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SOME APPLICATIONS OF THE GIROUX CORRESPONDENCE IN LOW-DIMENSIONAL TOPOLOGY

By

Cagri Karakurt

A DISSERTATION

Submitted to
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ABSTRACT

SOME APPLICATIONS OF THE GIROUX CORRESPONDENCE IN LOW-DIMENSIONAL TOPOLOGY

By

Cagri Karakurt

E. Giroux has showed that the study of contact structures is equivalent to that of open book decompositions on 3–manifolds. In this thesis, we discuss two applications of this correspondence in the low-dimensional topology.

In the first part, we show that every compact smooth 4-manifold X has a structure of a Broken Lefschetz Fibration. Furthermore, if $b_2^+(X) > 0$ then it also has a Broken Lefschetz Pencil structure with nonempty base locus. This improves a theorem of Auroux, Donaldson and Katzarkov, and our construction uses only the 4–dimensional handlebody theory.

In the second part, we study some invariants of contact structures that arise from the Giroux correspondence. We show that the Ozsváth-Szabó contact invariant $c^+(\xi) \in HF^+(-Y)$ of a contact 3-manifold (Y, ξ) can be calculated combinatorially if Y is the boundary of a certain type of plumbing X, and ξ is induced by a Stein structure on X. Our technique uses an algorithm of Ozsváth and Szabó to determine the Heegaard-Floer homology of such 3-manifolds. We discuss two important applications of this technique in contact topology. First, we show that it simplifies the calculation of the Ozsváth-Stipsicz-Szabó obstruction to admitting a planar open book. Then we define a numerical invariant of contact manifolds that respects a partial ordering induced by Stein cobordisms. We do a sample calculation showing that the invariant can get infinitely many distinct values.

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Chapter 1

Introduction

Mathematics is beautiful! One can appreciate its beauty even more when he/she sees results that connect seemingly different areas. In year 2002, E. Giroux announced such a result in the International Congress of Mathematics, [Gi]. His result established the equivalence two objects of different nature that appear on 3–manifolds. More precisely, he proved that there is a one to one correspondence between contact structures and open book decompositions on a closed oriented 3–manifold up to certain equivalence relations. The strength of his theorem comes from the fact that these two live in completely different realms; While contact structures are analytic objects that are defined by means of smooth 1–forms, open book decompositions are geometric in nature; They are fibrations over a circle that are defined in the complement of a link. Some of the remarkable applications of the Giroux correspondence are the proof of a 20 years old conjecture stated by Harer [H2] and proven by Giroux and Goodman [GG], the definition of a Floer theoretic invariant of contact structure given by Ozsváth and Szabó [OS2], and a symplectic compactification theorem due to Eliashberg [E4], see also [Et2] and [AO2], which was used in the proof of the Property P conjecture.

The aim of this thesis is to discuss two other applications of Giroux's correspondence. In the first part of the thesis, we discuss constructions of Lefschetz type fibrations on 4–manifolds. Open books naturally appear in the boundary of these objects and the Giroux correspondence will be used in an essential way to glue two such fibrations. In order to state our theorem in a precise form, we first need to recall some basic definitions and fix conventions:

Let X be an oriented 4-manifold and Σ be an oriented surface. We say a map $\pi : X \to \Sigma$ has a *Lefschetz singularity* at $p \in X$, if after choosing orientation preserving charts $(\mathbb{C}^2, 0) \hookrightarrow (X, p)$ and $\mathbb{C} \hookrightarrow \Sigma$, π is represented by the map $(z, w) \mapsto zw$. If π is represented by the map $(z, w) \mapsto z\bar{w}$ we call p an *achiral Lefschetz singularity* of π. We say $p \in X$ is a *base point* of π, if it is represented by the map $(z, w) \mapsto z/w$. We call a circle $Z \subset X$ *fold singularity* of π, if around each $p \in Z$ we can find charts $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}) \hookrightarrow (X, Z)$ where π is represented by the map

$$(t, x_1, x_2, x_3) \mapsto (t, x_1^2 + x_2^2 - x_3^2) \tag{1.1}$$

Definition 1.0.1. Let X^4 be a compact oriented 4-manifold. A *broken Lefschetz fibration* structure on X is a map $\pi : X \to S^2$ which is a submersion in the complement of disjoint union of finitely many points $\mathcal{P}$ and circles $\mathcal{C}$ in X, such that $\mathcal{P}$ are Lefschetz singularities of π, and $\mathcal{C}$ are fold singularities of π. When $\mathcal{C} = \emptyset$, we call π a *Lefschetz fibration.*

We call π a *broken Lefschetz pencil*, if there is a finite set of base points $\mathcal{B}$ in X (base locus) such that the restriction $\pi : X - \mathcal{B} \to S^2$ is a broken Lefchetz fibration. When $\mathcal{C} = \emptyset$ we call π a *Lefschetz pencil.*

Furthermore, If we allow achiral Lefschetz singularities among the points of $\mathcal{P}$ we will use the adjective "*achiral*" in front of all these definitions, e.g. *Achiral Broken Lefschetz Fibration.* Sometimes for brevity we will refer them simply by LF, BLF, ABLF, LP, BLP, ABLP.

When these singular fibrations defined on a 4-manifold with boundary one can describe three boundary behaviors: convex, concave and flat, see definition 3.1.1. By following a convention due to Akbulut and Ozbagci, we will call a convex Lefschetz fibration without achiral or fold singularities a *Positive Allowable Lefschetz Fibration*, or PALF for short, if every vanishing cycle is a non-separating curve on fiber. Now, we can state our first main theorem which was proven jointly with Akbulut [AK]:

Theorem 1.0.2. Every closed smooth oriented 4-manifold X admits a broken Lefschetz fibration. Furthermore if $b_2^+(X) > 0$ then X also admits a broken Lefschetz pencil structure (with nonempty base locus).

Remark 1.0.3. Previously by using analytic methods which generalize the technique of approximately holomorphic sections in [D], Auroux, Donaldson and Katzarkov proved that every closed smooth oriented 4-manifold with $b_2^+(X) > 0$ admits a broken Lefschetz pencil structure [ADK], so in particular they proved that, after blowing up, X admits broken Lefschetz fibration structure. More recently, by using the map singularity theory Baykur [B] and independently Lekili [L] proved that every closed smooth oriented 4-manifold X admits a broken Lefschetz fibration. Our proof and [L] give a stronger result, namely make the "fold singularities" of π global, i.e. around each singular circle Z we may assume that there are charts $(S^1 \times \mathbb{R}^3, S^1) \hookrightarrow (X, Z)$ so that π is represented by the map (1.1). Recently, in [GK2] there was an attempt by Gay and Kirby to give a topological proof of [ADK], which ended up producing a weaker version of Theorem 1.2 with "achiral broken Lefschetz fibration" conclusion. Here we prove this theorem by using 4-dimensional handlebody techniques. Also, a corollary to our proof is that, X can be decomposed as a union of a convex PALF and a concave BLF glued along their common (open book) boundaries. Lekili described a set of moves of (achiral) broken fibrations that does not change the ambient 4-manifold, [L]. One of the moves assert that an achiral singularity can be traded with three fold singularities. After the results of this thesis announced in [AK], Baykur wrote an appendix [B3] to [L] interpreting Lekili's moves in terms handlebodies. Therefore the combination [GK2], [L] and [B3] also gives a handlebody proof of theorem 1.0.2. Another recent remarkable result about broken fibrations is due to Williams [W] which says that Lekili's moves are sufficient to connect any two broken fibrations in the same homotopy class.

Our proof here proceeds by combining the PALF theory of [AO] with the decomposition (positron) technique of [AM], and it strongly suggests that linking a certain 4-dimensional handlebody calculus on ALF's to the Reidemeister moves of the knots in square bridge position. The philosophy here is that, just as a PALF $\mathcal{F}$ is a topological structure which is a primitive of a Stein manifold $W = |\mathcal{F}|$, an ALF is weaker topological structure which is a primitive of an "Almost Stein manifold" (for example, while the boundaries of PALF's are positive open books, the boundaries of ALF's are all open books). ALF's are more flexible objects, which are amenable to handle calculus. Our argument uses this flexibility. Unlike the alternative proof mentioned above, the positron technique does not create extra fold singularities.

In the second part of this thesis, we study some invariants contact structures arising from the Giroux correspondence and their relations. In [OS2], Ozsváth and Szabó defined a Heegaard-Floer homology class that reflects certain properties of a given contact structure using a compatible open book. In another direction, based on Giroux's work, Ozbagci and Etnyre [EtnO] defined an invariant, so-called support genus, which is the minimal page genus of an open book decomposition compatible with a fixed contact structure. Previously, Etnyre [Et] had found out that being supported by a genus zero open book puts some restrictions on intersection forms of symplectic fillings of a contact structure. His result was later improved by Ozsváth, Stipsicz and Szabó who showed that the image of the Ozsváth-Szabó contact invariant in the reduced version of Heegaard-Floer homology is actually an obstruction to be supported by a planar open book. More precisely, they proved the following.

Theorem 1.0.4. (Theorem 1.2 in [OSS]) Suppose that the contact structure ξ on Y is compatible with a planar open book decomposition. Then its contact invariant $c^+(\xi) \in HF^+(-Y)$ is contained in $U^d \cdot HF^+(-Y)$ for all $d \in \mathbb{N}$.

In spite of having useful corollaries, this theorem may not be easy to apply all the time because it involvolves calculation of the group HF^+ and identification of the contact invariant in this group. The former problem can be solved if we restrict our attention to a certain class of manifolds. In [OS1], Ozsváth and Szabó gave a purely combinatorial description of Heegaard-Floer homology groups HF^+ of some 3–manifolds which are given as the boundary of certain plumbings of disk bundles over sphere. We will show how to pin down the contact element within this combinatorial object.

To state our second main results, in what follows we shall assume that G is a negative definite plumbing tree with at most one bad vertex. Let $X(G)$ and $Y(G)$ be the 4– and 3–manifolds determined by the plumbing diagram respectively. Denote the set of all characteristic co-vectors of the lattice $H^2(X(G), \mathbb{Z})$ by $\text{Char}(G)$. We form the group $\mathbb{K}^+(G) = (\mathbb{Z}^{n \geq 0} \times \text{Char}(G))/\sim$ where the relation $\sim$ is to be described in Section 4.2. Recall from [OS2] that the Heegaard-Floer homology group HF^+ of any 3-manifold is equipped with an endomorphism U. In [OS3] (see also Section 4.2 below), Ozsváth and Szabó established the following isomorphism.

$$\text{Hom}\left(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>0} \times \text{Char}(G)}, \mathbb{Z}/2\mathbb{Z}\right) \simeq \text{Ker}(U) \subset HF^+(-Y(G)) \qquad (1.2)$$

Recall that if ξ is a contact structure, its Ozsváth-Szabó contact invariant $c^+(\xi)$ is a homogeneous element in $\text{Ker}(U) \subset HF^+(-Y(G))$. It is also known that $c^+(\xi)$ is non-zero if ξ is induced by a Stein filling. The following proposition pins down the image of contact invariant under the above isomorphism.

Proposition 1.0.5. Let J be a Stein structure on $X(G)$ and ξ be the induced contact structure on $Y(G)$. Under the identification described in equation 1.2, the contact invariant $c^+(\xi)$ is represented by the dual of the first Chern class $c_1(J) \in H^2(X, \mathbb{Z})$

.

Remark 1.0.6. This proposition can be generalized in several different directions. First, we may allow the graph G to have two bad vertices. In this case, the group on the left hand side of equation 1.2 gives only even degree elements in $\mathrm{Ker}(U) \subset HF^+(-Y(G))$. Second, the graph G which has at most one bad vertex could be semi-definite implying that $b_1(Y) = 1$, and we use the generalization of the Ozsváth-Szabó algorithm given in [R2]. Finally, keeping G positive definite, we may require J to be an $\omega-$tame almost complex structure on $X(G)$ for some symplectic structure ω which restricts positively on the set of complex tangencies of $Y(G)$ (i.e. $(X(G), \omega)$ forms a weak filling rather than a Stein filling for the corresponding contact structure on the boundary).

When combined with Theorem 1.0.4, Proposition 1.0.5 allows us to determine whether or not certain contact structures admit planar open books. Recall from [OS3] that the correction term for any spin^c structure $\mathfrak{t}$ of a rational homology 3-sphere Y is the minimal degree of any non-torsion class in $HF^+(Y, \mathfrak{t})$ coming from $HF^\infty(Y, \mathfrak{t})$.

Theorem 1.0.7. Assume the conditions in Proposition 1.0.5. Denote the correction term of the induced spin^c structure $\mathfrak{t}$ on $Y(G)$ by d. Also, let $d_3(\xi)$ be the 3-dimensional invariant of the contact structure ξ. Suppose that we have either

$d_3(\xi) \neq -d - 1/2$ or $\text{rank}(HF^+_d(-Y(G), \mathfrak{t})) > 1$ then ξ can not be supported by a planar open book.

Note that checking the conditions stated in this theorem is simply a combinatorial matter, [OSS] (see also Section 4.2 below). Corollary 1.7 of [OSS], which holds for arbitrary rational homology 3-spheres, implies the above statement when $d_3 \neq -d(\xi) - 1/2$. This could be taken as an evidence to conjecture that Theorem 1.0.7 also holds for every rational homology 3-sphere.

Remark 1.0.8. There is another version of Ozsváth-Szabó contact invariant $c(\xi)$ which lives in $\widehat{HF}(-Y)$ and is related to $c^+(\xi)$ by $\iota(c(\xi)) = c^+(\xi)$ where ι is the natural map $\iota : \widehat{HF}(-Y) \to HF^+(-Y)$. The invariant $c(\xi)$ can be calculated combinatorially as shown in [P] and [BP]. However, for the present applications the usage of the c^+ is essential.

The techniques of this thesis can also be used to study a natural partial ordering on contact 3–manifolds up to some equivalence. Following [E2], [EH] and [Ga], we write $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$ if there is a Stein cobordism from (Y_1, ξ_1) to (Y_2, ξ_2). Moreover, we write $(Y_1, \xi_1) \sim (Y_2, \xi_2)$ if these contact manifolds satisfy $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$ and $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$. Clearly, $\sim$ defines an equivalence relation on the set of contact manifolds, and $\preceq$ is a partial ordering on the equivalence classes. One can define a numerical invariant of contact manifolds that respects this partial ordering. Namely, if we let

$$\sigma(Y, \xi) = -\max\left\{d : c^+(\xi) \in U^d \cdot HF^+(-Y)\right\}$$

the naturality properties of the Ozváth-Szabó contact invariant (c.f. Section 4.1 below) imply that we have $\sigma(Y_1, \xi_1) \leq \sigma(Y_2, \xi_2)$ whenever $(Y_1, \xi_1) \preceq (Y_2, \xi_2)$. Note that σ invariant can be infinite. In fact, $\sigma(Y, \xi) = -\infty$ if (Y, ξ) admits a planar open book by Theorem 1.0.4. Clearly, if two contact manifolds have different σ-invariants, they lie in different $\sim$ equivalence classes. The following theorem tells that there are infinitely many such equivalence classes.

Theorem 1.0.9. Any negative integer can be realized as the σ invariant of a contact manifold.

In fact, we are going to obtain some contact manifolds with distinct σ invariants by doing Legendrian surgery on certain stabilizations of some torus knots. See Theorem 4.5.1 below.

The organization of this thesis is as follows. In Chapter 2, we discuss the basics of contact geometry and state the Giroux correspondence in a precise form. The remaining chapters are independent from each other. In Chapter 3, we prove the existence theorem (i.e. Theorem 1.0.2) of Broken Lefschetz fibrations on 4–manifolds. Chapter 4 is devoted to the study of the Ozsváth-Szabó contact invariant and the proofs of Proposition 1.0.5, Theorem 1.0.7 and Theorem 1.0.9.

Chapter 2

Preliminieries

2.1 Contact Structures

In this section we discuss basics of contact topology. For a more detailed discussion, reader can consult [Ga], or [OS].

Let Y be a closed oriented 3−manifold. A *contact structure* ξ on Y is a 2–plane distribution on the tangent bundle TY which is totally non-integrable. This condition means that locally one can express ξ as the kernel of a one form α such that $\alpha \wedge d\alpha$ is a positive multiple of the volume form. If such a 1–form exists globally, we call ξ a co-orientable contact structure. We always assume that our contact structures are co-orientable. The pair (Y, ξ) is called a *contact manifold.*

Equivalence of contact structures can be defined in several ways. One natural equivalence relation on the set of contact structures is homotopy. Two contact structures ξ_1 and ξ_2 on a 3−manifold Y are called *homotopic* if they are homotopic when regarded as 2−plane fields (i.e. sections of the Grassmannian bundle $Gr^2(TY) \to Y$). A stronger notion of equivalence is isotopy. Two contact structures ξ_1 and ξ_2 on a 3−manifold Y are called *isotopic* if there is a diffeomorphism $\phi : Y \to Y$ which is isopic to the identity diffeomorphism such that $\phi_*(\xi_1) = \xi_2$. By a theorem of Gray

[Gra], this is equivalent to that there is a homotopy between ξ_1 and ξ_2 through contact structures. Finally, one can talk about equivalence of contact manifolds. Two contact manifolds (Y_1, ξ_1) and (Y_2, ξ_2) are said to be *contactomorphic* if there is a diffeomorphism $\phi : Y_1 \to Y_2$ such that $\phi_*(\xi_1) = \phi_*(\xi_2)$.

Example 2.1.1. On $\mathbb{R}^3$, consider the one form $\alpha = dz + xdy$. It is easy to see that α satisfies the non-integrability condition, so the 2−plane field $\xi_{\text{std}} = \ker\alpha$ is a contact structure. We refer this as the *standard contact structure* on $\mathbb{R}^3$.

There is a basic dichotomy in the classification of contact structures. To state this, we need to give a few definitions first. Let (Y, ξ) be a contact manifold. An embedded curve $K \subset Y$ is called *Legendrian* if the tangent space T_pK of K lies in the contact plane ξ_p at every point $p \in K$. If K is a Legendrian knot, the contact structure induces a framing (i.e. a section of the normal bundle) on K, see section 2.2. If K is in addition null homologous, this framing is determined by the number twists it makes relative to the framing induced by a Seifert surface. This number is called the *Thurston-Bennequin* invariant of the Legendrian knot K, and it is denoted by $tb(K)$. Note that the Thurston-Bennequin invariant does not depend on the choice of a Seifert surface used in its definition. An embedded disk D is said to be *overtwisted* if ∂D is Legendrian with $tb(\partial D) = 0$. A contact manifold is called *overtwisted* if it contains an overtwisted disk, otherwise it is called *tight*. The isotopy classification of overtwisted contact structures is well-understood thanks to the following theorem of Eliashberg, [E3].

Theorem 2.1.2. Any homotopy class of 2−plane fields on a 3−manifold can be represented by an overtwisted contact structure. Two overtwisted contact structures are isotopic if and only if they are homotopic.

The classification of tight contact structures, however, is in general a difficult problem. The solution is known only for some special cases.

2.2 Legendrian Knots

The aim of this section is to recall some of the basic facts about Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$. Our plan is to define a special projection and show how elementary invariants of Legendrian knots can be determined by studying this projection.

Let L be a Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$. The image of L under the projection map $(x, y, z) \to (y, z)$ is called the *front projection* of L. It is easy to show that L is completely determined by its front projection. Indeed, the Legendrian condition implies that the x coordinate of any point on L can be recovered by the formula

$$x = -\frac{dz}{dy}. \tag{2.1}$$

Since the x coordinate of any point on K is finite, this formula rules out vertical tangencies in a front projection. Instead, we see cusps in front projections as in Figure 2.1. Note that cuspidal singularities disappear when we lift the knot to $\mathbb{R}^3$. Another notable point about front projections is that they allow only certain types of crossing. Indeed, Equation 2.1 tells that at any crossing the strand with more negative slope must be over the other strand.

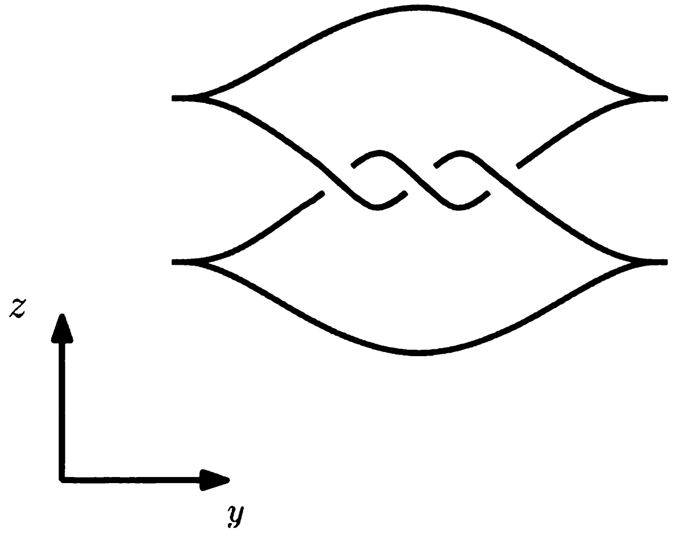

Figure 2.1: A Legendrian Trefoil

Equivalence of Legendrian knots can be defined in a natural way; We say that two Legendrian knots are *Legendrian isotopic* if they are isotopic through Legendrian knots. In $(\mathbb{R}^3, \xi_{\text{std}})$, any Legendrian isotopy can be broken into the elementary pieces shown in Figure 2.2. These will be called *Legendrian Reidemeister moves.* Of course, any Legendrian isotopy invariant should not be changed under these moves.

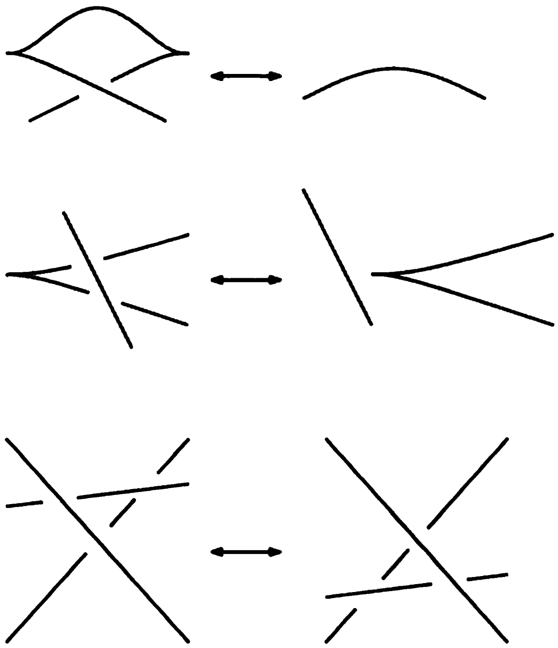

Figure 2.2: Legendrian Reidemeister moves

Let L be a Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$. Two classical Legendrian isotopy invariants of L can be formulated easily using the front projection of L. We denote the *Thurston-Bennequin invariant* of L by $\text{tb}(L)$ and the *rotation number* by $\text{rot}(L)$. These two invariants are given by the following formulae

$$\text{tb}(L) = w(L) + \frac{1}{2}c(L) \tag{2.2}$$

$$\text{rot}(L) = \frac{1}{2}(c_u(L) - c_d(L)), \tag{2.3}$$

where $w(L)$,$c(L)$, $c_d(L)$, and $c_d(L)$ denote the writhe, the total number of cusps, and the number of upward and downward cusps respectively. To define the last two we need to fix an orientation. So, $\text{rot}(L)$ depends on the orientation of L. It is easy to see that $\text{tb}(L)$ and $\text{rot}(L)$ do not change under Legendrian Reidemeister moves.

There is another interpretation of the Thurston-Bennequin invariant in terms of relative framings. Let L be a Legendrian knot in an arbitrary contact manifold. The contact planes induce a framing (i.e. a homotopy class of nowhere vanishing local vector field) on L as follows. The tangent bundle TL of the knot is a sub-bundle the restriction of the contact planes on L. The quotient ξ/TL is an orientable line bundle on a circle, so it is trivial. A section of this bundle defines a framing on L. We call this framing the *Thurston-Bennequin framing* of L. If L is also null-homologous there is another canonical framing on L coming from the normal vector field along the boundary of a Seifert surface. The difference of these two framings is an element of $\pi_1(SO(2)) = \mathbb{Z}$. The invariant $\text{tb}(L)$ is the number corresponding to the difference of Thurston-Bennequinf framing and Seifert framing.

2.3 Fillings

In this section we study several aspects of symplectic fillability in contact topology.

Let X be an oriented 4–manifold possibly with boundary. A *symplectic structure* ω is a 2–form which is closed and non-degenerate. Non-degeneracy in this context means that the exterior product of ω with itself is a positive multiple of the volume form. A vector field v on X is called a *symplectic dilation* if the Lie derivative of ω along v is a positive multiple of ω. A hypersurface Y in X is called convex (respectively concave) if locally, there exists a symplectic dilation which is positively (respectively negatively) transverse to Y.

The simplest way to obtain a symplectic manifold from a contact manifold is the process so-called *symplectization.* Let (Y, ξ) be a closed contact 3–manifold. Assume the contact structure is given as the kernel of a 1–form α. Let $X = Y \times [0, 1]$ and t denote the variable corresponding to $[0, 1]$. The two form $\omega_\alpha = d(e^t\alpha)$ can easily be seen to be a symplectic form on X. Note that the vector field ∂_t is a symplectic dilation making the boundary components $Y \times \{1\}$ and $Y \times \{0\}$ convex and concave respectively.

Let X be a complex manifold, and let $J : TX \to TX$ be the corresponding almost complex structure, let $J^* : T^*X \to T^*X$ be its dual map. To any function $\varphi : X \to \mathbb{R}$ we can associate a 2–form $\omega_\varphi := -dJ^*d\varphi$ and a symmetric 2-tensor $g_\varphi := \omega_\varphi(., J.)$. A function φ as above is called *strictly plurisubharmonic* if g_φ is a Riemannian metric. A compact complex manifold X with non-empty connected boundary is called a *Stein manifold* if it admits a strictly plurisubharmonic function which is constant on its boundary. In this case, the pair (J, φ) is called a *Stein structure* on X. Stein manifolds come equipped with many extra structures. The exact form ω_φ is non-degenerate, so it defines a symplectic structure. The gradient vector field $\nabla\varphi$ of the strictly plurisubharmonic function is a symplectic dilation, indeed the Lie derivative preserves the symplectic form $\mathcal{L}_{\nabla\varphi}(\omega_\varphi) = \omega_\varphi$, and it points outward on the boundary. This implies that the set of complex tangencies of the boundary form a contact structure on the boundary. This construction forms a model for a type of fillability. A contact manifold (Y, ξ) is called *Stein fillable* if there is a Stein manifold (X, J, φ) such that $Y = \partial X$ and $\xi = \ker(\iota_{\nabla\phi}(\omega_\phi))$.

Example 2.3.1. Let B^4 be the unit ball in the complex plane $\mathbb{C}^2$. Consider the function $\varphi(z_1, z_2) = |z_1|^2 + |z_2|^2$. It is easy to show that the associated 2–form ω_φ is the standard symplectic form $dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$ making (B^4, i, φ) a Stein manifold. Writing $z_k = x_k + iy_k$, $k = 1, 2$, we see that the 2–plane field $\ker(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)$ is a contact structure on $S^3 = \partial B^4$. We call this contact structure as the standard contact structure on S^3. Note that it restricts to the contact structure on $\mathbb{R}^3$ which has the same name.

There are two other notions of fillability whose definition involve only the essentials of the above construction. A contact manifold (Y, ξ) is said to be *strongly symplectically fillable* if there is a compact symplectic 4–manifold (X, ω) and a symplectic dilation v in the collar of the boundary such that $Y = \partial X$, $\xi = \ker(\iota_v(\omega))$ and v points outward on Y. We say (Y, ξ) is *weakly fillable* if there is a compact symplectic manifold (X, ω) such that $Y = \partial\omega$ and $\omega|_\xi > 0$. The following implications are clear.

Stein Fillable $\Rightarrow$ Strongly Fillable $\Rightarrow$ Weakly Fillable

The converses of the above implications are not true in general as shown by Eliashberg [E5], see also [DG2], and Ghiggini [Ghi2]. Fillability is a significant restriction on contact structures as the following theorem shows.

Theorem 2.3.2. (Eliashberg, Gromov) A weakly fillable contact srtucture is tight.

By slightly changing the definition of the strong fillability, it is possible to define another notion of filling. A contact manifold is said to be concave fillable if there is a compact symplectic 4–manifold (X, ω) and a symplectic dilation v in the collar of the boundary such that $Y = \partial X$, $\xi = \ker(\iota_v(\omega))$ and v points inward on Y. Admitting a concave filling is not as restrictive as admitting a convex filling.

Theorem 2.3.3. (Eliashberg, Gay, Etnyre-Honda) Every contact manifold admits a concave filling

2.4 Contact Surgery

The purpose of this section is to review two types of surgery operations that are also compatible with contact structures. We will describe these surgery operations as four-dimensional handle attachments. Let L be a Legendrian knot in a contact manifold (Y, ξ). As we saw in the previous section, the product manifold $Y \times [0, 1]$

can be equipped with a symplectic structure making one of the boundary components convex and the other component concave. Attach a four-dimensional two handle to the convex component along L with framing $\text{tb}(L) - 1$. In [Wei], Weinstein proves that there is a canonical way of extending the symplectic structure over the handle keeping the boundary component convex. We denote the new boundary component by $Y_{\text{tb}-1}(L)$. The resulting 4–manifold is a symplectic cobordism from Y to $Y_{\text{tb}-1}(L)$ where the sympectic cobordisms are directed from the concave components of the boundary to the convex components. This cobordism can be glued to any type of symplectic filling of Y on its concave side. The result of this gluing is a symplectic filling of $Y_{\text{tb}-1}(L)$ of the same type. The following theorem indicates this fact. It is due to Weinstein [Wei] for the symplectic fillings and Eliashberg [E] for the Stein fillings.

Theorem 2.4.1. If Y is Stein, strong or weakly fillable then so is $Y_{\text{tb}-1}(L)$.

Dually, a 2–handle attachment to the concave component $\partial Y \times [0,1]$ along a Legendrian knot L with framing $\text{tb}(L) + 1$ yields a symplectic cobordism from Y to $Y_{\text{tb}_1}(L)$, [DG]. Consequently, we have the following theorem.

Theorem 2.4.2. [DG] If Y is concave fillable then so is $Y_{\text{tb}+1}(L)$.

The $\text{tb} \pm 1$ surgeries are sufficient to generate all contact manifolds.

Theorem 2.4.3. [DG] Any contact manifold can be obtained from (S^3, ξ_{std}) by $\text{tb} \pm 1$ contact surgery on a Legendrian link.

2.5 Homotopy Invariants

We are going to mention two homotopy invariants of 2–plane fields of 3–manifolds in this section. Under favorable circumstances these invariants suffice to determine the homotopy type of a 2–plane field ξ. Then we will discuss the calculation of these invariants for contact structures using the contact surgery presentations. The reader is advised to turn to [G] for a complete discussion on homotopy invariants of 2–plane fields on 3–manifolds.

Let ξ_1 and ξ_2 be two (co-)oriented 2–plane fields on a closed oriented 3–manifold Y. We want to understand the obstruction to make ξ_1 and ξ_2 homotopic. Fix a trivialization τ of the tangent bundle TY. Passing to the unit normals, we may regard both ξ_1 and ξ_2 as maps $Y \to S^2$. Of course, these maps depend on τ. We consider the problem of making them homotopic. We can homotope ξ_1 to ξ_2 freely on the 1–skeleton $Y^{(1)}$ as $\pi_1(S^2) = \{1\}$. The obstruction to extend this homotopy to the 2-skeleton lives in $H^2(Y, \pi_2(S^2))$. When $H_1(Y, \mathbb{Z})$ has no 2–torsion, the obstruction can be detected by difference of the first Chern classes of ξ_1 and ξ_2. Here we regard ξ_1 and ξ_2 as complex line bundles where the almost complex structure is determined by the 90° rotation counter-clockwise.

When ξ is a contact structure on Y its first Chern class $c_1(\xi)$ can be calculated as follows. Obtain (Y, ξ) by attaching 2–handles along a Legendrian link in S^3. Let X be the corresponding 4–manifold. The first homology of Y is generated by the dual circles γ_i of the Legendrian handles L_i, and the relations between these generators are determined by the intersection form of X. The Poincare dual of $c_1(\xi)$ can be written as

$$PDc_1(\xi) = \sum rot(L_i)\gamma_i$$

When $H_1(Y, \mathbb{Z})$ has no 2–torsion, the set of homotopy classes of 2-plane fields with a fixed Chern class $c_1 \in H^2(Y, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/(d)$ where d is the divisibility of c_1. When c_1 is torsion, a numerical invariant distinguishes these classes. Assume that the contact manifold (Y, ξ) is the boundary of an almost complex manifold (X, J) such that $J\xi = \xi$. In [G], it is proved that every contact manifold can be expressed this way. In this case, we define the *3-dimensional invariant* of (Y, ξ) by

$$d_3(\xi) = \frac{c_1(X,J)^2 - 3\sigma(X) - 2\chi(X)}{4} \tag{2.4}$$

where $\sigma(X)$ and $\chi(X)$ denote the signature and the Euler characteristic of X respectively. The cup square of $c_1(X, J)$ is taken as follows: $c_1(X, J)^2 = c^2/n^2$ where n is the minimum positive integer such that $nc_1(Y, \xi) = 0$ and c is any relative cohomology class in $H^2(X, Y)$ that hits $nc_1(X, J)$ under the natural map. It is easy to see that the 3-dimensional invariant is independent of the choice of the almost complex manifold X since the right hand side of equation 2.4 is zero for closed almost complex manifolds.

When the contact manifold (Y, ξ) is obtained from (S^3) by Legendrian handle attachments where all the framings all equal to $tb - 1$, calculation of d_3 is especially simple. In this case, (Y, ξ) is naturally the boundary of a Stein 4-manifold (X, J) by Theorem 3.3.1. The first Chern class of this Stein manifold is represented by the cycle $\sum \mathrm{rot}(L_i)S_i$ where each L_i is the attaching circle of a Legendrian handle and S_i is the 2-cycle in X associated to L_i. See [DGS] and also section 3.3 below for the generalization of this argument in case $tb + 1$ surgeries present.

2.6 Open Book Decompositions and the Giroux Correspondence

Let Y be a closed oriented 3-manifold. An open book decomposition of Y is a pair (B, π) where B is a link in Y and π is a fibration over a circle defined on the complement of B over circle. The link B is called the *binding* and a typical fiber of π called a *page* of the the open book decomposition. An abstract open book is a pair (S, ϕ) where S is a compact surface with non-empty boundary, and ϕ is a diffeomorphism of S fixing the boundary pointwise. S is called the page and ϕ is called the *monodromy* of the open book.

An abstract open book determines a closed 3–manifold $Y_{(S,\phi)}$ as follows. First, construct the mapping torus of ϕ by identifying $(x, 1)$ with $(\phi(x), 0)$ in the product manifold $S \times [0, 1]$. Since ϕ restricts to the identity diffeomorphism near ∂S, the boundary of the mapping torus is a disjoint union of tori. Fill in each boundary component with a solid torus in a way that for every $y \in \partial S$, the circle $C_y = \{(y, t) : t \in [0, 1]\}$ for every $y \in \partial S$ bounds a disk. Clearly, $Y_{(S,\phi)}$ admits an open book decomposition with pages all diffeomorphic to S. Conversely, An open book decomposition (B, π) determines an abstract open book (S, ϕ) where $S = \pi^{-1}(\text{point})$ and ϕ is the first return map of π.

In this thesis, we use abstract open books and open book decompositions interchangeably, keeping in mind that the passage between these two notions is as in the paragraph above. Moreover, defining an abstract open book do we only specify the isotopy class of the monodromy as two isotopic monodromy maps yield diffeomorphic 3–manifolds.

Given an abstract open book (S, ϕ), we can construct its positive stabilization by adding a 1–handle to S and composing ϕ with a right handed Dehn twist along a curve that intersects the co-core of the 1–handle geometrically once. Using a left handed Dehn twist instead of a right handed one, we can define negative stabilizations. The 3–manifold determined by (S, ϕ) is unchanged by these two operations.

We say that a contact structure is compatible with an open book decomposition if the contact structure is given by a 1–form α such that $\alpha > 0$ on the binding and $d\alpha > 0$ on pages. In [TW], Thurston and Wilkhelnker constructed explicitly a contact structure on the 3–manifold $Y_{(S,\phi)}$ which is compatible with the open book (S, ϕ).

Theorem 2.6.1. (Thurston-Wilkhelnkemper) Every open book admits a compatible contact structure.

The relation between contact structures and compatible open book investigated further by Giroux, [Gi].

Theorem 2.6.2. (Giroux)

- Every open book admits a compatible contact structure.
- Two contact structures compatible with the same open book are isotopic.
- Two open book decompositions compatible with the same contact structure have a common positive stabilization.

Theorems 2.6.1 and 2.6.2 together establish a one to one correspondence between set of contact structures up to isotopy and the set of open book decompositions up to isotopy and positive stabilizations on a closed 3–manifold. This has very important consequences in contact geometry and low-dimensional topology. For example, we can characterize certain properties of contact structures in terms of the data that defines a compatible open book.

As stated in 2.6.2, positive stabilizations on an open book do not change the isotopy class of the isotopy type of the compatible contact structure. Negative stabilizations, on the other hand, change even change the homotopy type. In fact we will see in section 3.3 that negative stabilizations change the three dimensional invariant. Moreover, they make the contact structure overtwisted.

Theorem 2.6.3. (Giroux-Goodman) A contact structure is overtwisted if and only if it is compatible with an open book which is the negative stabilization of another open book.

There is another criterion of open books which determines whether the corresponding contact structure is Stein Fillable. The criterion uses the topological characterization of Stein manifolds, see Theorem 3.3.1. No such criterion is known for other types of fillings.

Theorem 2.6.4. A contact structure is Stein fillable if and only if it is compatible with an open book whose monodromy can be written as a product of right handed Dehn twists.

Chapter 3

Constructing Broken Lefschetz Fibrations on Four-Manifolds

The purpose of this chapter is to prove Theorem 1.0.2 saying that every 4–manifold admits a BLF structure. Our argument roughly goes as follows: We decompose the given 4–manifold into a convex and a concave fibrations in the sense of section 3.1. The convex fibration possibly has achiral singularities which will be repaired later. While the construction of the concave piece is essentially due to Gay and Kirby [GK2], we use Akbulut-Ozbagci algorithm to get the convex fibration [AO]. Though there are other techniques to construct convex ALFs, e.g. [GK2] and [EF], the usage of the Akbulut-Ozbagci algorithm is essential for the repairing process of the achiral singularities. Next, we turn to the problem of matching the open books on the boundaries of the concave and convex pieces. To do that, we use Eliashberg's classification of overtwisted contact structures (Theorem 2.1.2)and the Giroux correspondence (Theorem 2.6.2). Then we show the matching can be done keeping certain properties of the open books that are necessary in the next step. Finally, we use an interpretation of Akbulut-Matveyev positron move to get rid of the achiral singularities keeping boundary open books fixed so that one can glue the two pieces to obtain a globally defined BLF.

This chapter is organized as follows. In section 3.1, we define several boundary behaviors Lefschetz type fibrations on boundary. In section 3.2, we review the construction and the stabilization of concave fibrations. The next three sections are devoted to the review of Akbulut-Ozbagci algorithm to construct convex fibrations and its generalization into a slightly more general setting. We define almost Stein manifolds in section 3.3. A technical lemma recognizing if an open book decomposition has a destabilization is discussed in section 3.4. Then we relate almost Stein manifolds to convex achiral Lefschetz fibrations in section 3.5. We describe several moves of achiral Lefschetz fibrations that can be used to change the homotopy type of the corresponding contact structure on the boundary in section 3.6, and the open book matching argument is given in section 3.7. An important lemma for repairing achiral singularities is proved in section 3.8.2. Proof of Theorem 1.0.2 is given in section 3.9.

3.1 Introduction

First, we would like to alter the definition of open book decompositions to serve better our purpose in this section.

Definition 3.1.1. Let Y^3 be a closed oriented 3-manifold. An *open book decomposition* of Y is a map onto a 2 dimensional disk, $\pi : Y \to D^2$ such that the preimage of the interior of the disk is a disjoint union of solid tori and the following conditions hold:

- π is a fibration over $S^1 = \partial D^2$ away from these solid tori.
- Restriction of π to each of these tori is given by the projection map $S^1 \times D^2 \to D^2$.

For all $\lambda \in S^1$ the closure of the set $P_\lambda := \{\pi^{-1}(r\lambda) : r \in (0,1]\}$ is an oriented surface with non-empty boundary which will be referred as a *page* of the open book decomposition. The flow of a pullback of the unit tangent to $S^1 = \partial D^2$ defines a diffeomorphism of a fixed page fixing its boundary point wise. This diffeomorphism is called the *monodromy* of the open book decomposition.

Now let X be a 4-manifold with non-empty connected boundary. Let $f : X \to \Sigma$ be a achiral broken Lefschetz fibration where $\Sigma = S^2$ or D^2. The following is a description for several boundary behaviors of f. Some of the conditions we list here are redundant but we put them for contrasting from each other for a better understanding.

Definition 3.1.2. We say

1. f is *convex* if

   - $\Sigma = D^2$.
   - Fibers of f have non-empty boundary.
   - $f|_{\partial X} : \partial X \to D^2$ is an open book decomposition of ∂X.

2. f is *flat* if

   - $\Sigma = D^2$
   - Fibers of f are closed.
   - $f(\partial X) = S^1 = \partial D^2$.
   - $f|_{\partial X} : \partial X \to S^1$ is an honest fibration.

3. f is *concave* if

- $\Sigma = S^2 = D^2_+ \cup D^2_-$ (the upper and lower hemispheres).
- Fibers over D^2_+ are closed, while the fibers over D^2_- have non-empty boundary components.
- $f(\partial X) \subset D^2_-$.
- $f|_{\partial X} : \partial X \to D^2_-$ is an open book decomposition of ∂X.

These definitions are due to Gay and Kirby [GK2]. They also point out that one can glue a convex fibration and a concave fibration if the boundary open books match. Moreover, two flat fibrations can be glued together if they agree on the common boundary.

3.2 Concave Fibrations and Pencils

In this section, we illustrate several constructions of concave fibrations and pencils. The major part of these constructions is basically due to Gay and Kirby.

The easiest example of a concave fibration is obtained by restricting an Achiral Broken Lefschetz fibration on a closed manifold to a tubular neighborhood of a regular fiber union a section. Note that the complement of this set is a convex broken achiral lefschetz fibration. More generally, if we remove a convex ABLF from a ABLF (or ABLP) the remaining part is a concave ABLF (or ABLP). The following proposition uses these ideas to construct an interesting family of concave pencils. It was first appeared in a draft of [GK2]. Unfortunately, that draft is no longer available. We give a proof of the proposition for completeness.

Proposition 3.2.1. Let Σ be a closed oriented surface. Let N be an oriented D^2-bundle over Σ with positive Euler number. Then, N admits a concave pencil with no broken, achiral or Lefschetz singularity.

Proof. Let $\pi : N \to \Sigma$ be the bundle map . Denote its Euler number by $n := e(\pi) > 0$. Take a generic section $s : \Sigma \to N$. It intersects with the zero section at n distinct points. Let $D_1, D_2, \cdots, D_n$ be pairwise disjoint small disk neighborhoods of images of these points in Σ. The section s determines a trivialization τ of the complementary disk bundle such that the square part of the following diagram commutes.

$$\begin{array}{ccccc} N - \pi^{-1}(D_1 \cup \cdots \cup D_n) & \xrightarrow{\tau} & (\Sigma - (D_1 \cup \cdots \cup D_n)) \times D^2 & \xrightarrow{p_2} & D^2 = D^2_+ \\ \downarrow \pi & & \downarrow p_1 & & \\ \Sigma - (D_1 \cup \cdots \cup D_n) & \xrightarrow{\text{id}} & \Sigma - (D_1 \cup \cdots \cup D_n) & & \end{array}$$

We define the pencil map $f : N \to S^2 = D^2_+ \cup D^2_-$ by $f(x) = p_2 \circ \tau(x)$ for $x \in N - \pi^{-1}(D_1 \cup \cdots \cup D_n)$, and on each $\pi^{-1}(D_i) \cong D^2 \times D^2 \subset \mathbb{C}^2$, f is given by the usual pencil map $\mathbb{C}^2 \to \mathbb{C}P^1 \cong S^2$. It is an easy exercise to verify that these two maps agree on their common boundary and together they define a concave pencil on N.

A simpler explanation for this construction can be given via the following handlebody diagrams. We draw the usual handlebody diagram of N, disk bundle over a surface Σ of genus g with Euler number n. The diagram contains one 0-handle, $2g$ 1-handles and a 2-handle with framing n which is attached to the diagram in such a way that the attaching circle passes through each 1-handle exactly twice. To visualize a concave fibration, we blow up N at n distinct points as in the figure.

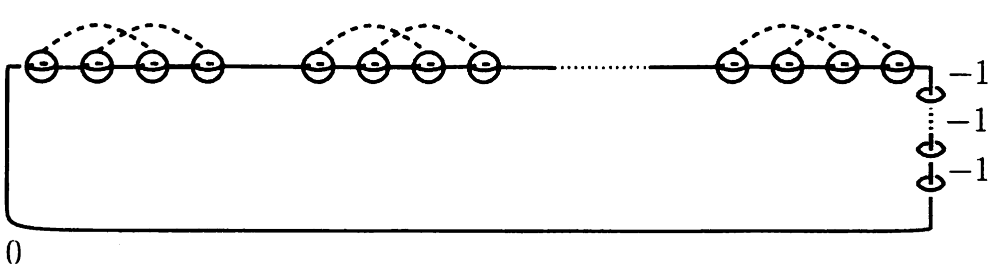

Figure 3.1: $N \sharp n \overline{\mathbb{C}P^2}$

Forgetting the exceptional spheres for a minute, we see a trivial fibration over disk with fiber genus g. To this fibration, we attach several 2-handles where the attaching circles of the 2-handle are sections of this fibration restricted to the boundary. By the text in the beginning of this section, the result of attaching these handles is a concave fibration. The boundary open book of this fibration has pages obtained from Σ by puncturing at n points. The monodromy of the open book is the product of left handed Dehn twists along curves parallel to boundary components of a page. The last two observations can be verified by the discussion in [EO] or by means of an easy exercise in handle slides: Anti blow up at each exceptional sphere. This creates n 0-framed 2-handles in the handlebody diagram. Replace each one them with a 1-handle. The resulting 4-manifold X admits a convex ALF with the same boundary open book. It is transparent from the handlebody diagram of X that our claims about the boundary open book are correct. Note that if we reverse the orientation of this manifold we get a convex PALF which we can glue to our concave fibration on $N\sharp n\overline{\mathbb{C}P^2}$ to get a Lefschetz fibration on the closed manifold $\overline{X} \cup N\sharp n\overline{\mathbb{C}P^2}$ where all the singularities lie in the convex side. This Lefschetz fibration is actually symplectic because it admits n sections of self intersection -1. Blowing down these sections and isolating N, we see that N admits a concave pencil and a compatible symplectic structure ω, making the boundary of N ω-concave. $\square$

An alternative construction of concave piece is provided by Gay and Kirby in [GK2] via round handle attachment. We are going to sketch this construction in proposition 3.2.2 but before doing that we need to review some basic facts about round handles and their relation with broken fibrations. For details, reader can consult [GK2] or [B2].

A round 1-handle is a copy of $S^1 \times D^1 \times D^2$ which can be attached to a 4-manifold along $S^1 \times \{-1\} \times D^2 \cup S^1 \times \{1\} \times D^2$. Up to diffeomorphism, all the attaching information is contained in a pair of framed links which will be referred as the attaching circles of a round round 1-handle. Moreover, we may choose the framing on one of the attaching circles to be 0. Every round 1-handle decomposes uniquely as a union of a 1-handle and a 2-handle passing over the 1-handle geometrically twice algebraically zero times. Therefore, in order to draw a round 1-handle to a handlebody diagram, it is enough to draw a 1-handle and a 2-handle as above. Attaching circles of the round 1-handle can be visualized by shrinking the legs of the 1-handle to pair of points.

Suppose, we have an (A)BLF or (A)BLP with boundary which is one of the three types listed in definition 3.1.2. We can attach a round 1–handle whose attaching circles are circles are sections of the fibration restricted to the boundary. Then, the fibration extends across the round 1–handle with a fold singularity on $S^1 \times \{0\} \times \{0\}$. Resulting object is an (A)BLF or (A)BLP with same type of boundary. Fibers on the boundary, i.e. pages, gain genus. We can visualize this as follows; Attaching circles of the round 1-handle intersects each page at a pair of points. We remove two small disks containing these points from each page and connect the boundary circles of these disks by a handle which is contained in the round 1-handle. The monodromy is composed with some power of Dehn twist along the boundary circle of one of the disks. This power is precisely the framing of round 1-handle.

Proposition 3.2.2. Let Σ be a closed oriented surface. The product $\Sigma \times D^2$ admits a concave Broken Lefschetz Fibration.

Proof. We start with the standard handlebody diagram of $\Sigma \times D^2$. We attach one 1-handle, two 2-handles: zero framed β_1 and -1 framed β_2, and a 3-handle as in the Figure 3.2. In the figure we did not show the whole diagram. 0-framed vertical arc is part of the unique 2-handle of $\Sigma \times D^2$. The framing on β_1 is chosen so that when we slide it over β_2 and cancel β_2 with the 1-handle, β_1 becomes a 0-framed unknot canceling the 3-handle. Therefore we did not change the diffeomorphism type of our manifold by these handle attachments.

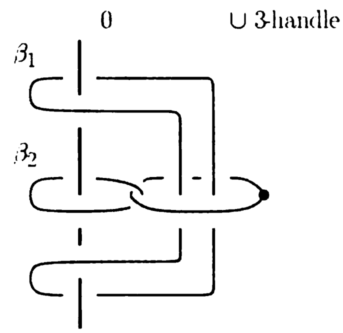

Figure 3.2: the passage from flat to concave fibrations

Now, we describe how the fibration structure extends across these handles. The 1-handle and β_1 forms a round 1-handle which is attached along a bisection of the boundary fibration. As we remarked above, the fibration extends as a broken fibration with one fold singularity inside the round handle. This broken fibration is still flat, i.e. it restricts to a surface bundle over S^1 on boundary. Note that the attaching the attaching circle of β_2 is a section of this surface bundle. Therefore, attaching β_2 turns the flat broken fibration into a concave broken fibration, creating new disk fibers over the southern hemisphere of base sphere. Pages are obtained by punching the fibers on boundary along attaching circle of β_2. Finally, we attach the 3-handle in such a way that its attaching sphere intersects each page along a properly embedded arc γ. Concave fibration extends across the 3-handle without forming any new singularity. The fibers over the southern hemisphere gain a 1-handle whereas pages lose a 1-handle (they are cut open along the arc γ.). □

The last topic we would like to discuss in this section is stabilization of concave fibrations. Recall that positive (resp. negative) stabilization of an open book amounts to plumbing a left (resp. right) handed Hopf band to its pages and composing its monodromy by a right handed (resp. left handed) Dehn twist along the core of the

Hopf band. If the open book is the boundary of a convex ALF positive (resp. negative) stabilization can be achieved by creating a canceling 1-2 handle pair where attaching balls of the 1-handle strung on binding and the 2-handle lies on a page with framing one less (resp. one more) than the framing induced by the page. Stabilization of concave fibrations is not that straight forward because 1-handles cannot be attached this way. Instead we create an extra canceling 2-3 handle pair. Moving the new 2-handle over the 1 handle creates a round 1-handle which can be attached to concave fibration, see Figure 3.3. It is evident from the picture that the resulting concave fibration has a new broken singularity, and a Lefschetz (resp. achiral) singularity inside.

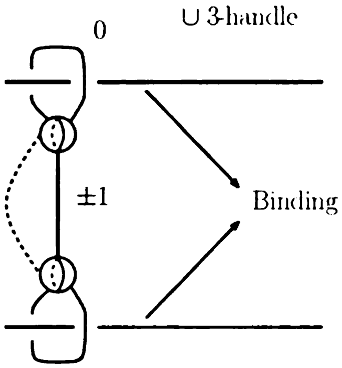

Figure 3.3: Stabilization of concave fibrations

It is possible to negatively stabilize concave fibrations without introducing achiral singularity inside. The trick goes as follows; we create another canceling 1-2 handle pair, where 2-handle has a good (-1) framing. We move bad (+1) framed 2-handle over the new 1-handle and regard them as a round -1-handle, see Figure 3.4.

We formed two fold singularities and a Lefschetz singularity inside of the concave fibration. One right handed and two left handed Hopf bands are plumbed to the boundary open book as in Figure 9 of [GK2].

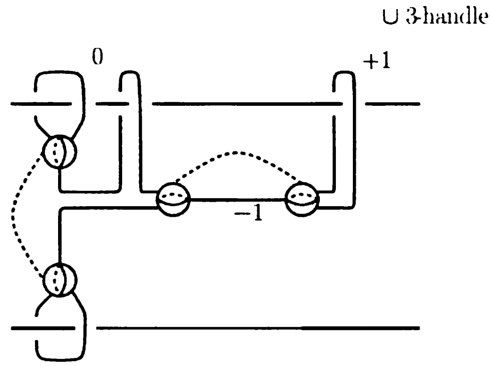

Figure 3.4: Negative stabilization of concave fibrations without achiral singularity

3.3 Almost Stein Manifolds

Here, we review some basic facts about Stein manifolds of real dimension 4. Afterwards, we will discuss their natural generalizations. For more discussion about Stein manifolds, reader can consult [OS].

In [E], Eliashberg gave the topological characterization of 4-dimensional Stein manifolds.

Theorem 3.3.1. (Eliashberg)

Let $X = B^4 \cup (1\text{-handles}) \cup (2\text{-handles})$ be a 4-dimensional handlebody with one 0-handle and no 3 or 4-handles. Then

1. the standard Stein structure on B^4 can be extended over the 1-handles so that the manifold $X_1 := B^4 \cup (1\text{-handles})$ is Stein.

2. If each 2-handle is attached to ∂X_1 along a Legendrian knot with framing one less than the Thurston-Bennequin framing then the complex structure on X_1 can be extended over the 2-handles making X a Stein manifold.

3. The handle decomposition of X is induced by a strictly plurisubharmonic function.

By abuse of language, we call a handle decomposition as in Theorem 3.3.1 as a Stein structure without specifying the almost complex structure or strictly plurisubharmonic function. These two objects should be understood to be constructed as in the proof in [E]. Now by using the characterization in Theorem 3.3.1 we give the following definition to generalize the Stein manifolds to a larger category.

Definition 3.3.2. Let X be a compact oriented 4−manifold. X is said to be *almost Stein* if it admits a proper Morse function $f : X \to \mathbb{R}$ with the following properties.

1. Critical points of f have index no bigger than 2.

2. Boundary ∂X is a level set for f.

3. Attaching circles of 2−handles associated to f are isotopic to some Legendrian curves in $(\partial X^{(1)}, \xi_0)$ such that the framing on any handle is either one less than or one greater than the Thurston-Bennequin framing. Here, $X^{(1)}$ is the union of 0− and 1−handles of X and ξ_0 is the unique Stein fillable contact structure on its boundary.

In Theorem 3.3.3 below we show that almost Stein manifolds exist in abudence. The same theorem, stated in different terms was proved in [GK]. In an almost Stein manifold the 2−handles, which are attached with framing $tb-1$, will be called a *good handle*, otherwise they will be called *defective*.

Theorem 3.3.3. Every 4-dimensional handlebody X with without 3 and 4-handles is an almost Stein manifold with at most one defective handle.

Proof. It is no loss of generality to assume that X has one 0-handle. By the first part in theorem 3.3.1, $X_1 = B^4 \cup$ (1-handles) has a Stein structure. We must show that 2–handles can be arranged to be attached with correct framings. Let h be a 2-handle in X. First, we isotope the attaching circle K of h to a Legendrian knot.

Say, the framing on K is n with respect to the Thurston-Bennequin framing. The number n can be increased arbitrarily by adding zigzags to the front projection of K. Decreasing n is tricky. Near a local minimum of front projection of K, we make Legendrian Reidemeister–1 move, see [G]. This operation does not change the Thurston-Bennequin framing. Next, we create a pair of canceling 1 and 2-handles, where the 2–handle has framing $+1$ as in the left hand side of the Figure 3.5. Sliding h over the new 2–handle, we can decrease its framing. Finally, we observe that the same canceling pair can be used to fix the framing of all 2-handles, leaving only one defective handle as desired.

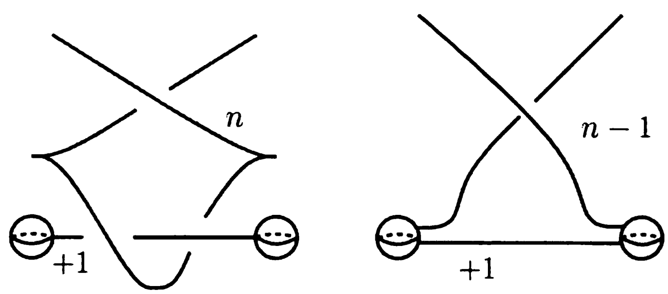

Figure 3.5: Framing fixing in almost Stein manifolds

□

Remark 3.3.4. 1. Creating a pair of 1-handle and a $+1$ framed 2-handle corresponds to negative stabilization of the contact structure ξ_0 on ∂X_1. This creates an overtwisted disk, and sliding a Legendrian knot over that disk increases its Thurston-Bennequin invariant. This observation was used by Gay and Kirby to prove Theorem 3.3.3. Only significant difference is that they apply Lutz twist to create an overtwisted disk.

2. Yet another proof for Theorem 3.3.3 using [AO] can be given via the correspondence between achiral Lefschetz fibrations and almost Stein structures which we state in a more precise form in Section 6 below. Harer [H] (see also [EF])

proved that every 4-dimensional handlebody X without any 3 or 4-handles admits an achiral Lefschetz fibration over disk with bounded fibers. This fibration naturally makes X an almost Stein manifold.

The main theorem in this section asserts that by carving certain properly embedded disk we can transform defective handles into good ones. The following example provides a model for that transformation.

Example 3.3.5. Let $X = D^2 \times S^2$. By [E], X does not admit a Stein structure. The simplest handlebody decomposition of X consists of one 0 and one 2−handle, where the 2−handle is attached to an unknot with zero framing, as in Figure 3.6. The obvious Legendrian realization of this unknot has front projection with two cusps. Thurston-Bennequin number of this Legendrian knot is -1 so, the 2−handle is attached with framing $tb + 1$. An alternative Legendrian realization of the unknot is drawn next to it. The curve γ is not a part of handlebody yet. We will show that the manifold $\widetilde{X} = X - D$ is Stein where D is a properly embedded disk in 0−handle whose boundary is γ.

Consider the Stein manifold on right hand side of Figure 3.6. It is an easy exercise to show this manifold is diffeomorphic to $\widetilde{X}$: Indeed, we can obtain a handlebody decomposition of $\widetilde{X}$ by putting a dot on γ (i.e. turning it into a 1-handle) in the previous figure. Create a canceling 1 and 2–handle pair, where the two handle has framing -1. Slide the new 1-handle over the 1-handle γ. Next slide two strands of original 2-handle over the new 2-handle, and then cancel the 1 and 2 pair. The resulting manifold is the smooth manifold described by the next figure.

Applying the procedure described in the Example 3.3.5 to each defective handle in an almost Stein manifold we arrive at the following conclusion.

Theorem 3.3.6. Every almost Stein manifold admits a Stein structure in the complement of finitely many properly embedded disks.

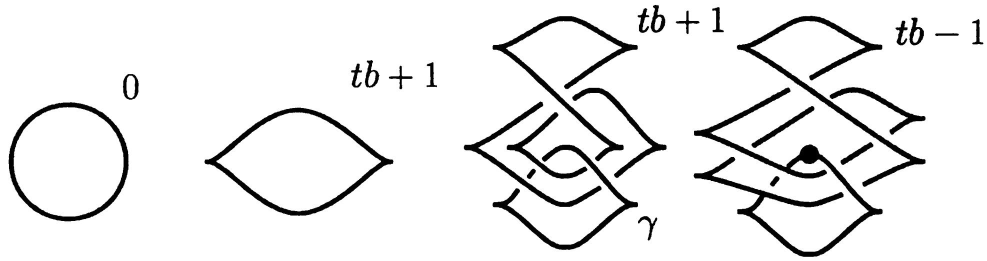

Figure 3.6: Repairing defective handles in an almost Stein manifold

By attaching a good handle and homotopically canceling this 1-handle created in the procedure, we see that *Steinification* of every almost Stein manifold can be made within its homotopy class. This was first observed by the first author and Matveyev in [AM] (e.g. positrons).

Theorem 3.3.7. (Akbulut-Matveyev) Every almost Stein manifold is homotopy equivalent to a Stein manifold.

Almost Stein manifolds do not carry as many extra structures as Stein manifolds. For example, they do not naturally admit a symplectic structure. An almost complex structure can be defined on an almost Stein manifold, but only in the complement of finitely many points each contained in a 2-handle. On the other hand, we still have a contact structure on the boundary defined by means of the set of complex tangencies. This contact structure will be denoted by ξ_X for an almost Stein manifold X.

Handlebody diagram inducing an almost Stein structure on X can easily be converted into a contact surgery diagram in the sense of Ding and Geiges [DG]. We replace each 1-handle with a circle with dot, a convenient notation first introduced in [A] (which means that the interior of the disk, which this circle bounds, is pushed into the 4-manifold interior, then its tubular neighborhood is removed). The only additional assumption is that we choose the circle to be a Legendrian unknot, and we put a negative twist to each 2-handle that passes over the 1-handle as in Figure 3.6. The twisting operation is done in order to match the conventions to calculate the relative framing with respect to Thurston-Bennequin framing. We convert our Legendrian handlebody diagram into a contact surgery diagram by erasing the dots on 1-handles and regarding them as $tb + 1$-framed 2-handles. Topologically, this operation corresponds to surgering some circles, but it does not affect the boundary contact manifold.

Homotopy invariants of ξ_X can easily be read off from the front projection of the Legendrian handlebody diagram inducing the almost Stein structure of X, see [G]. The Poincare dual of its first Chern class is represented by a chain $\sum k_i[\mu_i]$, where k_i is the rotation number of a Legendrian 2-handle and $[\mu_i]$ is the homology class represented by the meridian of that handle. On ∂X, the oriented 2-plane fields having the same first Chern class c_1 is a $\mathbb{Z}/d$ torsor, where d is the divisibility of c_1. Each element of this torsor can be distinguished by its 3-dimensional invariant $d_3 = (c_1(X)^2 - 3\sigma(X) - 2\chi(X))/4 + q$, provided c_1 is torsion. Here q is the number of defective handles, $c_1(X)$ is a cohomology class such that $c_1(X)(h) = \text{rot}(L)$ for every 2-handle h with Legendrian attaching circle L, and the cup square of $c_1(X)$ is taken in some appropriate sense. Let us call the operation of creating a canceling pair of a 1-handle and a $tb + 1$ framed 2-handle by *negative stabilization.* This operation does not have any topological effect but it does change the almost Stein structure on X and the contact structure on ∂X. The key observation here is if we have two almost Stein manifolds with same first Chern class of the contact structures on the boundary, then we can match the contact structures up to homotopy by negatively stabilizing one of the almost Stein manifolds.

3.4 Destabilizations

This section is devoted to a technical lemma. It will be used to determine the monodromy of a (p, q) torus knot but we believe that the lemma is important on its own as it provides a sufficient condition to destabilize an open book. It is analogous to subdividing contact cell decompositions as in [Gi] and product disc decompositions of sutured manifolds defined by Gabai [Gab].

Lemma 3.4.1. Let S be a page of an open book decomposition of the 3-sphere S^3. Suppose K is a non-separating knot on S with self linking number -1 with respect to the framing induced by S. Then S is isotopic rel K to plumbing of some surface $\widetilde{S}$ with a Hopf band H whose core is K. Moreover, the surface $\widetilde{S}$ is isotopic to S cut along a properly embedded arc γ that intersects transversely with K at precisely one point.

Proof. Since S has nonempty boundary and K is non-separating, the arc γ in the statement exists. Create two parallel copies of K on S, one to the left of K and other to the right. Call them K_1 and K_2 respectively. The union of these two curves is the boundary of a Hopf band H whose core is K

In the Figure 3.7 below we describe an isotopy of S in which it is apparent that H is plumbed to the rest of the surface. Starting at a little right of one of the components of $\gamma \cap \partial S$ we perform a finger move along a parallel copy of γ till we hit K_1. Next we turn right and continue the finger move along K_1 until we approach γ from its left. Finally, we perform the same isotopy from the other side but this time we turn left at K_2.

This isotopy isolates H from S. We can write $S = H \cup \widetilde{S}$ where the intersection $H \cap \widetilde{S}$ is a small neighborhood of the arc $\gamma \cap H$ in H. This is precisely the definition of plumbing.

To see why the last statement holds, we cut H along an arc from K_1 to K_2 intersecting with K exactly once. Clearly, we deplumbed H from $\widetilde{S}$. Now, pulling everything back, the assertion should be clear.

□

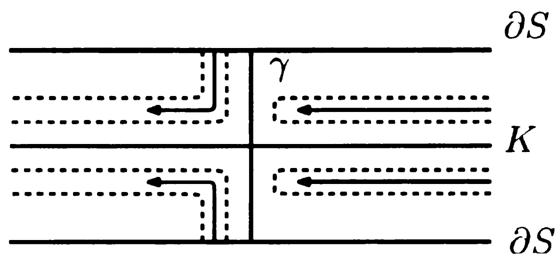

Figure 3.7: Isotopy by finger moves.

3.5 ALF-Almost Stein

In this section, we are going to review the algorithm of Akbulut and Ozbagci to establish a one to one correspondence between ALFs and almost Stein manifolds. Originally, in [AO] authors established this correspondence between PALFs and Stein manifolds but their arguments hold in our case as well.

Theorem 3.5.1. (Akbulut-Ozbagci) A 4-manifold is almost Stein if and only if it admits an ALF over disk with bounded fibers.

Combining this result with our previous observation, Theorem 3.3.3 we get the Harer's theorem [H];

Corollary 3.5.2. Every 4-dimensional handlebody without 3 or 4 handles admits an ALF over disk with fibers having non-empty boundary.

Proof. of theorem 3.5.1:

We will prove almost Stein implies ALF part only. First, we prove a special case where X has no 1-handles. By assumption, X is obtained by attaching several 2-handles along Legendrian link L in S^3 with standard contact structure. In the front projection of L, we replace all local maxima and minima with corners and then turn the picture 45° counter clockwise to get the square bridge diagram of L. This diagram allows us to find a (p, q) torus knot K and a Seifert surface Σ of K with the following properties:

1. Complement of K fibers over S^1 with fibers Σ, forming an open book decomposition of S^3, which supports the standard contact structure of S^3.

2. The open book decomposition in item (1) extends to a PALF of 4-ball B^4 with $(p-1)(q-1)$ vanishing cycles.

3. The link L can be put on Σ such that the surface framing agrees with the Thurston-Bennequin framing.

Once we achieve to find such K and Σ, PALF of B^4 in item (2) extends across the 2-handles as an ALF. Fibers are still diffeomorphic to Σ but the monodromy is composed with product of Dehn twists along components of L. Dehn twists are right handed for good handles and left handed for defective ones.

The original construction of K and Σ is due to Lyon [Ly]. Without loss of generality, we may assume that L is contained in the cube $[-1,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3 \subset S^3$. Say, L has p horizontal and q vertical segments. Choose numbers $1 > a_1 > a_2 > \cdots a_p > 0$, and $0 < b_1 < b_2 < \cdots < b_q < 1$, and a small constant $\varepsilon > 0$. For the i-th horizontal segment of L, we put the plate $\{1\} \times [0,1] \times [a_i - \varepsilon, a_i + \varepsilon]$. For the j-th vertical segment, we put the plate $\{-1\} \times [b_j - \varepsilon, b_j + \varepsilon] \times [0,1]$. We connect horizontal segments to vertical ones by a band making quarter twist. The result is a Seifert surface of a (p,q) torus knot with minimal genus.

Clearly, we can put L on the surface Σ constructed in this way. It is an easy exercise to verify that page framing on each component of L is its writhe minus the number of *south-west* corners. This is exactly the recipe for Thurston-Bennequin framing. If we project Σ to y-z plane, we get a (p,q)-grid. This grid has $(p-1)(q-1)$ holes in it. Take a loop enclosing one of these holes inside the grid and lift it to Σ. This curve has self linking number -1 with respect to the page framing. Hence, we can destabilize the Hopf band around the curve as in lemma 3.4.1, leaving $(p-1)(q-1)-1$

holes in the grid. If we continue to destabilize, we can verify (2) by explicitly finding the monodromy curves.

Finally, we saw that the monodromy of (p, q) torus knot is a product of right handed Dehn twists. Therefore, the contact structure it supports is Stein fillable. Eliashberg [E2] proved that every such contact structure on S^3 is isotopic to the standard one.

Remark 3.5.3. One problem in contact topology is to find an explicit open book decomposition supporting a given contact structure starting from its contact surgery diagram. As noticed by Stipsicz [St], the algorithm we described above provides an open book decomposition supporting a contact structure given by a surgery diagram if the framing coefficients ± 1. Ozbagci extended this result for the case of rational surgery in [O]. The idea of destabilizing the unnecessary Hopf bands in this algorithm is used by Arikan in [Ar] to find a smaller genus open book supporting the given contact structure.

We continue the proof by considering the general case. First, convert all the 1-handles into circles with dots as described at the end of section 3.3. Ignoring the dots for a minute, we apply our algorithm to this new Legendrian link we obtained to put it on a page of a (p, q) torus knot. Each component of the Legendrian link is a non-separating curve on a page. By an isotopy we can put those components which represent 1-handles in such a position that each one of them intersects every page exactly once. We push the disks bounded by these circles inside of 4-ball, each one of them is realized as a section, then scoop out tubular neighborhoods of these sections. The last operation creates holes on the fibers. Of course, the number of these holes is the same as the number of 1-handles. □

As an illustration, we apply the algorithm described in the proof to get ALF pictures of the last two Legendrian handlebody diagrams in Figure 3.6. In Figure

3.8 we put both diagrams on Seifert surface of a 7×7 torus link. The black curves represent 1-handles. That is why they are clasped to the binding. The monodromy of the ALF on the top picture, is a left handed Dehn twist along the red curve composed with the monodromy of the torus knot; whereas the monodromy of the PALF on the bottom picture, is a right handed Dehn twist along the blue curve composed with the monodromy of the torus link. The importance of this example is that it provides an interpretation of Akbulut-Matveyev positron move which concerns (almost) Stein manifolds in terms of (achiral) Lefschetz fibrations. We will return to this point in section 3.8.

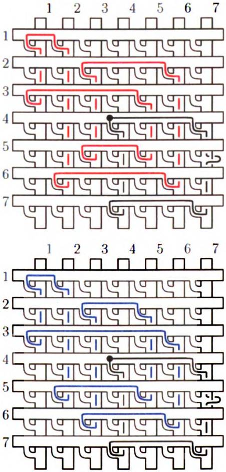

Figure 3.8: Two ALFs corresponding to the Almost Stein manifolds in Figure 3.6

3.6 Some ALF Moves and Their Effects

In this section, we are going to describe a set of moves that produce new ALFs out of a given one. We will describe the moves by means of a handlebody diagram in which 2-handles are drawn in square-bridge position. The corresponding ALF should be understood to be constructed as in Section 3.5. It is our hope that these moves can be extended to result in a uniqueness theorem for Lefschetz type fibrations.

Move 0:(*Refining*) Our first move is increasing the number of grids on fibers constructed as in section 3.5. It amounts to positively stabilizing the initial ALF several times. Most of the forthcoming moves require refining before we perform them. We are going to omit saying that we apply this move when it is clear from the content what sort of refining must be done.

Move 1:(*Isotopy*) Any sort of isotopy in the category of square-bridge knots is applicable provided that it does not change the Thurston-Bennequin framing. This includes *square-bridge Reidemeister moves* which we show in Figure 3.9.

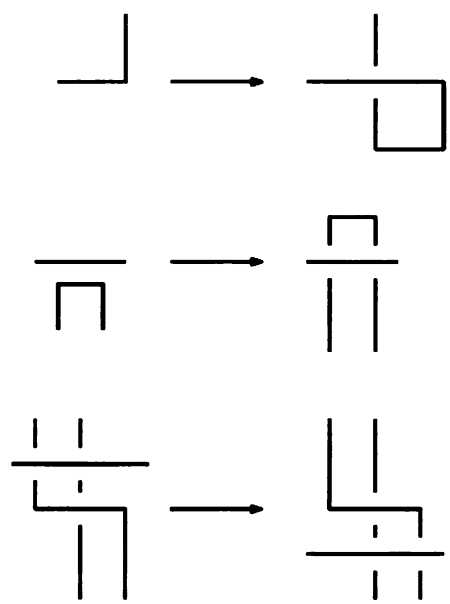

Figure 3.9: Square-bridge Reidemeister moves

Move 2:(*Stabilization*) We include both positive and negative stabilizations in our list of moves. As mentioned above, increasing the grid is a special type of positive stabilization. In general, we indicate positive (resp. negative) stabilizations by creating a pair of canceling 1-handle and a 2-handle where legs of the 1-handle strung on the binding and the 2-handle has framing $tb - 1$ (resp. $tb + 1$).

Move 3:(*Finger Move*) Let K be an attaching circle of a 2-handle lying on a page with framing either one less one more than the framing induced by the pages. Near K, we stabilize our ALF and make K run over the 1 handle of the stabilization without changing its framing see Figure 3.10. This move is called positive or negative finger move depending on what type of stabilization we made. Performing one positive and one negative finger move consecutively does not change the total space of the ALF. However, it adds or subtracts 2 to the number obtained by evaluating the first Chern class of the boundary contact structure on the meridian of K depending upon the orientation (see [EF] Lemma 10). Positive finger move has a nice interpretation in terms of square-bridge knots if the grid is refined enough. Given any horizontal or vertical segment of K, we create a small rectangle on top of it as in figure 3.11, then install this to our ALF. Clearly, this adds page framing one negative twist as we traverse the rectangle. Since the monodromy of a (p, q) torus links is a product of right handed Dehn twists only, there is no similar interpretation for left finger move in our language.

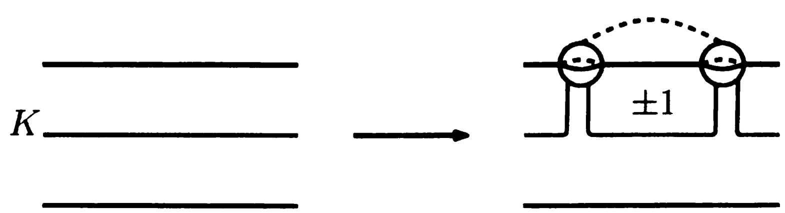

Figure 3.10: Positive and negative finger moves

Figure 3.11: Another interpretation of positive finger move

3.7 Matching the Open Books

In this section, we are going to prove a technical lemma to match open books of a convex ALF and a concave BLF.

Lemma 3.7.1. Let $X = X_1 \cup X_2$ where X_1 is a convex ALF and X_2 is a concave BLF constructed as in either proposition 3.2.1 or 3.2.2. Assume that the first homology group of $\partial X_1 = \partial X_2$ has no 2-torsion. Then Lefschetz type fibrations on X_1 and X_2 can be modified such that the boundary open books match. Moreover, we can arrange that every achiral vanishing cycle on the new convex ALF links twice with an unknot on a page.

Proof. Denote the induced open books on ∂X_1 and ∂X_2 be ob_1 and ob_2. The contact structures supported by these two open books will be denoted by ξ_1 and ξ_2. By the Giroux correspondence, see Theorem 2.6.2, we can match the open books up to positive stabilization if we can match the associated contact structures.

At this point, we would like to explain what we mean by matching the open books. Boundaries of convex and concave sides are identified by an orientation reversing diffeomorphism $f : \partial X_2 \to \partial X_1$. We apply Giroux's theorem to the open books f^*ob_1 and ob_2 to conclude that they have common positive stabilization if and only if the contact structures $f^*\xi_2$ and ξ_2 are isotopic. Note that stabilizing ob_2 requires a round handle attachment as in section 3.2. Consequently, new broken singularities formed in concave side. In addition, positive stabilization of f^*ob_1 means that we need to negatively stabilize ob_1, so, new achiral singularities are formed in convex side.

Now, we consider the problem of matching the contact structures $f^*\xi_1$ and ξ_2.

Before matching them up to isotopy, we study a more elementary problem; namely, matching them up to homotopy. To simplify the notation we drop f^* from $f^*\xi_1$ and f^*ob_1. As explained in [G], two oriented 2-plane distributions on a 3-manifold are homotopic if and only if they have the same Chern class and the 3-dimensional invariant d_3, in case the first homology has no 2-torsion.

We can make the Chern class of ξ_1 equal to $c_1(\xi_2)$ using techniques of Section 3.6. The Poincare dual of $c_1(\xi_1)$ is represented by a cycle $\sum k_i[\gamma_i]$ where γ_i is a meridian of a 2-handle. The handle could be good or defective but we know that its attaching circle is in square-bridge position. The number k_i is the number of up corners minus the number of down corners. We may increase or decrease this number by 2 applying a positive and a negative finger moves consecutively. It can be shown that in our examples $c_1(\xi_2) = 0$. Therefore we are done with matching Chern classes provided that $c_1(\xi_1)$ has correct parity. This is true because $c_1(\xi)$ is even for an arbitrary oriented 2-plane distribution on a 3-manifold M. To see this, fix a trivialization of the tangent bundle. Unit normal vector field to ξ defines a map $g : M \to S^2$ such that $\xi = g^*TS^2$. So $c_1(\xi) = g^*c_1(TS^2) = 2g^*PD([\text{point}])$.

To make the 3-dimensional invariants equal, we negatively stabilize one of ob_1 and ob_2 enough number of times. If we want to avoid achiral singularities in concave side, we need to form an extra round handle singularity and perform two positive stabilizations on ob_2.

Finally, we apply a theorem due to Eliashberg [E3], which says that two overtwisted contact structures on a 3-manifold are homotopic if and only if they are isotopic. To guarantee that both ξ_1 and ξ_2 are overtwisted we negatively stabilize both their open books once more.

□

3.8 Repairing Achiral Singularities

This section is devoted to a procedure to get rid of achiral singularities of a convex ALF without changing the total space and the boundary open book. This procedure is an adaptation of the Akbulut-Matveyev positron move [AM] to the Lefschetz type fibrations. The proof the main result is evident in the following example.

Example 3.8.1. We first describe our initial ALF. Consider 4-ball B^4 fibering over disk with disk fibers. This fibration induces an open book decomposition of the boundary $\partial B^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$. To visualize this open book, we indicate the binding as an unknot drawn as a rectangle in $\mathbb{R}^3$. Then, the meridian of this rectangle is a section of a fiber bundle over circle with disk fibers. Next, we negatively stabilize this open book to get a convex ALF with one defective handle. Recall that negative stabilization amounts to creating a pair of canceling 1 and 2-handles where the feet of 1-handles strung on the binding and the attaching circle of the 2-handle is on a page with framing one more than the surface framing. The main theorem requires a defective handle to pass twice over a 1-handle. This is why we change the picture by isotopy and carve 4-ball in such a way that the 2 handle satisfies the assumption of the main theorem. The square-bridge position of the final handlebody is given in Figure 3.12 on the left. A compatible ALF is constructed as in proof of Theorem 3.5.1 with the exception that the 1-handle of the canceling pair is indicated as a pair of 3-balls, on top of Figure 3.13. Now we perform a left twist along the 1-handle to repair the ALF. The square-bridge diagram and the corresponding PALF is shown on the right of Figure 3.12 and on the bottom of Figure 3.13. Note that two diagrams in Figure 3.13 differ by a Dehn twist along the curve labeled as α.

By applying this twist diffeomorphism to every defective handle, we obtain our PALFication lemma:

Lemma 3.8.2. Suppose that a 2-handlebody X admits an convex ALF constructed as in the proof of theorem 3.5.1, possibly with some extra stabilizations. Assume further that every framed link representing an achiral vanishing cycle passes over a 2-handle twice. Then, X also admits a PALF and there is a self diffeomorphism φ (isotopic to identity) of X which is also a fibration equivalence outside of achiral singular points.

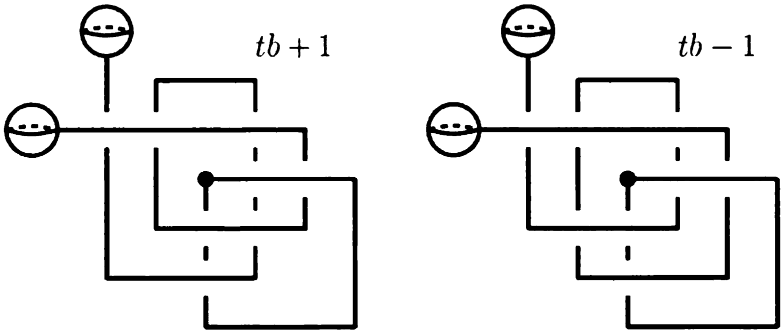

Figure 3.12: Repairing bad framings

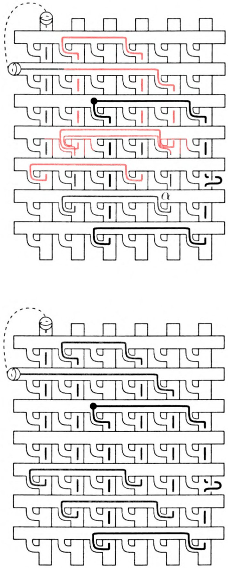

Figure 3.13: Alf picture of figure 3.12

3.9 Proof of the Main Theorem

Proof. of Theorem 1.0.2: Our first task is to find a concave fibration inside of X. If $b_2^+(X) > 0$, a neighborhood of a surface with positive self intersection can be equipped with a concave pencil by Proposition 3.2.1. Otherwise, we choose any self intersection zero surface in X and find a concave Broken Lefschetz Fibration on its neighborhood as in Proposition 3.2.2. This concave piece constructed in either way will be called X_2. Let X_1 be the complement X_2. By Lemma 13 in [GK] we may assume that X_1 has no 3 or 4-handles. In section 3.3, we proved that every such manifold is "Almost Stein", see Theorem 3.3.3. Thus by applying the Akbulut-Ozbagci algorithm which was described in section 3.5.1, we can find a convex Achiral Lefschetz Fibration on $X_1 \to D^2$ with bounded fibers. More specifically we get an ALF $\mathcal{F}$ with total space $|\mathcal{F}| = X_1$.

Having decomposed $X = X_1 \cup X_2$ as a convex ALF union concave BLF our next aim is to match the open books on the common boundary of these two submanifolds. In lemma 3.7.1 above, we described how techniques of [GK2] and [EF] can be modified to achieve this aim. Our version allows us to get the property that every achiral vanishing cycle (i.e. the frame knot representing it) links twice with an unknot sitting on a page of the open book (coming from $\partial\mathcal{F}$). This property will be crucial in the next step of our construction. This is the only non-explicit step in our proof because our arguments depend on Eliashberg's classification of overtwisted contact structures and Giroux's correspondence between contact structures and open book decompositions. During the application of the Giroux correspondence we need to stabilize both sides. Recall from section 3.2, we create more fold singularities when we positively stabilize the concave side. In order to use Eliashberg's theorem we need to negatively stabilize both sides in the first place making sure that the boundary contact structures are both overtwisted. We can negatively stabilize the concave piece without creating achiral singularities at the cost of two extra positive stabilizations. This way we collect all the achiral singularities in the convex side and all the fold singularities in the concave side.

In order to apply the "positron move" introduced in [AM], we isotope the unknots mentioned in the previous paragraph to meridians of the binding of the common open book on $\partial X_1 = \partial X_2$ and carve the disks bounded by these curves from X_1. This amounts to attaching a 2–handle to the concave side X_2 along a section. Yet another "*convex-concave decomposition*" of X is obtained this way. Moreover we did not lose the property that boundary open books match. Note that we would not be able to apply positron move if there was an achiral singularity in the concave side as we can carve only the convex fibrations and attach handles along sections only to concave fibrations. Now that we have all the defective handles go over a 1-handle twice, all the bad $(tb+1)$ framings can be can be turned into a good $(tb-1)$ framing by twisting these handles, see lemma 3.8.2. The twisting operation induces a self diffeomorphism ψ and a handle decomposition of the convex side X_1. The handlebody decomposition is naturally related to a PALF of X_1 and ψ sends nonsingular fibers of the previous ALF into fibers of this PALF. In particular, ψ induces a diffeomorphism (isotopic to identity) preserving the open books on the boundary ∂X_1. This means that we can glue our new PALF to the concave side and the result is still diffeomorphic to X. □

An easy corollary of this proof is:

Corollary 3.9.1. Every closed smooth oriented 4-manifold X admits a BLF (or BLP if $b_2^+ > 0$), by decomposing as a convex PALF and a concave BLF (or BLP) glued along the common (open book) boundary.

Chapter 4

Calculation of the Ozsváth-Szabó Contact Invariant

This chapter is organized as follows. In Section 4.1, basic properties of Heegaard-Floer homology and and contact invariant are briefly reviewed. Section 4.2 is devoted to the algorithm of Ozsváth and Szabó to determine the generators of Heegaard-Floer homology of 3-manifolds given by plumbing diagrams. Remarks given at the end of the section allow us to find relations easily by combinatorial means. We prove Proposition 1.0.5 and Theorem 1.0.7 in Section 4.3. A sample calculation is done in Section 4.4. Theorem 1.0.9 is proved in Section 4.5

4.1 Heegaard-Floer Homology and Contact Invariant

Let Y be a closed oriented 3-manifold and $\mathfrak{t}$ be a spin^C structure on Y. In [OS4] and [OS5], Ozsváth and Szabó define four versions of *Heegaard-Floer homology* groups

$\widehat{HF}(Y,\mathfrak{t})$, $HF^+(Y,\mathfrak{t})$, $HF^-(Y,\mathfrak{t})$,and $HF^\infty(Y,\mathfrak{t})$. These groups are all smooth invariants of $(Y,\mathfrak{t})$. When Y is a rational homology sphere, they admit absolute $\mathbb{Q}$-gradings. The groups HF^+, HF^-, and HF^∞ are also $\mathbb{Z}[U]$ modules where multiplication by U decreases degree by 2. Any spin^c cobordism $(X,\mathfrak{s})$ between $(Y_1,\mathfrak{t}_1)$ and $(Y_2,\mathfrak{t}_2)$ induces a homomorphism well defined up to sign

$$F^\circ_{X,\mathfrak{s}} : HF^\circ(Y_1,\mathfrak{t}_1) \to HF^\circ(Y_2,\mathfrak{t}_2)$$

Here HF° stands for any one of $\widehat{HF}$, HF^+, HF^-, or HF^∞. We work with $\mathbb{Z}/2\mathbb{Z}$ coefficients in order to avoid sign ambiguities. Also, we drop the spin^c structure from the notation when we direct sum over all spin^c structures.

Given any contact structure ξ on Y, Ozsváth and Szabó associate an element $c(\xi) \in \widehat{HF}(-Y)$ which is an invariant of isotopy class of ξ [OS2]. In this paper we are interested in the image $c^+(\xi)$ of $c(\xi)$ in $HF^+(Y)$ under the natural map. We list some of the properties of this element below.

1. $c^+(\xi)$ lies in the summand $HF^+(-Y,\mathfrak{t})$ where $\mathfrak{t}$ is the spin^c structure induced by ξ.

2. $c^+(\xi) = 0$ if ξ is overtwisted.

3. $c^+(\xi) \neq 0$ if ξ is Stein fillable.

4. $c^+(\xi) \in \mathrm{Ker}(U)$.

5. $c^+(\xi)$ is homogeneous. When Y is a rational homology sphere, it has degree $-d_3(\xi) - 1/2$, where $d_3(\xi)$ is the 3-dimensional invariant of ξ.

6. $c^+(\xi)$ is natural under Stein cobordisms: If W is a compact Stein manifold, $\partial W = Y' \cup -Y$, and ξ' and ξ are the induced contact structures, we can regard

W as a cobordism from $-Y'$ to $-Y$ and the induced map satisfies $F^+_W(c^+(\xi')) = c^+(\xi)$.

The contact invariant $c^+(\xi)$ is studied by Plamenevskaya in [P]. The following result will be used later in this paper when we prove our main theorem. We state it in a slightly more general form than in [P] but Plamanevskaya's proof is valid for our case as well.

Theorem 4.1.1. (Theorem 4 in [P]) Let X be a smooth compact 4-manifold with boundary $Y = \partial X$. Let J be a Stein structure on X that induce a spin^c structure $\mathfrak{s}_1$ on X and contact structure ξ_1 on Y. Let $\mathfrak{s}_2$ be another spin^c structure on X that does not necessarily come from a Stein structure. Suppose that $\mathfrak{s}_1|_Y = \mathfrak{s}_2|_Y$, but the spin^c structures $\mathfrak{s}_1$, $\mathfrak{s}_2$ are not isomorphic. We puncture X and regard it as a cobordism from Y to S^3. Then

1. $F^+_{X,\mathfrak{s}_2}(c^+(\xi_1)) = 0$
2. $F^+_{X,\mathfrak{s}_1}(c^+(\xi_1))$ is a generator of $HF^+_0(S^3)$.

Note that in this theorem if the spin^c structures $\mathfrak{s}_1|_Y$ and $\mathfrak{s}_2|_Y$ are not the same then conclusion (1) follows trivially.

Remark 4.1.2. Theorem 4.1.1 was later generalized by Ghiggini in [Ghi] where he requires J to be only an ω-tame almost complex structure for some symplectic structure ω on $X(G)$ that gives a strong filling for the boundary contact structure. In this paper, we work with rational homology spheres. For these manifolds, any weak filling can be perturbed into a strong filling, [OO].

4.2 The Algorithm

Let G be a weighted graph. For every vertex v of G, let $m(v)$ and $d(v)$ denote the weight of v and the number of edges connected to v respectively. A vertex v is said to be a *bad vertex* if $m(v) + d(v) > 0$. Enumerating all vertices of G, one can form the *intersection matrix* whose ith diagonal entry is $m(v_i)$ and $i-j$th entry is 1 if there is an edge between v_i and v_j, and is 0 otherwise. Throughout, we assume that G satisfies the following conditions.

1. G is a connected tree.
2. The intersection matrix of G is negative definite.
3. G has at most one bad vertex.

There is a 4-manifold $X(G)$ obtained by plumbing together disk bundles D_i, $i = 1 \cdots |G|$ over sphere where D_i is plumbed to D_j whenever there is an edge connecting v_i to v_j. Let $Y(G)$ be the boundary of $X(G)$. In [OS1], Ozsváth and Szabó give a purely combinatorial description of Heegaard-Floer homology group $HF^+(-Y(G))$. Here we repeat their description for completeness.

The second homology $H^2(X(G), \mathbb{Z})$ is a free module generated by vertices of G. Let Char(G) be the set all characteristic(co)vectors of this module, i.e. every element K of Char(G) satisfies $\langle K, v\rangle = m(v)$ (Mod 2) for every vertex v. Let $\mathcal{T}^+$ be the graded algebra $\mathbb{Z}/2\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}/2\mathbb{Z}[U]$ where the formal variable U has degree -2. Form the set $\mathbb{H}^+(G) \subset \mathrm{Hom}(\mathrm{Char}(G), \mathcal{T}^+)$ where any element ϕ of $\mathbb{H}^+(G)$ satisfies the following property; If K is a characteristic vector, v is a vertex, and n is an integer such that

$$\langle K, v\rangle + m(v) = 2n,$$

we have

$$U^{m+n}\phi(K+2\mathrm{PD}(v)) = U^m\phi(K) \text{ if } n>0,$$

or

$$U^m\phi(K+2\mathrm{PD}(v)) = U^{m-n}\phi(K) \text{ if } n>0.$$

The set of spin^c structures on $Y(G)$ gives rise to a natural splitting for $\mathbb{H}^+(G)$. For, if $\mathfrak{t}$ is a spin^c structure on $Y(G)$, one can consider the subset $\mathrm{Char}_{\mathfrak{t}}(Y(G))$ consisting of those characteristic vectors whose restriction on $Y(G)$ are $\mathfrak{t}$. The set $\mathbb{H}^+(G,\mathfrak{t})$ is the set of all maps in $\mathbb{H}^+(G)$ with support $\mathrm{Char}_{\mathfrak{t}}$. It is easy to see that $\mathbb{H}^+(G)=\bigoplus_{\mathfrak{t}}\mathbb{H}^+(G,\mathfrak{t})$.

The group $\mathbb{H}^+(G)$ is graded in the following way. An element $\phi\in\mathbb{H}^+(G)$ is said to be homogeneous of degree d if for every characteristic vector K with $\phi(K)\neq 0$, $\phi(K)\in\mathcal{T}^+$ is a homogeneous element with

$$\deg(\phi(K)) - \frac{K^2+|G|}{4} = d.$$

We are ready to describe the isomorphism relating $\mathbb{H}^+(G)$ to the Heegaard-Floer homology of $Y(G)$. Fix a spin^c structure $\mathfrak{t}$ on $-Y(G)$. Let K be a characteristic vector on $\mathrm{Char}_{\mathfrak{t}}(G)$. Puncture $X(G)$ and regard it as a cobordism form $-Y(G)$ to S^3. It is known that $X(G)$ and K induce a homomorphism

$$F_{X(G),K} : HF^+(-Y(G),\mathfrak{t}) \to HF^+(S^3)\simeq\mathcal{T}^+.$$

Now the map $T^+ : HF^+(-Y(G),\mathfrak{t})\to\mathbb{H}^+(G,\mathfrak{t})$ is defined by the rule $T^+(\xi)(K) = F_{X(G),K}(\xi)$

Theorem 4.2.1. (Theorem 2.1 in [OS1]) T^+ is a U-equivariant isomorphism preserving the absolute $\mathbb{Q}$-grading.

To simplify the calculations, we work with the dual of $\mathbb{H}^+(G)$. Let $\mathbb{K}^+$ be the subset of $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G)/\sim$, where the equivalence relation $\sim$ is defined as follows. Denote a typical element of $\mathbb{Z}^{\geq 0} \times \mathrm{Char}(G)$ by $U^m \otimes K$. Let v be a vertex and n be an integer such that

$$2n = \langle K, v \rangle + m(v)$$

then we have

$$U^{m+n} \otimes (K + 2\mathrm{PD}(v)) \sim U^m \otimes (K) \text{ if n} \geq 0$$

or

$$U^m \otimes (K + 2\mathrm{PD}(v) \sim U^{m-n} \otimes K \text{ if n} < 0.$$

Define a pairing $\mathbb{K}^+(G) \times \mathbb{H}^+(G) \to \mathbb{Z}$ by $(\phi, U^m \otimes K) \to (U^m \phi(K))_0$ where $()_0$ denotes the projection to 0th grade in $\mathcal{T}^+$. It is possible to show that this pairing is well defined and non-degenerate and hence it defines an isomorphism between $\mathbb{H}^+(G)$ and $\mathrm{Hom}(\mathbb{K}^+(G), \mathbb{Z})$. Using the duality map and isomorphism T^+ one can identify $\ker U^{n+1} \subset HF^+(-Y(G))$ as a quotient of $\mathbb{K}^+(G)$ for every $n \geq 0$.

Lemma 4.2.2. (Lemma 2.3 in [OS1]) Let B_n denote the set of characteristic vectors $B_n = \{K \in \mathrm{Char(G)} : \forall v \in G, |\langle K, v \rangle| \leq -m(v) + 2n\}$. The quotient map induces a surjection from

$$\bigcup_{i=0}^{n} U^i \otimes B_{n-i}$$

onto the quotient space

$$\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>n} \times \mathrm{Char}(G)}.$$

In turn, we have an identification

$$\mathrm{Hom}\left(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>n} \times \mathrm{Char(G)}}, \mathbb{Z}/2\mathbb{Z}\right) \simeq \ker U^{n+1} \subset \mathbb{H}^+(G, \mathbb{Z}). \qquad (4.1)$$

One should regard the above isomorphism as one between $\mathbb{Z}/2\mathbb{Z}[U]$ modules where the U action on the left hand side of equation 4.1 is defined by the following relation.

$$U.(U^p \otimes K)^*(U^r \otimes K') = \begin{cases} 1 & \text{if } U^p \otimes K \sim U^{r+1} \otimes K' \\ \\ 0 & \text{if } U^p \otimes K \not\sim U^{r+1} \otimes K' \end{cases} \quad (4.2)$$

Where $(U^p \otimes K)^*$ denotes the dual of $U^p \otimes K$.

Lemma 4.2.2 gives us a finite model for $\ker U^{n+1}$ for every $n \geq 0$. It is known that these groups stabilize to give HF^+. Therefore, one can understand HF^+ by studying the quotients $\mathbb{K}^+(G)/\mathbb{Z}^n \times \text{Char}(G)$ for all $n \geq 0$. The first quotient is well understood thanks to an algorithm of Ozsváth and Szabó . Below, we describe the algorithm and discuss a possible extension.

A characteristic vector K is called an *initial vector* if for every vertex v, we have

$$m(v) + 2 \leq \langle K, v \rangle \leq -m(v) \quad (4.3)$$

Start with an initial vector K_0. Form a sequence $(K_0, K_1, \cdots, K_n)$ of characteristic vectors as follows: K_{i+1} is obtained from K_i by adding $2\text{PD}(v)$ where v is a vertex with $\langle K_i, v \rangle = -m(v)$. The terminal vector K_n satisfies one of the following.

1. $m(v) \leq \langle K_n, v \rangle \leq -m(v) - 2$ for all v.

2. $\langle K_n, v \rangle > -m(v)$ for some v.

The sequence $(K_0, K_1, \cdots, K_n)$ is called a *full path*, and characteristic vector K_n is called the *terminal vector* of the full path. We say that a full path is called *good* if its terminal vector satisfies property (1) above and it is *bad* if the terminal vector satisfies (2). We list some of the properties of full paths, the reader can consult [OS1](especially proposition 3.1 in [OS1]) for proofs.

- Two characteristic vectors in B_0 are equivalent in $\mathbb{K}^+(G)$ if and only if there is a full path containing both of them where the set B_0 is defined as in lemma 4.2.2.
- If an initial vector K_0 has a good full path then any other full path starting with K_0 is good.
- If K_0 and K_0' are initial vectors having good full paths and $K_0 \neq K_0'$ then $K_0 \not\sim K_0'$ in $\mathbb{K}^+(G)$.
- A terminal vector K_n of a bad full path is equivalent to $U^m \otimes K'$ in $\mathbb{K}^+(G)$ for some $m > 0$ and $K' \in \mathrm{Char}(G)$. A terminal vector of good full path can not be equivalent to such an element of $\mathbb{H}^+(G)$.

Note that these properties allow us to find the generators of $\ker U$; They are simply the initial vectors having good full paths. In other words, we know the generators of the lowest grade subgroup of $HF^+(-Y(G))$. Recall from [OS3] that the lowest degree $d(Y, \mathfrak{t})$ of non-torsion elements in $HF^+(Y, \mathfrak{t})$ is called the *correction term* for a spin^c manifold $(Y, \mathfrak{t})$. The algorithm above provides us an efficient method to calculate the correction term $d(-Y(G), \mathfrak{t})$ for any spin^c structure $\mathfrak{t}$ (see Corollary 1.5 of [OS1])

$$d(-Y(G), \mathfrak{t}) = \min -\frac{K^2 + |G|}{4} \tag{4.4}$$

where the minimum is taken over all characteristic vectors admitting good full paths which induce the spin^c structure $\mathfrak{t}$.

The whole group $\mathbb{H}^+(G) \simeq HF^+(-Y(G))$ is determined by the relations amongst the generators of $\mathrm{Ker}(U)$. Given two characteristic vectors K_i, K_j admitting good full paths and inducing the same spin^c structure on $Y(G)$, a *relation* between K_1 and K_2 is a pair of integers (n, m) satisfying $U^n \otimes K_1 \sim U^m \otimes K_2$. If the non negative integers (n, m) are minimal with that property, we call the corresponding relation *minimal*. Here we describe a systematic method to find relations. Say K is a characteristic vector and n is a positive integer. We want to understand the equivalence class in $\mathbb{K}^+(G)$ containing $U^n \otimes K$. We define three operations that do not change this equivalence class.

(R1) $U^n \otimes K' \to U^n \otimes K$ where K' is obtained from K by applying the algorithm to find full paths.

(R2) $U^n \otimes K \to U^{n-1} \otimes (K + 2\mathrm{PD}(v))$ where v is a vertex with $\langle K, v\rangle + m(v) = -2$

(R3) $U^n \otimes K \to U^{n+1} \otimes (K + 2\mathrm{PD}(v))$ where v is a vertex with $\langle K_n, v\rangle + m(v) = 2$

Now assume that K is a characteristic initial vector which admits a good full path. In order to find particular representatives with small U-depths for the equivalence class containing $U^n \otimes K$ we apply **R1** then apply **R2** if possible else **R3**. Then we repeat the same procedure till it terminates at an element $U^r \otimes K'$. We call the vector part of this element as a *root vector* (the exponent m is determined by n and degrees of K and K'). A Root vector is not unique, it depends upon choices we made along the way; like the choice of the vertex at which we apply **R2** or **R3** is applied. However, the set of root vectors is a finite set which can be found easily and it can be used to establish relations amongst the generators of $\mathrm{Ker}(U)$. This simple observation will be useful when we do our calculations.

Proposition 4.2.3. Let K_1 and K_2 be two characteristic initial vectors admitting good full paths. Suppose n and m are non-negative integers such that the root vector sets of $U^n \otimes K_1$ and $U^m \otimes K_2$ intersect non trivially. Then we have $U^n \otimes K_1 \sim U^m \otimes K_2$.

Proof. Follows from the definitions. □

4.3 Main Theorem

Proof of Proposition 1.0.5. Let $\mathfrak{s}$ be the canonical spin^c structure and $\mathfrak{s}'$ be any other spin^c structure on $X(G)$. Note that $c_1(\mathfrak{s}) \not\sim c_1(\mathfrak{s}')$. Recall that the isomorphism $\text{Ker}(U) \simeq \text{Hom}\left(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>0} \times \text{Char}(G)}, \mathbb{Z}/2\mathbb{Z}\right)$ is given by means of the pairing

$$P : \text{Ker}(U) \times \text{Hom}\left(\frac{\mathbb{K}^+(G)}{\mathbb{Z}^{>0} \times \text{Char}(G)}, \mathbb{Z}/2\mathbb{Z}\right) \to \mathbb{Z}/2\mathbb{Z}$$

$$P(a, L) = (F^+_{X(G),L}(a))_0$$

In view of this observation, it is enough to show the following two equations hold.

$$(F^+_{X(G),\mathfrak{s}}(c(\xi)))_0 = 1 \tag{4.5}$$

$$(F^+_{X(G),\mathfrak{s}'}(c(\xi))_0 = 0 \tag{4.6}$$

These are simply the conclusions of Theorem 4.1.1.

□

Proof of Theorem 1.0.7. By Theorem 1.0.4 and Proposition 1.0.5, it is enough to show that $K \notin \text{Im}(U^k)$ for some $k \in \mathbb{N}$. Let $\{K_1, K_2, \cdots, K_r\}$ be the set of characteristic initial vectors admitting good paths such that $\deg(K_i) \leq$ K and $K_i|_{Y(G)} = \mathfrak{t}$ for all $i = 1 \cdots r$. Basic properties of contact invariant imply that this set is not empty if one of the assumptions is satisfied. It is known that on any rational homology sphere and for any spin^c structure, the Heegaard-Floer homology decomposes as $HF^+ = \mathcal{T}^+ \oplus HF_{\text{red}}$. So, one can find integers $n_0, n_1, \cdots, n_r$ such that

$$U^{n_0} \otimes K \sim U^{n_1} \otimes K_1 \sim \cdots U^{n_r} \otimes K_r.$$

Moreover, by choosing these numbers large enough, we can guarantee that the dual of $U^{n_0} \otimes K$ is the unique generator of the degree $-d_3(\xi) - 1/2 + 2n_0$ subspace of $HF^+(-Y)$. Then by equation 4.2, $K \notin \mathrm{Im}(U^{n_0})$.

□

4.4 Example

In this section, we shall illustrate an application Theorem 1.0.7 and show that certain Stein fillable contact structures do not admit planar open books. Obstructions to being supported by planar open books were known to exist before. The first such obstruction was found by Etnyre. It puts restrictions on intersection forms of symplectic fillings of planar open books.

Theorem 4.4.1. (Theorem 4.1 in [Et]) If X is a symplectic filling of a contact 3-manifold (Y, ξ) which is compatible with a planar open book decomposition then $b_+^2(X) = b_0^2(X) = 0$, the boundary of X is connected and the intersection form Q_X can be embedded into a matrix which is diagonalizable over integers.

Ozsváth Szabó and Stipsicz gave another obstruction. It is a consequence of Theorem 1.0.4 above though its statement has no reference to Floer homology.

Theorem 4.4.2. (Corollary 1.5 of [OSS]) Suppose that the contact 3-manifold (Y, ξ) with $c_1(s(\xi)) = 0$ admits a Stein filling (X, J) such that $c_1(X, J) \neq 0$. Then ξ is not supported by a planar open book decomposition.

Yet another criterion is stated in [OSS]. It partially implies Theorem 1.0.7 above.

Theorem 4.4.3. (Corollary 1.7 of [OSS]) Suppose that Y is a rational homology 3-sphere. The number of homotopy classes of 2-plane fields which admit contact structures which are both symplectically fillable and compatible with planar open book decompositions is bounded above by the number of elements in $H_1(Y;\mathbb{Z})$. More precisely, each spin^c structure $\mathfrak{s}$ is represented by at most one such 2−plane field, and moreover, the Hopf invariant of the corresponding 2−plane field must coincide with the correction term $d(-Y, s)$.

Below, we give examples of non-planar Stein fillable contact structures on a Seifert fibered space. Non-planarity of some of our examples do not follow from Theorem 4.4.1, Theorem 4.4.2 or Theorem 4.4.3.

Example 4.4.4. Consider the star shaped plumbing graph consisting of eight vertices where the central vertex has weight −4, a neighboring vertex has weight −3 and all the others are of weight −2 (See figure 4.1). The boundary 3-manifold Y is the Seifert fibered space

$$M(-4, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{6}, \frac{1}{3}).$$

The reason why we have so many self intersection −2 spheres is that we want to avoid L-spaces where the obstruction to admit planar open book does not exist. For topological characterization of L-spaces among Seifert fibered spaces see [LS].

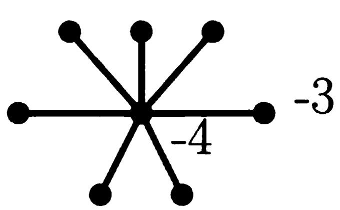

Figure 4.1: Plumbing description of Seifert fibered space $M(-4, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{3})$

It can be shown that the corresponding intersection form is negative definite,and it has determinant 128. Moreover it can be embedded into a symmetric matrix which is diagonalizable over integers. To see this, index the vertices so that the central one comes first and the weight -3 vertex is the last. Let $e_1, e_2, \cdots, e_{11}$ be a basis for $\mathbb{R}^{11}$ such that $e_i \cdot e_i = -1$ for all $i = 1, 2, \cdots, 11$. The embedding is defined by the following set of equations

$$
\begin{aligned}
v_1 &\rightarrow -e_1 - e_2 - e_3 - e_4 \\
v_2 &\rightarrow e_2 - e_7 \\
v_3 &\rightarrow e_2 + e_7 \\
v_4 &\rightarrow e_3 - e_8 \\
v_5 &\rightarrow e_3 + e_8 \\
v_6 &\rightarrow e_4 - e_9 \\
v_7 &\rightarrow e_4 + e_9 \\
v_8 &\rightarrow e_1 + e_{10} + e_{11}
\end{aligned}
$$

First, we calculate $HF^+(-Y, \mathfrak{t})$ for every spin^c structure $\mathfrak{t}$. For similar calculations, see [D], [R1], [R2], and Section 3.2 of [OS1]. Write any characteristic vector K in the form $K = [\langle K, v_1\rangle, \cdots, \langle K, v_8\rangle]$. There are 768 characteristic vectors satisfying equality 4.3, and 138 of them have good full paths. When we distribute these to spin^c structures of Y, we see that for 10 spin^c structures $\mathrm{Ker}(U)$ has rank 2, and the rank is 1 for the rest. In Table 4.1 below, we provide a table that shows HF^+ for these spin^c structures. Characteristic vectors on left are the generators of $\mathrm{Ker}(U)$. Ones that lie in the same box induce the same spin^c structure on boundary.

Next, we consider the obvious Stein structures that arise from the handlebody diagram associated to G. Following Eliashberg, we isotope the attaching circles of 2-handles into Legendrian position so that their framing become one less than the Thurston-Bennequin framing. For -2 framed unknots, there is unique way to do that. For the other unknots which correspond to v_1 and v_8 take Legendrian isotopes with rotation numbers i and j respectively where $i = -2, 0, 2$, and $j = -1, 1$. Call the resulting Stein structure as $J_{i,j}$ and the induced contact structure by $\xi_{i,j}$. See figure 4.2 for a picture of $J_{2,-1}$. The curve on left corresponds v_1 and the other represents v_8. They are both oriented counter clockwise. We omit the other unknots linking to v_1 in order not to complicate the picture. Note that the first Chern class of $J_{i,j}$ is given by the characteristic vector $K_{i,j} = [i, 0, 0, 0, 0, 0, 0, j]$. It is easy to verify that $d_3(\xi_{i,j}) + 1/2 = \text{degree}(K_{i,j}) = \dfrac{K_{i,j}^2 + |G|}{4}$. According to Theorem 1.0.7, the contact structures $\xi_{\pm 2, \pm 1}$ do not admit planar open books. By the algorithm given in [EtnO2] these contact structures do admit genus one open books, so their support genus are all one. One can not use Theorem 4.4.2 directly to get this conclusion because the Chern classes of the corresponding spin^c structures are all of order 4. Though Theorem 4.4.3 also implies our conclusion for $\xi_{2,1}$ and $\xi_{-2,-1}$, it doesn't apply to $\xi_{2,-1}$ or $\xi_{-2,1}$. So the latter two are the contact structures we promised at the beginning of the example.

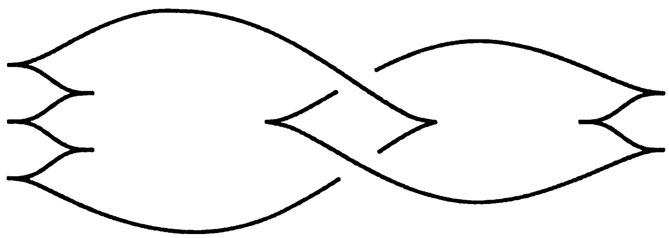

Figure 4.2: Legendrian handlebody diagram giving $J_{2,-1}$.

Remark 4.4.5. The main result of [EtnO] implies that the support genera of plumbings with at most two bad vertices are at most one. On the other hand the algorithm of Ozsváth and Szabó does not work if the number of bad vertices is greater than two. Therefore, the techniques used in this paper do not seem to be pushed further to find an example of a contact structure with support genus strictly greater than one. We are planning to turn this problem in a future project using a different approach.

4.5 Calculation of σ

In this section we shall prove Theorem 1.0.9 by calculating explicitly the σ invariant of a family of contact 3-manifolds. Our argument is based on a previous work of Rustamov, [R1].

For every positive integer n, consider the contact manifold (Y_n, ξ_n) obtained from (S^3, ξ_{std}) by doing Legendrian surgery on the $(2, 2n+1)$ torus knot L_n stabilized $2n-1$ times as in figure 4.3. Observe that the Thurston-Bennequin invariant of L_n is zero so the topological surgery coefficient is negative one. In fact, the 3-manifold Y_n is the Brieskorn homology sphere $\Sigma(2, 2n+1, 4n+3)$.

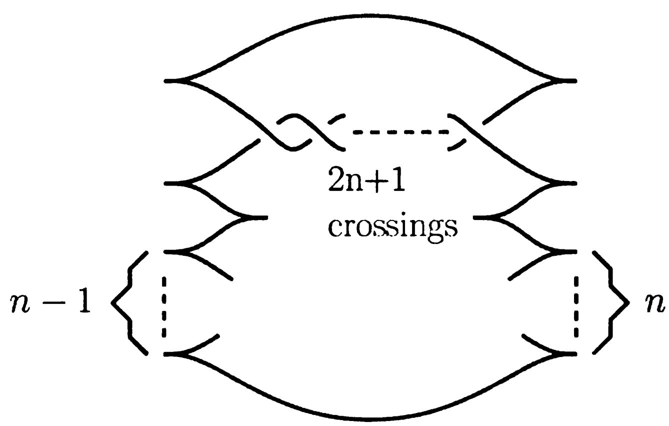

Figure 4.3: K_n

Theorem 4.5.1. $\sigma(Y_n, \xi_n) = p_n - 1$ where p_n is the nth element of the sequence $1, 1, 2, 2, 3, 3, \cdots$.

Clearly, Theorem 4.5.1 implies Theorem 1.0.9. Another immediate application of Theorem 4.5.1 is that (Y_n, ξ_n) can not be supported by a planar open book. This

Spin^c	Characteristic Vectors	Degree	Relation	$HF^+(-Y)$
1	$[2,0,0,0,0,0,-1]$ $[-2,0,0,0,0,0,0,3]$	7/8 7/8	$U\otimes K_1 = U\otimes K_2$	$\mathcal{T}_{-\frac{7}{8}}\oplus\mathbb{Z}_{-\frac{7}{8}}$
2	$[-2,0,0,0,0,0,1]$ $[0,0,0,0,0,0,0,3]$	7/8 7/8	$U\otimes K_1 = U\otimes K_2$	$\mathcal{T}_{-\frac{7}{8}}\oplus\mathbb{Z}_{-\frac{7}{8}}$
3	$[-2,0,0,0,0,0,-1]$ $[0,0,0,0,0,0,0,1]$	$-1/8$ 15/8	$U\otimes K_1 = U^2\otimes K_2$	$\mathcal{T}_{-\frac{15}{8}}\oplus\mathbb{Z}_{\frac{1}{8}}$
4	$[2,0,0,0,0,0,1]$ $[0,0,0,0,0,0,0,-1]$	$-1/8$ 15/8	$U\otimes K_1 = U^2\otimes K_2$	$\mathcal{T}_{-\frac{15}{8}}\oplus\mathbb{Z}_{\frac{1}{8}}$
$5+j$	$[-2,0,\cdots,\underbrace{2}_{j+2},\cdots,0,-1]$ $[0,0,\cdots,\underbrace{2}_{j+2},\cdots,0,1]$ $j=0,\cdots,5$	3/4 3/4	$U\otimes K_1 = U\otimes K_2$	$\mathcal{T}_{-\frac{3}{4}}\oplus\mathbb{Z}_{-\frac{3}{4}}$

Spin^c	Root Vectors
1	$[2,0,\cdots,0,\underbrace{-4}_{i},0,\cdots,0,-3]$ $i=2,\cdots,7$
2	$[0,0,0,0,0,0,0,-5]$, $[0,0,\cdots,0,\underbrace{-4}_{i},0,\cdots,0,1]$, $i=2,\cdots,7$
3	$[0,0,\cdots,0,\underbrace{-4}_{i},0,\cdots,0,-1]$, $i=2,\cdots,7$
4	$[-2,0,0,0,0,0,0,-3]$
$5+j$	$[2,0,\cdots,\underbrace{-4}_{i},\cdots,\underbrace{-2}_{j+2},\cdots,0,-1]$ $i=2,\cdots,7$ $j=0,\cdots,5$

Table 4.1: HF^+ of $M(-4,\frac{1}{2},\cdots,\frac{1}{2},\frac{1}{3})$ for 10 spin^c structures.

was first pointed out in [OSS]. Finally, combining this theorem with the fact that σ invariant respects the partial ordering coming from Stein cobordisms we have the following corollary.

Corollary 4.5.2. There is no Stein cobordism from (Y_n, ξ_n) to (Y_m, ξ_m) if $n > m+1$. In particular, one can not obtain (Y_m, ξ_m) from (Y_n, ξ_n) via Thurston-Bennequin minus one $(tb-1)$ surgery on a Legendrian link.

The above corollary should be compared to a classical result of Ding and Geiges in [DG], which was stated in Theorem 2.4.3 above, where it was proved that any two contact manifolds can be obtained from each other via a sequence of $(tb-1)$ or $(tb+1)$ contact surgeries. In fact, one can always choose such a sequence which contains at most one $tb+1$ surgery [DGS]. Therefore, the corollary tells us that the existence of $(tb+1)$ surgery is essential even though the contact manifolds in both ends are Stein fillable.

Proof of Theorem 4.5.1. Let V be the 4-manifold obtained by attaching a Weinstein 2-handle to a 4-ball along L_n. Eliashberg's theorem [E] tells that V admits a Stein structure. Let $\mathfrak{s}$ be the canonical spin^C structure on V, and denote the homology class determined by the 2-handle (for some orientation of L_n) by h. The way we stabilize L_n ensures that.

$$c_1(\mathfrak{s})(h) = \mathrm{rot}(L_n) = \pm 1 \tag{4.7}$$

Where $\mathrm{rot}(L_n)$ stands for the rotation number of L_n. Note that the sign of the rotation number depends on how we orient L_n. Next, V is blown-up $n+2$ times, and we do the handleslides indicated in figure 4.5. We see that the resulting 4-manifold is given by the plumbing graph G in figure 4.4. The manifold $X(G)$ is no longer Stein but it does admit a symplectic structure. Let $\mathfrak{s}'$ be the canonical spin^C structure on this symplectic manifold. Let e_i denote the homology class of the ith exceptional sphere. We have

$$c_1(\mathfrak{s}')(e_i) = 1 \quad i = 1, 2, \cdots, n+2. \tag{4.8}$$

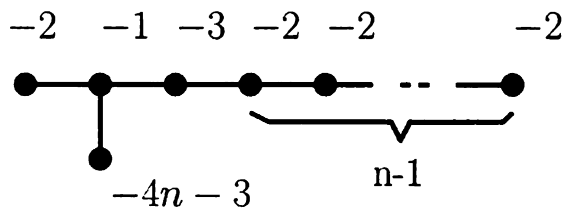

Figure 4.4: A Plumbing graph for Brieskorn homology sphere $\Sigma(2, 2n+1, 4n+3)$

Order the vertices of G so that first four are the ones with weight $-1, -2, -3$ and $-4n-3$ respectively, and all the remaining ones corresponding to -2s on right are ordered according to the distance from the root starting with the closest one. In [R1], Rustamov proves that

$$HF^+(-Y_n) = \mathcal{T}_0^+ \oplus \mathbb{Z}_{(0)}^{p_n} \oplus \bigoplus_{i=1}^{n-1} (\mathbb{Z}_{q_{n-i}}^{p_i} \oplus \mathbb{Z}_{q_{n-i}}^{p_i})$$

where $q_i = i(i+1)$ and $\mathbb{Z}_{(k)}^r = \mathbb{Z}[U]/U^r\mathbb{Z}[U]$ and U^{r-1} lies in degree k. More precisely, he shows that Ker$U \subset HF^+(-Y_n)$ is generated by the characteristic vectors

$$K_i = (1, 0, -1, -4n-3+2i), \quad i = 1, 2, \cdots, 2n.$$

He also proves that the minimal relations are given as follows:

$$U^{p_i} \otimes K_i \sim U^{p_i + q_{n-i}} \otimes K_{n+1} \tag{4.9}$$

$$U^{p_i} \otimes K_{n+i} \sim U^{p_i + q_{n-i}} \otimes K_n \tag{4.10}$$

where $i = 1, 2, \cdots, n$. Note that the characteristic vectors K_n and K_{n+1} are in the bottom level, and their degree is zero.

Our aim is to pin down the contact invariant $c^+(\xi_n)$ in $HF^+(-Y_n)$. Note that proposition 1.0.5 in the stated form can not be applied directly as it concerns Stein fillings of plumbed manifolds. However, as indicated in remark 1.0.6 it is also true for strong symplectic fillings. The only difference in the proof is that one uses Ghiggini's generalization [Ghi] of Plamenevskaya's theorem [P]. Alternatively, one can use the blow-up formula and handleslides invariance for this particular case to see that equations 4.5 and 4.6 hold. In any case, we see that the contact invariant $c^+(\xi_n)$ is represented by the first Chern class $c_1(\mathfrak{s}')$ of the canonical spin^c structure. In figure 4.5, we keep track of the homology classes in order to pin down the first Chern class. By equations 4.7 and 4.8, we have $c_1(\mathfrak{s}') = K_n$ or K_{n+1} depending on the orientation of L_n, but the contact invariant is in the image of $U^{p_n - 1}$ in any case.

□

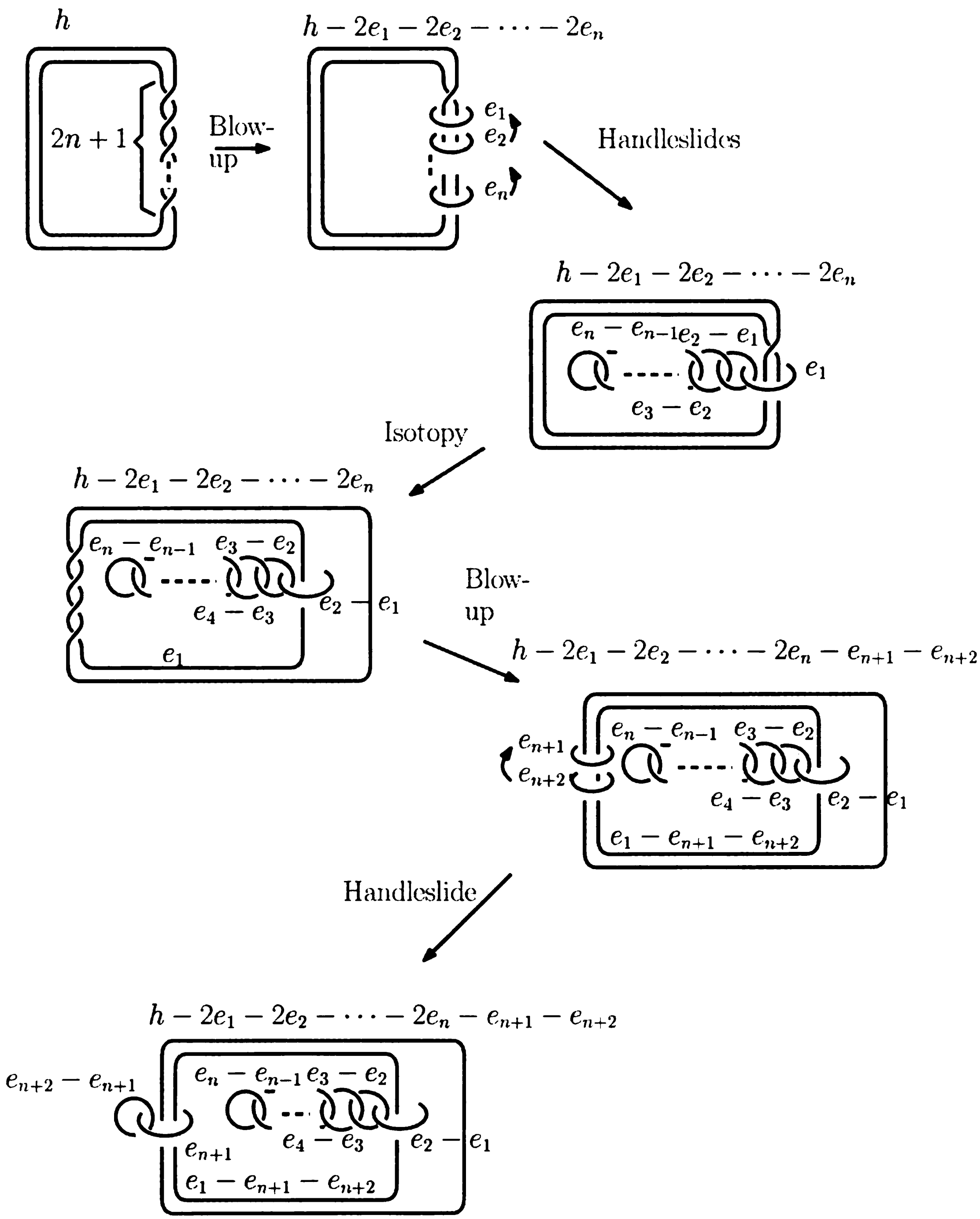

Figure 4.5: Sequence of Blow-ups from K_n to plumbing

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