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CONNECTING HISTORY AND CLASSROOM PRACTICE
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Beste Güçler

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Ph.D. degree in Mathematics Education

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DEVELOPMENT OF DISCOURSE ON LIMITS: CONNECTING HISTORY AND CLASSROOM PRACTICE THROUGH A COMMUNICATIONAL APPROACH TO LEARNING

By

Beste Güçler

A DISSERTATION

Submitted to
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in partial fulfillment of the requirements
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Mathematics Education

2010

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ABSTRACT

DEVELOPMENT OF DISCOURSE ON LIMITS: CONNECTING HISTORY AND CLASSROOM PRACTICE THROUGH A COMMUNICATIONAL APPROACH TO LEARNING

by

Beste Güçler

The notion of limit is considered to be the building block of many calculus concepts such as continuity, derivative and integral. On the other hand, the concept presents students with many challenges. This study views mathematics learning as initiation to the historically established mathematical discourse and uses a communicational approach developed by Sfard (2008) to explore the conceptual obstacles in learning limits. One of the goals of this study is to investigate how the discourse on limit and its underlying concepts is generated over history. This exploration goes in conjunction with the discursive analysis of the historical junctures that led to particular changes in the discourse on limits as mathematicians encountered conceptual obstacles. The study then focuses on one college-level calculus classroom to explore how the discourse on limits is generated by the instructor. This is followed by an investigation of students' discourse on limits at the end of their instruction. Finally, possible connections between the discourse on limits as generated over history and as generated in the classroom are explored to examine whether the communicational approach is useful to gain further insights about learning of limits.

The study revealed that the consideration of limit as a distinct object of mathematics (a number) obtained at the end of a process was challenging for mathematicians over history. The students in the study had difficulties distinguishing the

process aspect of limits from the realization of the concept as an end-state (a number), which is consistent with the historical development of the concept. Opportunities for addressing the differences between the consideration of limit as a process and limit as a number were present in the instructor's discourse. However, the distinct assumptions underlying each realization of limit remained implicit for students.

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To my mother and father…

ACKNOWLEDGMENTS

I am thankful for the help, encouragement, and guidance of many individuals throughout this arduous journey. I thank my father for showing enormous effort to understand the characteristics of a doctoral program and supporting me through my ups and downs. I thank my friends, especially Irini and Violeta, with whom we shared many ideas, emotions, experiences, and stories along the way.

I am grateful to Raven McCrory for all the support she provided during the dissertation process. Without her guidance, it would be impossible to bring this study to life. I also thank my committee members, Suzanne Wilson, Joan Ferrini-Mundy, and Vince Melfi, for their valuable input and support. I thank Anna Sfard for her help during the initial stages of my work and for the beautiful framework she developed. I thank Sharon Senk for always being there when I needed her guidance. For all their help, kindness and support, thanks to Jean Beland, Margaret Iding, Lisa Keller, and Jim Keller.

I also owe great debt to the participants of my study. Thanks to the instructor and the students who gave me access to their classroom to help me conduct my study. Without them, I would have no story to write about.

I am forever grateful to music and my favorite musicians for making me feel alive and giving me the strength to write. Last but not least, I thank my mother. Although you are long gone, you are always with me. It is because of you I embrace people and their stories with curiosity and love; it is because of you I can "feel" music and life; it is because of you I believe in education; it is because of you I am who I am. Thank You…

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CHAPTER I

INTRODUCTION

Starting with the calculus reform movement initiated in mid-eighties, research on undergraduate mathematics education has focused on improving curricular and pedagogical approaches to beginning calculus. Today, students in majors such as economics, engineering and physics are expected to be competent in various mathematical domains and calculus is one of the main courses required of students in all these majors. As a result, there also exists a relatively rich research base at the undergraduate level on student thinking about the content of elementary calculus concepts such as function, slope and derivative (Carlson, 1998; Monk, 1987, 1994; Monk & Nemirovsky, 1994; White & Mitchelmore, 1996; Zandieh, 2000).

The concept of limit has also been of particular interest for researchers since it is considered to be the building block of many fundamental calculus concepts such as continuity, derivative and integral. The notion of limit, however, presents major difficulties for students (Bezuidenhout, 2001; Cottrill et al., 1996; Tall & Vinner, 1981; White & Mitchelmore, 1996; Williams, 1991). These studies highlight that the formal understanding of the concept is unlikely to occur unless students first have an intuitive understanding of the concept. However, they also argue that the intuitive understanding of the concept relies heavily on the idea of continuous motion, which might hinder understanding of the other aspects of limit. In that respect, some of the representational tools (verbal, visual and symbolic) used by students while thinking about limit lead to additional difficulties (Bagni, 2004, Bezuidenhout, 2001; Cottrill et al., 1996; Williams, 1991). Therefore, the concept of limit presents the student with two challenges: the need to

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to make the transition from intuitive to formal understanding and the need to cope with the issue of compatibility of the conceptual and representational tools within the intuitively understood aspects of limit.

Considering mathematics learning as initiation to the historically established, patterned activity of doing mathematics, one of the goals of this study is to explore how the discourse on limit and its underlying concepts as generated over history. This exploration will go in conjunction with the discursive analysis of the historical junctures that led to particular changes in the discourse on limits as mathematicians encountered conceptual obstacles. In this study, such changes will be will be defined through elements of a communicational framework developed by Sfard (2008). Sfard's framework will then be used to explore how the discourse on limits is generated by the instructor in a college-level calculus classroom. This will be followed by an investigation of students' discourse on limits at the end of their instruction and what conceptual obstacles they encounter. Finally, possible connections between the discourse on limits as generated over history and as generated in classrooms will be explored to examine whether the communicational approach, in general, and a discursive analysis of historical junctures, in particular, help us gain further insights about learning of limits. More specifically, the study addresses the following questions: 1) How is the discourse on limits generated by the instructor in a beginning college-level calculus classroom? 2) Given the instructor's discourse on limits, how do students talk about limits in a beginning college-level calculus course? and 3) How do the elements of discourse on limits as generated over history compare and contrast with the discourse on limits generated in a beginning-level calculus course?

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The contribution of the study to educational research is two-fold: First, although research has identified many of the conceptual obstacles students have about limits, explanation of the nature of those obstacles remain incomplete. Some similarities between mathematicians' and students' struggles is implied by research but there is no elaboration on the principles underlying the transitions learners need to go through as they attend to different aspects of the limit notion. This study approaches the same problem by means of a different lens that emphasizes communication to examine whether it can provide further insights about the conceptual obstacles in learning of limits. I will use elements of Sfard's (2008) framework to investigate the historical development of discourse on limits with respect to possible roots of the conceptual obstacles faced by mathematicians. The study will then explore whether Sfard's (2008) framework can be a useful lens to gain more information on students' discourse on limits.

Second, although research about limits suggests that the intuitive aspects of the notion are perpetuated in calculus classrooms, there is no analysis of instruction in order to justify this claim. In this work, I will analyze one instructor's discourse on limits in a college-level calculus classroom and investigate the possible impacts of the instructor's discourse on students' thinking about limits. In this respect, the study is an attempt to fill out an important gap in the literature about teaching of limits at the undergraduate level.

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CHAPTER II

THE CONTEXT OF THE STUDY

This chapter starts with a brief description of the limit concept followed by a literature review on learning about limits. Next, the theoretical framework for the study is introduced and explained in terms of its main characteristics. Finally, particular elements of the theoretical framework that I use for the analysis of the historical development of limits are described.

Axioms, definitions, theorems and proofs presented in their formal representations are among the final products of mathematics. The processes with which mathematicians and learners of mathematics initially think about mathematical concepts, on the other hand, are informal and intuitive. Although such processes may be invisible in the mathematics curricula, the transition from the informal aspects of mathematical concepts to their rigorous formulations is by no means trivial. The historical development of a concept is a valuable tool in providing clues about where to look for the obstacles learners may face and the transitions they may go through as they tackle different aspects of the concept. The nature of those transitions and conceptual difficulties, however, may not be found in history itself but can be identified by means of a focused analysis of the historical development. This study uses elements of the *commognitive* framework (Sfard, 2008) for the analysis of the historical development of limit and its underlying concepts. In what follows, the notion of limit is briefly discussed and a summary of the research on its learning is given. Then the commognitive framework is introduced as a potentially useful lens that can help us gain more insights about the nature of the conceptual obstacles in the learning of limits.

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2.1 Limit: What is it and what do we know about its learning?

2.1.1. The notion of limit

The concept of limit is the foundation on which fundamental concepts of calculus are based. Limits are used to define the tangent to a curve, which leads to the notions of the derivative of a function and instantaneous rate of change. In that sense, limits are used to determine how functions vary. When used to define the behavior of Riemann sums at infinity, the notion leads to the concept of the integral of a function. In particular, limits are also used in determining whether sequences and series converge. Given these, it is impossible to talk about the essential concepts of calculus without limits. The informal definition of limit is often given in some form similar to the following:

> Let [a function] $f(x)$ be defined on an open interval about [the point] x_0, *except possibly for* x_0 *itself.* If $f(x)$ gets arbitrarily close to L (as close as we like) for all x sufficiently close to x_0, we say that f approaches the *limit* L as x approaches x_0, and write $\lim_{x \to x_0} f(x) = L$, which is read 'the limit of $f(x)$ as x approaches x_0 is L'. (Thomas, Weir, Hass & Giordano, 2008, p. 77)

A typical calculus-level formal definition, on the other hand, would be similar to the following:

> Let [a function] $f(x)$ be defined on an open interval about [the point] x_0, except possibly for x_0 itself. We say that the *limit of* $f(x)$ *as* x *approaches* x_0 *is the number* L, and write $\lim_{x \to x_0} f(x) = L$ if for every

number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. (Thomas et al., 2008, p. 92)

At the undergraduate level, students are introduced to the notion of limit in their preliminary calculus courses[1]. The textbooks designed for these courses often outline the content of calculus starting from reviews of basic notions such as real numbers, number line, functions and types of functions. Then the notions of limit and continuity are introduced. This is followed by discussions on derivative and then the integral. Finally, the notions of sequences and series are introduced with a particular focus on their behavior at infinity, which form the basis of the discussions about convergence and divergence[2].

2.1.2. Research on learning about limits

In what follows, the difficulties associated with the limit concept as pointed out by research on student learning will be explained.

Limit implies continuity. Bezuidenhout (2001) argues that this refers to the incorrect assumption that the existence of limit of a function at a given point is a sufficient condition for the continuity of the function at that point. The students who have this difficulty believe that if a function has a limit at a given point, then it must also be continuous at that point. For example, such students would think the limit does not exist at the point x=3 for the function in Figure 2.1 since it is not continuous at 3.

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[1] Students can also have familiarity with the limit concept from their high school courses such as AP Calculus and precalculus.

[2] See Thomas et al. (2008) and Hughes-Hallett et al. (2008) for their content outline for calculus.

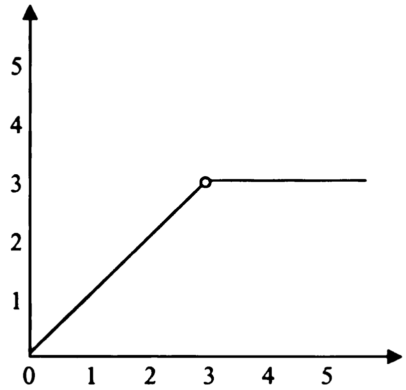

Figure 2.1: An example where the difficulty *limit implies continuity* can occur

Limit as the function's value. Bezuidenhout (2001) notes that the incorrect assumption limit implies continuity "may also originate from another misconception, namely, $\lim_{x \to 2} f(x) = 3$ implies $f(2) = 3$, so that f is then continuous at $x=2$" (p. 494). In this respect, students could view $\lim_{x \to 2} f(x)$ and $f(2)$ as the same thing. This view of limits is called "limit as the function's value" and it corresponds to the belief that "the limit of a function at a point means the value of the function at that point" (Cottrill et al, 1996, p. 178) Students having this difficulty would give 'limit does not exist' as the answer whenever the function is not defined at the point where the limit is taken. Their strategy while finding the limit of a function at a point is to evaluate the function's value at that point and give the result as the limit value. For example, for the function in Figure 2, such students would give 3 as the answer for the limit of the function at x=2.

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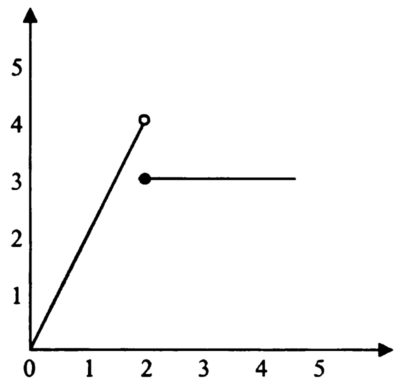

Figure 2.2: An example where the difficulty *limit as the function's value* can occur

Bezuidenhout (2001) argues that the procedures used in the calculation of limits such as the method of substitution "may sell the idea to some students that the value of the function at the point concerned is of primary importance, rather than the behavior of values of the function about the point" (p. 496).

Limit as a bound. Cornu (1991) mentions that limits can sometimes be interpreted as "an impassable limit which is reachable"; "a higher (or lower) limit"; "a maximum or minimum"; "a constraint, a ban, a rule"; "the end, the finish". (pp. 154-155). These utterances emphasize limit as a boundary. So limit as a bound refers to the idea that "limit is a number or point past which the function cannot go" (Williams, 1991, p. 221). Students who have this difficulty think that a function is bounded by a specific limit value or think of the absolute maximum/minimum values of the function, if they exist, as the limit for the function. They would also have difficulty working with horizontal asymptotes where the limit of a function at positive or negative infinity can be a number past which the function, in its whole domain, can go. For example, for the function in Figure 2.3, the limit at positive and negative infinity is equal to zero but the function also attains values greater than zero.

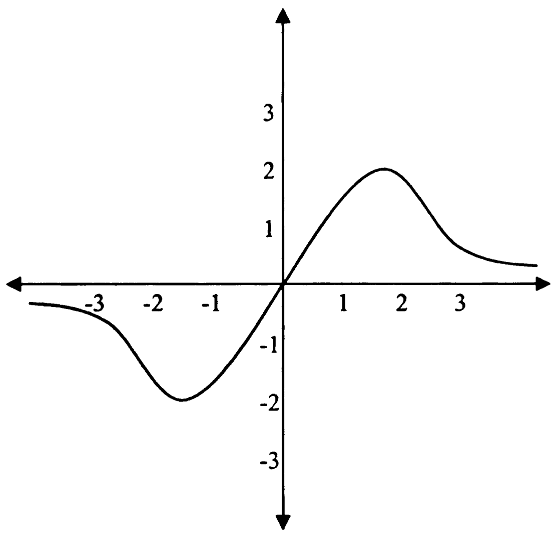

Figure 2.3: An example where the difficulty *limit as bound* can occur

Limit as monotonic. There are other interpretations of limits such as "monotonic and dynamic monotonic" which are based on formal teaching (Cornu, 1991, p. 155). Utterances such as "a convergent sequence is an increasing sequence bounded above (or decreasing bounded below)" and "a convergent sequence is an increasing (or decreasing) sequence which approaches a limit" (Cornu, 1991, p. 155) might lead to the expectation of monotonic behavior from the function in order to find its limit. This difficulty is related to the expectation of 'nice behavior' from the function. Research indicates that if a function is strictly increasing or strictly decreasing, students can more easily find the limit at a given point. This difficulty becomes evident when working with constant and piecewise functions (See Figures 2.4 and 2.5). The function in Figure 2.4 remains constant in its entire domain and so students could have difficulty considering it as increasing or decreasing. As a result, they might not be comfortable with the idea that the limit of that function at every point in its domain is equal to the same number 2. For the graph illustrated in Figure 2.5, the function is increasing but not strictly. For every *a, b* eV,

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$\in N$, the interval *(a,b)* is constant. Moreover, for every $a \in N$, the function jumps rather than increasing smoothly. Students who expect strictly increasing or strictly decreasing behavior from a function might have difficulties finding the limit of the function in Figure 2.5 at the points which lie between *(a,b)*, where the function remains a constant value. They also might think the limit at a given point $a \in N$ equals to both a and $a+1$ due to the discontinuous nature of the function at those points. In fact, Tall and Vinner (1981) report a similar finding for the case of sequences. They argue that given a sequence $\{1,0,1,0,1,0,\ldots\}$ students might think there are two separate sequences there instead of one. Therefore, students who perceive "limit as monotonic" might also have difficulty determining the convergence of a sequence where the subsequences can have different patterns from each other.

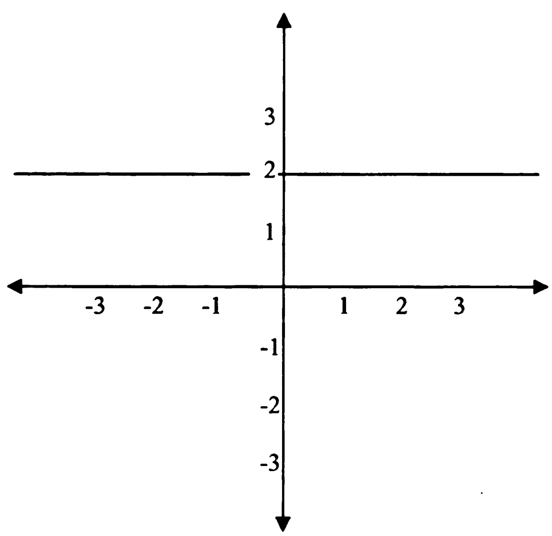

Figure 2.4: An example where the difficulty *limit as monotonic* can occur

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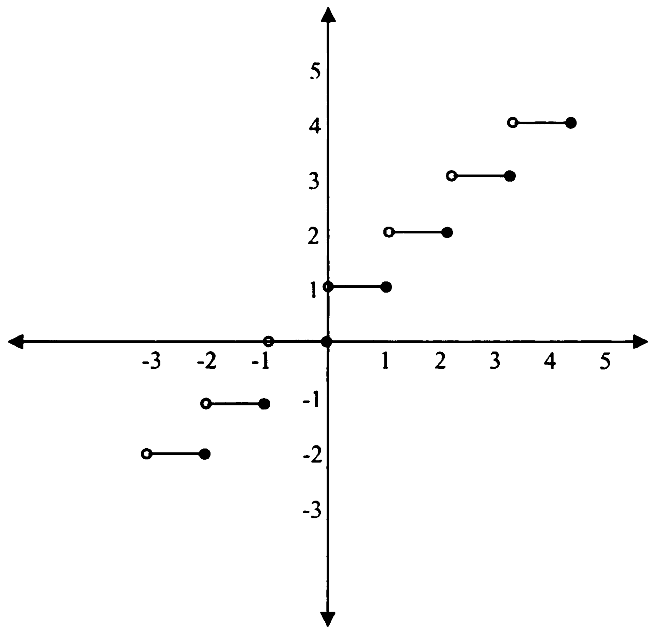

Figure 2.5: An example where the difficulty *limit as monotonic* can occur

Difficulties resulting from the dynamic approach to limit. The dynamic approach refers to using the intuitive idea of motion when working on limit related problems. Although it implies motion, the dynamic approach to limit is not necessarily a graphical approach. The common phrases such as *approaches*, *tends to*, *getting close to* all indicate motion-related processes and are considered among the verbal representations of the dynamic approach (Bagni, 2004). In his study on models of limit held by students in a second-semester calculus class, Williams (1991) found that

> In general, the words *approaching* or *getting close* were interpreted in one or both of two ways: as describing the physical process of evaluating a function at different numbers, which are chosen over time to be closer and closer to the value s [for $\lim_{x \to s} f(x)$], or as describing the mental process of

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> imagining the points on the graph moving closer and closer to the limit point. (p. 228)

Tall and Vinner (1981) also note that limits of functions of the form $\lim_{x \to a} f(x) = c$ are "often considered as a dynamic *process*, where x *approaches* a, causing f(x) to *get close* to c" (p. 160, italics in original). They argue that this intuitive approach to limits "is often so strong that the feeling of the student is a *dynamic* one: as x approaches a, so f(x) approaches c, with a definite feeling of motion" (Tall & Vinner, 1981, p. 161, italics in original).

Research has identified two difficulties resulting from the dynamic view of limits: limit as an approximation and limit as unreachable. In what follows, these terms will be explained and also exemplified.

Limit as approximation. One of the possible ways of using the dynamic approach to limits for $\lim_{x \to a} f(x)$ is investigating the behavior of a function around the limit point (as $x \to a$) by substituting successive x-values that are closer to the point a. In this case, responses such as 'when x approaches a, the values of $f(x)$ approach the limit L' often points to

> a confusion of the limiting process and the product resulting from that process. Such a confusion may go hand in hand with the erroneous view of limit as an approximation. One can also sense a dynamic character of the limit in the student's motivation. (Bezuidenhout, 2001, p. 492)

This approach is dynamic since it involves the movement of the points closer to the limit point and looking at the function's values near that point. It resembles the tabular representation of the function near the limit point. Some students see this approach as sufficie

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sufficient to determine the limit of a function at a given point without realizing that although the function can seem to be *approaching* or *getting close to* or *tending to* a limit value for those points, there are still infinitely many points near the limit point that remained unchecked (see Figure 2.6).

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
-0.0001	0.9999
0	1
0.0001	1.0001
0.001	1.001
0.01	1.01
0.1	1.1

Figure 2.6: An example where the difficulty limit as approximation can occur

Students having the dynamic view and who think of limit as an approximation argue that $\lim_{x \to 0} f(x)$ is equal to 1 since the function values approach 1 as the x values approach 0. On the other hand, it is not possible to conclude that the limit of this function is 1 since we do not have enough information about the behavior of the function between the points $x = -0.0001$ and $x = 0$ as well as between $x = 0.0001$ and $x = 0$.

Some students who consider limit as an approximation also round off the function's values at the successive points close to the limit point and claim it is reaching a value, which they would think is the limit value (Tall & Schwarzenberger, 1978). For example, for the function in Figure 2.6, students can just check the function values at the x values close to the limit value and round them to 1 to claim the limit value at $x = 0$ is equal to 1. "We surmise that at the root of such a misconception is the practice, both insi

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Limit as unreachable. The second idea related to the dynamic approach is mostly used when determining the limit of the function from a graph. Here the movement feature comes from the visualization of the graph. The basic logic behind this approach is to ask where the function approaches as x approaches the limit point. (Does $f(x) \rightarrow L$ as $x \rightarrow a$?) Here, some students confuse the idea 'x never reaches a' with 'f(x) never reaches L'. So, they think the limit is unreachable. These students have difficulty working with constant functions where the limit value is taken by the function at the limit point. They also have difficulty working with the continuous functions. Williams (1991) notes that although this view might suggest students' awareness of the irrelevance of the function value at the limit point, it might also suggest the consideration of taking the limit of continuous function as inappropriate. As a result, some students cannot accept continuous functions as having limits. He argues that "in general, reachability is not a characteristic of limits, but rather is a matter of continuity" (Williams, 1991, p. 228). According to Tall and Schwarzenberger (1978), the colloquial use of the words such as 'close' implies getting near to but not being coincident with. Given this, they argue that the informal notion of limit may carry for students the assumption that one can get close to the limit value but cannot reach it.

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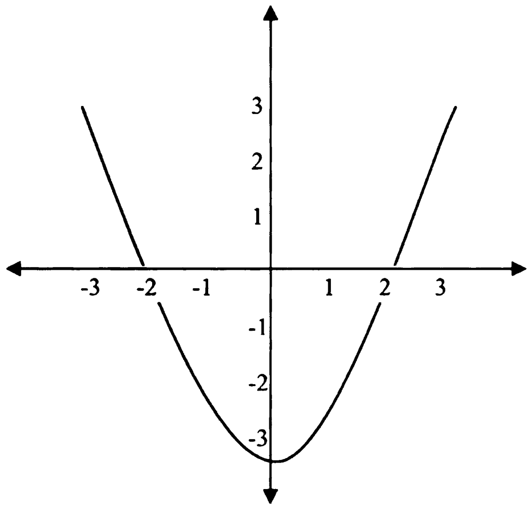

Figure 2.7: An example where the difficulty *limit as unreachable* can occur

For the graph in Figure 2.7, students who have the difficulty "limit as unreachable" would distinguish the function value at 0, which is f(0), from the function's value as it *approaches* x=0 . Therefore, they would be uncomfortable saying $\lim_{x \to 0} f(x)$ is *equal to* -1 since the function attains the value -1 at $x = 0$.

It is important to note that the consideration of "limit as unreachable" is not restricted to situations that involve graphing. The realization of limit as unreachable signals the separation of the process of approaching and the number that is approached (which can, in fact, be the limit). This distinction might also lead to confusion between plugging in and approaching the number. As a result, students might be comfortable computing $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$ by cancelling the common factor (x-5) since the function is not defined at $x = 5$ and so the values of x cannot reach that point. On the other hand, they might have difficulty understanding why they can plug 5 into the function when finding

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$\lim_{x \to 5}(x+5)$ since the function will then attain its value at the point $x = 5$[3]. Table 2.1 summarizes the student difficulties and the related assumptions about limit mentioned in the literature.

Table 2.1: Difficulties mentioned by the research on student thinking about limits

Student difficulties about limit	Assumption underlying the views
Limit implies continuity (Bezuidenhout, 2001)	If a function has a limit at a point, then it must be continuous at that point.
Limit as the function's value (Bezuidenhout, 2001)	When finding the limit of a function at a given point, it is enough to look at the function's value at that point.
Limit as a bound (Cornu, 1991; Williams, 1991)	A limit is a value past which the function cannot go. A limit is the absolute maximum (or minimum) value of a function.
Limit as monotonic (Cornu, 1991; Tall & Vinner, 1981)	A function (or sequence) has to be strictly increasing or strictly decreasing in order to have a limit.
Limit as approximation (Bezuidenhout, 2001; Parameswaran, 2007; Tall & Schwarzenberger, 1978)	In order to find the limit of a function at a point, it is sufficient to look at the behavior of the function at points successively closer to the limit point.
Limit as unreachable (Tall & Schwarzenberger, 1978; Williams, 1991)	A limit is a value that is approached but never reached.

These student difficulties about limits are mostly identified by empirical studies, which generally include surveys followed by individual student interviews. For example, Bezuidenhout's (2001) study focuses on first year students' understanding of limit of a function and continuity of a function at a point. He selected 100 students as the sample from a much larger population in three South African universities. The students were in engineering, physical sciences and in other majors that required service calculus courses. Among the 100 students who responded to an initial survey, 15 students were selected to

[3] See Szydlik (2000, p. 276) for the details of such student assumptions.

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participate in interviews. The difficulties "limit implies continuity" and "limit as the function's value" were mentioned frequently by the students who were interviewed.

In Williams' (1991) study, 341 students from two second-semester calculus classes were given a questionnaire about limits. Based on the student responses to the survey, students were selected for in-depth questioning. The first question of the survey included six statements to categorize students' views of limits as dynamic-theoretical; boundary; formal; unreachable; approximation; and dynamic-practical, respectively (See Appendix A for the survey). Among the 341 students, 36% stated that the statement that considered "limit as unreacheable" described best how they thought of limits. 30% chose the statement about the dynamic-theoretical characterization of limit as the best way they made sense of the concept. The statement about the formal definition was chosen as best by 19% of the students. The statements about "limit as boundary", "limit as approximation" and the dynamic-practical aspect were selected by 3%, 4%, and 5% of the students, respectively.

From 341 students, Williams (1991) then classified 50 volunteers in terms of the models of limit they held. He classified 24 students as having the view limit as dynamic; 20 students as having the view limit as unreachable; three students as having the view limit as a bound and one student as having the view limit as an approximation[4]. From those 50 students, he selected 10 students for treatment that consisted of five sessions over a period of seven weeks. He mentions that:

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[4] Williams (1991) couldn't classify two of the student responses since they were ambiguous.

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> All 10 students in the study expressed at some point a view of limit as dynamic, that is, involving motion along the graph, and plugging in points that over time get closer and closer to the value s as x approaches in the limit [for $\lim_{x \to s} f(x)$]. (Williams, 1991, p. 229)

He further notes that the problems students worked on in the treatment sessions targeted student difficulties through situations that could lead to cognitive conflicts. However, no real change occurred in students' views of limit as dynamic after treatment. "The stage was set for cognitive conflict, and in fact, some conflict did occur. What did not occur was real cognitive change" (Williams, 1991, p. 229). By the end of the final session, "no student was willing to give up the view that plugging in a finite number of values was essentially correct or that moving along the graph was a good way to view a limit problem" (Williams, 1991, p. 230).

> Limits tend to be seen as processes performed on functions, an idealized form of evaluating the function at a series of points successively closer to a given value. The dynamic element here is clear, and because the actual value of the function at the point of interest is irrelevant, the limit is never reached. The paradigm picture seems to be the classical geometric progression involved in walking halfway to a wall, then half the remaining distance, and so forth; students seem to be willing to accept the fact that we never reach the wall even though we may know exactly where the wall is. To most of the students, this was a compelling metaphor for limit… Still, it has been suggested that, although perhaps necessary, such a view

of limit does present a cognitive obstacle to further understanding.

(Williams, 1991, p. 230)

Szydlik's (2000) study focused on calculus students' beliefs about mathematics and the role of those beliefs in their understanding of the limit concept. She initially gave 577 second-semester calculus students a questionnaire about their convictions and beliefs about calculus and limits. Based on this questionnaire, she identified four groups of students, 27 of whom participated in a structured interview about limits. "All 27 students used a definition of limit as motion to explain their work on the limit problems; a few students used either a static or an infinitesimal definition as well" (Szydlik, 2000, p. 271). Nine of the 27 students agreed with statements that implied limit as a bound. "The majority of the students who held this conception believed that the limit is a value the function cannot exceed. They appeared to hold this view *globally*, often drawing or describing a horizontal asymptote." (Szydlik, 2000, p. 271). On the other hand, three students in the study thought of limit as a local bound. Students having this idea "believed that within a certain tolerance of the limiting value, the limit acts as a boundary; however, they did not think of the limit as a global boundary". (Szydlik, 2000, p. 271).

In Szydlik's (2000) study, students who view calculus as a collection of facts to be memorized and who do not follow the theory behind those facts

> often cannot give a coherent definition of limit of a function or explain why the formulas and procedures that they use to solve limit problems are valid. Many hold misconceptions of limit as a bound that cannot be crossed or as unreachable. (p. 273)

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The students who see calculus as consistent and logical, on the other hand, can have "access to formal definitions, power to solve limit problems, and concept images free of major inconsistencies" (Szydlik, 2000, p. 273). It is important to note, however, that Szydlik mentions that not all students holding these beliefs towards calculus have complete understanding of the limit notion. These findings are in conjunction with Sierpińska's (1987) findings which imply that "the students' attitudes towards knowledge, and mathematical knowledge, in particular, have a strong impact on their intuitions of infinity and limits" (p. 382).

2.1.3. Difficulties about the underlying concepts of limits

Some of these difficulties about limit come from the difficulties students have of the underlying concepts related to limit. According to Sierpińska (1987), the obstacles related to the notions of scientific knowledge, infinity, function and real number form the basis of the epistemological obstacles about limits. Carlson (1998) and Vinner & Dreyfus (1989) argue that some students perceive functions merely consisting of algebraic rules. Similarly, Bezuidenhout (2001) and Tall & Vinner (1981) found that students mostly rely on rules and graphs of functions while trying to find the limit of the function at a given point. Williams (1991) and Szydlik (2000) also mention students' faith in graphing as a means of understanding the behavior of functions when finding limits

Students' notions of infinitely small and infinitely large can also play roles in their understanding of the limit concept. According to Parameswaran (2007), "it is common practice in real life as well as in classroom that one often 'ignores' negligible quantities and rounds off numbers to convenient significant digits" (p. 194). He investigated how such practices affect students' understanding of the limit concept. He used ques

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used questions involving infinitely large and infinitely small quantities (infinitesimals) to explore the student difficulties such as "limit as approximation" and the dynamic view of limit. In order to do this, he investigated students' experiences in arithmetic where they represent decimals and do finite approximations such as rounding off real numbers to relevant decimal numbers. Parameswaran (2007) found that students use approximations before taking the limit of the function if the function includes infinitesimal quantities. Another strategy students used when working on such limit problems was to round the very small quantities to zero.

> ...the students identify what they perceive as 'large numbers' with infinity and 'small numbers' with zero. Also, in our experiments, they unwittingly rounded off very small parameter values occurring in the definition of the function in question to a convenient number close to it... Most of the students in our samples view limiting as a process of *approximation* when very minute quantities are involved in the definition of the function...They tend to approximate the given function by changing or ignoring quantities appearing in its definition which they perceive as 'small' constants to zero. (Parameswaran, 2007, p. 209, italics in original)

In her study of 31 pre-calculus students Sierpińska (1987) found that some students think of infinity as a large finite number. Tall and Schwarzenberger's (1978) note that the students in their study had idiosyncratic views of infinity. Some thought about infinity as a symbol that represents what is unreachable; some defined it as the biggest number that exists; and others thought about it as the endpoint of the real numbers.

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Continuity is also one of the problematic terms that cause difficulties in students' understanding of the limit concept. Daily uses of the term in phrases such as "the rail is continuously welded" or "it rained continuously all day" often result in associating continuity with having no gaps or breaks (Tall & Vinner, 1981, p.164). Such uses, together with the initial uses of the term continuous functions result in a "reinforcement of the intuitive idea that the graph has 'no gaps' and may be drawn freely without lifting the pencil from the paper" (Tall & Vinner, 1981, p. 165).

> This viewpoint is often reinforced by teacher's attempts to give a simple insight into the notion of continuity by speaking of the graph "being in one piece" or "drawn without taking the pencil off the paper", thereby confusing the mathematical notions of continuity and connectedness.
>
> (Cornu, 1991, pp. 156-157)

Tall and Vinner (1981) gave 41 students a questionnaire that involved different functions. The students were asked to decide which of the given functions were continuous and to explain their reasoning. They found that most of the students who gave the correct answers gave those for the incorrect reasons. For example, students might say a given function is continuous because "it was given by a single formula" or "it is all in one piece" (Tall & Vinner, 1981, p. 167). Similarly, students might say a given function is discontinuous because "the graph is not in one piece" or "it is not given by a single formula" (Tall & Vinner, 1981, p. 167).

There is further evidence that students' early notions of continuity, geometric motion and very small quantities (infinitesimals) can lead to problems with respect to their notions of function, infinity and limit (Carlson, 1998; Cottrill et al., 1996; Tall, 1950;

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1980; Vinner & Dreyfus, 1989). It is possibly due to the interplay of all these notions that there are many different colloquial and idiosyncratic interpretations of the word limit (Cornu, 1991). As a result, everyday uses of the terms *limit*, *approaching*, or *tending to* can be problematic for students since they are used differently in the theory of limits (Bagni, 2004). In addition to this, mathematical notations like "$f(x) \to c$ as $x \to a$", which is verbalized as "$f(x)$ approaches c as x approaches a" entails a feeling of motion and hinders students' understanding of the formal definition (Tall & Vinner, 1981, p.155). The main difficulty for students when dealing with the $\varepsilon - \delta$ definition results from the static character of the formal theory and the dynamic character of the intuitive approach (Bagni, 2004). Bezuidenhout (2001) argues that the formal approach is extremely difficult to understand for students who have the difficulties mentioned above. Similarly, Williams (1991) argues that the dynamic approach to limit is likely to hinder student understanding of the formal approach where the idea is to make *f(x)* as close to the limit value as we want by making x close enough to the limit point.

These researchers consider the formal definition of limits as an important element of understanding limits conceptually. Although they acknowledge that the dynamic view seems to be useful, and perhaps inevitable, in making sense of the limit concept intuitively at the initial stages of learning, they also highlight that the lack of familiarity with the assumptions of the formal theory results with difficulties with respect to particular applications of the concept[5] (Bezuidenhout, 2001; Tall, 1980; Tall &

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[5] On the other hand, Parameswaran (2007) mentions that the introductory calculus courses should be informal and intuitive and notes that "the precise, formal definition of the concept of limit is so complex

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Schwarzenberger, 1978; Williams, 1991). "Limiting processes are not always intuitive, and so a formal framework provides a powerful tool for thinking about and evaluating some limits" (Szydlik, 2000, p. 259).

2.1.4. Textbooks, teaching and attitudes towards mathematics

Some researchers would argue that the intuitive aspects of limit are perpetuated in teaching and textbooks with their emphases on the visual representations of functions as graphs or numerical approximations, which are based on the natural perception of continuity. Cornu (1991) mentions that "in teaching mathematics, certain aspects of the limit concept are given greater emphasis which are revealed by a review of the curriculum, the textbooks and examinations" (p. 153). According to him, instead of focusing on limit conceptually, textbooks may overemphasize "equalities, the notion of absolute value, the idea of sufficient condition and, above all, on *operations*: the limit of a sum, a product, and so on" (Cornu, 1991, p. 153, italics in original). He argues that the textbooks' focus on algebra and calculation reflects a bias favoring operations and procedures over the analytical aspects of the limit concept.

Similarly, Bezuidenhout' (2001) argues that students' understanding of limits and its underlying concepts seems to be based on isolated procedures and the conceptual link among those are missing.

> Such a situation may be mainly due to a learning and teaching approach that emphasizes to a large extent the procedural aspects of the calculus, and neglects a solid grounding in the understanding of the conceptual

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and counterintuitive that it fails to bring out readily the simple and intuitively obvious ideas which led to it in the first place" (p. 194).

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> underpinnings of the calculus. Moreover, the stereotyped exercises that are a feature of several calculus texts often encourage an instrumental approach, rather than a relational understanding of the calculus concepts. Taking into account the procedure-oriented nature of some calculus texts, it should not be considered as strange if a student confuses manipulative skills with a real understanding of calculus content. (p. 498)

Williams (1991) considers students' attitudes toward practicality and mathematical truth, which hinder their appreciation of formal thinking, as another element that can affect learning about limits. He mentions that attitudes toward practicality can result from students' classroom experiences and the current curriculum provides little motivation with respect to formal thinking and notes that the students in his study

> ...often considered the ease and practicality of a model of limit more important than mathematical formality. This is particularly true in the sense that models of limit that allow them to deal with the realities of limits in the classroom, the kind they see on tests, tend to be seen as sufficient for the purposes of most students. It was noted by several students that neither formal or dynamic models of limit figure heavily in the procedures students use to work problems from their calculus class; their procedural knowledge (e.g., substituting values into continuous functions, factoring and cancelling, using conjugates, employing L'Hôpital's rule) is largely separate from their conceptual knowledge. (Williams, 1991, p. 233)

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Parameswaran (2007) states that "it is typical for calculus books to motivate the notion of limits graphically" (p. 212). He then talks about an example in a typical calculus textbook in which $\lim_{x \to 0} \frac{\sin x}{x}$ is first motivated by the tabulating some values of the function around x=0 and then drawing the graph of the function. He points out that although the textbook author mentions explicitly that the table only allows one to guess the limit value but not to prove it, "the students seem to develop the idea that limit is no different from a process of approximation" (p. 213).

2.1.5. Possible links between historical development and student learning of limits

Cornu (1991) distinguishes between didactical and cognitive obstacles in learning mathematics. Didactical obstacles "occur because of the nature of the teaching and the teacher", whereas epistemological obstacles "occur because of the nature of mathematical concepts themselves" (Cornu, 1991, p. 158). He identifies four epistemological obstacles in the historical development of the limit concept: the failure to link geometry with numbers, the notion of infinitely large and infinitely small, the metaphysical aspect of the notion of limit, and the question of whether the limit is attained or not. By considering some of the conceptual obstacles about limits as epistemological, he acknowledges that the difficulties faced by students might also result from the nature of the limit concept besides the teaching approaches. In fact, some of the conceptual obstacles students face as they learn the concept may be identical to those mathematicians faced over the historical development of limits. For example, Williams (1991) mentions "limit as boundary" and "limit as unreachable" as common student difficulties about limit (See Section 2.1.2). These views of limit are mathematically incorrect and were also problematized by Lagrange and other mathematicians when they debated "whether a variable can g

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variable can go beyond the limit and whether a variable can definitely reach the limit" (Schubring, 2005, p. 293). Given this, some researchers consider bringing some elements of the historical development of limits to classrooms as potentially useful in student learning. For example, given the findings of his study, Williams (1991) comments that

> Just as students' informal limit models tend to parallel those of the mathematical community prior to Cauchy, it is possible that only by appreciating the sorts of problems that motivated Cauchy's work will students be motivated to understand its implications. Perhaps this is to say that the very historical and cultural contexts that lent vitality to the original work are the best medium through which to approach the understanding of that work. (p. 235)

Similarly, Bagni (2004) highlights that the historical development of visual, verbal and symbolic representations of limits might parallel those of students'. He then notes that this could help design teaching to overcome some of the conceptual obstacles and to help students develop the different registers required by the static and the dynamic views of limit[6].

At this point, it is also important to note that research on learning about limits and infinity is, so far, directed mainly by a cognitivist framework and primarily focuses on the notion of misconceptions. According to Sfard (2001), the cognitivist framework is based on the metaphor *learning as acquisition*, which considers learning "as the storage of information in the form of mental representations" (p.20). It considers understanding as relating new knowledge to prior knowledge by refining the existing mental representations. By doing so, it highlights the individual nature of learning, viewing it as the acquisition of the necessary mental schemes. Although this framework significantly enhanced our understanding of the conceptual obstacles associated with the learning of limit, another outlook on the issue might be needed since we cannot have direct access to abstract constructs such as mental schemes, intuition and (mis)conceptions. Therefore, a cognitive framework may not offer sufficient tools of analysis with which we can explore how learners make sense of the limit concept. This study uses the *commognitive* framework (Sfard, 2008), which is based on the metaphor *learning as participation* and views learning as becoming a participant in a discourse. Basing learning processes on social foundations, this framework considers discourse as its central unit of analysis in which "the language of *mental schemes*, *misconceptions*, and *cognitive conflict* seems to be giving way to a discourse on *activities*, *patterns of interaction* and *communicational failures*" (Sfard, Forman & Kieran, 2001, p. 1, italics in original).

2.2 Commognitive framework

2.2.1. The general tenets

One of the highlights of the commognitive framework is the interrelationship between communication and thinking. By defining thinking as the individualized form of communication, Sfard (2008) argues that the "cognitive processes and interpersonal communication processes are thus but different manifestations of basically the same phenomenon" (p. 83). Given this, the term *commognitive* entails the combination of the

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[6] On the other hand, Bagni (2004) warns that the introduction of the problems faced by mathematicians in history would not necessarily help students with their difficulties. In this respect, he seems to also consider the epistemological nature of some of the conceptual obstacles associated with the limit concept.

terms *cognitive* and *communicational*. From this perspective, developmental transformations are "the result of two complementary processes, that of *individualization of the collective* and that of *communalization of the individual*" (Sfard, 2008, p. 80, italics in original). That the communal and individual aspects of discourse development are intertwined processes can be explained as follows: Individual learning, which is defined as participation in a discourse (e.g., mathematical discourse) whose rules are communally agreed upon, is an example of *individualization of the collective*. On the other hand, as individually formed ideas get accepted by the larger discourse community, it is also possible for the individual to affect the development of a discourse on a broader range. This is an example of *communalization of the individual*. By being processes rather than static entities, discourses construct and reconstruct themselves in the interplay of individualization and communalization, and are thus the "medium and the carrier of both continuity and developmental change" (Sfard, 2008, p. 118). Therefore, the study of human development can be considered equivalent to the study of the development of discourses, where discourses are defined as "the different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93).

The commognitive framework views mathematics as a particular type of discourse which is distinguishable by its *word use*, *visual mediators*, *routines,* and *narratives*. Sfard (2008) differentiates between two types of discourse: colloquial discourses are non-specialized, everyday discourses, and literate discourses are the "discourses mediated mainly by symbolic artifacts created specifically for the sake of communication" (p. 299). Number or quantity related words can frequently be found in colloquial discourses but "mathematical discourses as practiced in schools or in academia dictate their own, more disciplined uses of these words" (Sfard, 2008, p. 133). Unlike colloquial discourses, the objects of the mathematical discourse "are featured as something that can perhaps be 'represented' with visual means, but never really shown" (Sfard, 2008, p.135). Given the abstract nature of mathematical concepts and that students are expected to participate in the literate mathematical discourse, *word use* is a critical element of the discourse because possible differences in participants' use of those words can hinder mathematical communication.

Visual mediators refer to the visible objects created and operated upon for the sake of communication. Colloquial discourses are "often mediated by the images of concrete objects" whereas scientific and mathematical discourses are primarily mediated by symbolic artifacts (Sfard, 2008, p. 147). In mathematics, such artifacts consist of algebraic symbols as well as the conventionally or idiosyncratically created diagrams, graphs, tables and icons.

Routines refer to the set of metarules[7] that define repetitive patterns in a discourse. Routines can be idiosyncratic. For example, a student's repetitive patterns while doing mathematics might differ from that of another student or mathematicians. The routines that are accepted as valid and enacted extensively by the experts of the community are called *norms*. Although both routines and norms consist of metarules that characterize repetitive patterns in a discourse, not every metarule that is enacted or endorsed can be considered a norm. A metarule must satisfy the following in order to be

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[7] Metarules refer to the rules that characterize the patterns in the activity of the participants of a discourse. See Section 2.2.2 for a more detailed discussion of the notion.

a norm: It must be enacted widely within the community of discourse, and it must be endorsed by the majority of the community, "especially by those within the community who count as experts" (Sfard, 2008, p. 204).

Narrative is "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures" (Sfard, 2008, p. 134, italics in original). Axioms, definitions and theorems are among the endorsed narratives of mathematics. Narratives of a given discourse that are endorsed by the majority of the discourse community, in particular by "experts", are considered as "true". The endorsed narratives of an individual, however, can be different than those endorsed by the mathematics community. For the analysis of an individual's discourse, an endorsed narrative refers to what one considers to be true in relation to the routines one uses to substantiate those narratives. The idiosyncratic nature of endorsed narratives results from the idiosyncratic nature of the routines. Mathematical learning takes place as the endorsed narratives and routines of an individual become compatible with those of the experts.

Sfard (2008) notes that "*mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begins*" (p.129, italics in original) and so it is "a multilayered recursive structure of discourses-about-discourse, and its objects therefore are, in themselves, discursive constructs" (p. 161). The generative power of mathematical discourses, like any other discourse, is obtained through recursion with which "we can turn one discursive act into the object of another" (Sfard, 2008, p. 103) and create metastatements, that is, statements about statements.

Sfard (2008) argues that discourses develop in terms of expansions and compressions. Discursive expansions are considered as endogenous "when there is an increase in *the amount of complexity* of discursive routines" and as exogenous "when there is a *proliferation of new discourses*" (Sfard, 2008, p. 119, italics in original). Discursive compressions, on the other hand, result from reaching to a metalevel by means of *objectification*. Objectification occurs through *reification* and *alienation*. Reification "is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44), whereas alienation refers to "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 295). Through objectification, we identify the commonalities between different processes within a discourse and we unify many lower-level phenomena under one name. This new metalevel discourse *subsumes* the lower-level, independently existing discourses and "enables us to express in the new language everything that can be said in any of the original discourses with their own signifiers" (Sfard, 2008, p. 122). Therefore, the objectified discourse is more abstract than any single one of the discourses it subsumes. Objectification increases the effectiveness of our communication and is also a means of formalization of the mathematical discourse, especially through the use of symbolic artifacts. However, since objectification replaces the talk about processes with the talk about objects in an impersonal way[8], it hides the discursive layers that constitute the objects and also the metaphorical nature of the objects we speak about. By doing so, it

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[8] Note that changing the *talk* about processes to the *talk* about objects necessitates changes in the word use of a discourse. Therefore, Sfard (2008) considers degree of objectification as crucial factor in the analysis

also blurs the differences between the lower-level discourses that are subsumed, which can be of significant importance, especially at the beginning stages of learning. This is probably the reason why the insiders of the mathematical discourse (e.g., mathematicians, mathematics teachers) “lose the ability to see as different what children cannot see as the same” (Sfard, 2008, p. 59).

When human development is considered equivalent to the development of discourses, the study of the historical development of mathematical concepts become equivalent to the study of the evolution of the mathematical discourse about the concepts. Since mathematics is a patterned, historically established metadiscursive activity, such evolution consists of the development of thinking about the objects. Therefore, although the historical processes of object creation follows a different sequence than students' individualization of those objects, the communal aspect of mathematizing contains the narratives and the routines students need to adapt to as they become participants in the mathematical discourse. Moreover, the historical development of mathematical concepts includes the junctures that enable the growth of the discourse through the interplay of expansion and compression, which can be useful in the exploration of the conceptual obstacles associated with the concepts. In this study, I will discuss the conceptual obstacles associated with limit and its underlying concepts (infinitesimals and infinity) through the analysis of the historical junctures that necessitated particular changes in the previous mathematical discourse about the concepts. Next, I describe the elements of the commognitive framework I will use to analyze those historical junctures in Chapter III.

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of word use. The details of how to analyze of word use, visual mediators, routines and endorsed narratives will be discussed in Chapter IV (design of the study).

2.2.2. Metarules and metaphors in discourse development

It was mentioned in Section 2.2.1 that discourses develop through the interplay of expansions and compressions. More often than not, such junctures in the development of a discourse result in changes in the metarules, also called metadiscursive rules, of the existing discourse in order to extend it further. Unlike object-level rules, which "take the form of narratives on the objects of the discourse", metarules "*define patterns in the activity of the discursants trying to produce and substantiate object-level narratives*" (Sfard, 2008, p. 201). "A metarule in one mathematical discourse will give rise to an object-level rule as soon as the present metadiscourse turns into a full-fledged part of the mathematics itself" (Sfard, 2008, p. 202). For example, one of the metarules of arithmetic 'If we add an even number with another even number, the sum is an even number', becomes an object-level rule '$2n+2m=2(n+m)$ for all $n,m\in R$'[9] in the algebraic discourse when expressed as the relation between the algebraic objects n and m. The tacit nature of the metarules of mathematics is amplified by objectification and symbol use. Through reification and alienation, mathematical statements reach their timeless forms, making it seem like mathematics exists independently of the creators of those statements. Therefore, among the reasons metarules are mostly tacit are the metadiscursive and metaphorical nature of mathematical objects. By being discourses about discourses, metadiscursive statements hide the discursive layers they consist of and the metaphors they are based on. Metaphors help us create new discourses through usage

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[9] Note that the metarule and the object-level rule mentioned here are justified by the experts of mathematics as true. As a result, they are among the endorsed narratives of mathematics. The former is a meta-level narrative whereas the latter is an object-level narrative. Said differently, one can talk about rules as well as narratives of a discourse as being meta-level and object-level.

of words from familiar contexts when making sense of an unfamiliar context. Thus, the mechanism of metaphor can be thought of as "the action of 'transplanting' words from one discourse to another" (Sfard, 2008, p. 39). Although metaphors are crucial mechanisms with which we build and expand discourses, "words that have been transferred from one discourse to another cannot be incorporated to the new discourse without some bending of the old rules" (Sfard, 2008, p. 75). Given this, the exploration of the metaphors that govern different layers of a mathematical discourse becomes a central part of the exploration of metarules[10] in the development of the discourse. The analysis of junctures in the development of a discourse with respect to the changes in the metarules can give us information regarding the transitions learners need to go through as they participate in the extended discourse. In what follows, an example of such a juncture will be given in the domain of arithmetic.

When we work with positive integers, the metaphor underlying multiplication is repeated addition. We make sense of multiplication by means of addition and consider the product 2×3 as $2 + 2 + 2$ (two added to itself three times) or $3 + 3$ (three added to itself two times). Therefore, in this case, 'we multiply by adding repeatedly' is the metarule of multiplication. This metarule also leads to another metarule, that 'multiplication always makes bigger'. This means that whenever we multiply two positive integers different than 1, the product is bigger than either of those integers. When we multiply two positive rational numbers, on the other hand, the discourse on multiplication has to go through some change. Considering $\frac{1}{2} \times \frac{1}{3}$ as adding $\frac{1}{2}$ to itself

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[10] Note that *using* metaphors consistently in a given discourse is a type of metarule.

$\frac{1}{3}$ times does not make sense. Given this, the metarule, that multiplication is repeated addition, has to change. We can think of multiplication of positive rational numbers by considering each number as the side length of a rectangle and think about multiplication in terms of the area of that rectangle. Moreover, since $\frac{1}{2} \times \frac{1}{3}$ is equal to $\frac{1}{6}$, which is a smaller number than both of the numbers we multiply, the metarule 'multiplication always makes bigger' also needs to be abandoned. The visualization of multiplication as hopping on the number line, which works for the case of positive integers, needs to be replaced by the visualizations of multiplication as area of rectangles[11]. It is these types of junctures that will be elaborated on in this study since they require changes in the metarules of the previously existing discourse to extend the discourse further. It should be noted that the discourse on multiplication of positive rational numbers subsumes the discourse on the multiplication of positive integers. Given this, the metarules, visual mediators, and endorsed narratives, as well as the object-level rules that are valid for the former are also valid for the latter. In contrast, not all the metarules, visual mediators and endorsed narratives of the discourse on multiplication of positive integers are necessarily valid for the more general, subsuming discourse on the multiplication of positive rational numbers.

By being based on the tacit metarules and metaphors governing the discourse, developmental junctures eventually require changes in the endorsed narratives of the

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[11] Of course, the discourse about multiplication goes through yet other transitions when negative numbers get into the picture.

discourse. In that sense, they lead to the changes that are essential in order for the learner to participate in the new aspects of the extended mathematical discourse.

Summary

In this chapter, I described the limit concept and summarized the literature about its learning. Existing research on the learning of limits mostly characterizes the difficulties students have about limits through misconceptions based on the assumptions of a cognitivist framework. From this perspective, students need to change and refine their mental schemes in order to overcome the conceptual obstacles they have about limits. I argued that we do not have direct access to constructs such as (mis)conceptions and mental schema. In other words, a cognitive framework may not offer sufficient tools of analysis with which we can explore how learners make sense of the limit concept. I then introduced Sfard's (2008) *commognitive* framework as an alternative lens to investigate development of discourse on limits and student difficulties associated with the concept. From this perspective, students need to change elements of their *discourse* on limits in order to overcome their conceptual obstacles. Sfard (2008) highlights word use, visual mediators, routines, and narratives as the main tools of analyses to learn more about one's mathematical discourse. In this study, I will investigate whether such an analysis enhances our knowledge of student learning about limits.

Existing research on limits also points to some possible links between the historical development of limits and student learning. However, it does not elaborate on the principles underlying the transitions learners need to go through as they attend to different aspects of the limit notion. By viewing developmental processes as resulting from *individualization of the communal* and *communalization of the individual*, I will concentrate on historical, and therefore communal, development of discourse on infinity, infinitesimals, and limit in the following chapter. More specifically, I will explore the historical development of limit related concepts through the commognitive framework, with a particular focus on the junctures that resulted in changes in the metarules and metaphors of the previously existing discourse on limits. The purpose of this investigation is to gain more information about the nature of the conceptual obstacles related to limits over history. Later in the study (Chapter VII), I will examine whether such a consideration of historical development can be useful to learn more about individual student learning.

CHAPTER III

HISTORICAL DEVELOPMENT OF THE DISCOURSE ON INFINITY, INFINITESIMALS AND LIMITS

In this chapter, I explore the historical development of infinity, infinitesimals and limits. While doing so, I rely on historical documents as well as research on the historical development of those concepts. When investigating the historical development of discourse about those concepts, I pay attention to particular elements of the commognitive framework, namely word use (objectification), metarules and metaphors[1], that characterized particular realizations of infinity, infinitesimals and limits. I also use these elements to identify the historical junctures that led to changes in the metadiscursive rules as the discourse on these notions extended.

Although the historical developments of the concepts of infinity, intinitesimals and limits are intertwined, I will first focus on them separately. I will then argue that there were two types of historical junctures that led to changes in the metadiscursive rules of the discourse on these concepts. The first type led to the objectification of the ideas about infinity, infinitesimals and limits. The second type, which occurred in the development of discourse on limits, led to an alternative realization of limit and ultimately resulted in the elimination of motion as an idea underlying this notion. I will argue that these two types of junctures form the bases of the changes in the metadiscursive rules in the historical development of discourse on these three concepts.

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[1] Recall that using metaphors when making sense of a mathematical concept is a type of metarule (See Section 2.2.2)

In order for the reader to follow the historical development of these concepts clearly, I present the timeline for the mathematicians who will be addressed in this chapter in Table 3.1.

Table 3.1: Timeline of the mathematicians discussed in this chapter over history

Ancient times	Renaissance period	17th and 18th centuries	19th and 20th centuries
Eudoxus (405-355 B.C.)	Viete (1540-1603)	Newton (1643-1727)	Weierstrass (1815-1897)
Aristotle (384-322 B.C.)	Descartes (1596-1650)	Leibniz (1646-1716)	Dedekind (1831-1916)
Euclid (325-265 B.C.)	Cavalieri (1598-1647)	Euler (1707-1783)	Cantor (1845-1918)
Archimedes (287-212 B.C.)	Fermat (1601-1665)	Lagrange (1736-1813)	Robinson (1918-1974)
	Wallis (1616-1703)	Cauchy (1789-1857)	

3.1. The notion of infinity

Historically, the notions of potential infinity and actual infinity have been of interest for philosophers, mathematicians and scientists since ancient times. Lakoff and Nunez (2000) note that "outside mathematics, a process is seen as infinite if it continues (iterates) indefinitely without stopping" (p. 156). On the other hand, there can be no direct experience with the notion of infinity in real-life since our environment is restricted by finiteness. Therefore, Aristotle (384-322 B.C) considered infinity only as potential: "the non-limited *possibility* to increase an interval or to divide it" (Fischbein, Tirosh & Hess, 1979, p. 3) and rejected the notion of actual infinity to avoid Zeno's paradoxes[2] and the inconsistencies in the existing discourse about potential infinity. Fischbein (2001) notes that, in the case of potential infinity,

> we deal with a dynamic form of infinity when we consider processes, which are, at every moment, finite, but continue endlessly. We cannot conceive the entire set of natural numbers, but we can conceive the idea

that after every natural number, no matter how big, there is another natural number. (p. 310)

The dynamic nature of visualizing infinite continuation without an endpoint is also mentioned by Lakoff and Nunez (2000). They note that we make sense of such continuation by motion that goes on and on forever. With this metaphor, we talk about continuous processes without end by thinking of them as "infinite iterative processes, processes that iterate without end but in which each iteration has an endpoint and a result" (Lakoff & Nunez, 2000, p. 157). Thus, this metaphor enables us make sense of infinite processes by means of infinitely many step-by step processes that are discrete. This is the metaphor with which we realize potential infinity. In mathematics, we use the notion of potential infinity whenever we write down the elements of a given sequence, for example $\frac{1}{n}$, as $1, \frac{1}{2}, \frac{1}{3}, \ldots$ or when we write the decimals of $\sqrt{3}$ in terms of ones, tenths, hundredths, and so on. Similarly, Tall (1992) argues that using words such as 'x tending to infinity' for the notations like '$x \to \infty$' also represent infinity as a potentiality. Therefore, mathematics makes frequent use of the notion of potential infinity through word use and symbolization. At this point, it is important to note that, despite its common use in the mathematical discourse, potential infinity is not objectified. When we think about infinity as potential, we talk about it as a *process* but not as a distinct mathematical object. Its use in the mathematical discourse is equivalent to its use in the *colloquial discourses*, "which are also known as *everyday* or *spontaneous* because they often develop as if by themselves, as a by-product of repetitive day-to-day actions" (Sfard,

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[2] For the description of Zeno's paradox and its variations, see (Fischbein, 2001).

2008, p. 132). Aristotle's notion of potential infinity continued for centuries and it was not until the 19th century that the mathematical distinction between potential and actual infinity was made explicit: "In XVIII century...the difference among a very big number and on 'infinity' was neglected and it seemed self-evident that a theorem true for every n was true for n infinite, too" (Kline, 1991, I, p. 506 as cited in Bagni, 1997, p. 210).

Cantor's (1845-1918) work in the 19th century replaced the notion of potential infinity – the idea that infinity resides beyond any given number (positive or negative) – with the notion of actual infinity, realized as an accumulation point. Using Cantor's words:

> Mathematical infinity...is crescent beyond every limit or indefinitely decrescent, and it is a quantity that remains *finite*. I call it *improper infinity* [potential infinity]. Moreover, recently, another kind of infinity...took place...By that...the infinity is considered as concentrated in a certain point. When infinity occurs in this form, I call it *proper infinity* [actual infinity]. (Bottazzini, Freguglia & Toti Rigatelli, 1992, p. 428 as cited in Bagni, 1997, p. 210).

Here Cantor talks about *improper*, or potential, infinity similar to Fischbein's (2001) arguments that although the process continues endlessly, it is considered as finite at every given moment. Cantor uses the word p*roper*, or actual, infinity, on the other hand, to refer to an end-state. Given this, Cantor realized actual infinity as a distinct entity but not as a process and therefore objectified the notion of infinity through reification. This, on the other hand, required changes in both the existing discourse about potential infinity and also in the metarules of mathematical activities such as counting and measuring. "The shift of attention, from the natural numbers as a potentially infinite collection going on and on, to a single entity the *set* N, given axiomatically, leads to considering the relationships with other infinite sets" (Tall, 2001, p. 212, italics in original). By introducing the notion of one-to-one correspondence in determining the equivalence of the cardinality of sets, Cantor eventually transformed the metarules of counting: "If we have to compare two infinite sets, we should not count their elements as we count finite groups of objects. We have to determine the equivalence-or non-equivalence-of these sets by *formal means*" (Fischbein, 2001, p. 310). The acceptance of infinity as actually existing required the acceptance of "the strange proposition that the whole may be equivalent to some of its parts" (Fischbein, Tirosh & Hess, 1979, p.4). For example, we can come up with a one-to-one correspondence between the set of positive even numbers and natural numbers, which in Cantorian terms, means that the cardinality of these two sets are equal to each other. This, on the other hand, also means that the set of positive even numbers, which is contained in the set of natural numbers, has equal cardinality with the set of which it is a subset. Therefore, a previous metarule of counting that "if we find the cardinality of a proper subset of a given set, the result is a number less than the cardinality of the larger set" had to change in the case of infinite sets. Moreover, Cantor also showed that the cardinality of the set of natural numbers and the cardinality of the set of points on a number line are not equivalent to each other since there could be no one-to-one correspondence between these two sets. Therefore, although both sets contained infinitely many numbers, they had distinct cardinalities. This meant that more than one type of infinity existed. In that sense, although infinity was objectified as an ultimate state and an existing 'number' ∞, the symbolic equation $5 = 5$, which made perfect sense for the number 5 as an object of mathematics, was meaningless for the case of infinity: infinity was not equal to infinity! Instead, we used a family of notations $\aleph_i$ to denote different infinities. Lakoff and Nunez (2000) also mention that the famous mathematician Hardy warns us not to consider ∞ as a number in the usual sense "because mathematicians have devised notions and ways of thinking, talking, writing, in which ∞ is a number with respect to enumeration, though not calculation" (p. 165).

Besides counting, the notion of actual infinity also required changing some of the metarules of arithmetic. Tall (2001), highlights that neither the subtraction nor the division of the infinite cardinals can be uniquely defined. On the other hand, the introduction of the notion of actual infinity results in a proliferation of other mathematical discourses. Tall (1992) considers cardinal infinity, ordinal infinity and non-standard (measuring) infinity among the three notions of infinity[3] used in mathematics today. The fact that "natural numbers are not only used for *counting*, but also for putting a set into an order" (Tall, 2001, p. 216) eventually led to the definition of order relationships and the creation of ordinal numbers for which addition was not commutative. As a result, the metarule of adding quantities regardless of their order, which is valid for real numbers and the cardinal numbers does not hold for ordinal numbers. "So strange were these ideas to the mathematics community when first announced that Kronecker prevented the initial publication of Cantor's theory of infinite cardinals…" (Tall, 2001, p. 218).

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[3] Cardinal infinity extends counting by means of comparison of sets. Ordinal infinity is conceptualized in terms of comparison of ordered sets. Measuring infinity extends measuring from real numbers to larger ordered field (Tall, 1992).

Fischbein, Tirosh and Hess (1979) argue that "the contradictory nature of infinity can be pushed to higher levels but cannot be completely eliminated...even with the most sophisticated mathematical tools" (p.4). Lakoff and Nunez (2000) seem to agree with this since they claim that the metaphor with which we realize actual infinity is a *special* "case of a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result" (p. 158). They call this metaphor the *Basic Metaphor of Infinity* and argue that it turns potential infinity into actual infinity in terms of a largest 'number' ∞. By means of this metaphor, proof by induction "needs no longer be considered as potentially infinite process" but becomes a three-step procedure following Peano's axiom[4] (Tall, 2001, p. 210). As a result, actual infinity also changes the metarules of proving.

It is probably true that even the symbolization of the notion of infinity in mathematics cannot keep us away from the conceptual obstacles associated with it.

> The common mathematical notion for infinity-'...'– as in the sequence '$1+\frac{1}{2}+\frac{1}{4}+...$'– does not even distinguish between potential and actual infinity. If it is potential infinity, the sum only gives an endless sequence of partial sums always less than 2; if it is actual infinity, the sum is exactly 2. (Lakoff & Nunez, 2000, p. 180)

On the other hand, actual infinity is now a meaningful, "non-contradictory concept, consistent with the totality of the other mathematical concepts" (Fischbein, Tirosh &

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[4] Tall (2001, p. 211) mentions that to prove a statement *P(n)* by induction, it is enough to (a) show *P(1)* is true, (b) show the truth of *P(k)* implies the truth of *P(k+1)*, and (c) quote the induction axiom, which states that if (a) and (b) are true, then *P(n)* is true for all *n* in the set of natural numbers.

Hess, 1979, p. 3). Therefore, we accept the notion's mathematical *reality*. In that respect, the mathematical answer for $1+\frac{1}{2}+\frac{1}{4}+\ldots$ is *exactly* 2. The difficulty Lakoff and Nunez addresses above, therefore, results from the conceptual but not the mathematical aspect of infinity.

Summary

The objectification of infinity as a distinct mathematical entity did not occur till the 19th century. This required changing the talk (and word use) about potential infinity, which is a process, to the talk about actual infinity, which is an accumulation point. Such a transition necessitated changes in the metarules of counting, measuring, arithmetic and proving. The notion of actual infinity also gave rise to new types of 'numbers', namely, cardinal, ordinal and measuring (non-standard) infinities. Therefore, the development of actual infinity led to an exogenous expansion in the mathematical discourse since it ignited the proliferation of different mathematical discourses. Fischbein's (2001) and Tall's (2001) arguments support the idea that the discursive expansion about the notion of cardinals and ordinals took place only after, not before, the objectification and formalization of actual infinity by means of an axiomatic system.

Although the notions of potential and actual infinity are distinct from each other mathematically, they are both used frequently in the mathematical discourse[5]. Lakoff and Nunez (2000) argue that the metaphors we use for potential and actual infinity are

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[5] It should be noted, however, that the notion of actual infinity created quite a chaos in the mathematics community before its mathematical existence was accepted (Bagni, 1997; Fischbein, Tirosh & Hess, 1979; Tall, 2001).

variations of each other and are primarily based on our realization of the infinite processes through iterative processes that have an end and an ultimate state.

3.2. The notion of infinitesimals

The concept of infinity, which is mostly associated with the notion of infinitely large, also brings with it the notion of infinitely small. The term infinitesimal entails infinitely small quantities, generally negligible, that are not real constants. Infinitesimals are also called indivisibles, differentials, evanescent quantities and infinitely small magnitudes (Kleiner, 2001). As mentioned in the preceding section, the discussions about infinity go back to Aristotle and the first integral-like approach given to area related problems was given by Eudoxus (405-355 B.C). Eudoxus suggested an approach that seemed to have characteristics similar to an infinite process. This approach was called the method of exhaustion, which led to the implicit notion of infinitesimals. The method assumed infinite divisibility of magnitudes and was primarily based on the proposition

> If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind. (Eves, 1983, p. 289)

A magnitude being "less than any preassigned magnitude of the same kind" formed the initial underpinnings of an infinitesimal quantity. Although this method was handled by Euclid (about 325-265 B.C.) and Archimedes (287-212 B.C.) later, it was Archimedes who tied the problems of finding area to the explicit use of the concept of infinity. He initiated the idea that an area could be composed of infinitely many geometrical lines but he could not form a solid explanation of the vaguely defined concept of infinitesimals. The mathematics of this period is considered as *static* since it lacked the consideration of motion and change and was based on the axiomatic structure of geometry. Although the initial underpinnings of infinitesimals were present in this era (around 300 B.C.), they could not be justified with the geometric foundation. Since the idea of infinitesimals relied too much on intuition and was not based on the solid foundation of mathematics, which was geometry with its axiomatic/deductive structure in this period, it was considered unsound. Therefore, it was discarded by the ancient Greek mathematicians and was not emphasized assertively by Archimedes.

In 1635, Cavalieri (1598-1647) applied indivisibles or fixed infinitesimals successfully to problems in the mensuration of areas and volumes which brought infinitesimals back into discussion (Boyer, 1970). He used the geometrical approach, which still dominated the renaissance mathematics, and found the integrals of n^{th}-degree polynomials accurately. Yet, a revolution was about to come as mathematicians like Viete, Descartes, Wallis and Fermat recognized the use of algebra as an aid to geometry. This led to the tendency towards the symbolic-algebraic over geometric by the end of the renaissance period. Descartes' (1596-1650) work, published shortly after Cavalieri's, changed the course of infinitesimal analysis once again and initiated the period called the arithmetization of geometry[6].

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[6] This, together with the discovery of non-Euclidean geometry, marked the stagnation of geometry for about a century and a half.

The mathematics of the late renaissance and 17^{th} century relied on the dynamic[7] approach and overshadowed the static approach given to mathematics since the ancient Greeks. These changes enabled the reevaluation and reanalysis of the infinitesimal calculus into what we now know as calculus. The use of infinite-series expansions for the generalization of analysis not only to polynomial functions but also to rational, irrational, algebraic and transcendental functions required a new infinite analysis. This was recognized first by Newton in 1665-66 and then independently by Leibniz in 1673-76 (Boyer, 1970).

Newton's main contribution was to justify that the infinite processes were as respectable as the algebraic ones. His approach to infinite processes was mainly dynamic and relied on the notion of incremental change because of the physical nature of his problems at hand. However, since mathematicians have been deeply skeptical of the concept of infinitesimals due to their intuitive and non-rigorous characteristics, Newton avoided using 'infinitely little' but used the term 'evanescent' while discussing fluxions[8]. Historically, the strongest criticism of Newton's calculus came from Berkeley in 1734. Berkeley's reaction was primarily based on Newton's implicit use of infinitesimals in calculus: "And what are these same evanescent Increments? They are neither finite Quantities, nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?" (Berkeley, 1734, as cited in Jesseph, 1993, p. 199).

Leibniz, on the other hand, gave importance to appropriating notations and was able to give the correct rule for differentiation for the product of two quantities. Yet, he

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7 The consideration of motion and change in mathematics through physical problems.

did not consider these quantities as variables one of which depended on the other, so he missed the idea of differentiation of a variable with respect to another variable. Leibniz' proof of the product rule was as follows:

> $d(xy) = (x + dx)(y + dy) - xy$ where dx and dy are the differentials or infinitely small differences of x and y. So Leibniz states that "the quantity $dxdy$ ...is infinitely small in comparison with the rest, and hence can be disregarded" (Edwards, 1979, p.255).

Boyer (1970) argues that Leibniz and his disciples could not make clear what they meant by infinitely small change and could not justify the elimination of quantities that were infinitely small compared to others. According to him, Leibniz' calculus was a failure compared to Newton's from this logical viewpoint. It should be noted that both Newton and Leibniz worked on the calculus of instantaneous change. Therefore, they were implicitly using limit *as a process*, but not as an explicitly defined concept, as they obtained the tangent line at a point through a sequence of the secant lines passing from that point (Lakoff & Nunez, 2000). On the other hand, Newton used a geometric approach in the process whereas Leibniz relied more on arithmetic.

Euler (1707-1783) shone in 18th century with his work on the possibilities inherent in the infinite power series. By means of symbol manipulation, Euler showed that "what is true for convergent series is true for divergent series, what is true for finite quantities is true for infinitely large and infinitely small quantities" (Kleiner, 1991, p. 295). To Euler, the transition from finite differences to the limit method (limit as a

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[8] Newton referred to a varying (flowing) quantity as a fluent. He used the term fluxion to refer to the instantaneous rate of change of a fluid.

process) was straightforward and was based on his consideration of continuity of real numbers:

> Both infinitely small and infinitely large quantities often occur in series of numbers. Since there are finite numbers mixed in these series, it is clearer than daylight how, according to the laws of continuity, one passes from finite quantities to infinitely small and to infinitely large quantities (Euler, 2000, p. 90).

On the other hand, infinitely small quantities were intentionally banned from Lagrange's lectures since he considered them as concepts that lacked adequate foundational basis (Schubring, 2005, p. 290).

Infinitesimals went on being under attack till the beginning of the 20th century[9]. On the other hand, "they still continued to flourish in the practical world of engineering and science …, representing not a fixed infinitesimal quantity, but as a variable that could become 'arbitrarily small'" (Tall & Tirosh, 2001, p. 130). In mid 20th century, Abraham Robinson (1918-1974) introduced his theory of non-standard analysis, in which "infinitesimals were formulated on a logical basis" (Tall & Tirosh, 2001, p. 130). This still did not solve the debate as to whether infinitesimals can be considered as logically sound mathematical objects since the new formulation of infinitesimals brought its relevant incompatibilities with the existing mathematical discourses. This debate continues into the 21st century.

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[9] See the next section about limit for further details.

It was mentioned before that the introduction of different types of infinities changes the metarules of counting and measuring. Cardinal numbers, for example, require a different way of counting for infinite sets than finite sets through the notion of one-to-one correspondence. Recognition of non-standard infinity, which Tall (1992) refers to as the *measuring infinity*, marks a change in the notion of infinity and the metarules of measuring and is based on the non-standard analysis of Robinson. "To explain such a theory requires the formal interpretation of the notion of an infinitesimal" (Tall, 1980a, p. 274). Measuring infinity considers a point as a ball of infinitesimal size, by which "we discover a theory that allows both the indivisibility of 'points' and also infinite divisibility of a line" (Tall, 1980a, p. 274). In that sense, a point which is considered dimensionless is now a ball with an infinitesimal radius. Tall (1980b) also mentions that non-standard analysis, through the reformulation of infinitesimals, leads to new types of numbers, namely *hyperreal* or *superreal* numbers.

Summary

Although the objectification of infinitesimals is absent before the 20th century, the period that precedes Robinson's non-standard analysis highlights the conceptual obstacles regarding the acceptance of infinitesimals as mathematically justifiable objects. Sfard (2008) notes that "not every metarule, whether enacted or endorsed, is a norm. In order for a rule to be a norm, it must be widely enacted within a community" and it must also "be endorsed by everybody [as true], and especially by those within the community who count as experts" (p. 204). Given this, although the uses of infinitesimals had been present, the narratives about them were not endorsed by the majority of the mathematics community for a long time.

Initially, the reason that hindered the acceptance of infinitesimals as valid mathematical objects resulted from the almost mystical nature of the utterances about infinitesimals (e.g., evanescent quantities or vanishing increments). Moreover, the use of geometry by its dynamic representations, such as curves and graphs, but not by means of its formal-deductive structure (Euclidean geometry) entailed continuous motion in space, which was considered intuitive and sensuous. A more important reason, however, was the way infinitesimals contradicted the endorsed narratives about real numbers. Lakoff and Nunez (2000) mention that infinitesimals do not obey the arithmetic rules of real numbers since they violate the Archimedean Principle[10] and commensurability[11]. Did Robinson's objectification of infinitesimals as particular types of numbers solve these problems?

> To this day the debate continues. Although the infinite cardinals are generally accepted by the mathematical community, there are mathematicians who fully embrace the theory of infinitesimals in non-standard analysis, those who deny their existence and assert the pre-eminence of standard analysis, and even a greater number who do not agonise over the foundational problems and simply get on using mathematics for practical purposes. (Tall & Tirosh, 2001, p. 130)

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[10] "Archimedean Principle: Given numbers A and B (where A is less than B) corresponding to the magnitudes of two line segments, there is some natural number n such that A times n is greater than B" (Lakoff & Nunez, 200, p. 298).

[11] "Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure" (Heath, 1956, p. 10). In modern sense, two non-zero quantities A and B are commensurable if there exists a quantity C such that $A=mC$ and $B=nC$ for non-zero whole numbers m and n.

Interestingly, the reformulation of infinitesimals by means of logic "invoked the axiom of choice to assert that such entities existed without being able to give a specific finite construction" (Tall & Tirosh, 2001, p. 130). That is to say, the formal theory of infinitesimals assumes the existence of the concept but cannot prove or construct their existence. In that respect, it seems impossible to justify the notion based on the solid foundations of mathematics. However, the discourse on infinitesimals continues to lead to the proliferation of new mathematical discourses such as measuring infinities, hyperreal and superreal numbers.

3.3. The notion of limit

Being the founders of calculus, Newton and Leibniz both used infinitesimals in their theories as they worked on incremental change. By obtaining the tangent line at a point through the use of a sequence of secant lines, they were using the notion of limit as a process. The dynamic mathematics influencing the renaissance period relied heavily on the metaphor of continuous motion. On the other hand, the symbolism of arithmetic and its use for expanding the discourse on functions made arithmetic a better candidate for 'the' foundation of mathematics than geometry. Given these, it is not surprising that Newton was criticized more than Leibniz or Euler. Although they all referred to infinitesimals in their work, Newton also used geometry in the dynamic, and hence the most intuitive, manner. Leibniz, by relying on arithmetic, probably avoided some criticism.

Using the notion of limit as a process is referred to as the *limit method* in the historical documents. After Newton and Leibniz, mathematicians such as MacLaurin and d'Alembert kept on using this method on their problems. Lagrange opposed them on their use of the limit method:

> MacLaurin and d'Lambert used the idea of limits; but one can observe the subtangent is not strictly the limit of subsecants, because there is nothing to prevent the subsecants from further increasing when it has become a subtangent. True limits… are quantities which one cannot go beyond, although they can be approached as close as one wishes. (Lagrange, 1799, as cited in Schubring, 2005, p. 293)

Lagrange's arguments were primarily based on

> the lacking of the concept of absolute value…so that it seems as if the variable goes beyond the limit; the criticism is also at the problem, which has always remained controversial, whether a variable can definitely reach the limit or is only allowed to come close to it at any rate (Schubring, 2005, p. 293).

Although Lagrange uses words like "true limits…are quantities", it was not until Cauchy (1789-1857) that the notion of limit was objectified. Lagrange seems to talk about limit as a "subtangent", which is "the limit of subsecants". In that sense, he considers limits as the quantities obtained through the limit process and does not explicitly define them. It should also be noted that Lagrange's word use "[true limits]…can be approached as close as one wishes" entail very small or infinitely small increments as well as motion. Therefore, the limit method makes use of the notion of infinitesimals and is based on the metaphor of continuous motion.

The discourse of calculus went through a fundamental change with Cauchy. He realized the necessity of establishing a theory of limits, which required the explicit definition of the concept. He selected the fundamental concepts of calculus like limit, convergence, derivative and integral and created the grand design of calculus where limit became the concept on which the others were based and the concept of derivative came before the concept of integral (Kleiner, 1991, 2001). The college level calculus that we teach today mostly follows the outline of Cauchy. Table 3.2 shows the comparison of the historical development of the calculus concepts with Cauchy's design.

Table 3.2: Comparison of the historical development of calculus with Cauchy's framework

Historical development	Design of Cauchy
Area and integral	Infinity
Infinity and Infinitesimals	Limit
Series (finite, infinite) and sequences	Derivative
Derivative	Integral
Limit	Series and sequences

One of the reasons underlying Cauchy's revolutionary departure from the established practice was his opposition to Lagrange, whose foundation of calculus was based on algebra. Cauchy wanted to eliminate algebra as a basis of calculus and wanted his methods to have the rigor demanded in geometry (Kleiner, 1991, 2001). Cauchy defined limit as follows:

> When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing form it by as little as one could wish, that fixed value is called the limit of all the others (Kitcher, 1983, p.247).

His definition of infinitesimals was:

When the successive absolute values of a variable decrease indefinitely in such a way as to become less then any given quantity, that variable becomes what is called an infinitesimal. Such a variable has zero for its limit (Kitcher, 1983, p.247).

An analysis of Cauchy's word use reveals that he objectified the notion of limit through reification by referring to limit as a "fixed value", that is a distinct mathematical object. Note also that he uses the word "approaching", which is based on the metaphor of continuous motion. Finally, the phrases "absolute values of a variable decrease indefinitely" and "differing from [a value] as little as one could wish" entail the use of infinitely small quantities, namely, infinitesimals. Therefore, Cauchy's definition of limit was based on infinitesimals and the continuous motion metaphor, which were both problematic for mathematicians of his time. The dynamic interpretation of limit was considered intuitive by the community since terms like *tending to* have a "connotation of desire, of aspiration. Numbers do not tend" (Fischbein, 1994, p. 239).

Since Cauchy based all of his calculus on the concept of limit, a precise definition of limit became of crucial importance. Weierstrass (1815-1897) and Dedekind [12](1831-1916) attempted to 'remedy' Cauchy's definitions by finding "a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis" (Dedekind, 1963, p.1 as cited in Kleiner, 1991). These mathematicians wanted to replace Cauchy's kinematic approach with the algebraic-arithmetic approach. The goal was to reconceptualize calculus as arithmetic by eliminating spatial intuition. In order to do this,

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12 Bolzano and Hilbert were also among the mathematicians of 19th century who favored arithmetization of analysis.

> Natural continuity had to be eliminated from the concepts of space, planes, lines, curves, and geometric figures. Geometry had to be reconceptualized in terms of sets of discrete points, which were in turn to be conceptualized purely in terms of numbers: points on a line as individual numbers...The idea of a function as a curve in terms of the motion of a point had to be completely replaced. There could be no motion, no direction, no approaching a point. All these ideas had to be reconceptualized in purely static terms using only real numbers. The geometric idea of approaching a limit had to be replaced by static constraints on numbers alone, with no geometry and no motion. This is necessary for characterizing calculus purely in terms of arithmetic. (Lakoff & Nunez, 2000, p. 308)

Weierstrass accomplished this full agenda by considering space as sets of points. This led to the consideration of the points on a line as numbers, which then led to the realization of "continuity for a function as the preservation of closeness" (Lakoff & Nunez, 2000, p. 322). In that sense, the distance between points in motion was replaced by the distance between numbers. The result was Weierstrass' ultimate definition of limit:

> Let a function f be defined on an open interval containing a, except possibly for a itself, and let L be a number. Then $\lim_{x \to a} f(x) = L$ if and only if for any number $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Having eliminated the metaphor of continuous motion associated with infinitesimals and geometry, and being a logical-deductive system that had arithmetic as its foundation, this definition seemed to provide the precision mathematicians were looking for[13]. Secondly, the definition was strong enough to explain anomalous cases that violated the geometric and dynamic conceptions of functions as curves. By doing so, it was believed that this new paradigm was generalizable to a broader number of situations[14]. On the other hand, some mathematicians argued that this definition wiped out all the intuitive tools with which we make sense of the concept. Note that the formal definition of limit is not constructive since it does not enable us to find what the limit of a function is but to prove that the limit we initially hypothesize is indeed the limit of the function at a particular point. That may be why the dynamic approach is still widely used both by mathematicians and the students as they make sense of the notion.

Summary

The objectification of the notion of limit initiated by Cauchy requires the consideration of limit as a thing: a particular value obtained from the limiting process. Hence, Cauchy gives a definition of limit by reification[15]. His definition of limit is based on the metaphor of continuous motion and infinitesimals. Weierstrass and Dedekind's attempts to 'remedy' Cauchy's definition result from the incompatibility of these two notions with the discourse on previously existing concepts of mathematics that can be described merely by means of algebra. The formal definition of limit Weierstrass introduces changes the metaphor of natural continuity to the metaphor of discreteness. By

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13 At least till the collapse of the search for foundations (See Hersh, 1997; Lakatos, 1976).

14 However, Lakoff and Nunez (2000) argue that, since natural continuity uses a different conceptual metaphor than the continuity of Weierstrass, the latter is neither a formalization nor a generalization of the former.

15 Note that reification is part of objectification and changes the talk about processes to the talk about products (See Section 2.2.1).

considering numbers as *sets* of discrete points on the number line, he eliminates natural continuity and therefore geometric motion and time from the discourse on limits. It is important to point out that such a shift also requires a change in the definition of functions as curves as well as the phrases like 'tending to' or 'approaching' since these utterances entail continuous motion. It is by means of arithmetization, and thus the discretization, of calculus that function becomes a type of correspondence between two sets and that the distance between two points in space becomes the absolute value of the difference between two numbers.

3.4. Historical junctures in the development of discourse on infinity, infinitesimals and limits

In the previous sections about infinity, infinitesimals and limit, the historical development of these concepts were discussed with a focus on some elements of the commognitive framework such as word use (objectification), metarules and metaphors. In this section, I identify the historical junctures in the development of discourse on these concepts that resulted in changes in the metarules of the discourse in order to extend it further. It was mentioned that the exploration of metaphors that govern different layers of a discourse is a central part of the exploration of the metarules in discourse development (See Section 2.2.2). Table 3.3 shows the junctures that transformed the metarules in the development of infinity, infinitesimals and limits over history.

I highlight two types of junctures in the historical development of discourse on these three concepts: one led to the objectification of each concept; and one led to an alternative realization of the limit concept by the elimination of motion in space. Note that for infinity and limits, the objectification initially took place by means of reification, that is, by changing the talk about a process to the talk about a product.

Table 3.3: Historical junctures in the development of discourse on infinity, infinitesimals and limits over history

Concept	Infinity	Infinitesimals	Limits
Juncture(s)	Cantor's objectification of potential infinity as actual infinity	Robinson's formulation of infinitesimals on a logical basis	[1] Cauchy's objectification of limit [2] Weierstrass' introduction of the formal definition of limit
Changing metaphor(s)	Realization of infinity as an indefinite process through continuous motion is changed to realization of infinity as an end-state, ultimate result or an accumulation point.	Realization of infinitesimals as evanescent or diminishing quantities is changed to realization of infinitesimals as variables that can be made arbitrarily small.	[1] Realization of limit as a process is changed to realization of limit as a fixed value obtained as a result of the process. [2] Continuous/spatial motion and infinitesimals in Cauchy's definition are eliminated. Motion is replaced by the distance between discrete numbers.
Changing metarules	Counting, measuring, proving (proof by induction) and properties of arithmetic such as addition and division.	Measuring (a point is a ball with an infinitesimal radius)	Representing functions (as algebraic rules but not as graphs of curves), points in space (as discrete points on the number line) and geometrical objects (arithmetization of geometry).

Potential infinity and the limit method were originally realized as processes in their historical development. Actual infinity and limit, however, were realized as end-states or numbers obtained at the end of the processes of going on forever and limit method, respectively. The objectification of infinitesimals was slightly different than infinity and limits in that the mathematical justification of the concept is still under scrutiny. Moreover, their initial realization was not in terms of a process but in terms of very small quantities that could be eliminated. I consider Robinson's approach to the concept through logic a historical juncture since it enabled a formal theory of infinitesimals as *objects* of mathematics. However, the existence of infinitesimals cannot still be justified through constructive methods of mathematics but rather is assumed.

I argue that the second type of historical juncture took place in the historical development of discourse on limits. By revising Cauchy's definition of limit, Weierstrass also changed centuries-held metaphors about motion and continuity. Weierstrass' formal definition of limit eliminated infinitesimals from the previously existing discourse on limits and replaced the metaphor of natural continuity with the metaphor of discreteness. He discarded motion and time from his discourse on limits and geometry and offered realizations of points as numbers on the number line and distances between points in space as the distances between numbers.

As discourses develop and expand, the metarules and metaphors underlying the mathematical concepts can change. Although an expanded discourse on a mathematical concept subsumes the preceding discourse and enables generalization, some aspects of the former version of the concept are lost during such transitions. The metaphor that changes might be the most natural attribute of the concept with which learners initially make sense of the concept. Therefore, the junctures in discourse development leading to transformations in the metarules can be of significant importance for learners and also might explain some of their difficulties. Changes in metaphors and metarules eventually result in changes in the word use and the endorsed narratives of the discourse as the metadiscursive rules take the form of object-level rules (See Section 2.2.2). As a result, they enable the proliferation of discourses. For example, the objectification of infinity resulted in a proliferation of discourses leading to concepts such as cardinal, ordinal and measuring infinities. Therefore, historical junctures may highlight some of the transitions learners need to go through as they participate in the expanded mathematical discourse.

Summary

In this chapter, the development of discourse on infinity, infinitesimals and limits over history was explored by means of some elements of the commognitive framework.

The exploration revealed the objectification of the notions as important milestones in discourse development. Historically, objectification of mathematical concepts changed the talk about processes to the talk about products or end states. Such changes also resulted in transformations in particular metaphors and metarules of mathematics.

It was mentioned in Chapter II that, according to Sfard (2008), developmental changes take place in the interplay of two processes: *individualization of the communal and communalization of the individual.* The historical development of limit reveals how a mathematical idea generated by an individual mathematician gains acceptance collectively (e.g., Cantor's objectification of infinity as actual infinity). From the commognitive lens, this is an example of *communalization of the individual.* One instance where the second process, *individualization of the communal,* takes place is student learning. Note that Sfard (2008) considers learning as participation in the communally agreed upon discourse on mathematics. Therefore, a question that will be pursued later in the study is how and whether the investigation of historical development of limits through the commognitive lens is useful to gain more information about teaching and learning of limits. In particular, can the historical junctures that led to changes in the metarules of the discourse on limits be useful to explain some of the transitions students go through as they learn about limits? These issues will be addressed in Chapter VII of the study.

CHAPTER IV

DESIGN OF THE STUDY

In this chapter, I first present the specific research questions for the study. I then describe the overall design of the study including the participants, data collection and data analysis methods.

4.1. Specific research questions

The study addresses the following questions: 1) How is the discourse on limits generated by the instructor in a beginning college-level calculus classroom? 2) Given the instructor's discourse on limits how do students talk about limits in a beginning college-level calculus course? and 3) How do the elements of discourse on limits as generated over history compare and contrast with the discourse on limits generated in a beginning-level calculus course?

4.2. The participants

Participants for this study consisted of one calculus instructor and his section of undergraduate students who were taking a beginning-level calculus course in a large Midwestern University. The course addresses the fundamental concepts of calculus such as limits, differentiation and integrals. The course is fast-paced and loaded in terms of the number of topics covered. It is structured to focus more on the concepts and their applications than proofs.

While selecting the classroom to observe, I initially formed a complete list of instructors who were teaching the course in Spring 09. I then selected the instructors whose teaching schedules enabled me to observe their classrooms. I sent five instructors, who were randomly selected, an e-mail that briefly described my research interests and asked whether they would like to participate in my study. One instructor responded to the e-mail and expressed his willingness to let me observe his classroom. He wanted to learn how I planned to conduct my study so we met before the beginning of the semester. During that meeting, he gave me some information about the syllabus, the textbook and the students enrolled in his classroom. I told him I was interested in both the teaching and learning of limits. Initially, I suggested only audio-taping the class not to disturb the flow of the lessons. However, the instructor suggested video-taping. I informed him about the diagnostic survey I wanted to give students at the end of the unit on limits. He provided a schedule for finishing the discussion on limits and suggested giving the surveys at the end of the last lesson, a review session for the first exam. During the meeting, the instructor also mentioned he planned to go over the formal definition of limit[1] as well as proofs of some basic theorems and facts about limit in the classroom to motivate the students who might take higher-level calculus classes in the future.

There were 31 students registered to the instructor's section. During the period of classroom observations, the number of students attending the class ranged between 17 and 23. The class was very diverse in terms of the majors of students. Table 4.1 shows the distribution of students across their majors. There were 18 first-year; nine second-year; three third-year students and one fourth-year student enrolled in the class. The whole section was asked to take a diagnostic survey at the end of the unit on limits.

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[1] I did not give the instructor any directives about what to teach and did not in any way intervene in his teaching method or topics covered.

Table 4.1: Distribution of students across their majors

Department	Number of students	Department	Number of students
Computer engineering	2	Statistics	1
Electrical engineering	1	Accounting	1
Civil engineering	2	Finance	4
Engineering (no preference)	2	Biochemistry and molecular biology	2
Computer Science	2	GBA-Prelaw	1
Premedical	3	Marketing	2
Residential College	1	Economics	2
Mathematics	2	Ecological sciences and agriculture	1
Physics	1	Asian language	1

Based on the responses given to the survey, I interviewed four students to further explore their discourse on limits[2]. Three of the students who participated in the individual interview session had not taken calculus before; this was the first time they were introduced to the limit concept. One student took a calculus class during high-school. Having prior knowledge about limits was not necessarily problematic for the study since students' familiarity with colloquial or some of the literate aspects of limits was anticipated, though not required. The focus of the empirical part of the study was, given their previous knowledge and also how the notion was introduced in the classroom, how students worked on and talked about particular limit problems.

4.3. Data collection

The primary sources of data for this study consisted of field notes as well as video tapes that were taken during the classroom observations; responses to a diagnostic survey given to students; and task-based interviews including students' written work. The textbook students used in their class (*Thomas' Calculus*, 11th edition) and informal discussions with the instructor with respect to his mediation of students' use of the

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2 How the students were recruited will be discussed in Section 4.3.

textbook (e.g., assigned readings or homework problems) were considered as supplementary data.

The data for the classroom observations were collected in eight days over a period of two and a half weeks[3]. I video-taped the classes in which the instructor talked about limits and continuity starting from the first week of Spring 09 semester. Overall, the observation data consisted of eight 50-minute lessons. While video-taping, I only focused on the instructor and did not video-tape the students. I took field notes during classroom observations and used them to keep track of the number of students attending the class as well as the questions they asked to the instructor. I also used the field notes to keep a record of definitions and problems the instructor presented in the class. My role in the classroom was a participant observer. I did not interfere with the flow of the class during instruction. However, I helped some students before and after class if they asked me some questions about limits. Since I wanted to interview some of the students at the end of the unit on limits, providing such help was useful to establish a relationship with them and get ideas about their difficulties about limits. I transcribed the video-taped lessons both with respect to what the instructor said and what he did in the classroom. Therefore, for the analysis of the instructor's discourse, my data consisted of video-tapes of eight lessons and their transcripts as well as the field notes taken during classroom observations.

On the last day of the classroom observations, I gave all students a diagnostic survey. That lesson was the review session for the exam and by that time, the instructor

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[3] The class met three times a week.

had completed his discussions on limits and continuity. He gave me 10 minutes at the end of the class to administer the survey to the students. There were 23 students in the class that day and all of them agreed to take the survey. I used the diagnostic survey (a) to select the students for an individual interview session, and (b) to analyze student responses with respect to the instructor's discourse[4]. The questions in the diagnostic survey (See Appendix A) were taken from Williams (2001) since his classification of views related to limit is widely endorsed by research on student learning about limit. The first question of the survey included six statements about limits and asked students to decide whether the statements were true or false. Each of the six statements in the survey was related to a different view of limit, some of which were difficulties addressed by research on student learning. The second question then asked them which of the six statements best described their understanding of limits. The third question asked students to describe what they understood a limit to be. The final question asked students to give a rigorous (formal) definition of limit, if possible.

While selecting the students for interview sessions, I focused on their responses to the first and the second questions of the survey. I initially grouped all students with respect to the six statements they chose as best for the second question. I planned to interview one student for each of the six different views of limit they chose as best describing how they thought about limits. Since I wanted variety in terms of students' views of limit, I then recorded the number of correct responses students in each group gave for Question 1. Table 4.2 is a cross tabulation of questions 1 and 2, and shows the

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4 In this section, I only talk about how the students for the interview session were selected. The details of the analysis of the diagnostic survey in regard to the instructor's discourse will be discussed in Section 4.4.

classification of responses for the selection of students for the interview session. Its use is explained below.

Table 4.2: Classification of student responses for Question 1 and Question 2 of the survey

Number of correct responses given to Question1	Number of students selecting the statements as best in Question 2. Total number of students = 23						
	I. Statement1 (N=11)	II. Statement2 (N=0)	III. Statement3 (N=1)	IV. Statement4 (N=6)	V. Statement5 (N=1)	VI. Statement6 (N=3)	VII. "None" (N=1)
All correct	2	0	0	0	0	0	0
5 correct	5	0	1	0	0	1	0
4 correct	4	0	0	0	0	1	0
3 correct	0	0	0	2	1	0	1
2 correct	0	0	0	3	0	1	0
1 correct	0	0	0	1	0	0	0
0 correct	0	0	0	0	0	0	0

Legend: shaded cell = Initial emailing; boxed/hash-marked cell = Final selection

As an example, Table 4.2 shows that of the 11 students who selected statement 1 ("A limit describes how a function moves as x moves toward a certain point") as the best definition of limit, two categorized correctly all 6 statements in question 1, while 4 categorized four correctly and two incorrectly. I initially e-mailed six students from Columns I, III, IV, V, VI and VII, respectively[5] (See Table 4.2). The students I e-mailed are from the shaded cell in each column. The final selection of students is shown with hash-marks in the relevant cells. The selection process proceeded as follows.

In the e-mails I briefly described my study, provided some information about the interview sessions and offered a $25 gift card from a bookstore for their participation. Only one student, from Column III, agreed to be interviewed. The other students did not want to participate (participation was voluntary). I then sent another set of e-mails to six

[5] Note that none of the students chose Statement 2 as best.

students: two students with four correct answers from Column I, two students with three correct answers from Column IV, and the two remaining students from Column VI (See Table 4.2). Only the student from Column VI with five correct responses agreed to be interviewed. At that point, I had one student selected from Column III and one from Column VI. The students who were in Columns V and VII did not want to participate and there was no student in Column II.

I repeated the procedure again, this time e-mailing all of the remaining students from Columns I and IV. One of the students from Column I with five correct responses agreed to participate in the interview session. At that stage, I had three students who considered three different statements related to limits as best describing their realization of limit. On the other hand, the students' responses to the six statements were mostly accurate[6]. In order to find candidates who would likely have many of the difficulties indicated by the literature, I went to the classroom and looked for volunteers. Together with the instructor, we encouraged students to participate since the experience could contribute to their learning. As a further motivation (besides the gift card), I also suggested tutoring the students at their convenience (such as before the exam). Only one student agreed to participate. She was from Column I and had five correct responses to Question 1 (See Table 4.2). I wrote my e-mail on the board and asked students to contact me in case they wanted to be interviewed but did not receive any response.

Getting students' acceptance to volunteer for the study lasted for about two and a half weeks. In the end, I decided to go with the four who volunteered. Unfortunately, I could not recruit the students with many incorrect responses to the survey to participate in the study. However, the interview sessions revealed that some of the four students who seemed to have grasped the idea behind limits had many of the difficulties indicated by research when they worked on tasks that targeted those obstacles[7].

The questions in the interview session were designed to probe students' realizations of limits and also investigate in further detail the conceptual obstacles addressed by research on learning about limits. For the interview sessions, I initially formed a pool of problems, which consisted of ten questions. Some of those questions were taken directly from research on student learning on limits and limit related concepts; the others I developed considering the instructor's discourse on limits. For example, the instructor told me he planned to go over the formal definition so I added problems to the pool that were about the formal view of limit. By the time I interviewed the students, I reduced the number of questions to six (See Appendix B). There were two reasons for the elimination of those tasks: (a) It was unlikely for students to work on all of the problems in about an hour, and (b) given the instructor's discourse in the class, some of the problems turned out to be redundant or completely unfamiliar to students. For example, one of the questions I eliminated was about computing limits of a variety of functions represented algebraically. The instructor did not only go through similar examples in the class but he also assigned these types of problems as homework. Therefore, it seemed that this problem was redundant since it was going to assess students' computational

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[6] It should be noted the term "accuracy" is considered here only in terms of students' responses to the six statements in the diagnostic survey and does not necessarily imply students' realizations of limits were accurate in general.

[7] The details of the student discourse on limits will be discussed in Chapter VI.

skills more than their mathematical reasoning. Another problem I eliminated was taken from research on student learning about infinity and required familiarity with some early underpinnings of sequences. Since the instructor did not talk about infinity as a separate concept in the class and since he did not give any elementary examples of sequences, I concluded students would be too unfamiliar with the question. The other two questions I eliminated were different versions of similar problems that were already in the remaining six questions.

I conducted the interviews with the participants individually over a one week period. The interviews lasted between 53 and 76 minutes. I audio-taped the interviews and transcribed them with respect to what the students said and did. In order to keep track of what the students did, I took notes during the interviews as students worked on the problems. Besides the transcriptions, interview data also consisted of students' written work. To sum up, for the analysis of students' discourse on limits, my data consisted of responses to a diagnostic survey as well as four audio-taped interview sessions, transcripts of the sessions and written student work.

4.4. Data analysis

4.4.1. Analysis of the instructor's discourse

I used the transcribed classroom observations to analyze the discourse of the instructor with respect to the four elements of discourse from the commognitive framework: word use, visual mediators, routines, and narratives. During the process of transcription, I created a list of common words the instructor used and categorized them as signifying functions, infinity, infinitesimals, limits, motion and proximity. When the transcription of the eight lessons was completed in terms of what the instructor said and what he did, I color-coded the words I initially identified depending on their categorization. Every category was coded with a different color in the transcribed text. The words were color-coded based only on their mathematical use in the class. For example, I coded the word *at* as signifying proximity in the instructor's discourse (e.g., "f is continuous if you can find the limit *at a* by plugging in") but I did not code it when he used it in a colloquial manner (e.g., "now, let us look *at* this example").

The next step was to identify the utterances in which the instructor talked about limits. In order to do that, I first pulled out all the sentences including the word "limit", which was already color-coded, from the transcripts of the video tapes. However, there were also utterances about limit in which the instructor did not explicitly utter the word "limit". In some of the cases where the instructor did not utter the word "limit", he described the behavior of function values as the x values approached the limit point. In some others, he referred to a previously mentioned limit (e.g., "It [the limit] does not exist"; "L [the limit] is equal to three"; "The answer [the limit] is one."). I considered such utterances as related to limits as well. An utterance about limit was formed by a sentence, part of a sentence or multiple sentences that conveyed a particular idea about limit that could be interpreted just by reading. About 85% of the 775 utterances I identified as related to limits consisted of a single sentence. The remaining utterances either consisted of less than (about 2%) or more than (about 13%) a whole sentence depending on the purposes of the study. The only context in which an utterance consisted of some part of a sentence was when the instructor attended to multiple limit notations in a single sentence. During the transcription process, I noted that the instructor used a variety of words when he attended to the limit notation although he and the textbook suggested using a single word, namely "approaches" when reading the notation. The existence of a family of words associated with the same notation in the instructor's discourse led me consider each instance in which he referred to the limit notation as a distinct utterance in order to look for the patterns in his word use. In these cases, an utterance could be less than one sentence if the instructor addressed more than one limit notation. For example, the following explanation was considered as consisting of three utterances since the instructor referred to three different limit notations.

> "If the limit as *x approaches c* from the right is *L* [first utterance about $\lim_{x \to c^+} f(x) = L$] and the limit as *x approaches c* from the left is *K* [second utterance about $\lim_{x \to c^-} f(x) = K$], and if *K* is not equal to *L*, then the limit as *x approaches c* of *f* of *x* does not exist [third utterance about $\lim_{x \to c} f(x) = \text{does not exist}$]".

The only contexts in which an utterance was considered as consisting of more than one sentence were (a) when the instructor described the behavior of the function values in relation to the *x* values, (b) when he asked what a given limit was equal to and immediately gave the answer following his question, and (c) when the idea the instructor communicated about limits could be understood only together with the sentences preceding his conclusive statement about limits. While identifying an utterance in these contexts, I looked whether the utterance conveyed a complete idea about limits. Each of the following examples was considered as a single utterance about limits although they consisted of two sentences:

Example 1: "So really what this [the function] is trying to do, it is approaching a number very very small. A small negative or a small positive".

Example 2: "What is the limit of *f* of *x* as *x* approaches six? You get an undefined".

Example 3: "As *x* approaches, say *c*, of the function one. What is that limit?"

Example 4: "When x approaches one, what do the function values do? They get closer to two".

Example 5: "What I want to do is to talk about another tool that is useful for computing limits. This is called the sandwich theorem".

Example 6: "What is our conclusion that we want here? Here is our conclusion: *f* of *x* minus *L* [the limit] is less than epsilon".

Once the utterances about limit were identified, they were coded into four categories with respect to the degree of objectification: colloquial, operational, objectified, and both operational and objectified[8]. The identification of these categories in word use was based on the commognitive framework (degree of objectification) (See Chapter III). Colloquial word use referred to talking about limits in everyday sense. Operational word use referred to talking about limits as a process, whereas objectified word use referred to talking about limit as an end result of the limit process or as a number. There were also utterances in which the instructor talked about limits both in an

[8] The details of these categories will be further discussed in the word use section of the chapter on instructor's discourse.

operational and objectified manner. These four categories (colloquial, operational, objectified and both operational and objectified) and color-coding were used to explore the patterns of word use in the instructor's discourse. The exploration of patterns in the instructor's word use was then used to identify the contexts in which he altered his word use from one categorization to the other or where his word use remained consistent, (e.g., strictly objectified) without any shifts. As a result, there were four contexts in which the instructor's word use was analyzed in detail: (a) informal definition of limit, (b) formal definition of limit, (c) computing limits, and (d) continuity.

For the analysis of visual mediators, an inventory of all the visual mediators the instructor used was created from the transcripts, which also included the snapshots of everything he wrote and drew on the board. Those mediators were then classified in four categories: written words; drawn pictures of geometric shapes; graphs; and symbolic representation.

Routines correspond to the set of metarules that describe repetitive actions of the discursants (Sfard, 2008). Note that the repetitive nature of routines requires them to be applied consistently in similar situations. Therefore, not all actions count as routines unless they are consistently used in analogous contexts. For the investigation of routines, I mainly focused on what the instructor did[9] throughout the eight lessons. There were many possible routines that could be elaborated on over the entire observation period (e.g., assigning homework at the end of the class). For the purposes of the study, however, only the routines emerging from the transcripts that were most relevant to the

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[9] Note that the transcripts of the instructor's discourse included both his words and actions in the classroom.

analyses of the word use and visual mediators were reported. Some of the routines in the instructor's discourse consisted of repeated mathematical procedures (algebra-based routines) he utilized in the classroom in regard to limits. Some other routines, such as graphing, were repeated actions emerging also from the analysis of word use and visual mediators. The main focus during the analysis of routines was to identify the *when* and the *how* of a routine. *How* of a routine can be thought of as the "*course of action* or *procedure*", whereas *when* of a routine refers to the instances "in which the discursants would deem this performance as appropriate" (Sfard, 2008, p. 208, italics in original). The *when* of a routine embodies the applicability and closure conditions (Sfard, 2008). Applicability conditions enable the exploration of the situations that trigger the application of a particular routine. Closure conditions characterize the circumstances under which a performer considers her routine as successfully completed. Since routines are used to substantiate mathematical narratives, closure conditions mark the end of a routine and are followed by the closing statements after the implementation the routine.

Narrative is "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures" (Sfard, 2008, p. 134, italics in original). It was mentioned in Section 2.2.2 that "a metarule in one mathematical discourse will give rise to an object-level rule as soon as the present metadiscourse turns into a full-fledged part of the mathematics itself" (Sfard, 2008, p. 202). Object-level rules, once endorsed by the community, form the object-level narratives of mathematical discourse and are known as mathematical facts. The meta-level narratives, on the other hand, characterize the metarules related to the object-level narratives. For the instructor's discourse, most of the endorsed narratives were object-level in the form of a definition, theorem or rule about limits. For example, he endorsed the following object-level narrative about the limit of the sum of two functions:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, where L and M are real numbers, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

In this study, the instructor's object-level narratives were not reported because this would result in restating all the facts about limits that are widely endorsed by the mathematical community that can also be found in a beginning calculus textbook. Instead, the focus was on the meta-level narratives that were most relevant to the instructor's word use, visual mediators and routines in the classroom (e.g., "limit is a number", and "limit is a process", etc.).

4.4.2. Analysis of the diagnostic survey and student interviews with respect to the instructor's discourse

The diagnostic survey taken from Williams (2001) was used to select students for the interview sessions and to gain information on students' discourse on limits at the end of their lessons (See Appendix A). How the survey was used to select students for the interviews was discussed in Section 4.3. Here, I talk about the analysis of the survey with respect to the instructor's discourse.

Recall that the first question of the survey consisted of six statements about limits students chose as true or false. In the second question, students picked one of those statements as best describing their view of limit. For the third question, they provided their own definitions of what a limit is. For the fourth and final question, they provided a rigorous (formal) definition of limit.

According to Williams (1991), the six statements in the first question of the survey correspond to six views about limit. He categorizes these views as dynamic-theoretical; boundary; formal; unreachable; approximation; and dynamic-practical, respectively. The statements describing limit as dynamic-theoretical, formal, and dynamic-practical (Statements 1, 3, and 6, respectively) give general information whether the students' realizations of limit is based on motion or not. The statements describing limit as boundary, unreachable, and approximation (Statements 2, 4, 5, respectively), on the other hand, are used whether students have some of the difficulties identified by research on student learning about limits (See Section 2.1.2, Table 2.1).

For the analysis of the first two questions of the survey, I initially recorded the responses students gave for each of these statements as true or false. I then made a cross-comparison of the responses with respect to the statements students chose as true. For example, I looked at how many of the students marking Statement 1 as true also marked Statements 2, 3, 4, 5, 6 as true. The purpose here was to look at the range of responses as well as how and whether students considered different views of limit as related to each other. I then made a cross-comparison of the statements students chose as true in regard to the statements students chose as best describing their views of limit. For example, I looked at how many of the students marking Statement 1 as true considered it as the best statement; how many of those students considered Statement 2 as the best statement, etc. The purpose of this part was to gain information about the views that dominated students' realizations of limit and what other views of limit were connected to those realizations. I then analyzed students' responses for each of the six statements (some of which were difficulties addressed by research) in relation to the instructor's discourse. The purpose of this was to examine how instructor's word use, visual mediators, routines, and endorsed narratives compared and contrasted with the view of limit indicated by the statements.

For the third and fourth questions, I assigned numbers corresponding to each student's response. I also included which of the six statements students chose as best were closest to their descriptions. I then explored student responses in relation to their word use, visual mediators and endorsed narratives[10]. While doing so, I investigated whether students referred to limit as a process (operational word use) or as an end-state (objectified word use). I also looked whether students' word use was dynamic (based on motion) or static (based on proximity by means of distance). If students' word use was operational, they endorsed the narrative "limit is a process"; if their word use was objectified, they endorsed the narrative "limit is a number". In case students used any visual mediators in their descriptions, such as the limit notation, I focused on their word use when talking about the notation and whether it was based on dynamic or static word use. I then compared and contrasted students' word use, visual mediators and endorsed narratives with the instructor's. In cases instructor's word use when talking about a visual mediator was reflected in students' discourse, I also examined how the instructor used that visual mediator (such as graphs or symbolic notation).

Missing in the diagnostic survey were student difficulties such as "limit implies continuity", "limit as the function's value", and "limit as monotonic" (See Section 2.1.2, Table 2.1). The interview questions were designed to address all of the difficulties mentioned by research and also provide more information about students' discourse on

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[10] The diagnostic survey was not a context in which students' routines could be analyzed since such analysis requires the exploration of repetitive patterns in a discourse. In order for such patterns to emerge,

limits. I analyzed students' discourse in the interviews, again, in terms of (a) elements of the instructor's discourse on limits (word use, visual mediators, routines, and endorsed narratives), and (b) the difficulties indicated by research on student learning about limits.

While analyzing student responses in relation to the elements of the instructor's discourse, I focused on five mathematical contexts that emerged from the analysis of the instructor's discourse in relation to his word use. Those contexts were: (a) dynamic aspect of limits, (b) formal aspect of limits, (c) limit notation, (d) infinity, and (d) continuity. I used the transcripts of what the students said and did to explore each instance of talk about these five contexts. I then compiled all the utterances for each student in regard to these contexts in a separate document. The document also included each student's routines (e.g., graphing, plugging in the limit point to the function, etc) and visual mediators (e.g., graphs and symbolic notation) that emerged from the transcripts. I then looked at the general characteristics of each student's word use (degree of objectification as well as use of dynamic and static vocabulary), visual mediators, routines, and endorsed narratives. I then compared and contrasted these four elements of students' discourse with those of the instructor's.

While analyzing students' discourse with respect to research on learning about limits, I looked for the instances in the interview transcripts where students showed signs of having the difficulties mentioned in the literature (See Section 2.1.2, Table 2.1). I compiled each student's utterances in such instances in another document, categorizing them with respect to the six difficulties identified by the literature (See Table 2.1).

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students' discourse over a period of time needs to be observed in multiple contexts. This element of students' discourse was investigated during the interview sessions.

After this, I focused on the similarities and differences between the elements of students' and instructor's discourse and explored how and whether the instructor's discourse was reflected in students' discourse.

4.4.3. Analysis of the classroom discourse with respect to the historical development of limits

In Chapter III, the historical development of infinity, infinitesimals and limit was described with a focus on particular elements of the commognitive framework: word use (objectification), metarules and metaphors. In the same chapter, I also identified the historical junctures in the development of discourse on these concepts that resulted in changes in the metarules in order to extend it further (See Section 3.4). One of the goals of this study was to explore whether the historical development of limits through the commognitive lens could help us gain more information about student learning in today's calculus classrooms. In order to address this question, I examined the instructor's and students' discourse on limits in relation to the historical development of limit related concepts.

While analyzing the instructor's discourse with respect to the development of discourse on limits over history, I compared the ordering and introduction of topics related to limits with the historical development of the related concepts. Besides this, the focus of the analysis of instructor's discourse in terms of the historical development was to compare and contrast the word use (objectification), metarules and metaphors in the instructor's discourse with those in the discourse on limits as generated over history.

While examining students' discourse in regard to the historical development of limit related concepts, I first explored the contexts in which the experts' conceptual obstacles were similar to or different from those of the students' in the study. I then focused on stu

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focused on students' realizations of the informal and the formal definition of limit as well as infinity and compared and contrasted the word use (objectification), metarules, and metaphors in students' discourse with those in the discourse on limits as generated over history. Finally, I looked at whether the historical junctures I identified through the commognitive framework were reflected in the instructor's and the students' discourse on limits.

In what follows, I will first present the findings of the analysis of the instructor's discourse on limits (Chapter V). Next, I will talk about students' discourse on limits at the end of their instruction (Chapter VI). Last, I will examine the classroom discourse in relation to the historical development of limit related concepts (Chapter VII).

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CHAPTER V

THE INSTRUCTOR'S DISCOURSE ON LIMITS

In this section, I first describe the general characteristics of the lessons taught by the instructor. I then analyze the instructor's discourse on limits with respect to word use, visual mediators, routines, and endorsed narratives using the commognitive framework. I use a pseudonym for the instructor and refer to him as Jason throughout the study.

A big portion of this chapter will be devoted to the analysis of word use. While exploring Jason's word use, I focus on the degree of objectification in his utterances on limits. The degrees of objectification in his word use are classified as colloquial, operational, objectified, and both operational and objectified. I then concentrate on particular mathematical contexts to elaborate further on Jason's word use. Those contexts are the informal and the formal definition of limit, computing limits (limits at a point, limits at infinity and infinite limits), and continuity.

The second element of Jason's discourse I attend to is the visual mediators he used in the class. I talk about four types of visual mediators: written words, drawn pictures of geometric shapes, graphs and symbolic notation. The discussion about the visual mediators is followed by the discussion of the routines (metarules that underlie the repetitive patterns) in Jason's discourse. I talk only about the routines that are most related to his word use and visual mediators: algebra-based routines, geometry-based routines, using continuous motion as a metaphor, and using discreteness as a metaphor. Finally, I describe the narratives that Jason endorsed in the classroom based on the analyses of the previous elements of his discourse.

Jason covered limits during eight 50-minute lessons, one of which was a review session for the exam. There were 31 students registered for the course. Jason's mode of teaching was lecture and there was little, if any, discussion among students. He started the limit chapter on the first day of the semester and did not review or cover functions and basic algebra. In the first lesson, he gave students information about homework, textbook, syllabus and also exams. He told students they did not have to know calculus for the course but they needed to have solid algebra knowledge. He mentioned this consistently throughout the following lessons. He put a lot of emphasis on homework, which he expected students to submit every week. He considered homework as the most important part of the class in order for students to keep the pace of the class and also get feedback on their work. He further noted that the exam problems would be directly from or slight variations of the homework problems.

Jason assigned the homework problems directly from the textbook (*Thomas' Calculus, 2008, 11th edition)*. He considered the textbook as the most important resource for the class. He mentioned that students could use either the 5th or the 11th edition of the book. He said that the media upgrade, which is the 11th edition, contained CDs that had lectures and mentioned this as possibly useful for some of the students. He also noted that he would sometimes follow the textbook very closely and sometimes he would deviate from it to give students extra examples. It should be noted, however, that Jason did not allow calculators for the exam; did not use any technology in the classroom and did not assign any problems that would require students to use technology as homework problems. Table 5.1 shows the topics of each of the eight lessons and the corresponding textbook section.

Table 5.1: Topic outline for Jason's lessons

Lesson number and date	Topics covered	Textbook section
Lesson1 01/12/09	• Introduction: Homework, exam, attendance, calculator policy; syllabus and textbook information • Rates of change and limits • Average rates of change (average speed) • Instantaneous rate of change (instantaneous speed) • Limit (as the instantaneous rate of change)	Section 2.1
Lesson2 01/14/09	• Formal definition of rate of change • Geometrical interpretation (slope, secant lines, tangent line) • Instantaneous rate of change • What is a limit? Informal/intuitive definition • Computing limits (polynomials, rational functions, constant functions) • Factor theorem for polynomials • Finding the limits from a given graph	Section 2.1
Lesson3 01/16/09	• Limit laws • Computing limits by using the limit laws • Rule (Theorem 2 in the book): You can find the limits of polynomials by plugging in. • (Theorem 3 in the book) We can find the limit of rational functions by plugging in as long as the denominator is not zero. • Some applications of the theorems: computing limits	Section 2.2
Lesson4 01/21/09	• Long division (initiated from a homework problem) • Sandwich theorem • An application of the sandwich theorem: computing the limit of a function sandwiched in between two functions • The precise/formal definition of a limit • One proof problem f(x)=2x-1; a=2; L=3 (algebraic) • Geometric explanation of the proof problem	Section 2.2 and Section 2.3
Lesson5 01/23/09	• Refining the concept of limit and more about the sandwich theorem • Proof using sandwich theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ • Computing other limits derived from combinations of $\frac{\sin\theta}{\theta}$ • Finding limits from graphs of functions (as an introduction to two-sided limits) • Theorem (Theorem 6 in the book) If the right hand limit and the left hand limit exist at a point, then the limit exists at that point. • Limits at infinity • Computing limits at infinity (rational functions) • Horizontal asymptotes (definition and some applications in the form of computing limits of functions at infinity)	Section 2.4

Table 5.1 cont'd: Topic outline for Jason's lessons

Lesson number and date	Topics covered	Textbook section
Lesson6 01/26/09	• Homework problem about one-sided limits that include absolute value • Horizontal asymptote applications: finding horizontal asymptotes of given functions • Oblique/slanted asymptotes (Long division of polynomials) • Graphing functions focusing on their behavior at positive and negative infinity; right-hand-side and left-hand-side of a vertical asymptote (Book's terminology: dominant terms) • Vertical asymptote (definition and applications)	Section 2.4 and Section 2.5
Lesson7 01/28/09	• Continuous functions • Intuitive definition of continuous functions • Mathematical definition of continuous functions • Examples of continuous functions • Showing that a function is continuous (two examples) • Properties of continuous functions • Theorem (Example 6 in the book): Polynomials are continuous. Rational functions are continuous when the denominator isn't equal to zero. –without proof • Theorem(Theorem 10 in the book): Continuity of a composite function-without proof • Fact(not in the book): Sin and Cos are continuous-without proof • Continuous extension/maximal continuous extension examples	Section 2.6
Lesson8 01/30/09	• Intermediate value theorem (won't be on the exam)-theoretical definition and an example • Computing limits of various functions (rational, trigonometric, involving absolute value, limits at infinity) • Graphing a rational function using dominant terms (oblique asymptote, vertical asymptote –computing these limits) • Continuous extension/maximal continuous extension of a rational function and a trigonometric function • Discussion: Can a function have an oblique asymptote and a horizontal asymptote at the same time?	Review session

5.1. Word use

The words Jason used when talking about limit of a function mainly consisted of the words related to motion, proximity, infinitesimals, infinity, and continuity. Figure 5.1 shows the list of common words Jason used in his discourse on limits and the related notions those words signify. The list includes the words that Jason used most frequently when referring to limit related concepts and was generated during the process of transcribing the video-taped classroom sessions.

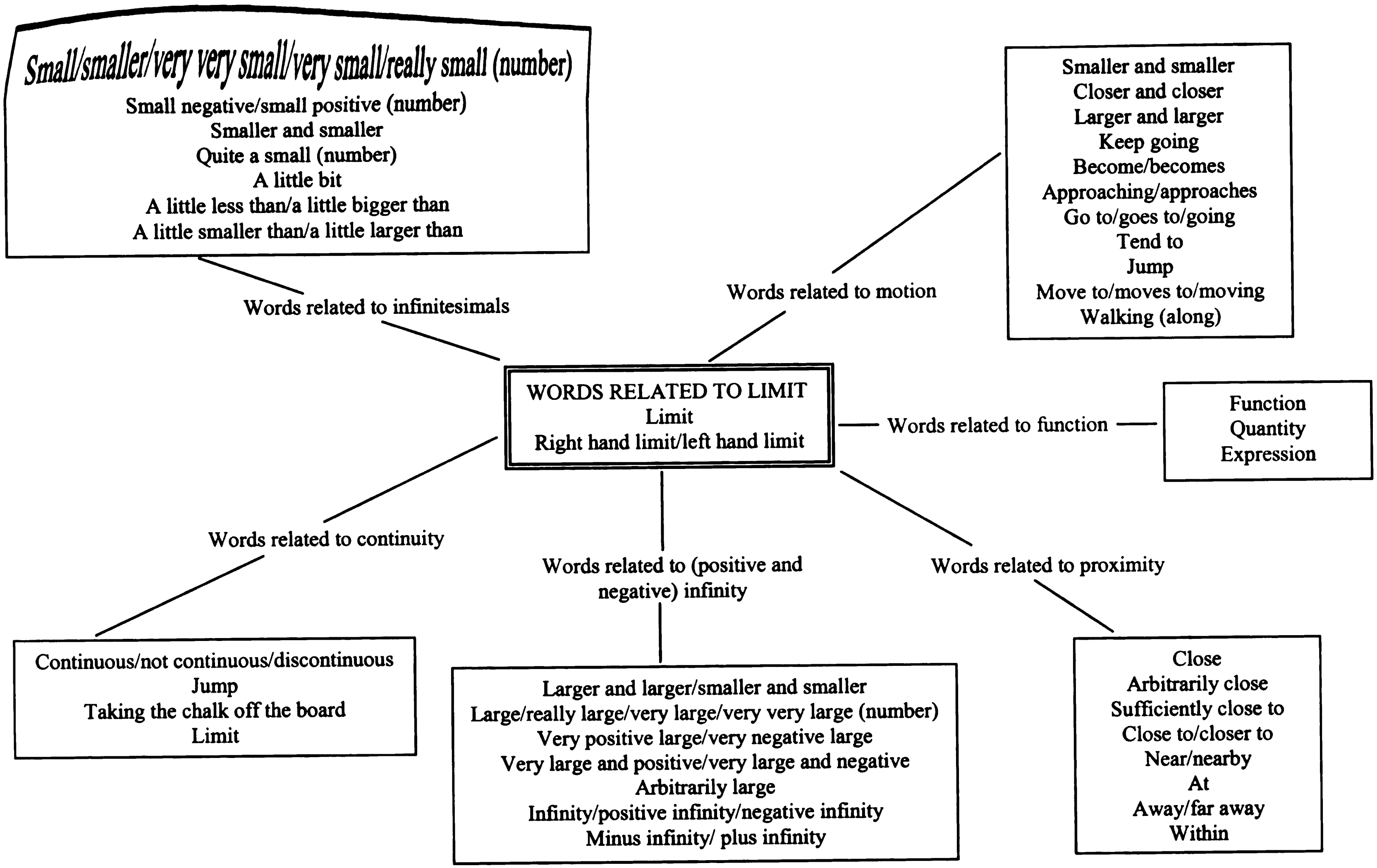

Figure 5.1: Common words used in Jason's discourse on limits

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The analysis of Jason's discourse revealed that the degree of objectification in his discourse varied depending on the context of word use. His utterances were classified as colloquial, (mathematical) operational and (mathematical) objectified. Jason's utterances were considered colloquial if he used limits in everyday sense; operational if he referred to limit as a process and based his arguments mainly on dynamic motion; and objectified (or structural) if he referred to limit as a number, that is, as a distinct mathematical object obtained at the end of the limiting process.

The classification of a particular utterance as operational or objectified turned out to be complex in the context of reading the notation $\lim_{x \to a} f(x) = L$ [1]. Mathematically, we read $\lim_{x \to a} f(x) = L$ as "the function *f(x) approaches the limit L as x approaches a*" (Hughes-Hallett et al., 2008; Thomas et al., 2008). In this respect, verbalizing $x \to a$ using the word *approaches* is a part of the objectified literate discourse on limits and although the word *approaches* signifies motion, its use in this context is not necessarily based on motion; it is how we read the notation. Therefore, Jason's utterances when reading the notation in cases where he used the word *approaches* were classified as objectified if he referred to L as a number at the end of the notation. His use of the word *approaches* was classified as operational when he used it outside the context of reading the limit notation. His utterances when reading the notation were classified as operational if: (a) he used different words that signify motion for verbalizing $x \to a$, for example, "as x gets closer and closer to a"; "as x becomes larger and larger" or "as x becomes a",

[1] This will be discussed more in detail in the section about visual mediators.

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and (b) he used the word "*approaches*" while reading $x \to a$ but did not refer L as a number that is obtained at the end of the limiting process. Jason sometimes read $\lim_{x \to a} f(x) = L$ as "the limit of f of x at a is [the number] L". Such utterances were classified as objectified since they were not based on dynamic motion and the end product, which is the limit L, was used as a distinct number.

Overall, there were 775 utterances about limit throughout the eight lessons. Given this large number of utterances, it is not feasible to discuss every utterance. However, Jason's colloquial, operational and objectified uses of limit related words will be exemplified during the discussions that follow.

5.1.1 Colloquial word use

Jason used limits in the colloquial sense only twice and he did so in the first lesson where he talked about average and instantaneous rate of change. When addressing average rate of change in terms of speed, he asked "What is the speed limit on Grand River [Ave]? Probably twenty five miles per hour" (Jason, 12 January, 2009, Lesson 1). Later in the lesson, when he wanted to connect the notion of limit to instantaneous rate of change, he mentioned the title of the lesson and said "I am trying to get to the second word that we had in the title today. Rates of change…and the second word was limits". This was considered a colloquial use of the word limit since Jason referred to limit as a word he wrote on the board but not as a concept. Besides these, he did not use limits in everyday sense but used the term mathematically.

5.1.2 Operational word use

Operational use of limits results from the consideration of limit as a process, which is consistent with the dynamic view of limit based on continuous motion. In most of these cases, Jason did not use the word limit but described the behavior of function values (*f(x)* approaching *L*) as *x* approaches a. In this respect, he referred to the limiting process instead of referring to limit as a fixed value obtained from that process. While doing so, he frequently used words that signify dynamic motion (see Figure 5.1, words related to motion). 127 of Jason's 775 utterances about limit were classified as operational.

Operational word use contains two elements: use of words that signify motion and description of the process of obtaining the limit rather than referring to the limit as a number. Table 5.2 shows some examples of Jason's operational word use. Jason explicitly mentioned once that the mathematical way to deal with the "process of getting closer and closer" (See Table 5.2, [1])[2] is limits. In the context of continuity, while determining the continuity of a function at a particular point from a given graph, he looked at the function values on the left hand side and the right hand side of the limit point and compared them with the function's actual value at the limit point. He referred to this as a "limiting process" (Table 5.2, [16]). In these utterances, Jason used the word limit but referred to it as a process rather than the end result of the process.

When Jason used words operationally, he referred to $x \to a$ using words that signify motion such as "*x* get(s) closer and closer to" (Table 5.2, [3], [4]); "*x* goes to" (Table 5.2, [7], [13]); "*x* gets/becomes smaller and smaller/larger and larger" (Table 5.2, [11], [14]); and "*x* becomes very very large" (Table 5.2, [15]). He also mentioned "walking on" (Table 5.2, [12]) and "moving along" (Table 5.2, [17]) the *x*-axis while determining the related function values in his discussions about particular limits.

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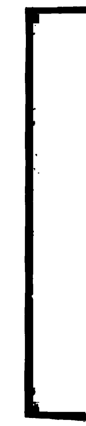
Table 5.2: Examples of Jason's operational word use

	Utterance	Context of use
[1]	First of all, this **process of getting closer and closer**, mathematically the way you want to deal with this is **limits**.	Instantaneous rate of change
[2]	When x is zero point nine nine nine, then f of x is one point nine nine nine; and when x is one point zero zero zero zero zero zero zero one, then we get two point zero zero zero, very close to two.	$\lim_{x\to 1} \frac{x^2-1}{x-1}$
[3]	We have some kind of function and what we are looking for is what happens to this expression here when x gets **closer and closer to** three.	$\lim_{x\to 3} \frac{3x^2-1}{x}$
[4]	If I get **closer and closer**, as x gets **closer and closer to** five, this quantity gets **closer and closer to** six.	$\lim_{x\to 5} x+1$
[5]	If we are less than six, the function tries to get **smaller and smaller** [referring to negative infinity].	$\lim_{x\to 6} f(x)$
[6]	If we are a little bigger than six, it tries to get **larger and larger** [referring to positive infinity].	$\lim_{x\to 6} f(x)$
[7]	So as x **goes to** infinity, the numerator **goes to** two.	$\lim_{x\to\infty} \frac{2x^2-3}{7x+4}$
[8]	And if x is smaller than one, it will look like it will **tend to** negative infinity.	$\lim_{x\to 1} \frac{1}{x-1}$
[9]	So as x approaches one, it **becomes** really large if x is larger than one.	$\lim_{x\to 1} \frac{1}{x-1}$
[10]	So what are we doing here? Denominator **is going to** zero; how about the numerator? It **is going** twenty five minus five and it **approaches** twenty.	$\lim_{x\to 5} \frac{x^2-5}{x-5}$
[11]	So as I graph this, as x **becomes smaller and smaller** [referring to 0], one over **x becomes larger and larger.**	$\lim_{x\to 0} \sin(\frac{1}{x})$
[12]	So I just keep **walking on** this axis [referring to the x-axis] the function values, in this case, they get **closer and closer to** zero.	$\lim_{x\to\infty} \frac{1}{x}$
[13]	If x **goes to** infinity, then seven over x **approaches** zero.	$\lim_{x\to\infty} \frac{2x^2-3}{7x+4}$
[14]	Well, if we consider this, what happens as x **becomes larger and larger**?	$\lim_{x\to\infty} \frac{\sin x}{x}$
[15]	What I want to do is I want, just like here, try to describe what the function does as x **becomes** very very large.	$\lim_{x\to\infty} (2+\frac{\sin x}{x})$
[16]	If I start drawing like this and I **approach** say here [referring to the left hand side of the limit point] I have to make sure actually I can get to that point [the limit point]. So this here, drawing like this, you can think of it as a **limiting process**.	Continuity
[17]	I **move along** the x axis along my function values.	Continuity
[18]	So I want to end up, I want to get **closer and closer to** the function value at this point.	Continuity

[2] (Table 5.2, [1]) refers to utterance number 1 in Table 5.2.

There were cases in which Jason's word use was operational but did not necessarily include words signifying motion. For example, in [2] (See Table 5.2), although he didn't use a word signifying motion, he assigned successive values for x and investigated the behavior of the function at those points to obtain the limit, which is described as the dynamic view of limit in literature.

Another element that characterizes operational word use is whether an utterance describes the process of obtaining a limit or the end result of the process, in which case limit is a distinct value. If it is the former, the word use is operational, if it is the latter, the word use is objectified. For example, in the context of computing $\lim_{x\to 1}\frac{x^2-1}{x-1}$, Jason noted that the function values got "very close to two" (Table 5.2, [2]) but did not mention the limit as being *equal to* the number two in that utterance. Similarly, in the context of $\lim_{x\to 5} x+1$, he said "this quantity gets closer and closer to six" (Table 5.2, [4]) but this utterance does not consider the limit of the quantity as equal to six. In the context of finding $\lim_{x\to 6} f(x)$ from a given graph (see Figure 5.2), Jason talked about the function values as getting "smaller and smaller" [5] when x is less than six and as getting "larger and larger" [6] when x is greater than six. However, he did not objectify the process of getting "smaller and smaller" and "larger and larger" with negative infinity and positive infinity, respectively. Therefore, these two utterances describe the process of obtaining the limit rather than the end result of that process.



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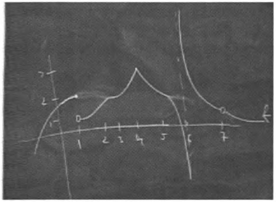

Figure 5.2: Jason's hand-written example for finding limits from a given graph

Symbolically, the operational word use seems to describe the processes of x values approaching a ($x \to a$) and the function values approaching the limit ($f(x) \to L$). In this respect, the limit L is approached but is not referred to as a distinct value that is obtained. Jason's utterances in which the function "get(s) closer and closer to" [4], [12], [18]; "becomes really large" [9]; "tends to" [8]; "is going to" [10]; "goes to" [7] and "approaches" [10], [13] highlight that his focus is on the process of the function values *approaching* L rather than the limit being *equal to* L.

5.1.3 Objectified word use

An utterance about limits is considered objectified if (a) the word *limit* is uttered explicitly to signify a mathematical object or a number that is obtained at the end of the limiting process or (b) the word limit is not explicitly uttered but the behavior of the function around the limit point is described through words that signify proximity or distance, which is consistent with the formal definition of limits. Throughout the eight lessons, 634 of Jason's 775 utterances about limit were classified as objectified. Table 5.3 shows some examples of Jason's objectified word use on limits.

In Jason's objectified discourse, limits were particular mathematical objects that could be found (See Table 5.3, [6], [19]) and computed (Table 5.3, [4], [6], [13], [23]) having some "properties" (Table 5.3, [3]) and "rules" (Table 5.3, [7]). Moreover, they could also help define other mathematical concepts such as continuity (Table 5.3, [20]).

Table 5.3: Examples of Jason's objectified word use

	Utterance	Context of use
[1]	What does it mean that **the limit** of the function as x approaches x zero **is L?**	$\lim_{x \to a} f(x) = L$
[2]	**This limit is one.**	$\lim_{x \to 2} \frac{1}{x-1}$
[3]	So what I want to do is to look at some **properties** of limits.	Limit laws
[4]	I want to **compute limits** from **limits** that I know already.	Limit laws
[5]	x approaches say c of the function one... **What is that limit? One.**	$\lim_{x \to c} 1$
[6]	If p of x is a polynomial, then you can **find the limits** by plugging in.	$\lim_{x \to c} P(x)$
[7]	...we can use our **limit rules** and **compute a number** and we know the **limit exists.**	When does a limit not exist?
[8]	...and suppose I know that the **limit** as x approaches c of h of x **is L.**	$\lim_{x \to c} h(x) = L$
[9]	Well if the function values of f **get arbitrarily close to L** as long as x is **sufficiently close to** x zero.	Informal definition of limit
[10]	This means x is **sufficiently close to** x zero and this means f of x is **arbitrarily close to L.**	Formal definition of limit
[11]	We want to say that the function values should be no further than epsilon **away** from **the limit.**	Formal definition of limit
[12]	The limit...the function values should be **close to the limit.**	$\lvert f(x) - L \rvert < \varepsilon$
[13]	You want to **compute the limit** as x approaches two of f of x and that should be **equal to three.**	$\lim_{x \to 2} f(x) = 3$
[14]	I say I want to be **within** one over two hundred **close to three** [the limit], how **close** do we have to be to two in order to insure that?	$\lim_{x \to 2} f(x) = 3$
[15]	What is the **limit** as x approaches one of the function? **That is one.**	$\lim_{x \to 1} f(x)$
[16]	So if the **left hand** and the **right hand limit** exists but are not equal, then the **limit does not exist.**	Right hand and left hand limits
[17]	So the horizontal asymptote just simply means that **the limit at infinity exists and equals a number b**...	Horizontal asymptote
[18]	Infinity is not really a number. So technically, **this limit doesn't exist.**	$\lim_{x \to 0^+} \frac{1}{x}$
[19]	Let's look at this. f is continuous if you can **find the limit** at a by plugging in.	$\lim_{x \to a} f(x) = f(a)$
[20]	Well the definition of a continuous function is given in terms of the **limit.**	Continuous functions
[21]	**What is this limit?** It's one; we have **computed** this.	$\lim_{x \to 0} \frac{\sin x}{x}$
[22]	So the **limit** x approaches two from the left... What is the **answer**? **Negative two.**	$\lim_{x \to 2^-} \frac{2x-4}{\lvert x-2 \rvert}$
[23]	If I need to **compute limits,** all I need to know is what the function does **near** when x is equal to zero.	$\lim_{x \to 0} \frac{(2+x)^2 - 4}{x}$

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Jason's consideration of limit as an object was also apparent when he referred to limits as the "answer" of a limiting process (Table 5.3, [22]) and when he uttered phrases like "what is this limit?" (Table 5.3, [5], [21]) Once objectified, the limit also leads to other mathematical objects such as "the left hand and the right hand limit" (Table 5.3, [16]) that could again be computed.

Jason frequently talked about limits being equal to a number in his objectified discourse (Table 5.3, [2], [5], [13], [15], [17]). He noted that the limit "exists" if it is equal to a number (Table 5.3, [7], [17]); it "does not exist" if the right hand limit is not equal to the left hand limit (Table 5.3, [16]) or the limit is equal to infinity (Table 5.3, [18]). The instances where Jason read the limit notation (Table 5.3, [1], [5], [8], [15]) were also considered a part of his objectified discourse as long as he referred to limit as an end product (a number or infinity) of the limiting process.

When talking about limits in an objectified manner, Jason's words that signify motion gave way to words signifying proximity in terms of distance (see Figure 5.1, words related to proximity). When describing the behavior of the function around the limit point, he started talking about the function values being "arbitrarily close to L (the limit)" as the x values are "sufficiently close to" (Table 5.3, [9], [10]) or "near" x_0 (Table 5.3, [23]). Jason also mentioned the function values being "no further than epsilon away from the limit" (Table 5.3, [11]); "close to the limit" (Table 5.3, [12]) or "within one over two hundred (epsilon) close to three (the limit value)" (Table 5.3, [14]) instead of using words signifying motion such as "closer and closer to" (See Table 5.2, [2], [3]) or "become" (Table 5.2, [14]).

The only case when Jason used a word signifying motion in his objectified discourse on limits was when he referred to the limit notation. In these cases, he used the word "approaches" (Table 5.3 [1], [5], [8], [15]) while reading notations of the form $\lim_{x \to a} f(x) = L$. It was mentioned before that his use of the word *approaches* was considered as objectified since this was how Jason and the textbook for the course described how to read the limit notation. Moreover, he talked about the limit as a number when he read the notation whereas he did not explicitly utter the word "limit" when he used *approaches* in an operational manner.

5.1.4 Operational and objectified word use

There were twelve instances where Jason's utterances about limits were classified as both operational and objectified. In such cases, Jason used a combination of words that signify motion, which is consistent with the operational use of limit, together with words that signify limit as objectified (See Table 5.4).

For example, he mentioned that if the function values get "closer and closer to some number L, then we call that the limit" (Table 5.4, [1]). Here, he used "closer and closer to", which he explicitly identified as a process before (see Table 5.2, [1]), together with the end result of that process - "the limit". On the other hand, in this utterance, the limit value is not obtained; the function values get "closer and closer" to it. Except this, all his utterances that were categorized as both operational and objectified include the word "goes to" (Table 5.4, [2-12]) in which he used the word limit as a distinct mathematical object or a number but considered $x \to a$ as a process. For instance, in [7] (See Table 5.4), Jason explicitly uttered the word limit and stated two as the answer of the limit problem. However, he used the word "goes to" instead of "approaches" while reading the limit notation.

Table 5.4: Examples of Jason's operational and objectified word use

	Utterance	Context of use
[1]	There was a little if in our definition if you read it carefully, and it says if it gets **closer and closer to some number L**, then we call that **the limit**. If it doesn't, we say it is not defined.	Informal definition of limit
[2]	Well, if we have some sort of expression or function, we can try to **take a limit**, say when x **goes to** zero	What is a limit?
[3]	Now I want to **compute the limit** as theta **goes** to zero.	$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$
[4]	Let's **compute the limits** as theta **goes** to zero of those two outside functions here.	$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$
[5]	So what is **the limit** as x **goes** to infinity of the numerator?	$\lim_{x \to \infty} \frac{5x^2 - 3x + 1}{2x^2 - 5}$
[6]	Well, how do we do it? What is a horizontal asymptote? We say that this function has a horizontal asymptote y equals b if **the limit** as x **goes** to plus infinity or as x **goes** to negative infinity of this quantity is b.	Horizontal asymptote $f(x) = 2 + \frac{\sin x}{x}$
[7]	What is **the limit** of two as x **goes** to infinity? Two.	$\lim_{x \to \infty} (2 + \frac{\sin x}{x})$
[8]	The **limit** of sine x over x as x **goes** to infinity? One.	$\lim_{x \to \infty} (2 + \frac{\sin x}{x})$
[9]	I converted a limit from negative infinity to a **limit** where t **goes** to infinity.	$2 + \lim_{t \to \infty} \frac{\sin(-t)}{-t}$
[10]	So the **limit**, to make it precise here, as x **goes** to infinity is zero of the denominator.	$\lim_{x \to \infty} \frac{2x^2 - 3}{7x + 4}$
[11]	How about the **limit** as x **goes** to negative infinity of two x squared minus x plus one divided by three x plus x squared minus five.	$\lim_{x \to -\infty} \frac{2x^2 - x + 1}{3x + x^2 - 5}$
[12]	Because in order to have an oblique asymptote, **the limit** as **goes** to infinity is what? It is plus or minus infinity	Oblique asymptote

At first, it seems that "goes to" and "approaches" are similar phrases that can be used while reading the limit notation. However, the textbook and Jason mentioned the notation $\lim_{x \to a} f(x) = L$ is read as *the limit as x approaches a is L* and eight of the eleven utterances in which Jason used "goes to" took place in the context of reading the limit notation when *x* approached infinity. These observations suggest that Jason did not use these two phrases as synonyms. Jason used "goes to" only three times in his utterances about limit at a point (Table 5.4, [2-4]). In contrast, he used the phrase eight times when he talked about limits at infinity. Such operational word use was also consistent with his overall utterances in the context of limit at infinity and is likely to result from Jason's consideration of infinity as *potential* infinity instead of *actual* infinity[3].

Table 5.5 and Figure 5.3 summarize the categorization of Jason's utterances throughout the eight lessons. Table 5.5 shows the count and percentage for each of the four categories across all the lessons. Figure 5.3 excludes the utterances in Lesson 1, and thus all the colloquial utterances, because of the sparseness of limit-related discourse in Lesson 1.

Table 5.5: Categorization of Jason's limit related utterances in four categories

	Lesson number	L1	L2	L3	L4	L5	L6	L7	L8	**Total**
Limit related utterances	Colloquial	2	0	0	0	0	0	0	0	**2**
	Operational	3	21	15	2	20	47	3	16	**127**
	Objectified	3	44	117	87	125	107	87	64	**634**
	Operational and objectified	0	1	1	0	3	5	0	2	**12**
	Total	**8**	**66**	**133**	**89**	**148**	**159**	**90**	**82**	**775**
Percentages	Colloquial	25.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	**0.26**
	Operational	37.50	31.82	11.28	2.25	13.51	29.56	3.33	19.51	**16.39**
	Objectified	37.50	66.67	87.97	97.75	84.46	67.30	96.67	78.05	**81.81**
	Operational and objectified	0.00	1.52	0.75	0.00	2.03	3.14	0.00	2.44	**1.55**

3 This will be discussed more in detail in Section 1.5.3.2 (limit at infinity) and Section 1.5.3.3 (infinite limits).

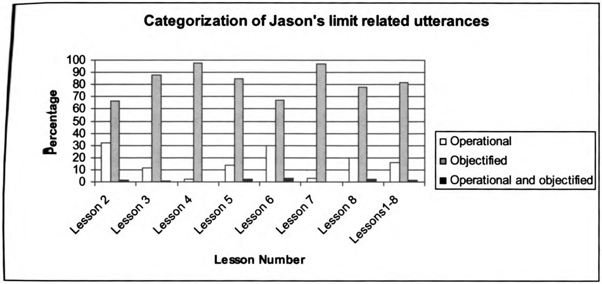

Figure 5.3: Categorization of Jason's limit related utterances as operational, objectified or both in lessons 2-8.

5.1.5 Mathematical context of use

It is important to note that Jason often used a combination of operational and objectified utterances in the same mathematical context. In this section, some of his word use in different mathematical contexts will be described in further detail. For the sake of simplicity, I use "context" rather than the longer "mathematical context" in the remainder of this section. The contexts that are fundamental to the realizations of limit were identified as: the informal definition of limit; the formal definition of limit; computing a limit (at a point and at infinity); infinite limits; and continuity. The purposes of this section can be summarized as follows:

- To exemplify the existence of shifts in word use (objectified, operational and/or both) in Jason's discourse in the same context.
- To describe Jason's informal and formal definition of limit and to explore the differences in his word use in these contexts.

- To describe Jason's word use when he talked about limits at infinity and infinite limits and to draw attention to his going back and forth between potential and actual infinity in these contexts.
- To explore Jason's use of infinitesimal related words in these contexts.
- To describe Jason's informal and precise definition of continuity and to explore the differences in his word use in these contexts.

5.1.5.1 Informal definition of limit

Jason introduced the informal definition of limit in the second lesson, right after his discussions of average and instantaneous rate of change. He initially defined instantaneous rate of change as the limit of average rate of changes "over smaller and smaller intervals". He then defined what a limit is and gave the informal definition of the concept (See Table 5.6 and Figure 5.4). Although his discussions of instantaneous rate of change contained words that signify infinitesimals, such as "smaller and smaller" or "very small" intervals, he did not use infinitesimals in the informal definition of limit.

During his definition of the informal aspect of limit, Jason referred to the behavior of the function both in an operational and an objectified manner. He said that the function values "should get closer and closer to L" as the x values get "closer and closer to" x_0 (Table 5.6, [4]), which describes the limiting process by means of continuous motion. Note that although Jason verbally mentioned this process, what he wrote on the board was the objectified version of the function's behavior: "we say that the limit is L" (Table 5.6, [8]) if the function values get "arbitrarily close to L for all x sufficiently close to x_0" (Table 5.6, [6]).

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Table 5.6: Jason's utterances about the informal definition of limit

What is said	What is done	Type of utterance
[1] What is a limit?	He writes "f is defined on an interval about x_0" on the board and starts drawing a graph.	Objectified
a) x zero is here. I have some function and I want to make sure it is defined at least near x zero.		
b) So if I take a small interval here, this function is defined.	He draws an open interval around x_0.	
c) I don't care what it does somewhere away from x zero.	He shows the point on the left of x_0 where the function has an asymptotic behavior.	
d) So we don't ask it to be defined at x zero but at least nearby.	He writes "except possibly at x_0" on the board.	
[2] Then we say as x approaches x zero of the function f of x equals some number L if...(does not finish his sentence)	He writes $\lim_{x \to x_0} f(x)$ and then pauses.	Objectified
[3] What does it mean that the limit of the function as x approaches x zero is L?	He shows the notation and turns back to the graph he drew.	Objectified
[4] It means that the function value, if I get closer and closer to x zero, it should approach some number L.	He puts L on the y-axis in the graph he drew (See Figure 5.4).	Operational
[5] It should get closer and closer to L.		Operational
[6] I want to say it gets arbitrarily close to L for all x sufficiently close to x zero.	He writes these on the board (See Figure 5.4).	Objectified
[7] So this is what I would want to call the intuitive definition of a limit.		Objectified
[8] So we say the limit is L if I can make the function values to be arbitrarily close to L if I choose my values of x sufficiently close to x zero.		Objectified

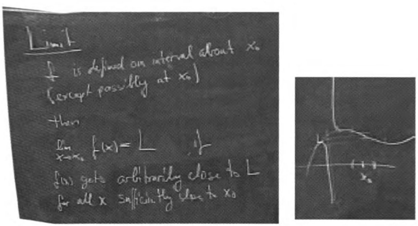

Figure 5.4: Jason's informal definition of limit (hand-written)

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In such utterances, he did not base his arguments on continuous motion but on proximity through words that signify distance, which is consistent with the formal definition of limit. Therefore, it seems that Jason used elements from both definitions to introduce the limit notion.

Only two of his eight utterances about the informal definition of limit were operational. However, his utterances following this definition about evaluating the limits of particular functions and finding limits from a given graph (see Table 5.1 for the topic outline for this lesson) were often operational. In fact, Lesson 2 is the lesson that contains the highest proportion of operational utterances throughout the eight lessons (See Table 5.5 and Figure 5.3).

5.1.5.2 Formal definition of limit

Jason introduced the formal definition of limit in the fourth lesson and also worked on an example where he proved $\lim_{x \to 2} 2x - 1 = 3$. Before talking about the formal definition, he provided the rationale for the need for a precise definition of a limit (see Table 5.7). Jason mentioned that the ways they computed limits "at least convinced us what these limits are" (Table 5.7, [4]) but noted that they would need a "precise definition" in order to make sure those techniques work (Table 5.7, [5-5a]). After this, he said that students were going to have some homework problems about the formal definition but the topic was not going to be in the exam. He wanted students to consider this as a challenge and encouraged them to try to do the related homework problems. He also told students at the end of the class that the homework problems about the precise definition would not be graded. In fact, he mentioned this definition being very abstract while working on the proof example.

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Table 5.7: The rationale for the precise definition of limit

	What is said	What is done	Type of utterance
[1]	I want to spend the rest of the time in this class today by looking at the formal definition of the limit…		Objectified
[2]	So what I want to do here is I want to write down the precise definition of a limit.	He writes “precise definition of limit” on the board.	Objectified
[3]	Why would we need such a thing as precise definition of a limit?		Objectified
[4]	Well, we kind of said what a limit is in words and it helped us to at least convince us what these limits are that we have been computing in class, say of polynomials or things like this.		Objectified
[5]	It kind of made sense those laws of limits that I wrote down but to really make sure these work in mathematics, we have to prove those things.		Objectified
a)	In order to prove something, we need a precise definition…		
[6]	So that is why we need the precise definition of a limit.	All his comments up to here are verbal.	Objectified

Table 5.8 shows Jason’s utterances about the formal definition of limit (See also Figure 5.5).

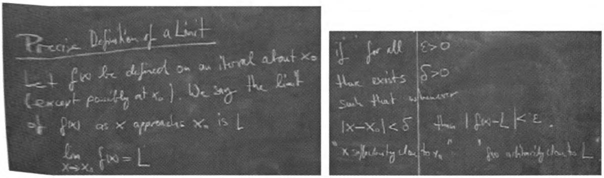

Figure 5.5: Jason's precise definition of limit (hand-written)

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Table 5.8: Jason's precise definition of limit

	What is said	What is done	Type of utterance
[7]	So how do we do a precise definition of a limit?		Objectified
[8]	We need a function f and we need it to be defined in…my words that I said before when we used for the loose definition of the limit I said *near* x zero.	He emphasizes the word "near".	Objectified
	a) Let's make this maybe more precise. On an interval about x zero except possibly for x zero.	He writes these on the board.	
[9]	So we have seen before limits are very interesting if our function is not defined at the point.		Objectified
[10]	Then we say that the limit of f of x as x approaches x zero is L.	He writes these on the board.	Objectified
[11]	And let's see in math language how do I write this? I say the limit as x approaches x zero of f of x equals L.	He writes these symbolically on the board (See Figure 5.5).	Objectified
[12]	We say that is the case if…in my words before I said well if the function values of f are arbitrarily close to L as long as x is sufficiently close to x zero.	He emphasizes the word "sufficiently".	Objectified
[13]	So a little more formally, in math language, I say if for all epsilon greater than zero, this is going to play the role of measuring how close we are to L in our function values, there exists a delta greater than zero, the delta is going to play the role to measure saying that we are *sufficiently* close to x zero.	He writes $\lvert x - x_0 \rvert < \delta$ and then $\lvert f(x) - L \rvert < \varepsilon$ on the board.	Objectified
[14]	So let's put that in words… whenever x is sufficiently close to x zero so that means that the difference is no more than delta then the function values should be close to L.	He writes "sufficiently close to x_0" under $\lvert x - x_0 \rvert < \delta$ on the board.	Objectified
[15]	So how do we write this? The difference of the function values from L should be less than epsilon.	He writes "f(x) arbitrarily close to L" under $\lvert f(x) - L \rvert < \varepsilon$ on the board. (See Figure 5.5)	Objectified
[16]	Now this looks somewhat complicated. So let me maybe decipher this. This means x is sufficiently close to x zero and this means f of x is arbitrarily close to L.	He first shows $\lvert x - x_0 \rvert < \delta$ and then $\lvert f(x) - L \rvert < \varepsilon$	Objectified
[17]	For any epsilon, I should be able to do this and for any one of them then there exists this delta.	He shows $\delta > 0$	Objectified
[18]	That means if x is sufficiently close, delta close to x zero, then the limit…the function values should be close to the limit.	He first shows $\lvert x - x_0 \rvert < \delta$ and then $\lvert f(x) - L \rvert < \varepsilon$	Objectified
[19]	So let's maybe do one example and prove that a limit exists in this way.		Objectified

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Overall, there were 19 utterances about the formal definition of limit and all of them were categorized as objectified. The words Jason used when talking about the precise definition of limit differed from those he used when talking about the informal definition in two aspects. First, his words signifying motion were replaced with words that signify proximity. Second, the presence of symbolism in the formal definition required Jason to associate symbolic representations with particular words. In what follows, these two aspects will be described in more detail.

Jason's words about the informal, or in his words "intuitive" (Table 5.6, [7]) and "loose" (Table 5.8, [8]), definition of limit addressed the limiting process and were based on continuous motion. Utterances in which the function values "approach" or "get closer and closer to *L*" (Table 5.6, [4], [5]) as the *x* values "get closer and closer to x_0" (Table 5.6, [4]) are examples of Jason's operational word use when he talked about the informal definition of limit. In contrast, his words about the formal or "precise" (Table 5.8, [7]) definition addressed proximity and were based on distance measured by means of absolute values. He mentioned that ε would "play the role of measuring how close we are to L in our function values" and δ would "play the role to measure saying that we are sufficiently close to x zero" (Table 5.8, [13]). He also talked about the closeness of the *x* values to x_0; and the *f(x)* values to *L* in terms of the "difference" $|x - x_0|$ being no bigger than δ [14] and the "difference" $|f(x) - L|$ being less than ε [15], respectively. In summary, the operational terminology such as getting closer and closer to (Table 5.2, [1]), approaching (Table 5.2, [10]), becoming (Table 5.2, [9]) and going (Table 5.2, [7]) were rep

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were replaced by being "sufficiently close" (Table 5.8, [13], [14], [16], [18]), "arbitrarily close" (Table 5.8, [12], [16]) and "close" (Table 5.8, [18]) in the formal definition[4].

Indeed, Lesson 4 stands out as the lesson which had the highest proportion of objectified utterances about limits (see Figure 5.3). This is in contrast with Lesson 2, where Jason introduced the informal definition of limit and worked on some examples, which had the highest proportion of operational utterances about limits. This signals a difference in word use in these two contexts. However, this shift in vocabulary was not addressed by Jason in the classroom. Instead, he tried to connect these definitions when he mentioned $|x - x_0| < \delta$ can be read as "x is sufficiently close to x_0" and $|f(x) - L| < \varepsilon$ can be read as "f(x) is arbitrarily close to L" (Table 5.8, [16] and Table 5.6, [8]). This brings us to the second aspect of the word use in the formal definition.

The difference in word use between the informal and the formal language also resulted from the translation of symbolic representations to words. Jason went back and forth between the "math language" (Table 5.8, [11], [13]) and "words" (Table 5.8, [12], [14]) more frequently while discussing the formal definition than the informal definition. The informal definition contains $\lim_{x \to x_0} f(x) = L$, which is expressed as "the limit of the function as x approaches x zero is L" (Table 5.6, [3]) in words. The formal definition, on the other hand, also requires the explanation of what ε and δ refer to and what $|x - x_0| < \delta$ and $|f(x) - L| < \varepsilon$ mean. Jason considered this translation as complicated and

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4 Jason used *sufficiently close* and *arbitrarily close* in the informal definition of limit as well (Figure 5.4) but his word use was not consistent with this terminology when he worked on examples following the informal definition. When describing the behavior of the function in those examples, he often referred to the limiting process using words signifying motion as illustrated in Table 5.2.

therefore wanted to "decipher" (Table 5.8, [16]) the formal definition by using elements of the informal definition.

Jason did not utter any infinitesimals related word in the context of the formal definition of limit.

5.1.5.3 Computing a limit

Throughout the eight lessons, Jason computed 64 different limits. Among those, 43 were limits computed at a point; 10 were limits computed at infinity; and 11 were infinite limits. Infinite limits refer to the limits computed at a point or infinity whose result is positive or negative infinity rather than a real number L. The reason infinite limits were considered as a distinct category of computing limits results from Jason's different word use between infinite limits and limits at infinity when referring to infinity.

It was mentioned before that, although Jason's word use about the informal and formal definition of limit was mostly objectified, his description of a function's behavior while computing limits was mainly operational. In what follows, his discourse in three different contexts will be examined in detail to both justify the existence of mixed utterance use and to explore Jason's discourse about infinity and infinitesimals.

5.1.5.3.1 Limit at a point

The majority of the limit computation problems Jason worked on were computing the limit of a function at a given point. For the purposes of this study, the limits at a point for which the limit is equal to L, where L is a real number were considered separately from the limits at a point for which the limit is equal to plus or negative infinity (infinite limits) since Jason's word use differed significantly in these contexts.

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When addressing limit at a point, Jason often talked about limit as objectified, but his word use on the behavior of the function was primarily operational. Table 5.9 shows Jason's utterances when he computed $\lim_{x \to 2} \frac{1}{x-1}$. In this computation, Jason used a combination of objectified and operational utterances. He considered the limit of the function as a distinct number obtained at the end of the limiting process when he said "this limit is one" (Table 5.9, [5]).

Table 5.9: Jason's utterances in the context of computing $\lim_{x \to 2} \frac{1}{x-1}$.

What is said	What is done	Type of utterance
[1] What is this limit?	He writes $\lim_{x \to 2} \frac{1}{x-1}$ on the board. (Some students say it is one.)	Objectified
[2] Let's see. If x gets closer and closer to two, this quantity gets closer and closer to one over two minus one.	He shows $x \to 2$ and then shows $\frac{1}{x-1}$. He says these verbally and doesn't write anything on the board.	Operational
[3] It is very close to one over one.		Operational
[4] So the closer x gets to two, the closer this will get to one.	He shows $\frac{1}{x-1}$.	Operational
[5] This limit is one.	He writes 1 near the question; no graph is drawn.	Objectified

When he talked about the function's behavior near the limit point, on the other hand, he mainly used words signifying continuous motion and considered limit as a process. He referred to $x \to 2$ as x getting "closer" (Table 5.9, [4]) and "closer and closer" to two (Table 5.9, [2]) instead of the notational language of "x approaches two" (See Table 5.6, [2]). He also talked about $f(x) \to L$ in an operational manner when he said the function values get "closer" (Table 5.9, [4]) and "closer and closer to" (Table 5.9, [2]) one. Jason worked on many of the limit computation problems in similar ways, although he sometimes used different words signifying motion when describing the behavior of the functio

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function such as “becoming”, “moving”, and getting “smaller and smaller” (See Figure 5.1 and Table 5.2 for motion related/operational word use). What is not visible in the above example is his use of infinitesimal related words, which often took place when he talked about right hand and the left hand limits. Table 5.10 shows Jason’s utterances when he computed $\lim_{x \to 1} f(x)$ from a given graph.

Table 5.10: Jason’s utterances in the context of computing $\lim_{x \to 1} f(x)$ from a graph.

What is said	What is done	Type of utterance
[1] The limit as x approaches one of f of x. I claim it doesn’t exist.		Objectified
a) Why would that be? Let’s think about the definition.		
[2] What we want is that we get closer and closer to one, it should approach some number.	He shows x=1 on the graph.	Operational
[3] But if it is a little less than one, then it looks like we are getting close to two.	He puts a dot to the left hand side of x=1 (See Figure 5.6) and shows the point f(x)=2.	Operational
[4] If we are a little bigger than one, it looks like the function wants to be equal to one. a) It can’t make up its mind.	He puts a dot on the right hand side of x=1 and shows f(x)=1 on the graph	Objectified
[5] So there is no single number that this function gets closer and closer to.		Operational
[6] So what we say here, the limit is undefined.	He writes $\lim_{x \to 1} f(x) =$ undefined.	Objectified

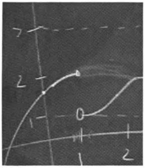

Figure 5.6: Computing limits from a given graph (hand-written)

Here, we see that Jason talked about the x values getting "closer and closer to" one (Table 5.10, [2]) and he looked at what number the function values were approaching (Table 5.10, [2]) or getting "closer and closer to" (Table 5.10, [5]). In this respect, he again used the dynamic view of limit when explaining how the function behaved near the limit point. Besides this, he drew two points very close to the left hand side and the right hand side of the limit point $x=1$ (See Figure 5.6) and mentioned being "a little less than one" [3] and "a little bigger than one" [4], respectively. This suggests infinitesimal related word use[5] since he used these phrases to refer to the points that are very close to the limit point, which are of very small proximity to the limit point. Jason consistently used infinitesimal related words when he talked about the right hand and the left hand limits at a given point. In the context of limit at a point for which the answer is a real number, Jason used infinitesimals for finding $\lim_{x \to 1} f(x)$ from a graph (See Table 5.10), when he made the transition from average rate of change to instantaneous rate of change in Lesson 1, and when he computed $\lim_{x \to 1} \frac{\sqrt{2x}(x-1)}{|x-1|}$ in Lesson 6. During his introduction to instantaneous rate of change, he mentioned the time intervals getting "smaller and smaller" and being "a very small number" (Jason, January 12, 2009, Lesson 1) when he referred to $h \to 0$ in $\frac{f(x+h)-f(h)}{h}$. When looking at the right hand and the left hand limits of the function $\frac{\sqrt{2x}(x-1)}{|x-1|}$ at 1, he mentioned the x values being "a little larger than one" and "a little smaller/less than one" (Jason, January 26, 2009, Lesson 6). He did not

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5 Infinitesimals refer to very small quantities that are often negligible (See Chapter III, Section 3.2).

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use infinitesimal related words in the context of limit at a point besides these instances. Further details about his infinitesimal related word use[6] will be discussed in the following sections about infinite limits and limits at infinity.

5.1.5.3.2 Limit at infinity

Limit at infinity is a limit computed at positive or negative infinity. Throughout the eight lessons, all limits Jason computed at infinity had a real number L as the answer except for one. For one infinite limit Jason computed, the answer was equal to infinity. This problem was categorized as "infinite limit" rather than "limit at infinity" because how Jason referred to infinity changed between these two contexts.

Jason introduced the notion in Lesson 5 under the title *limits at infinity*. After this, he started computing $\lim_{x \to \infty} \frac{1}{x}$ (See Table 5.11). Note that the title, limit *at* infinity (Figure 5.7 and Table 5.11 [1]) signifies that the distance from the x values to infinity is zero; that is, it treats infinity as a distinct entity that can be reached. This view of infinity is referred to as the actual infinity and was discussed previously in Chapter III. Reading part of the limit notation $x \to \infty$ as *x approaches infinity* (Table 5.11, [2]), however, talks about potential infinity that cannot be reached. The only reason this particular utterance was considered objectified is because both Jason and the textbook defined this as how to read the notation. Jason's consideration of infinity as potential in the context of limit at infinity, on the other hand, is not restricted to how he read the notation. He associated $x \to \infty$ with also the x values getting "larger and larger" (Table 5.11, [3]) and with

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[6] See Figure 5.1 for a complete list of infinitesimal related words in Jason's discourse.

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“walking on this axis [x-axis]” (Table 5.11, [4]) as he extended the x-axis he initially drew to represent infinity.

Table 5.11: Jason's utterances when computing $\lim_{x\to\infty} \frac{1}{x}$.

What is said	What is done	Type of utterance
[1] So now, I want to talk about some other types of limits. I can also look at limits at infinity.	He writes “limits at ∞”.	Objectified
a) So this horizontal eight, it means infinity.	He writes “infinity” near ∞.	
[2] What is the idea behind this? I want to say something like the limit as x approaches infinity of say the function one over x.	He writes $\lim_{x\to\infty} \frac{1}{x}$.	Objectified
a) What should that be? Does that make any sense? Let’s just graph the function one over x to get some idea.		
b) This is what the function one over x looks like if I graph it.	He starts drawing the graph of the function.	
[3] So, if x approaches infinity, what should that mean? That means if I take x and make x larger and larger…(does not finish his sentence)	He puts his hand on a point on the x axis and moves along the x axis towards the right with his hand.	Operational
[4] So I just keep walking on this axis, the function values, in this case, they get closer and closer to zero.	He extends the initial graph he drew towards positive infinity and also extends the x axis (See Figure 5.7). He then moves his hands along the y values on the graph and then writes 0 as the answer for the limit.	Operational
[5] So this [the limit], I want to say, is equal to zero.	He shows $\lim_{x\to\infty} \frac{1}{x}$.	Objectified

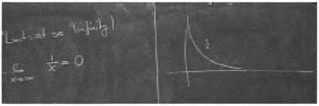

Figure 5.7: Jason's hand-drawn graph of $f(x) = \frac{1}{x}$

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These utterances are consistent with the view of infinity as potential: a *continuous* process that goes on indefinitely. His bodily gestures of extending his arms and moving along the function with his hands can also be thought of as further evidence of expressing the process of approaching infinity by dynamic means. (See Table 5.11, what is done)

In this example, Jason described the behavior of the function in dynamic and motion related words, which refers to limit as a process (Table 5.11, [4]). Yet, he also said that the limit "is equal to zero" (Table 5.11, [5]), which refers to limit as the result of that process. Therefore, we can still see his use of mixed utterances in the context of limits at infinity.

Overall, Jason worked on ten problems about computing the limit at infinity. Some of those were tied to the idea of finding the horizontal asymptotes of a given function. Table 5.12 shows the limits Jason computed at infinity and also how he talked about the x values approaching infinity in the limit notation.

Besides the word *approaches*, Jason frequently talked about $x \to \infty$ using the phrase *goes to* (Table 5.12, [8], [10], [13], [19-21]). Note that "going to" infinity invokes motion towards infinity, which is not reached, and is compatible with the view of infinity as potential. Moreover, Jason's other utterances in which the x values made/get/become larger and larger (Table 5.12, [2], [5], [16], [18]) further support this dynamic process of approaching infinity. On the other hand, Jason also used the word *at* when talking about $x \to \infty$ (Table 5.12, [6], [12], [15]), which is more compatible with the view of actual infinity (See Section 3.1) since it refers to infinity as a discrete point. In the context of limit at infinity, Jason uttered the word *number* referring to (negative) infinity only once, when he said "as x becomes larger and larger but a large negative number" (Table 5.12,

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[16]). Except for this instance in which Jason talked about infinity as a number and the instances he used the word *at*, he talked about infinity as a potentiality (process) rather than a distinct entity. Therefore, his word use about infinity was mainly operational than objectified.

Table 5.12: Jason's utterances about the x values when he computed limits at infinity

Limits at infinity			
Lesson Number	Context of use		How $x \to \infty$ is referred to
Lesson 5	$\lim_{x\to\infty} \frac{1}{x}$	[1] [2]	Limit as x **approaches** infinity... If I take x and make x **larger and larger**...
	$\lim_{x\to\infty} \frac{1}{x-5}$	[3]	Limit as x **approaches** infinity...
	$\lim_{x\to\infty} \frac{1}{x}+5$		No infinity related word is uttered.
	$\lim_{x\to\infty} 5$	[4] [5] [6]	Limit as x **approaches** infinity... ...when x gets **larger and larger**... ...the limit **at** infinity...
	$\lim_{x\to\infty} \frac{5x^2-3x+1}{2x^2-5}$	[7] [8]	Limit as x **approaches** infinity... Limit as x **goes to** infinity...
Lesson 6	$\lim_{x\to\infty} (2+\frac{\sin x}{x})$	[9] [10] [11] [12]	Limit as x **approaches** infinity... Limit as x **goes to** infinity... We know what it [the limit] is **at** infinity. Limit **at** positive infinity...
	$\lim_{x\to-\infty} (2+\frac{\sin x}{x})$	[13] [14] [15] [16]	Limit as x **goes to** negative infinity ... Limit as x **approaches** negative infinity... Limit **at** negative infinity... As x **becomes larger and larger** but a large negative **number**...
	$\lim_{x\to\infty} \frac{\sin x}{x}$	[17] [18] [19]	Limit as x **approaches** infinity ... ...as x **becomes larger and larger**... ...as x **goes to** infinity...
Lesson 8	$\lim_{x\to-\infty} \frac{2x^2-x+1}{3x+x^2-5}$	[20]	Limit as x **goes to** negative infinity...
	$\lim_{x\to\infty} \frac{x^2-1}{x+2}$	[21]	Limit as x **goes to** infinity...

In the context of limits at infinity, Jason used infinitesimals only once when he computed $\lim_{x\to\infty} \frac{\sin x}{x}$. Table 5.13 shows his utterances when he worked on this limit.

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Table 5.13: Jason's utterances when he computed $\lim_{x\to\infty}\frac{\sin x}{x}$

What is said	What is done	Type of utterance
[1] How about the limit as x approaches infinity of sine x over x?	He writes $\lim_{x\to\infty}\frac{\sin x}{x}$	Objectified
[2] Well, if we consider this, what happens as x becomes larger and larger?	He shows $\frac{\sin x}{x}$ when he says "this".	Operational
a) The sine is a number that stays between negative one and one.		
[3] The larger x gets, the smaller this will be for sure.	He shows $\frac{\sin x}{x}$ when he says "this".	Operational
[4] So this limit is zero.	He writes $\lim_{x\to\infty}\frac{\sin x}{x}=0$	Objectified

Here, Jason mentioned that as the *x* values "become larger and larger" [2], $\frac{\sin x}{x}$ gets "smaller" [3]. He used the word *smaller* to talk about small quantities that are very close to zero since he then said the limit of $\frac{\sin x}{x}$ would be zero [4]. It was mentioned in Chapter III that Cauchy defined infinitesimals as follows:

> When the successive absolute values of a variable decrease indefinitely in such a way as to become less then any given quantity, that variable becomes what is called an infinitesimal. Such a variable has zero for its limit (Kitcher, 1983, p.247).

Jason's utterance about the process of $\frac{\sin x}{x}$ getting "smaller" as the x values getting "larger" [3] clearly describes the function as decreasing indefinitely and having zero as its limit. Therefore, Jason used infinitesimals in this particular problem. He did not use any oth

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any other infinitesimal related utterance in the context of limit at infinity[7] where the limit is equal to a real number L.

5.1.5.3.3 Infinite limits

Infinite limit is a limit computed either at a point or infinity that has plus or minus infinity as the answer. Although Jason introduced vertical asymptotes in Lesson 6, he worked on some problems about infinite limits in the preceding lessons (See Table 5.14).

Overall, there were 11 cases in which Jason computed infinite limits. Table 5.14 shows how he referred to infinity and the limit in each of those cases. He mentioned function values being/getting/becoming a (very) large positive or negative number (Table 5.14, [5], [7], [15-17], [20]); and being/becoming an arbitrarily large (negative) number (Table 5.14, [11], [13]) when he talked about infinite limits. In this respect, he referred to infinity as a number more frequently than he did in the context of limits at infinity. When referring to infinity, Jason also mentioned the function values getting larger and larger/smaller and smaller (Table 5.14, [1-2]); becoming larger and larger (Table 5.14, [10]); becoming really/very/arbitrarily large (Table 5.14, [3], [6], [20-21]); getting large/really large (Table 5.14, [14-15]); tending to/going to/approaching to plus or negative infinity (Table 5.14, [4], [20-22]); becoming arbitrarily large (Table 5.14, [12]); and getting closer and closer to a line (slanted asymptote) (Table 5.14, [9]).

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[7] Jason also used infinitesimals when he worked on $\lim_{x\to\infty} \frac{2x^2-3}{7x+4}$ but this limit was considered as an infinite limit in this study.

Table 5

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Table 5.14: Jason's utterances about infinity in the context of infinite limits

Lesson Number	Context of use	Infinite limits: How $f(x) \to \infty$ is referred to	Infinite limits: How $\lim_{x \to a} f(x) = \pm\infty$ is referred to
Lesson 2	$\lim_{x \to 6} f(x)$ from a graph	[1] If we are a little bigger than six, it [the function] tries to get larger and larger [2] If we are less than six, the function tries to get smaller and smaller.	[23] The limit is not defined. [24] ...the limit does not have to exist.
Lesson 3	$\lim_{x \to 1} \frac{1}{x-1}$	[3] ...it [the function] becomes really large if x is larger than one. [4] ...and if x is smaller than one... it will tend to negative infinity.	[25] ... it just does not turn to any fixed number. So the limit is undefined.
	$\lim_{x \to 5} \frac{x^2-5}{x-5}$	[5] ... we could get a very large positive number if x is just a little bigger than five. [6] ...and if I divide this [twenty] by a very small number, I get something really large. So this [the function] will become very large. [7] ...choose x to be a little smaller than five... it [the function] will be a very large negative number.	[26] So this [the limit] is undefined.
Lesson 6	$\lim_{x \to \infty} \frac{2x^2-3}{7x+4}$	[8] The denominator? If x goes to infinity, then seven over x approaches zero. [9] It [the function] will get closer and closer to a line.	[27] So the limit...as x goes to infinity of the denominator is zero. [28] The limit [of the function] does not exist.
	$\lim_{x \to 0} \frac{1}{x}$	[10] ... one over x, becomes larger and larger. [11] It's an arbitrarily large number. [12] ...as x approaches zero from the right, f of x becomes arbitrarily large and is positive. [13] If we look to the left...it becomes an arbitrarily large negative number. [14] [The function] gets very large and positive or very large and negative.	[29] It [the limit] is undefined. [30] The [right hand] limit equals plus infinity. [31] Infinity is not really a number. So technically, this limit doesn't exist. [32] The [left hand] limit is just negative infinity.
	$\lim_{x \to 2} \frac{1}{x-2}$	[15] (approaching from the right of two)... we get a positive number that is very large.	[33] This [right hand] limit will be plus infinity or in other words, this quantity will get arbitrarily large. [34] It [left hand limit] is going to be negative infinity.
	$\lim_{x \to -\frac{4}{7}} \frac{\frac{-115}{49}}{7x+4}$	[16] We have a negative number, we divide it by a very small positive number, it makes it a very large negative number. [17] If x approaches negative four over seven from the left...I get a large positive number.	[35] This [right hand] will be negative infinity. [36] So this [left hand] limit in the end will be plus infinity.
	$\lim_{x \to 1^+} \frac{x+1}{x^2-1}$	[18] It [the function] is going to become very large; arbitrarily large	[37] So we say it [the limit] is plus infinity.

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Table 5.14 cont'd: Jason's utterances about infinity in the context of infinite limits

Infinite limits			
Lesson Number	Context of use	How $f(x) \to \infty$ is referred to	How $\lim_{x \to a} f(x) = \pm\infty$ is referred to
Lesson 7	$\lim_{x \to 0} \frac{1}{x}$	No infinity related word is uttered.	[38] The limit does not exist.
	$\lim_{x \to 1} \frac{x+1}{x^2 - 1}$	No infinity related word is uttered.	[39] So this [limit] does not exist. [40] At plus one, the limit does not exist.
Lesson 8	$\lim_{x \to -2} \frac{x^2 - 1}{x+2}$	[19] When x is close to negative two, this thing becomes really large. [20] So as the denominator gets a small negative number, this [the remainder] becomes a very large negative number. So it will approach negative infinity. [21] ...as x approaches two from the left, my function goes to negative infinity. [22] If I come negative two from the right, it goes to plus infinity.	[41] ...the limit doesn't exist. It is plus or minus infinity if I come from the left or the right. [42] So the answer [left hand limit] is just negative infinity. [43] Similarly here [right hand limit]... you will get plus infinity.

In all these instances where he described $f(x) \to \infty$, Jason's word use was operational since he was talking about limit as a process. His arguments about infinity were mainly based on potential infinity except possibly for the cases where he referred to it as a number. It should also be mentioned that in these cases, he rarely uttered the word *infinity* explicitly and when he did (Table 5.14, [4], [20-22]), the function values were considered to tend/go to/approach infinity, suggesting continuous motion towards infinity and therefore, potential infinity.

When talking about limit in an objectified manner, that is, as the result of the limiting process, we see Jason talking about the limit as being plus or negative infinity (Table 5.14, [30], [32-37], [42], [43]). Therefore, he uttered the word *infinity* more often when talking about the limit as the end of the limiting process. He was more likely to think about actual infinity rather than potential infinity in these instances where he mentioned limit as being plus or negative infinity since then infinity was signified as the end r

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end result of the process. This can further be supported with the fact that Jason wrote $\lim_{x \to a} f(x) = \pm\infty$ for each of these problems when representing the limit using the notation. By doing so, the limit was considered as *equal to* infinity but not as a process that went on and on as his earlier utterances suggested. Jason considered an infinite limit as "not defined" (Table 5.14, [23]); "undefined" (Table 5.14, [25-26, 29]); and "does not exist" (Table 5.14, [24], [28], [31], [38-39]). So for him, $\lim_{x \to a} f(x) = \pm\infty$ implied that the limit did not exist or was undefined. This was also clear when he said "infinity is not really a number. So technically, this limit does not exist" (Table 5.14, [31]). Note, however, that he used utterances that signified infinity as a large/very large/arbitrarily large number during his discussions on the behavior of the function values. Therefore, his word use about infinity was not consistent between the contexts in which he talked about the function values approaching infinity and the limit being equal to plus or negative infinity.

In summary, Jason used a combination of operational and objectified utterances in the same context when he addressed limits at infinity and infinite limits. Moreover, his consideration of infinity as operational (potential) and objectified (actual) depended on whether he was addressing the behavior of the x values as well as the function values or the limit of the function.

Infinite limits turned out to be the most important context for gaining information about Jason's word use on infinitesimals. It was mentioned before that Jason used infinitesima

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infinitesimals three times in the context of limit at a point[8] and once in the context of limits at infinity[9]. Except these, all remaining seven cases in which Jason used infinitesimal related words took place in the context of infinite limits. Table 5.15 shows Jason's utterances about infinitesimals in the context of infinite limits.

While working on infinite limits, Jason frequently mentioned "a very small (positive/negative) number (Table 5.15, [1], [5-7], [12], [14-17]). When talking about x values approaching the limit point from the right, he mentioned being "a little" or "slightly bigger than" the limit point (Table 5.15, [1], [4], [13]). When the talked about the x values approaching the limit point from the left, he mentioned being "smaller/a little smaller than" the limit point (Table 5.15, [2], [7]). He also said the function values could be "something very very small" (Table 5.15, [3]); a "small negative or small positive (number)" (Table 5.15, [3], [18]); "quite a small number" (Table 5.15, [9]); and could get "smaller and smaller" (Table 5.15, [10]).

When Jason used an infinitesimal related word for the function values or specific parts of a function such as the denominator, he considered the values to be very close to zero. For example, he said "This [the denominator] will be a very small number, say zero point zero zero zero one" (Table 5.15, [5]). Such quantities had zero for their limits. Jason mentioned this explicitly when he said "...dividing seven by a very large number is very small. The limit will be zero" (Table 5.15, [8]) and "So this limit is zero" (Table 5.13, [4]).

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[8] When he computed $\lim_{x \to 1} f(x)$ from a graph (See Table 5.10); when he computed $\lim_{x \to 1} \frac{\sqrt{2x}(x-1)}{|x-1|}$; and during the transition from average rate of change to instantaneous rate of change.

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Table 5.15: Jason's infinitesimal related word use in the context of infinite limits

Lesson Number	Context of use		Infinitesimals related utterances
Lesson 3	$\lim_{x \to 1} \frac{1}{x-1}$	[1]	Well, if x is a little larger than one, then this quantity is one over a very small positive number.
		[2]	... and if x is smaller than one, then this denominator here will be a really small negative number.
	$\lim_{x \to 5} \frac{x^2-5}{x-5}$	[3]	So really what this is trying to do, it is approaching to a number twenty over something very very small. Maybe a small negative or a small positive.
		[4]	... we could get a very large positive number if x is just a little bigger than five
		[5]	So x is a little bigger than five then what is the denominator? This will be a very small number, say zero point zero zero zero one.
		[6]	...and if I divide this [twenty] by a very small number, I get something really large.
		[7]	...choose x to be a little smaller than five... this [the denominator will be a very small negative number.
	$\lim_{x \to \infty} \frac{2x^2-3}{7x+4}$	[8]	Seven over x approaches zero because x becomes a larger and larger number, dividing seven by a very large number is very small. The limit will be zero.
		[9]	Similarly, here, four over x squared as x becomes larger and larger... that is quite a small number.
		[10]	It [$4/x^2$] gets smaller and smaller.
		[11]	When x is very large... this one [the remainder] will be very small.
	$\lim_{x \to 2} \frac{1}{x-2}$	[12]	What do we have in the denominator? We have a positive number that is very small.
Lesson 6	$\lim_{x \to -\frac{4}{7}} \frac{\frac{-115}{49}}{7x+4}$	[13]	If x approaches negative four over seven, but it is slightly bigger than that...
		[14]	So if the denominator is positive but very small number it means we divide by a very small positive number, the numerator is negative.
		[15]	We have a negative number, we divide it by a very small positive number, it makes it a very large negative number
		[16]	If x approaches negative four over seven from the left... I have a negative number, and I divide it by a very small negative number.
	$\lim_{x \to 1^+} \frac{x+1}{x^2-1}$	[17]	So this will approach a number that is two divided by a very small positive number.
Lesson 8	$\lim_{x \to -2} \frac{x^2-1}{x+2}$	[18]	So as the denominator gets a small negative number, this becomes a very large negative number.

Except for $\lim_{x \to \infty} \frac{2x^2-3}{7x+4}$, all infinitesimal related words uttered in the context of infinite limits took place when determining the function's behavior on the right hand side

[9] When he computed $\lim_{x \to \infty} \frac{\sin x}{x}$ (See Table 5.13).

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and the left hand side of the limit point. Jason did not use the terminology of *approaching from the left/right* until Lesson 6, where he introduced right hand and the left hand limits. Instead, he mentioned the x values being *a little larger/bigger than* or being *smaller/a little smaller than* the limit point (Table 5.15, [1-2], [5], [7]).

The analysis of word use also revealed that Jason used the same term, *getting smaller and smaller*, for both infinitesimal quantities that are negligibly small (whose limit is zero) and for negative infinity depending on the context. For example, in the context of computing $\lim_{x \to \infty} \frac{2x^2 - 3}{7x + 4}$, he said that , $\frac{4}{x^2}$, which he obtained after dividing every term by the highest power of x, "gets smaller and smaller" (Table 5.15, [10]). Here, Jason used the term to refer to an infinitesimal quantity since the limit of $\frac{4}{x^2}$ at infinity is equal to zero. Similarly, he referred to infinitesimals when he talked about the "time intervals getting smaller and smaller" (Jason, January 12, 2009, Lesson 1) in the context of instantaneous rate of change to talk about the h values approaching 0 when working on $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. In the context of computing limits from a graph (See Figure 5.2) where he talked about $\lim_{x \to 6} f(x)$, however, he also said "if we are a little less than six, the function tries to get smaller and smaller" (Table 5.14, [2]). Here, he used *smaller and smaller* to talk about the left hand limit of the function at the point 6, which is negative infinity. Besides this instance, he did not use *smaller and smaller* to signify negative infinity but used "a very large negative number" (Table 5.14, [5], [16], [20]) or "arbitrarily large negative number" (Table 5.14, [13]) instead.

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5.1.5.4 Continuity

Jason introduced continuity in Lesson 7 (See Table 5.1 for the topic outline). He mentioned that he was going to give one intuitive and one mathematical definition of the notion. He then told students that they would need to use the mathematical definition to answer questions in the exam.

Table 5.16: Jason's utterances about the intuitive definition of continuity

What is said	What is done[10]
[1] So what is a continuous function?	He writes continuous functions on the board.
[2] A continuous function is a function, and I am just going to say it in words, that I can graph without taking the chalk off the board.	
[3] So this is really all that we need to know, from an intuitive perspective, about continuous functions.	
[4] Continuous just means I can graph without taking the chalk off the board.	He states these verbally.

After giving students an intuitive definition of continuity, he drew graphs of arbitrary functions and discussed their continuity based on whether he could graph them "without taking the chalk off the board" (Table 5.16, [2], [4]). While doing so, he moved his hand along the graphs of the functions and mentioned that approaching and graphing like this could be thought as a "limiting process" (See Table 5.2, [16]). Therefore, although he did not utter any limit related word during his discussions of the intuitive definition of continuity, he verbally stated that he relied on the limiting process as well as whether he was taking the chalk off the board when determining the continuity of a given function from a graph. He then introduced the "precise" definition of continuity (See Table 5.17 and Figure 5.8).

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Table 5.17: Jason's utterances about the precise definition of continuity

	What is said	What is done	Type of utterance
[1]	So let's try to make this precise. A function f of x is continuous at a point x equals a if...	He writes these on the board but does not finish his sentence.	
[2]	So what I want to say is what does it mean to be continuous at this point?		
[3]	The limit has to exist and it has to equal to the function value.		Objectified
[4]	So I need the limit to exist.		Objectified
[5]	So the limit as x approaches a of f of x has to exist and equal to f of a.	He writes $\lim_{x \to a} f(x) = f(a)$ on the board (See Figure 5.8).	Objectified
[6]	Just writing this already implies that it [the limit] exists because it equals a number.		Objectified
[7]	So in particular, f is defined at a and the limit exists.	He writes these besides the definition in parenthesis.	Objectified

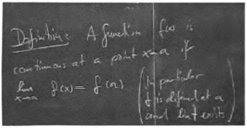

Figure 5.8: Jason's precise definition of continuity (hand-written)

Whereas Jason's informal definition of continuity was based on dynamic motion where limit was considered as a process, his precise definition of the notion considered limit as a number (Table 5.17, [6]), which is equal to the function value (Table 5.17, [3], [5]). Given this, in context of the precise[11] definition of continuity, his utterances about limit were mainly objectified. In fact, Lesson 7 was the lesson that had the second highest proportion of objectified utterances about limit (See Figure 5.3).

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[10] Note that Jason did not use any limit related utterance here. Therefore, the types of utterances were not reported.

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It should be noted that Jason did not talk about continuity in any of the lessons prior to Lesson 7 although his operational word use about limits relied on continuous motion. The precise definition of continuity was the first context in which he explicitly connected these two concepts. Later in the class, he also said

> How can I tell if a function is continuous or not? A question about continuity is a question about limits. If I ask to see if a function is continuous, we need to check that for each point a the limit exists and equals to the function value. (Jason, 28 January, 2009, Lesson 7)

After this, he talked about properties of continuous functions, which he noted would be the same as limit laws because "the definition of a continuous function is given in terms of the limit" (Jason, 28 January, 2009, Lesson 7).

Jason talked about plugging in[12] as one of the possible ways of computing a limit in the lessons prior to Lesson 7. In Lesson 7, however, he seemed to put extra emphasis on plugging in as a means of computing the limit of a continuous function. For example, while showing $f(x)=\dfrac{x+1}{x^2+1}$ is continuous, he showed for every number a,

$\lim_{x\to a} f(x)=\dfrac{a+1}{a^2+1}=f(a)$ and mentioned that he found that limit by plugging in. He then turned back to the precise definition of continuity and described the relationship between continuous functions and plugging in as illustrated in Table 5.18.

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[11] Jason did not introduce the formal definition of continuity that involves ε and δ. Therefore, the mathematical definition he gave was referred to as "precise" instead of "formal" to be compatible with Jason's own description of the definition (See Table 5.17, [1]).

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Table 5.18: The relationship between computing the limit of a continuous function and plugging in

What is said	What is done	Type of utterance
[1] In fact, we found this limit by plugging in.	He shows $\lim_{x \to a} f(x) = \dfrac{a+1}{a^2+1} = f(a)$	Objectified
[2] A continuous function is a function where we can always find limits by plugging in.		Objectified
a) That is really what this definition says.	He goes back to the precise definition of limit he wrote on the board (See Figure 8xx)	
[3] f is continuous if you can find the limit at a by plugging in.	He first shows $\lim_{x \to a} f(x)$ and then $f(a)$ in the definition.	Objectified
a) That is exactly what the definition says.		
[4] In other words, function f is continuous at a if we can find the limit of f at x equals a by plugging in.	He writes these on the board.	Objectified
a) These are great functions.		
[5] Continuous functions are exactly the ones that we can find the limit by plugging in.		Objectified

Jason did not mention plugging in when he introduced the precise definition of continuity (see Table 5.17). However, after showing that the function $f(x) = \dfrac{x+1}{x^2+1}$ is continuous, he revisited this definition and explained how plugging in was embedded in the definition (Table 5.18, [1-5]). Some of his utterances in this context also exemplified Jason's word use when referring to the limit notation in the context of continuity. He used "limit at a" (Table 5.18, [3]) or limit of the function "at x equals a" (Table 5.18, [4]) when referring to $\lim_{x \to a} f(x)$. In the context of continuity, Jason only used the words *approaches* (the limit as *x approaches a* of a function) and *at* (the limit of a function *at* the point *a*) when

[12] In the context of evaluating a limit $\lim_{x \to a} f(x)$, plugging in refers to evaluating the function value at the point *a*, which would be equal to *f(a)*. So plugging in can be thought of as inserting the number *a* into the

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talking about the limit notation. He did not refer to $x \to a$ in the limit notation as x values "getting closer and closer to"; "going to" or "becoming" *a* as he did in other contexts discussed before. Note that to determine the continuity of a function at a given point, one first needs to find the limit of the function as x 'approaches' that point and then check if the limit value is equal to the function value 'at' that point. In this respect, the function's behavior *at* the limit point is relevant to the discussions about continuity. This, together with Jason's explanation about continuous functions as the functions whose limits can be found by plugging in might be the reason why he used *approaches*, which is how to read the limit notation, together with the word *at* in his discussions about continuity. Plugging in the limit value to the function is one of Jason's routines and will be further discussed in Section 5.3.1. The student difficulties "limit as the function's value" and "limit implies continuity" (See Table 2.1) are tied to the routine of plugging in since they result in checking the function value at the limit point to determine its limit. Students might generalize Jason's routine of plugging in for continuous functions to functions in general[13]. However, Jason also explicitly mentioned that, in general, the function's value at the limit value is not related to the limit of the function at that point both in this lesson and in the preceding lessons:

> To compute the limit as x approaches zero, what the function does at the point x equals zero is irrelevant. What only matters is what happens nearby. (Jason, 28 January, 2009, Lesson 7)

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function *f(x)*.

[13] Students' discourse with respect to the instructor's discourse will be discussed in Chapter VII.

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> The function value is something completely different. And here we see something again that the actual value of the function at the point where we are looking at has very little to do with the limit. In fact, it has nothing to do with the limit. (Jason, 23 January, 2009, Lesson 5)

Jason did not use any infinitesimal related word in the context of continuity.

Summary of word use

The limit, if it exists, is an obtained value at the end of the limiting process. Utterances that consider limit as a process, on the other hand, do not emphasize this ultimate result as the limit. Instead, through the use of words that signify continuous motion, it treats limit as a process that goes on and on and therefore never actually reached. Commognitive framework enables the exploration of this issue through considering objectification in the instructor's word use.

The analysis of Jason's discourse on limits reveals the existence of shifts in objectified and operational word use in the contexts of informal definition of limit; computing limits, infinity and continuity. The context in which his utterances were consistently objectified was the formal definition of limit. Although the majority of Jason's utterances about limits were objectified (See Table 5.5 and Figure 5.3), shifts in word use in a given limit related context might be problematic for students[14]. The instructor can move flexibly among these utterances and distinguish their similarities and differences depending on context but students might not yet be able to participate in the limit related discourse in similar ways. Therefore, switching between the operational and objectified word use, and treating limits as end products and as processes in the same context

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context, could contribute to one of the common student difficulties with limit, "limit as unreachable" (See Table 2.1). Talking about limit as a process in which the function values "approach", "get closer and closer to" or "become" a number L suggests that one can get close to the limit but cannot reach it (See Section 2.1.2).

A similar issue arises in Jason's discourse on infinity. We see that, depending on whether he computes a limit at infinity or he works on infinite limits, how he refers to the notion changes between actual (end-state) and potential infinity (process). Again, as the instructor, he is able to work with different realizations of infinity depending on context. However, students are unlikely to do that since they are relatively new participants to such mathematical discourse. On one hand, we see Jason talking about infinity as potential, especially in the context of referring to $x \to \infty$ in the limit notation. His utterances about the x values "getting larger and larger", "going to" or "becoming" infinity consider infinity as an ongoing process without an end state. On the other hand, we see him talking about infinity as actual in the context of infinite limits and when he writes the limit of a particular function as being *equal to* infinity. His utterances about the $f(x)$ values being an "arbitrarily large number", "very large number" are examples in which he talks about infinity as an actual number. The instructor's shifts in word use in the context of infinity may result in students' realization of infinity as a process and contribute to their realization of "limit as unreachable". It is also possible that, based on Jason's utterances on limit where he talks about infinity as a number might result in students' consideration of infinity as a finite number similar to the findings of Sierpińska (1987) (See Section 2.1.3). It should be noted, however, that Jason explicitly mentioned

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[14] Student thinking about limits in relation to the instructor's discourse will be discussed in Chapter VI.

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infinity is not a number, and stated that if the limit of a function is infinity, then the limit is undefined.

5.2. Visual mediators

Visual mediators refer to the visible objects created and operated upon for the sake of communication. In mathematics, visual mediators can take many forms such as manipulatives, tables, figures, diagrams, graphs, symbols, etc. There were four types of visual mediators identified in Jason's discourse as he communicated with his students: (a) written words, (b) drawn pictures of geometrical shapes, (c) graphs, and (d) mathematical symbols. Among these, graphs and symbolic notations were more common means of visual communication.

5.2.1. Written words

Written words correspond to what Jason wrote on the board besides mathematical symbols when he talked about limits. The analysis of his word use was conducted in the previous section. What will be highlighted here is the difference between his written and spoken words in particular contexts.

It was mentioned in the preceding sections that Jason used a combination of operational and objectified utterances in the same context (such as the informal definition of limit and computing limits). The analysis also revealed that his operational utterances mostly occurred when he used words only verbally whereas his objectified utterances often occurred when he wrote words on the board. For example, in the context of informal definition of limit, Jason wrote the following on the board:

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$\lim_{x \to x_0} f(x) = L$ if $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0 (See Figure 5.4).

Here, Jason mentioned the function values getting "arbitrarily close" to the limit value as the x values get "sufficiently close" to the limit point, and so used the words in an objectified manner. When he explained what this definition meant verbally, on the other hand, he mentioned the function values "approaching" or getting "closer and closer" to the limit value (Table 5.6, [4], [5]) as the x values got "closer and closer to" the limit point (Table 5.6, [4]). By doing so, he used the words in an operational manner. In fact, this difference between the words uttered and written can also be seen in the context of computing limits (See Tables 5.9, 5.10, 5.11 and 5.13, what is said and what is done) and in the context of continuity (See table 5.16, [2]). In those instances, Jason used words operationally when he communicated his ideas verbally and when he talked about the graphs that he drew, whereas he used them in an objectified way when he wrote the final arguments on the board. The context where his written and spoken words were most consistent with each other was the formal definition of limit. Note that, in that case, he wrote and spoke at the same time (See Table 5.8, what is said, what is done) and his word use was objectified throughout his explanations.

In summary, there seems to be a difference in word use depending on the means of communication Jason chose. When he presented his ideas visually, writing words on the board, he was more formal. When he further explained ideas related to limit verbally, he was less precise and less formal. Moreover, he rarely, if ever, wrote on the board any of his operational utterances. This signaled a difference in the mode of endorsement of narratives about limits, which was likely to be a result of Jason's alternating positions in the classroo

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the classroom. On one hand, being a mathematician, he was careful presenting the mathematical ideas accurately. On the other hand, as an instructor, he was aware when students did not follow his arguments. It is often in these instances that he was more flexible in his word use and relied more on informal utterances (as well as graphs) to help his students.

5.2.2. Drawn pictures of geometric shapes

There were only three occasions where Jason drew pictures of geometric shapes to explain the related concepts. In Lesson 1, he drew a falling rock when he talked about average rate of change. In Lesson 5, he drew the unit circle and formed triangles within the circle. He later used the similarity of these triangles to find the boundary functions for $\frac{\sin\theta}{\theta}$ before computing $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$. Finally, in Lesson 7, he drew the unit circle again, this time to talk about the fact that $\sin\theta$ and $\cos\theta$ are continuous. Besides these, he did not draw any diagrams or figures but frequently used graphs of functions.

5.2.3. Graphs

Jason used graphs of functions as he communicated his ideas about limits. He drew graphs in three different settings: (a) when he computed the limit of a function, (b) when he explained a particular definition, theorem or fact about limits, and (c) when he solved a problem that specifically asked to draw the graph of a given function.

In the context of computing a limit, Jason utilized graphs in two ways. He sometimes determined limits of functions directly from the graphs that he drew without utilizing any algebra-based technique to compute the limit. He also used them for further explanation or clarification after computing a limit algebraically. When Jason only used graphs to determine the limit of a function, his approach was considered as graphical.

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When he initially solved a limit problem by means of algebraic manipulations and then drew the graph to support his arguments, his approach was considered both algebraic and graphical. When Jason only relied on algebraic manipulations when computing a limit, his approach was identified as algebraic. Table 5.19 summarizes these approaches throughout the eight lessons in the context of computing limits.

Table 5.19: Types of visual mediators used in the context of computing a limit

Lesson Number[15]	Graphical approach	Algebraic approach	Algebraic and graphical approach	Number of graphs drawn
Lesson 2	6	3	1	3
Lesson 3	1	10	0	1
Lesson 4	0	1	2	2
Lesson 5	8	6	1	3
Lesson 6	1	9	1	2
Lesson 7	0	6	0	0
Lesson 8	0	8	0	0
Total	**16**	**43**	**5**	**12**

Note: The number of graphs Jason drew in the context of computing a limit is less than the number of instances he relied on a graphical approach while computing limits. This results from the fact that Jason sometimes computed multiple limits using a single graph he drew. Note also that the nonexistence of graphs in Lesson 7 and Lesson 8 in the context of computing limits does not mean Jason did not use any graphs in those lessons; it just means he did not use graphs while computing limits.

Jason used a graphical approach for 16 of the 64 limit computation problems that he worked on. He used a combination of algebraic and graphical approach for five problems. In those cases, the graphs were drawn only after the limits were initially computed by an algebraic method. The remaining 43 problems were solved using only an algebraic approach[16]. Therefore, in the context of computing a limit, Jason's primary visual mediators were symbolic rather than graphic.

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[15] Jason did not compute any limits in Lesson 1 (See Table 5.1 for the topic outline)

[16] Some of the common algebraic approaches Jason used will be discussed in the section about routines.

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Besides computing limits, Jason also used graphs when explaining a definition, theorem or fact about limits; and when solving a problem that specifically asked for the graph of a function using dominant terms[17]. There were 21 such graphs identified throughout the eight lessons. 20 of those were utilized to explain limit related facts and one was drawn as the solution of a problem about graphing a function using dominant terms.

In the cases where Jason utilized graphs to explain the endorsed narratives about limits such as definitions and theorems, the graphs were mostly drawn when he introduced them for the first time. In the remaining instances, Jason drew graphs for further elaboration after realizing the students were not clear about the mathematical ideas he communicated previously. Table 5.20 shows some examples of the graphs drawn in the context of introducing/explaining/elaborating on a limit related idea.

Most important for the purposes of the study is the graph Jason drew when he proved $\lim_{x \to 2} 2x - 1 = 3$ (See Table 5.20, the last example). He worked on this problem right after he introduced the formal definition of limit and initially solved the problem using only the algebraic approach that involved ε and δ. The students were mostly silent and asked some clarifying questions about the algebraic solution of the problem. It was only then Jason started drawing the graph as "another way that we can play this game about limits" (Jason, January 21, 2009, Lesson 4).

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[17] Graphing a function using dominant terms refers to graphing the function's behavior at positive and negative infinity (horizontal asymptotes) as well as its behavior at the points where it is undefined (vertical asymptotes). This requires the evaluation of the limit of the function at positive and negative infinity as well as at the points where the function is undefined while graphing. Jason told students they do not have to draw the graph accurately except for the dominant terms since drawing a precise graph of a function would be discussed during the chapter on derivatives.

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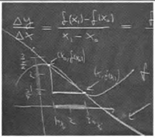
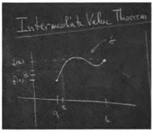
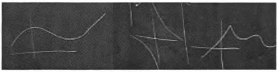
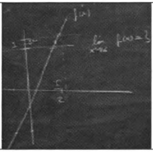
Table 5.20: Examples of Jason's hand-drawn graphs besides the context of computing limits

Graph drawn	Context of use	Purpose
	Average rate of change	"To think about the [difference] quotient geometrically" (Jason, January 12, 2009). "To represent speed as the slope of the secant line" (Jason, January 12, 2009).
	Informal definition of limit (See Figure 5.4)	To explain what it means for a function to be defined near x_0.
	Sandwich theorem	To explain the theorem visually.
	Intermediate value theorem	To explain the theorem visually.
	Continuity	To introduce the mathematical definition of continuity based on the intuitive aspects of continuity.
	Proving that $\lim_{x \to 2} 2x - 1 = 3$	To elaborate on students' confusion about the algebraic solution of the problem. Jason presented this as another way to think about the problem.

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Note that Jason did not represent the formal definition of limit visually when he defined it (See Figure 5.5), so this was the only instance in which he addressed what ε and δ corresponded to on a graph. As will be discussed later, this graph turned out to be the most useful mediator with which one of the participants in the study made sense of the limit concept.

5.2.4. Symbolic notation

Mathematical symbols were the primary visual mediators Jason used both in the context of computing limits (See Table 5.19) and also in other limit related contexts. He solved most of the limit computation problems by means of algebraic manipulations and represented definitions, theorems and facts about limits using mathematical notations consisting of symbols (as well as written words). The characteristics of the most common algebraic approaches Jason used will be discussed in the following section about routines. Here, the limit notation and how Jason addressed the notation will be examined.

It was mentioned in the section about word use that Jason introduced the notation $\lim_{x \to a} f(x) = L$ which he addressed as *the limit as x approaches a of f of x equals some number L* (see Table 5.6, [2]). In fact, he used the word *approaches* for about 70% of the 197 instances where he attended to the notation throughout the lessons. It was noted, however, that he also used other words besides *approaches* when he talked about the limit notation. During the discussions on Jason's operational word use (see Section 5.1.2), he was reported to refer to $x \to a$ in the notation also as x "gets closer and closer to" a (Table 5.2, [3], [4]) and as x "goes to" a, (Table 5.4, [2-4]) when a was a real number. Jason also used the word "at" ten times when he attended to the limit notation, where *a* w

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where a was a real number. Five of those utterances took place in the context of continuity (See Table 5.18).

In the context of limits at infinity, he referred to $x \to \infty$ in the limit notation as x "goes to" infinity[18]; x values "getting/becoming larger and larger" as well as "limit at positive/negative infinity" (See Table 5.12). When he wrote $\lim_{x \to a} f(x) = \pm\infty$, on the other hand, he talked about the limit as "equal to" or "being" plus/minus infinity (See Table 5.14).

In both $x \to \infty$ and $\lim_{x \to a} f(x) = \pm\infty$, the symbol ∞ signified infinity. In Jason's discourse, however, ∞ in $x \to \infty$ was often associated with potential infinity whereas ∞ in $\lim_{x \to a} f(x) = \pm\infty$ was often associated with actual infinity (see Section 5.1.5.3.2 and Section 5.1.5.3.3). So the same symbol ∞ was used to signify two different aspects of the concept of infinity. In the context of limits at infinity, the symbol indicated an infinite process; in the context of infinite limits, the same symbol indicated the end result of the process.

To summarize, although both Jason and the textbook read the notation using the word *approaches*[19], the arrow in the notation $\lim_{x \to a} f(x) = L$ signified a family of words such as "approaches", "goes to", "at", "becoming", "getting closer and closer to" in Jason's discourse on limits.

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[18] In Jason's discourse, eight of the eleven occurrences of the phrase "goes to" took place in the context of limit at infinity (see Section 1.4, Table 5.4).

[19] Mathematically, we read $\lim_{x \to a} f(x) = L$ as the function *f(x) approaches the limit L as x approaches a* (Hughes-Hallett et al., 2008; Thomas et al., 2008).

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Summary of visual mediators

The different ways Jason addressed the limit notation is another aspect of his discourse that might impact student learning. Referring to the arrow sign in the limit notation as "approaching", "going to" and "getting closer and closer to" the limit point signifies the process of moving towards the limit point without reaching it whereas the word "at" would signify reaching the limit point. In addition to this, talking about the function values "getting closer and closer to", "approaching" or "becoming" the limit value L considers limit as an ongoing process whereas the limit of the function being represented as equal to L in the notation considers it as a number obtained at the end of that process. Given this, it might be difficult for students to distinguish the process from the product in cases when Jason attended to the limit notation.

Having a family of words associated with the same symbol also occurred when Jason referred to infinity in the limit notation. In the context of limits at infinity, the ∞ symbol in the limit notation signified potential infinity, that is infinity as a process, when Jason addressed it as the x values "getting/becoming larger and larger", "approaching" or "going to" infinity. In the context of infinite limits, however, the same symbol was associated with actual infinity, an end-state, when he talked about limit as being "equal to" infinity. Therefore, the way Jason attended to the limit notation contributed to his use of mixed utterances about limits in the same context.

Jason did not rely mainly on graphs when he determined the limits of functions as the literature on learning about limits suggested (See Section 2.1.3). He used graphs more often to introduce a definition, theorem or fact about limits than to compute limits of functions. In fact, he only drew 12 graphs for the 64 limit computation problems that he worked on.

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worked on. He used graphs as an aid for teaching the main ideas related to limit and for addressing students' confusions about the algebraic solutions of the problems than making sense of every function by means of graphs, but his primary mode of representation was symbolic, not graphical.

Jason used a variety of functions in his discussions on limits and drew a variety of graphs for those functions. In this respect, he represented functions both algebraically and graphically but he did not use a tabular representation. He gave examples of continuous functions, constant functions as well as functions with removable and jump discontinuities. Moreover, he used not only polynomials but also rational functions and trigonometric functions as he computed limits. This finding is not consistent with the research on student learning arguing that instruction heavily relies on continuous functions (like polynomials) and graphs while teaching limits (Bezuidenhout, 2001; Parameswaran, 2007).

5.3. Routines

It was mentioned in Section 2.2.1 that metarules describe the patterns of the discursants as they construct and substantiate object-level narratives, that is, narratives about the objects of mathematics. Routines refer to the set of metarules that describe repetitive actions of the discursants. Sfard (2008) distinguishes between *how* of a routine from *when* of a routine. *How* of a routine can be thought of as the "*course of action* or *procedure*", whereas *when* of a routine refers to the instances "in which the discursants would deem this performance as appropriate" (Sfard, 2008, p. 208, italics in original). The *when* of a routine can be further split into two conditions: applicability and closure. Applicability conditions characterize the "circumstances in which the routine course of action is like

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action is likely to take place" (Sfard, 2008, p. 209). Applicability conditions also help determine the routine *prompts*, which refer to particular aspects of situations that are likely to trigger the application of a routine. Closure conditions characterize the "circumstances that the performer is likely to interpret as signaling a completion of performance" (Sfard, 2008, p. 209).

In what follows, the routines identified in Jason's discourse that are relevant to his word use and visual mediators will be presented with a focus on the prompt(s) as well as the how and the when of the routines. It should be noted that reporting a thorough list of the instances in which a particular routine is likely to appear (*when* of a routine) is complex, "if not altogether unworkable" (Sfard, 2008, p. 209). Given this, *when* of the routines described in the study should, by no means, be considered as a complete list of the circumstances in which the routine takes place. Instead, they represent the most clearly identified instances when Jason performed a specific course of action. There were four types of routines prominent in Jason's discourse on limits: algebra-based routines, geometry-based routines, using the metaphor of continuous motion, and using the metaphor of discreteness.

5.3.1 Algebra-based routines

Algebra-based routines refer to the algebraic techniques Jason utilized when computing the limit of a function. They constituted the main procedures with which he substantiated the narrative that limit is a specific number, if it exits. Table 5.21 shows the list of the algebra-based routines.

Table 5.

Prompt
Compute the limit of a function

Table 5.21: Algebra-based routines in Jason's discourse

Prompt	Routine	How	When (Applicability)	When (Closure)
Compute the limit of a function	[1] Plugging in	The function's value at the limit point is evaluated.	The denominator is non-zero at the limit point. The function is continuous. The limit is computed at a point.	The limit is equal to L. The limit does not exist/is undefined.
	[2] Cancelling out the common factor	The numerator and the denominator are simplified and their common factors are cancelled.	The numerator and the denominator of a rational function are both zero at the limit point.	The limit is equal to L. The limit does not exist/is undefined.
	[3] Using limit laws	Laws for the limit of a sum, difference, product or quotient is applied.	The function is composed of basic expressions whose limits are computed in the class. Can be applied *only* when the limit of each basic expression that make up the function exists.	The limit is equal to L. The limit does not exist/is undefined.
	[4] Multiplying by the conjugate	The numerator and the denominator are multiplied by the conjugate of the expression in the denominator.	The function contains an expression of the form $(a \pm \sqrt{bf(x)})$ and the numerator and the denominator are both zero at the limit point.	Expression is of the form $a^2\text{-}bf(x)$. The limit is equal to L. The limit does not exist/is undefined.
	[5] Getting rid of the absolute value sign	The expression $\lvert u(x)\rvert$ is converted to $u(x)$ or $-u(x)$ depending on the sign of $u(x)$ near the limit point.	The function whose limit is taken contains an expression in absolute value.	Absolute value signs are removed. The limit is equal to L. The limit does not exist/is undefined.
	[6] Dividing by the largest power	The numerator and the denominator of a rational function are divided by the largest power of the polynomial term in the function.	The limit is computed at infinity and the rational function is of the form $\frac{\infty}{\infty}$.	The limit is equal to L. The limit does not exist/is undefined.
	[7] Substitution (changing variables)	A different variable is assigned for some part of the function.	A limit at negative infinity is converted to a limit at positive infinity. When working with trigonometric functions such as the sine function.	The limit is equal to L. The limit does not exist/is undefined.
	[8] Long division	Long division is performed on a rational function where the degree of the numerator is higher than the degree of the denominator.	The limit is computed at infinity. The degree of the numerator is higher than the degree of the denominator. When finding the oblique asymptote of a function.	The limit is equal to L. The limit does not exist/is undefined. The function gets close to a line.

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There were eight distinct routines that were classified as algebra-based routines. However, this does not mean Jason only relied on one while determining the limit of a function. For example, routines such as *multiplying by the conjugate*, *getting rid of the absolute value sign*, *dividing by the largest power*, and *substitution* were often followed by *cancelling out the common factor* and/or *plugging in* for the same limit computation problem. Therefore, Jason sometimes used a combination of routines depending on the problem. Since all those routines took place in the context of computing a limit, however, they all had the same prompt and similar closures such as 'the limit is a number' or 'the limit does not exist/ is undefined' (Table 5.21).

The algebra-based routines [2-8] (Table 5.21) often occurred when *plugging in* did not work. When computing limits in the class, Jason's first attempt was to plug in the limit point to the functions formulated algebraically. He also mentioned that in the case of particular functions, such as continuous functions and for the functions whose denominators are non-zero at the limit point, one could compute the limit directly by *plugging in*. For example, when computing $\lim_{x \to 1} \frac{x^2 - 5}{x - 5}$ he said "the denominator is not zero. So we can just find the limit by plugging in" (Jason, 16 January, 2009, Lesson 3). In a similar fashion, for $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$, he said "so here, we cannot find the limit by plugging in simply because both the numerator and the denominator are zero. So the denominator is zero; that means we cannot find limits by plugging in" (Jason, 23 January, 2009, Lesson 5). Although Jason considered infinity not as a number (see Section 5.1.5.3.3), he also mentioned *plugging in* when he was finding $\lim_{x \to \infty} \frac{2x^2 - 3x^3 + 5}{4x^2 + 3x - x^3}$ at infinity: "What this mean

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means is we want to formally plug in infinity here. We cannot really do this. So that is why we come up with some way to write the limit as x approaches infinity" (Jason, 23 January, 2009, Lesson 5). In the context of computing a limit, many of the algebraic routines (Table 5.21, [2-8]), therefore, were introduced as alternative procedures to find a limit when the initial routine *plugging in* did not work.

5.3.2 Geometry-based routine (Graphing)

It was mentioned in Section 5.2 (visual mediators) that Jason used drawn pictures of geometric shapes and graphs as visual mediators, which both convey a limit-related idea geometrically. For the purposes of this section, however, drawing pictures of geometric shapes was not considered a routine since it occurred only three times (see Section 5.2.2) throughout the eight lessons and their use did not seem to follow a repetitive pattern. Thus, only graphing will be considered as a geometry-based routine in Jason's discourse, whose characteristics in terms of *when* and *how* are summarized in Table 5.22.

The somewhat apparent purpose of graphs is to provide visual aids for the communication of mathematical ideas. Graphing, as a routine however, helps us gain more information about the instances where Jason needed such aid in his discourse on limits. In that respect, the *when* and *how* of graphing also inform us about the diverse reasons Jason used graphs.

Unlike the algebra-based routines, graphing was a routine that was initiated by various prompts[20]. Besides its utilization in the context of computing the limit of a given function, it was also used to determine the behavior of a function, introduce or explain theore

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theorems and definitions about limits and to address students' confusions in the classroom (Table 5.22). There were also instances in which the problem Jason worked on particularly asked that graphs of a function be drawn. Although the closures of all these situations usually ended with the graph that was intended to be drawn (Table 5.22), graphing was used to substantiate a variety of narratives rather than a single one due to the different prompts that triggered the application of this routine.

Table 5.22: Graphing as a geometry-based routine in Jason's discourse

Prompt	How	When (Applicability)	When (Closure)
Compute the limit of a function	Graphs of arbitrary functions are drawn.	The limits are to be determined only from the graph drawn.	The limit is equal to *L*. The limit does not exist/is undefined.
	Graphs of particular functions formulated algebraically are drawn.	The limit is computed at infinity or the function is a trigonometric function.	
Determining the behavior of a function	Graphs of particular functions formulated algebraically are drawn as visual aids.	The function is a trigonometric function.	The graphs are drawn.
Graph a function using dominant terms	Graphs of functions are drawn after determining their horizontal, vertical and oblique asymptotes.	When the question explicitly asks a function formulated algebraically to be graphed.	The graphs are drawn with a particular focus on the functions' behavior at positive/negative infinity and the points where the function is undefined.
Introducing and/or explaining theorems/ definitions about limits	Graphs of arbitrary functions are drawn.	Informal definition of limit, sandwich theorem, intermediate value theorem.	The graphs are drawn and the theorems/definitions are completely stated or written on the board.
		The graphs are either drawn before or in conjunction with the symbolic notation used for the theorem/definition.	
Addressing students' confusions	Graphs of arbitrary or specific functions are drawn to exemplify the instructor's previous arguments.	Proving an explanation that a limit is equal to a number.	The graphs are drawn.
		When the students are silent or ask questions after an explanation.	Students confirm they follow the instructor's arguments.

[20] See Section 5.2.3 for further details about the contexts in which Jason utilized graphs.

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For example, when graphing was used to compute the limit of a function, as was the case for $\lim_{x \to \infty} \frac{1}{x} = 0$ (see Figure 5.7), it was used to substantiate the narrative 'the limit of the function one over x as x approaches infinity is equal to zero' or possibly 'the limit (if it exits) is a number' in general. On the other hand, when graphing was used to introduce and explain a theorem about limits, say the intermediate value theorem, the narrative that Jason endorsed was follows:

> If f is continuous on $[a, b]$ then f assumes every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then there is a c in $[a, b]$ so that $f(c)=y$. (See Figure 5.9)

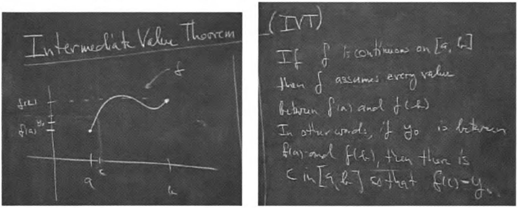

Figure 5.9: Jason's introduction of the intermediate value theorem (hand-written)

5.3.3 Using the metaphor of continuous motion

This routine refers to Jason's use of the metaphor of continuous motion[21] in his discourse. His utilization of this metarule remained implicit in the class since he did not explicitly mention he was using a particular metaphor when talking about limits. Instead, this routine emerged from his use of words signifying motion (See Figure 5.1) in the cont

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context of informal definition of limit, computing limits, and continuity as well as his routine of graphing. Note that using words signifying motion when talking about limit related concepts support their realization as a process and, therefore, is closely related to Jason's operational word use. Another reason this routine remained implicit for students results from the fact that Jason's operational utterances occurred when he used words only verbally whereas his objectified utterances took place when he wrote the words on the board (See Section 5.2.1). In other words, he did not make his words signifying motion visible to students. Table 5.23 shows the characteristics of the routine of using dynamic motion as a metaphor in Jason's discourse.

Table 5.23: Jason's routine of using the metaphor of continuous motion

Prompt	How	When (Applicability)	When (Closure)
Compute the limit of a function	Words signifying motion (See Figure 5.1) when referring to x values, infinity and function values are uttered (not written).	Determining the behavior of a function at the limit point *before* writing its limit as *equal to L* on the board.	(Referring to limit) The function values approach/get closer and closer to/tend to L.
	Words signifying motion are uttered (not written) as/after drawing the graph of the function.	Referring to infinity and the limit value in the limit notation When the students are silent or ask questions after an explanation.	(Referring to infinity) The function values get/become larger and larger/ smaller and smaller.
Introducing informal aspect of limits	Words signifying motion when referring to x and the function values are uttered (not written).	Informal definition of limit	"The function value, if I get closer and closer to x zero, it should approach some number L" (Table 5.6, [4]).
	Words signifying motion are uttered (not written) as/after drawing the graph of arbitrary functions.	Informal definition of continuity When the students are silent or ask questions after an explanation.	"A continuous function is a function, and I am just going to say it in words, that I can graph without taking the chalk off the board" (Table 5.16, [2]).

[21] Using a metaphor consistently in a given discourse is a type of metarule (See Section 2.2.2).

5.3.4 Using the metaphor of discreteness

This routine took place only when Jason talked about the formal definition of limit[22] and a proof problem and refers to the elimination of words signifying motion in his discourse. In the metaphor of discreteness, he used words signifying proximity (See Figure 5.1) through distance. Recall that Jason's word use was consistently objectified in the context of the formal definition of limit (See Section 5.1.5.2). His utilization of this metarule also remained implicit in the class since he did not explicitly mention he was using a metaphor when talking about the formal definition. Table 5.24 shows the characteristics of the routine of using discreteness as a metaphor in Jason's discourse.

Table 5.24: Jason's routine of using the metaphor of discreteness

Prompt	How	When (Applicability)	When (Closure)
Introducing formal aspect of limits	Words signifying proximity (See Figure 5.1) when referring to the *x* and function values are uttered (and written) (See Figure 5.5).	Formal definition of limit	"...whenever x is sufficiently close to x zero so that means that the difference is no more than delta then the function values should be close to L" (Table 5.8, [14]).
	Symbols instead of graphs that signify motion are used.	Proving that the limit of a function is equal to a number using the formal definition of limit	"...if x is sufficiently close, delta close to x zero,...the function values should be close to the limit" (Table 5.8, [18]).

In general, the metaphor underlying the informal definition of limit and the behavior of the function values near the limit point is based on continuous motion, which considers limit as a process. The metaphor behind the formal definition, however, is based on discreteness and the elimination of motion, and therefore time, from the informal definition of limit (See Chapter III). Using distinct metaphors was present in Jason's

22 Although Jason relied on this metarule in only one context, he repeatedly used the metaphor of discreteness in that context. Therefore, his (implicit) use of the metaphor was considered as a routine.

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discourse as metarules when he talked about different aspects of limits. However, these metarules remained implicit and he did not draw students' attention to his shifts in word use and also on the distinct metaphors related to realizations of limit.

Summary of routines

In Jason's discourse, algebra-based routines often resulted in objectified word use whereas graphing and the verbal statements about limit as a process resulted in operational word use. Jason's routines were used to substantiate the narratives that limit is a number; limit is a process as well as other narratives of the form of definitions, theorems and rules.

Jason relied more on algebra-based routines than graphing in the context of computing a limit (see Table 5.19). Being based on algebraic manipulations and symbolic notation, algebra-based routines (Table 5.21) also helped determine the instances in which Jason's objectified word use took place when he computed limits. The geometric-based routine, which is identified in the study as graphing, was more often used for explaining a definition, theorem or fact about limits. Although Jason clearly mentioned that the limit of a function at a point may be distinct from the function's value at that point, his frequent use of plugging in as the initial attempt to compute the limit of a function links the function value with the limit value in a way that could easily contribute to confusion of the two.

Jason implicitly used distinct metaphors as metarules in his discourse on limits. One was based on dynamic motion whereas the other was based on the static aspect of limit. These different metaphors also supported the realization of limit as a process or a number. Jason's shifts in word use and his utilization of different metarules remained impl

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implicit for his students since he did not explicitly attend to these aspects of his discourse in the class.

The analysis of Jason's routines also revealed that his word use was closely related to his means of communication. His operational word use took place only when he communicated his ideas verbally whereas his objectified word use took place when he wrote his ideas on the board. It was discussed in Section 5.2.1 that such a difference in the means of communication might also have resulted from Jason's dual positions in the classroom (being a mathematician and being a teacher).

5.4. Endorsed narratives

Endorsed narratives are the last discursive feature under consideration in this study. *Narrative* is "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures" (Sfard, 2008, p. 134, italics in original). Once endorsed, narratives are considered as true and ultimately are known as mathematical facts. Some examples of endorsed narratives of mathematics are axioms, definitions and theorems. The construction and substantiation of narratives, however, are not uniquely defined. There is a variety of ways in which narratives can be substantiated depending on the context and also on a person's familiarity with the mathematical discourse. For example, in the mathematics community, narratives are often endorsed by means of proofs, which are mainly based on deductive reasoning. In contrast, empirical evidence and routines such as trial and error can be used to endorse narratives by students, especially at their initial stages of learning. Therefore, with respect to an individual's discourse, *endorsed narratives* refer to what that

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In his discourse, Jason endorsed many narratives both related to limit and also to algebraic properties such as exponent rules and performing long division. Given this, there were many definitions, theorems and rules he addressed in the classroom. There were also many instances in which he explicitly designated a particular sentence as true or untrue about limits, infinity and continuity. The goal of this section is not to give a comprehensive list of every endorsed narrative but to focus on the ones that were most significant based on the information gathered on Jason's word use, visual mediators and the routines with which he substantiated those narratives.

Limit is a number

The narrative Jason most frequently endorsed in his discourse was the description of limit as a number. This narrative was often endorsed in the context of computing limits as well as the formal definition of limit and was mainly substantiated by algebra-based and geometry-based routines discussed in Section 5.3.2. The limit of a given function existed when it was equal to a number. In fact, Jason explicitly mentioned that the limit "exists" if it is equal to a number (Table 5.3, [7], [17]); it "does not exist" if the right hand limit is not equal to the left hand limit (Table 5.3, [16]) or the limit is equal to infinity (Table 5.3, [18]).

Limit is a process

Limit is a process was another narrative endorsed by Jason, though not as frequently as *limit is a number*. It was mentioned in the analyses of his operational word use (Section 5.1.2) and routines (Section 5.3) that Jason relied on dynamic motion, grap

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graphs and verbal statements while substantiating this narrative. Jason endorsed this narrative in the context of instantaneous rate of change; computing a limit; informal definition of limit and also continuity. While substantiating this narrative, he used words such as “getting closer and closer to”, “goes to” and “becoming” (See Table 5.2) that are based on continuous motion and sometimes used graphs of functions as visual mediators to talk about the limit as a process. There were only two instances, however, where he explicitly mentioned the word “process” (See Table 5.2, [1], [16]). Therefore, this endorsed narrative was mainly inferred through the elements of his discourse in which he described a process when finding the limit of a function.

Infinity is not a number

Jason’s analysis of word use in the context of limits at infinity and infinite limits revealed that he talked about infinity as a potentiality and as actual interchangeably (See Sections 5.1.5.3.2 and 5.1.5.3.3). It was also mentioned that he referred to infinity as a number in the context of infinite limits as he considered it as the end product of an indefinite process (See Table 5.14). Moreover, he talked about “plugging in” infinity as the initial attempt to work on limits and infinity, treating it as a distinct entity that, like a number, can be put in the function (See Section 5.3.1). It was noted, however, that he then mentioned one could not do that since infinity is not a number (Table 5.14, [31]). This narrative was further endorsed when he considered infinite limits as non-existing. Therefore, although he explicitly endorsed the narrative that *infinity is not a number* and substantiated it in the context of limits at infinity, some of his word use and routines in the context of infinite limits seemed to suggest otherwise.

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The function's value at the limit point is irrelevant to the limit value

Plugging in was a common routine in Jason's discourse in the context of computing limits and continuity (See Sections 5.1.5.3 and 5.1.5.4) He often emphasized that the limits of continuous functions could be found by plugging in, which would be equal to the function's value by the definition of continuity. On the other hand, he explicitly stated that, in general, the function value is "something completely different" than the limit value, and "has nothing to do with the limit" (See Section 5.1.5.4). In his discourse, he substantiated this narrative through graphs and examples of functions with removable and jump discontinuities for which the limit value, if it existed, was different than the function value.

Summary of endorsed narratives

In all the contexts he worked on, the primary narrative Jason endorsed was that limit is a number. His word use when talking about the behavior of the x and $f(x)$ values, on the other hand, considered limit as a process. Jason explicitly endorsed limit as a process only twice in his discourse on limits. However, many of the limit computation problems in which he described the behavior of the function values implicitly endorsed limit as process. Therefore, although in different frequency, Jason endorsed limit both as a number and as a process.

Similarly, Jason explicitly endorsed the narrative that infinity is not a number. On the other hand, his word use in the context of infinite limits also showed that he referred to infinity as a number (5.1.5.3.3). He also treated infinity as a number when he said we would "want to formally plug in infinity" to find $\lim_{x \to \infty} \frac{2x^2 - 3x^3 + 5}{4x^2 + 3x - x^3}$ (See Section 5.3.1).

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Although he then said we cannot do that, he implicitly endorsed the narrative that infinity is a number in the context of infinite limits.

To sum up, Jason endorsed different narratives in his discourse on infinity and limit depending on the context, which can lead to confusion whether to consider infinity and limit as a process or as an end state.

Summary of Jason's discourse on limits

In this chapter, I investigated one instructor's discourse on limit related concepts by focusing on his word use, visual mediators, routines, and endorsed narratives. Research on learning about limits brings forward many of the conceptual obstacles students have about limits and discusses the possible links between those obstacles and the teaching of calculus (See Section 2.1.4). This research adds to the body of literature by particularly focusing on the *teaching* of limits. In what follows, I summarize in general terms what we have learned about one instructor's discourse[23] on limits while teaching the notion to beginning-level calculus students.

Jason's word use on limit and infinity revealed that he referred to them as both a process and an end state depending on the context. Although he flexibly uses limit and infinity as a process or product and distinguishes the characteristics of each realization depending on the context, students might be unlikely to notice the characteristics underlying these differences[24]. This might result from two factors: (a) Jason did not make the instances where he shifted his word use from operational to objectified (and vice

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[23] For more detailed results in regard to word use, visual mediators, routines and endorsed narratives, see the summaries at the end of the sections 5.1, 5.2, 5.3, and 5.4, respectively.

[24] The results of the diagnostic survey and the interviews given to students at the end of their instruction will be discussed in the next chapter.

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versa) explicit in the classroom, and (b) given that this was the first time the majority of students were introduced to the limit notion, they were relatively new to the discourse on limits and did not have the experience to notice the important subtleties underlying each realization of limit and infinity.

The contexts in which Jason shifted his word use seem to support the existence of possible links between the instructor's discourse and the historical development of limit related concepts[25]. Note that Jason consistently used words signifying motion with a consideration of limit as a process in the context of informal definition of limit and computing limits (See Sections 5.1.5.1 and 5.1.5.3). On the other hand, he did not use any motion related word and did not mention limit as a process when he talked about the formal definition of limit (See Section 5.1.5.2). Moreover, he used the metaphor of dynamic motion in the context of the informal definition of limit whereas he used the metaphor of discreteness in the context of the formal definition of limit. These are consistent with the realizations of the informal and formal definitions of limits as developed over history[26] (See Chapter III). Similar to his shifts in word use, however, the utilization of these metaphors remained implicit for the students as he did not address it in the class.

The analysis of the visual mediators Jason used in the class showed that he did not rely too much on graphs in the context of determining the limit of a function. However, he used graphs frequently when explaining a limit related definition, theorem or fact.

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[25] Instructor's discourse in relation to the historical development of limit related concepts will be examined in Chapter VII.

[26] A detailed analysis of the instructor's discourse with respect to the historical development of limit related concepts will be given in Chapter VII.

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Therefore, rather than considering graphs as a basic tool with which to make sense of the behavior of a given function, Jason mainly used them as teaching aids during the introduction of limit related ideas. The analysis of written words as a visual mediator revealed that Jason's word use was always objectified when he wrote on the board. His operational word use only took place when he communicated his ideas verbally. This suggests a degree of precision depending on the context. Jason was more precise and careful when he wrote ideas on the board. He was less precise and less formal when he addressed limit related ideas about which the students were likely to be confused. Such a shift seems to signal Jason's mode of endorsement of limit related narratives depending on his position in the classroom. Being a mathematician, he wanted to make sure he conveyed the mathematical ideas correctly. Being a teacher, on the other hand, he lowered the degree of precision and talked about limits in an intuitive manner to enhance student learning.

Another important visual mediator Jason used in the classroom was the symbolic limit notation. In fact, this aspect of his discourse, together with his word use, provided significant information how Jason talked about limit and infinity. When talking about the arrow in the limit notation $\lim_{x \to a} f(x) = L$, Jason used a family of words such as "approaches", "goes to", "gets closer and closer to", which refer to the *process* of the x values moving toward the limit point a without reaching it. This is ambivalent in some cases, since the word "approaches" is the canonical, endorsed way of reading the symbol even in the formal definition. On the other hand, he also used the word "at" when referring to the arrow, which considered the x values as reaching the limit point. Similarly, when talking about the function values, Jason used "approach", "become",

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“get closer and closer to”, which are about the *process* of moving towards the limit L without reaching it. When he completed writing the limit notation using the equal sign, however, he referred to the limit as reachable (the limit is equal to L). Therefore, Jason’s word use in regard to the limit notation might inevitably lead to confusion with respect to the consideration of limit as a process or as a product (number). The same issue was apparent also for the symbol ∞ signifying infinity. In the context of limits at infinity, Jason talked about infinity as a process, or potential infinity, when he mentioned the x values “getting/becoming larger and larger”, “goes to”, and “approaches”. In the context of infinite limits he associated the same symbol with an end result, or actual infinity, when he talked about the limit of a function being “equal to” infinity. Such elements of Jason’s discourse are likely to trigger the dynamic view of limit that is based on continuous motion as well as the incorrect realization of limit such as “limit as unreachable” (See Section 2.1.2).

The analysis of routines in Jason’s discourse showed that he often utilized algebra-based routines while computing the limit of a function (See Table 5.21). He used many algebra-based routines but most of those routines took place in cases when the routine *plugging in*, that is plugging the limit point a into the function $f(x)$, did not work. Said differently, *plugging in*, if applicable, was the initial routine Jason encouraged students to use when computing a limit. There were many instances in which this routine did not work. Jason used a variety of functions whose limits could not be found by plugging in. Moreover, Jason explicitly addressed the relationship between the function value at the limit point and the limit value when he talked about continuity. He mentioned that the value a function attains at the limit point is irrelevant to its limit value (See

Section 5.1.5.4). On the other hand, he often emphasized that the limit of a continuous function could be found just by *plugging in.* As a result, although Jason provided opportunities for students to distinguish between the function value at the limit point and the limit value, some of his word use and his routine of *plugging in* might support the student difficulty "limit as the function's value" (See Section 2.1.2).

In all of the contexts he worked on, "limit is a number" was the main narrative Jason endorsed most explicitly and frequently. The shifts in the word use and some of his routines, such as talking about limit as a process only verbally, however, also led to the endorsement of the narrative "limit is a process". Jason referred to limit as a process when describing the behavior of a function near the limit point. Although he referred to limit as a number at the end of every such description, it might not be clear for students when the process gives way to the end result of the process in the context of limits. Similarly, although Jason explicitly endorsed that "infinity is not a number", his treatment of the notion as a number in the context of infinite limits might lead to a similar confusion in students' realizations of infinity in terms of its realization as a process or an end state.

Given the possible implications of Jason's discourse on student learning, it is important to investigate how students talk about limits at the end of their instruction. The classroom observations did not provide much information about students' discourse on limits since there were very few occasions in which students talked in the classroom. The format of instruction was lecture and students talked in the class only to ask clarifying questions and to correct some computational mistakes Jason made. Therefore, the diagnostic survey and individual interview sessions were the means by which I had some access to the students' discourse on limits. In the following chapter, I will discuss students' discourse on limits in relation to the instructor's discourse.

CHAPTER VI

STUDENT DISCOURSE ON LIMITS IN RELATION TO THE INSTRUCTOR'S DISCOURSE

In this chapter, I explore students' discourse on limits at the end of their instruction. I first present the results of the diagnostic survey I gave to 23 students. I then focus on four students who participated in an individual interview session in which they worked on questions targeting the conceptual obstacles in learning of limits. While reporting on the survey and the interview sessions, I mainly focus on how and whether the students' discourse on limits is similar to or different from the instructor's discourse. For the interview sessions, I only elaborate on the instances which highlight students' discourse on limits in relation to the instructor's discourse. The purpose of this chapter is to explore the links between students' and the instructor's discourse on limits.

It was not possible to gain much information on students' discourse on limits based on the video-taped classroom observations since the instructor's mode of teaching was lectures. During the period of eight lessons, there was no student-student interaction and few instances of student-teacher interaction. Although the instructor encouraged students to ask questions in the classroom, he did not facilitate any student discussion. Students interacted with the instructor (Jason) when they asked clarifying questions, when they did not follow Jason's explanations and when they corrected a few computational mistakes the instructor made in the class. Therefore, the diagnostic survey and the individual interview sessions were the means by which I had access to how students made sense of the limit concept.

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6.1. Responses to the diagnostic survey

I gave the diagnostic survey to Jason's class at the end of the unit on limits. In seven of the eight observed lessons, Jason covered ideas related to limits and continuity. The eighth lesson was a review session before the exam. I administered the survey to students during the last ten minutes of the eighth lesson. It was mentioned before that, although 31 students enrolled in the class, the number of students attending the class ranged between 17 and 23 in the period of the classroom observations. There were 23 students present in the class on the day I gave the diagnostic survey and all of the students agreed to take the survey. The purpose of the survey was two-fold. First, I wanted information on students' discourse on limits at the end of their instruction. Second, I used the survey to select the students for the individual interview sessions.

The questions in the diagnostic survey (See Appendix A) were taken from Williams (2001) since his classification of views related to limit is widely endorsed in research on student learning. The first question of the survey included six statements (See Table 6.1) about limits and asked students to decide whether the statements were true or false. The second question then asked them which of the six statements best described their understanding of limits. The third question asked students to describe what they understood a limit to be. The final question asked students to give a rigorous (formal) definition of limit, if possible.

Note that the views of limit such as boundary, unreachable and approximation (See Table 6.1) are among the conceptual obstacles students have when thinking about limits (See Section 2.1.2, Table 2.1). Hence, the diagnostic survey was useful for assessing whether participants in the study were likely to have those difficulties.

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Table 6.1: Statements in the first question of the diagnostic survey and the corresponding views of limit (Williams, 1991; 2001)

Statements	Description	View of limit
Statement 1	A limit describes how a function moves as x moves toward a certain point.	Dynamic-theoretical
Statement 2	A limit is a number or point past which the function cannot go.	Boundary
Statement 3	A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.	Formal
Statement 4	A limit is a number or point the function gets close to but never reaches.	Unreachable
Statement 5	A limit is an approximation that can be made as accurate as you wish.	Approximation
Statement 6	A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.	Dynamic-practical

The difficulties not addressed by the diagnostic survey but mentioned by research on learning about limits were intended to be examined during the individual interview sessions. Those difficulties were "limit implies continuity", "limit as the function's value", and "limit as monotonic" (See Section 2.1.2, Table 2.1). The diagnostic survey was also useful for identifying if students' realizations of limits were mainly dynamic, that is based on motion, or static, that is based on the assumptions of the formal definition of limit. The identification of students' realizations of limits was based not only on the responses they gave to Questions I and II but also on how they described limit in their own words (Question III, See Appendix A). In what follows, I elaborate on student responses for each question in the survey

6.1.1. Questions I and II

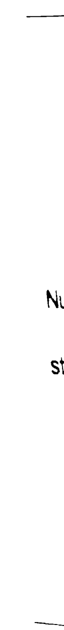
There were six statements in the first question of the survey describing a different view of limit (See Table 6.1). The students were asked to decide whether the statements were true or false. Although each statement focused on a distinct view of limit, the views were not necessarily non-overlapping. For example, views of limit as unreachable and as an approximation are identified as difficulties resulting from the dynamic view of limit by the research on student learning (See Section 2.1.2). Table 6.2 shows the responses students gave to the first question of the survey. Table 6.3 and Figure 6.1 show the cross-comparison of student responses given to the six statements in Question 1 with respect to the statements they chose as true.

Table 6.2: Students' responses to the first question of the diagnostic survey

Statements	View of limit	Number of student responses (N=23) True	False
1. A limit describes how a function moves as *x* moves toward a certain point.	Dynamic-theoretical	20	3
2. A limit is a number or point past which the function cannot go.	Boundary	6	17
3. A limit is a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values.	Formal	16	7
4. A limit is a number or point the function gets close to but never reaches.	Unreachable	13	10
5. A limit is an approximation that can be made as accurate as you wish.	Approximation	12	11
6. A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.	Dynamic-practical	9	14

Table 6.3: Cross-comparison of student responses with respect to the statements they chose as true

		Number of students choosing the statements as true					
		Statement1 (S1) N=20	Statement2 (S2) N=6	Statement3 (S3) N=16	Statement4 (S4) N=13	Statement5 (S5) N=12	Statement6 (S6) N=9
Number of students choosing the statements as true	Statement1 N=20	20	4	13	10	10	8
	Statement2 N=6	4	6	4	6	3	3
	Statement3 N=16	13	4	16	8	7	6
	Statement4 N=13	10	6	8	13	7	4
	Statement5 N=12	10	3	7	7	12	5
	Statement6 N=9	8	3	6	4	5	9



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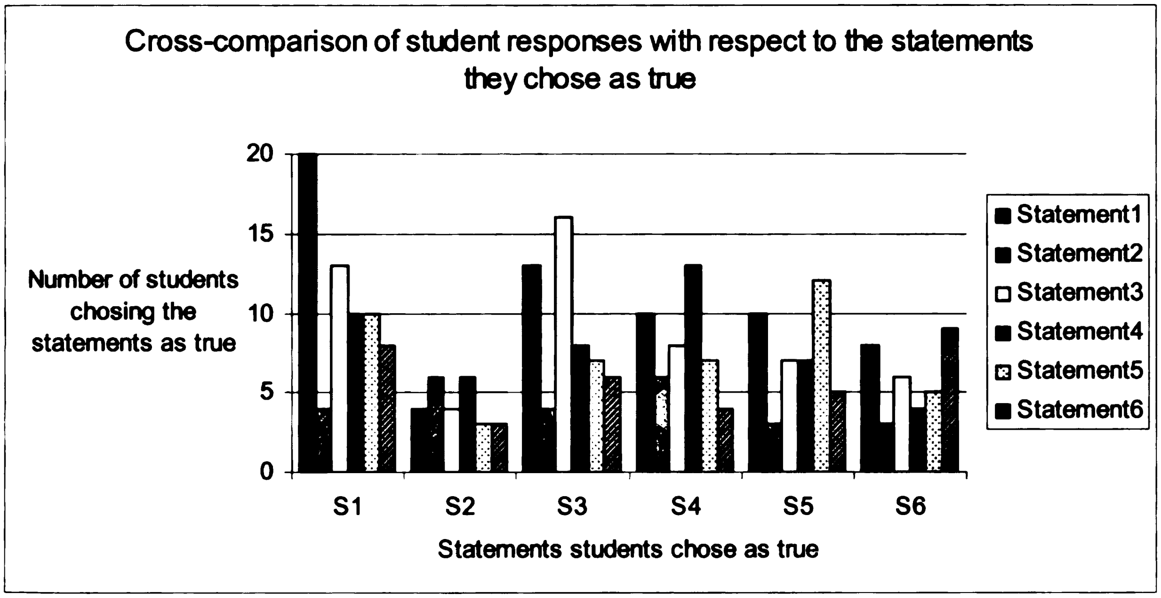

Figure 6.1: Cross-comparison of student responses with respect to the statements they chose as true

Statement 1 was considered as true by 20 of the students and was the most popular statement students considered as best describing their realizations of limit (See Table 6.2). The statement clearly entailed a dynamic view of limit since it involved the movement of the function values as the x values moved toward the limit point. That this statement addresses limit as a process can further be supported by the fact that it views limit as a means for *describing* the behavior of a function rather than talking about limit as a specific *number*. Such a view is consistent with Jason's explanations of the behavior of functions in the context of computing limits (See Section 5.1.5.3) and his routine of using the metaphor of continuous motion in that context (See Section 5.3.4). It was mentioned in Chapter V that Jason's description of a function's behavior while computing limits was mainly based on operational word use than objectified[1]. Therefore,

[1] Operational word use refers to talking about limit as a process whereas objectified word use refers to talking about limit as a number obtained at the end of that process (See Sections 5.1.2 and 5.1.3).

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given the context of the classroom, I did not consider responding to this statement as true as incorrect since this was how Jason investigated the behavior of a function near the limit point before reaching a conclusion about the limit of the function at that point. Said differently, the statement had theoretical validity when exploring the behavior of a function near the limit point.

Statement 3, which corresponds to the formal view of limits, was considered as true by 16 of the 23 students and was the second most popular statement students chose as true in the survey (See Table 6.2). In fact, given that Statement 3 is based on the static aspect of limit in contrast to Statement 1, which is based on the dynamic aspect of limit, it is interesting that 13 of the 20 students who marked Statement 1 as true also marked Statement 3 as true (See Table 6.3 and Figure 6.1). Note also that, unlike the first statement, third statement talks about limit as a *number* but not as a process. Jason explicitly referred to limit as a number while introducing the informal and the formal definition of limit. He also frequently referred to limits as numbers (if they existed) when talking about the final answers of the problems in the context of computing a limit (See Section 5.3.1, Table 5.21, closure conditions). Jason's persistent referral to limits as numbers (except when he talked about the function values near the limit point) and students' familiarity with the formal definition of limit (See Figure 5.4 and 5.5) support students' consideration of Statement 3 as true. Table 6.4 and Figure 6.2 show the cross-comparison of student responses to the second question in the survey with respect to the statements students chose as best describing their realizations of limit.

Table 6.4: Cross-comparison of students' responses with respect to the statements they chose as best describing their realizations of limit

		Number of students choosing the statements as best describing their realizations of limit						
		S1 N=11	S2 N=0	S3 N=1	S4 N=6	S5 N=1	S6 N=3	None N=1
Number of students choosing the statements as true	Statement1 N=20	11	0	1	4	1	3	0
	Statement2 N=6	1	0	0	4	0	1	0
	Statement3 N=16	9	0	1	2	0	3	1
	Statement4 N=13	5	0	0	6	0	1	1
	Statement5 N=12	3	0	1	4	1	2	1
	Statement6 N=9	2	0	0	3	1	3	0

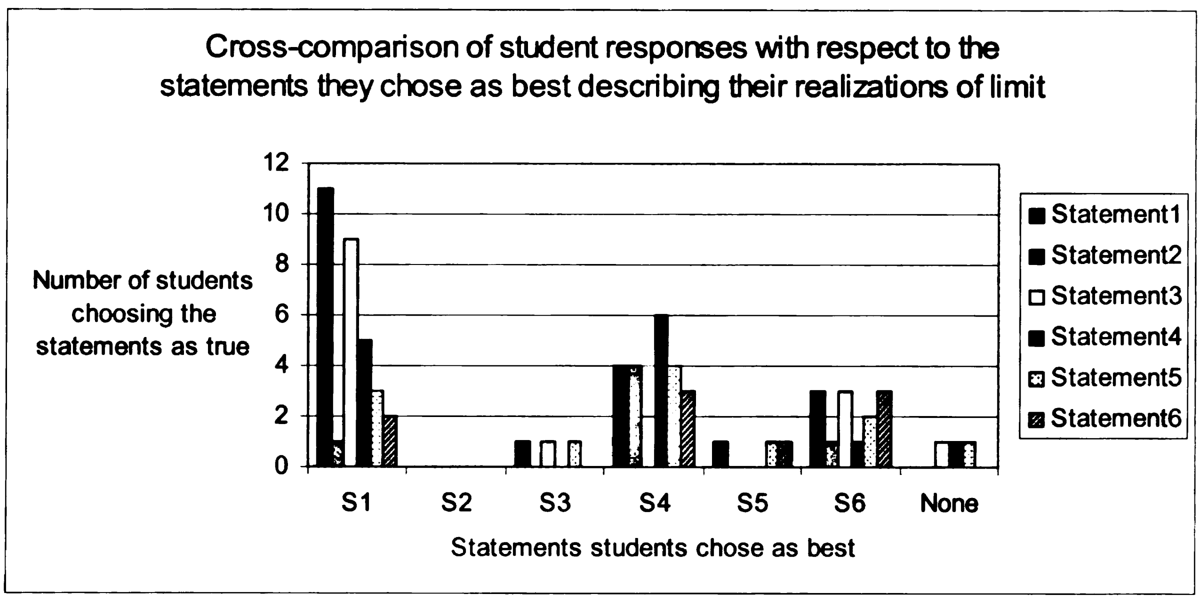

Figure 6.2: Cross-comparison of student responses with respect to the statements they chose as best describing their realizations of limit

9 out of 13 students who marked Statement 3 as true selected Statement 1 as best describing their views of limit (See Table 6.4). Only one student who marked Statement 1 as true considered Statement 3 as the best statement. 11 of the 20 students who chose Statement 1 as true considered it as the best statement with which they make sense of limits (See Table 6.4 and Figure 6.2). Therefore, student responses given for these

statements revealed that, although the instructor can clearly distinguish between limit as a process and limit as a number depending on the context, students considered both realizations as true about limits with a clear preference for the dynamic (process) view.

Statement 6 is the other statement in the survey[2] that describes limit as a process. Similar to Statement 1, Statement 6 describes *how* a limit is obtained rather than *what* the limit is and so does not refer to limit as a *number*. The dynamic aspect in the statement comes from plugging successive numbers to the function as the numbers get closer and closer to the limit point. At first glance, one would expect majority of students to consider this statement as true since this statement has similar characteristics with Statement 1-the most popular choice as best describing students' views of limit (See Table 6.4). However, nine of the 23 students marked Statement 6 as true (See Table 6.2). This is consistent with the fact that plugging in values to the function as they get successively closer to the limit point was a strategy Jason did not utilize in the classroom. There was only one instance during the classroom observations in which Jason mentioned plugging points closer and closer to the limit value (See Table 5.2, [2]) when finding the limit of a function. While doing so, however, all his discussion was verbal; he did not actually plug in points and compute the function values on the board. Except for this instance, he did not employ the dynamic procedure as described in Statement 6 while working on limits. In addition to this, Jason did not use any tabular representation of functions throughout the eight lessons. Such representation of functions is likely to trigger the idea that one can find the limit of a function by just looking at the values that

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[2] Statement 6: A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached (Williams, 1991; 2001).

are close to the limit point. There was no evidence in Jason's discourse to support such an approach for finding a limit. The three students who chose Statement 6 as best describing their views of limit (See Table 6.4 and Figure 6.2) were likely to focus on the dynamic language of the statement and the process of "getting closer and closer to" the limit point since 8 out of 9 students who chose Statement 6 as true also marked Statement 1 as true (See Table 6.3 and Figure 6.1).

Research on learning about limits highlights the close relationship between the dynamic view of limit as realized through a description similar to Statement 6 and the student difficulty "limit as approximation"[3] (Bezuidenhout, 2001). In other words, students who consider plugging in values successively closer to the limit point can view limit as an approximation. Statement 5 of the diagnostic survey[4] considers limit as an approximation and the statement was marked as true by 12 students (See Table 6.2). Although five of these students also marked Statement 6 as true, ten marked Statement 1 as true (See Table 6.3). Therefore, some form dynamic view of limits was related to the view "limit as approximation". That the majority of students thinking about "limit as approximation" chose Statement 1 but not Statement 6 as true might again result from the fact that Jason did not utilize the procedure implied by Statement 6 in the class. Only one student chose Statement 5 as best describing his realization of limits (See Table 6.4 and Figure 6.2).

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[3] See Section 2.1.2 for the details of the student difficulty "limit as approximation".

[4] Statement 5: A limit is an approximation that can be made as accurate as you wish (Williams, 1991; 2001).

"Limit as unreachable" is another student difficulty resulting from the dynamic view of limits[5] indicated by the literature (Tall & Vinner, 1981; Williams, 1991). According to Tall and Schwarzenberger (1978), the colloquial use of the words such as *close to* implies getting near to but not being coincident with. Given this, they argue that the informal notion of limit may carry for students the assumption that one can get close to the limit value but cannot reach it. Statement 4 of the diagnostic survey[6] described limit as a number that cannot be reached and was considered as true by 13 students (See Table 6.2). 10 of these students also marked Statement 1 as true, suggesting that "limit as unreachable" is a difficulty based on dynamic view (See Table 6.3). This can further be supported by the fact that the 13 students who chose Statement 4 as true chose Statement 4 (six students) and Statement 1 (five students) as best describing their realizations of limits (See Table 6.4 and Figure 6.2).

Statement 1 is based on the idea that limit is a process and talks about limit as a descriptor of how a function moves as the x values move toward the limit point (See Table 6.1). Such consideration of limit is consistent with the informal aspect of limit Jason employed in the classroom when he investigated the behavior of a function near the limit point. Jason never endorsed the narrative "limit is unreachable" but some characteristics of his discourse on limits supported this view of limit. It was in the context of computing limits that he switched between operational and objectified word use treating limits (and also infinity) as processes and as end products (numbers) depending on the problems that he worked on (See Sections 5.1.5.3.1 and 5.1.5.3.2). When

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[5] See Section 2.1.2 for the details of the student difficulty "limit as unreachable".

discussing the behavior of a function near the limit point, Jason mainly explored what value *f(x) approached* as the x values *approached* the limit point. While doing so, he also used other words besides "approaches" such as "getting closer and closer to", "tending to", and "becoming" (See Section 5.1.2). Talking about limit as a process in which the function values "approach", "get closer and closer to" or "become" a number L carries an implication that one can get close to the limit but cannot reach it.

The different ways Jason addressed the limit notation is another aspect of his discourse that supports a view of "limit as unreachable". Throughout the eight lessons, Jason referred to the arrow sign in the limit notation $\lim_{x \to a} f(x) = L$ as the x values "approaching", "going to", and "getting closer and closer to" the limit point, which signifies the process of moving towards the limit point without reaching it (See Section 5.2.4). This is consistent with some students' views that the x values do not reach the limit point with the idea that the function values can never reach the limit value. In the context of continuity (See Section 5.1.5.4), Jason also mentioned that the function value at the limit point is irrelevant to the limit value; "what only matters is what happens nearby [the limit point]" (Jason, 28 January, 2009). This might be another reason for students' conclusion that the function values cannot be reached while determining the limit.

To sum up, student responses to the fourth statement in the survey reveal the connection between the view "limit as unreachable" and the dynamic-theoretical aspect of limit. The instances in which Jason shifted his word use in the context of computing

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[6] Statement 4: A limit is a number or point the function gets close to but never reaches (Williams, 1991; 2001).

limits support such a realization of limit. In those instances, Jason investigated the behavior of the function near the limit point and, hence, considered limit as a process. Although at the end of all such processes he referred to limit as a number, he did not explicitly address process and product aspects of limits in his discourse leaving an explanation for the shifts in word use in the classroom.

The final statement that will be elaborated on in Question I of the survey is Statement 2, which describes limit as a bound[7]. Students with this view think that a function is bounded by a specific limit value or that the absolute maximum/minimum values of the function are the limits for the function (See Section 2.1.2). This statement was marked as true by six students in the classroom (See Table 6.2). Compared to the other statements in the survey, Statement 2 was selected by the least number of students as true and was not selected by any of the students as best describing how they make sense of limits (See Table 6.4 and Figure 6.2). It is difficult to contemplate whether Jason's discourse on limits played any role in students' consideration of limit as a bound. This view of limit is often based on the colloquial use of the word *limit* than the mathematical aspects of the concept. Everyday uses of the word in phrases such as "the speed limit is 25mph", "we reached the city limit", and "we have to limit our expenses" could result in students' realization of limit as a constraint or a boundary. There was no evidence in Jason's discourse of support for this view of limit. Note, however, that all of the students who marked Statement 2 as true also marked Statement 4 ("limit as unreachable") as true (See Table 6.3 and Figure 6.1) and four of the six students who marked Statement 2 as true selected Statement 4 as the best statement to describe their views of limit (See Table 6.4 and Figure 6.2). Therefore, these students were likely thinking of "limit as bound" as connected to the view "limit as unreachable", making this a difficulty related to a dynamic view[8].

6.1.2. Questions III and IV

The third question in the survey asked students to describe what they understood a limit to be using their own words. Table 6.5 shows the student responses for Question III. One student did not give a response for Question III (Table 6.5, [5]) and one mentioned that limit was not a clear idea for him (Table 6.5, [8]). One student did not mention *limit* or the limit value L and used an arrow to represent the x values approaching the number one requires (Table 6.5, [7]). However, this student did not use the word "approaches" but just relied on the arrow to communicate her ideas about limit. Therefore, although it seemed that she was describing x values approaching the limit point, her description did not provide clear evidence about her view of limit. These responses (Table 6.5, [5], [7], [8]) were not descriptive enough to gain information with respect to the students' realizations of limit.

Three students used elements of the formal view of limit as indicated by the survey (See Table 6.1, Statement 3) using words such as "arbitrarily close" (Table 6.5, [6], [10], [15]). The first student also mentioned x values "sufficiently approaching" s (Table 6.5, [6]). In the formal view of limit, motion is eliminated and the x values *are* sufficiently *close* to s rather than *approaching* s.

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[7] Statement 2: A limit is a number or point past which the function cannot go (Williams, 1991; 2001).

[8] The converse of the argument was not necessarily true. Only 6 of the 13 students who marked Statement 4 as true marked Statement 2 as true (See Table 6.3 and Figure 6.1) and none of those 13 students selected Statement 2 as best describing their realizations of limit (See Table 6.4 and Figure 6.2).

Table 6.5: Students' responses to the third question of the diagnostic survey[9]

Response No.	Describe what it means to say that the limit of a function *f* as $x \to s$ is some number *L*. (N=23)	Best statement describing students' view of limit (See Table 6.1)
[1]	As *x* gets closer and closer to the value *s* then y values of *f(x)* get closer and closer to *L*	Statement 3
[2]	When *x* approaches to *s*, the relative value on y-axis will be close to *L*	Statement 1
[3]	That the function won't reach L as *x* approaches *s*.	Statement 4
[4]	Limit is while *x* approaches to an initial interval of the function, it may be closer to the value at the function at that point. It's approximately close; however, never reaches.	"None"
[5]	No response is given.	Statement 6
[6]	If *x* sufficiently approaches s, *f* arbitrarily close to *L*.	Statement 6
[7]	A number $x \to$ the number you require	Statement 5
[8]	A limit is maybe not so clear of an idea to me.	Statement 4
[9]	Functions have limits. Limits are the point or number the *f(x)* can never reach but can get as closer as it can.	Statement 4
[10]	Limit is when *x* approaches arbitrarily close to a given number. There is a *y*-values counterpart with it.	Statement 4
[11]	What value in the *y* coordinate is approached when *x* approaches *s*.	Statement 6
[12]	*L* would be the point of the *y* value of the function *f* as it approached the point *s*.	Statement 1
[13]	A limit describes a function as it gets closer and closer to a point. As *x* approaches a number *y* approaches the limit.	Statement 1
[14]	This means that as *x* approaches some number *s* the limit is some number *L*.	Statement 1
[15]	As *x* approaches *s* it will be arbitrarily close to *L* but never reaches *L*.	Statement 4
[16]	A limit is some number *L* that a function can get really close to but never actually reaches. If you say $\lim_{x \to s} f(x) = L$ you are saying as you approach *s* on *f(x)* you get *L*.	Statement 4
[17]	A limit is a description of what a function comes as it approaches certain values, i.e. $\lim_{x \to 0} 2x - 4 = \infty$	Statement 1
[18]	I understand limit to be the *y*-value a function gets close to as the *x* value approaches a number.	Statement 1
[19]	A limit is a value that the function approaches as *x* approaches a certain value.	Statement 1
[20]	To me a limit is just a point that a function approaches at a given *x*-value.	Statement 1
[21]	As *x* gets closer to *s* the # [number] gets closer to *L*.	Statement 1
[22]	As *x* approaches *s* the *y* values get close to *L*.	Statement 1
[23]	$\underset{x \to s}{f(x)} = L$ means that as the function's *x* values become closer and closer to *s*, the *y* value of the function becomes closer and closer to *L*.	Statement 1

9 Any grammatical, symbolic or mathematical errors in the sentences are preserved in order to keep the

This student used a combination of words from both the formal and the dynamic aspects of limit and chose Statement 5 ("limit as approximation") as the best statement describing her view of the concept (Table 6.5, [6]).

The second student using elements of the formal aspect considered the x values (but not the function values) "arbitrarily" approaching a given number and mainly described limit as a process (Table 6.5, [10]). Similar to the first student, she used both the formal and dynamic views related to limit since she did not talk about being *arbitrarily close* but "arbitrarily approaching". She did not consider limit as a number but as something that happens "*when* x approaches" a number (Table 6.5, [10], emphasis added). As a result, despite her attempt to use the formal aspect of limit, this student's description of limit was based on the realization of limit as a process.

The third student who used elements of the formal aspect of limit mentioned the value L, without referring to it as a number (Table 6.5, [15]). He also used the word "approaches" when describing the behavior of the x values. Therefore, there were elements of both the formal and the dynamic aspects of limit in his description. The existence of the dynamic view of limit in his response can further be supported by the fact that he considered the value L as unreachable. In fact, this student and the second student chose Statement 4 ("limit as unreachable") (See Table 6.5, [10], [15]) as best describing their view of limit and they both marked Statement 3 (formal view of limit) as false in the survey. On the other hand, all three students marked Statement 1, the dynamic-theoretical view (See Table 6.1), as true. Hence, student responses [6], [10], [15] (Table 6.5) were considered as dynamic but not formal.

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originality of student responses.

The six students mentioned above were the cases in which it was hard to infer realizations of limit directly from definitions either due to a lack of response or a mixture of word use. All of the remaining 17 students clearly used some version of the dynamic view when describing their realizations of limit. The only student who marked Statement 3 (formal view of limit) as best describing his view of limit was among those 17 students (See Table 6.4 and Table 6.5, [1]). Only four of the 17 students referred to limit as a "number" or "the y value of the function" (Table 6.5, [9], [12], [14], [16]). The remaining 13 students described limit as a process. Overall, there were four students referring to limit as objectified (number), 16 students[10] referring to limit as operational (process), and three students whose descriptions of limit could not be classified.

Some of the 16 students referring to limit as operational described the process of the function values approaching the limit value as x values got closer and closer to the limit point. Some others considered limit as a descriptor of how a function behaves (e.g., Table 6.5, [13], [17]). Therefore, although Jason's word use in the classroom was mainly objectified rather than operational, students adopted his operational word use. Jason referred to limit as a process only verbally, not writing any of his operational word use on the board (See Section 5.3.3). On the other hand, student responses for the third question of the survey show that the instructor's verbal comments played a dominant role in students' descriptions of limit. Jason's word use shifted from operational to objectified (and vice versa) in the contexts of informal definition of limit, computing a limit, and the limit notation. It was argued in Chapter V that students were unlikely to distinguish the

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[10] The total number of students viewing limit as a process was 16, including the three students who used a combination of formal and dynamic view in their description.

process aspect of limit from the product aspect in these contexts since such shifts remained implicit in Jason's discourse

Another aspect of Jason's discourse that supported the realization of limit as a process and remained implicit for students was his routine of using the metaphor of continuous motion in the context of computing limits and the informal aspect of limits (See Section 5.3.4). Although he also used the metaphor of discreteness when talking about the formal aspect of limits (See Section 5.3.5), students in the study only relied on the metaphor of continuous motion in their discourse.

Jason's shifts in word use and his utilization of the metaphor of continuous motion are consistent with the fact that, no matter how frequently and explicitly he endorsed the narrative *limit is a number* (See Section 5.4), students mainly endorsed the narrative *limit is a process* in their responses to the survey. Throughout the eight lessons, Jason explicitly endorsed the narrative *limit is a process* only twice (See Section 5.4). However, he endorsed it implicitly every time he computed the limit of a function by investigating the function's behavior near the limit point. Despite his referral to limit as a number (if it existed) at the end of every computation, the notion mainly remained as a process for students.

Responses to Question III also revealed five students' consideration of "limit as unreachable" (Table 6.5, [3-4], [9], [15-16]). These students explicitly mentioned that the function values cannot reach limit value L and all of them described limit using elements of the dynamic view. Moreover, all five students marked Statement 4 (See Table 6.1), which described "limit as unreachable", as true and four of them selected it as the best statement with which they made sense of limits (Table 6.5). One student chose none of the statements as best describing her view of limits[11].

The fourth question in the survey asked students to give a rigorous definition of limits, if possible. While administering the survey, I told students that a rigorous definition of limit is what Jason referred to as the precise (or formal) definition of limit in the class. In spite of this, majority of the students left this question unanswered. Only five of the 23 students provided a response for Question IV of the survey. Table 6.6 shows student responses for the question.

Table 6.6: Students' responses to the fourth question of the diagnostic survey[12]

Response No.	If possible, write down a rigorous definition of limit. (N=23)
[1]	$\underset{x \to s}{f(x) = L}$
[2]	"limit" is what is used to describe the number that we would try our best to get on y-axis when x approaches to another number.
[3]	Limit is approaching a value that usually cannot be defined or = in a normal equation.
[4]	As x approaches a number, y approaches a number
[5]	A limit is a value that a function can never reach, however, it only comes arbitrarily close to it.

One of the students considered limit as a "value a function can never reach" (Table 6.6, [5]) when providing a rigorous definition of limit. Three students talked about limit as a process (Table 6.6, [2-4]) and one gave the symbolic notation as a rigorous definition of limit (Table 6.6, [1]). This question did not provide additional information in terms of students' realizations of limit both because there were very few student responses and also the descriptions students provided had similar characteristics to those they provided

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11 This student marked "none" for Question II of the survey (See Table 6.4).

12 Any grammatical, symbolic or mathematical errors in the sentences are preserved in order to keep the originality of student responses.

for Question III. The dynamic view (Table 6.6, [2-5]) and the consideration of "limit as unreachable" (Table 6.6, [5]) were again common themes in students' definitions.

Students' lack of responses for this question could result from their unfamiliarity with the word "rigorous". Jason used the term "precise" when talking about the formal definition of limit in the class. Although I mentioned that those two terms mean the same thing while administering the survey, it is possible students did not realize these terms as the same. Another reason this question was left unanswered by the majority of students is that Jason presented the formal definition of limit as optional, a personal challenge rather than a required topic for the exam (See Section 5.1.5.2). Therefore, it is possible that students did not attend to the formal aspect of limit carefully or forgot the complicated statements including ε and δ as well as the existential quantifiers such as "for all" and "there exists" in the formal definition.

Summary

The responses given to the diagnostic survey revealed that the students in the study had a dynamic view of limits at the end of their instruction. 16 of 20 students who had the dynamic view of limit described limit as a process[13]. Four students mentioned limit as a number but still used the dynamic view in their descriptions. This result was consistent with the fact that the statement describing limit by dynamic-theoretical means (Statement 1 in Question I, see Table 6.2) was marked as true by 20 of the 23 students taking the survey. Dynamic view of limit refers to the consideration of the concept by means of continuous motion and results in the realization of limit as a process. The instructor's

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[13] Three students' explanations of limit were not explanatory enough to conclude about their descriptions of limit (See Section 6.1.2).

routines, and shifts in word use when computing limits, referring to the limit notation and talking about the informal aspects are consistent with students' realization of limit as a process.

The statement which described the formal view of limit was the next popular statement students chose as true (Statement 3 in Question I, see Table 6.2). The three students who incorporated elements of the formal definition to their description of limit did so inaccurately since they mainly relied on the dynamic aspect of limit rather than the discrete or static aspect in their explanations. Said differently, the formal view of limit was considered as true by the majority of students but was not employed in their descriptions of limit. They had seen the formal (precise) definition which Jason introduced in the classroom, but perhaps did not grasp it implications, instead relying on motion in their descriptions of limit. In the end, only four of the students mentioned limit as a number, the objectified view that dominated Jason's discourse.

Statement 4 and Statement 5 in the first question of the diagnostic survey were also chosen as true by the majority of students (See Table 6.2). Statement 4 assessed students' view of "limit as unreachable" whereas Statement 5 assessed the view of "limit as approximation". These views are addressed by research on learning limits as student difficulties resulting from the dynamic aspect of limit (Tall & Schwarzenberger, 1978; Williams, 1991). Students' responses to these statements, together with their definitions of limit are in accord with their consideration of limit as a process.

Some of Jason's discourse on limits possibly had a direct impact on student learning. Consistent with Jason's discourse, the majority of the students marked the dynamic-practical aspect of limit as false (Statement 6 in Question I, see Table, 6.2).

Recall that Jason did not find limits of functions by plugging in points successively close to the limit value. Similarly, his introduction of the precise or the formal definition of limit led to some familiarity with the words associated with the formal aspect of limit. Such familiarity supports students' consideration of the statement describing the formal view as true. The view "limit as boundary" did not seem to be related to Jason's discourse on limits in the classroom. Instead, this realization of limit was more closely related to the colloquial use of the word *limit* (See Section 6.1.1).

Some other elements of Jason's discourse on limit, however, support students' consideration of limit as a process. His routine of using the metaphor of continuous motion and switching between the operational and objectified word use in the contexts of informal definition of limit, computing limits, and referring to the limit notation are consistent with students' realizations of limit as a process based on dynamic motion. Jason talked about limit as a process when he investigated the behavior of functions near the limit point. Although he referred to limit as a number after determining the behavior of the functions, students mentioned the process but not the end result of the process in their own descriptions of limit.

6.2. Responses to the individual interview sessions

In this section, I explore four students' discourse on limits by focusing on (a) elements of the instructor's discourse on limits, and (b) the difficulties indicated by research on student learning about limits. I use pseudonyms – Amy, Jessica, Harry, and Keith – for the students' names. Among these students, Amy, Harry and Keith were from the list of students I initially identified to interview. Jessica volunteered and I interviewed her as well (See Section 4.2). All four students responded correctly to five of the six statements in Question I of the diagnostic survey (See Table 6.1). However, many of the difficulties identified by research on learning about limits were found in students' discourse during the interview sessions.

The questions in the interview session were designed to probe students' realizations of limits. The diagnostic survey did not address student difficulties such as "limit implies continuity", "limit as the function's value", and "limit as monotonic" (See Section 2.1.2, Table 2.1) nor did it provide information regarding students' views of continuity and infinity. The questions in the interview sessions were useful to focus on ideas not addressed by the survey. The questions also provided contexts in which student difficulties "limit as unreachable", "limit as boundary", and "limit as approximation" (See Section 2.1.2, Table 2.1) were investigated in further detail. Figure 6.3 shows the problems in the interview sessions[14].

I started the interview sessions asking some general questions to students in terms of their background as well as how and whether they utilized the textbook in their learning. Table 6.7 provides some general information about the students obtained both through the diagnostic survey and the interview sessions. Note that none of the students used their textbook for reviewing material or preparing for the exam[15].

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[14] The interview session problems as shown in Figure 6.3 are also included in Appendix B.

[15] This was the primary reason the analysis of the textbook with respect to its discourse on limits was considered beyond the scope of this study.

1) If $f(x)=\begin{cases} x & \text{when } x \ge 0 \\ x^3 & \text{when } x < 0 \end{cases}$, what can you say about

$\lim_{x\to 0} f(x)$, $\lim_{x\to -\infty} f(x)$, $\lim_{x\to \infty} f(x)$? Is the function continuous at 0? How about negative and positive infinity?

2) If $f(x)=\begin{cases} x^2+1 & \text{when } x \le -1 \\ x-1 & \text{when } x > -1 \end{cases}$, what can you say about

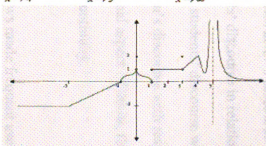
$\lim_{x\to -1} f(x)$ $\lim_{x\to -\infty} f(x)$, $\lim_{x\to \infty} f(x)$? Is the function continuous at -1? How about negative and positive infinity?

3) Given the following graph of a function f, what can you say about the following limits? Explain your reasoning.

$\lim_{x\to -\infty} f(x)=?$ $\lim_{x\to -5} f(x)=?$ $\lim_{x\to -1} f(x)=?$

$\lim_{x\to 0} f(x)=?$ $\lim_{x\to 1} f(x)=?$ $\lim_{x\to 2} f(x)=?$ $\lim_{x\to 3} f(x)=?$

$\lim_{x\to 4} f(x)=?$ $\lim_{x\to 5} f(x)=?$ $\lim_{x\to \infty} f(x)=?$



Find all the points where $f(x)$ is discontinuous. Why is the function discontinuous at those points?

4) A student was given a function F and asked to find the limit of F as x approached 0. He plugged in numbers on each side of 0 and made the following table:

x	$F(x)$
-.1	0.9
-.01	0.99
-.001	0.999
-.0001	0.9999
-.00001	0.99999
-.000001	0.999999
-.0000001	0.9999999
-.00000001	0.99999999
.1	1.1
.01	1.01
.001	1.001
.0001	1.0001
.00001	1.00001
.000001	1.000001
.0000001	1.0000001
.00000001	1.00000001
.000000001	1.000000001

a) What can you say about $\lim_{x\to 0} F(x)$?

b) What do you think about the student's method of finding this limit? Is he correct? Can we find the limit of functions using this method? Why/Why not?

c) Given $F(x)=x+1+\frac{1}{10^{20}x}$, what can you say about $\lim_{x\to 0} F(x)$?

5) Which statements below must be true if f is a function for which $\lim_{x\to 2} f(x)=3$?

a) f is continuous at the point $x=2$

b) f is defined at $x=2$

c) $f(2)=3$

d) $\lim_{h\to 0}\{f(2+h)-3\}=0$

e) For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if $0<|x-2|<\delta$, then $|f(x)-3|<\varepsilon$

f) None of the above mentioned statements

6) Let a function $f(x)$ be defined on an open interval about [the point] x_0, except possibly for x_0 itself. We say that the *limit of $f(x)$ as x approaches to x_0 is the number L*, and write $\lim_{x\to x_0} f(x)=L$ if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$. (Thomas et al., 2008, p. 91)

a) Please explain in your words what this definition means for you.
b) How is this definition similar/different than the informal definition of limit you used in your class?
c) In your opinion, what is the purpose of this definition?

Figure 6.3: Interview session problems

Table 6.7: General information about the students participating in the interview sessions

	Amy	Jessica	Harry	Keith
Statement best describing their view of limit in the survey (See Table 6.1)	Statement 1	Statement 1	Statement 6	Statement 3
Major	Sociology-Residential College	Finance	Computer engineering	Computer science
Year in the university	First year	First year	Second year	Second year
Did you take calculus in high school?	No	Yes	No	No
Did you take calculus before during your undergraduate study?	No	No	No	Yes (Once)
Do you use your textbook besides the homework problems?	Not at all	Never	Never	No
Do you use the lecture notes for reviewing material or preparing for the exam?	No	Yes	No	No
What sources do you use for reviewing material or preparing for the exam?	Reviews the homework problems and asks friends	Uses lecture notes	Watches Math TV from YouTube and asks friends	Reviews the homework problems

They only used the textbook to work on the assigned homework problems. Only one of the students used the lecture notes whereas the other students relied on other sources, such as friends and internet-based lessons, while studying for the course.

6.2.1. Students' discourse with respect to the instructor's discourse on limits

In this section, I investigate the students' discourse in relation to the instructor's discourse on limits. The contexts in which the students' discourse will be discussed in this section emerged from the analysis of Jason's discourse with respect to his word use (See Chapter V). Those contexts are (a) informal aspect of limits, (b) formal aspect of limits, (c) limit notation, (d) infinity, and (e) continuity.

6.2.1.1. Informal aspect of limits

All of the students participating in the interviews made frequent use of the dynamic view of limit as they worked on the limit computation problems and talked about limits. Two of the students, Amy and Jessica, considered the statement in the diagnostic survey that described limit as a process as the best statement describing their view of limits[16] (See Table 6.7). Harry picked the statement that described limit as an approximation as the best statement, also a dynamic view of limit. Keith was the only student who considered the formal view of limit as the best statement with which he made sense of limits. During the interview, however, he talked about limit as a process and relied on the informal aspects of limit.

During the interviews, students computed limits and wrote those limits as *equal to* particular numbers. On the other hand, none of the students explicitly referred to limit as a number. Instead, they described the behavior of the function values approaching the limit value as the x values approached the limit point. This was consistent with Jason's operational word use and his routine of using continuous motion in the class. Jason's discussions of limit as a process (operational word use) only took place when he communicated his ideas verbally (See Section 5.2.1). He did not write any of his operational utterances on the board. The students, however, talked about the instructor's investigation of the behavior of the function as x approached the limit point as a definition of limit. In fact, Amy thought that finding limits in this manner "fit the definition he [the instructor] presented to us in class" (Amy, 3 March, 2009). Note that the instructor's informal definition of limit (See Table 5.6 and Figure 5.4) referred to limit as a number (objectified word use). His treatment of limit as a process mainly took

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[16] During the interview, Jessica mentioned that it was a combination of the dynamic view and the formal view that she made sense of limits. She said she chose Statement 1 (See table 6.1) because it was simpler.

place when he computed limits. Therefore, he did not technically present the process aspect of limit as a definition to students in the class.

When investigating the behavior of a function near the limit point, students' terminology was similar to the instructor's, frequently using words signifying motion:

> Amy: Coming from the right, it [the function value] is approaching zero and coming in from the left, it is approaching zero.
>
> Jessica: As x moves towards zero, the y values change.
>
> Harry: It [the function value] is approaching; it gets closer and closer to one.
>
> Keith: ...the limit is a value as it is approaching a certain number.

Note that although Keith uttered the word "number", he did not refer to limit as being equal to the number. Instead, he mentioned limit as it was approaching that number, which was the function's y value. Jessica explicitly mentioned that limit was not just a number and used the word "proceed" to describe limit as a process.

> Jessica: The limit is not only a particular number.
>
> Researcher: The limit is not a particular number. What is it then?
>
> Jessica: It is a moving proceed [sic] I think.
>
> Researcher: When you say proceed, what do you mean?
>
> Jessica: x approaches c; it is not equal to c.

Students talked about limit as a process also when they attended to the right hand and the left hand side of functions. All of the students computed right hand and left hand limits at some point during the interview but did not utter the word "limit". Instead, they mentioned "approaching from the left/right". The frequency with which they utilized this approach depended on the context. Keith and Jessica consistently used left and right limits throughout the questions whereas Amy and Harry did not attend to both sides when the function was represented algebraically.

Graphing was the primary routine three students utilized when investigating the behavior of a given function. The only problem involving a graph in the interview session was Question 3 (See Figure 6.3). Keith and Jessica drew graphs while working on every problem in the interview. Harry drew graphs for Questions 1, 2 and 4 (part c) of the interview (See Figure 6.3). These students used graphs not only to compute the limit of a function at a particular point but also to communicate their thinking about limits and to provide examples or counter-examples to some of the arguments in the questions. On the other hand, they expressed a dynamic view of limit (using the metaphor of continuous motion) and treated limit as a process every time they talked about their graphs. Amy did not draw any graphs during the interview due to a particular view of limit ("limit implies continuity"), which will be discussed later in the chapter. She mainly attended to the function's value while computing limits and did not use graphing as a routine to communicate her ideas.

In summary, all of the students' word use about limit was mainly operational rather than objectified since they talked about limit as a process but not as a number. This was in contrast with Jason's discourse since his word use was mainly objectified. The students considered the x values approaching the limit point and the function values approaching the limit as elements of the dynamic view, which was based on continuous motion. The way they explored the behavior of a function near the limit point was consistent with the instructor's routine of computing limits (See Section 5.3.3). On the other hand, unlike the instructor, students could not refer to limit as a number at the end of the limiting process during the interview. It was mentioned in Section 5.4 that Jason explicitly endorsed the narrative *limit is a process* only twice whereas he endorsed the narrative *limit is a number* consistently in the classroom. The students in the interview sessions, however, endorsed the former narrative about limits.

6.2.1.2. Formal aspect of limits

The formal definition of limit Jason introduced in the classroom provided a context in which to highlight the static aspect of limit. During his discussions on the formal definition, his word use about limits was consistently objectified. Moreover, he did not utter words signifying motion but used words signifying proximity instead (See Section 5.1.5.2). While doing so, he used the metaphor of discreteness as a routine (See Section 5.3.4). Question 5 (part e) and Question 6 of the interview session problems (See Figure 6.3) were used to gain information on students' view of the formal definition of limit and how they talked about this definition as being similar to or different from the informal definition of limit Jason introduced in the classroom. In Question 6, I used the textbook's definition of formal definition of limit, which was similar to Jason's formal definition of limit (See Figure 5.5), since the textbook's definition refers explicitly to ε and δ as numbers.

All of the four students marked the statement in the diagnostic survey that described limit by formal means as true (See Statement 3 in Table 6.1). However, they all relied on operational word use while talking about the formal definition of limit in the context of Question 6 of the interview session. After reading the formal definition in Question 6 (See Figure 6.3), Amy split it into three parts as follows:

Part 1. Let a function $f(x)$ be defined on an open interval about [the point] x_0, except possibly for x_0 itself.

Part 2. We say that the *limit of* $f(x)$ *as* x *approaches to* x_0 *is the number* L, and write

$$\lim_{x \to x_0} f(x) = L$$

Part 3. if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. (Thomas et al., 2008, p. 91)

> Amy: [Referring to Part 1] So it says like it has to be defined which makes me think that it has to be continuous which kind of goes along with what I already know about limits...[Referring to Part 2] So this part...it goes along with my informal definition like really well...[Referring to Part 3] For every number...there exists a corresponding delta such that...for me goes along with that proof which it is kind of like 'whatever'.

Amy considered the first part of the formal definition as describing the continuity of the function at the limit point. The second part, which basically described how to read and symbolically represent the limit notation, matched her dynamic or informal view of limit. The third part, on the other hand, represented the part of the definition that she ignored. When she was asked to explain why Part 3 did not make sense to her, she said

> When we did the proofs of the limits, he [the instructor] was not going to test us on it and he said that we would never really have to do them. So I didn't pay attention... I completely disregarded it after that homework assignment. The way he presented the material to us by saying that we

would never have to know it for a test just makes me not pay any close attention to it. (Amy, 3 March, 2009)

Jason's mentioning of the formal definition as a personal challenge and his explicit statement that the homework problems would not be graded resulted in Amy's ignorance of the formal definition as a relevant part of the course. When she was asked to explain how she thought of the informal and formal definitions as similar or different, she mentioned that Part 2 "is the informal definition" (Amy, 3 March, 2009) and did not say more. In Amy's discourse, the static aspect of limit was missing. Dynamic view of limit through the assumption of continuity was how she talked about the formal definition.

Harry realized Question 5 (part e) in the interview session (See Figure 6.3) as related to the formal definition but did not want to work on it saying he "hated those symbols" (Harry, 27 February, 2009). He showed some effort to work on the definition in Question 6.

> So what I recall from the epsilon and delta is that difference between epsilon is proportional to the delta. So if there is a difference of let's say two and an x value, it will be proportional to the y value. If there was like two x, the y value would be four. If it was four x, the y value would be eight. It is like proportional. (Harry, 27 February, 2009)

Note that he used the term "difference", a word signifying proximity instead of motion, and also referred to some sort of dependence. The dependence he described by means of proportionality seemed, however, to express y values as dependent on the x values rather than δ being dependent on ε[17]. When describing how he thought about this definition as similar to or different from the informal definition, Harry only commented on the sentence where the limit notation and how to read the notation was introduced. He said "they are actually very similar. When Jason [the instructor] did it, he said the limit as x approaches a of f of x equal to L...He explained that L was the y value as x is approaching a" (Harry, 27 February, 2009). Harry did not comment on the part of the definition including ε and δ. Harry realized the formal definition of limit as indicating a type of dependence but, similar to Amy, he relied on the dynamic view of limit while talking about the formal definition. Amy and Harry both talked about the part of the definition that introduced how to read and write the limit notation as related to the informal definition of limit. As a result, they concluded that the two definitions were similar.

Unlike Amy and Harry, Jessica and Keith used graphs as visual mediators when explaining how they made sense of the formal definition. They both drew correct graphs representing the relationships between ε and δ. They both referred to ε as related to the difference between the function values and the limit value, and δ as related to the difference between the x values and the limit point. They also mentioned that the graph the instructor drew when he worked on a proof problem (See Table 5.20, the last graph) was very useful for them while thinking about limits. Jessica split the definition into four parts as follows.

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[17] In the formal definition, δ (which is related to the difference between x values and the limit point) depends on ε (which is related to the difference between the function values and the limit value).

Part 1. Let a function $f(x)$ be defined on an open interval about [the point] x_0, except possibly for x_0 itself.

Part 2. We say that the *limit of* $f(x)$ *as* x *approaches to* x_0 *is the number* L, and write

$$\lim_{x \to x_0} f(x) = L$$

Part 3. if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$

Part 4. such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. (Thomas et al., 2008, p. 91)

She said Part 1 of the definition "is like the requirement...like it should be an open interval and except for *x* zero" (Jessica, 5 March, 2009). She wrote "moving" near the second part of the definition. She described Part 3 of the definition as relation and said "it shows that there is a relationship between...what happened on the *x* axis and what will happen on the *y* axis. This is the relation part" (Jessica, 5 March, 2009). She referred to Part 4 of the definition "this part just wrote words into another form. But when you see here [shows $|x - x_0|$]...it tells you...it should...move both sides" (Jessica, 5 March, 2009). When asked how the informal definition was similar to or different from the formal definition, Jessica said that they both mention approaching particular points but that the formal definition was more confusing since "there are three things to figure out: *L*, epsilon and delta" (Jessica, 5 March, 2009). Note that she connected these two definitions by means of motion. Still, Jessica considered the formal view, together with the informal view of limit as best describing how she made sense of the concept.

Keith was the only student who considered the statement that described limit by means of a formal view (See Statement 3 in Table 6.1) as best describing his realization of limits in the diagnostic survey. During the interview, he said "the informal approach helped in give me the right answers and working through it and like you know the quick methods…But I thought that the formal definition provided a much better understanding of what the actual definition was" (Keith, 3 March, 2009). However, Keith was primarily referring to the graph the instructor drew for the proof problem as the most useful tool to make sense of the formal definition. He said he followed the arguments about the formal definition visually but was initially confused about the symbolism included in the definition when he worked on Question 5 (part e) (See Figure 6.3).

> When we went over the formal definition, we didn't really discuss what these symbols were [shows ε and δ]…When we went over the definition, like I got it visually. I got it visually what we are doing and it helped immensely. But when we were using…like I don't really remember what these symbols are…(Keith, 3 March, 2009)

On the other hand, in the context of Question 6 of the interview session, Keith showed a lot of effort to make more sense of the symbolism. Without any guidance and using only the graph he initially drew for the formal definition, he was able to realize ε in relation to the difference between the function values and the limit value, and δ in relation to the difference between the x values and the limit point. However, he defined $|x - x_0|$ as equal to δ and $|f(x) - L|$ as equal to ε: "it almost seems like x minus x zero should be equal to sigma [he means delta] and f of x minus the limit should be equal to epsilon" (Keith, 3

March, 2009). When asked how he thought the formal and the informal definitions as similar to or different from each other, Keith said

> [Referring to the formal definition] It is similar because it is kind of saying the same thing. It is different in the fact that it has all these symbols and it's much more...Using the visualization, that the informal definition and formal definition I believe should be taught simultaneously kind of together. Because the formal definition allows me, for me anyways, to visualize it and see. (Keith, 3 March, 2009)

Jessica and Keith were more elaborate in their responses in relation to the formal definition of limit. However, the presence of the dynamic view of limit was apparent both in their word use and also in their routine of graphing throughout the interview. It was mentioned in Section 5.2.3 that the instructor drew a graph representing the formal definition of limit only after noticing that students did not follow his discussion on the proof problem that he worked on. That graph seemed to have enhanced Jessica's and Keith's realization of the formal aspect of limit.

Overall, all of the four students' discourse on the formal definition of limit was consistent with the instructor's in that they talked about the formal definition as similar to the informal definition. Recall that both the textbook's and Jason's definitions of limit included an explanation of how to read and represent the limit notation. It was because of the existence of the word *approaches* that all students related the formal view of limit to the informal one. The students' discourse on the formal definition differed from the instructor's in that students' word use about limit was not objectified and they used the metaphor of continuous motion instead of the metaphor of discreteness. Unlike their instructor, students did not use the phrases *arbitrarily close* and *sufficiently close*. Jason's word use (See Section 5.1.5.2) was most consistent in the context of the formal definition of limit in which he did not utter any motion related word and used words in an objectified way. Jessica and Keith were able to talk about the *difference* between values in the context of the formal definition. However, they connected difference with motion but not with discreteness. Therefore, although the context of formal definition gave students an opportunity to attend to the static aspect of limit, their realizations remained dynamic.

6.2.1.3. Limit notation

All of the students used words signifying motion when attending to the limit notation, which was similar to how the instructor talked about the notation (See Section 5.2.4). At some point during the interview, Jessica wrote $x \to a$ on the paper and said "this arrow means getting closer and closer to that particular point" (Jessica, 5 March, 2009). On the other hand, unlike the instructor, the ways students talked about the notation were often incomplete or inaccurate. Table 6.8 shows some examples of students' word use when addressing particular limit notations during the interviews.

Students used words such as "approaches" (Table 6.8, [1], [4], [6], [8], [10-11]), "goes to" (Table 6.8, [2]) and "at" (Table 6.7, [3], [7]) when referring to the arrow in the limit notations of the form $\lim_{x \to a} f(x) = L$. There were few occasions in which they talked about the function values or limit after describing the x values approaching a (Table 6.8, [5], [7], [11]). In the remaining cases, students mentioned the behavior of the x values and directly wrote the limit values on the paper without explicitly talking about them as numbers.

Table 6.8: Some examples of students' word use when referring to the limit notation

	Context (See Figure 6.3)	Notation	Word use
Amy	Question 1	$\lim_{x\to 0} f(x)$	[1] "Limit as *x* approaches zero…"
	Question 3	$\lim_{x\to -5} f(x)$	[2] "*x* goes to negative five…"
	Question 3	$\lim_{x\to 2} f(x)$	[3] "The limit at *x* equals two"
Jessica	Question 1	$\lim_{x\to 0} f(x)$	[4] "When *x* approach [sic] zero…"
	Question 4	$\lim_{x\to 0^-} F(x) = 1$	[5] "*x* negative to zero, it [the limit] will be one".
Harry	Question 2	$\lim_{x\to -1} f(x)$	[6] "The limit as *x* approaches negative one…"
	Question 3	$\lim_{x\to -5} f(x) = -2$	[7] "At negative five, it [the function value] is negative two".
	Question 3	$\lim_{x\to \infty} f(x)$	[8] "*x* approaches infinity…" [9] "When *x* is infinity…"
Keith	Question 2	$\lim_{x\to -1^-} f(x)$	[10] "As the limit approaches negative one from the negative side…"
	Question 3	$\lim_{x\to -\infty} f(x)$	[11] "As *x* approaches negative infinity, it [the function value] would equal two".

It was mentioned in Section 5.1 about word use that the instructor introduced the notation $\lim_{x\to a} f(x) = L$ which he addressed as "the limit as *x* approaches *a* of *f* of *x* equals some number *L*" (see Table 5.6, [2]). The students did not refer to the notation in a similar manner since (a) they rarely uttered the word "limit" when talking about the notation and (b) they rarely referred to limit as a number. There were also some instances where the students confused which quantities were approaching what or they thought the *x* values could reach the limit point. For example, Keith once stated that the "limit approaches negative one [the limit point] from the negative side" (Table 6.8, [11]) instead of talking about the *x* values approaching the limit point. Amy and Harry mentioned the *x* values being equal to the limit point when they said "limit at *x* equals two" (Table 6.8, [3]) and "when *x* is infinity" (Table 6.8, [9]), respectively.

Overall, students talked about the notation for x approaching the limit point in ways similar to the instructor's talk. However, many of the students' utterances regarding the notation were incomplete or inaccurate. Compared to the frequency with which they determined limits during the interview, they rarely verbalized the limit notation. They also did not talk about limit as a number both in the context of the limit notation and the limit computation problems although many of them consistently wrote the answer as a number.

6.2.1.4. Infinity

None of the students attempted to plug in infinity, treating infinity as a number, when they worked on Question 1 and Question 2 of the interview sessions (See Figure 6.3). On the other hand, some of their explanations and routines suggested otherwise. When I asked Amy why she did not plug infinity into the function value for the first question, she said "because infinity is not a number. It is not a defined value. So you can't just plug infinity… It cannot give me any answer to anything" (Amy, 3 March, 2009). When determining the limit at infinity[18] for the second question, however, she mentioned approaching infinity from the right and the left side as if it were a number.

Jessica explicitly talked about infinity as a number. She used the word "unlimited" for infinity. When working on Question 4 (part c) of the interview problems (See Figure 6.3), she incorrectly[19] attempted to use the limit law for the sum $\lim_{x\to 0} x + \lim_{x\to 0} \frac{1}{10^{20}x}$. She said the first limit would be zero and the second limit

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[18] Limit at infinity is a limit computed at positive or negative infinity.

[19] In order to apply the limit law for a sum, both limits have to exist and equal a number.

would be "unlimited". She wrote 0 for the former limit and $+\infty$ for the second limit and concluded that the answer would be $+\infty$.

Jessica: Because zero is a number and you add another number, it will be the number itself.

Researcher: Okay. Is positive infinity a number?

Jessica: Yeah. You can see it as a number...Zero add any number like five, it will be five itself. So when it [zero] is added to positive unlimited, it will be unlimited itself.

Researcher: So what does this symbol represent? [I show ∞] How do you think about infinity?

Jessica: Infinity? Unlimited.

Researcher: Is unlimited a number?

Jessica: Yes. You always can find another number larger than this number [she shows $+\infty$] and in this one [she shows $-\infty$], you can always find a number smaller than this number.

Researcher: Okay but you still think about it as a number in itself?

Jessica: Yes.

Since Jessica thought of infinity as a number, she also considered infinite limits[20] as existing limits. Although the instructor wrote $+\infty$ or $-\infty$ as the answer to the limit computation problems, he considered infinite limits as "undefined" or "does not exist" (See Table 5.14). Jessica's view of limit as a number suggested that she considered infinity as an end result rather than a process. However, she extended the graphs she drew and also the graph in Question 3 (See Figure 6.3) when determining the limits of functions at infinity. The instructor talked about infinity as potential, an ongoing process that never ended, and used the metaphor of continuous motion in the context of computing limits at infinity (See Sections 5.1.5.3.2 and 5.3.3). He talked about infinity as an end state in the context of infinite limits (See Section 5.1.5.3.3). For the latter type of limits, Jason indeed uttered the word "number" when referring to infinity (See Table 5.14). Jessica's extension of the graphs with the assumption of continuous motion when determining limits at infinity suggested that she realized infinity as potential in those instances. Her reference to infinity as a number in the context of infinite limits, however, showed that she considered it as an end state. Therefore, Jessica's word use about infinity resembled the instructor's discourse on the notion.

Harry also extended the graphs of functions, including Question 3, when he worked on limits at infinity. Unlike Jessica, he mentioned that infinity is not a number: "infinity is not a number because infinity is like… you can't define infinity; it's an endless… it can't begin somewhere and it can't finish somewhere" (Harry, 27 February, 2009). Although it seemed that Harry was about to call infinity an "endless process", he did not complete that sentence. His extension of the graphs for limits at infinity signaled the consideration of infinity as a potential but he seemed to relate infinity to being undefined or a state of no beginning and no ending.

Keith extended graphs saying that "it [the function] goes off" (Keith, 3 March, 2009). While doing so, he thought of infinity as potential. On the other hand, he

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[20] Infinite limit is a limit computed either at a point or infinity that has plus or minus infinity as the answer.

expressed confusion whether to consider an infinite limit as undefined or as being equal to infinity.

> Keith: I would think that they are different. I am still not entirely sure… I see them as different because undefined and infinity I view as two different things. Infinity would be continuously going and going and going; undefined would be closer to like something divided by zero or you know something that mathematically just doesn't make sense and so…I guess I am leaned more towards infinity.
>
> Researcher: So when you get infinity as an answer for the limit, do you think the limit exists overall or not?
>
> Keith: I view them as completely different. In undefined, I would say that the limit does not exist but in infinity I would say the limit exists as infinity.

Keith's consideration of infinity as potential and use of dynamic motion was apparent when he described it as "continuously going and going and going". Yet, he also described an infinite limit as existing. This was similar to Jessica's view of infinity and was different than the instructor's discourse on infinite limits.

During the interviews, students' word use and routine of extending graphs in the context of limit at infinity were in accord with the instructor's word use (talking about infinity as potential) and routine of using the metaphor of continuous motion. Their word use was different from the instructor's in that none of the students referred to an infinite limit as undefined or nonexistent. Only Jessica mentioned infinity as a number, which suggested that she viewed it as an end state rather than a process in the context of infinite limits. The instructor's reference to infinity as a number (See Table 5.14) is consistent with Jessica's view of infinite limits. Amy and Harry were explicit not to consider infinity as a number. However, Amy explored the right and side and left hand side of infinity, and thus treated it as a number, in the context of limit at infinity. In the interview, students worked on some limit computation problems which they answered as "does not exist" but none of the students wrote "does not exist" or "undefined" for infinite limits. When working on infinite limits in the classroom, Jason consistently wrote $\lim_{x \to a} f(x) = \pm\infty$ on the board. At the same time, he verbally mentioned that the limit did not exist or was undefined. In fact, he explicitly endorsed the narrative that *infinity is not a number* in the class (See Section 5.4). However, his use of the symbol ∞ as being *equal to* the limit seemed to lead some confusion for students whether to consider an infinite limit as an existing or an undefined limit.

6.2.1.5. Continuity

When talking about continuity of functions at particular points, none of the students uttered the word "limit" during the interviews. Amy talked about points of removable discontinuity as "jumps" and attended to the "open circles", or holes in graphs when exploring continuity of functions: "it [the function] is discontinuous because it has an open circle [referring to the y value where the function does not attain its limit value], which means that the function jumps essentially" (Amy, 3 March, 2009).

Jessica's arguments about continuity were in close relationship to connectedness. When talking about why she thought the function in Question 2 (See Figure 6.3) was discontinuous at $x = -1$, she said "you just need to focus on the point [she shows $f(-1)$] that connects both continuous functions". Similarly, when she explained why the function in Question 3 (See Figure 6.3) was not continuous at $x = -5$, she said "it [the function] is discontinuous because we cannot find a connection or link between these two graphs [she shows the graph on the right hand and the left hand side of $x = -5$]" (Jessica, 5 March, 2009). Besides her arguments about connectedness, she also attended to the instances where the function jumped.

Harry also used the word "jump" when talking about continuity of a function at a given point: "it is not continuous because the value [*y* value] jumps". He later mentioned taking his hand off the paper: "so the way I see if a function is continuous…Am I taking my hand off the paper? I had to take it off now so it is discontinuous" (Harry, 27 February, 2009). He also looked whether the function was defined at the limit point.

Keith's main routine when determining the continuity of a function was to find the instances where he took his hand off the graphs of the functions: "It [the function] is continuous because you would not have to pick up your pencil. I know there is a more mathematical reason but I can't remember it now". (Keith, 3 March, 2009). Keith was aware that this explanation lacked mathematical precision but could not think of a different way than the intuitive approach of tracing graphs and reporting the points where he picked up his pencil as points of discontinuity. Besides this, he also talked about jumps: "It [the function] is not continuous because there is a jump" (Keith, 3 March, 2009).

Students' views of continuity were in accord with the instructor's intuitive approach. Amy attended to the holes in a graph of a function whereas Harry and Keith mentioned taking their hands off the graph. Jessica's arguments about continuity showed that she was relating the notion to connectedness. All these students relied on motion and operational word use as they traced the graphs with their hands or pencils. It was mentioned in Section 5.1.5.4 that Jason determined functions' continuity based on whether he could graph them "without taking the chalk off the board" (Table 5.16, [2], [4]). Therefore, similar to the students, he used the metaphor of continuous motion (See Section 5.3.3). Although he also introduced the precise definition of continuity based on limits and introduced properties of continuous functions via their properties of limit, none of the students talked about continuity in relation to limits during the interview.

6.2.2. Students' discourse with respect to student difficulties indicated by research on learning about limits

In this section, I talk about the responses students gave to the interview problems with respect to the difficulties identified by research on student learning about limits. Those difficulties were discussed in detail in Section 2.1.2 and were summarized in Table 2.1. In what follows, I focus on each of the six difficulties addressed by the literature and whether the students in the study showed signs of those difficulties during the interview sessions. While doing so, I again pay attention to the instances in which the students' discourse on limits is similar to or different from the instructor's discourse.

6.2.2.1. Limit implies continuity

This difficulty refers to the idea that if a function has a limit at a point, then it must be continuous at that point (Bezuidenhout, 2001). The possible instances of this view of limit are found in Questions 2, 3 and 5 in the interview (See Figure 6.3). Among the four students, only Amy had this difficulty during the interview. Her main routine when finding the limit of a function at a given point was to plug in the limit point to the function. In case the function value was not defined or there was a hole in the graph, she said the limit did not exist at that point.

> [When working on $\lim_{x \to -1} f(x)$ in Question 3] The fact that it is an open hole [shows the graph of the function at -1] is throwing me off because it means the function is not continuous. Limits are only defined for continuous functions I believe.
>
> [When working on $\lim_{x \to 3} f(x)$ in Question 3] I am going to say it [the limit] does not exit because the point is open so it is not continuous. (Amy, 3 March, 2009)

Amy also marked part (a) of Question 5 (See Figure 6.3) as true mentioning that "so what I know about limits is that they can only exist if the function is continuous at that particular point [the limit point]" (Amy, 3 March, 2009). Therefore, Amy clearly thought that limit implies continuity. The other students did not show any sign of the difficulty throughout the interview. It was mentioned in Section 5.3.1 that the instructor used algebra-based routines frequently when computing limits. His first approach was to plug the limit point into the function. In case this approach did not provide information about the limit, he then utilized other algebraic routines (See Table 5.21). However, Amy applied *plugging in* as the sole routine when computing limits throughout the interview.

6.2.2.2. Limit as the function's value

This view of limit refers to the idea that when finding the limit of a function at a given point, it is enough to look at the function's value at that point (Bezuidenhout, 2001). This view of limit is in close relation to the difficulty "limit implies continuity" and the routine of *plugging in* the limit point to the function to find the limit of the function at that point.

Possible instances of this view of limit were in Questions 1, 2, 3 and 5 (See Figure 6.3). Amy also had this difficulty. Her initial and only routine when computing limits was to compute the function value at a limit point. If she obtained a number, she reported it as the limit value; if she did not obtain a value or if the function did not attain its limit value, she mentioned that the limit did not exist. She said "every time I see a number, all I want to do is plug it in. That is all I can remember doing with numbers...like doing the substitution value" (Amy, 3 March, 2009).

During the interview, Harry also attended frequently to the functions' y values when finding limits of continuous functions. Yet, he seemed to differentiate between the limit value and the function's value at the limit point. "[When working on $\lim_{x \to 3} f(x)$ in Question 3] f of three would be equal to two but if you did a limit as x approaches three, it [the limit] would be one" (Harry, 27 February, 2009). Therefore, he did not talk about limit as the function's value.

Jessica and Keith consistently attended to the right hand and the left hand limits and whether they were equal to each other when finding limits. Even if they utilized plugging in, they did not generalize this routine as applicable to all functions and showed no sign of considering limit as the function's value.

One of the endorsed narratives in the instructor's discourse was that *the function's value at a limit point is irrelevant to the limit value* (See Section 5.4). Jason said the function values would have nothing to do with the limit value in the context of continuity (See Section 5.1.5.4) but he also frequently mentioned that one could find the limit of continuous functions by *plugging in.* Although Jason used different function types for which the routine *plugging in* could not be used when determining limits, the routine supported Amy's and Harry's focus on the function values when computing limits.

6.2.2.3. Limit as a bound

Considering limit as a bound refers to the idea that "limit is a number or point past which the function cannot go" (Williams, 1991, p. 221). It was mentioned in Section 6.1.2 that this view of limit is somewhat independent than the other views of limit in that it seems to be based on the colloquial use of the word *limit* than the mathematical aspects of the concept. Everyday use of the term limit might result in realizations of limit as a constraint or a bound. As a result, students can report the absolute maximum or minimum value of a function near the limit point as the limit value. None of the students showed signs of this difficulty during the interview since they did not refer to limit as a boundary or a constraint, and did not report the maximum or minimum value of functions as the limit values.

6.2.2.4. Limit as monotonic

This view of limit is based on formal teaching (Cornu, 1991) and is based on the expectation of 'nice behavior' from a function. In other words, students might assume that a function has to be strictly increasing or strictly decreasing in order to have a limit. Students having this conceptual obstacle could have difficulty finding limits of constant functions, piecewise functions and also sequences[21] in which the subsequences converge to different values (Tall and Vinner, 1981). Questions 2 and 3 (See Figure 6.3) were sites for revealing this view. None of the four students showed any sign of this difficulty during the interview. The instructor used different types of functions when computing limits giving students an opportunity to consider limits of constant and piecewise functions. Student responses to Question 1 and 2 were consistent with the instructor's discourse.

6.2.2.5. Limit as unreachable

This view considers limit as a value that is approached but not reached and is based on the dynamic view of limit (Tall & Schwarzenberger, 1978; Williams, 1991). Students who have this view of limit have difficulty finding the limits of continuous functions where the functions attain their limit value at the limit point. Possible instances in which this view of limit could be observed included Questions 1, 3 and 4 of the interview sessions.

It was mentioned that Amy had the difficulties "limit implies continuity" and "limit as the function's value" during the interview. Since she said only continuous functions would have limits and applied the routine of *plugging in* the limit points to functions when computing limits, it was unlikely that she would consider a limit as unreachable. However, she talked about asymptotes as exceptions to "normal graphs or regular limit problems" (Amy, 3 March, 2009). According to her, in the case of the asymptotes, a function's y value and the limit would not be reached.

Jessica and Harry clearly stated that a function could attain its limit value and did not show any sign of the difficulty during the interview. Keith initially had this difficulty when he talked about limit as *approaching* a number (but not being *equal to* that number) and when he determined the limit of the function in Question 3 at the limit point $x = -5$. The function was continuous at that limit point and so attained its limit value. Keith was

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[21] Limits of sequences are beyond the scope of this study.

uncomfortable when reporting the limit value for this problem since "it looks so much like plugging in the number. It looks like you are evaluating *f* of *x* at negative five but I know that is not true" (Keith, 3 March, 2009).

Keith did not consider the routine of *plugging in* as suitable when finding a limit of a function since he was aware that functions did not have to attain their limit values, which was a narrative the instructor endorsed in the classroom (See Section 5.4). Indeed, Keith immediately determined the limit of the function in Question 3 at the limit point $x = -1$, where the function had a limit but did not attain its limit value.

> Researcher: So what bothers you about this limit [I show $\lim_{x \to -5} f(x) = ?$ in Question 3]?
>
> Keith: When it [the function] gets the value, it feels like you are just plugging that in and I know that is not correct. It [$\lim_{x \to -5} f(x)$] was throwing me off because the point was actually defined.
>
> Researcher: You were quick when finding this limit though [I show $\lim_{x \to -1} f(x) = ?$ in Question 3].
>
> Keith: In that case you are not really plugging negative one, you are plugging in a number very very close to negative one so it is going to approach this value [the limit value]. When it [the function value] is not there, then I don't feel like plugging in anymore".

Keith confused the idea that *x* values approach the limit point without reaching it with the idea that the function values approach the limit value without reaching it. Therefore, he talked about limit as unreachable at the initial stage of working on Question 3. When finding the limit of the function in the same question at the limit point $x = 2$ (See Figure 6.3), on the other hand, he gave the function's value as the limit value (the function was constant near $x = 2$). When I asked him why he was not uncomfortable that the function attained its limit value at that point, he mentioned that he remembered the instructor giving examples of constant functions, where the limit value was equal to the function's value at the limit point. This provided an opportunity for Keith to think more about his initial response for limits of continuous functions. In the end, Keith realized focusing only on the right hand and the left hand limits and whether they are equal at the limit point when computing limits. He did not have any difficulty working with continuous functions after this instance.

Note that, when referring to the limit notation $\lim_{x \to a} f(x) = L$, the instructor used phrases such as "gets closer and closer to", "approaches", "goes to" and "becomes" when talking about $x \to a$ (See Section 5.2.4), which suggested that the x values approach the limit point a but never reach a. Similarly, when talking about the behavior of a function near the limit point, the instructor mentioned function values "approaching", "getting closer and closer to", and "going to" the limit value (See Section 5.1.2). Such operational word use for the function values is in accord with Keith's view that the function values approach the limit value without reaching it. The instructor's operational word use in the context of computing limits and the limit notation might, in fact, have a larger impact given that 13 of the 23 students in the study marked the statement about "limit as unreachable" as true in the diagnostic survey (See Table 6.2).

6.2.2.6. Limit as approximation

The assumption underlying this view of limit is that in order to find the limit of a function at a point, it is sufficient to look at the behavior of the function at points successively closer to the limit point (Bezuidenhout, 2001; Tall & Schwarzenberger, 1978). Parameswaran (2007) argues that considering limit as an approximation might also result from the common classroom practice of rounding numbers to convenient significant digits. Similar to "limit as unreachable", this realization of limit is based on the dynamic view of limit. Question 4 of the interview session (See Figure 6.3) provided the most useful context to gain more information about the view "limit as approximation".

When working on part (a) of Question 4, all of the students initially stated that the function values approached the value 1. Part (a) of the question included one student's method of plugging in successive values into the function to find its limit at the limit point. The student represented the related x and y values in tabular form.

> Amy: It [the function] gets closer and closer to one.
>
> Jessica: It [the function] approaches one from both sides.
>
> Harry: It [the function] is approaching...it gets closer and closer to one. It is a continuous function.
>
> Keith: I would say that it [the limit] would approach one. When he takes the numbers below zero, they get closer and closer to one; as he takes the numbers above zero, they get closer and closer to one. Yeah, in both cases they get closer and closer to one. So that is why I would say one.

All students used dynamic views based on the metaphor of continuous motion and used words operationally when talking about the limit in part (a). Note that Harry directly assumed continuity of the function from its tabular representation and that the students talked about limit as a process rather than a number.

When working on part (b) of the question, which asked whether the student's method could be used to find the limit, Amy and Keith were skeptical of the student's method "since we never did it in the class" (Amy, 3 March, 2009), "because I was never taught it" (Keith, 3 March, 2009). Jessica referred to the student's method as follows:

> We can't use this method to prove the result of that limit but we can find the limit and check the answers. I mean you have plan one, plan two. And you can solve the problem by plan one and if you want to check the answer of plan one, it is right. You can use this way [plugging in successive points into the function] to check it. (Jessica, 5 March, 2009)

Harry chose the statement which considered limit as an approximation as best describing his view of limit in the diagnostic survey (See Statement 5 in Table 6.1 and Table 6.6). He said that the student's method was correct but was ineffective since there were too many numbers plugged into the function and the numbers looked complicated. He explicitly mentioned that "the student tried to approximate" (Harry, 27 February, 2009) and if the student worked with "easier" numbers[22], the method would be more effective. In fact, Harry used a simpler version of the method when finding $\lim_{x \to 3} f(x)$ in Question 3

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[22] Easier numbers referred to plugging in numbers such as -1, -0.5, 0.5, 1 rather than -0.00001 or 0.00000001 into the function when computing its limit at $x = 1$ (See Question 4 in Figure 6.3). Harry's comments about such easy numbers might be connected to Parameswaran's (2007) idea of rounding numbers to their convenient significant digits.

(See Figure 6.3). In that particular case, he looked at the function's behavior from the right and the left side of the limit point by plugging in only one value from each side. Besides that instance, however, he did not utilize this method during the interview.

When working on part (c) of Question 4, which included a function represented algebraically and which had exactly the same y values for the x values plugged into the function in part (a), all of the students plugged the limit point into the function and mentioned that the function's limit was infinity. Therefore, they obtained two different limit values for a function: its tabular representation suggested that its limit was 1 but its algebraic representation suggested that the limit was infinity. All of the students realized this conflict during the interview. Keith was skeptical of the student's method in part (a) but did not choose one answer over the other although he said "I can see how the student's method can be wrong" (Keith, 3 March, 2009). Amy and Harry thought their computations of the limit in part (c) might have been wrong and concluded that the limit was one. Jessica was initially undecided about which answer to choose but then remembered that she considered the student's method as a checking mechanism and concluded that the limit was not defined for the function in part (c). At that point, she referred to the student's method in part (a) as "just showing the tendency. It does not show what really happens [at the limit point]" (Jessica, 5 March, 2009).

Except for Harry, students did not refer to limit as an approximation. Amy and Keith explicitly mentioned that finding limits by plugging in successive values into a function was a method the instructor did not use in the classroom. As a result, these students were skeptical of this method. Despite such skepticism, however, students' consideration of limit as a process by means of dynamic motion was so strong that when dealing with the conflict presented to them in part (c) of Question 4, they were more suspicious of their own algebraic computations than how the limit was determined in part (a) of the question. It was mentioned in Sections 5.2 that the instructor did not use tabular representations of functions during the classroom observations. Except for one instance (See Table 5.2, [2]), he also did not mention plugging in successive values into the function to compute its limit at the limit point[23]. Students' confusion whether to consider limit as an approximation or not did not seem to result from the approximation aspect of the difficulty (except for Harry) but from the dynamic view of limit and the consideration of limit as a process.

Summary

The results of the interview sessions, in conjunction with the findings of the diagnostic survey, suggest that students primarily used the dynamic view of limit and considered limit as a process. Their discourse on the informal aspect of limit was similar to the instructor's since students used words operationally when investigating functions' behavior near the limit points. In addition to this, students used the metaphor of continuous motion as a metarule for realizing limits, which was consistent with the instructor's routine in the context of informal aspect of limit (See Section 5.3.3). On the other hand, students' discourse was different from the instructor's with respect to the endorsed narratives. Although Jason endorsed the narrative *limit is a number* at the end of each limit computation problem, students endorsed the narrative *limit is a process* since

[23] In that one case, his comments about investigating the behavior of the function were only verbal; he did not actually plug in those points into the function.

they rarely talked about limit as a number during the diagnostic survey and the interview sessions.

The formal definition Jason introduced in the classroom provided an opportunity for students to learn about the static aspect of limit, which supported the consideration of limit as an end state. Although students accepted statements about a formal view of limit as true in the diagnostic survey and the interview sessions, their word use about the formal definition remained operational than objectified. The four students who participated in the interview session talked about the similarity of the formal definition with the informal view of limit, paying attention to the part of the definition that introduced how to read and write the limit notation. For them, the notation entailed movement, and connected well with their informal realization of limit. Jason's word use was consistently objectified in the context of the formal definition. Moreover, he used words signifying proximity instead of motion (See Section 5.1.5.2) and used the metaphor of discreteness (See Section 5.3.4) instead of the metaphor of continuous motion. Students, however, used words operationally and relied on motion when talking about the formal definition during the interview.

Students used graphs as visual mediators throughout the interview to make sense of functions when computing limits whereas Jason often used them to introduce a definition, theorem or fact about limits (See Section 5.2.3). His primary mode of representing functions was symbolic, not graphical. The limit notation was another context in which students' discourse was analyzed. Similar to Jason, students used a family of words such as "approaches" "goes to", and "at" when talking about the arrow in the limit notation. Unlike the instructor, students' utterances regarding the notation were often incomplete or inaccurate and they did not refer to limit as a number despite their use of the equal sign to write $\lim_{x \to a} f(x) = L$.

Computing limits, referring to the limit notation and the informal definition of limit were the contexts in which Jason shifted his word use between operational and objectified. Although he did not write the operational words he uttered on the board (See Section 5.2.1), students in the study relied on the spoken rather than written aspects of his word use when forming their realizations of limit. Jason's investigation of the behavior of functions near the limit and his operational word use in that context were consistent with students' realizations of limit as a process.

Students' confusions about process and the end result of a process were also apparent when they talked about infinity. During the interview sessions, the students who endorsed the narrative *infinity is not a number* treated it as a number depending on the context. When working on limits at infinity, students considered infinity as potential, that is, a process that goes on and on, which was similar to how Jason talked about infinity in the same context (See Section 5.1.5.3.2). Students' routine of extending graphs of functions at positive and negative infinity supported their realization of infinity as potential. Extending graphs of functions was a routine Jason did not utilize in the classroom but his word use was consistent with students' use of the metaphor of continuous motion in the context of limits at infinity. When working with infinite limits, students talked about infinity as an end state, similar to Jason's discourse. Jason talked about infinity as a "number" or an end state in the context of infinite limits (See Section 5.1.5.3.3). His word use differed from students' in that he considered an infinite limit as undefined whereas students considered it as being equal to infinity. The way Jason used the limit notation as a visual mediator might have played some role in students' realization of an infinite limit as an existing limit. In the context of infinite limits, he first wrote the limit as being *equal to* $\pm\infty$ and then said that "the limit does not exist" or "it is undefined" (See Table 5.14).

Students' responses in the interview sessions showed that they focused on the intuitive but not the precise definition of continuity the instructor introduced in the class. Unlike Jason, none of the students talked about continuity in relation to limits. However, they seemed to adopt the phrases such as "jump" and "taking the hand off the graph" that Jason used in the classroom when talking about continuity (See Section 5.1.5.4). Jason mentioned that one could find limits of continuous functions by plugging in although he explicitly endorsed the narrative *the function's value at the limit point is irrelevant to the limit value* (See Section 5.4). Amy, however, considered Jason's routine of *plugging in* as the main method to compute a limit and considered it as the function's value. She also thought that in order to have a limit, a function had to be continuous. As a result, she had the difficulties "limit implies continuity" and "limit as the function's value" during the interview. *Plugging in* was a routine Jason utilized in the classroom as the first method of finding the limit of a function presented algebraically (See Section 5.3.1). On the other hand, he also used different algebra-based routines in case *plugging in* did not work. Given the dominance of the dynamic view of limit in students' realizations, however, it is likely that some students would consider *plugging in* as the only means of computing a limit.

The majority of the students agreed with the view "limit as unreachable" in the diagnostic survey (See Statement 4, Table 6.2). The instructor's shifts in word use when referring to the limit notation as well as his routine of using continuous motion in the context of computing limits were consistent with students' realization of limit as unreachable. The phrases "approaches", "gets closer and closer to", "becomes", and "goes to" suggest moving towards the limit value without reaching it. As a result, students could confuse the idea that x values approach the limit point without reaching it with the idea that function values approach the limit value without reaching it. During the interview sessions, Amy and Keith showed signs of the difficulty "limit as unreachable". Amy stated that, for the case of asymptotes, a function cannot reach its limit value whereas Keith talked about limit as *approaching* a value and was initially uncomfortable when finding the limits of continuous functions.

Jason did not use tabular representations of functions formed by plugging in successive values to the function when computing limits. That contributed to students' skepticism about such a method as being suitable for determining limits. Despite the skepticism, Harry had the difficulty "limit as approximation" at some point of the interview. The instructor's use of different types of functions, such as piecewise, discontinuous, and trigonometric functions, helped students work comfortably with functions other than polynomials during the interview. Consistent with Jason's discourse, the participants in the interviews did not have the view "limit as bound" and "limit as monotonic".

Overall, the contexts in which students struggled with limits coincided with the contexts in which the instructor shifted his word use. Recall that Jason's utterances about limits were mainly objectified (See Table 5.5) and his operational word use took place when he communicated his ideas just verbally, without writing on the board (See Section

5.2.1). Although Jason's operational utterances constituted only about 16% of his total utterances (See Table 5.5), the dominance of operational word use in students' discourse suggest that they adopted the spoken aspects of Jason's discourse in which he was less formal and less precise. This can further be supported by the fact that students talked about both limit and continuity through the intuitive definitions Jason provided in spoken discourse rather than the precise definitions he wrote on the board.

Intuitive aspects of limits result from the dynamic view based on the metaphor of continuous motion. Students' frequent use of this metaphor as a metarule for realizing limits supports their consideration of limit as a process. In fact, limit was not objectified till 17th century by mathematicians and the metaphor of continuous motion was also present in their discourse till 19th century (See Section 3.3). Therefore, it is possible that learners intuitively realize the concept as a process at the early stages of their learning. In the following chapter, I explore the classroom discourse on limits in relation to the historical development of limits to examine how the instructor's and the students' discourse compare and contrast with those of the mathematicians' over history.

CHAPTER VII

CLASSROOM DISCOURSE ON LIMITS IN RELATION TO THE HISTORICAL DEVELOPMENT OF LIMIT RELATED CONCEPTS THROUGH THE COMMOGNITIVE LENS

In this chapter, I explore the classroom discourse on limits with respect to the historical development of infinity, infinitesimals and limits. In Chapter III, the historical development of these concepts was described with a focus on particular elements of the commognitive framework: word use (objectification), metarules and metaphors. I then identified the historical junctures in the development of discourse on limits that resulted in changes in the metarules[1], also called metadiscursive rules, of the existing discourse in order to extend it further (See Sections 2.2.2 and 3.4).

The commognitive framework considers developmental transformations as resulting from the interplay of individualization and communalization since

> ...mathematical discourse is a historically established activity practiced and extended by one generation after another and taught in schools for the sake of further continuation. Mathematics students are this supposed to join this activity rather than invent their own, idiosyncratic version. (Sfard, 2008, p. 203)

Therefore, we can think of learning mathematics as the individualization of the communal activity of doing mathematics. Teachers are among the sources that enable the communication of historically established mathematical discourse to students and play important roles in the process of *individualization of the communal.*

[1] I also argued that the exploration of metaphors is an important part of the exploration of metarules since using metaphors for the realizations of mathematical concepts is a type of metarule (See Chapter III).

It should be noted that "the order of things in the processes of discourse individualization is different from that in historical processes of object-creation" (Sfard, 2008, p. 177). For example, in this study, students were immediately introduced to limit in an objectified manner (as a number) although the objectification[2] of the concept took centuries for mathematicians. On the other hand, the findings of the study suggested that, just because they were exposed to an already objectified mathematical concept did not necessarily mean that students could objectify the concept in their discourse. The participants in the study had difficulties considering limit as a number, a distinct mathematical object, at the end of their instruction (See Chapter VI). Thus, although the order of individualization processes could differ from historical processes of object creation, the realization of limit as an object of mathematics seems to be challenging for both students and, historically, for mathematicians.

For the case of limit and infinity, objectification historically resulted in changing the discourse about processes to discourse about end-states. It was not till 17th century that Cauchy objectified limit as a number and it was not till 19th century that Cantor objectified infinity as an end state (See Sections 3.1 and 3.3). The realization of infinitesimals as objects of mathematics is still under question (See Section 3.2). One of the benefits of using commognitive framework as a lens to focus on the conditions and assumptions of object creation over history is that it enables us to acknowledge objectification as quite a complex phenomenon in the development of mathematical discourse. I considered objectification of infinity, infinitesimals and limit as among important junctures in the development of discourse on limits over history because

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[2] Objectification of a concept changes the talk about a process to the talk about a product or an end state

objectification of these concepts also resulted in changes in the metaphors and metarules of the previously existing discourse (See Section 3.4). A question that will be pursued later in the chapter is whether these junctures can also be important in the development of students' discourse on limits.

In what follows, I examine the instructor's and students' discourse on limits with respect to the historical development of limit related concepts. While doing so, I compare and contrast the metarules and metaphors leading to the objectification of the concepts over history with those reflected in the instructor's and students' discourse. I also look for similarities and differences between the order and characteristics of object creation over history and in this beginning-level calculus course.

7.1. The instructor's discourse with respect to the historical development of limit related concepts

If we look at the order in which the instructor (Jason) introduced the concepts during the eight observed lessons, we see some elements of the historical development of limit related notions playing out in the classroom. Jason started his discussions of limits by first introducing average rate of change and then defining instantaneous rate of change as the limit of average rate of changes of a function as the time intervals got smaller and smaller (See Table 5.1 and Section 5.1.5.1). While doing so, he also used graphs to represent the average rate of changes with secant lines and the instantaneous rate of change with a tangent line, which was the limit of the secant lines. This resembles closely Newton (1643-1727) and Leibniz's (1646-1716) approaches to the limit concept since they both considered limit as a process when they obtained a tangent line at a point

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through reification (See Section 2.2.1).

through the use of secant lines (See Sections 3.2 and 3.3). Besides this, the sequencing of the calculus concepts in both the curriculum[3] and the classroom teaching (See Table 5.1) followed Cauchy's outline of calculus concepts. In Cauchy's design, the notion of limit precedes derivatives and integrals as well as series and sequences (See Table 3.2). Although Cauchy objectified "limit", referring to it as a number obtained at the end of the limit process, his definition of the concept relied on dynamic motion and infinitesimal use (See Section 3.3). Jason's introduction of limits, together with his operational word use when he described the behavior of functions near the limit points suggest that his focus was more on the intuitive aspects of the limit notion than the formal aspects. On the other hand, Jason also chose to introduce the formal definition of limit and solved a proof problem, which might be uncommon for a beginning-level calculus course. He considered the formal definition as optional but noted that if students took further mathematics courses, like analysis, this definition would come up again (See Section 5.1.5.2).

In what follows, I examine how and whether the metarules (in particular, using specific metaphors) related to infinity and limit[4] in the historical junctures I identified in Chapter III (See Table 3.3) were reflected in the instructor's discourse.

Historically, potential infinity is consistent with Aristotle's (384-322 B.C.) view of infinity as an indefinite process. Actual infinity, however, is consistent with Cantor's

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[3] The order Jason presented the topics was also the order they were presented in the textbook (See Thomas et al., 2008 for their content outline for calculus).

[4] Jason could not possibly address, and was not expected to address, the metaphors and metarules related to infinitesimals in his discourse since mathematically sophisticated version of the theory is much beyond the scope of a beginning calculus course. However, since he used infinitesimals in his discourse on limits, this signaled that he relied on the intuitive and dynamic aspect of limit in those cases than the formal aspect of

(1845-1918) view of the concept as an end state (See Section 3.1).

In Chapter III, I considered Cantor's objectification of infinity as the main historical juncture in the historical development of the concept (See Table 3.3). This juncture led to changes in the previous discourse on infinity and changed the metarules of counting, measuring, and some properties of arithmetic such as addition and division. Moreover, using the metaphor of indefinitely continuous processes, a metarule for realizing infinity, was changed to using the metaphor of infinitely iterative steps, each with an end result (See Section 3.1 and Table 3.3).

Jason talked about infinity as potential and actual depending on the context. In the context of limits at infinity, he talked about infinity as a continuous process that went on and on (See Section 5.1.5.3.2, Table 5.11); in the context of infinite limits, he talked about it as an end state (See Section 5.1.5.3.3, Table 5.14). That he referred to infinity as an end state was also visible when he wrote limits as *equal to* plus or minus infinity (See Section 5.1.5.3.3). Recall, however that he explicitly endorsed the narrative *infinity is not a number* in the class (See Section 5.4). Jason talked about infinity consistent with both Aristotle's and Cantor's realizations of the concept but he did not call students' attention to the shifts in his word use and did not address using distinct metaphors as different metarules when making sense of infinity. In other words, although objectification of infinity was present in his discourse, it remained implicit for students.

I have identified two historical junctures in the development of discourse on limit over history. The first juncture was the objectification of limit by Cauchy (See Table 3.3). In his definition, Cauchy described limit as a number obtained at the end of the limiting

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the concept. His use of infinitesimals took place only when he talked about limits in an informal manner, consistent with their use in the historical development of the limit concept (See Section 3.2).

process. On the other hand, his description of the concept was dynamic and based on the assumption of continuous motion (See Section 3.3). The second juncture was Weierstrass' introduction of the formal definition of limit which eliminated spatial motion and continuity from the previously existing discourse on limits (See Table 3.3). By arithmetization of geometry, Weierstrass replaced using the metaphor of continuous motion, a metarule for realizing limit, with using the metaphor of discreteness (distance between discrete numbers) (See Section 3.3). The second juncture transformed some other metarules of mathematics. For example, elimination of motion also resulted in realizations of functions as algebraic rules but not as graphs of curves.

Jason was careful and accurate when he described the informal and the formal definitions of limit. His definition of the informal aspect of limit corresponded to the formal aspect of the concept in that he did not use words signifying motion but used words signifying proximity such as "arbitrarily close" and "sufficiently close" (See Figure 5.4). When he utilized the informal definition to compute limits, however, his word use was operational. His descriptions of the behavior of a given function near the limit point made frequent use of continuous motion and treated limit as a process. At the end of every limit computation problem, however, Jason talked about limit as a number. The instructor's realization of limit as a number together with his routine of using continuous motion as a metaphor for the process aspect of the concept are consistent with Cauchy's realization of limit. Similar to infinity, objectification of limit was present in Jason's discourse but it remained implicit for the students since Jason did not address when the process aspect of limit ends and the product aspect begins.

When talking about the formal definition of limit, Jason's word use was consistently objectified and his words signifying motion gave way to words signifying proximity in terms of distance (See Section 5.1.5.2), which is consistent with Weierstrass' realization of limit. As a mathematician, Jason was aware of the metarule of using the metaphor of discreteness since motion was eliminated in his discourse when he talked about the formal definition. On the other hand, he did not draw students' attention to the changing metarules and his shift to objectified word use in this context. Therefore, similar to infinity and informal aspect of limit, the metarules remained tacit for students in the classroom. Moreover, rather than drawing students' attention to the *differences* between these two aspects of limits, he tried to find ways to talk about how they were *similar* (See Section 5.1.5.2 for how Jason connected the formal definition to the informal definition of limit).

In terms of metarules related to the uses of the definitions, Jason was more explicit. He mentioned that the informal definition "convinced us what these limits are that we have been computing in the class…" (See Table 5.7, [4]) and he made explicit that "to really make sure these [limit computation methods] work in mathematics, we have to prove those things. In order to prove something, we need a precise definition" (Table 5.7, [5-5a]). Therefore, Jason clearly mentioned that the informal definition was different than the precise or formal definition of limit in that, one could not prove the existence of a limit with the former[5].

[5] However, he did not address that in order to prove the limit of a given function as a number L, one needs to first hypothesize that number. Said differently, the formal definition of limit does not enable a constructive proof; one can only prove or disprove an already guessed number L to be the limit of a function with the precise definition.

In summary, the historical junctures changing the metarules of infinity and limit coincided with the contexts in which Jason shifted his word use between operational and objectified in his discourse. He talked about infinity and limit as processes or end states depending on the context. In each of the contexts, his reasons for utilization or exclusion of the metaphor of continuous motion remained implicit for students, supporting the tacit nature of metarules in his discourse. It was mentioned in Chapters V and VI that the formal definition of limit provided a context in which Jason could attend to the static aspect of limit, which supported its realization as an end state (a number). Although Cauchy's realization of limit also provides a context in which it is possible to talk about limit as a number, as Jason did, the findings of the diagnostic survey and interview sessions indicated that the dynamic element in the informal definition supported students' consideration of limit as a process[6]. Jason's consistent referral to limits as objectified and his elimination of motion related words in his discourse on the formal definition showed some evidence that he was aware of the metarules related to the formal realization of limit. In fact, he used the metaphor of continuous motion in the context of informal definition and the metaphor of discreteness in the context of the formal definition (See Sections 5.3.3. and 5.3.4). However, there was no evidence in his discourse that he considered attending to these distinct metarules as pedagogically relevant in the classroom. Instead, he talked about how the definitions were similar to each other. Whether calling students' attention to the changing metarules related to infinity and limit

[6] Recall that students associated informal aspect of limit with motion and rarely referred to limit as a number in the diagnostic survey and during the interview sessions (See Chapter VI).

could be useful for student learning will be discussed in the next section where I examine students' discourse in relation to the historical development of the concepts over history.

7.2. Students' discourse with respect to the historical development of limit related concepts

Students may develop many idiosyncratic ways to justify their narratives of mathematics, which are not necessarily compatible with the narratives endorsed by the mathematical community. For the case of limit and infinity, however, there are similarities between some of the difficulties addressed by research on learning about limits and the historical development of limit. Note that "limit as bound", and "limit as unreachable" are among the conceptual obstacles students have about limits (See Section 2.1.2, Table 2.1). It was mentioned in Chapter III that Lagrange (1736-1813) opposed some mathematicians' use of the limit method:

> MacLaurin and d'Lambert used the idea of limits; but one can observe the subtangent is not strictly the limit of subsecants, because there is nothing to prevent the subsecants from further increasing when it has become a subtangent. True limits… are quantities which one cannot go beyond, although they can be approached as close as one wishes. (Lagrange, 1799, as cited in Schubring, 2005, p. 293)

Schubring (2005) notes that Lagrange's arguments were based on

> the lacking of the concept of absolute value…so that it seems as if the variable goes beyond the limit; the criticism is also at the problem, which has always remained controversial, whether a variable can definitely reach the limit or is only allowed to come close to it at any rate (p. 293).

In these arguments, we see Lagrange problematizing whether limit is a bound when he says "true limits…are quantities which cannot go beyond…" and also the elements of the debate whether a limit is reachable. Therefore, the conceptual obstacles "limit as bound" and "limit as unreachable" were also present in the development of discourse on limits over history. In this study, students' responses to the diagnostic survey and the interview problems did not reveal any difficulties in terms of the realization of limit as a bound (See Section 6.2.2.3). The responses, on the other hand, provided evidence in terms of the realization of limit as unreachable, which is based on the dynamic view of limit (See Sections 6.1.1, 6.1.2, and 6.2.2.5).

It was discussed in Chapter VI that the students in the study frequently used the dynamic view of limit based on the metaphor of continuous motion. Such realization of limits is consistent with the historical development of limit in that (a) the initial underpinnings of the concept emerged from physical problems that were based on motion, and (b) it was not till 19th century that spatial motion was eliminated from the existing discourse on limits. Said differently, similar to the students in the study, the dynamic view of limit was also the intuitive and initial view with which mathematicians made sense of the concept historically. Using continuous motion was also a metarule that shaped mathematicians' realization of infinity as an indefinite process for centuries. Students in this study showed similar views of infinity when they talked about it as potential in the context of limit at infinity (See Section 6.2.1.4). Therefore, objectification of limit and infinity was challenging both for students in the study and mathematicians over history. Once objectified, mathematicians were able to separate process from product, which was not necessarily the case for the students in the study.

It was argued in Section 5.5 that, although the instructor can flexibly use limit and infinity as a process or product and distinguish the characteristics of each realization depending on the context, students might be unlikely to notice those characteristics or realize the differences and similarities underlying these concepts. The findings of the diagnostic survey and student interviews support this claim for the participants in this study. Both in the diagnostic survey and interviews, students talked about limit as a process or a descriptor of how a function behaves and rarely talked about it as a number (See Sections 6.1.2 and 6.2.1). In other words, they did not objectify limit in their discourse. At the end of their instruction, these students did not make the transition from the realization of limit as a process to its realization as a number as introduced by Cauchy, which is one of the historical junctures I identified in the historical development of limit (See Table 3.3). The students also did not move from a discourse on limits based on the metarule of using dynamic motion to one based on the metarule of using a static and discrete realization of the concept, which was the contribution of Weierstrass. I identified his introduction of the formal definition of limit as a second juncture in the historical development (See Table 3.3).

The interview sessions provided some information in regard to students' realizations of infinity. Students talked about infinity as a process and an end result depending on the context. When finding limits at infinity, they talked about the concept as an ongoing process[7] (See Section 6.2.1.4), which is consistent with Aristotle's view of infinity as potential. When working on infinite limits, they talked about infinity as an end state but none of the students considered an infinite limit as undefined. This signaled that they treated infinity as an existing entity, although two of the participants explicitly said infinity is not a number. One participant talked about infinity as a number in the context of infinite limits and one mentioned the limit existed as infinity (See Section 6.2.1.4). I identified Cantor's objectification of infinity as an accumulation point as a historical juncture in the development of discourse on infinity. Students in the study seemed to have some aspects of this transition present in their discourse. Although they talked about infinity as a process when the x values approached infinity, they treated the concept differently when they obtained infinity as the limit of a given function. Recall that the instructor considered an infinite limit as undefined or not existing (See Section 5.1.5.3.3) but the students in the study were not comfortable considering an infinite limit as undefined. Besides this difference, their discourse on infinity was similar to the instructor's in that they talked about it as potential in the context of limits at infinity (consistent with Aristotle's realization) and as an end state in the context of infinite limits (consistent with Cantor's realization).

The contexts in which students struggled most during the interview sessions about infinity and limit coincided with the contexts in which the instructor's word shifts took place (See Section 5.1). Those contexts also coincided with the historical junctures that changed the existing metarules about infinity and limit to extend the discourse on limits. Changes in the metarules historically led to the objectification of limit related concepts and eventually to the changes in the endorsed narratives of calculus.

Both the informal and the formal definition of limit are contexts in which it is possible to talk about limit as objectified (a number). The first definition is compatible

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[7] Limit at infinity was a context in which students extended graphs of functions at infinity and implicitly

with Cauchy's realization whereas the second one is compatible with Weierstrass' realization of limit[8]. It was mentioned in Chapter VI that the metarule of using continuous motion dominated students' discourse on limits and supported their realization of limit as a process. In other words, students focused on the process element in the informal aspect of limit more than the end result of the process. Therefore, it seemed necessary that, in order for them to realize limit as a distinct object of mathematics (a number), their assumption of dynamic motion needed to be challenged. The formal definition of limit was a context in which it was possible for Jason to highlight the elimination of motion from the discourse on limits. On the other hand, his word use and the metarules, which were consistent with the historical junctures in the development of limit, remained implicit for the students in the class.

Similar to limit, the historical juncture leading to the objectification of infinity with the distinct metarules related to different realizations of the concept were reflected in Jason's discourse. However, the word use, metarules and metaphors remained implicit for the students since he did not address when the process aspect of infinity ends and the product aspect begins.

Summary

More often than not, the junctures resulting in discursive transformations require changes in the metadiscursive rules of defining, substantiating and recording narratives of mathematics. Note that since learners of mathematics are expected to join in the

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used the metaphor of continuous motion (See Section 6.2.1.4).

[8] Most importantly, since they describe the same mathematical concept, both definitions are also compatible with each other. They are similar in that they characterize limit as a number. They differ, however, in their description of the "process" aspect of limits. Cauchy acknowledges the process through motion whereas Weierstrass eliminates it and uses absolute values of differences between particular numbers instead.

historically established activity of mathematics, "gradual modification of metarules that govern the students' mathematical discourse is one of the goals of school learning" (Sfard, 2008, p. 2008). What makes such modification difficult for students is that "metarules are mostly tacit" (Sfard, 2008, p. 221). In Chapter III, I focused on some elements of the commognitive framework such as word use, metarules and metaphors while investigating the historical development of limit related concepts to identify the tacit metarules explicitly as the realizations of infinitesimals, infinity and limit evolved through history. I identified the historical junctures in the development of discourse when they led to changes in the metarules of the existing mathematical discourse[9]. Such junctures often resulted in the objectification of limit related concepts.

The exploration of the instructor's discourse in terms of the historical junctures I identified in Chapter III (See Table 3.3) revealed that the contexts in which Jason shifted his word use coincided with the junctures that changed the metarules of previously existing discourse on infinity and limit over history. The instructor talked about the informal and formal definition of limit in different ways, which were compatible with the historical development of limit, but did not draw attention to his change of word use and did not address the metarules these definitions are based on in the classroom. Similarly, he talked about infinity as potential and actual depending on the context but did not highlight how those contexts and the distinct realizations of infinity differed from each other.

The examination of the students' discourse in terms of the historical junctures revealed that some aspects of limits and infinity students struggled with were consistent

[9] In this study, I only identified the junctures changing the metarules of existing discourse on limits over

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with the transitions mathematicians went through over history. Students could not distinguish process from product and they also were not aware of the elimination of dynamic motion in the formal definition of limit. As a result, they struggled with some of the interview session problems. Therefore, the junctures in the development of discourse on limits over history also seemed to be critical in the development of students' discourse.

It is important to note that I do not consider the historical transitions as identical to those of students' since students may have many other realizations of limits that are not present in the development of discourse over history. Students are introduced to the elements of the discourse on limits in a very compact manner and in a short period of time so they do not have as much chance to reflect and realize different aspects of the concepts. On the other hand, some of the difficulties about limits seem to be common for both mathematicians over history and students. The objectification of infinity and limit, the dominance of the dynamic view as well as the realization of "limit as bound" and "limit as unreachable" are present in both the experts' and the students' discourse on limits as conceptual obstacles.

Limit related concepts inherently present learners of mathematics with difficulties both over history and in today's classrooms. It is due to this fact that the analysis of historical development of limits through the commognitive framework has some useful implications for classroom discourse on limits. Word use, metaphors and metarules are some of the elements of mathematical discourse highlighted by the commognitive framework that might remain tacit both in the historical development and also in the instructor's discourse on limits. Students could benefit from those elements of discourse

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history but one can also talk about such junctures in the development of an individual's discourse.

being explicitly addressed in the classroom since the historical contexts in which experts struggled about limits were compatible with the contexts in which students in the study struggled when working with limit related concepts. Therefore, it is likely that the historical junctures during which mathematicians changed and revised their previously existing realizations of limits might be useful for student learning. Said differently, historical junctures changing the metarules of previously existing discourse on limits can form some of the junctures students have to go through as they participate in the discourse on limits.

CHAPTER VIII

SUMMARY, DISCUSSION AND CONCLUSIONS

In this chapter, I return to the questions that motivated this study and guided my analyses. Some educational researchers interested in the concept of limit have investigated the historical development of limit related concepts. Others have focused on student learning and investigated the conceptual obstacles students face as they learn about limits. A missing link in this chain in literature is the teaching of limits at the undergraduate level. Therefore, one of the contributions of this study is to provide information about teaching of limits by focusing on one instructor's discourse.

The literature on historical development of limits and student learning is primarily based on the assumptions of a cognitive framework that highlight the nature of difficulties about limits in terms of misconceptions. According to Sfard (2001), the cognitivist framework is based on the metaphor *learning as acquisition*, which considers learning "as the storage of information in the form of mental representations" (p.20). From the point of cognitivist framework, understanding is defined as relating new knowledge to prior knowledge by refining the existing mental representations. By doing so, this view highlights the individual nature of learning and views it as the acquisition of the necessary mental schemes one either possesses or not. In this study, I used the commognitive framework developed by Sfard (2008) to explore teaching and learning of limits. There were two reasons I used this different lens to work on the issues related to learning of limits: (a) By viewing thinking as an individualized form of communication, this framework highlights the importance of social constructions in individuals' thinking. Mathematics learning is considered as participation in mathematical discourse, whose rules are determined communally throughout history. As a result, obstacles learners face may be attributed to communicational rather than cognitive difficulties. (b) From this perspective, the three elements considered in the study – historical development, teaching, and learning of mathematics – are viewed as interrelated and connected elements of discourse development. Given these presuppositions, I explored students' discourse on limits and the historical development of limit related concepts through the commognitive framework. Therefore, another contribution of this study to educational research is that it pilots an analytical tool with which to explore students' discourse on and the historical development of limits.

I pursued three specific research questions in this study: 1) How is the discourse on limits generated by the instructor in a beginning college-level calculus classroom? 2) Given the instructor's discourse on limits, how do students talk about limits in a beginning college-level calculus course? and 3) How do the elements of discourse on limits as generated over history compare and contrast with the discourse on limits generated in a beginning-level calculus course? I addressed these questions in Chapters V, VI, and VII, respectively. Here, I provide a brief summary for each question to set up my later comments for discussion.

8.1. Summary of findings

For my first research question, I analyzed one instructor's discourse on limits by using the four elements of the commognitive framework: word use, visual mediators, routines, and endorsed narratives. The findings for this question indicated that the instructor shifted his word use depending on particular mathematical contexts about limits. Those contexts were identified as the informal and formal definition of limit, computing limits and symbolic notation. The instructor's alternating word use signified infinity and limits as processes or as end states in his discourse. The instructor did not address such shifts in his word use in the classroom, and did not make specific the differences between the realizations of limit as a process and as a number.

The visual mediators the instructor utilized in the classroom mainly consisted of graphs of functions and symbolic notation. The relationship between the limit notation $\lim_{x \to a} f(x) = L$ and the instructor's word use provided interesting results in regard to realizations of infinity and limits as processes or as products. Although he talked about L as a number, his referral to the arrow in the notation using words such as "approaches", "gets closer and closer to", "goes to", "becomes" supported the realization of limit as a process. Similarly, when referring to the symbol ∞ in the limit notation, he talked about infinity as potential (a process) in the context of limits at infinity whereas he talked about it as actual (an end state) in the context of infinite limits.

The instructor referred to limit as a process when he investigated the behavior of a given function near the limit point. On the other hand, each time he talked about infinity and limits as processes, his utterances were spoken rather than written. The primary narrative the instructor endorsed in the classroom was *limit is a number*. He explicitly endorsed the narrative *limit is a process* only twice during the eight observed classroom periods and he did so only verbally. His language, however, along with the conventional reading of the symbol $\lim_{x \to a} f(x) = L$ that includes the word "approaches", suggested an implicit endorsement of the narrative *limit is a process*.

The only context in which the instructor's word use, visual mediators, routines and endorsed narratives about limits were most consistent with each other was the formal d

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definition of limit. In this context, he did not utter any words signifying motion but uttered words signifying proximity. By doing so, he changed his use of the metaphor of continuous motion to the metaphor of discreteness. However, similar to his shifts in word use, the instructor's utilization of distinct metaphors remained implicit for students since he did not mention the metaphors he used in the classroom.

For my second research question, I explored how students in the instructor's section talked about limits at the end of the unit on limits. I analyzed students' discourse with respect to (a) the instructor's discourse on limits, and (b) research on student learning about limits. The findings of this part of the study indicated that the contexts in which the instructor shifted his word use coincided with the contexts in which students had difficulty talking about limits. When making sense of the limit concept, students mainly adopted the instructor's spoken words than written ones. They frequently endorsed the narrative *limit is a process*, consistent with the instructor's discourse when investigating the behavior of the function near the limit point. On the other hand, students rarely, if ever, talked about limit as a number. The findings also indicated that students used graphs as visual mediators more frequently than the instructor when computing limits. In terms of symbolic notation, students' utterances were either inaccurate or incomplete. Similar to the instructor, they used the metaphor of continuous motion when referring to the limit notation and investigating the behavior of functions near the limit points. Unlike the instructor, they did not use the metaphor of discreteness in their discourse on limits. As a result, they could not talk about the differences between the informal and the formal aspects of limits.

When analyzing students' discourse in relation to research on student learning about limits, I first investigated whether students showed signs of the difficulties indicated by the literature. I then explored whether elements of the instructor's discourse were reflected in students' realizations of limits. The findings for this part of the study indicated that some elements of the instructor's discourse (e.g., using a variety of functions, introducing the formal definition, not computing limits by plugging in successive values near the limit point) supported students' accurate responses for different realizations of limit. This was also reflected in the fact that students did not hold some of the difficulties reported in the literature (e.g., "limit as bound" and "limit as monotonic") and these were the very ideas absent from the instructor's discourse. Yet, some other elements of his discourse (shifts in word use and routines) supported students' realization of limit as a process rather than a number.

For the third question, I examined the instructor's and the students' discourse on limits in relation to the historical development of limit related concepts through the commognitive lens. In particular, I explored whether the historical junctures I identified in the development of discourse over history were reflected in classroom discourse on limits. The findings here indicated that the contexts in which the instructor shifted his word use coincided with the historical junctures that resulted in changes in the metarules of the discourse on limits. Those contexts were the informal and the formal definition of limit as well as infinity. The changing metarules related to the concepts over history were reflected in the instructor's use of words and different metaphors. However, he did not explicitly address the metaphors and did not attend to the metarules underlying different realizations of limit and infinity. Therefore, the study provided evidence with respect to the tacit nature of metarules in the classroom.

The exploration of students' discourse with respect to the historical development of limits indicated that some of the difficulties addressed by research on student learning were also present in experts' discourse over history. Those difficulties were "limit as bound", and "limit as unreachable". In addition, similar to students' discourse, the discourse on limits over history was based on the metaphor of continuous motion for centuries before limits were considered as distinct objects of mathematics. Objectification of limits were challenging both for the mathematicians over history and for the students in the study.

Most of the junctures I identified in the historical development of limit related concepts signaled objectification as the milestones leading to the expansion of the previously existing discourse on limits. I identified the elimination of spatial motion and continuity from the discourse on limits as another type of historical juncture in the historical development of discourse. The students in the study did not show signs of objectification and elimination of motion in their discourse on limits. They talked about limit as a process but not as a distinct mathematical object (a number). Moreover, their view of limit was based on the metaphor of continuous motion and not discreteness. On the other hand, the students who participated in the interview sessions showed signs of objectification of infinity as an end-state in their discourse.

In sum, the contexts in which students struggled most in the diagnostic survey and the interviews coincided with the historical junctures that changed the metarules about limits. Those contexts also coincided with those in which the instructor's shifts in word use took place. Therefore, the junctures in the development of discourse on limits over history also seemed to be critical for the development of students' discourse in the study.

8.2. Discussion

The study illustrates the usefulness of the commognitive framework to analyze features of mathematical discourse. The exploration of discursive patterns, which could not be revealed through a cognitive approach, enables the analysis of the connection and conflicts among the elements of an individual's discourse (e.g., connection between words and symbolic notation; words and routines; routines and endorsed narratives; written and spoken words, etc). For example, although the instructor in the study endorsed the narrative *limit is a number*, some of his routines when investigating the behavior of functions near the limit points supported the narrative *limit is a process*.

The commognitive framework also enables the identification of the communicational failures through the analysis and comparison of word use, visual mediators, routines and endorsed narratives of multiple discursants. For example, although the instructor's word use on limits was predominantly objectified, students' word use was predominantly operational. Similarly, although the instructor often used graphs as visual mediators to explain definitions and theorems about limits, students often used graphs to identify the behavior of functions.

The use of commognitive framework highlighted an important and complex relation between the limit notation and word use. Students in the study used dynamic motion when making sense of the limit notation $\lim_{x \to a} f(x) = L$ and connected it to the informal definition of limit. They did not focus on the equal sign and the limit L, a distinct number, as much as they focused on the x values approaching a and the y values appro

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approaching *L*. Note that although the notation supports talking about the limit as a number, it does not necessarily support the elimination of motion. The conventional way to read the notation includes the word "approaches", a word signifying motion. Therefore, even in a formal definition of limit, the symbolic notation and the words associated with it might result in conflicts with respect to the consideration of limit as a number or process. In this study, the instructor and students used a family of words besides "approaches" that signified motion (e.g., "goes to", "gets closer and closer to", etc) when talking about the limit notation. Given that the students' utterances about the limit notation were often inaccurate or incomplete, it seems crucial that the relationship between the symbols and the words associated with those symbols are made clear in the classroom.

The relation between the written and spoken words in the classroom discourse was another finding of the study revealed through the commognitive framework. Although the instructor's operational word use corresponded to a small portion of his overall utterances about limits, the students in the study primarily focused on operational word use in the instructor's spoken discourse and disregarded his written discourse when forming their realizations of limit. Students' lack of attention to the written aspects of the classroom discourse was also consistent with their approach to the textbook. The students participating in the interview sessions mentioned that they never used the textbook for reviewing material. Three of those four students did not even use the lecture notes while reviewing material or preparing for the exam. This suggests that students relied on what they heard, saw and recalled from the classroom while forming their realizations of limit.

It is possible that students did not utilize the written aspects of the instructor's and the textbook's discourse due to their inexperience with the material. Only four of the students attending to the instructor's section took calculus in high school; this was the first time the remaining students learned about the topic. In addition, students learned the ideas related to limits in only a two and a half week period over eight lessons. As a result, it might be hard for them to follow the precise and objectified word use about limits in the textbook and the instructor's written discourse. Instead, they focused on the informal and intuitive aspects of limits in the instructor's spoken discourse. This signals that they only attended to the elements of the instructor's discourse that helped them cope with the requirements of the course.

Students' coping strategies were clearer when they talked about how they prepared for the exam. Although they did not use the textbook and the lecture notes, all of the students in the interview session stated they revisited the homework problems the instructor assigned from the textbook when preparing for the exam. One participant (Amy) explicitly stated that she ignored the instructor's explanations unless that particular material would be tested on the exam. Therefore, students seemed to focus on learning how to solve particular problems for the exam than learning about the conceptual meanings of limits.

The parallels between the development of discourse on limits over history and students' discourse support the idea that it might be inevitable for students to realize limit as a process at the early stages of their learning. Williams (1991) notes that

> Just as students' informal limit models tend to parallel those of the mathematical community prior to Cauchy, it is possible that only by

appreciating the sorts of problems that motivated Cauchy's work will students be motivated to understand its implications. Perhaps this is to say that the very historical and cultural contexts that lent vitality to the original work are the best medium through which to approach the understanding of that work. (p. 235)

An important issue that remains to be addressed is whether or at what point students need to make the historical transitions in their discourse on limits. The study does not suggest that the dynamic view of limit should be discarded from the classroom discourse on limits since it can be the most useful tool with which to initially make sense of the concept. In fact, both the informal and the formal definition of limit[1] are contexts in which it is possible to talk about limit as objectified (a number). However, in this study, students' reliance on dynamic motion did not support their realization of limit as a number. Instead, they talked about limit as a process. Therefore, it seemed necessary that their metarule of using continuous motion while making sense of limit needed to be challenged at some point in order to support the objectification of limit (as a number) in their discourse.

The formal definition of limit was a context in which it was possible for the instructor to highlight the elimination of motion from the discourse on limits and challenge students' assumptions of continuous motion. On the other hand, his shifts in word use (signifying proximity instead of motion) and use of the metaphor of discreteness remained implicit for the students in the class. It seems important that the metarules behind the historical junctures, which were reflected in the instructor's discourse when talking about the informal and the formal definition, are explicitly mentioned in the classroom in order for students to distinguish the process aspect of limit from the product aspect. Although students might not need to know about the existential quantifiers, the symbolism, and proofs associated with the formal definition, familiarity with the metarules and metaphors of the formal theory might enhance their realization of limit as a number but not as a process or "happening".

Note that the course was designed to address informal aspects of the concepts and their applications more than their formal aspects and justifications. One possible reason for this structure might be due to the diversity of students required to take the course with respect to their majors. In this study, the 31 students enrolled in the classroom were from 18 different majors. 18 of those students were first-year; nine of them second-year; three of them third-year and one of them was a fourth-year student. Although such diversity and inexperienced body of students make it difficult for the instructor to attend to all aspects of calculus concepts in detail[2], these characteristics of the classroom also make it vital that the main ideas about limits are communicated effectively.

Among the most striking results of this study is how the instructor's accurate use of words, his endorsed narrative that *limit is a number*, and his utilization of different metaphors as metarules for realizing limits remained invisible for the students. Being a mathematician, the instructor communicated the limit related ideas, which were compatible with the development of the discourse on limits over history, precisely.

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[1] The former definition is consistent with Cauchy's approach whereas the latter is consistent with Weierstrass' approach (See Section 3.3).

[2] In fact, it is possible that the instructor's introduction of the formal definition was atypical for this particular course.

On a pedagogical dimension, however, his communication with his students did not result in shared meanings about limit related concepts. The existence of such gaps in the mathematical communication between the instructor and the students might have resulted from the participation structure in the class. In the period of observations, the "talk" in the classroom was unidirectional: from instructor to students. There were few instances of student-teacher interaction and no student-student interaction. This hindered students' active involvement in meaning making in the classroom.

A second possible reason contributing to the communicational difficulties can be tied to the instructor's level of explicitness in his discourse. It was mentioned that students learned most of the ideas about limit in a compact manner and in a short period of time. In order for students to "decipher" the objectified discourse on limits in the required time, it might be necessary for the instructor to be explicit about his word use, visual mediators, endorsed narratives, routines (including use of metaphors), and, most importantly, the connections among these elements of his discourse. That the contexts in which the instructor shifted his word use coincided with the contexts in which students struggled most in the study (which also coincided with the historical junctures changing the metarules and metaphors of limits) highlights the importance of explicitness regarding the elements of the mathematical discourse in the classroom.

8.3. Conclusions

In this section, I discuss the implications and limitations of the study. I then provide a list of further questions and ideas this study encourages us to think more about.

8.3.1. Implications of the study

This study contributes to educational research in several ways. It pilots an analytical lens, the commognitive framework (Sfard, 2008), for analyzing classroom discourse in a beginning-level calculus course by focusing on the limit concept. The study also takes a step to fill an important gap in the literature by exploring the teaching of calculus at the undergraduate level. In addition, commognitive framework offers tools, namely word use (through objectification), metarules and metaphors, to interpret classroom discourse in relation to the historical development of mathematical concepts.

The utilization of the commognitive framework provided revealing information with respect to the patterns in word use, visual mediators, routines, and endorsed narratives of mathematical discourse on limits. The framework also pointed to the dynamic relationship among these four elements of discourse and how the relationships are formed and reformed depending on the mathematical context. In particular, the commognitive framework highlighted the significance of word use in mathematical communication. A cognitivist framework would not help us recognize the discursive patterns and the importance of mathematical contexts in the development of mathematical discourse since its approach to learning is mostly context-independent. Therefore, one of the implications of the study is that the commognitive framework is a promising lens to investigate the teaching and learning of other mathematical concepts.

Besides these theoretical implications, the study also has some practical implications for the teaching and learning of limits. First, although it took mathematicians centuries to develop and objectify mathematical ideas, students can be immediately presented with the objectified version of the notions. Since objectification hides the layers of mathematical discourse, it may be important for teachers to unpack the objectified discourse so that students can have access to the main ideas. The unpacking of the objectified mathematical discourse can be made possible by an explicit focus on the word use, visual mediators, routines, endorsed narratives of the discourse and their relation to one another. Word use and routines, the metarules characterizing the repetitive actions of participants, seem to be the main elements of discourse that can remain tacit in the classroom. Therefore, it might be important for teachers to talk about *when* a routine is implemented (e.g., mentioning that we use the metaphor of continuous motion when realizing limits informally) as much as *how* a routine is implemented (e.g., uttering words signifying motion when talking about limits informally).

Second, since mathematics learning is participating in the mathematical discourse, active interaction can promote the discussion and exploration of different uses of mathematical words and routines in the classroom. Such an approach can give teachers and students chances to compare and contrast their individual realizations with the communally agreed upon realizations of the mathematical concepts. By doing so, it can also contribute to the unpacking of words and metarules related to the concepts.

Third, the junctures in the historical development changing the metarules of mathematical concepts might be useful for teaching. In this study, such junctures seemed to be critical for students since the contexts in which mathematicians tackled with limits over history coincided with the contexts students struggled. In case there are such parallels between student learning and historical development of other mathematical concepts[3], awareness of the historical junctures can be quite important for teachers to recognize and address some of the student difficulties in their classrooms.

8.3.2. Limitations of the study

The study is conducted as a case study. It is an attempt to investigate the teaching and learning of limits at the undergraduate level by focusing on one classroom. Therefore, the findings of the study may not be applicable beyond this context. In other words, I do not claim that it is possible to generalize these findings to a population of undergraduate calculus classrooms. Note also that the application of a particular lens, in this case the commognitive framework, brings with it many presuppositions leading to that decision (e.g., consideration of mathematical knowledge as socially constructed, consideration of thinking as individualized form of communication, acknowledging the communicational nature of student difficulties, etc.) Since educational research is interpretive, I also acknowledge that another researcher using a different lens (e.g., who operates from a cognitivist framework) could see the classroom in different ways than I did.

8.3.3. Further questions

The study raises additional questions and ideas for further research. First, the analysis of curricular materials in terms of their discourse on limits and infinity is needed to complement how the mathematical ideas developed throughout history are reflected in today's classrooms. In the context of undergraduate mathematics classrooms, textbooks are among the main curricular sources with which instructors organize their lessons. For the utilization of the textbook by students, the study raises important questions: How can

[3] In fact, there is also evidence there are similarities between some of students' conceptual obstacles about

we encourage students to interact with the textbook except for end-of-section problems? How can the instructors find ways of incorporating textbooks so that students think of the textbook as a significant resource for learning? The textbook authors have more chances to think about and revise their word use, visual mediators, routines and narratives of mathematics. As a result, the textbook's discourse can be more consistent than instructors' given that the instructors of undergraduate mathematics are not necessarily trained in pedagogy of mathematics.

The study is an initial attempt to explore the teaching of calculus at the undergraduate level. More research on the teaching of mathematics in undergraduate setting is crucial since calculus is a requirement for many majors (e.g., economics, engineering, physics, and mathematics), and students might not be introduced to any of the calculus concepts prior to their university courses.

Finally, since the implementation of the commognitive framework for limits proved to be a useful lens to gain more information about student learning, repetition of the study in other undergraduate classrooms and for other mathematical concepts seems to be a promising way to enhance our knowledge of classroom discourse.

Additional questions raised by the study include: Would the investigation of historical junctures resulting in changes in the metarules of other concepts of mathematics besides limits be useful to gain information about the conceptual obstacles students face as they learn about those concepts? Would a teaching approach that explicitly focuses on word use, metarules and metaphors related to the mathematical concepts make a difference in students' realizations of the concepts? What are the

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functions and the historical development of functions (Sfard, 1992).

conditions under which students seem to be "ready" for making discursive changes as reflected in the historical development of mathematical concepts?

APPENDIX A

DIAGNOSTIC SURVEY

(Williams, 2001, p. 348)

I. Please mark the following six statements about limits as being true or false:

1. T F A limit describes how a function moves as x moves toward a certain point.

2. T F A limit is a number or point past which a function cannot go.

3. T F A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.

4 T F A limit is a number or point the function gets close to but never reaches.

5. T F A limit is an approximation that can be made as accurate as you wish.

6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

II. Which of the above statements best describes a limit as you understand it? (Circle one)

1 2 3 4 5 6 None

III. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \rightarrow s$ is some number L.

IV. If possible, write down a rigorous definition of limit.

APPENDIX B

INTERVIEW SESSION PROBLEMS

1) If $f(x)=\begin{cases} x & when \quad x \geq 0 \\ x^3 & when \quad x < 0 \end{cases}$, what can you say about $\lim_{x\to 0} f(x)$, $\lim_{x\to -\infty} f(x)$, $\lim_{x\to \infty} f(x)$?

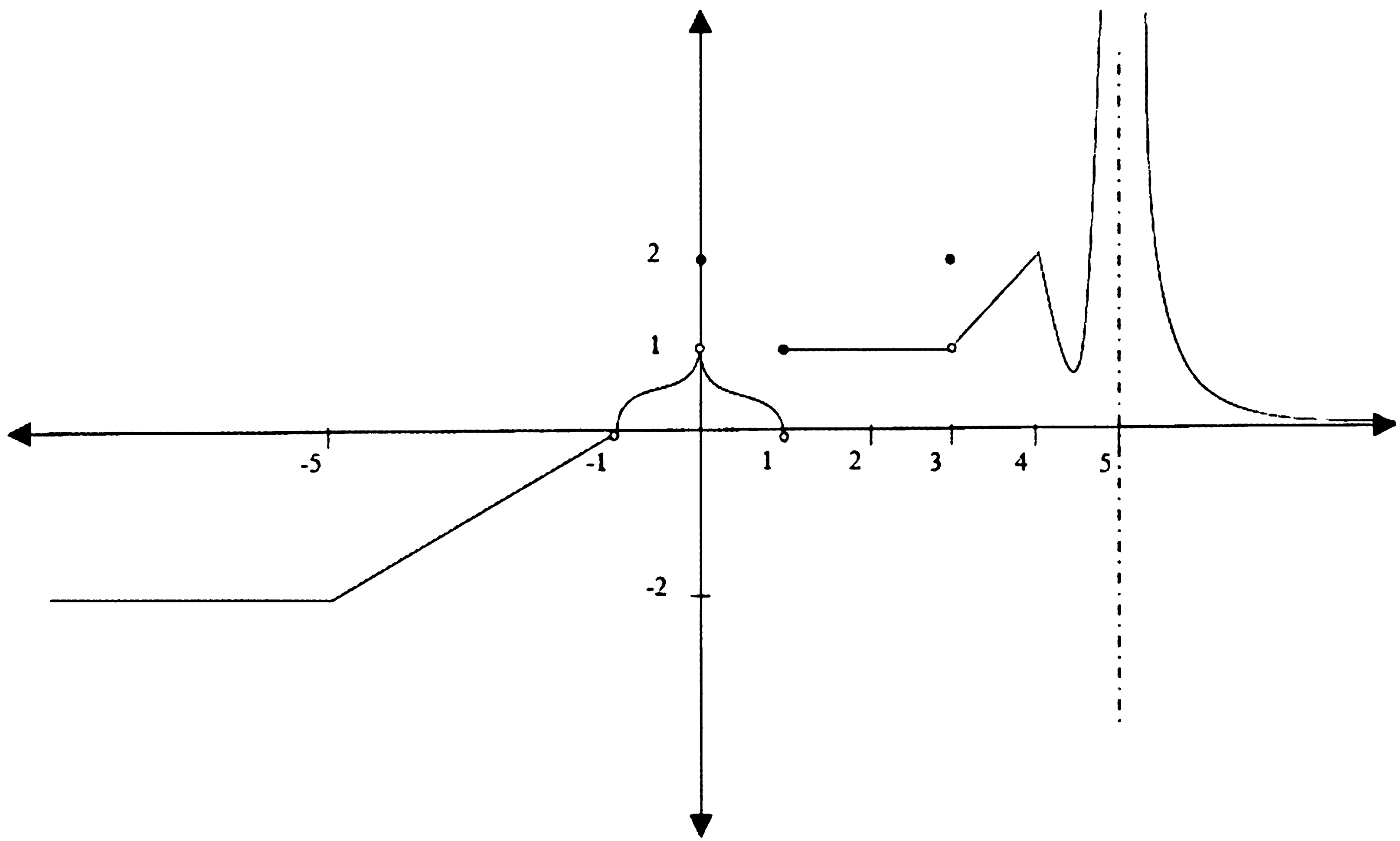
Is the function continuous at 0? How about negative and positive infinity?

2) If $f(x)=\begin{cases} x^2+1 & when \quad x \leq -1 \\ x-1 & when \quad x > -1 \end{cases}$, what can you say about $\lim_{x\to -1} f(x)$ $\lim_{x\to -\infty} f(x)$, $\lim_{x\to \infty} f(x)$?

Is the function continuous at -1? How about negative and positive infinity?

3) Given the following graph of a function f, what can you say about the following limits? Explain your reasoning. $\lim_{x\to-\infty} f(x) = ?$

$\lim_{x\to-5} f(x) = ?$ $\lim_{x\to-1} f(x) = ?$ $\lim_{x\to0} f(x) = ?$ $\lim_{x\to1} f(x) = ?$ $\lim_{x\to2} f(x) = ?$ $\lim_{x\to3} f(x) = ?$ $\lim_{x\to4} f(x) = ?$ $\lim_{x\to5} f(x) = ?$ $\lim_{x\to\infty} f(x) = ?$



Find all the points where $f(x)$ is discontinuous. Why is the function discontinuous at those points?

4) A student was given a function F and asked to find the limit of F as x approached 0. He plugged in numbers on each side of 0 and made the following table:

x	$F(x)$
-.1	0.9
-.01	0.99
-.001	0.999
-.0001	0.9999
-.00001	0.99999
-.000001	0.999999
-.0000001	0.9999999
-.00000001	0.99999999
.1	1.1
.01	1.01
.001	1.001
.0001	1.0001
.00001	1.00001
.000001	1.000001
.0000001	1.0000001
.00000001	1.00000001

a) What can you say about $\lim_{x \to 0} F(x)$?
b) What do you think about the student's method of finding this limit? Is he correct? Can we find the limit of functions using this method? Why/Why not?
c) Given $F(x) = x + 1 + \frac{1}{10^{20} x}$, what can you say about $\lim_{x \to 0} F(x)$?

(Williams, 1991, p. 224)

5) Which statements below must be true if f is a function for which $\lim_{x\to 2} f(x) = 3$?

a) f is continuous at the point $x = 2$
b) f is defined at $x = 2$
c) $f(2) = 3$
d) $\lim_{h\to 0} \{f(2+h) - 3\} = 0$
e) For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|f(x) - 3| < \varepsilon$
f) None of the above mentioned statements

6) Let a function $f(x)$ be defined on an open interval about [the point] x_0, except possibly for x_0 itself. We say that the *limit of* $f(x)$ *as* x *approaches to* x_0 *is the number* L, and write $\lim_{x\to x_0} f(x) = L$ if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. (Thomas et al., 2008, p. 91)

a) Please explain in your words what this definition means for you.
b) How is this definition similar/different than the informal definition of limit you used in your class?
c) In your opinion, what is the purpose of this definition?

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