial-state feedback control (5.28)-(5.29) is given by

$$\dot{z} = \psi(z, x) \tag{5.30}$$

$$\dot{x} = Ax + B\phi(z, x, \gamma(\vartheta, x)) \tag{5.31}$$

$$\dot{\vartheta} = \Gamma(\vartheta, x) \tag{5.32}$$

In this step of the analysis we combined the slow dynamics of (5.21) with those of (5.18)-(5.20). The closed-loop system (5.30)-(5.32) is required to satisfy the following assumption.

Assumption 5.4. *The origin* $(x = 0, \vartheta = 0)$ *is an asymptotically stable equilibrium point of the closed-loop system (5.30)-(5.32).*

We will elaborate on the implication of this assumption in Section 5.4. In the second step of the analysis, we bring-in the high-gain observer (5.22) to estimate x. The dynamic output feedback controller is given by equations (5.28)-(5.29), with x replaced with $\hat{x}$.

5.3 Performance Recovery

The purpose of the following analysis is to show that under the stated assumptions, the origin ($z = 0, x = 0, \vartheta_a = 0, \hat{z} = 0, \hat{x} = 0$) of the closed-loop system under output feedback is asymptotically stable for sufficiently small ϵ. For the closed-loop analysis, the observer dynamics are replaced by the equivalent dynamics of the scaled estimation error [2]

$$D(\epsilon)\eta = x - \hat{x} \tag{5.33}$$

where

$$D(\epsilon) = \left[\epsilon^{r-1}, \epsilon^{r-2}, \cdots, 1\right]$$

The closed-loop system under the output feedback controller can be written as

$$\dot{z} = \psi(z, x) \tag{5.34}$$

$$\dot{x} = Ax + B\phi(z, x, \gamma(\vartheta, x - D(\epsilon)\eta)) \tag{5.35}$$

$$\dot{\vartheta} = \Gamma(\vartheta, x - D(\epsilon)\eta) \tag{5.36}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon B\delta(z, x, \vartheta, D(\epsilon)\eta) \tag{5.37}$$

where the matrix $\frac{1}{\epsilon}A_0 =: (A-HC)$ is Hurwitz and $\delta(z, x, \vartheta, D(\epsilon)\eta) = \phi(z, x, \gamma(\vartheta, \hat{x})) - \phi_0(\hat{x}, \gamma(\vartheta, \hat{x}))$.

5.3.1 Recovery of the Boundedness and Convergence of Trajectories

Let $\mathcal{R}$ be the region of attraction of the closed-loop system (5.30)-(5.32). Let the initial states be $(z(0), x(0), \vartheta(0)) = (x_0, z_0, \vartheta_0) \in \mathcal{S}$, and $\hat{x}(0) = \hat{x}_0 \in \mathcal{Q}$, where $\mathcal{S}$ is any compact set in the interior of $\mathcal{R}$, and $\mathcal{Q}$ is any compact subset of R^r. With this formulation, the system (5.34)-(5.37) fits in the framework of [2]. In particular, the system (5.34)-(5.37) is in standadrd singularly perturbed form, and $\eta = 0$ is the unique solution of (5.37) when $\epsilon = 0$. Substituting $\eta = 0$ in (5.34)-(5.36) yields the reduced system exactly as the closed-loop system (5.30)-(5.32). For convenience, we write the system (5.34)-(5.36) as

$$\dot{\chi} = f_r(\chi, D(\epsilon)\eta) \tag{5.38}$$

where $\chi = [z^T, x^T, \vartheta^T]^T$, and $\chi(0) = [z_0^T, x_0^T, \vartheta_0^T]^T$. The reduced system can now be written as

$$\dot{\chi} = f_r(\chi, 0) \tag{5.39}$$

The boundary-layer system, obtained by applying to (5.37) the change of time variable $\tau = t/\epsilon$ then setting $\epsilon = 0$, is given by

$$\frac{d\eta}{d\tau} = A_0\eta \tag{5.40}$$

Let $(\chi(t,\epsilon), \eta(t,\epsilon))$ denote the trajectory of the system (5.34)-(5.37) starting from $(\chi(0), \eta(0))$. The following result is due to [2, Theorems 1, 2 & 3].

Theorem 5.1. *Consider the closed loop system (5.34)-(5.37). Let Assmuptions 5.1 - 5.4 hold. Then, the following results hold.*

- *There exists $\epsilon_1^* > 0$ such that, $\forall \epsilon \in (0, \epsilon_1^*]$, the trajectories (χ, η) starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t > 0$.*

- *Given any $\xi > 0$, there exists $\epsilon_2^* = \epsilon_2^*(\xi) > 0$ and $T_1 = T_1(\xi)$ such that,*

$\forall \epsilon \in (0, \epsilon_2^*]$, *we have*

$$\|\chi(t,\epsilon)\| + \|\eta(t,\epsilon)\| \leq \xi, \quad \forall t \geq T_1 \tag{5.41}$$

- *Let* $\chi_r(t)$ *be the solution of (5.39) starting from* $\chi(0)$. *Then, given any* $\xi > 0$, *there exists* $\epsilon_3^* > 0$ *such that,* $\forall \epsilon \in (0, \epsilon_3^*]$, *we have*

$$\|\chi(t,\epsilon) - \chi_r(t)\| \leq \xi, \quad \forall t > 0 \tag{5.42}$$

5.3.2 Recovery of the Asymptotic Stability of the Origin

Theorem 5.1 guarantees that the trajectories of the system (5.34)-(5.37), starting in $\mathcal{S} \times \mathcal{Q}$, enter a small ball of radius $\xi > 0$ around the origin $(\chi, \eta) = (0, 0)$ after a finite time and stay thereafter. We now show asymptotic stability.

Assumption 5.5. *The origin* $(x = 0, \vartheta = 0)$ *is an exponentially stable equilibrium point of the closed-loop system (5.30)-(5.32).*

The following result is due to [2, Theorem 5].

Theorem 5.2. *Suppose the function* $f_r(\chi, 0)$ *is continuously differentiable around the origin. Let Assumptions 5.1 - 5.5 hold. Then, under the hypotheses of Theorem 1, there exists* $\epsilon_4^* > 0$ *such that,* $\forall \epsilon \in (0, \epsilon_4^*]$, *the origin of the system (5.34)-(5.37) is exponentially stable and* $\mathcal{S} \times \mathcal{Q}$ *is a subset of its region of attraction.*

5.4 Discussion on the main results

The results in Theorem 5.1 guarantee that the trajectories of the closed-loop system under the output feedback (5.34)-(5.37), starting in $\mathcal{S} \times \mathcal{Q}$, are bounded and asymptotically converge to the trajectories of the closed-loop system (5.30)-(5.32). Notice, however, that the system (5.30)-(5.32) is different from the closed-loop system under state feedback (5.18)-(5.20), in that the former consists of not only the state feedback control (5.16)-(5.17) but, in addition, contains the slow part (i.e. Equation (5.21)) of the full-order observer (5.21)-(5.22). Thus, the trajectories of the closed-loop system (5.34)-(5.37) do not necessarily converge to the trajectories of the closed-loop system under state feedback (5.18)-(5.20). The result of Theorem 5.2 establishes that, under prescribed conditions, the exponential stability properties of the origin of the closed-loop system (5.30)-(5.32) can be recovered under the output feedback, in the presence of modeling error in the function $\phi(z, x, u)$.

The uncertainity in the function $\psi(z, x)$ is implicitly taken care of by assuming that the origin of the closed-loop system (5.30)-(5.32) is asymptotically stable. In other words, any uncertainity in the function $\psi(z, x)$, which does not destroy the asymptotic stability of the closed-loop system (5.30)-(5.32), is allowed.

In contrast to [28], we do not require the functions $\phi(z, x, u)$ and $\psi(z, x)$ to be *globally* Lipschitz in x and z. Our work showed the need for global boundedness of these functions in x, in order to avoid the effect of peaking, an issue that was not addressed in [28]. Our results hold on any compact sets and, therefore, are not just limited to global results presented in [28]. Furthermore, we allow for uncertainty in the nonlinear functions $\phi(z, x, u)$ and $\psi(z, x)$, whereas the state observer synthesis in [28] assumes perfect knowledge of these functions.

5.5 Example

Consider the nonlinear system

$$\dot{z} = -z + \theta_1 x_1 \tag{5.43}$$

$$\dot{x}_1 = x_2 \tag{5.44}$$

$$\dot{x}_2 = z + \theta_2 x_2^3 + \theta_3 u \tag{5.45}$$

$$y = x_1 \tag{5.46}$$

The system (5.43)-(5.46) is in the normal form (5.1)-(5.3), with

$$\psi(z,x) = -z + \theta_1 x_1,$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = [0\ 1]^T, \ C = [1 \quad 0],$$

$$\phi(z,x,u) = z + \theta_2 x_2^3 + \theta_3 u$$

in which θ_1, θ_2 and θ_3 are some positive constants. This system can be globally stabilized by the state feedback controller

$$u = \frac{1}{\theta_3}(-z - \theta_2 x_2^3 - x_1 - x_2) \tag{5.47}$$

We compare the performance of three control designs. *Design 1* uses the state feedback control (5.47). In *Design 2*, we modify (5.47) as a prtial-state feedback control

$$u = \frac{1}{\theta_3}(-\hat{z} - \theta_2 x_2^3 - x_1 - x_2) \tag{5.48}$$

where $\hat{z}$ is provided by the observer

$$\dot{\hat{z}} = -\hat{z} + \theta_1 y \tag{5.49}$$

Design 3 incorporates the output feedback controller given by

$$u = \frac{1}{\theta_3}(-\hat{z} - sat\ (\theta_2 \hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)) \tag{5.50}$$

where $\hat{z}$, $\hat{x}_1$, and $\hat{x}_2$ are provided by the full-order high-gain observer that comprises of (5.49) together with

$$\dot{\hat{x}}_1 = \hat{x}_2 + g_1(x_1 - \hat{x}_1)/\epsilon \tag{5.51}$$

$$\dot{\hat{x}}_2 = g_2(x_1 - \hat{x}_1)/\epsilon^2 \tag{5.52}$$

in which g_1 and g_2 are chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. In the simulation, we use the numerical values: $\theta_1 = \theta_2 = \theta_3 = 1$, $g_1 = 1$, $g_2 = 2$ and $\epsilon = 0.1$.

Figures 5.1 and 5.2 show the performance of the closed-loop system under the three control designs, with $\epsilon = 0.1$, and $\epsilon = 0.01$, repectively. The states z, x_1, and x_2, exhibit the expected transient behavior; namely, the response under output feedback (Design 3) approaches the response under the partial-state feedback (Design 2) as ϵ decreases from 0.1 (in Figure 5.1) to 0.01 (in Figure 5.2). In fact, in the latter case, the response under output feedback is indistinguishable from the same under partial-state feedback. As expected, the trajectories under output feedback (Design 3) do not approach to those under state feedback (Design 1). Figure 5.3 shows the estimation errors $\eta_{x_2} = x_2 - \hat{x}_2$, and $\eta_z = z - \hat{z}$, under output feedback (Design 3), with $\epsilon = 0.01$. The estimation errors $\eta_{x_1} = x_1 - \hat{x}_1$ (not shown in Figure 5.3), and η_{x_2} converge rapidly to zero, in contrast to the estimation error η_z, which converges to zero, slowly. As the value of ϵ decreases, the speed of convergence of the estimation errors η_{x_1}, and η_{x_2} increases.

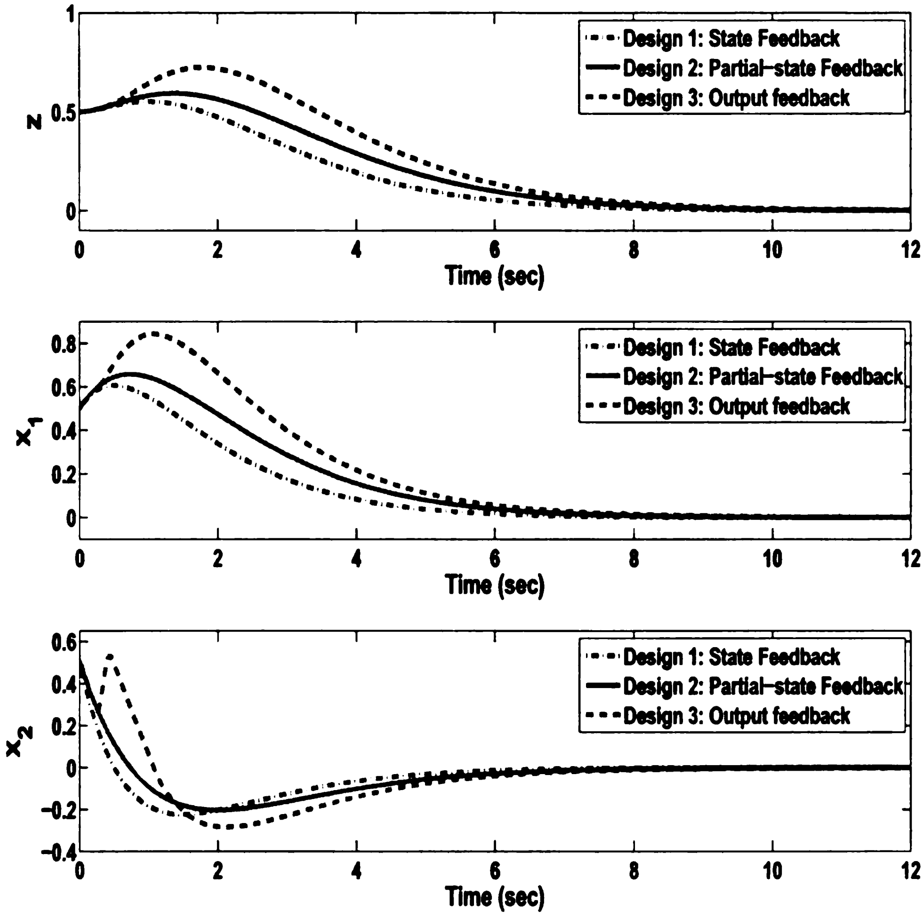


Figure 5.1: Performance under state feedback, partial-state feedback, and output feedback, with $\epsilon = 0.1$

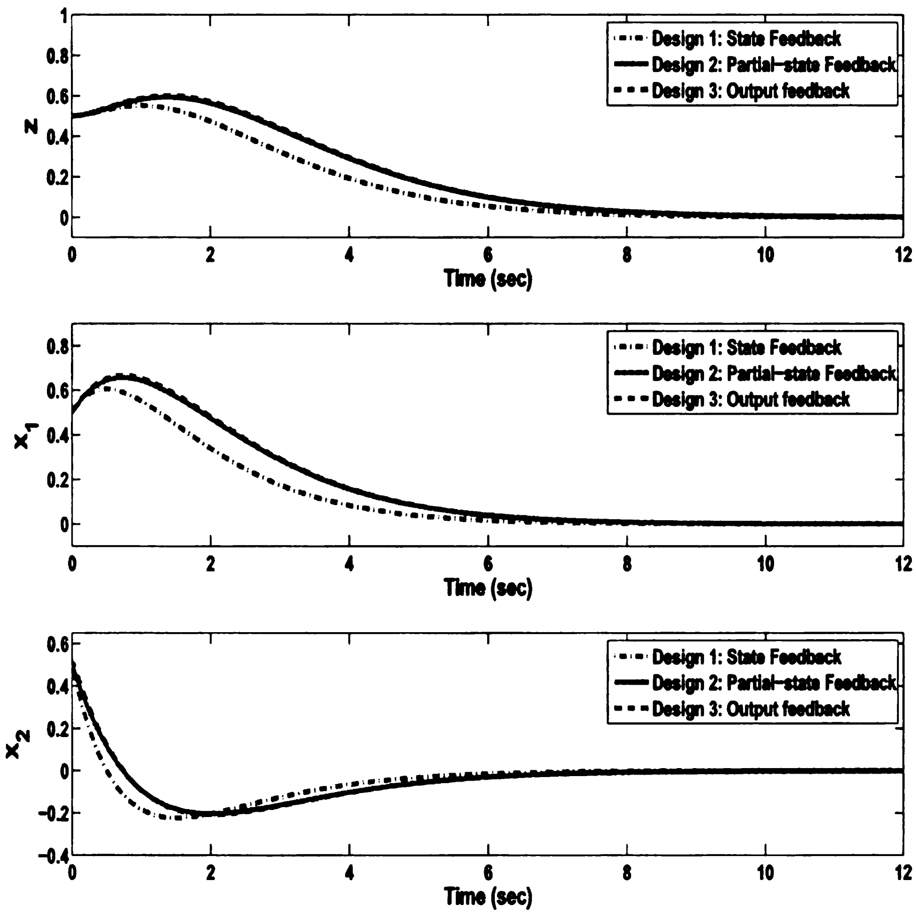
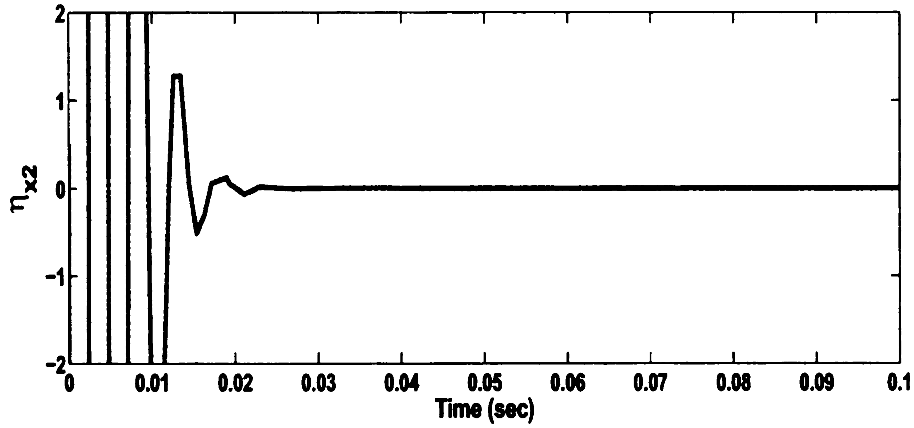


Figure 5.2: Performance under state feedback, partial-state feedback, and output feedback, with $\epsilon = 0.01$



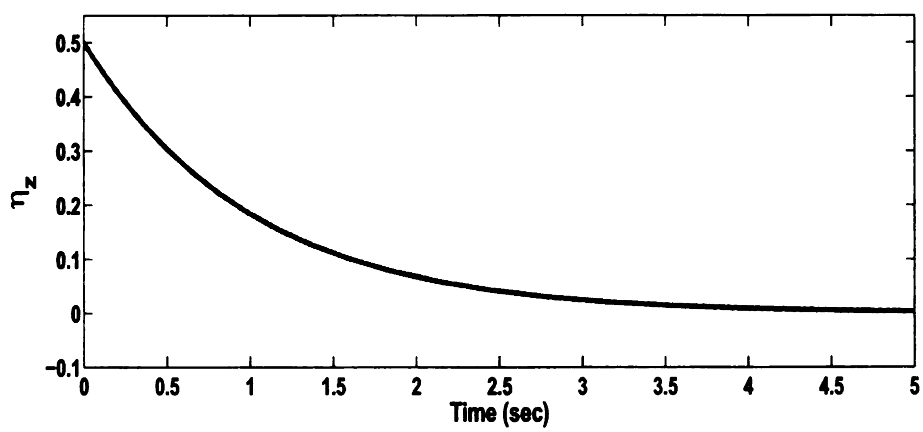


Figure 5.3: Estimation errors $\eta_{x_2} = x_2 - \hat{x}_2$, and $\eta_z = z - \hat{z}$, with $\epsilon = 0.01$

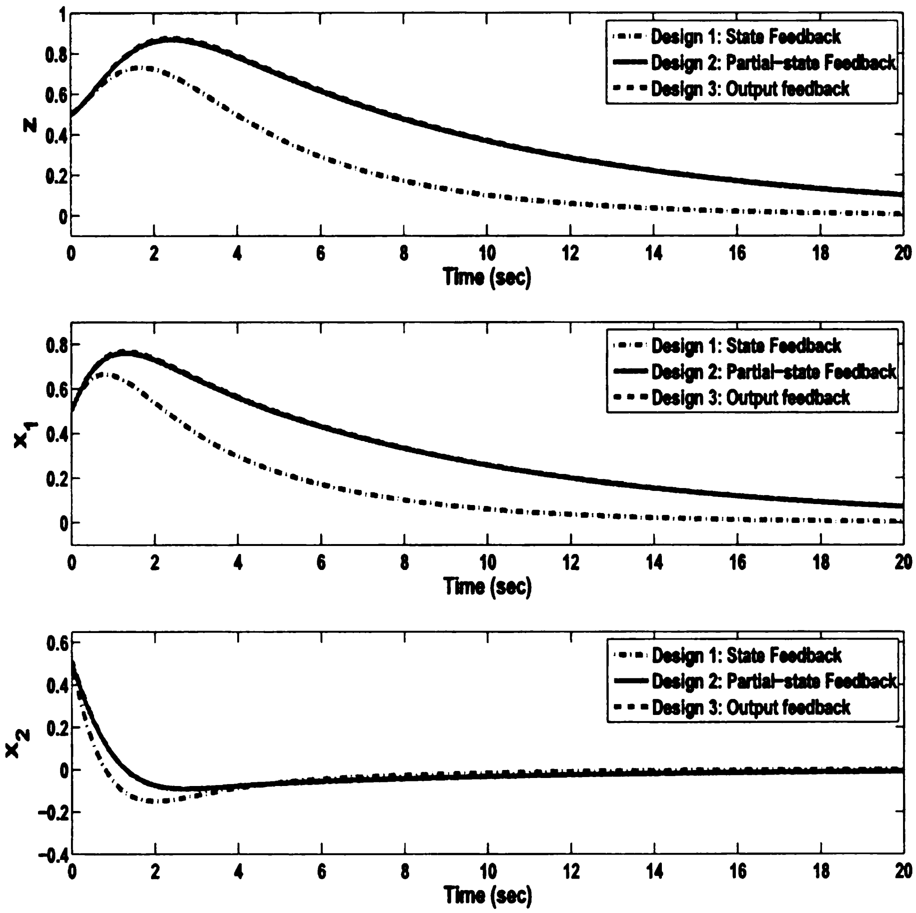


Figure 5.4: Closed-loop Performance under model uncertainty, with $\epsilon = 0.01$, $\theta_1 = 1.25$, $\theta_3 = 0.75$

Figure 5.4 shows the closed-loop performance under the three control designs, when the coefficients θ_1, and θ_3 are perturbed from 1 to 1.25 and 0.75, respectively. It can be seen that the output feedback (Design 3) recovers the performance of the partial-state feedback (Design 2), in the presence of the model perturbation.

5.6 Conclusions

In this chapter, we considered the problem of state estimation of a minimum-phase nonlinear system using a full-order high-gain observer. The observer comprises of two components, a slow open-loop observer that estimates the state z of the internal dynamics, and a fast observer that estimates the state x. From a dynamic state feedback controller, we define a dynamic partial-state feedback controller that assumes availability of x but uses the estimate $\hat{z}$. It is shown that the performance of such a dynamic partial-state feedback control can be recovered by the dynamic output feedback control using a sufficiently fast high-gain observer, in the presence of model uncertainty. The observer design approach is based on the two-time scale observer design of Esafandiari and Khalil [13], and the performance recovery is shown using the separation principle of Atassi and Khalil [2]. The performance of the observer design is demonstrated by simulation.

Chapter 6

Conclusions

This dissertation concentrates on the problem of output regulation for a class of minimum-phase nonlinear systems, with emphasis on improving the transient performance. We have extended the technique of *conditional servocompensators* in a sliding mode control framework [53] to more general feedback controllers by using *Lyapunov redesign* and *saturated high-gain feedback*. The issue of transient performance is significant in output regulation problem because conventional approaches to designing servocompensators often result in poor transient performance. A conditional servocompensator, in contrast, provides servo action only in a neighborhood of the zero-error manifold while acting as a stable system otherwise, thus, leading to improvement in the transient response while achieving zero steady-state regulation error.

The striking feature of our approach is the flexibility of starting with any stabilizing state feedback controller and then including a conditional servocompensator to achieve zero steady-state regulation error without degrading the transient performnace. We have proved that the trajectories of the closed-loop system under saturated high-gain feedback control with a conditional servocompensator approach those of a closed-loop system under saturated high-gain feedback control without a servocompensator. Analytical results are provided for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally. A précis of the results presented in this dissertation is given in the following section.

6.1 Synopsis of Results

In Chapter 2, we considered the problem of state feedback regulation of nonlinear systems using conditional servocompensators. We used the Lyapunov redesign and saturated high-gain feedback approach to design the stabilizing compensator. We showed that the inclusion of conditional servocompensators in the Lyapunov redesign framework enables us to achieve zero steady-state regulation error, in the presence of time-varying exogenous signals that are generated by a known exosystem. Analytical results are provided for regional and semi-global output regulation, and for the performance recovery of a saturated high-gain feedback controller without a servocompensator. Advantages of the proposed framework over the conventional approach were shown by simulation.

In Chapter 3, we considered the output regulation problem for a class of minimum-phase input-output linearizable nonlinear systems. The state feedback control design of Chapter 2 is specialized to partial state feedback control design for this class of systems. This partial state feedback controller can be viewed as an intermediate step towards the output feedback controller of Chapter 3, which is implemented using a reduced-order high-gain observer. We also proved that the output feedback controller with conditional servocompensator recovers the performance of a state feedback controller that does not include any servocompensator. We have included simulation results for two examples to demonstrate the advantages of the proposed framework over the conventional servocompensator design approaches.

In Chapter 4, we considered the output regulation problem of linear systems subject to control constraints. The presence of saturation in the input channel imposes strong limitations to the achievable control objectives such as transient performance. We applied the Lyapunov-redesign-conditional-servocompensator approach of Chapter 2 to the linear output regulation problem under input constraints, to achieve desirable transient performance. Because of the control constraint, the mechanism of solving the stabilization problem through Algebraic Riccati Equation (ARE) necessitates the use of full-state feedback. Therefore, the output feedback control is implemented using a two-time-scale full-order observer design of Esfandiari & Khalil [13] and the performance recovery is shown using the separation principle of Atassi & Khalil [1, 3]. Advantages of the proposed approach over the conventional approach presented in Lin *et al.* [37] were shown by simulation.

In Chapter 5, we considered the problem of state estimation of a minimum-phase nonlinear system using a full-order high-gain observer. The motivation comes from the desire of extending the methodology of Chapter 4 to nonlinear systems, which necessitates the development of a full-order nonlinear high-gain observer. A rather similar situation arises in applications that incorporate optimal stabilizing controllers, where the output feedback control can only be implemented using a full-order observer.

The observer designed in Chapter 5 comprises two components, a slow open-loop observer that estimates the state of the internal dynamics, and a fast observer that estimates the state of the external dynamics, in the presence of model uncertainty. The observer design approach is based on the two-time scale observer design of Esafandiari and Khalil [13], and the performance recovery is shown using the separation principle of Atassi and Khalil [2]. The performance of the observer design is demonstrated by simulation.

6.2 Future Work

One special feature of the proposed Lyapunov redesign + saturated high-gain feedback framework is that it allows us to start with any stabilizing controller and then include a conditional servocompensator by modifying the original controller to achieve the desired control objectives. Identifying and implementing stabilizing controllers with some built-in special features such as robustness properties, optimality etc. in this framework, and investigating to see if the overall performance of such nonlinear control designs can be improved when applied to the nonlinear servomechanisms, is an interesting line of future work.

A promising direction of future research work would be understanding whether the proposed framework allows the flexibility of incorporating controller designs that extend the class of systems considered in this dissertation, or relax the assumptions we have made. Of particular significance, in this regard, are the controller designs for the output regulation of nonminimum phase nonlinear systems [50] and the controller designs that incorporate adaptive or nonlinear internal models [51, 9, 5, 45, 46].

The saturated high-gain feedback designs naturally accomodate the applications with control constraints, however, with a significant trade-off between the region of attraction and the speed of convergence. An interesting, yet chellenging, direction of future research would be investigating the output regulation of constrained nonlinear systems, with the goal of semi-global regulation, when the control level is fixed apriori.

Last but not the least, further research work needs to be done on understanding how to tune the controller parameters in order to achieve specific control objectives, and to identifying possible limitations on the achievable performance.

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