TOPICS IN LINK HOMOLOGY

By

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ABSTRACT

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We prove two results about mutation invariance of link homology theories: We show that Khovanov's universal $\mathfrak{sl}(2)$ homology is invariant under mutation and that the reduced $\mathfrak{sl}(n)$ homology defined by Khovanov and Rozansky is invariant under component-preserving positive mutation when n is odd. We also give a relationship between the Khovanov homology of a closed positive 3-braid and the Khovanov homology of the braid after adding a number of full twists.

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Chapter 1

Introduction

This thesis is focused on properties of link homology theories. Link homology theories can be viewed as refinements of polynomial invariants associated to links.

In the 1980s, the discovery of the Jones polynomial [15] sparked an interest in polynomial invariants of knots, leading to various generalizations of the Jones and the Alexander polynomials such as the HOMFLY-PT polynomial [12] and Reshetikhin and Turaev's invariant for links colored by Lie algebra representations [40].

Beginning in the late 1990s, various corresponding *link homology theories* were introduced, starting with Khovanov's categorification of the Jones polynomial [24] and knot Floer homology, a link homology theory discovered by Oszváth and Szabó [18] and independently by Rasmussen [39], whose graded Euler characteristic recovers the Alexander polynomial. The idea of categorifying link polynomials has been especially fruitful in constructing such link homology theories.

The goal of categorifying a mathematical object is to lift the construction one (categorical) dimension up, so that the original object can be recovered as a "shadow" of the categorified object, hopefully explaining connections that seemed mysterious in the decategorified world. As an example, consider the natural numbers. Natural numbers are ubiquitous because we use them to count things; mathematically speaking, they are cardinalities of finite sets. But by identifying all sets of the same cardinality, we lose the ability to talk about relationships between the objects we are counting. A reasonable categorification of the natural numbers is thus the category of finite sets with equality weakened to bijection and addition and multiplication corresponding to disjoint union and direct product. A different categorification, which requires us to make fewer choices on the level of objects, is the category of finite-dimensional vector spaces over a fixed field, where the number assigned to a vector space is simply its dimension. An advantage of vector spaces over sets is that this categorification naturally extends to a categorification of the integers by passing from vector spaces to chain complexes, where the underlying integer can be recovered by taking the Euler characteristic of the complex. Natural operations on the integers readily carry over into the categorified setting: Addition corresponds to direct sums, multiplication to tensor products and negation corresponds to an odd shift in homological degree.

Polynomials are collections of integers living in various degrees, hence what we are looking for in a categorification of a link polynomial is a chain complex of graded vector spaces (or more generally graded modules over a ring) for each diagram of a link. But there should be additional structure: At the very least, every isomorphism of link diagrams, that is every sequence of Reidemeister moves, should induce an isomorphism (in a suitable sense) of graded chain complexes. In fact, we are asking for more: Every cobordism of links, represented by a sequence of Reidemeister moves and handle attachments, should correspond to a chain map which, up to homotopy, is independent of the presentation of the cobordism. In practice, this is often only true projectively, that is up to a sign.

Further structure is revealed by climbing down one step on the dimensional ladder. In the case of link polynomials, we associate a module to a circle with an even number of marked points (thought of as a diagram for a two-sphere with marked points) together with multilinear maps corresponding to all the ways of gluing disks with marked points on the boundary together to form a new such disk. This is called a planar algebra. To every tangle diagram, we associate an element of the module associated to its boundary; in particular, the module assigned to a circle without marked points is free of rank 1 and the element of this module associated to a link recovers the link polynomial. This allows us to compute the invariant of the link by cutting and pasting: once we know the value of the invariant on the basic building blocks, i.e. the crossings, we can use the planar algebra structure to compute the invariant for arbitrary links and tangles. We may also need to take orientations into account, which is done by turning marked points into arrows that can point inward or outward.

Similar structure can often be found in the categorified setting. To a circle with marked points or arrows, we associate a triangulated category, usually the homotopy category of chain complexes over some additive category, together with tensor products corresponding to gluing of disks. To every tangle diagram we associate an object of the category associated to its boundary and to every cobordism of tangles we associate a morphism in the corresponding category. This allows the calculation of the invariant by the same cut-and-paste approach described above.

A major part of this thesis is concerned with the effect mutation has on link homologies. Mutation is a topological operation that transforms a link into another link by cutting out a ball that meets the link in four points, applying an involution of the ball that preserves the set of intersection points of the link with the boundary of the ball and then gluing the ball back in. This operation has no effect on many link polynomials, a fact that is usually remarkably easy to see. The situation is more complicated in the case of link homology theories, which often detect mutation of links even when the underlying link polynomial does not. We study two of those link homology theories. The first is universal Khovanov homology, a generalization of Khovanov's original link homology theory categorifying the Jones polynomial. An advantage of working in this more general setting is that it contains enough information to recover various spectral sequences that can be used to define Rasmussen's s-invariant and other similar invariants that are useful to study the slice genus of knots. We succeed in showing invariance of universal Khovanov homology when working in characteristic 2. As an immediate corollary, we get invariance of a certain analog of the s-invariant, but we do not know whether it coincides with Rasmussen's original invariant. The second theory that we study is $\mathfrak{sl}(n)$ homology, a theory defined by Khovanov and Rozansky as a categorification of the $\mathfrak{sl}(n)$ polynomial. We show that, for odd n, the reduced version of this theory is invariant under mutation when imposing certain restrictions on the kinds of mutations that we allow.

The last chapter of the thesis investigates the Khovanov homology of links represented by 3-braids. We describe an explicit relationship between the Khovanov homology of the closure of a positive 3-braid and closure of the same 3-braid after adding a number of full twists. Every closed 3-braid can be written in this form and all but three 3-braids can be represented (up to conjugation) as the product of an alternating braid and a number of twists, whose Khovanov homology is determined by the Jones polynomial and the signature.

We point out that although we will happily call the theories we are considering "homology theories," we will rarely actually take homology and rather work with the underlying chain complexes (up to homotopy) instead. The reason for this is two-fold. One, the tangle invariants are chain complexes over an additive category that is in general not abelian. Two, experience has taught us that the proper setting to study homological algebra in is the derived category and homology may forget some of the information that is present in the derived category. The homotopy categories that our tangle invariants live in are in some sense analogs of the derived category. This can be made precise in the case of links, where we are working in the homotopy category of chain complexes of free modules over some graded polynomial ring, which is (at least in the bounded above case) equivalent to the derived category of graded modules over the ring (see for example [46], Theorem 10.4.8). The difference between "free" and projective objects vanishes here because of the Quillen-Suslin Theorem.

Finally, we remark that it has become common in the literature to refer to the theories as "homology theories" despite the fact that they are, strictly speaking, cohomology theories. We adopt this convention and do not worry about the distinction unless we are dealing with (co)homology of topological spaces. This is justified because the notions are identical up to re-indexing. All chain complexes we will consider are cohomological, that is the differential increases (co)homological degree.

The organization of this dissertation is as follows: In chapter 2, we review the definitions of the link homology theories that are the topic of this dissertation, specifically (universal) Khovanov homology and $\mathfrak{sl}(n)$ homology. We then investigate the behavior of these theories under mutation in chapter 3, showing that universal Khovanov homology of knots is invariant in characteristic 2 and that for odd n, reduced Khovanov-Rozansky homology of knots is invariant under a particular type of mutation. In chapter 4, we analyze the effect that adding a twist has on the Khovanov homology of a positive 3-braid.

Chapter 2

Definitions

In this chapter, we review the definitions we will use throughout the thesis. Whenever definitions differ from the definitions in the literature, we show that both are equivalent.

2.1 Mutation

Conway mutation is the process of decomposing a link L as the union of two 2-tangles $L = T \cup T'$ and then regluing in a certain way. Diagrammatically, we may assume that one of the tangles (the 'inner' tangle T) lies inside a unit circle with endpoints equally spaced as in Figure 2.1. Mutation consists of one of the following transformations R of the inside tangle, followed by regluing: reflection along the x-axis (R_x), reflection along the y-axis (R_y) or rotation about the origin by 180 degrees (R_z). In other words, the mutant is given by $L' = R(T) \cup T'$. When taking orientations into account, we can distinguish two types of mutation (see for example Kirk and Livingston [17]).

Definition 2.1.1. Mutation of an oriented link is called *positive* if orientations match when regluing, i.e. if $L' = R(T) \cup T'$ as an oriented link and it is called *negative* if the orientation

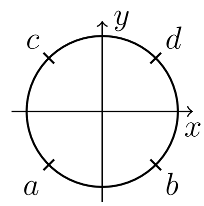
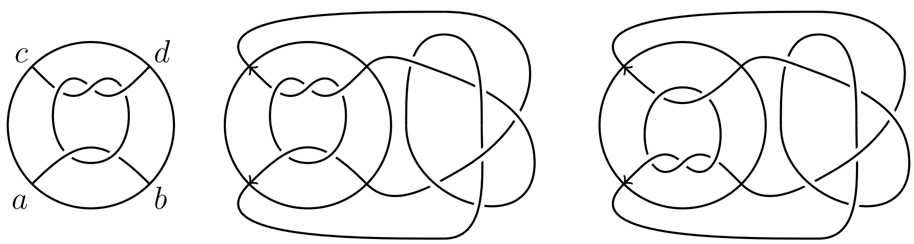


Figure 2.1: Placement and labeling of the endpoints of the inner tangle



(a) T (b) 11^n_{42} (c) 11^n_{34}

Figure 2.2: A 2-tangle with orientation-reversing symmetry and the Kinoshita-Terasaka - Conway mutant pair

of the inner tangle needs to be reversed before regluing, i.e. if $L' = -R(T) \cup T'$ as an oriented link, where $-R(T)$ denotes $R(T)$ with orientations reversed.

As an example, consider the two knots in Figure 2.2. 11^n_{34} is a positive mutant of 11^n_{42} since rotation about the x-axis preserves the orientations of the ends of T. It is also a negative mutant, as can be seen by considering rotation about the z-axis.

There are 16 mutant pairs with 11 or fewer crossings, see [30]. It can be checked that all of them can be realized by negative mutation. Among the 16 pairs, we found 5 that can be realized on the tangle T depicted in Figure 2.2(a): $(11^a_{57}, 11^a_{231})$, $(11^a_{251}, 11^a_{253})$, $(11^n_{34}, 11^n_{42})$, $(11^n_{76}, 11^n_{78})$ and $(11^n_{151}, 11^n_{152})$. $R_y(T)$ is isotopic to T but with orientations reversed, therefore these 5 mutant pairs can be realized by both positive and negative mutation. In particular, our proof applies to the famous Kinoshita-Terasaka - Conway pair, illustrated in Figure 2.2(b) and (c).

Definition 2.1.2. Mutation of a link is called *component-preserving* if a and $R(a)$ lie on the same component of the original link (or equivalently, on the same component of the mutant).

Note that knot mutation is always component-preserving. If positive mutation is component-preserving, then a and $R(a)$ are either both incoming or both outgoing edges, hence all 4 endpoints lie on the same component of the link. We refer to this component as the *component of the mutation*.

2.2 Enriched Categories

We will adopt the language of enriched categories ([9]) for our description of universal $\mathfrak{sl}(2)$ homology. An *enriched category* $\mathcal{A}$ over a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, or $\mathcal{V}$-*category*, is a collection of objects together with an assignment of an object $\mathcal{A}(A, B)$ of $\mathcal{V}$ to every pair of objects A and B, an identity $I \to \mathcal{A}(A, A)$ for every object A and composition $\mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple of objects A, B and C. These are required to satisfy certain natural axioms, see [9] for details. Most of the constructions below can be carried out in much greater generality, but for simplicity, we restrict ourselves to two examples of categories $\mathcal{V}$: The category R-mod of R-modules over a commutative ring R and the category R-gmod of graded R-modules, where R is a graded commutative ring. Morphisms in R-gmod are required to be of degree 0. We introduce a grading shift functor $\{\cdot\} : R\text{-gmod} \to R\text{-gmod}$ such that if $x \in A$, then the corresponding element $y \in A\{k\}$ satisfies $\deg y = \deg x + k$.

An *ideal* $\mathcal{I}$ in an R-mod-category $\mathcal{A}$ is a collection of submodules $\mathcal{I}(A, B) \leq \mathcal{A}(A, B)$ such that $\rho\psi\phi \in \mathcal{I}(A, D)$ for $\phi \in \mathcal{A}(A, B)$, $\psi \in \mathcal{I}(B, C)$ and $\rho \in \mathcal{A}(C, D)$. It is easy to see that given an ideal $\mathcal{I}$ in an R-mod-category $\mathcal{A}$, there is a quotient category $\mathcal{A}/\mathcal{I}$ whose objects are the objects of $\mathcal{A}$ and morphism spaces are $(\mathcal{A}/\mathcal{I})(A, B) = \mathcal{A}(A, B)/\mathcal{I}(A, B)$.

Let G be a directed multigraph. The *free* R-mod-*category* $\mathcal{A}$ on G has objects the vertices of G and morphism spaces $\mathcal{A}(A, B)$ the free R-module generated by all paths from A to B in G. Composition is given by concatenation of paths using the identification $R \otimes_R R \cong R$. If R is a graded ring and we assign a degree to each edge of G, then the free R-mod category on G has the structure of an R-gmod-category, which we will refer to as the *free* R-gmod-*category* on G. More precisely, a path $\phi_n \cdots \phi_1$ from A to B corresponds to a summand $R\{\deg \phi_1 + \cdots + \deg \phi_n\}$ in $\mathcal{A}(A, B)$. An ideal $\mathcal{I}$ of $\mathcal{A}$ is called *homogeneous* if for each A and B, the homogeneous parts of elements of $\mathcal{I}(A, B)$ are also in $\mathcal{I}(A, B)$. It is straightforward to see that the ideal generated by homogeneous elements is homogeneous and that the quotient category by a homogeneous ideal is again an R-gmod-category.

If $\mathcal{A}$ is an R-gmod-category, we define $\overline{\mathcal{A}}$ to be the R-gmod category whose objects are formal shifts $A\{k\}$ of objects A of $\mathcal{A}$, with morphism spaces given by $\overline{\mathcal{A}}(A\{k\}, B\{l\}) = \mathcal{A}(A, B)\{l - k\}$.

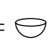
Given a $\mathcal{V}$-category $\mathcal{A}$, where $\mathcal{V}$ is either R-mod or R-gmod, we define an underlying category $\mathcal{A}_0$, whose objects coincide with the objects of $\mathcal{A}$ and whose morphisms $A \to B$ are (degree-0) elements of the (graded) R-module $\mathcal{A}(A, B)$. In both cases $\mathcal{A}_0$ is a pre-additive category.

For any pre-additive category $\mathcal{C}$ we define its additive closure $\mathbf{Mat}(\mathcal{C})$ to be the category whose objects are (possibly empty) formal direct sums of objects of $\mathcal{C}$ and whose morphisms are matrices of morphisms of $\mathcal{C}$ with the appropriate source and target objects. Composition is given by matrix multiplication. Clearly, $\mathbf{Mat}(\mathcal{C})$ is additive and $\mathcal{C}$ embeds into $\mathbf{Mat}(\mathcal{C})$.

As a simple example, consider the graph G with one vertex and no edges. Let $\mathcal{A}$ be the free R-gmod-category on G. $\mathcal{A}$ is the graded free R-module R of rank 1, regarded as an R-gmod-category with one object. Its underlying category $\mathcal{A}_0$ is the full subcategory of R-gmod on the object R. $\overline{\mathcal{A}}_0$ is the category of free graded R-modules of rank 1, whereas $\mathbf{Mat}(\overline{\mathcal{A}}_0)$ is the category of finitely generated free graded R-modules, which we denote by R-fgmod.

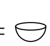
2.3 Universal $\mathfrak{sl}(2)$ Khovanov Homology

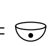
We now define universal $\mathfrak{sl}(2)$ homology. It is also known as equivariant $\mathfrak{sl}(2)$ homology since its underlying Frobenius system can be obtained by considering the $U(2)$-equivariant cohomology ring of the 2-sphere ([25]). We begin by describing the Frobenius system underlying the construction the theory (a Frobenius system is a Frobenius extension together with a choice of counit and comultiplication). This corresponds to the system $\mathcal{F}_5$ in [25]. Let R be the ring $\mathbb{Z}[h,t]$ and A be the free R-module generated by  and . We define a unit $\iota : R \to A$, a counit $\epsilon : A \to R$ and a multiplication map $m : A \otimes_R A \to A$ by

$\iota(1) =$ 

$\epsilon($  $) = 0$

$\epsilon($  $) = 1$

$m($   $) =$ 

$m($   $) = m($   $) =$ 

$m($   $) = h$  $+ t$ 

Graphically, we may represent h, t, ι, ϵ and m by , , , and , respectively. The composition $\epsilon \circ m$ defines a symmetric bilinear form on A, giving an identification of A with its dual module A^*. We use the notation for the image of under this identification and for the image of . Note that this is consistent with our previous notation for ϵ. If V and W are finitely generated free modules, then the map $W^* \otimes V^* \to (V \otimes W)^*$, $f \otimes g \mapsto (v \otimes w \mapsto f(w) \cdot g(v))$ gives a canonical identification of $W^* \otimes V^*$ with $(V \otimes W)^*$. A comultiplication map $\Delta : A \to A \otimes_R A$, graphically represented by , is obtained from $m^* : A^* \to (A \otimes_R A)^*$ using the identifications $A^* \cong A$ coming from the bilinear form $\epsilon \circ m$ and the canonical identification $(A \otimes_R A)^* \cong A^* \otimes_R A^*$. One readily checks that

$$\Delta(\ ) = \ \ + \ \ - h \ \ $$

$$\Delta(\ ) = \ \ + t \ \ $$

Remark 2.3.1. Our treatment differs from that of Khovanov [25] in that we used self-duality (with respect to the identification of A and A^* coming from the bilinear form) of the Frobenius system as an axiom. If one uses the identification $A \to A^*$, $\ \mapsto \ - h\ $, $\ \mapsto\ $, then the Frobenius system is only "almost self-dual", as described in [25] in example 2 at the end of the section entitled "A universal rank two Frobenius system."

The ring R becomes a graded ring by setting $\deg h = 2$ and $\deg t = 4$. We will refer to this grading as *q-grading*. By setting the q-degree of the generator to be -1 and the q-degree of the generator to be 1, we turn A into a graded R-module. One readily checks that $\deg \iota = \deg \epsilon = -1$ and $\deg m = \deg \Delta = 1$. Note that the grading always corresponds to the negative Euler characteristic of the associated picture.

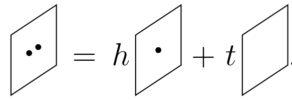
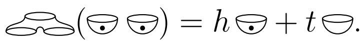
The Frobenius system above defines a symmetric monoidal functor from the symmetric monoidal category **Cob** with objects closed 1-manifolds and morphisms 2-dimensional cobordisms modulo isotopy (with the monoidal structure given by disjoint union) to the category R-mod of R-modules.

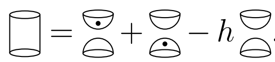
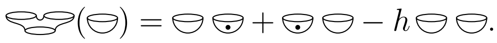
Associated to a link diagram D with n crossings is an n-dimensional cube of resolutions, whose vertices are the resolutions of the diagram and whose edges are split or merge cobordisms corresponding to changing the resolution of a crossing from the 0-smoothing to the 1-smoothing. Adjusting signs so that faces anti-commute and flattening the cube, we obtain a complex in $\mathbf{K}^{\mathrm{b}}(R\text{-mod})$, the homotopy category of bounded complexes of R-modules. This universal Khovanov complex $Kh(D)$ is an invariant of the knot in the sense that if two diagrams D and D' represent the same knot, then $Kh(D) \cong Kh(D')$ in $\mathbf{K}^{\mathrm{b}}(R\text{-mod})$. This theory can be endowed with q-gradings, as explained for tangles below. Khovanov's original theory can be recovered as $Kh(D)/(h = 0, t = 0)$.

This theory can be generalized to tangles along the lines of Bar-Natan [1]. Let D be the diagram of a tangle, which we assume to lie inside a topological disk B and to have boundary $E := \partial D \subset B$. We now define an R-gmod-category $\mathbf{Kob}(E)$ whose objects are compact one-manifolds properly embedded in B with boundary E and whose morphisms spaces are R-modules freely generated by equivalence classes of dotted cobordisms between them. We require a cobordism $f : X \to Y$ to be properly embedded in $B \times [0, 1]$ and its boundary to be $E \times [0, 1] \cup X \times \{0\} \cup Y \times \{1\}$. Furthermore, we define the degree of a generator E to be $\chi(f) - \frac{1}{2}|E|$, where dots contribute 2 to the Euler characteristic as above. The equivalence relation on cobordisms is generated by isotopy and the graphical relations from above. Explicitly:

- The value of a sphere without dots is 0, and the value of a sphere with one dot is 1.

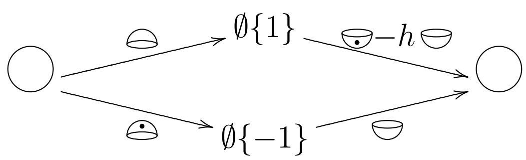
This reflects how the counit ϵ acts on generators.

- (Two-dot relation) Locally, . This corresponds to the equation 

- (Neck-cutting relation) Locally, . This corresponds to the equation 

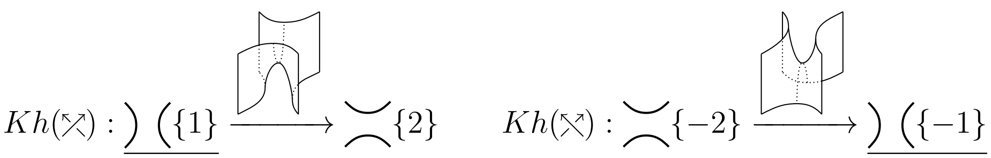
Let $\mathbf{Kob}'(E)$ be the skeleton of the full subcategory of $\mathbf{Kob}(E)$ on one-manifolds without closed loops (two objects in an enriched category $\mathcal{A}$ are considered isomorphic if the corresponding objects in $\mathcal{A}_0$ are isomorphic). $\mathbf{Kob}'(E)$ has one object for every isotopy class of such one-manifolds.

Let $\mathbf{mKob}(E) = \mathbf{Mat}(\overline{\mathbf{Kob}(E)}_0)$ and $\mathbf{mKob}'(E) = \mathbf{Mat}(\overline{\mathbf{Kob}'(E)}_0)$. Each object of $\mathbf{mKob}(E)$ is isomorphic to an object of $\mathbf{mKob}'(E)$ via the delooping isomorphism below (compare Bar-Natan [3]), hence the two categories are equivalent.



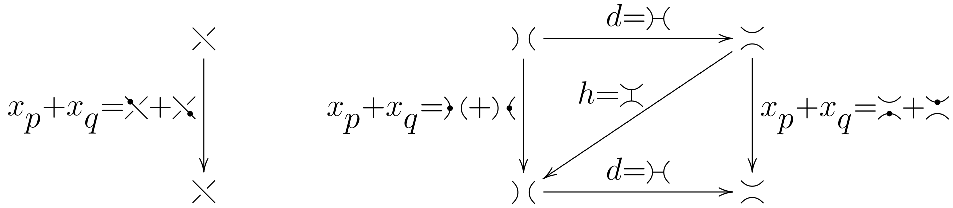
Gluing gives rise to a tensor product. Specifically, let B_1 and B_2 be two disks which share a boundary arc and E_1 and E_2 be a collection of points on the boundary of B_1 and B_2 such that $E_1 \cap B_2 = E_2 \cap B_1 = \{x_1, \dots x_n\}$. If X_1 and X_2 are objects of $\mathbf{Kob}(E_1)$ and $\mathbf{Kob}(E_2)$, respectively, we define the tensor product $X_1 \otimes_{R[x_1,\dots x_n]} X_2 \in \mathbf{Kob}((E_1 \cup E_2) \cap \partial(B_1 \cup B_2))$ to be $X_1 \cup X_2$. Similarly, for morphisms $f_i : X_i \to Y_i$ $(i = 1, 2)$, we define $f_1 \otimes_{R[x_1,\dots x_n]} f_2 : X \to Y$, where $X = X_1 \otimes_{R[x_1,\dots x_n]} X_2$ and $Y = Y_1 \otimes_{R[x_1,\dots x_n]} Y_2$, to be $f_1 \cup f_2$.

We can now define the Khovanov complex $Kh(D) \in \mathbf{K}^{\mathrm{b}}(\mathbf{mKob}(E))$ for any tangle diagram D inside a disk B with endpoints E by specifying the Khovanov complex of a single crossing and then extending to arbitrary tangles using the tensor product above. The complexes for a single positive and a single negative crossing are shown below. Underlined chain objects represent 0 homological height.



We use the two-dimensional pictures $)\!\!-\!\!($ and $\asymp$ to represent the cobordisms in the differentials of $Kh(\times)$ and $Kh(\times)$, respectively. Since the categories $\mathbf{K}^{\mathrm{b}}(\mathbf{mKob}(E))$ and $\mathbf{K}^{\mathrm{b}}(\mathbf{mKob'}(E))$ are equivalent with an explicit equivalence induced by the delooping functor, we will often regard $Kh(D)$ simply as an object of $\mathbf{K}^{\mathrm{b}}(\mathbf{mKob'}(E))$, in particular we consider $Kh(D)$ of a link diagram D to be an object of $\mathbf{K}^{\mathrm{b}}(R\text{-fgmod})$.

Every edge of a tangle diagram, which we represent by a point p on the edge, induces a chain morphism $x_p : Kh(D) \to Kh(D)$: For every resolution of D, we place a dot on the identity cobordism on the component that p lies on. Graphically, we represent this morphism by a dot placed on the diagram at p. Up to sign (and homotopy), this morphism only depends on the component of the link that p lies on, as can be shown by the explicit homotopy below.



Next we define the s-invariant. Let $\mathcal{F}C$ be a filtered chain complex with an increasing filtration $0 = \mathcal{F}^a C \subseteq \mathcal{F}^{a+1} C \subseteq \cdots \subseteq \mathcal{F}^b C = C$ (this is the opposite convention from [38] since we fixed the dot to be of degree 2 rather than -2). This filtration induces a filtration on the homology of C by setting $\mathcal{F}^i H^*(C) = \{[v] \mid v \in \mathcal{F}^i C\}$, whose associated graded complex is the E_∞-page of the spectral sequence associated to the filtration. Given a non-trivial homology class $\alpha \in H^*(C)$, $s(\alpha)$ is the grading of the image of α in the associated graded complex. More explicitly, $s(\alpha) = s_C(\alpha) = \min\{i \in \mathbb{Z} \mid \exists v \in \mathcal{F}^i C : [v] = \alpha\}$. We also set $s(0) = -\infty$.

Let $\tilde{R}$ be a particular ring with distinguished elements $\tilde{h}, \tilde{t} \in \tilde{R}$. We endow $\tilde{R}$ with an R-module structure, where h acts by $\tilde{h}$ and t acts by $\tilde{t}$ and define $Kh_{\tilde{h},\tilde{t}}(D; \tilde{R}) = Kh(D) \otimes_R \tilde{R}$. The grading on $Kh(D)$ induces a filtration on $Kh_{\tilde{h},\tilde{t}}(D; \tilde{R})$. We now restrict to the case that D represents a knot diagram. In the case $\tilde{R} = \mathbb{Q}$, $\tilde{h} = 0$, $\tilde{t} = 1$, the homology of $Kh_{\tilde{h},\tilde{h}}(D; \tilde{R})$ is isomorphic to two copies of $\mathbb{Q}$, generated by two classes $\alpha_{\min}$ and $\alpha_{\max}$ such that $s(\alpha_{\min}) + 1 = s(\alpha_{\max}) - 1$. This is Rasmussen's s-invariant [38], which we denote by $s(D)$. Similarly, we can set $\tilde{R} = \mathbb{Z}_2$, $\tilde{h} = 1$, $\tilde{t} = 0$. Turner [41] shows that the homology of $Kh(D)$ is generated over $\mathbb{Z}_2$ by two generators $\alpha_{\min}$ and $\alpha_{\max}$. The discussion in the last section of [25] applies over any field, thus we can define a "$\mathbb{Z}_2$-Rasmussen invariant" by $s_2(D) := s(\alpha_{\min}) + 1 = s(\alpha_{\max}) - 1$.

2.4 Khovanov-Rozansky homology

$\mathfrak{sl}(n)$ homology, defined by Khovanov and Rozansky in [21], is a categorification of the $\mathfrak{sl}(n)$ polynomial, a certain specialization of the HOMFLY-PT polynomial that can be obtained from the fundamental n-dimensional representation of $U_q(\mathfrak{sl}(n))$. As noted implicitly by Gornik [11] and later used by Rasmussen in [37] (see also Krasner [19] and Wu [48]), the definitions make sense in a more general context: To any polynomial $p \in \mathbb{Q}[x]$, one can assign a homology theory that conjecturally only depends on the multiplicities of the (complex) roots of it derivative $p'(x)$. $\mathfrak{sl}(n)$ homology is recovered by setting $p(x) = \frac{1}{n+1}x^{n+1}$. For odd n, we establish invariance under positive mutation, that is mutation that respects the orientations of both 2-tangles involved in it.

Our definitions closely follow [37], but note that we work with $\mathbb{Z}_2$-graded matrix factorizations instead of $\mathbb{Z}$-graded ones in order to get a stronger version of invariance under Reidemeister moves.

A matrix factorization over a commutative ring R with potential $w \in R$ is a free $\mathbb{Z}_2$-graded module C^* equipped with a differential $d = (d_0, d_1)$ such that $d^2 = w \cdot I_C$. Following [37], we use the notation

$$C^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} C^0 \, .$$

Morphisms are simply degree-0 maps between matrix factorizations which commute with the differential. We say that two morphisms of matrix factorizations $\phi, \psi : C \to C'$ are homotopic if $\phi - \psi = d_{C'}h + hd_C$ for some degree-1 homotopy $h : C \to C'$. Thus we can define a category of matrix factorizations over R with potential w and morphisms considered up to homotopy. For a graded ring R, whose grading we will call *q-grading*, we also introduce a notion of graded matrix factorizations with homogeneous potential w by requiring that both d^0 and d^1 be homogeneous of q-degree $\frac{1}{2}\deg w$. Morphisms between graded matrix factorization are required to have q-degree 0, whereas homotopies must have q-degree $-\frac{1}{2}\deg w$. The corresponding homotopy category of graded matrix factorizations will be denoted by $\mathbf{hmf}_w(R)$. For the two different gradings in $\mathbf{hmf}_w(R)$ we introduce two types of grading shifts: A shift in the $\mathbb{Z}_2$-grading coming from matrix factorizations will be denoted by $\langle\cdot\rangle$ and a shift in q-grading by $\{\cdot\}$. We follow the convention that $R\{n\}$ has a single generator in q-degree n. Note that if $\phi : A \to B$ has q-degree d, then the q-degree of $\phi : A\{k_A\} \to B\{k_B\}$ is $d + k_B - k_A$.

An important class of matrix factorizations is the class of Koszul factorizations, which we will briefly describe here. For a more detailed treatment, we refer the reader to Section 2.2 of [23] (but note that we switched the order of the arguments of K in order to be consistent with [22] and [37]). If $u, v \in R$, then $K(u; v)$ is the factorization

$$R\left\{\frac{\deg v - \deg u}{2}\right\} \underset{u}{\overset{v}{\rightleftarrows}} R$$

We will sometimes write $K_R(u; v)$ to clarify which ring we are working over. For $\mathbf{u} = (u_1, \ldots, u_n)^T$, $\mathbf{v} = (v_1, \ldots, v_n)^T$ we define $K(\mathbf{u}, \mathbf{v}) = \bigotimes_{k=1}^n K(u_k; v_k)$. This is a matrix factorization with potential $\sum_{k=1}^n u_k v_k$. We will also use the notation

$$K(\mathbf{u}, \mathbf{v}) = \begin{Bmatrix} u_1 & v_1 \\ \vdots & \vdots \\ u_n & v_n \end{Bmatrix}$$

If we are not interested in $\mathbf{u}$, we may apply arbitrary row transformations to $\mathbf{v}$: for an invertible matrix X, $K(\mathbf{u}, \mathbf{v}) \cong K\left((X^{-1})^t \mathbf{u}, X\mathbf{v}\right)$. We describe order-two Koszul matrix factorizations explicitly, thereby fixing a sign convention for the tensor product of matrix factorizations:

$$\begin{Bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{Bmatrix} = R\{k_1\} \oplus R\{k_2\} \underset{\begin{pmatrix} v_2 & u_1 \\ v_1 & -u_2 \end{pmatrix}}{\overset{\begin{pmatrix} u_2 & u_1 \\ v_1 & -v_2 \end{pmatrix}}{\rightleftarrows}} R\{k_1 + k_2\} \oplus R$$

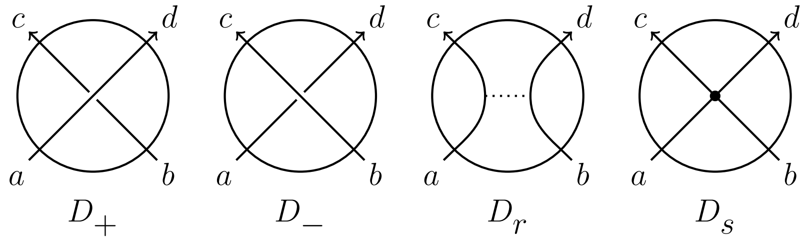


Figure 2.3: Positive crossing, negative crossing, oriented smoothing and singular crossing. The dotted line connecting the two arcs of the oriented smoothing illustrates that we consider both arcs to be on the same component of the smoothing.

Here $k_1 = \frac{\deg v_1 - \deg u_1}{2} = \deg v_1 - \frac{\deg w}{2}$ and $k_2 = \frac{\deg v_2 - \deg u_2}{2} = \deg v_2 - \frac{\deg w}{2}$.

Our definition of Khovanov-Rozansky homology closely follows Rasmussen [37], whose definitions we amend slightly for technical reasons. We also restrict ourselves to connected diagrams. To any diagram of a (possibly singular) oriented tangle, which we allow to contain any of the diagrams depicted in Figure 2.3 as subdiagrams, Rasmussen defines two rings, which depend only on the underlying 4-valent graph obtained by replacing all of those diagrams by a vertex. The *edge ring* $R(D)$ is the polynomial ring over $\mathbb{Q}$ generated by variables x_e, where e runs over all edges of the diagram, subject to a relation of the form $x_a + x_b - x_c - x_d$ for each vertex of the underlying 4-valent graph. By setting $\deg x_e = 2$ for each edge e of D, $R(D)$ becomes a graded ring. The *external ring* $R_{\text{ext}}(D)$ (called R_e in [37]) is the subring of $R(D)$ generated by the variables associated to the endpoints of D. Lemma 2.5 in [37] shows that if we associate the variables x_i ($i \in \{1, 2, \ldots, k\}$) to the incoming edges of D and y_i to the outgoing edges, then $R_{\text{ext}}(D) \cong \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\left(\sum_i y_i - \sum_i x_i\right)$.

Fix a polynomial $p \in \mathbb{Q}[x]$. If p is not homogeneous, we will disregard q-gradings below. To each tangle diagram D, we associate a complex $C_p(D)$ of matrix factorizations over $R(D)$, which we consider to be an object of the category $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_w(R_{\text{ext}}(D)))$, where $\mathbf{K}^{\mathrm{b}}(\mathcal{C})$ denotes the homotopy category of bounded complexes over the additive category $\mathcal{C}$

and $w = \sum_i p(y_i) - \sum_i p(x_i)$ where x_i and y_i are associated to the incoming and outgoing edges as above. $C_p(D)$ is first defined on the diagrams shown in Figure 2.3. In each case $R := R(D) = R_{\text{ext}}(D) = \mathbb{Q}[x_a, x_b, x_c, x_d]/(x_a + x_b = x_c + x_d)$. We set

$$\begin{aligned}
C_p(D_r) &= K(x_c - x_a; *)\langle 1\rangle = K(*; x_c - x_a)\{n-1\},\\
C_p(D_s) &= K(*; x_c x_d - x_a x_b)\{-1\},\\
C_p(D_+) &= K(*; x_c x_d - x_a x_b)[-1] \xrightarrow{d_+} K(*; x_c - x_a) \text{ and}\\
C_p(D_-) &= K(*; x_c - x_a) \xrightarrow{d_-} K(*; x_c x_d - x_a x_b)\{-2\}[1].
\end{aligned}$$

Here $*$ is of course determined by the potential in each case; we postpone the definitions of d_+ and d_- until we need them in Lemma 3.3.7.

This definition is extended to arbitrary tangle diagrams by the formula

$$C_p(D) = \bigotimes_i C_p(D_i) \otimes_{R(D_i)} R(D), \tag{2.1}$$

where D_i runs over all crossings of D and the big tensor product is taken over $R(D)$. As indicated above, we usually view $C_p(D)$ as a matrix factorization over the smaller ring $R_{\text{ext}}(D)$.

Rasmussen shows (Lemma 2.8 in [37])

Proposition 2.4.1. *If D is obtained from D_1 and D_2 by taking their disjoint union and identifying external edges labeled $x_1, \ldots, x_k$ in both diagrams, then*

$$R(D) \cong R(D_1) \otimes_{\mathbb{Q}[x_1,\ldots,x_k]} R(D_2) \text{ and}$$

$$C_p(D) \cong C_p(D_1) \otimes_{\mathbb{Q}[x_1,\ldots,x_k]} C_p(D_2).$$

To define reduced Khovanov-Rozansky homology of a link with respect to a marked component, we pick an edge on the marked component, which we label by x. We view $C_p(D)$ as a complex of matrix factorization over $\mathbb{Q}[x]$, i.e. as an object of $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_0(\mathbb{Q}[x]))$. Alternatively, we may consider the diagram D° obtained from D by cutting it open at the marked edge. Let x and y be the labels of the incoming and outgoing edge of D°, respectively. Then $C_p(D^{\circ})$ is a complex of matrix factorization with potential $p(y)-p(x)=0$ over the ring $R_{\mathrm{ext}}(D^{\circ}) = \mathbb{Q}[x,y]/(y-x) \cong \mathbb{Q}[x]$ and $C_p(D) \cong C_p(D^{\circ})$ as objects of $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_0(\mathbb{Q}[x]))$. Define the reduced complex

$$\widehat{C}_p(D) := C_p(D^{\circ})/(x) \in \mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_0(\mathbb{Q}))$$

(we use $\widehat{\cdot}$ rather than $\bar{\cdot}$ in order to avoid confusion with the involution $\bar{\cdot}$ to be defined later).

Since we are working over a field and matrix factorizations with potential 0 are simply $\mathbb{Z}_2$-graded chain complexes, the category $\mathbf{hmf}_0(\mathbb{Q})$ is equivalent to the category of $\mathbb{Z}_2 \oplus \mathbb{Z}$-graded $\mathbb{Q}$-vector spaces by Proposition 2.4.4 below. Hence the category $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_0(\mathbb{Q})$ is equivalent to the category $\mathbf{K}^{\mathrm{b}}(\mathbb{Z}_2 \oplus \mathbb{Z}$-graded $\mathbb{Q}$-vector spaces), which in turn is equivalent to the category of $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{Q}$-vector spaces (bounded with respect to the second $\mathbb{Z}$ summand) by Proposition 2.4.3. Reduced Khovanov-Rozansky homology is the image of $\widehat{C}_p(D)\{(n-1)w\}$ under this equivalence of categories, where w is the writhe of D. Since the theory, is supported in only one of the two $\mathbb{Z}_2$-gradings, we may view it as a bigraded vector space. We use the notation $\widehat{H}_p^{i,j}(D)$ for the subspace of this vector space in q-degree i and homological degree j.

Proposition 2.4.2. *The definition of reduced $\mathfrak{sl}(n)$ homology above is equivalent to Khovanov and Rozansky's original definition in* [21].

Proof. Other than working with $\mathbb{Z}_2$-graded matrix factorization rather than $\mathbb{Z}$-graded ones, our definitions coincide with Rasmussen's in [37]. Equivalence to the the original formulation thus follows directly from Proposition 3.12 in [37]. $\square$

The following two propositions are well-known in the finitely generated case. We verify that proof carries over to the infinitely generated setting.

Proposition 2.4.3. *Let* $C = \ldots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \ldots$ *be a chain complex over* $\mathbb{Q}$ *(with not necessarily finitely generated chain groups). Then C is homotopy equivalent to a complex with zero differential (its cohomology).*

Proof. As usual, let $Z^k := \ker(d^k)$ and $B^{k+1} := \operatorname{im}(d^k)$. Since vector spaces are free as modules, the short exact sequences $0 \to Z^k \to C^k \xrightarrow{d^k} B^{k+1} \to 0$ and $0 \to B^k \to Z^k \to H^k(C) \to 0$ split. It is easy to check that with respect to the decomposition $C^k \cong Z^k \oplus B^{k+1} \cong B^k \oplus H^k(C) \oplus B^{k+1}$, C decomposes as a direct sum of chain complexes $0 \to H^k(C) \to 0$ and $0 \to B^k \xrightarrow{\mathrm{id}} B^k \to 0$. The Proposition now follows from the fact that the latter chain complex is homotopy equivalent to the zero complex. $\square$

Proposition 2.4.4. *Any $\mathbb{Z}_2$-graded chain complex* $C^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} C^0$ *is homotopy equivalent to a $\mathbb{Z}_2$-graded chain complex with zero differential.*

Proof. Arguing as in the proof of the previous Proposition, we may decompose C as a direct sum of $H^1(C) \rightleftarrows 0$, $0 \rightleftarrows H^0(C)$, $B^1 \underset{\mathrm{id}}{\overset{0}{\rightleftarrows}} B^1$ and $B^0 \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} B^0$, where the latter two complexes are homotopy equivalent to zero complexes. $\square$

Chapter 3

Link Homology and Mutation

Understanding the behavior of Khovanov homology under Conway mutation has been an active area of study. Wehrli [44] demonstrated that unlike the Jones polynomial, Khovanov homology detects mutation of links. Bar-Natan [2] showed that for a pair of mutant knots (or, more generally, two links that are related by component-preserving mutation) there are two spectral sequences with identical E_2 pages converging to the Khovanov homologies of the knots. Champanerkar and Kofman [8] relate Khovanov homology to a (mutation-invariant) matroid obtained from the Tait graph of a knot diagram. The question remains open, but with coefficients in $\mathbb{Z}_2$ it was solved independently by Bloom [6] and Wehrli [45]. In fact, Bloom proves the more general result that odd Khovanov homology (see Ozsváth, Rasmussen and Szabó [36]) is invariant under arbitrary mutation of links. A similar statement cannot hold for the original Khovanov homology, as we know from Wehrli's example in [44]. Recently, Kronheimer, Mrowka and Ruberman [13] showed that the total rank of instanton knot homology is invariant under genus-2 mutation, which implies invariance under Conway mutation.

In this chapter, we prove two mutation invariance results about the previously defined link homology theories. First, we show mutation invariance of the universal $\mathfrak{sl}(2)$ homology of knots with coefficients in $\mathbb{Z}_2$, which immediately implies invariance of a variant of Rasmussen's s-invariant. We then show that reduced $\mathfrak{sl}(n)$ homology is invariant under *positive* mutation if n is odd.

$\mathfrak{sl}(3)$ homology can be defined over the integers using webs and foams ([26]) and there is a universal theory similar to universal $\mathfrak{sl}(2)$ homology ([33]). This theory is invariant under arbitrary mutation when working with coefficients in $\mathbb{Z}_2[a,b,c]$ and a reduced version of the theory is invariant under positive mutation when working over $\mathbb{Z}[b]$. This can be shown using similar methods, but we will not discuss the case in this thesis. For $n > 3$, $\mathfrak{sl}(n)$ homology has only been defined over the rational numbers. If a suitable definition over $\mathbb{Z}$ is found, we expect the observed pattern to continue: $\mathfrak{sl}(n)$ homology should be invariant when working over $\mathbb{Z}_2$ and a reduced version should be invariant under positive mutation for odd n.

3.1 Lemma about invariance of mapping cones

The following lemma is at the heart of the proof. We will use it to show that invariance under mutation is essentially a property of the category associated to 2-tangles. The functors $\mathcal{F}$ and $\mathcal{G}$ are necessary to account for grading shifts; we suggest that the reader think of them as identity functors and of f as a natural transformation in the center of the category.

Lemma 3.1.1. *Let $\mathcal{C}$ be an additive category and let $\mathcal{F}$, $\mathcal{G}$ and $\bar{\cdot}$ be additive endofunctors of $\mathcal{C}$, where $\bar{\cdot} : \mathcal{C} \to \mathcal{C}$ is required to be the identity on objects and an involution on morphisms. Furthermore, let $f : \mathcal{F} \Rightarrow \mathcal{G}$ be a natural transformation from $\mathcal{F}$ to $\mathcal{G}$ and let $\partial : \mathcal{C}(A, B) \to \mathcal{C}(\mathcal{G}A, \mathcal{F}B)$ be an operation defined on the* Hom*-sets of $\mathcal{C}$ with the following properties*

(1) *∂ is $\mathbb{Z}$-linear, i.e. for $\phi, \psi \in \mathcal{C}(A,B)$, $\partial(\phi - \psi) = \partial\phi - \partial\psi$.*

(2) *For $\phi \in \mathcal{C}(A,B)$, $\mathcal{G}(\phi - \bar{\phi}) = f_B\, \partial\phi$ and $\mathcal{F}(\phi - \bar{\phi}) = \partial\phi\, f_A$.*

(3) *Composable morphisms $\phi \in \mathcal{C}(A,B)$ and $\psi \in \mathcal{C}(B,C)$ satisfy a perturbed Leibniz rule: $\partial(\psi\,\phi) = \partial\psi\,\mathcal{G}\phi + \mathcal{F}\bar{\psi}\,\partial\phi$.*

If C is a chain complex over $\mathcal{C}$ with differential d, then f gives rise to a chain morphism $f_C : \mathcal{F}C \to \mathcal{G}C$. Let $\bar{C}$ be the chain complex obtained by applying $\bar{\cdot}$ to the differential of C. Then the mapping cones $\mathrm{Cone}(f_C)$ *and* $\mathrm{Cone}(f_{\bar{C}})$ *are isomorphic.*

Remark 3.1.2. To get a more symmetric statement, we could add the condition $\partial(\psi\,\phi) = \partial\psi\,\mathcal{G}\bar{\phi} + \mathcal{F}\psi\,\partial\phi$ to (3). This is unnecessary, however, since this condition follows from the other conditions by $(\partial\psi\,\mathcal{G}\phi + \mathcal{F}\bar{\psi}\,\partial\phi) - (\partial\psi\,\mathcal{G}\bar{\phi} + \mathcal{F}\psi\,\partial\phi) = \partial\psi\,\mathcal{G}(\phi - \bar{\phi}) - \mathcal{F}(\psi - \bar{\psi})\,\partial\phi = \partial\psi\, f\,\partial\phi - \partial\psi\, f\,\partial\phi = 0$.

Proof. We adopt the following conventions for the mapping cone. For any chain complex A, let $\varepsilon_A : A \to A$ be the identity in even homological heights and the negative of the identity in odd heights. Note that ε commutes with morphisms of even homological degree and anti-commutes with morphisms of odd degree. Then the mapping cone $\mathrm{Cone}(f_C)$ is given by $\mathcal{F}C[-1] \oplus \mathcal{G}C$ with differential $\begin{pmatrix} \mathcal{F}d & \\ f_C \circ \varepsilon_{\mathcal{F}C} & \mathcal{G}d \end{pmatrix}$. Since $\mathcal{F}d$ here has degree 1, it is easy to check that this defines a differential.

We claim that the horizontal arrows in Figure 3.1 give rise to an isomorphism between $\mathrm{Cone}(f_C)$ and $\mathrm{Cone}(f_{\bar{C}})$. $\begin{pmatrix} I & \partial d\,\varepsilon \\ & I \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} I & -\partial d\,\varepsilon \\ & I \end{pmatrix}$, so it remains to check that it defines a chain morphism, i.e. that $\begin{pmatrix} \mathcal{F}\bar{d} & \\ f\varepsilon & \mathcal{G}\bar{d} \end{pmatrix} \begin{pmatrix} I & \partial d\,\varepsilon \\ & I \end{pmatrix} = \begin{pmatrix} I & \partial d\,\varepsilon \\ & I \end{pmatrix} \begin{pmatrix} \mathcal{F}d & \\ f\varepsilon & \mathcal{G}d \end{pmatrix}$. Since $\mathcal{F}d : \mathcal{F}C[-1] \to \mathcal{F}C[-1]$, $\mathcal{G}d : \mathcal{G}C[-1] \to \mathcal{G}C[-1]$ and

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}C[-1] \oplus \mathcal{G}C \arrow[r, "\begin{pmatrix} I & \partial d \circ \varepsilon_{\mathcal{G}C} \\ & I \end{pmatrix}"] & \mathcal{F}C[-1] \oplus \mathcal{G}C \\
\mathcal{F}C[-1] \oplus \mathcal{G}C \arrow[u, "\begin{pmatrix} \mathcal{F}d & \\ f_C \circ \varepsilon_{\mathcal{F}C} & \mathcal{G}d \end{pmatrix}"] \arrow[r, "\begin{pmatrix} I & \partial d \circ \varepsilon_{\mathcal{G}C} \\ & I \end{pmatrix}"'] & \mathcal{F}C[-1] \oplus \mathcal{G}C \arrow[u, "\begin{pmatrix} \mathcal{F}\bar{d} & \\ f_{\bar{C}} \circ \varepsilon_{\mathcal{F}\bar{C}} & \mathcal{G}\bar{d} \end{pmatrix}"']
\end{tikzcd}
$$

Figure 3.1: Proof of Lemma 3.1.1

$f_C : \mathcal{F}C[-1] \to \mathcal{G}C$ each have homological degree -1 as they are part of the differential on Cone f_C, they anti-commute with ε. Similarly, $\partial d : \mathcal{G}C \to \mathcal{F}C[-1]$ has homological degree 0 as it is part of the chain morphism, so it commutes with ε.

The claim now follows from $\mathcal{F}\bar{d} = \mathcal{F}d - \partial d\, f = \mathcal{F}d + \partial d\, \varepsilon f \varepsilon$, from $\mathcal{F}\bar{d}\,\partial d\, \varepsilon = \mathcal{F}\bar{d}\,\partial d\, \varepsilon - \partial(d^2)\, \varepsilon = \mathcal{F}\bar{d}\,\partial d\, \varepsilon - \partial d\, \mathcal{G}d\, \varepsilon - \mathcal{F}\bar{d}\,\partial d\, \varepsilon = -\partial d\, \mathcal{G}d\, \varepsilon = \partial d\, \varepsilon\, \mathcal{G}d$ and from $f \varepsilon \partial d\, \varepsilon + \mathcal{G}\bar{d} = f\, \partial d + \mathcal{G}\bar{d} = \mathcal{G}d$. $\square$

3.2 Mutation invariance of $\mathfrak{sl}(2)$ homology

In this section, we show that universal $\mathfrak{sl}(2)$ homology over the ring $\mathbb{Z}_2[h,t]$ is invariant under mutation of knots. This generalizes an earlier result of Wehrli [45], who considered (in our notation) the case $h = 0$. It also generalizes a special case of Bloom's proof of mutation invariance of odd Khovanov homology [6]. An advantage of working in this generality is that we immediately obtain that the $\mathbb{Z}_2$-Rasmussen invariant is preserved by mutation. We do not know whether this invariant is identical to the invariant defined by Rasmussen in [38], but we expect that it shares all its properties, in particular that it gives a lower bound on the slice genus.

Since we work in characteristic 2 throughout the section, let us define $R_2 := R \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2[h,t]$ and $Kh_2(D) := Kh(D) \otimes \mathbb{Z}_2$, which we may view as an object of $\mathbf{K}^{\mathrm{b}}(\mathbf{mKob}(E) \otimes \mathbb{Z}_2)$ or $\mathbf{K}^{\mathrm{b}}(\mathbf{mKob}'(E) \otimes \mathbb{Z}_2)$. We will prove that if D and D' are two knot diagrams related by mutation, then $Kh_2(D) \cong Kh_2(D')$.

Let $\mathbf{Kob}(4) = \mathbf{Kob}(E)$, $\mathbf{Kob}'(4) = \mathbf{Kob}'(E)$, $\mathbf{mKob}(4) = \mathbf{mKob}(E)$ and $\mathbf{mKob}'(4) = \mathbf{mKob}'(E)$ where E is the set of four points on the boundary of the unit disc in north-east, north-west, south-west and south-east position. We will describe the category $\mathbf{Kob}'(4)$ explicitly.

Proposition 3.2.1. *The* $\mathbb{Z}[h,t]$-gmod-*category* $\mathbf{Kob}'(4)$ *has objects* $)($ *and* $\asymp$ *and morphism spaces*

$$\mathbf{Kob}'(4)()(,)() = \mathbb{Z}[h,t]\langle)(,\ \dot{)}(,\ )\dot{(},\ \dot{)}\dot{(} \rangle$$

$$\mathbf{Kob}'(4)()(,\asymp) = \mathbb{Z}[h,t]\langle \rangle\!\langle,\ \rangle\!\!\cdot\!\!\langle \rangle$$

$$\mathbf{Kob}'(4)(\asymp,)() = \mathbb{Z}[h,t]\langle \Join,\ \dot{\Join} \rangle$$

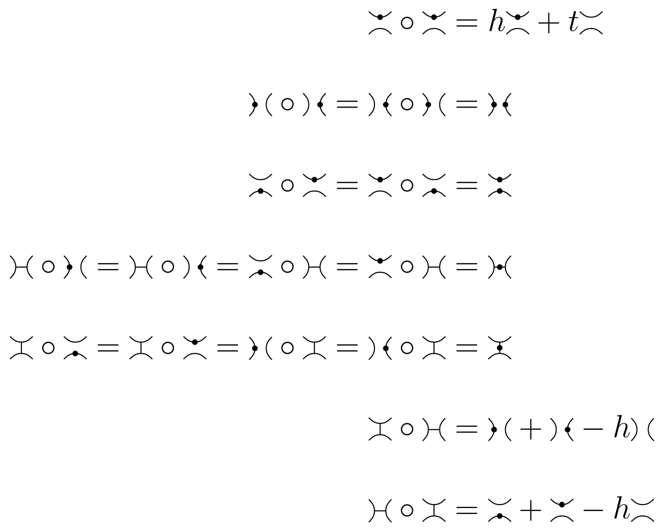
$$\mathbf{Kob}'(4)(\asymp,\asymp) = \mathbb{Z}[h,t]\langle \asymp,\ \underset{\cdot}{\asymp},\ \overset{\cdot}{\asymp},\ \dot{\underset{\cdot}{\asymp}} \rangle$$

Here $\deg)(= \deg \asymp = 0$, $\deg \dot{)}(= \deg)\dot{(} = \deg \underset{\cdot}{\asymp} = \deg \overset{\cdot}{\asymp} = 2$, $\deg \dot{)}\dot{(} = \deg \dot{\underset{\cdot}{\asymp}} = 4$, $\deg \rangle\!\langle = \deg \Join = 1$ *and* $\deg \rangle\!\!\cdot\!\!\langle = \deg \dot{\Join} = 3$. *The identity morphisms of* $\mathbf{Kob}'(4)$ *are* $)(:)(\to)($ *and* $\asymp : \asymp \to \asymp$. *In addition, the following relations hold:*

$$\dot{)}(\circ \dot{)}(= h\dot{)}(+ t)(\tag{3.1}$$

$$)\dot{(} \circ)\dot{(} = h)\dot{(} + t)(\tag{3.2}$$

$$\underset{\cdot}{\asymp} \circ \underset{\cdot}{\asymp} = h\underset{\cdot}{\asymp} + t\asymp \tag{3.3}$$



(3.4)

(3.5)

(3.6)

(3.7)

(3.8)

(3.9)

(3.10)

(3.11)

Proof. Let α be a morphism in $\mathbf{Kob}'(4)$. We can represent α as a linear combination of cobordisms without closed components. By the neck-cutting relation, we can further reduce α to a linear combination of dotted cobordisms of genus 0. If the source and the target of α are identical, we can additionally represent it as a linear combination of dotted sheets. Finally, if there is a component with more than one dot on it, we can use the two-dot relation to reduce the number of dots. Thus α can be represented as a linear combination of the cobordisms listed above.

Relations (3.1) through (3.4) now are consequences of the two-dot relation, (3.5) through (3.8) follow from isotopy and (3.9) and (3.10) come from the neck-cutting relation. One readily checks that these relations determine composition completely. It can be seen from closing up the tangle in the two possible ways that there are no non-trivial relations between the generators of the category. $\square$

Corollary 3.2.2. $\mathbf{Kob}(4)'$ *is isomorphic to the free* $\mathbb{Z}[h]$-gmod-*category generated by* $)($,

)(, $\asymp$, $\asymp$,)(and $\asymp$ modulo the ideal generated by relations (3.1)–(3.2), (3.3)–(3.4), (3.9), (3.10) and the left hand sides of equations (3.5) through (3.8).

Proof. It is clear that the relations in Proposition 3.2.1 imply those in the corollary, we must show that the converse is true. Define $t :)(\to)($ by $t =)(\circ)(- h)(=)(\circ)(- h)($ and $t : \asymp \to \asymp$ by $t = \asymp \circ \asymp - h\asymp = \asymp \circ \asymp - h\asymp$. In order to prove the corollary, we need to show that t is in the center of the category. We will show that t commutes with)(below; commutativity with the other generators follows similarly.

$$t \circ)(= \asymp \circ \asymp \circ)(- h\asymp \circ)(=)(\circ)(\circ)(- h)(\circ)(=)(\circ t$$

$\square$

Given a category described by generators and relations as above, we would like to define an operation ∂ by specifying its value on generators and then extending via the perturbed Leibniz rule from Lemma 3.1.1(3). We have to make sure this yields a well-defined operation.

Lemma 3.2.3. *Let $\mathcal{A}$ be an R-*gmod*-category given by generators and relations, which we write as $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{I}$, where $\tilde{\mathcal{A}}$ is the free R-*gmod*-category corresponding to the generators and $\mathcal{I}$ is the homogeneous ideal generated by the relations. For any generator ϕ from A to B, which we view as an element of $\tilde{\mathcal{A}}(A,B)$, fix a value $\bar{\phi} \in \tilde{\mathcal{A}}(A,B)$ of the same degree and a value $\partial\phi \in \tilde{\mathcal{A}}(A,B)$ of degree $\deg\phi - 2$. $\bar{\cdot}$ extends uniquely to a R-*gmod*-functor $\bar{\cdot} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$, which is the identity on objects. We require that $\bar{\cdot}$ be an involution.*

(1) *There is a unique R-linear extension of ∂ (of degree -2) satisfying $\partial(\psi\,\phi) = \partial\psi\,\phi + \bar{\psi}\,\partial\phi$ (compare Lemma 3.1.1(3)).*

(2) *If for all relations r, both $\bar{r}$ and ∂r lie in $\mathcal{I}$, then for any morphism $\psi \in \mathcal{I}$, $\partial\psi \in \mathcal{I}$. Hence ∂ descends to a well-defined operation on $\mathcal{A}$.*

(3) *For all objects A of $\mathcal{A}$, choose an element f_A of $\mathcal{A}(A, A)$ of degree 2 such that $\phi - \bar{\phi} = f_B\,\partial\phi = \partial\phi\, f_A$ holds for every generator $\phi : A \to B$. Let $\mathcal{C} = \mathbf{Mat}(\overline{\mathcal{A}}_0)$ and define $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ to be the shift functor $\{2\}$ and $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ the identity functor. Then f gives rise to a natural transformation $f : \mathcal{F} \Rightarrow \mathcal{G}$ and the category $\mathcal{C}$ together with $\bar{\cdot}$ and ∂ satisfies all hypotheses of Lemma 3.1.1.*

Proof. (1) We first note that if 1_A is the identity morphism on an object A (i.e. the generator of the free graded R-module $\tilde{\mathcal{A}}(A, A)$ corresponding to the empty path), then $\partial 1_A = \partial(1_A \circ 1_A) = \partial 1_A \circ 1_A + \bar{1}_A \circ \partial 1_A = 2\partial 1_A$ (note that $\bar{1}_A = 1_A$ since $\bar{\cdot}$ is a functor), hence $\partial 1_A = 0$. We can use R-bilinearity and the formula given for $\partial(\psi\,\phi)$ to define ∂ on $\tilde{\mathcal{A}}$. To show well-definedness, one readily checks that the formula for $\partial(\psi\,\phi)$ trivially holds if ψ or ϕ are identity morphisms and that for the composition of three morphisms $\phi\psi\rho$,

$$\partial((\rho\psi)\phi) = \partial(\rho(\psi\phi)) = \partial\rho\,\psi\,\phi + \bar{\rho}\,\partial\psi\,\phi + \bar{\rho}\,\bar{\psi}\,\partial\phi.$$

(2) Every element in $\mathcal{I}$ can be written as the sum of elements of the form $\phi\psi\rho$, where ψ is a relation. It follows from the computation in (1) that $\partial(\phi\psi\rho) \in \mathcal{I}$, hence $\mathcal{I}$ is closed under ∂.

(3) We constructed ϕ to satisfy conditions (1) and (3) of Lemma 3.1.1. Condition (2) holds for generators by the assumption and can be extended to arbitrary morphisms of $\mathcal{C}$: If $f\,\partial\phi = \mathcal{G}(\phi - \bar{\phi})$ and $f\,\partial\psi = \mathcal{G}(\psi - \bar{\psi})$, then $f\,\partial(\psi\,\phi) = f\,\partial\psi\,\mathcal{G}\phi + f\,\mathcal{F}\bar{\psi}\,\partial\phi = f\,\partial\psi\,\mathcal{G}\phi + \mathcal{G}\bar{\psi}\,f\,\partial\phi = \mathcal{G}(\psi - \bar{\psi})\,\mathcal{G}\phi + \mathcal{G}\bar{\psi}\,\mathcal{G}(\phi - \bar{\phi}) = \mathcal{G}(\psi\,\phi - \bar{\psi}\,\bar{\phi})$. In the same way, one can

show that $\partial\phi\, f = \mathcal{F}(\phi - \bar{\phi})$. $\square$

We now define a differential on $\mathbf{Kob}(4)' \otimes \mathbb{Z}_2$ of degree -2 by setting $\partial \dot{)}(= \partial)\dot{(} =)($, $\partial \underset{\cdot}{\asymp} = \partial \overset{\cdot}{\asymp} = \asymp$, $\partial)\!\!-\!\!(= 0$ and $\partial \asymp\!\!| = 0$. We do not specify $\bar{\cdot}$ yet, but we require that $\{\overline{\dot{)}(}, \overline{\overset{\cdot}{\asymp}}\} = \{\dot{)}(,)\dot{(}\}$, $\{\overline{\underset{\cdot}{\asymp}}, \overline{\overset{\cdot}{\asymp}}\} = \{\underset{\cdot}{\asymp}, \overset{\cdot}{\asymp}\}$, $\overline{)\!\!-\!\!(} =)\!\!-\!\!($ and $\overline{\asymp\!\!|} = \asymp\!\!|$. We set $f_{)(} = \dot{)}(+ \overline{\dot{)}(}$ and $f_{\asymp} = \underset{\cdot}{\asymp} + \overline{\underset{\cdot}{\asymp}}$.

We are now ready to verify the hypotheses of Lemma 3.2.3. For every relation r mentioned in Corollary 3.2.2, we must show that $\bar{r}$ and ∂r are zero when viewed as morphisms in $\mathbf{Kob}(4)' \otimes \mathbb{Z}_2$. It is immediately clear that this is the case for $\bar{r}$. The relations (3.1)–(3.2), (3.5) and (3.9) can be written as $r_1 = \dot{)}(\circ \dot{)}(+)\dot{(} \circ)\dot{(} + h\dot{)}(+ h)\dot{(}$, $r_2 =)\dot{(} \circ \dot{)}(+ \dot{)}(\circ)\dot{(}$, $r_3 = \asymp\!\!| \circ)\!\!-\!\!(+ \dot{)}(+)\dot{(} + h)($, respectively; (3.7) is equivalent to the set of relations $r_{\alpha,\beta} =)\!\!-\!\!(\circ \alpha + \beta \circ)\!\!-\!\!($ where $\alpha \in \{\dot{)}(,)\dot{(}\}$ and $\beta \in \{\underset{\cdot}{\asymp}, \overset{\cdot}{\asymp}\}$.

One readily computes that in $\mathbf{Kob}(4)' \otimes \mathbb{Z}_2$, $\partial r_1 = \dot{)}(+ \overline{\dot{)}(} +)\dot{(} + \overline{)\dot{(}} + 2h = 0$, $\partial r_2 = \dot{)}(+ \overline{)\dot{(}} +)\dot{(} + \overline{\dot{)}(} = 0$, $\partial r_3 = 0 + 2 + 0 = 0$ and $\partial r_{\alpha,\beta} =)\!\!-\!\!(+)\!\!-\!\!(= 0$.

We will also verify that for all generators $\phi \in \mathbf{Kob}(4)'(A, B)$, $\phi - \bar{\phi} = f_B\, \partial\phi = \partial\phi\, f_A$. This is clear for $\phi =)\!\!-\!\!($ and $\phi = \asymp\!\!|$ since in these cases, $\bar{\phi} = \phi$ and $\partial\phi = 0$. If ϕ is one of $\dot{)}($, $)\dot{(}$, $\underset{\cdot}{\asymp}$ or $\overset{\cdot}{\asymp}$, then $\partial\phi = 1$ and our definition of f implies that $\phi - \bar{\phi} = f$.

Mutation of the inner tangle is given by one of the involutions R_x, R_y and R_z, where R_x and R_y are reflections along the x-axis and y-axis, respectively and $R_z = R_y \circ R_x$ is rotation by 180° around the origin. R_x, R_y and R_z induce functors on $\mathbf{Kob}(4)$, $\mathbf{Kob}'(4)$, $\mathbf{mKob}(4)$ and $\mathbf{mKob}'(4)$, which we all denote by $\mathcal{R}_x$, $\mathcal{R}_y$ and $\mathcal{R}_z$. It is easy to see that these functors commute with the delooping functor $\mathbf{mKob}(4) \to \mathbf{mKob}'(4)$ (compare [45]).

On $\mathbf{Kob}'(4)$, the functors $\mathcal{R}_x$, $\mathcal{R}_y$ and $\mathcal{R}_z$ are identity functors on objects and are given by $\mathcal{R}_x)\!\!-\!\!(= \mathcal{R}_y)\!\!-\!\!(= \mathcal{R}_z)\!\!-\!\!(=)\!\!-\!\!($, $\mathcal{R}_x \asymp\!\!| = \mathcal{R}_y \asymp\!\!| = \mathcal{R}_z \asymp\!\!| = \asymp\!\!|$, $\mathcal{R}_x \dot{)}(= \mathcal{R}_y)\dot{(} = \mathcal{R}_z)\dot{(} = \dot{)}($,

$\mathcal{R}_x)\!\!\cdot\!(= \mathcal{R}_y)\!\cdot\!(= \mathcal{R}_z)\!\cdot\!(=)\!\cdot\!($, $\mathcal{R}_x \asymp = \mathcal{R}_y \asymp = \mathcal{R}_z \asymp = \asymp$ and $\mathcal{R}_x \asymp = \mathcal{R}_y \asymp = \mathcal{R}_z \asymp = \asymp$ on generating morphisms. Thus they satisfy all the conditions we imposed on $\bar{\cdot}$ and ∂ above, so Lemma 3.1.1 applies and

$$\mathrm{Cone}\left(f : Kh_2(D)\{2\} \to Kh_2(D)\right) \cong \mathrm{Cone}\left(f : Kh_2(\bar{D})\{2\} \to Kh_2(\bar{D})\right).$$

We have only shown this isomorphism over $\mathbb{Z}_2[h]$ so far. To show that it holds over R_2 we must verify that the isomorphism commutes with t. But this follows immediately from the fact that t is in the center of the category as shown in the proof of Corollary 3.2.2.

Before we can prove the theorem we need to ensure that the category $\mathbf{K}^{\mathrm{b}}(R_2\text{-fgmod})$ is well-behaved.

Definition 3.2.4. An additive category $\mathcal{C}$ is called *Krull-Schmidt* if every object of $\mathcal{C}$ is the direct sum of finitely many indecomposable objects and if $\bigoplus_{i \in I} X_i \cong \bigoplus_{j \in J} Y_j$ for indecomposable objects X_i and Y_j implies that there is a bijection $\phi : I \to J$ such that $X_i \cong Y_{\phi(i)}$.

Proposition 3.2.5. *The category* $\mathbf{K}^{\mathrm{b}}(R_2\text{-fgmod})$ *is Krull-Schmidt.*

Proof. We first show that idempotents in $\mathbf{K}^{\mathrm{b}}(R_2\text{-fgmod})$ split. Let $e : C \to C$ be such an idempotent. Proposition 3.2 in [7] shows that idempotents split in any triangulated category that has arbitrary direct sums (in particular, infinite direct sums). We cannot use this statement directly, since we need to stay in a category of complexes of finitely generated modules. However, we may use their construction to define a complex C_e as a totalization of the complex of complexes

$$\cdots C \xrightarrow{1+e} C \xrightarrow{e} C \xrightarrow{1+e} C$$

by successively taking appropriate mapping cones. It is easy to see that C_e is a complex of finitely generated free graded R_2-modules that is bounded above but not bounded below. C_e gives rise to a splitting of e, i.e. there are maps $r : C \to C_e$ and $i : C_e \to C$ such that $r \circ i \simeq \mathrm{id}$ and $i \circ r \simeq e$. The homology of C_e is supported in finitely many degrees, say $H^k(C_e) \cong 0$ for $k \leq N$. Write C_e as

$$\cdots C_e^{n-2} \xrightarrow{d^{n-2}} C_e^{n-1} \xrightarrow{d^{n-1}} C_e^n$$

and truncate it to get a complex C'_e given by

$$\ker d^N \longrightarrow C_e^N \xrightarrow{d^N} \cdots C_e^{n-1} \xrightarrow{d^{n-1}} C_e^n,$$

which is quasi-isomorphic to C_e via the evident quasi-isomorphism $C_e \to C'_e$. Since R_2-gmod has enough projectives, the functor $\mathbf{K}^-(R_2\text{-fgmod}) \to \mathbf{D}^-(R_2\text{-gmod})$ from the bounded above homotopy category of projectives to the bounded above derived category is an equivalence of categories. Hence C_e and C'_e are isomorphic in $\mathbf{K}^-(R_2\text{-fgmod})$ and maps $r' : C \to C'_e$ and $i' : C'_e \to C$ with $r' \circ i' \simeq \mathrm{id}$ and $i' \circ r' \simeq e$ give rise to a splitting of e.

Next we show that the endomorphism rings of indecomposable objects of $\mathbf{K}^{\mathrm{b}}(R_2\text{-fgmod})$ are local. As a $\mathbb{Z}_2$-vector space, $\mathrm{End}(C)$ is finite-dimensional since for all k and l, the space $R_2\text{-fgmod}(R_2\{k\}, R_2\{l\}))$ is finite-dimensional. If $x \in \mathrm{End}(C)$, then there are natural numbers $m < n$ such that $x^n = x^m$. Thus $x^{2m(n-m)} = x^{m(n-m-1)+m+m(n-m)} = x^{m(n-m-1)+m}$ implies that $x^{m(n-m)}$ is an idempotent, hence it is either an isomorphism or 0. In the first case, x is a unit in $\mathrm{End}(C)$; in the second case, $(1+x)^{2^k} = 1 + x^{2^k} = 1$ for $2^k \geq m(n-m)$, so $1+x$ is a unit.

Finally, we note that $\mathrm{End}(C)$ is finite-dimensional for any object C, hence any object admits a decomposition into finitely many indecomposables. The standard proof of the Krull-Schmidt theorem now applies (see for example Theorem 7.5 in [27]). $\square$

Lemma 3.2.6. *Let $\mathcal{C}$ be a Krull-Schmidt category and $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ be an autofunctor of $\mathcal{C}$ such that $\mathcal{F}^2 X \ncong X$ for all non-zero objects $X \in \mathcal{C}$. Then $X \oplus \mathcal{F}X \cong Y \oplus \mathcal{F}Y$ implies that $X \cong Y$.*

Proof. X and Y split into the same number n of indecomposable objects; we prove the lemma by induction on n. If X is the zero object, then so is Y and hence $X \cong Y$. Now suppose the lemma holds for all $n' < n$. Write $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{i=1}^n Y_i$, so by assumption $\bigoplus_i X_i \oplus \bigoplus_i \mathcal{F}X_i \cong \bigoplus_i Y_i \oplus \bigoplus_i \mathcal{F}Y_i$. We claim that $X_j \cong Y_k$ for some j and k. Otherwise, each X_i would have to be isomorphic to one $\mathcal{F}Y_i$ and each Y_i isomorphic to one $\mathcal{F}X_i$, which would imply $X \cong \mathcal{F}Y$ and $Y \cong \mathcal{F}X$ and therefore $X \cong \mathcal{F}^2 X$, a contradiction. Write $X \cong X_j \oplus X'$ and $Y \cong Y_k \oplus Y'$ in the obvious way and notice that by induction hypothesis, $X' \cong Y'$, hence $X \cong Y$. $\square$

Theorem 3.2.7. *Let K and K' be two knots related by mutation. Specifically, let D and D' be 2-tangles such that $D \cup D'$ represents K and $\bar{D} \cup D'$ represents K'. Then $Kh_2(K) \cong Kh_2(K')$.*

Proof. By the discussion above,

$$\begin{aligned} &\operatorname{Cone}\big(f : Kh_2(D)\{2\} \to Kh_2(D)\big) \\ \cong &\operatorname{Cone}\big(f : Kh_2(\bar{D})\{2\} \to Kh_2(\bar{D})\big) \end{aligned}$$

Tensoring both sides with $Kh_2(D')$ we obtain an isomorphism

$$\begin{aligned} &\operatorname{Cone}\big(f \otimes 1 : Kh_2(K)\{2\} \to Kh_2(K)\big) \\ \cong &\operatorname{Cone}\Big(f \otimes 1 : Kh_2(K')\{2\} \to Kh_2(K')\Big) \end{aligned}$$

Since $f \otimes 1$ is the difference $x_p - x_q$, where p and q are opposite endpoints of the tangle D. After closing up p and q lie on the same component, hence $f \otimes 1$ is null-homotopic. It follows that

$$Kh_2(K)[-1]\{2\} \oplus Kh_2(K) \cong Kh_2(K')[-1]\{2\} \oplus Kh_2(K')$$

Applying Lemma 3.2.6 with $\mathcal{F}A = A[-1]\{2\}$, we see that $Kh_2(K) \cong Kh_2(K')$. □

Corollary 3.2.8. *The $\mathbb{Z}_2$-Rasmussen invariant s_2 is invariant under mutation of knots.*

Proof. It is clear that the definition of $s_2(K)$ only depends on the homotopy type of the complex $Kh_2(K)$. □

3.3 Mutation invariance of $\mathfrak{sl}(n)$ homology

This section is devoted to proving the following result.

Theorem 3.3.1. *If L and L' are two links related by component-preserving positive mutation and n is odd, then their reduced $\mathfrak{sl}(n)$ homologies are isomorphic (reduced with respect to the component of the mutation, defined after Definition 2.1.2). More generally, let $p(x) = \sum_k a_{2k}x^{2k}$ be a polynomial with only even powers of x, then the reduced Khovanov-Rozansky homologies of L and L' associated to this polynomial are isomorphic.*

Using Rasmussen's spectral sequence from HOMFLY-PT homology to $\mathfrak{sl}(n)$ homology, and the fact that HOMFLY-PT homology of knots is finite-dimensional, we get the following corollary.

Corollary 3.3.2. *If K and K' are two knots related by positive mutation, their HOMFLY-PT homologies are isomorphic.*

We prove the theorem by first showing that the Khovanov-Rozansky complex of the inner 2-tangle can be built out of the complexes assigned to two basic diagrams: a pair of arcs and a singular crossing. For the Khovanov-Rozansky complex, we follow Rasmussen's definitions from [37], since Khovanov and Rozansky's original definitions are not general enough to serve our purpose. We then derive a criterion for a certain mapping cone of this complex to be invariant under reflection, which turns out to be the case for odd n in the case of positive mutation. Closing up the tangle, we see that the mapping cone computes reduced Khovanov-Rozansky homology.

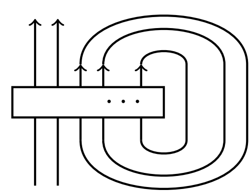
Finally, we note that our result is consistent with calculations for the Kinoshita-Terasaka knot and the Conway knot carried out by Mackaay and Vaz [34].

We quickly summarize the organization of the section. In subsection 3.3.1, we reduce the problem to the case of mutation of a 2-tangle in what we call braid form. In subsection 3.3.2, we investigate how the Khovanov-Rozansky complex behaves under positive mutation. In subsection 3.3.3, we show how to represent the Khovanov-Rozansky complex of a 2-tangle in braid-form as a complex over a particularly simple category. In subsection 3.3.4, we derive a general criterion for when a chain complex over an additive category is isomorphic to its image under a certain involution functor and show how this criterion applies to the problem at hand. In subsection 3.3.5, we combine the results from the previous subsections to prove Theorem 3.3.1.

3.3.1 Topological considerations

In this subsection, we show that we may assume that the inner tangle is presented in a specific form.

Definition 3.3.3. We say that a 2-tangle is in braid form if it is represented in the following way, where the rectangle represents an open braid.



Theorem 3.3.4. *Let L be an oriented link and L' be a mutant of L obtained by positive mutation. Then the mutation can be represented on a diagram whose inner tangle is given in braid form by a transformation of type R_y.*

The following two lemmas immediately imply the Theorem.

Lemma 3.3.5. *We may assume that the endpoints of the inner tangle are oriented as in Figure 3.2(a) and that the transformation of the inner tangle is of type R_y.*

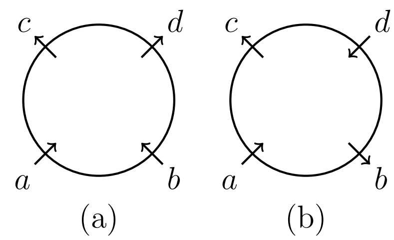


Figure 3.2: The two possible orientations of the endpoints of inner tangle

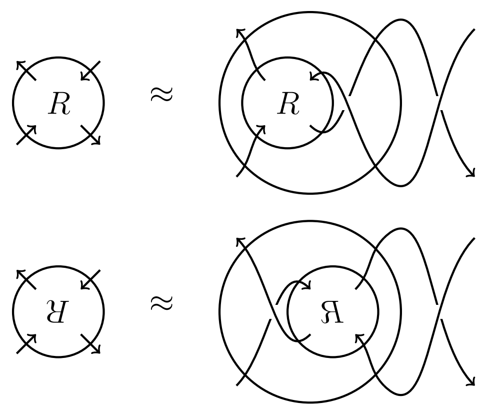
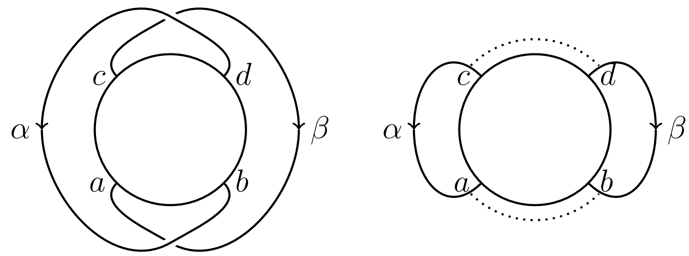


Figure 3.3: R_z mutation on a tangle of type (b) is equivalent to R_y mutation on a tangle of type (a)

Lemma 3.3.6. *Any 2-tangle with endpoints oriented as in Figure 3.2(a) can be represented by a diagram in braid form.*

Proof (of Lemma 3.3.5). If the tangle has two adjacent endpoints with the same orientation, it is isotopic to a tangle with endpoints as depicted in Figure 3.2(a) and the only positive mutation is of type R_y. Otherwise we are in case (b) of Figure 3.2, where the only positive mutation is of type R_z. But we can realize this type of mutation by R_y-mutation on a tangle of type (a), as illustrated in Figure 3.3. □

Proof (of Lemma 3.3.6). The proof uses a slight variation of the Yamada-Vogel [43, 49] algorithm to prove an analog of Alexander's Theorem for 2-tangles. We follow Birman and Brendle's exposition of the proof [5].



(a) (b)

Figure 3.4: A closure of the tangle and its Seifert picture

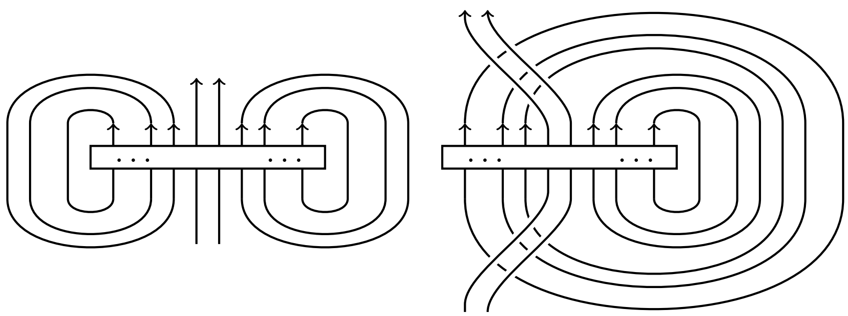


Figure 3.5: Transforming the diagram into braid form

Close the tangle by two arcs α from c to a and β from d to b as in Figure 3.4(a). The algorithm works by repeatedly performing a Reidemeister II move in a small neighborhood of a so-called *reducing arc*. The algorithm is performed on the Seifert picture of the link diagram, which is depicted in Figure 3.4(b). A reducing arc is an arc connecting an incoherently oriented pair of Seifert circles that intersects the Seifert picture only at its endpoints. Since the Seifert circles that α and β belong to are coherently oriented, the the unbounded region of the Seifert picture in Figure 3.4(b) cannot contain a reducing arc. Hence we may push the reducing arc into the circle. The algorithm now gives us a tangle diagram whose Seifert circles and Seifert arcs (from a to c and from b to d) are coherently oriented. This implies that all Seifert circles lie nested inside each other to the left of the left arc and to the right of the right arc, in other words it can be represented by a diagram in the form illustrated on the left of Figure 3.5. But this can be easily transformed into braid form, as seen on the right of Figure 3.5.

□

3.3.2 Behavior of the Khovanov-Rozansky chain complex under reflection

Lemma 3.3.7. *Let D be an oriented (possibly singular) tangle diagram and $\bar{D}$ be the reflection of D. Label the endpoints of D by $e_0, e_1, \ldots, e_{2k-1}$, and the corresponding endpoints of $\bar{D}$ by $\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_{2k-1}$. Then $C_p(\bar{D}) = \phi(C_p(D))$, where $\phi : R(D) \to R(\bar{D})$ is the ring homomorphism given by $\phi(x_{e_i}) = -x_{\tilde{e}_i}$.*

Proof. If D is one of the diagrams shown in Figure 2.3, $C_p(D)$ is one of the following complexes of matrix factorizations.

$C_p(D_+)$:

$$\begin{array}{ccc} R\{1-n\} & \underset{\frac{w}{x_c-x_a}}{\overset{x_c-x_a}{\rightleftarrows}} & R \\ {\scriptstyle x_c-x_b}\uparrow & & \uparrow{\scriptstyle 1} \\ R\{3-n\} & \underset{\frac{w}{(x_c-x_a)(x_c-x_b)}}{\overset{(x_c-x_a)(x_c-x_b)}{\rightleftarrows}} & R \end{array}$$

$C_p(D_-)$:

$$\begin{array}{ccc} R\{1-n\} & \underset{\frac{w}{(x_c-x_a)(x_c-x_b)}}{\overset{(x_c-x_a)(x_c-x_b)}{\rightleftarrows}} & R\{-2\} \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle x_c-x_b} \\ R\{1-n\} & \underset{\frac{w}{x_c-x_a}}{\overset{x_c-x_a}{\rightleftarrows}} & R \end{array}$$

$C_p(D_r)$:

$$R \underset{\frac{w}{x_c-x_a}}{\overset{x_c-x_a}{\rightleftarrows}} R\{n-1\}$$

$C_p(D_s)$:

$$R\{2-n\} \underset{\frac{w}{(x_c-x_a)(x_c-x_b)}}{\overset{(x_c-x_a)(x_c-x_b)}{\rightleftarrows}} R\{-1\}$$

Note that $\phi(x_c - x_a) = -x_{\tilde{d}} + x_{\tilde{b}} = x_{\tilde{c}} - x_{\tilde{a}}$, $\phi(x_c - x_b) = -x_{\tilde{d}} + x_{\tilde{a}} = x_{\tilde{c}} - x_{\tilde{b}}$ and $\phi(w) = \phi\left(\sum_k a_{2k}(x_c^{2k} + x_d^{2k} - x_a^{2k} - x_b^{2k})\right) = \sum_k a_{2k}(x_{\tilde{c}}^{2k} + x_{\tilde{d}}^{2k} - x_{\tilde{a}}^{2k} - x_{\tilde{b}}^{2k}) = \tilde{w}$.

Hence all maps in the above diagrams are mapped by ϕ to the same maps with $x_{\tilde{a}}$, $x_{\tilde{b}}$, $x_{\tilde{c}}$ and $x_{\tilde{d}}$ in place of x_a, x_b, x_c and x_d, respectively, that is ϕ maps $C_p(D)$ to $C_p(\bar{D})$.

The general case follows from (2.1): It is clear that by taking the internal edges of D into consideration, we can extend ϕ to an isomorphism between $R(D)$ and $R(\bar{D})$. Hence we get isomorphisms $C_p(D_i) \otimes_{R(D_i)} R(D) \cong C_p(\bar{D}_i) \otimes_{R(\bar{D}_i)} R(\bar{D})$, which in turn induce an isomorphism $C_p(D) \cong C_p(\bar{D})$. $\square$

In light of the Lemma, we will simply denote the homomorphism ϕ by $\bar{\cdot}$.

3.3.3 Khovanov-Rozansky Homology of 2-tangles

In this subsection, we investigate the Khovanov-Rozansky homology of 2-tangles in braid form. Denote the variables corresponding to the endpoints a, b, c and d of the tangle by x_a, x_b, x_c and x_d, respectively. The complex associated to such a tangle is a complex of graded matrix factorizations over the ring $R = \mathbb{Q}[x_a, x_b, x_c, x_d]/(x_a + x_b = x_c + x_d)$ with potential $w = p(x_c) + p(x_d) - p(x_a) - p(x_b)$. Let $\mathbf{hmf}_2$ denote the full subcategory of $\mathbf{hmf}_w(R)$ whose objects are direct sums of shifts of $C_p(D_r)$ and $C_p(D_s)$.

Theorem 3.3.8. *Let D a connected diagram of a 2-tangle in braid-form. Then $C_p(D)$ is isomorphic in $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_w(R))$ to an object of $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_2)$.*

Before proving the theorem, we need to recall an important tool for dealing with matrix factorizations: 'excluding a variable'. We quote Theorem 2.2 from [23].

Theorem 3.3.9. *Let R be a graded polynomial ring over $\mathbb{Q}$ and and $u, v \in R[y]$ two polynomials, with v being monic in y. Furthermore, let $\bar{w} \in R$ and M be a graded matrix factorization over $R[y]$ with potential $w = \bar{w} - uv$. Then $M/(v)$ and $K(u; v) \otimes M$ are iso-*

morphic as objects of $\mathbf{hmf}_w(R)$*. We say that we* exclude *the variable* y *to obtain* $M/(v)$ *from* $K(u;v) \otimes M$*.*

The Theorem is only stated for ungraded matrix factorization in [23], but it is trivial to check that the quotient map $K(u;v) \otimes M \to M/(v)$ constructed in the proof is of degree 0. □

We will also need another well-known result about Koszul matrix factorizations; this is, for example, the $n = 2$ special case of Theorem 2.1 in [23].

Theorem 3.3.10. *Let* R *be a graded polynomial ring over* $\mathbb{Q}$ *and* $v_1, v_2 \in R$ *be relatively prime. Then any two Koszul matrix factorizations of the form* $\begin{Bmatrix} * & v_1 \\ * & v_2 \end{Bmatrix}$ *with the same potential are isomorphic.*

In the same spirit, we show that a matrix factorization that is almost the direct sum of two order-two Koszul matrix factorizations can be transformed into an honest direct sum.

Theorem 3.3.11. *Let* R *be a graded polynomial ring and* $\dot{R} = R\{\dot{k}\}$ *and* $\ddot{R} = R\{\ddot{k}\}$ *be free* R*-modules of rank* 1*, then any graded matrix factorization of the form*

$$\dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} \underset{U}{\overset{V}{\rightleftarrows}} \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R}$$

with

$$U = \begin{pmatrix} b & * & 0 & * \\ a & * & 0 & * \\ 0 & * & d & * \\ 0 & * & c & * \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} * & * & * & * \\ a & -b & 0 & 0 \\ * & * & * & * \\ 0 & 0 & c & -d \end{pmatrix},$$

where $\gcd(a,b) = \gcd(c,d) = 1$ *and* $k_x = \deg x - \frac{\deg w}{2}$ *for* $x \in \{a,b,c,d\}$*, is isomorphic to* *a matrix factorization of the form*

$$\begin{Bmatrix} * & a \\ * & b \end{Bmatrix}\{\dot{k}\} \oplus \begin{Bmatrix} * & c \\ * & d \end{Bmatrix}\{\ddot{k}\}$$

Proof. Let

$$U = \begin{pmatrix} b & * & 0 & b_2 \\ a & * & 0 & a_2 \\ 0 & d_1 & d & * \\ 0 & c_1 & c & * \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} * & * & c_2 & -d_2 \\ a & -b & 0 & 0 \\ a_1 & -b_1 & * & * \\ 0 & 0 & c & -d \end{pmatrix}.$$

Computing the lower left and the upper right quadrant of $UV = wI$, we see that

$$\begin{pmatrix} d_1 & d \\ c_1 & c \end{pmatrix} \begin{pmatrix} a & -b \\ a_1 & -b_1 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} b & b_2 \\ a & a_2 \end{pmatrix} \begin{pmatrix} c_2 & -d_2 \\ c & -d \end{pmatrix} = 0.$$

Since $\gcd(a, b) = \gcd(c, d) = 1$, the rank of each of these matrices is at least 1, so none of them can have rank 2. Hence $0 = \det \begin{pmatrix} a & -b \\ a_1 & -b_1 \end{pmatrix} = -b_1 a + a_1 b$ and there exists an $\alpha \in R$ such that $a_1 = \alpha_1 a$ and $b_1 = \alpha_1 b$. Similarly, we can find an $\alpha_2 \in R$, such that $a_2 = \alpha_2 a$ and $b_2 = \alpha_2 b$, as well as $\beta_i \in R$ ($i \in \{1, 2\}$) such that $c_i = \beta_i c$ and $d_i = \beta_i d$. The fact that the two matrix products above are 0 implies that $\beta_i = -\alpha_i$. We now perform a change of basis,

$$\begin{tikzcd} \dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} \arrow[r, shift left, "V"] \arrow[d, equal] & \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R} \arrow[l, shift left, "U"] \arrow[d, shift left, "X"] \\ \dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} \arrow[r, shift left, "V'"] & \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R} \arrow[l, shift left, "U'"] \arrow[u, shift left, "X^{-1}"] \end{tikzcd}$$

where

$$X = \begin{pmatrix} 1 & & & \alpha_2 \\ & 1 & & \\ & -\alpha_1 & 1 & \\ & & & 1 \end{pmatrix}, \; U' = \begin{pmatrix} b & * & 0 & 0 \\ a & * & 0 & 0 \\ 0 & 0 & d & * \\ 0 & 0 & c & * \end{pmatrix} \text{ and } V' = \begin{pmatrix} * & * & 0 & 0 \\ a & -b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & c & -d \end{pmatrix},$$

the lower row being exactly the desired direct sum of Koszul matrix factorizations.

We still need to verify that C is of degree 0: We have $\deg \alpha_1 = \deg a_1 - \deg a = (\frac{\deg w}{2} + k_a + \ddot{k} - k_c - k_d - \dot{k}) - (k_a + \frac{\deg w}{2}) = \ddot{k} - \dot{k} - k_c - k_d$ and $\deg \alpha_2 = \deg a_2 - \deg a = (\frac{\deg w}{2} + \dot{k} - k_b - \ddot{k}) - (k_a + \frac{\deg w}{2}) = \dot{k} - \ddot{k} - k_a - k_b$, which implies $\deg(-\alpha_1 : \dot{R} \to \ddot{R}\{k_c + k_d\}) = \deg(\alpha_2 : \ddot{R} \to \dot{R}\{k_a + k_b\}) = 0$. □

The following proposition is an analog of Lemma 4.10 and Propositions 4.3–4.6 in [37] and Lemma 3 and Propositions 4–7 in [22]. Unfortunately, we cannot deduce it from any of the previous results: The theory introduced in [21] is weaker than what we consider here (in the $\mathfrak{sl}(2)$ case, this is the difference between Khovanov Homology and Bar-Natan's universal variant [1]). We also cannot use the results in [37], which are only shown to hold up to a notion of quasi-isomorphism. However, the proofs in Rasmussen' paper can be modified to apply to our situation.

Proposition 3.3.12. *The following isomorphisms hold in the homotopy category of matrix factorizations over the external ring corresponding to the diagrams.*

(a) Let D be a diagram of a fully resolved tangle, and D' be a diagram obtained from D by replacing a smoothing of type D_r. (See Figure 2.3) by a pair of arcs without increasing the number of components. Then $C_p(D) \cong C_p(D')$.

(b) Up to grading shifts, $C_p(D_O)$ is isomorphic to a direct sum of n copies of $C_p(D_A)$.

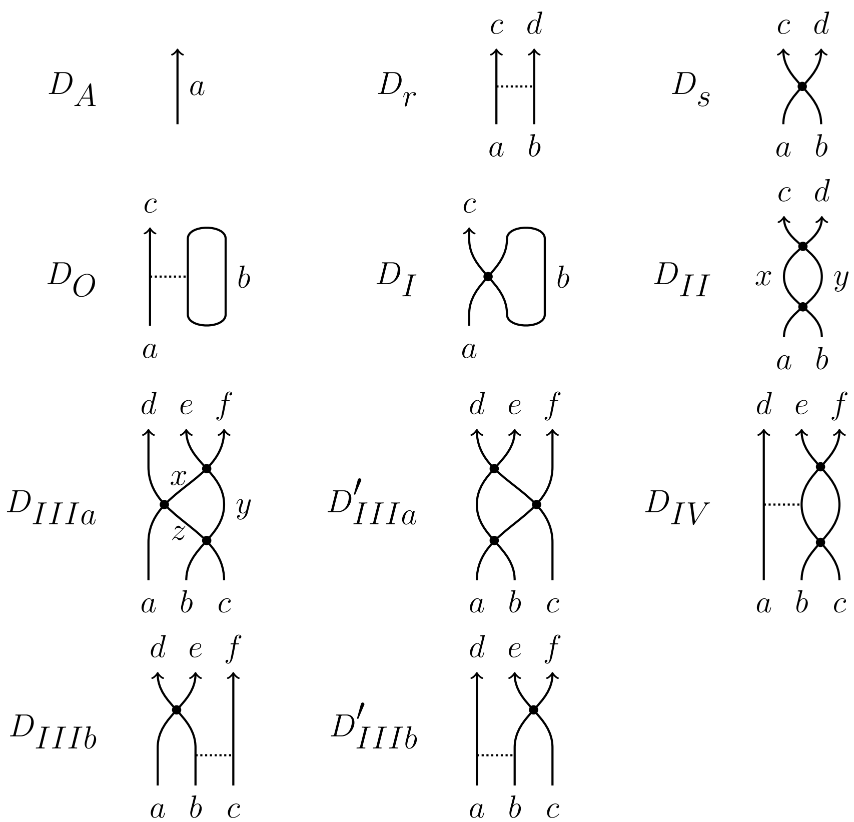


Figure 3.6: Singular braid diagrams

(c) Up to grading shifts, $C_p(D_I)$ is isomorphic to a direct sum of $n-1$ copies of $C_p(D_A)$.

(d) Up to grading shifts, $C_p(D_{II}) \cong C_p(D_s) \oplus C_p(D_s)$.

(e) Up to grading shifts, $C_p(D_{IIIa}) \oplus C_p(D_{IIIb}) \cong C_p(D'_{IIIa}) \oplus C_p(D'_{IIIb})$.

(f) Up to grading shifts, $C_p(D_{IV}) \cong C_p(D'_{IIIb}) \oplus C_p(D'_{IIIb})$.

Proof. (a) Since D and D' are connected, their external rings $R_{\text{ext}}(D)$ and $R_{\text{ext}}(D')$ are identical. Since $R(D') = R(D)/(x_a = x_c)$ and $R(D) \cong R(D')[x]$ by Lemma 2.4 in [37], x_a and x_c are different elements of $R(D)$. If x_a and x_c were both linear combinations of external edges, then their difference $x_c - x_a$ would be a linear combination of external edges as well. But $x_c - x_a \neq 0 \in R(D)$ and $x_c - x_a = 0 \in R(D')$, which contradicts $R_{\text{ext}}(D) = R_{\text{ext}}(D')$. Assume w.l.o.g. that x_c is not a linear combination of external edges. Since $K(*; x_c - x_a)$ appears as a factor of $C_p(D)$, we may exclude the variable x_c to obtain $C_p(D')$.

(b) We have $R(D_O) = \mathbb{Q}[x_a, x_b, x_c]/(x_c - x_a)$, $R_{\text{ext}}(D_O) = \mathbb{Q}[x_a, x_c]/(x_c - x_a)$ and $R(D_A) = \mathbb{Q}[x_a]$, hence

$$C_p(D_O) = K\left(x_c - x_a; \frac{p(x_c) + p(x_b) - p(x_a) - p(x_b)}{x_c - x_a}\right)\langle 1\rangle = K(0; p'(x_a) - p'(x_b))\langle 1\rangle.$$

Excluding the variable x_b, we obtain

$$C_p(D_O) \cong \bigoplus_{i=0}^{n-1} C_p(D_A)\langle 1\rangle\{2i\}.$$

(c) As in part (b), we have $R(D_s) = \mathbb{Q}[x_a, x_b, x_c]/(x_c - x_a)$, $R_{\text{ext}}(D_s) = \mathbb{Q}[x_a, x_c]/$

$(x_c - x_a)$ and $R(D_A) = \mathbb{Q}[x_a]$, so

$$\begin{aligned}
C_p(D_I) &= K\left(\frac{p(x_c)+p(x_b)-p(x_a)-p(x_b)}{(x_c-x_a)(x_c-x_b)}; (x_c - x_a)(x_c - x_b)\right)\{-1\} \\
&= K\left(\frac{p'(x_a)-p'(x_b)}{x_a-x_b}; 0\right)\{-1\} \\
&= K\left(0; \frac{p'(x_a)-p'(x_b)}{x_a-x_b}\right)\langle 1\rangle\{2-n\},
\end{aligned}$$

so once again we may exclude x_b to get

$$C_p(D_I) \cong \bigoplus_{i=0}^{n-2} C_p(D_A)\langle 1\rangle\{2-n+2i\}.$$

(d) Choose labels as in Figure 3.6 and set $x := x_x$ and $y := x_y$. As matrix factorizations over $R(D_{II})$,

$$\begin{aligned}
C_p(D_{II})\{2\} &= \begin{Bmatrix} * & (x-x_a)(x-x_b) \\ * & (x_c-x)(x_c-y) \end{Bmatrix} \\
&= \begin{Bmatrix} * & (x-x_a)(x-x_b) \\ * & (x_c-x)(x-x_d) \end{Bmatrix} \\
&\cong \begin{Bmatrix} * & (x-x_a)(x-x_b) \\ * & (x_c-x)(x-x_d)+(x-x_a)(x-x_b) \end{Bmatrix} \\
&= \begin{Bmatrix} * & (x-x_a)(x-x_b) \\ * & (x_c-x_a)(x_c-x_b) \end{Bmatrix}
\end{aligned}$$

Let $R = R_{\text{ext}}(D_{II}) = R_{\text{ext}}(D_s)$. Excluding the variable x, we get a matrix factorization $K(\alpha + \beta x; (x_c - x_a)(x_c - x_b))$ over the ring $R' = R[x]/(x^2 = (x_a + x_b)x - x_a x_b)$ whose potential $(\alpha + \beta x)(x_c - x_a)(x_c - x_b)$ has to lie in R, hence $\beta = 0$. As a graded R-module,

$R' \cong R \oplus R\{2\}$, so $C_p(D_{II}) \cong K_R(\alpha; (x_c - x_a)(x_c - x_b))\{-2\} \oplus K_R(\alpha; (x_c - x_a)(x_c - x_b)) \cong C_p(D_s)\{-1\} \oplus C_p(D_s)\{1\}$.

(e) Choose labels as in Figure 3.6, and set $x := x_x$, $y := x_y$ and $z := x_z$. Let $R = R_{\text{ext}}(D_{IIIa})$, and note that $R(D_{IIIa}) \cong R[x]$. As matrix factorizations over $R(D_{IIIa})$,

$$\begin{aligned}
C_p(D_{IIIa})\{3\} &= \left\{\begin{matrix} * & (x_d - x_a)(x_d - z) \\ * & (x_e - x)(x_e - y) \\ * & (z - x_b)(z - x_c) \end{matrix}\right\} \\
&= \left\{\begin{matrix} * & (x_d - x_a)(x_a - x) \\ * & (x_e - x)(x - x_f) \\ * & (x + x_d - x_a - x_b)(x + x_d - x_a - x_c) \end{matrix}\right\} \\
&\cong \left\{\begin{matrix} * & (x_d - x_a)(x_a - x) \\ * & (x_e - x)(x - x_f) \\ * & x_a x_b + x_b x_c + x_c x_a - x_c x_d - x_d x_e - x_e x_f \end{matrix}\right\},
\end{aligned}$$

where the last line is obtained from the previous one by adding the top right and the center right entry to the bottom right entry. Let $p = x_e + x_f$, $q = x_e x_f$, $\alpha = x_d - x_a$ and $\beta = x_a x_b + x_b x_c + x_c x_a - x_c x_d - x_d x_e - x_e x_f$, so that the last line reads

$$C_p(D_{IIIa})\{3\} \cong \left\{\begin{matrix} * & a x_a - a x \\ * & -x^2 + px - q \\ * & b \end{matrix}\right\}$$

Using the second row to exclude the variable x, we obtain an order-two Koszul matrix factorization over the ring $R' = R(D_{IIIa})/(x^2 = px - q)$, which is given explicitly (with respect to the standard decomposition of R' as a free R-module of rank two) as $C_p(D_{IIIa}) \cong$

$R_1 \underset{U}{\overset{V}{\rightleftarrows}} R_0$, where

$$R_1 = R\{3-n\} \oplus R\{5-n\} \oplus R\{3-n\} \oplus R\{5-n\},$$

$$R_0 = R\{6-2n\} \oplus R\{8-2n\} \oplus R \oplus R\{2\},$$

$$A = \begin{pmatrix} \beta & 0 & * & * \\ 0 & \beta & * & * \\ \alpha x_a & \alpha q & * & * \\ -\alpha & \alpha(x_a - p) & * & * \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ \alpha x_a & \alpha q & -\beta & 0 \\ -\alpha & \alpha(x_a - p) & 0 & -\beta \end{pmatrix}.$$

We apply the following change of basis

$$\begin{tikzcd} R_1 \arrow[r, shift left, "V"] \arrow[d, shift left, "X"] & R_0 \arrow[l, shift left, "U"] \arrow[d, shift left, "X"] \\ R_1' \arrow[r, shift left, "V'"] \arrow[u, shift left, "X^{-1}"] & R_0' \arrow[l, shift left, "U'"] \arrow[u, shift left, "X^{-1}"] \end{tikzcd} \text{, where } X = \begin{pmatrix} 1 & p - x_a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_a \end{pmatrix},$$

$$R_1' = R\{3-n\} \oplus R\{5-n\} \oplus R\{5-n\} \oplus R\{3-n\} \text{ and}$$

$$R_0' = R\{6-2n\} \oplus R\{2\} \oplus R\{8-2n\} \oplus R.$$

C is of q-degree 0; a straightforward computation shows that

$$U' = \begin{pmatrix} \beta & * & 0 & * \\ -\alpha & * & 0 & * \\ 0 & * & \beta & * \\ 0 & * & \alpha(x_a - x_e)(x_a - x_f) & * \end{pmatrix} \quad \text{and}$$

$$V' = \begin{pmatrix} * & * & * & * \\ -\alpha & -\beta & 0 & 0 \\ * & * & * & * \\ 0 & 0 & \alpha(x_a - x_e)(x_a - x_f) & -\beta \end{pmatrix}.$$

We compute $\gcd(\alpha, \beta) = \gcd(\alpha, (x_d - x_c)\alpha - \beta) = \gcd((x_d - x_a), (x_b - x_e)(x_b - x_f)) = 1$, hence by symmetry $\gcd(x_a - x_e, \beta) = 1$ and $\gcd(x_a - x_f, \beta) = 1$ as well. Therefore, $\gcd(\alpha, \beta) = \gcd(\alpha(x_a - x_e)(x_a - x_f), \beta) = 1$, so we may apply Theorem 3.3.11 to get

$$\begin{aligned} C_p(D_{IIIa})\{3\} &\cong \begin{Bmatrix} * & -\alpha \\ * & \beta \end{Bmatrix}\{2\} \oplus \begin{Bmatrix} * & \alpha(x_a - x_e)(x_a - x_f) \\ * & \beta \end{Bmatrix} \\ &\cong \begin{Bmatrix} * & -\alpha \\ * & \beta + (x_e + x_f - x_a)\alpha \end{Bmatrix}\{2\} \oplus \begin{Bmatrix} * & \alpha(x_a - x_e)(x_a - x_f) - x_a^2\beta \\ * & \beta \end{Bmatrix} \\ &\cong \begin{Bmatrix} * & x_a - x_d \\ * & x_b x_c - x_e x_f \end{Bmatrix}\{2\} \oplus \begin{Bmatrix} * & x_a x_b x_c - x_d x_e x_f \\ * & \beta \end{Bmatrix} \end{aligned}$$

It is easy to see that the first summand is isomorphic to $C_p(D'_{IIIb})\langle 1\rangle\{3\}$. Denote the second summand by $\Upsilon\{3\}$, so that we have $C_p(D_{IIIa}) \cong C_p(D'_{IIIb})\langle 1\rangle \oplus \Upsilon$. By Lemma 3.3.7, reflection along the middle strand is given by the ring homomorphism $\bar{\cdot} : R \to R$, $\bar{x}_a = -x_c$, $\bar{x}_b = -x_b$, $\bar{x}_c = -x_a$, $\bar{x}_d = -x_f$, $\bar{x}_e = -x_e$ and $\bar{x}_f = -x_d$. Since $\bar{\Upsilon} \cong \Upsilon$ by Theorem 3.3.10 under this isomorphism, we obtain that $C_p(D'_{IIIa}) \cong C_p(D_{IIIb})\langle 1\rangle \oplus \Upsilon$, which implies claim (e).

(f) This follows immediately from (a) and (d). $\square$

We will collectively refer to diagrams of type D_r and D_s as *resolved crossings.*

Proof. (of Theorem 3.3.8) We will prove the theorem by repeatedly reducing $C_p(D)$ according to Proposition 3.3.12. At each step, we get a complex of matrix factorizations whose underlying graded object is $\bigoplus_i C_p(D_i)$ for some collection of singular diagrams in braid form. Following Wu [47], we define a complexity function on singular braids by $i(D) = \sum_j i_j$ where j runs over all resolved crossings in the diagram and i_j is 1 for an oriented smoothing and one plus the number of strands to the left of the crossing for a singular crossing. We show that each step of the reduction process decreases either the maximum complexity of diagrams D_i or the number of diagrams that have maximum complexity. This reduction can be performed as long as the maximum complexity is greater than 1. The only connected diagrams of complexity 1 are D_r and D_s, so if the maximum complexity is 1, $C_p(D)$ is the direct sum of shifts of $C_p(D_r)$ and $C_p(D_s)$ and the Lemma follows. To perform the reduction, choose a diagram of maximum complexity. The Lemma below guarantees that either D_{IIIa} or one of the diagrams on the left hand side of Proposition 3.3.12(a)-(d) or (f) is a subdiagram of D. In the latter case we can simply replace the complex on the left hand side by the the one on the right-hand side; notice that this reduces the number of diagrams of this complexity. If there is a subdiagram of type D_{IIIa}, we are given a complex of the form

$$\dots C^{k-1} \xrightarrow{\begin{pmatrix}\alpha \\ \beta\end{pmatrix}} C^k \oplus C_p(D_{IIIa}) \xrightarrow{\begin{pmatrix}\gamma & \delta\end{pmatrix}} C^{k+1} \dots,$$

which is (up to a grading shift) isomorphic in $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_w(R))$ to

$$\dots C^{k-1} \oplus C_p(D_{IIIb}) \xrightarrow{\begin{pmatrix}\alpha \\ \beta \\ 1\end{pmatrix}} C^k \oplus C_p(D_{IIIa}) \oplus C_p(D_{IIIb}) \xrightarrow{\begin{pmatrix}\gamma & \delta & 0\end{pmatrix}} C^{k+1} \dots,$$

which is in turn isomorphic to

$$\dots C^{k-1} \oplus C_p(D_{IIIb}) \to C^k \oplus C_p(D'_{IIIa}) \oplus C_p(D'_{IIIb}) \to C^{k+1} \dots,$$

so we once again were able to reduce the number of diagrams of the given complexity. $\square$

Lemma 3.3.13. *If D is a connected singular (open) braid diagram of complexity greater than* 1*, then it contains at least one of the following subdiagrams:*

(i) A resolved crossing of type D_r or D_s in rightmost position which is the only resolved crossing in this column,

(ii) a diagram D_r which has the property that D stays connected when D_r is removed,

(iii) a diagram of type D_{II}, D_{III} or D_{IV}.

Proof. We prove the lemma by induction on the braid index. If the braid index is 2 and $i(D) > 1$, then we either have a subdiagram of type D_r, which can be removed without disconnecting the diagram, or we have at least two subdiagrams of type D_s and none of type D_r, so we can find D_{II} as a subdiagram. If the braid index is greater than 2, we may assume that there are at least two resolved crossings in rightmost position. We may also assume that we have no subdiagrams of type D_r in rightmost position, since we could remove them without disconnecting the diagram. If two such singular crossings are adjacent, we have found D_{II} as a subdiagram. Otherwise choose the topmost two such singular crossings and apply the induction hypothesis to the part of the braid between those two singular crossings, giving us either a subdiagram of the required type inside this part of the braid or, potentially after performing an isotopy, a diagram of type D_{III} or D_{IV}. $\square$

3.3.4 Mutation invariance of the inner tangle

Lemma 3.3.14. *Let* $R = \mathbb{Q}[x_a, x_b, x_c, x_d]/(x_a + x_b = x_c + x_d)$ *and let* $\bar{\cdot}$ *be the ring homomorphism defined by* $\bar{x}_a = -x_b$, $\bar{x}_b = -x_a$, $\bar{x}_c = -x_d$ *and* $\bar{x}_d = -x_c$, *which induces an involution functor on* $\mathbf{hmf}_2$. *Let* $\mathcal{F}$ *be the grading shift functor* $\{2\}$ *and* $\mathcal{G}$ *be the identity functor. Then there is a differential* ∂ *on the morphism spaces of* $\mathbf{hmf}_2$ *satisfying the hypothesis of Lemma 3.1.1*

Proof. $\bar{\cdot} : R \to R$ is well-defined since $\overline{x_a + x_b} = -x_b - x_a = -x_d - x_c = \overline{x_c + x_d}$. R is isomorphic to the polynomial ring in x_a, x_b and x_c. Substituting $x_b = -x_a$ in any expression of the form $r - \bar{r}$, we obtain 0, hence $r - \bar{r}$ is divisible by $x_a + x_b$ and we may define ∂ on R by $\partial r = \frac{r - \bar{r}}{x_a + x_b}$. Viewing the ring R as an additive category with one element, it is straightforward to check that ∂ satisfies the hypothesis of Lemma 3.1.1.

The differential descends to a differential on $\mathbf{hmf}_2$. First note that objects in $\mathbf{hmf}_2$ are direct sums of one-term Koszul factorizations $K(u; v)$ with potential $w = p(x_c) + p(x_d) - p(x_a) - p(x_b)$. It follows from the proof of the one-crossing case of Lemma 3.3.7 that $\partial w = 0$ and $\partial v = 0$ for the two choices of v, that is $v = x_c - x_a$ and $v = (x_c - x_a)(x_c - x_b)$. This implies $0 = \partial w = \partial u\, v + \bar{u}\, \partial v = \partial u\, v$, hence $\partial v = 0$ since R does not have zero divisors. We define the differential of a morphism of such matrix factorizations,

$$\begin{tikzcd}
R\{\frac{\deg v' - \deg u'}{2}\} \arrow[r, shift left, "v'"] & R \arrow[l, shift left, "u'"] \\
R\{\frac{\deg v - \deg u}{2} + \deg z\} \arrow[u, "y"] \arrow[r, shift left, "v"] & R\{\deg z\} \arrow[l, shift left, "u"] \arrow[u, "z"']
\end{tikzcd}$$

to simply be

$$\begin{tikzcd} R\{\frac{\deg v' - \deg u'}{2} + 2\} \arrow[r, shift left, "v'"] & R\{2\} \arrow[l, shift left, "u'"] \\ R\{\frac{\deg v - \deg u}{2} + \deg z\} \arrow[u, "\partial y"] \arrow[r, shift left, "v"] & R\{\deg z\} \arrow[l, shift left, "u"] \arrow[u, "\partial z"'] \end{tikzcd}$$

This is a morphism of matrix factorizations since $\partial y\, u = \partial(yu) = \partial(u' z) = u' \,\partial z$ and $\partial z\, v = \partial(zv) = \partial(v' y) = v' \,\partial y$.

Since any null-homotopic morphism

$$\begin{tikzcd} R\{\frac{\deg v' - \deg u'}{2}\} \arrow[r, shift left, "v'"] & R \arrow[l, shift left, "u'"] \\ R\{\frac{\deg w}{2} + \deg h\} \arrow[u, "u'h + kv"] \arrow[r, shift left, "v"] & R\{\deg h + \deg u\} \arrow[l, shift left, "u"] \arrow[u, "hu + v'k"'] \end{tikzcd}$$

is sent to the null-homotopic morphism

$$\begin{tikzcd} R\{\frac{\deg v' - \deg u'}{2} + 2\} \arrow[r, shift left, "v'"] & R\{2\} \arrow[l, shift left, "u'"] \\ R\{\frac{\deg w}{2} + \deg h\} \arrow[u, "u' \partial h + \partial k\, v"] \arrow[r, shift left, "v"] & R\{\deg h + \deg u\} \arrow[l, shift left, "u"] \arrow[u, "\partial h\, u + v' \partial k"'] \end{tikzcd}$$

∂ descends to a differential on $\mathbf{hmf}_2$.

The natural transformation ϕ is given by

$$\begin{tikzcd} R\{\frac{\deg v - \deg u}{2}\} \arrow[r, shift left, "v"] & R \arrow[l, shift left, "u"] \\ R\{\frac{\deg v - \deg u}{2} + 2\} \arrow[u, "x_a + x_b"] \arrow[r, shift left, "v"] & R\{2\} \arrow[l, shift left, "u"] \arrow[u, "x_a + x_b"'] \end{tikzcd}$$

Since we can view (representatives of) morphisms in $\mathbf{hmf}_2$ as pairs of elements of R, the fact that R satisfies the hypothesis of Lemma 3.1.1 implies that $\mathbf{hmf}_2$ does as well. $\square$

3.3.5 Proof of invariance

Before we can finish the proof, we need to borrow another Lemma from [37].

Lemma 3.3.15. *(Lemma 5.16 in* [37]*) Let D be the diagram of a single crossing with endpoints as in Figure 3.2(a). Then the maps $x_b : C_p(D)\{2\} \to C_p(D)$ and $x_c : C_p(D)\{2\} \to C_p(D)$ are homotopic. Since $x_a + x_b = x_c + x_d$, this of course implies that x_a and x_d are homotopic as well.*

Proof. Let $d_+ : C_p(D_r) \to C_p(D_s)$ be the differential of a positive crossing and $d_- : C_p(D_s) \to C_p(D_r)$ be the differential of a negative crossing. Clearly, $d_-d_+ = x_c - x_b : C_p(D_r) \to C_p(D_r)$ and $d_+d_- = x_c - x_b : C_p(D_s) \to C_p(D_s)$, so $d_\mp$ is a null-homotopy for $x_c - x_b : C_p(D_\pm) \to C_p(D_\pm)$. We ignored q-gradings above, the reader can easily check that the proof applies in the graded setting as well. $\square$

We are now ready to prove Theorem 3.3.1. Given a pair of mutants L_1 and L_2, we may assume, by Theorem 3.3.4, that the mutation is realized as a mutation of type R_z whose inner tangle diagram D is in braid form. By Theorem 3.3.8, there is an object C in $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_2)$ such that $C_p(D) \cong C$ in $\mathbf{K}^{\mathrm{b}}(\mathbf{hmf}_w(R))$. Applying the ring isomorphism $\bar{\cdot}$, we obtain an isomorphism $\overline{C_p(D)} \cong \bar{C}$, hence by Lemma 3.3.7, $C_p(\bar{D}) \cong \bar{C}$. Applying Lemma 3.1.1, we obtain that $\mathrm{Cone}(x_a + x_b : C\{2\} \to C)$ is isomorphic in $\mathbf{hmf}_2$, and hence in $\mathbf{hmf}_w(R)$ to $\mathrm{Cone}(x_a + x_b : \bar{C}\{2\} \to \bar{C})$. Taking the tensor product over $\mathbb{Q}[x_b, x_c, x_d]$ with the complex associated to the outer tangle, we get an isomorphism

$$\mathrm{Cone}(x_a + x_b : C_p(L_1^\circ)\{2\} \to C_p(L_1^\circ)) \cong \mathrm{Cone}(x_a + x_b : C_p(L_2^\circ)\{2\} \to C_p(L_2^\circ))$$

by Proposition 2.4.1, where L_1° and L_2° denote L_1 and L_2 cut open at a, respectively. Because we consider only positive mutation, x_a and x_b lie on the same component of both L_1 and L_2, so x_a and x_b are homotopic by repeated application of Lemma 3.3.15. Hence we get an isomorphism

$$\operatorname{Cone}(2x_a : C_p(L_1^\circ)\{2\} \to C_p(L_1^\circ)) \cong \operatorname{Cone}(2x_a : C_p(L_2^\circ)\{2\} \to C_p(L_2^\circ))$$

and thus

$$\widehat{C}_p(L_1)[-1]\{2\} \oplus \widehat{C}_p(L_1) \cong \widehat{C}_p(L_2)[-1]\{2\} \oplus \widehat{C}_p(L_2).$$

$C_p(L)$ is generally infinitely-generated, but since its definition involves only finitely many generators of the form $\mathbb{Q}[x_1, \ldots, x_n]$, where the x_k all have q-degree 2, it is finite-dimensional in every q-degree. Thus the isomorphism types of $\widehat{C}_p(L_1)$ and $\widehat{C}_p(L_2)$ are determined by their Poincaré Laurent power series $\widehat{\chi}_p(L_k) \in \mathbb{Z}[t^{-1}, t][q^{-1}, q]]$,

$$\widehat{\chi}_p(L_k) = \sum_{i,j} q^i t^j \dim(\widehat{H}^{i,j}(L_k)).$$

Thus the above equivalence translates to

$$(q^2 t^{-1} + 1)\widehat{\chi}_p(L_1) = (q^2 t^{-1} + 1)\widehat{\chi}_p(L_2),$$

which implies $\widehat{\chi}_p(L_1) = \widehat{\chi}_p(L_2)$ and thus

$$\widehat{C}_p(L_1) \cong \widehat{C}_p(L_2).$$

□

Proof (of Corollary 3.3.2). This follows directly from Theorem 1 in [37], which asserts that for sufficiently large n, the $\mathfrak{sl}(n)$ homology of a knot is a regraded version of its HOMFLY-PT homology. It is clear that we can recover the triple grading of HOMFLY-PT homology by choosing n large enough. □

Chapter 4

Khovanov Homology of 3-braids

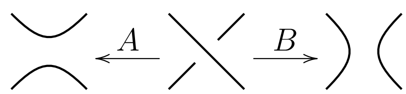
According to Garside's solution of the word problem for B_n (see for example Birman [4], Theorem 2.5), each element of B_n has a unique representative of the form $\Delta^n s$ (which can be determined algorithmically), where s is a positive braid and Δ is the central element of B_n, geometrically corresponding to a half twist.

From this point of view, it is natural to study how knot invariants change under insertion and deletion of powers of Δ. In chapter, we consider the case of the Khovanov homology of 3-braids. We show (Theorem 4.4.1) that if s is a positive 3-braid, then the Khovanov homologies of the closures of s and $\Delta^{2k}s$ are related by a distinguished triangle. If in addition $k \geq 0$, we show that the Khovanov homology of the closure of $\Delta^{2k}s$ is determined by the Khovanov homology of the closure of s.

The result is a generalization of earlier work of Turner [42], who computed the Khovanov homology of $(3,p)$ torus links.

4.1 Khovanov Homology

As before, we work in the setting of universal $\mathfrak{sl}(2)$ homology. In contrast to the previous treatment, we will allow arbitrary signs ($+$ or $-$) to be assigned to each crossing of the tangle. We do not demand that these signs be induced by an orientation of the tangle. The Khovanov complex of a single positive crossing (X_+) is given as a chain object as $Kh(X_+) = Kh(X_A)\{1\} \oplus Kh(X_B)[1]\{2\}$, where X_A is the A-smoothing and X_B the B-smoothing of the crossing:



The Khovanov complex of a negative crossing $X\_$ is $Kh(X\_) = Kh(X_A)[-1]\{-2\} \oplus Kh(X_B)\{-1\}$ as an chain object. Clearly $Kh(X_+) = Kh(X\_)[1]\{3\}$.

We also introduce the notation $[\![\cdot]\!]$ for the Khovanov complex $Kh(\cdot)$ considered as an object of the appropriate category of chain complexes (not the homotopy category). We write the tensor product arising from gluing two braids together simply as concatenation.

4.2 Simplifying chain complexes

Since we are studying Khovanov complexes in the more general setting of complexes over a (not necessarily abelian) additive category, we do not have the usual tools of homological algebra at our disposal. Instead of taking homology, we will reduce complexes using a specific type of a strong deformation retract, which allows us to state a result reminiscent of the spectral sequence induced by a double complex.

Definition 4.2.1. A chain map $G : C \to \hat{C}$ is called a *strong deformation retract* if there is a chain map $F : \hat{C} \to C$ a homotopy map $h : C \to \hat{C}[-1]$ such that $GF = I$, $FG = I - dh - hd$ and $hF = 0 = Gh$. If in addition $h^2 = 0$ we will call G a *special deformation retract* (Note: this is not a standard definition).

Remark 4.2.2. If idempotents in the underlying additive category split, then it is easy to see that this definition is equivalent to C being isomorphic to a direct sum of $\hat{C}$ and a complex whose differential is the identity. The projections onto the three direct summands of C are given by FG, dh and hd; note that dh and hd are idempotents since $hdh = h(I - hd - FG) = h$.

Proposition 4.2.3. *The property of being a special deformation retract is closed under composition.*

Proof. It is well-known that strong deformation retracts are closed under composition, so we only need to show that $h^2 = 0$. Let C_1, C_2 and C_3 be chain complexes. For $i = 1, 2$, let $G_i : C_i \to C_{i+1}$, $F_i : C_{i+1} \to C_i$, $h_i : C_i \to C_{i+1}[-1]$ such that $G_i F_i = I$, $F_i G_i = I - d_i h_i - h_i d_i$, $h_i F_i = 0 = G_i h_i$. Then

$$h^2 = h_1^2 - h_1 F_1 h_2 G_1 - F_1 h_2 G_1 h_1 + F_1 h_2 G_1 F_1 h_2 G_1 = F_1 h_2^2 G_1 = 0.$$

□

The following theorem is essentially a homotopy version of the spectral sequence of a double complex, but avoids the problem that it is in general not possible to reconstruct the integral homology from the E_∞ page. We adopt the following conventions. A double complex is an object in a bigraded additive category with a horizontal differential d of bidegree $(1, 0)$ and a “diagonal” differential of bidegree $(0, 1)$, in particular $d^2 = 0$ and

$f^2 = 0$. We require that differentials anti-commute, i.e. $df + fd = 0$.

$$
\begin{tikzcd}
 & & & \iddots & \iddots \\
 & \cdots \arrow[r] & C_i^{j+1} \arrow[r, "f_i^{j+1}"] \arrow[ur] & C_{i+1}^{j+1} \arrow[r] \arrow[ur] & \cdots \\
\cdots \arrow[r] & C_i^j \arrow[r, "f_i^j"] \arrow[ur, "d_i^j"] & C_{i+1}^j \arrow[r] \arrow[ur, "d_{i+1}^j"'] & \cdots & \\
\iddots \arrow[ur] & \iddots \arrow[ur] & & &
\end{tikzcd}
$$

The total complex is given by the direct sums $\bigoplus_{i+j=s} C_i^j$ over the columns in the above picture and differential $d + f$. Since we are not interested in the vertical and diagonal chain complexes, we simply refer to the total complex as the double complex.

Theorem 4.2.4. *If $C = C_i^j$ is a (bounded) double complex and $G_i : C_i \to \hat{C}_i$ are special deformation retracts with inverses F_i and associated homotopy maps h_i, then (the total complex of) C is homotopy equivalent to $\hat{C}$, which is given by $\bigoplus_{i+j=s} C_i^j$ and has differential $\hat{d} + G(f + fhf + \dots)F$ In fact, this homotopy equivalence is a special deformation retract in the sense of Definition 4.2.1.*

Proof. It is convenient to formally define $\frac{I}{I-x} = I + x + x^2 + \dots$. We first need to show that the map $\hat{d} + Gf\frac{I}{I-hf}F = \hat{d} + G\frac{I}{I-fh}fF$ does indeed define a differential, i.e. that its square is 0. Clearly, $fd = -df$, $Gd = \hat{d}G$, $F\hat{d} = dF$, $FG = I + dh + hd$ and $f^2 = 0$.

Therefore

$$\begin{aligned}
&\left(\hat{d}+G\tfrac{I}{I-fh}fF\right)\left(\hat{d}+Gf\tfrac{I}{I-hf}\right)\\
=\;& G\left(df\tfrac{I}{I-hf}+\tfrac{I}{I-fh}fd+\tfrac{I}{I-fh}f(I+dh+hd)f\tfrac{I}{I-hf}\right)F\\
=\;& G\tfrac{I}{I-fh}\left((I-fh)df+fd(I-hf)+f(dh+hd)f\right)\tfrac{I}{I-hf}F\\
=\;& G\tfrac{I}{I-fh}\left(df-fhdf+fd-fdhf+fdhf+fhdf\right)\tfrac{I}{I-hf}F=0
\end{aligned}$$

We will show that the following picture defines a homotopy equivalence.

$$\begin{tikzcd}
C \arrow[rr, shift left, "d+f"] \arrow[d, shift left, "G\frac{I}{I-fh}"] & & C[1] \arrow[ll, shift left, "-h-hfh-..."] \arrow[d, shift left, "G\frac{I}{I-fh}"] \\
\hat{C} \arrow[u, shift left, "\frac{I}{I-hf}F"] \arrow[rr, "\hat{d}+G(f+fhf+...)F"'] & & \hat{C}[1] \arrow[u, shift left, "\frac{I}{I-hf}F"]
\end{tikzcd}$$

The upward-pointing arrows define a chain map since

$$\begin{aligned}
&(d+f)\tfrac{I}{I-hf}F-\tfrac{I}{I-hf}F\left(\hat{d}+Gf\tfrac{I}{I-hf}F\right)\\
=\;& \tfrac{I}{I-hf}\left((I-hf)(d+f)-d(I-hf)-FGf\right)\tfrac{I}{I-hf}F\\
=\;& \tfrac{I}{I-hf}\left(d+f-hfd-hf^2-d+dhf-f-dhf-hdf\right)\tfrac{I}{I-hf}F=0
\end{aligned}$$

Similarly, we get a chain map from the downward-facing arrows because

$$\begin{aligned}
&G\tfrac{I}{I-fh}(d+f)-\left(\hat{d}+G\tfrac{I}{I-fh}fF\right)G\tfrac{I}{I-fh}\\
=\;& G\tfrac{I}{I-fh}\left((d+f)(I-fh)-(I-fh)d-fFG\right)\tfrac{I}{I-fh}
\end{aligned}$$

$$= \; G\frac{I}{I-fh}\left(d+f-dfh-f^2h-d+fhd-f-fdh-fhd\right)\frac{I}{I-fh}=0$$

Finally, we need to show that

$$\begin{aligned}
& \frac{I}{I-hf}FG\frac{I}{I-fh}-I-(d+f)h\frac{I}{I-fh}-\frac{I}{I-hf}h(d+f) \\
= \; & \frac{I}{I-hf}\big(FG-(I-hf)(I-fh) \\
& -(I-hf)(d+f)h-h(d+f)(I-fh)\big)\frac{I}{I-fh} \\
= \; & \frac{I}{I-hf}\big(I+dh+hd-I+hf+fh-hf^2h \\
& -dh-fh+hfdh+hf^2h-hd-hf+hdfh+hf^2h\big)\frac{I}{I-fh}=0
\end{aligned}$$

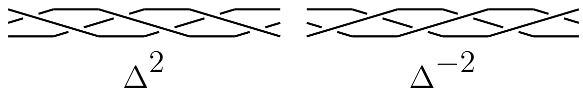
$\square$

4.3 Three-Braids

We are interested in the effect of adding a number of twists to the Khovanov homology (or more precisely, the homotopy type of the Khovanov complex) of the closure of a 3-braid. One can view braids as links in a standardly embedded thickened annulus. Adding twists then corresponds to switching to a non-standard embedding of the annulus. Any smoothing of a closed 3-braid in an annulus that is not the trivial 3-braid yields isotopic links regardless of the chosen embedding. This fact can be exploited to give a recursive formula for the Jones polynomial, namely $J(\Delta^2 s)-t^6J(s)=(-\sqrt{t})^{w(s)}(J(\circ\circ\circ)-t^6J(\Delta^2))$. The goal of this chapter is to establish a similar relationship for Khovanov homology. We will restrict ourselves to positive 3-braids.

In the following we will always represent a full positive twist Δ^2 by the braid $(\sigma_1\sigma_2)^3$

and a full negative twist Δ^{-2} by the braid $(\sigma_1\sigma_2)^{-3} = (\sigma_2^{-1}\sigma_1^{-1})^3$.



If s is a positive 3-braids of length n, we will compare s with $\Delta^{2k}s$ ($k \in \mathbb{Z}$). In light of the previous discussion, $C^k := Kh(\Delta^{2k}s)$ can be viewed as a tensor product of $Kh(\Delta^{2k})$ and $Kh(s)$, which is a double complex whose diagonals are $C^k_X := Kh(\Delta^{2k}s_X)[b(X)]\{2b(X)+n\}$ with differential d, where s_X is the smoothing corresponding to $X \in \{A, B\}^n$ and $b(X)$ is the number of Bs occurring in X. Horizontal maps between smoothings X and X' are given by $f^k_{X,X'}$, which is induced by a crossing change on s. Let $C^k_{\bar{A}^n}$ be the subcomplex of C^k whose underlying graded object is $\bigoplus_{X\in\{A,B\}^n\setminus\{A^n\}} C^k_X$ and C^k_A be the quotient complex $C^k/C^k_{\bar{A}^n}$, which we identify with $C^k_{A^n}$. We may think of $C^k_{\bar{A}^n}$ as a cube of partial resolutions for $C^k = Kh(\Delta^{2k}s)$ with the A^n-vertex removed.

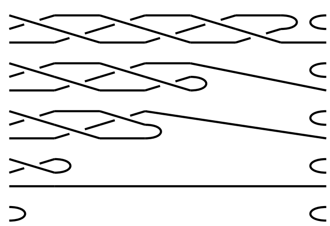


Figure 4.1: Transforming $\Delta^2 s_X$ into s_X. All other cases, negative twist, first B-smoothing on the bottom are analogous.

In order to apply Theorem 4.2.4, we need to establish special deformation retractions from all the C^k_X. If $X = A^n$, then C^k_A retracts to a complex $\tilde{C}^k_A$, which is (for positive k) supported in gradings between 0 and $4k$ by Corollary 4.4.2 below. If X is not the all-A smoothing, then the braid $\Delta^{2k}s_X$ is isotopic to s_X, with an explicit isotopy given by Figure 4.1. Note that the isotopy consists of only Reidemeister I and II moves, which induce

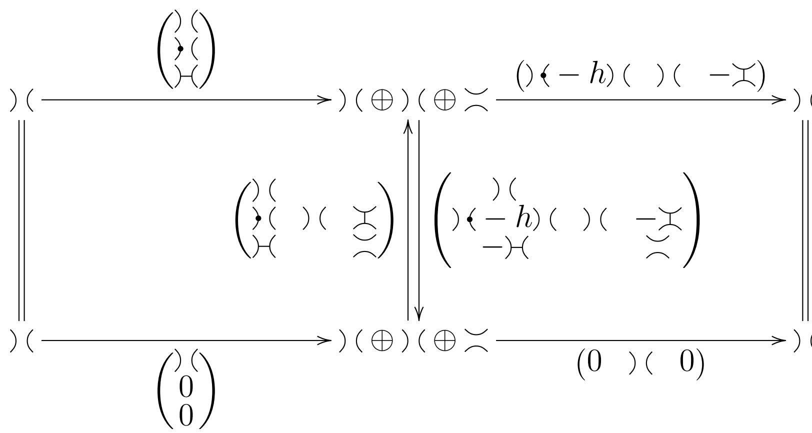


Figure 4.2: Reidemeister II moves induce special deformation retracts

special deformation retracts. This is obvious for Reidemeister I moves, since homotopies are of homological degree -1 and the complex is supported in two adjacent homological heights. For Reidemeister II, the top row of Figure 4.2 shows the complex $[\![\asymp]\!]$ after delooping, which is isomorphic to a direct sum of $[\![\asymp]\!]$ and a complex with identity differential, as seen in the figure. We leave it to the reader to work out grading shifts. In order to be able to perform this isotopy, we need to change $4k$ positive crossings to negative crossings, thus we get a retraction $G : C^k_X \to \tilde{C}^k_X$. where $\tilde{C}^k_X := Kh(s_X)[b(X) + 4k]\{2b(X) + n + 12k\}$. Note that $\tilde{C}^k_X = C^0_X[4k]\{12k\}$.

Theorem 4.2.4 now implies that there is a reduced complex $\tilde{C}^k$, which is the direct sum of the $\tilde{C}^k_X$s with differential $d + \sum_k G(fh)^k fF$. As for C^k, we define a subcomplex $\tilde{C}^k_{\bar{A}}$ and a quotient complex $\tilde{C}^k_A = \tilde{C}^k / C^k_{\bar{A}}$.

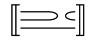
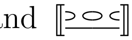
Proposition 4.3.1. *Let $m > 1$. Then, in the notation of Theorem 4.2.4, $G(fh)^m fF = 0$ if $k \geq 0$. If $k < 0$, $G(fh)^m fF = 0$ only on $\tilde{C}^k_{\bar{A}}$.*

Proof. The chain morphism $G(fh)^m fF$ has (homological) degree 1 since it is a differential. Since all the $\tilde{C}^k_X$ $(X \neq A^n)$ are supported in grading $b(X) + 4k$, a degree-1 morphism that travels along more than one edge of the cube is 0 if it starts at any place other then $\tilde{C}^k_A$. If $k \geq 0$, then $\tilde{C}^k_A$ is supported in gradings $\leq 4k$, so the morphisms originating at $\tilde{C}^k_A$ are also 0 if $m > 1$. $\square$

Proposition 4.3.2. *Restricted to* $C^k_{\bar{A}}$, $Gf^k_{X,X'}F = \pm f^0_{X,X'}[4k]\{12k\}$, *where* $f^k_{X,X'} : C^k_X \to C^k_{X'}$ *is an edge belonging to the differential of* C^k *as above.*

Proof. This is trivial if there is a B-smoothing to the left of the crossing that is being changed, since it does not affect the sequence of Reidemeister moves performed on Δ^{2k}. If changing the smoothing changes the left-most B-smoothing from being on the bottom to being on the top, $Gf^k_{X,X'}F$ is induced by a morphism $[\![\ ]\!] \to [\![\ ]\!]$ of q-degree 1. The space of such morphisms is one-dimensional, so $Gf^k_{X,X'}F$ is an integer multiple of the morphism induced by the saddle $g = [\![\ ]\!] \to [\![\ ]\!]$, say $Gf^k_{X,X'}F = mg$. As a special case of Theorem 4.2.4, the cone over the closure of $Gf^k_{X,X'}F$ is homotopy equivalent to the cone over $f^k_{X,X'}$, which computes the closure of $[\![\ ]\!]$, that is an unknot. If m were neither 1 nor -1, then with coefficients in $\mathbb{Z}/m\mathbb{Z}$, $Gf^k_{X,X'}F = 0$, so the homology of the (mod-m) Khovanov complex of the unknot would have rank $4 + 2 = 6$, a contradiction. The case where $Gf^k_{X,X'}F : [\![\ ]\!] \to [\![\ ]\!]$ is completely analogous.

If the left-most B-smoothing stays on the top or on the bottom, say without loss of generality on the top, it is easy to calculate $Gf^k_{X,X'}F : [\![\ ]\!] \to [\![\ ]\!]$ explicitly. Notice that by neck-cutting, the morphism $f^k_{X,X'} : [\![\Delta^{2k}\ ]\!] \to [\![\Delta^{2k}\ ]\!]$ can be written as $I_{\Delta^{2k}} f_1 + I_{\Delta^{2k}} f_2 - h\, I_{\Delta^{2k}} f_3$, where $f_1 =$, $f_2 =$ and $f_3 =$ (the middle parts of these cobordisms correspond to , and , respectively). Clearly, $I_{\Delta^{2k}} f_2 \circ F = F \circ f_2$ and $I_{\Delta^{2k}} f_3 \circ F = F \circ f_3$. Since sliding a dot across a crossing gives a chain morphism that is homotopic to the negative of the original one, it is easy to see that $I_{\Delta^{2k}} f_1 \circ$

$F \simeq F \circ f_1$. Thus $Gf^k_{X,X'}F = G \circ \left(I_{\Delta^{2k}} f_1 \circ F + I_{\Delta^{2k}} f_2 \circ F - h\, I_{\Delta^{2k}} f_3 \circ F \right) \simeq G \circ \left(F \circ f_1 + F \circ f_2 - h\, F \circ f_3 \right) = GFf^0_{X,X'} = \pm f^0_{X,X'}$.  and  are supported in a single homological grading, hence the notions of homotopy and equality coincide and $Gf^k_{X,X'}F = \pm f^0_{X,X'}$. □

Proposition 4.3.3. *The following isomorphisms of complexes hold:*

(a) $\tilde{C}^k_A \cong C^0_A[4k]\{12k\}$.

(b) If $k > 0$*, then* $\tilde{C}^k$ *is isomorphic to the mapping cone*

$$\mathrm{Cone}\left(f : \tilde{C}^k_A \to C^0_A[4k]\{12k\} \right),$$

where f *is induced by* $Gf^k_{A,X}F : \tilde{C}^k_A \to \tilde{C}^k_X \cong C^0_X[4k]\{12k\}$ *for all* $X \in \{A, B\}^n$ *with* $b(X) = 1$.

Proof. We prove both parts in parallel. In each case, the two complexes agree up to signs. More precisely, the complexes on the left are given as cubes with vertices $\tilde{C}^k_X$ (and with the A^n vertex removed in case (a)), whose edges are the chain morphisms $Gf^k_{X,X'}F$. By Proposition 4.3.1, the corresponding morphisms on the right hand side are $\epsilon_{X,X'} Gf^k_{X,X'}F$ with $\epsilon_{X,X'} \in \mathbb{Z}^* = \{\pm 1\}$. We claim that when viewing this cube (possibly with a vertex and all its adjacent cells removed) as a simplicial complex, ϵ defines a 1-cocycle. This requires us to show that if a face of the cube has vertices X, X', $\tilde{X}'$ and X'' that $\epsilon_{X,X'}\epsilon_{X',X''} = \epsilon_{X,\tilde{X}'}\epsilon_{\tilde{X}',X''}$, which will follow once we've shown that $Gf^k_{X'',X'}F \circ Gf^k_{X,X'}F \neq 0$. If $X \neq A^n$, then this morphism is a shift of $\pm f^0_{X'',X'} f^0_{X,X'}$, which cannot be zero as a composition of two saddle cobordisms. If $X = A^n$, we will argue by contradiction, so assume that $Gf^k_{X'',X'}F \circ Gf^k_{A^n,X'}F = Gf^k_{X'',X'} f^k_{A^n,X'}F = 0$. Extend s by σ_1 on the right and note that the above equality implies that $Gf^k_{X''B,X'B}f^k_{A^nB,X'B}F \simeq 0$. Again, up to grading shift this morphism is given by $f^0_{X''B,X'B}f^0_{A^nB,X'B} : C^0_{A^nB} \to C^0_{X''B}$. As before, this morphism is not zero, and it cannot be homotopic to zero either, since $C^0_{A^nB}$ and $C^0_{X''B}$ are supported in a single homological grading, which is the desired contradiction.

Since the simplicial complex is contractible in both cases and thus has trivial first cohomology, ϵ is a coboundary and there exists a 0-cochain η such that $\partial\eta = \epsilon$ and thus $\epsilon_{X,X'} = \eta_X\eta_{X'}$. Hence η gives the desired isomorphism of complexes. $\square$

4.4 Khovanov homology after adding twists

We are now ready to prove the promised relationship between $Kh(s)$ and $Kh(\Delta^{2k}s)$.

Theorem 4.4.1. *Let $\Delta = \sigma_1\sigma_2\sigma_1 \in B_3$ be a half-twist and let $\widehat{s}$ denote the closure of the 3-braid s. Suppose that all three-braids are oriented in the natural way and define*

$$r(s) = \begin{cases} \epsilon & \text{if } s = \epsilon \\ \sigma_i & \text{if } s \text{ only contains } \sigma_i\text{'s} \\ \sigma_1\sigma_2 & \text{if } s \text{ contains both } \sigma_1\text{'s and } \sigma_2\text{'s} \end{cases}$$

(a) There is a chain complex Y_s such that

$$Y_s \oplus Kh(\widehat{r(s)})\{|s| - |r(s)|\} \cong Kh(\widehat{s})$$

(b) For any $k \in \mathbb{Z}$, there is a distinguished triangle

$$Y_s[4k]\{12k\} \longrightarrow Kh(\widehat{\Delta^{2k}s}) \longrightarrow Kh(\widehat{\Delta^{2k}r(s)})\{|s| - |r(s)|\} \longrightarrow Y_s[4k+1]\{12k\}.$$

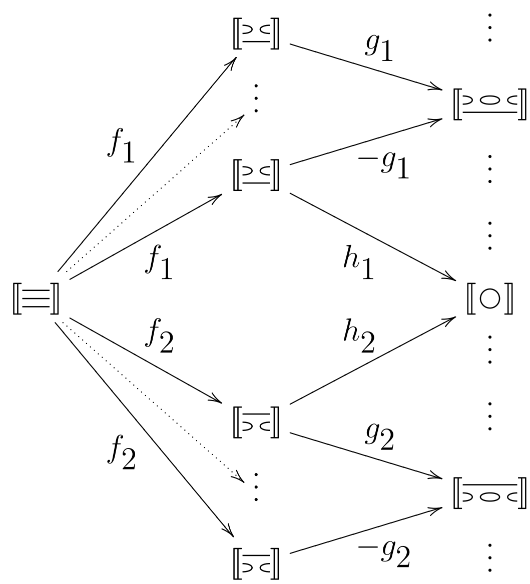


Figure 4.3: The Khovanov complex near a corner of the Khovanov cube

(c) If $k \geq 0$, then

$$Kh(\widehat{\Delta^{2k} s}) \cong Y_s[4k]\{12k\} \oplus Kh(\widehat{\Delta^{2k} r(s)})\{|s| - |r(s)|\}$$

Proof. Introduce morphisms f_i, g_i and h_i as shown in Figure 4.3. The diagrams represent smoothings of the closure of the braid even though they are depicted as smoothings of braid diagrams. We will decompose $C := C^0$ into $A \oplus B$ as shown in Figure 4.4.

Now, $z \in C_0 = [\![\equiv]\!]$ implies

$$dz = ((f_1 z, \ldots f_1 z), (f_2 z, \ldots f_2 z)) \in A_1.$$

Similarly,

$$d((x, \ldots x), (y, \ldots y)) = ((g_1 x - g_1 x, \ldots), (h_1 x - h_2 y, \ldots), (g_2 y - g_2 y, \ldots)) \in A_2.$$

$$
\begin{aligned}
A_0 &= C_0 \\
A_1 &= \left\{((x,\dots x),(y,\dots y)) \in [\![\asymp]\!]^m \oplus [\![\asymp]\!]^n\right\} \\
A_2 &= \left\{((0,\dots 0),(y,\dots y),(0,\dots 0)) \in [\![\asymp\bigcirc\asymp]\!]^{\binom{m}{2}} \oplus [\![\bigcirc]\!]^{mn} \oplus [\![\asymp\bigcirc\asymp]\!]^{\binom{n}{2}}\right\} \\
A_i &= 0 \qquad (i > 2) \\
B_0 &= 0 \\
B_1 &= \left\{((x_1,\dots x_{m-1},0),(y_1,\dots y_{m-1},0)) \in [\![\asymp]\!]^m \oplus [\![\asymp]\!]^n\right\} \\
B_2 &= \Big\{\Big((x_1,\dots x_{\binom{m}{2}}),(y_1,\dots y_{mn-1},0),(z_1,\dots z_{\binom{n}{2}})\Big) \\
&\qquad \in [\![\asymp\bigcirc\asymp]\!]^{\binom{m}{2}} \oplus [\![\bigcirc]\!]^{mn} \oplus [\![\asymp\bigcirc\asymp]\!]^{\binom{n}{2}}\Big\} \\
B_i &= C_i \qquad (i > 2)
\end{aligned}
$$

Figure 4.4: Decomposing the Khovanov complex of a closed braids into a direct sum

We claim that $d|_{A_1} : A_1 \to A_2$ is surjective, which implies $dA_2 \subseteq A_3 = 0$ since $d^2 = 0$. It is clearly sufficient to consider the case $n = m = 1$, where we consider the homology of the Khovanov complex of an unknot. Since $A_1 = C_1$ for $n = m = 1$ and the second homology of the complex associated to the unknot is trivial, $\mathrm{im}(d|_{A_1}) = \ker(d|_{A_2}) = A_2$, which implies the claim. Hence $dA \subseteq A$.

If $z = ((x_1,\dots x_m),(y_1,\dots y_m)) \in B_1$, then $dz = ((\dots),(\dots,h_1 0 - h_2 0),(\dots)) \in B_2$. Thus $dB \subseteq B$. We can now set $Y_s := B$ to prove the claim.

For part (b), notice that we can find a similar subcomplex $\tilde{B}$ in $\tilde{C} := \tilde{C}^k$. By Proposition 4.3.3, $\tilde{B} \cong B[4k]\{12k\}$, furthermore $\tilde{C}/\tilde{B} \cong \tilde{A}\{|s| - |r(s)|\}$, where $\tilde{A}$ is the reduced complex corresponding to $\Delta^{2k} r(s)$. Thus the short exact sequence of complexes $0 \to B[4k]\{12k\} \to \tilde{C} \to \tilde{A}\{|s| - |r(s)|\} \to 0$ gives rise to the desired distinguished triangle.

For part (c), we can construct $\tilde{A}$ in the same way as A in part (a), and we get a similar decomposition $\tilde{C} = \tilde{A}\{|s| - |r(s)|\} \oplus B[4k]\{12k\}$ that carries over to homology. $\square$

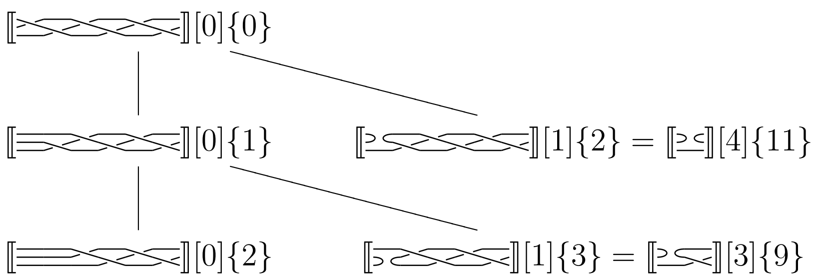


Figure 4.5: (Partial) tree of resolutions of Δ^2

Corollary 4.4.2. *For $k \geq 1$, the homotopy type of the Khovanov complex of Δ^{2k} is supported in homological gradings between 0 and $4k$.*

Proof. For $k = 1$, the proof follows from Figure 4.5 and the fact that a tangle with n positive crossings is supported in gradings between 0 and n by repeated application of Theorem 4.2.4. For the induction step, notice that $(\sigma_1\sigma_2)^4$ is supported in gradings between 0 and $8 \leq 4k+4$, so for $s = \Delta^{2k}$, Y_s is supported between 0 and $4k$ by induction hypothesis, so Y_s lies between 4 and $4k+4$ and $\Delta^{2k}r(s)$ lies between 0 and 4, thus by Theorem 4.4.1(c), Δ^{2k+1} is supported in gradings between 0 and $4k + 4$. $\square$

Remark 4.4.3. We are using Corollary 4.4.2 to show Theorem 4.4.1, which might appear to be circular reasoning. However, since we only use Theorem 4.4.1 for $k - 1$ to show Corollary 4.4.2 for k, we can use induction to show Theorem 4.4.1 and Corollary 4.4.2 simultaneously.

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