

TORSION IN H-SPACES OF LOW RANK

Thesis for the Degree of Ph.D.
MICHIGAN STATE UNIVERSITY
KAI KIE DAI
1971

This is to certify that the
thesis entitled

Torsion in H-spaces of Low Rank

presented by

Kai K. Dai

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in *mathematics*

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8/5/71

ABSTRACT

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By

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Our discussion concerns a path connected H-space X that is also a finite CW complex. By a theorem of Hopf the reduced cohomology of X with rational coefficients is an exterior algebra on odd dimensional generators. The number of generators is called the rank of X . If $H^i(X; \mathbb{Z})$ contains an element of order p^r ($r \geq 1$), for some i and some prime p , then X is said to have p -torsion. In particular, if $r > 1$, then X is said to have higher p -torsion. It has been shown by W. Browder that if X has rank one, then X has no p -torsion for any odd p . In addition, it is a long standing conjecture that ΩX , the loop space of X , is torsion free.

The principal results of this thesis are the following:

- (1) If X has rank two, then X has no p -torsion for $p \geq 5$.

- (2) If X is 1-connected and has rank 2, then X has no p -torsion for $p \geq 3$.
- (3) If X has rank 2, then ΩX is torsion free and X has no higher p -torsion for any prime p .
- (4) If X has rank 2 and has no 2-torsion, then X has 3-torsion if and only if $H^2(X; \mathbb{Z})$ has an element of order 3.
- (5) (a) If X has rank less than or equal to 5 and has no 2-torsion, then X has no p -torsion for $p > 5$.
 (b) If X is 1-connected, has rank less than or equal to 4, and has no 2-torsion, then X has no p -torsion for $p > 3$.
- (6) If X is associative (i.e., there exists a multiplication that is associative), 1-connected, and has rank less than or equal to 5, then X has no p -torsion for $p > 5$.

TORSION IN H-SPACES OF LOW RANK

By

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A Thesis

Submitted to
Michigan State University

in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1971

27770

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis advisor, Ronald C. O'Neill, for his patient guidance and enthusiastic encouragement throughout the work in the thesis and my graduate career.

I also would like to thank the department of mathematics of Dartmouth College for their hospitality and for providing me all kinds of facilities during my visit there in 1970-71, when this thesis was written.

My thanks and appreciation go to Mary Starr for the excellent typing of this thesis.

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CHAPTER I

Introduction

A topological space X is an H-space if and only if there exist a distinguished base point $e \in X$, the unit element, and a continuous map $m : X \times X \rightarrow X$, called the multiplication, such that $m(x, e) = m(e, x) = x$ for all $x \in X$. Lie groups and topological groups are H-spaces. There are H-spaces that are not topological groups; e.g., S^7 . A finite H-complex is an H-space that is also a finite CW complex (for the definition of CW complex, see [41; p.401]). A classical theorem of Hopf says that if X is a path connected finite H-complex, then its reduced cohomology with rational coefficients is an exterior algebra on odd dimensional generators:

$$H^*(X; \mathbb{Q}) \cong \wedge(x_{n_1}, \dots, x_{n_i}),$$

where $\dim x_{n_j} = n_j = \text{odd}$. If $H^*(X; \mathbb{Q})$ is generated by $x_{n_1}, \dots, x_{n_i}$ in dimensions $n_1, \dots, n_i$, respectively, with $n_j \leq n_{j+1}$, then X is said to have rank i and type $(n_1, \dots, n_i)$. If the integral cohomology of X ,

$H^*(X)$, contains an element of order p^r ($r \geq 1$) for some prime p , then X is said to have p -torsion. In particular, if $r > 1$, then X is said to have higher p -torsion.

The extension to finite H-complexes of known topological properties of compact Lie groups provides the principal motivation for the study of finite H-complexes. A good example of such a theorem is the Hopf theorem quoted above. In Hopf's original paper [28] essentially only Lie groups were treated, but the paper gave rise to the concept of H-complex since a continuous multiplication was the only property required. Prior to the paper of Hilton-Roitberg [27] it was a standing conjecture that if X is a finite H-complex, then there exists a Lie group G_X such that X has the homotopy type of $G_X \times S^7 \times \dots \times S^7$. However, this conjecture was demolished by P. Hilton and J. Roitberg by showing that the 10-manifold $M_{7\omega}$, the total space of the principal S^3 -bundle over S^7 classified by 7ω , where ω classifies $Sp(2) \rightarrow S^7$, is an H-space but not of the homotopy type of any Lie group. Thus the classification problem of finite H-complexes

will not reduce to that of Lie groups. Precisely, finite H-complexes are classified up to H-equivalence: An H-map of two H-spaces X and Y with multiplications m and n , respectively, is a map $h : X \rightarrow Y$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X \times X & \xrightarrow{m} & X \\ \downarrow h \times h & & \downarrow h \\ Y \times Y & \xrightarrow{n} & Y \end{array} .$$

Two H-spaces X and Y are H-equivalent if there exists a homotopy equivalence $h : X \rightarrow Y$ which is an H-map.

One approach to the classification problem is to investigate how much such a complex must look homologically like a Lie group, i.e., to investigate all possible finite H-complexes up to H-equivalence by their rank.

If X is a path connected finite H-complex of rank one, then by the result of W. Browder [15], X has the homotopy type of S^1 , S^3 , S^7 , RP^3 , or RP^7 . In [38] it is shown that the set of homotopy classes of multiplications on a finite H-complex X is in one-one correspondence with $[X \wedge X, X]$, where $X \wedge X$ means the smash product of X , i.e., the identification space

obtained from $X \times X$ by identifying $X \times \{e\} \cup \{e\} \times X$ to a single point, and $[X \wedge X, X]$ means the set of homotopy classes of maps from $X \wedge X$ to X . This set could be infinite. For S^1 there is only one H-space structure since $[S^1 \wedge S^1, S^1] \cong [S^2, S^1] \cong \pi_2(S^1) = 0$. There are 12 and 120 homotopy classes on S^3 and S^7 , respectively. Since each multiplication is H-equivalent to its transpose or opposite, the number of non-equivalent H-structures is 6 or 60, respectively. For the projective spaces the number of non-equivalent H-structures has not yet been settled.

If X is a connected finite H-complex of rank two and if X is torsion free, then X has the homotopy type of $S^1 \times S^1$, $S^1 \times S^3$, $S^1 \times S^3$, $S^3 \times S^3$, $SU(3)$, $S^3 \times S^7$, $Sp(2)$, $E_{2\omega}$, $E_{3\omega}$, $E_{4\omega}$, $E_{5\omega}$, $E_{6\omega}$, $E_{7\omega}$, or $S^7 \times S^7$. This was proved independently by P. Hilton and J. Roitberg [26], M. Curtis and G. Mislin, E. Thomas, and A. Zabrodsky [unpublished]. The number of homotopy classes of multiplications on products of spheres can easily be computed and the number of homotopy classes of multiplications on $SU(3)$ and $Sp(2)$ is $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ and $2^{20} \cdot 3 \cdot 5^5 \cdot 7$, respectively [34].

The classification of arcwise connected finite H-complexes of rank two with torsion is not yet complete. In fact, it is an open problem that if X is a 1-connected finite H-complex with or without 2-torsion, then X has no p -torsion for $p > 5$. A related problem is a long standing conjecture that if X is a 1-connected finite H-complex, then ΩX , the loop space of X , is torsion free.

Torsion plays an important role in homotopy classification of finite H-complexes. The purpose of this thesis is to solve some of the problems on torsion in finite H-complexes.

CHAPTER II

Preliminaries and Statements of Results

The requisite background for the proofs of the results in this thesis is sketched in the first seven sections and the principal results of the thesis are given in the last section.

§1. On the structure of Hopf algebras.

A good reference for this section is [33].

Let R be a commutative ring with unit. A graded R algebra consists of a graded R module $A = \{A^q\}$ and a homomorphism of degree 0

$$\mu : A \otimes A \rightarrow A$$

called the product of the algebra (μ then maps $A^p \otimes A^q$ into A^{p+q} for all p and q). For $a, a' \in A$ we write $aa' = \mu(a \otimes a')$. The product is associative if $(aa')a'' = a(a'a'')$ for all $a, a', a'' \in A$ and is commutative if $aa' = (-1)^{\deg a \deg a'} a'a$ for all $a, a' \in A$.

A graded R coalgebra consists of a graded R module $A = \{A^q\}$ and a homomorphism of degree 0

$$d : A \rightarrow A \otimes A$$

called the coproduct of the algebra (so d maps A^q into $\bigoplus_{i+j=q} A^i \otimes A^j$ for all q). The coproduct is said to be associative if

$$(d \otimes 1)d = (1 \otimes d)d : A \rightarrow A \otimes A \otimes A$$

and is said to be commutative if $Td = d$, where

$T : A \otimes A \rightarrow A \otimes A$ is the homomorphism $T(a \otimes a') = (-1)^{\deg a \deg a'} a' \otimes a$. A connit for the coalgebra

is a homomorphism $\epsilon : A \rightarrow R$ such that each of the composites in the diagram,

$$\begin{array}{ccccc} A & \xrightarrow{d} & A \otimes A & \begin{array}{l} \nearrow \epsilon \otimes 1 \\ \searrow 1 \otimes \epsilon \end{array} & \begin{array}{l} R \otimes A \\ A \otimes R \end{array} \\ & & & & \begin{array}{l} \approx \\ \approx \end{array} \\ & & & & A, \end{array}$$

is the identity map.

A Hopf algebra over R is a graded R algebra A which is also a coalgebra whose coproduct

$$d : A \rightarrow A \otimes A$$

is a homomorphism of graded R algebras. A Hopf algebra A is said to be connected if A^0 is the free R module

generated by a unit element 1 for the algebra and the homomorphism $\epsilon : A \rightarrow R$ defined by $\epsilon(\alpha 1) = \alpha$ for $\alpha \in R$ is a counit for the coalgebra.

Let X be a connected finite H-complex with multiplication m and K a field. It is well-known (cf. [36, p.49]) that $H_*(X;K)$ is a connected Hopf algebra with product induced by the multiplication and the coproduct induced by the diagonal map. Dually, $H^*(X;K)$ is a connected Hopf algebra with product induced by the diagonal map and coproduct induced by the multiplication.

If A is a graded coalgebra with coproduct $d : A \rightarrow A \otimes A$, then an element y of A which is not in A^0 is called primitive if $d(y) = y \otimes 1 + 1 \otimes y$. Thus for any space Y , whether Y is an H-space or not, an element y in $H_*(Y;K)$ is called primitive if and only if the homomorphism Δ_* induced by the diagonal map has the following property:

$$\Delta_*(y) = y \otimes 1 + 1 \otimes y.$$

Primitivity of y in $H_*(Y;K)$ has nothing to do with the multiplication. Also if X is a finite H-complex with multiplication m , then an element y in

$H^*(X;K)$ is called primitive if and only if the homomorphism m^* induced by the multiplication m has the following property:

$$m^*(y) = 1 \otimes y + y \otimes 1.$$

Notice that primitivity of y in $H^*(X;K)$ depends on the multiplication m .

An element in a graded algebra is called indecomposable if and only if it cannot be written as a product of lower nonzero dimensional elements.

Let X be a connected finite H -complex and K be a field. For simplification let A be either $H^*(X;K)$ or $H_*(X;K)$ and A^* be its vector space dual. Then we have:

Lemma 1.1. [33; § 3]. Let $P(A)$ denote the subspace of primitive elements of A and $Q(A)$ the subspace of indecomposable elements of A . Then $P(A^*) \cong (Q(A))^*$.

Lemma 1.2. [33; § 7]. If K is perfect, then A is isomorphic as an algebra with a tensor product $A_1 \otimes \dots \otimes A_n$, where A_i , $1 \leq i \leq n$ is a Hopf algebra with a single generator x_i .

Lemma 1.3. [33; § 7]. If A is associative and if K of characteristic zero, then A is isomorphic to an exterior algebra on odd dimensional generators.

Lemma 1.4. [33; § 4]. Suppose that the field K is of characteristic $p \neq 0$. Then there is an exact sequence:

$$0 \rightarrow P(K(\xi A)) \rightarrow P(A) \rightarrow Q(A),$$

where $P(A)$ denotes the subspace of primitive elements of A , $Q(A)$, the subspace of indecomposable elements of A , and ξA , the image of the homomorphism $\xi : A_n \rightarrow A_{pn}$ defined by $\xi(x) = x^p$ for all $x \in A_n$; $K(\xi A)$ is the subalgebra generated over K by ξA . In other words, if a primitive element of A is decomposable, then it is a p th power.

§2. On the types of a connected finite H-complex.

Let X be a connected finite H-complex. There is a relationship between torsion in X and the possible types of X . The results listed in this section will be used to determine the existence of certain p -torsion in X .

Lemma 2.1. [1]. Let X be a connected finite H -complex of rank one, then X has type (1) , (3) , or (7) .

Lemma 2.2. [29; Theorem 1.1]. Let X be a connected finite H -complex of rank 2 having no 2-torsion. Then the type of X is $(1,1)$, $(1,3)$, $(1,7)$, $(3,3)$, $(3,5)$, $(3,7)$, or $(7,7)$.

We remark here that Lemma 2.2 was first proved partially by J. Adams [4] and completed by Douglas-Sigrist [24] and was also proved independently by J. Hubbuck [29].

Lemma 2.3. [29; Theorem 1.1]. Let X be a connected finite H -complex of rank less than or equal to 5 having no 2-torsion. Then the type of X is a union of sets taken from (1) , (3) , (7) , $(3,5)$, $(3,7)$, $(3,5,7)$, $(3,7,11)$, $(3,5,7,9)$, $(3,7,11,15)$, $(3,5,7,9,11)$, or $(3,7,11,15,19)$.

The method in proving Lemma 2.3 is by studying the projective plane of the finite H -complex defined in [43] and using a result by W. Browder and E. Thomas on the cohomology of the projective plane of X [18].

The machinery is K-theory and Adams operations on K-theory, which, when a space has no torsion, "represent" the Steenrod power operations on cohomology [6; Theorem 6.5].

Later, with similar techniques and a result by T. Sugawara and H. Toda [45], J. Hubbuck computed all possible types of a connected finite H-complex without 2-torsion when the dimension of the generator of $H^*(X; \mathbb{Z}_2)$ of highest degree is not of the form $2^{s+1}-1$:

Lemma 2.4. [32; Theorem 1.2]. Let X be a connected finite H-complex without 2-torsion. If the dimension of the generator of $H^*(X; \mathbb{Z}_2)$ of highest degree is not of the form $2^{s+1}-1$, then $H^*(X; \mathbb{Z}_2)$ is isomorphic as a Hopf algebra to $H^*(G; \mathbb{Z}_2)$, where G is one of the Lie groups $U(n)$, $SU(n+1)$, $S^1 \times Sp(n-1)$, or $Sp(n)$, and n is the rank of X .

We remark here that in Lemma 2.3 and Lemma 2.4 it is originally assumed that $H^*(X; \mathbb{Q})$ is primitively generated, i.e., there exists a set of generators which are primitive, but this hypothesis is superfluous by a result due to C. Curjel that X has a multiplication such that $H^*(X; \mathbb{Q})$ is primitively generated; cf. [22; §8].

For a 1-connected associative finite H-complex (i.e., there exists a multiplication which is associative) the types have also been determined for rank less than or equal to 5 by various people listed below:

Lemma 2.5. Let X be a 1-connected associative finite H-complex. If X is of rank one, then the type of X is (3) [1 or 15]. If X is of rank 2, then the type of X is (3,3), (3,5), (3,7), or (3,11) [40]. If X is of rank 3, then the type of X is (3,3,3), (3,3,7), (3,5,7), (3,3,11), or (3,7,11) [37]. If X is of rank 4, then the type of X is (3,3,3,3), (3,3,3,5), (3,3,5,5), (3,3,3,7), (3,3,5,7), (3,3,7,7), (3,5,7,9), (3,3,3,11), (3,3,5,11), (3,3,7,11), (3,7,7,11), (3,3,11,11), (3,7,11,15), or (3,11,15,23) [29 or 44]. If X is of rank 5, then the type of X is (3,3,3,3,3), (3,3,3,3,5), (3,3,3,5,5), (3,3,3,3,7), (3,3,3,5,7), (3,3,5,5,7), (3,3,3,7,7), (3,3,5,7,7), (3,3,5,7,9), (3,5,5,7,9), (3,3,3,3,11), (3,3,3,5,11), (3,3,3,7,11), (3,3,5,7,11), (3,3,7,7,11), (3,5,7,9,11), (3,3,3,11,11), (3,3,7,11,11), (3,3,7,11,15), (3,5,7,11,15), (3,7,9,11,15), (3,7,11,11,15), (3,7,11,15,19), or (3,3,11,15,23) [25].

We remark here that the main technique used in the proof above is the results by A. Clark in the paper "On π_3 of finite dimensional H-spaces,"

appeared in Annals of Mathematics in 1963, and some extensions of them.

§3. On a relation between generators in $H^*(X;Z_p)$ and types of X .

Lemma 3.1. [13; Theorem 4.7]. Let X be a path connected finite H-complex. If $H^*(X;Z_p)$ has a generator in dimension $2m-1$, then $H^*(X;Q)$ has a generator in dimension $2mp^k-1$, $\infty > k \geq 0$. If $H^*(X;Z_p)$ has a generator in dimension $2m$, then $H^*(X;Q)$ has a generator in dimension $2mp^k-1$, $\infty > k \geq 1$.

Lemma 3.2. [13; Lemma 6.4]. Let X be a connected finite H-complex. Let s be the smallest integer for which $H^s(X)$ has p -torsion. Then $s = 2n$, and if $p \neq 2$, then $H_s(X;Z_p)$ has a primitive element. Further, if $p = 2$, and if n is even, then $H_s(X;Z_2)$ has a primitive element.

Lemma 3.3. [8 and 12]. Let X be a connected finite H-complex. A necessary and sufficient condition that X has no p -torsion is that its cohomology with Z_p coefficients is an exterior algebra on odd dimensional generators.

§4. Some facts about torsion in H-spaces.

The p-dimension of a space X is the largest integer t such that $H_t(X; \mathbb{Z}_p) \neq 0$. Similarly, the rational dimension of X is the largest t such that $H_t(X; \mathbb{Q}) \neq 0$.

Lemma 4.1. [12; Theorem 7.1]. Let X be a path connected finite H-complex. Then for all p the p-dimension of X equals the rational dimension of X .

Lemma 4.2. [12; Corollary 7.2]. Let X be as in Lemma 4.1 above. Then $H_t(X) \cong \mathbb{Z}$ and $H_{t-1}(X)$ is free.

For simply connected, path connected finite H-complexes, we have:

Lemma 4.3. [12; Theorem 6.11]. Let X be a 1-connected finite H-complex. Then $\pi_1(X) = \pi_2(X) = 0$.

The techniques used in §3 and §4 above are a close study of Hopf algebra structure, Bockstein spectral sequence, biprimitive spectral sequence, Serre spectral sequence, and Leray-Cartan spectral sequence (on the covering spaces of H-spaces). The reader is referred to the well-written papers quoted above and the paper by J. Milnor and J. Moore [33].

§5. On a relation between the homology of X and the homology of ΩX .

Even when X is a finite H-complex, ΩX will not be a finite H-complex unless X has the homotopy type of $K(G, 2)$, where $K(G, 2)$ is the Eilenberg-MacLane complex and G is finite free abelian [17]. A theorem by W. Browder states that:

Lemma 5.1. [13; Theorem 5.15]. Let X be a path connected, simply connected H-space. Suppose $H^*(X; K) = \wedge(x_1, \dots, x_m, \dots)$, an exterior algebra on generators $x_1, x_2, \dots$, $\dim x_i = 2n_i + 1$, K a field. Then $H_*(\Omega X; K) = K[y_1, \dots, y_m, \dots]$, a polynomial algebra on generators $y_1, \dots, y_m, \dots$ with $\dim y_i = 2n_i$.

We remark here that the lemma above is true even without the hypothesis that X is an H-space. The proof can be shown by using Serre's theory of classes of abelian groups or by using the spectral sequence of the cobar construction.

Lemma 5.2. [20; Theorem 4.1]. Let X be a path connected, simply connected H-space of finite homological type and let s_i be the homology mod p suspension in degree i :

$$s_i : Q(H_i(\Omega X; Z_p)) \rightarrow P(H_{i+1}(X; Z_p))$$

from the subspace of indecomposable elements of $H_i(\Omega X; Z_p)$ to the subspace of primitive elements of $H_{i+1}(X; Z_p)$. Then if $p=2$, s_i is a monomorphism unless $i = 2^q(2^k m + 2) - 2$ for $q > 0$, $k > 0$, and $Q(H_m(\Omega X; Z_2)) \neq 0$ and s_i is an epimorphism unless $i = 2^k m + 1$ for $k > 0$ and $Q(H_m(\Omega X; Z_2)) \neq 0$.

We remark here that the above result is a slight improvement over that of [13; Theorem 5.13]. The technique of proof is an application of Eilenberg-Moore spectral sequence.

Lemma 5.3. [19]. If X is 1-connected finite H-complex and if ΩX is torsion free, then X has no higher torsion.

Lemma 5.4. [13; Theorem 6.6]. Let X be a path connected finite H-complex. Suppose that ΩX has no torsion. If s is the smallest integer for which $H^s(X; Z)$ has p -torsion, then $s = 2n$, and $n \equiv 1 \pmod{p}$.

§6. On localization of H-complexes.

One of the powerful techniques in attacking finite H-complex problems is the concept of localization

of H-complexes. Localization of CW complexes has been studied by several mathematicians, notably by J. Adams [2], D.W. Anderson [5], Bonsfield-Kan [10], Curtis-Mislin [23], Mimura-Nishida-Toad [35], D. Sullivan [46], and others. A relatively complete list of references can be found in [35]. For a description of the construction of $X_{(p)}$, the localization of X at the prime p , see one of the above references. The principal results on $X_{(p)}$ are listed below:

Lemma 6.1. [35; Theorem 2.4]. The correspondence $X \rightarrow X_{(p)}$ is a functor from the homotopy category of 1-connected CW complexes of finite type to the homotopy category of 1-connected countable CW complexes.

Lemma 6.2. [35; Theorem 2.5]. Let X be a 1-connected CW complex of finite type. Then $H_*(X_{(p)}) \cong H_*(X) \otimes Q_{(p)}$, where $Q_{(p)}$ denotes the integers localized at p , (i.e., the ring of rationals whose denominators when reduced to the simplest form are prime to p .)

Lemma 6.3. [35; Theorem 7.1]. Let X be an H-space. Then $X_{(p)}$ is also an H-space.

Lemma 6.4. [35; Proposition 2.2]. Let X be a 2-connected CW complex of finite type. Then $(\Omega X)_{(p)}$ has the homotopy type of $\Omega(X_{(p)})$.

§7. The cohomology of covering spaces of H-spaces.

Let X be a connected finite H-complex. Then in [39] it is shown that $\tilde{X}$, the universal covering space of X , is a 1-connected finite H-complex. Let $\bar{X}$ be a covering space of X . If $\bar{X} \xrightarrow{\pi} X$ is the covering projection, then we can convert $\bar{X} \xrightarrow{\pi} X$ to the fibration sequence $\bar{X} \xrightarrow{i} X' \xrightarrow{f} K(G, 1)$, where i is the inclusion map and X' has the homotopy type of X and consider the Leray-Cartan spectral sequence of this fibration sequence. We have:

Lemma 7.1. [11]. Let p be an odd prime. Then $H^*(\bar{X}; \mathbb{Z}) \cong A \otimes E$ as rings, where $A = \pi^*(H^*(X; \mathbb{Z}_p)) \cong H^*(X; \mathbb{Z}_p)/I$, I is the ideal generated by $f^*(H^*(K(G, 1); \mathbb{Z}_p))$ and E is the exterior algebra on n generators $x_1, \dots, x_n$, where the dimension of x_i is $2p^{r_i} - 1$, and $2p^{r_1}, \dots, 2p^{r_n}$ are the dimensions of a system of generators of the kernel of f^* . If $p=2$, then the same result holds, but the isomorphism is only as modules.

Lemma 7.2. [11]. As in Lemma 7.1 above if $\bar{X} = \tilde{X}$, then the ideal I is generated by $H^1(X; \mathbb{Z}_p)$ and the subspace B of $H^2(X; \mathbb{Z}_p)$, where each element of B lies in the image of the Bockstein homomorphism $\beta_{p^s} : H^1(X; \mathbb{Z}_p) \rightarrow H^2(X; \mathbb{Z}_p)$ for every s . The dimension of the x_i 's is determined by writing the algebra generated by B as the tensor product of polynomial rings and truncated polynomial rings. For each truncated polynomial ring on one generator we get one x_i whose dimension is the height of the generator minus one.

§8. Statements of Results.

Throughout this section, X denotes a path connected finite H-complex and ΩX denotes the loop space of X . Also assume that if X is a 1-connected finite H-complex of rank 2 having 2-torsion, then

$$H^*(X; \mathbb{Z}_2) \cong H^*(G_2; \mathbb{Z}_2),$$

where G_2 is the exceptional Lie group whose cohomology with $\mathbb{Z}_2$ coefficients is:

$$H^*(G_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \wedge (Sq^2 x_3),$$

the tensor product of a polynomial algebra on one generator of dimension 3 truncated at height 4 and an exterior algebra on one generator of dimension 5.

We remark here that this hypothesis might be superfluous as the proof of it was claimed [unpublished] by J.R. Hubbuck during the International Conference on H-spaces at Neuchatel in Switzerland in 1970.

As mentioned in Chapter one it has been shown by W. Browder that if X is of rank one, then X has no odd torsion. If X is of rank 2, we have:

Theorem 1. If X has rank 2, then X has p -torsion for $p \geq 5$.

Theorem 2. If X is 1-connected and has rank 2, then X has no odd torsion.

Torsion in X is closely related to torsion in ΩX . A relation of them is the following:

Lemma 3. Let p be a prime. If X has no p -torsion, then ΩX has no p -torsion.

For most known examples of 1-connected finite H-complex X , ΩX has been shown to be torsion free; thus it is a long standing conjecture that ΩX is torsion free. In required to this we have:

Theorem 4. If X has rank 2, then ΩX is torsion free.

Recall that if $H^i(X; \mathbb{Z})$ has an element of order p^2 for some i and some prime p , then we say that X has higher p -torsion.

Corollary 5. If X is 1-connected and has rank 2, then X has no higher p -torsion for any p .

If X has rank 2, then from Theorem 1 above we have that X has no p -torsion for $p \geq 5$. In the absence of 2-torsion we have an interesting restriction on the presence of 3-torsion.

Corollary 6. If X has rank 2 and has no 2-torsion, then X has 3-torsion if and only if $H^2(X; \mathbb{Z})$ has an element of order 3.

This completes the study of torsion in the rank 2 case. For ranks higher than 2 we have:

Theorem 7. (i) If X has rank less than or equal to 5 and if X has no 2-torsion, then X has no p -torsion for $p \geq 7$.
 (ii) If X is 1-connected, has rank less than or equal to 4, and has no 2-torsion, then X has no p -torsion for $p \geq 5$.

The center of $SU(n)$ is isomorphic to Z_n , a cyclic group of order n [3]. Thus for any prime p the corresponding projective group $SU(p)/Z_p$ has p -torsion. However, these groups are not simply connected. For 1-connected finite H-complexes having no 2-torsion we have Theorem 7 above. For 1-connected associative finite H-complexes we have:

Theorem 8. Let X be a 1-connected associative finite H-complex of rank less than or equal to 5. Then X has no p -torsion for $p \geq 7$. Further, ΩX has no p -torsion for $p \geq 7$.

For a connected topological group or loop space G with $H^*(G; Z_p)$ finitely generated, if p is an odd prime and if $H_*(G; Z)$ has p -torsion, then $H^*(G; Z_p)$ is not primitively generated. This is a result by W. Browder [16; Theorem 1]. A recent result of J. Hubbuck [31; Corollary 1.3] states that if Y is a 1-connected homotopy commutative and homotopy associative H-space and if $H^*(Y; Z)$ has no p -torsion where p is an odd prime, then $H^*(Y; Z_p)$ is primitively generated if and only if the ring $H^*(Y; Z_p)$ is isomorphic to $Z_p[y_1, \dots, y_m, \dots] \otimes \wedge(x_1, \dots, x_n, \dots)$, a tensor product

of a polynomial algebra on generators y_i all having dimension 2 and an exterior algebra on odd dimensional generators x_j .

As a further application of the techniques of this thesis, we give a simple proof of Hubbuck's result quoted above for a 1-connected finite H-complex X when X has rank less than or equal to 4 and has no 2-torsion.

CHAPTER III

Proofs of Results

Throughout this chapter X denotes a path connected finite H-complex and ΩX denotes the loop space of X . We also assume that if X is 1-connected finite H-complex of rank 2, and if X has 2-torsion,

$$H^*(X; \mathbb{Z}_2) \cong H^*(G_2; \mathbb{Z}_2),$$

where the cohomology of G_2 is stated in §8 of Chapter II. Again we remark here that this hypothesis might be superfluous as the proof of it was claimed [unpublished] by J.R. Hubbuck during the International Conference on H-spaces at Heuchatel in Switzerland in 1970.

Theorem 1. If X has rank 2, then X has no p -torsion for $p \geq 5$.

Proof. We divide into two cases, namely,

(i) X has no 2-torsion, and (ii) X has 2-torsion.

(i) If X has no 2-torsion, then the type of X is $(1,1), (1,3), (1,7), (3,3), (3,5), (3,7)$, or $(7,7)$ by Lemma 2.2 in Chapter II.

Suppose that X has p -torsion for $p \geq 5$. If s is the smallest integer for which $H^s(X; \mathbb{Z})$ has p -torsion, then by Lemma 3.2 in Chapter II, $s=2m$ and, since $p \neq 2$, $H_s(X; \mathbb{Z}_p)$ has a primitive element. From Lemma 1.1 in Chapter II we have $P(H_{2m}(X; \mathbb{Z}_p)) \cong (Q(H^{2m}(X; \mathbb{Z}_p)))^*$, since $H^*(X; \mathbb{Z}_p)$ and $H_*(X; \mathbb{Z}_p)$ are dual to each other. Thus $H^{2m}(X; \mathbb{Z}_p)$ has an indecomposable element; hence a generator. By Lemma 3.1 in Chapter II $H^*(X; \mathbb{Q})$ has a generator in dimension $2mp^k - 1$, for some k , $0 < k < \infty$. We have $2m \geq 2$, $p \geq 5$, and $k \geq 1$; so $2mp^k - 1 \geq 9$. This contradicts the possible types of X which are $(1,1), (1,3), (1,7), (3,3), (3,5), (3,7)$, and $(7,7)$.

- (ii) If X has 2-torsion, then again we consider two cases, (a) X is simply connected, and (b) X is not simply connected.

(a) If X has 2-torsion and is simply connected, then $H^*(X; \mathbb{Z}_2) \cong H^*(G_2; \mathbb{Z}_2)$, where the exceptional Lie group G_2 has mod 2 cohomology:

$$H^*(G_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_3]/(x_3^4) \otimes \wedge(S_q^2 x_3),$$

the tensor product of a polynomial algebra on one generator of dimension 3 truncated at height 4 and an exterior algebra on one generator of dimension 5. Since $H^{14}(G_2; \mathbb{Z}_2) \neq 0$ and $H^i(G_2; \mathbb{Z}_2) = 0$ for $i > 14$, we have $H^{14}(X; \mathbb{Z}_2) \neq 0$ and $H^i(X; \mathbb{Z}_2) = 0$ for $i > 14$. By Lemma 4.1 in Chapter II we have $H^i(X; \mathbb{Q}) \neq 0$ and $H^i(X; \mathbb{Q}) = 0$ for $i > 14$. Now, if X has p -torsion for $p \geq 5$, then by the argument used in part (i) above we see that there is a generator for $H^*(X; \mathbb{Q})$ in dimension $2mp^k - 1$, where $2m$ is the smallest integer for which $H^{2m}(X; \mathbb{Z})$ has p -torsion and $0 < k < \infty$. Since X is simply connected, we have $\pi_1(X) = \pi_2(X) = 0$ by Lemma 4.3 in Chapter II. This implies that $H^1(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) = 0$. Thus, $2m \geq 4$, $p \geq 5$, $k \geq 1$, and so $2mp^k - 1 \geq 19$. This contradicts the fact that $H^i(X; \mathbb{Q}) = 0$ for $i > 14$.

(b) If X has 2-torsion and is not simply connected, then consider the universal covering space $\tilde{X}$ of X . First, observe that X does not have type $(1,1)$. If X has type $(1,1)$, then $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$

and $H^i(X;Z) = 0$ for $i > 2$ by Lemma 4.2 in Chapter II. Since X has 2-torsion, $H^{2n}(X;Z)$ contains an element of order 2^r ($r \geq 1$), where $2n$ is the smallest integer for which $H^{2n}(X;Z)$ has 2-torsion. Since $2n \geq 2$, we have a contradiction; hence X does not have type $(1,1)$. Also notice that $\tilde{X}$ is simply connected finite H-complex [see §7 in Chapter II]. From Lemma 7.2 in Chapter II we have that if $(n_1, \dots, n_i)$ is the type of $\tilde{X}$, then the possible types of X are $(n_1, \dots, n_i)$ and $(1, \dots, 1, n_1, \dots, n_i)$. Let $\tilde{X}$ have type (n_1) , i.e., $\tilde{X}$ has rank 1. Then by Lemma 2.1 in Chapter II we have $n_1 = 3$ or 7 . This implies that X has type $(1,3)$ or $(1,7)$, in which case there is no p -torsion for $p \geq 5$ by the same argument used in (i) above. Now let $\tilde{X}$ have rank 2. From Lemma 2.2 in Chapter II we see that if $\tilde{X}$ has no 2-torsion, then $\tilde{X}$ has type $(3,3), (3,5), (3,7)$, or $(7,7)$. So the type of X is $(3,3), (3,5), (3,7)$, or $(7,7)$. As reasoned in (i) above, X has no p -torsion for $p \geq 5$ when $\tilde{X}$ has no 2-torsion. If $\tilde{X}$ has 2-torsion, then again we have:

$$H^*(\tilde{X};Z_2) \cong H^*(G_2;Z_2).$$

X has no p -torsion for $p \geq 7$ because if it does $H^*(X;Q)$ will have a generator in a dimension at least

$2(1)(7) - 1 = 13$. This implies that the type of X is $(1,13)$ since the top dimension of $H^*(X;Q)$ is 14. But then $\tilde{X}$ is of rank one, a contradiction to our hypothesis that X is of rank 2. Suppose X has 5-torsion. Then $H^*(X;Q)$ has a generator in dimension $2m5^k - 1$ for $m \geq 1$, $k \geq 1$, where $2m$ is the smallest integer for which $H^{2m}(X;Z)$ has 5-torsion. The only dimension to consider is 9 since the top dimension is 14. Again this implies that the type of X is $(5,9)$. Now, X has 2-torsion, so let $2n$ be the smallest dimension in which $H^*(X;Z)$ has 2-torsion. We see that $2n$ is not 2, 4, or 8 since then $H^*(X;Q)$ will have a generator in dimension $2(1)2^k - 1$, $2(2)2^k - 1$, or $2(4)2^k - 1$, respectively, which is not 5 or 9. Thus 6 and 10 are the only remaining possibilities. In either case $H^3(X;Z_2) = H^4(X;Z_2) = 0$. But $H^3(\tilde{X};Z_2) \cong H^3(G_2;Z_2) \cong Z_2$ so by Lemma 7.2 in Chapter II, we have a contradiction. This completes the proof of the theorem.

Remark. The theorem above says that X has no p -torsion for $p \geq 5$. In fact, X may have 2-torsion; simply let $X = G_2$. Also, X may have 3-torsion. Consider $SU(3)$. Its center is Z_3 [3]. Let X be the corresponding projective group $PSU(3)$. It is obvious that X is of rank 2 and $H^2(X;Z) \cong Z_3$.

Theorem 2. If X has rank 2 and is 1-connected, then X has no odd torsion.

Proof. We divide into two cases, namely,

(i) X has no 2-torsion, and (ii) X has 2-torsion.

(i) Let X have no 2-torsion. Since X is simply connected, we have $\pi_1(X) = \pi_2(X) = 0$ by Lemma 4.3 in Chapter II. This implies that $H^1(X) = H^2(X) = 0$; hence the type of X is $(3,3), (3,5), (3,7)$, or $(7,7)$, by Lemma 2.2 in Chapter II. Suppose that X has p -torsion for $p \geq 3$. Let s be the smallest integer for which $H^s(X; \mathbb{Z})$ has 3-torsion. Then by Lemma 3.2 in Chapter II, $s=2m$ and $H_s(X; \mathbb{Z}_p)$ has a primitive element. From Lemma 1.1 in Chapter II we have that $P(H_{2m}(X; \mathbb{Z}_p)) \cong (Q(H^{2m}(X; \mathbb{Z}_p)))^*$, since $H^*(X; \mathbb{Z}_p)$ and $H_*(X; \mathbb{Z}_p)$ are dual to each other. Thus $H^{2m}(X; \mathbb{Z}_p)$ has an indecomposable element; hence a generator. By Lemma 3.1 in Chapter II, $H^*(X; \mathbb{Q})$ has a generator in dimension $2mp^k - 1$, for some k , $0 < k < \infty$. We have that $2m \geq 4$

since X is simply connected, $p \geq 3$, and $k \geq 1$; so $2mp^k - 1 \geq 4 \cdot 3 - 1 = 11$. This contradicts the possible types of X which are $(3,3)$, $(3,5)$, $(3,7)$, and $(7,7)$.

- (ii) Suppose that X has 2-torsion. From Theorem 1 X has no p -torsion for $p \geq 5$. Thus all we need to show is that X has no 3-torsion. Suppose the contrary holds. Then X has 2-torsion and 3-torsion. Consider $X_{(3)}$, the localization of X at the prime 3. By Lemma 6.1 in Chapter II, $X_{(3)}$ is a 1-connected CW complex and by Lemma 6.3 in Chapter II we see that $X_{(3)}$ is a 1-connected finite H-complex. Since X is of rank 2, $X_{(3)}$ is of rank 2 by Lemma 6.2 in Chapter II. Also by Lemma 6.2 in Chapter II we have that $X_{(3)}$ has only 3-torsion which implies that $X_{(3)}$ has no 2-torsion. Now, by the result in part (i) above we see that this is a contradiction. Thus X has no p -torsion for $p \geq 3$ and this completes the proof of the theorem.

Lemma 3. Let p be a prime and X be 1-connected. If X has no p -torsion, then ΩX has no p -torsion.

Proof. Suppose that ΩX has p -torsion for some prime p . Then by the Universal Coefficient Theorem for homology:

$$H_n(\Omega X; \mathbb{Z}_p) \cong (H_n(\Omega X; \mathbb{Z}) \otimes \mathbb{Z}_p) \oplus (H_{n-1}(\Omega X; \mathbb{Z}) * \mathbb{Z}_p)$$

we have that if $H_n(\Omega X; \mathbb{Z})$ has p -torsion, then $H_n(\Omega X; \mathbb{Z}_p) \neq 0$ and $H_{n+1}(\Omega X; \mathbb{Z}_p) \neq 0$. This means that $H_i(\Omega X; \mathbb{Z}_p) \neq 0$ for some positive odd integer i . This implies that $H_*(\Omega X; \mathbb{Z}_p) \neq \mathbb{Z}_p[y_1, \dots, y_m, \dots]$, where $\dim y_i = 2n_i$; hence $H^*(X; \mathbb{Z}_p) \neq \wedge(x_1, \dots, x_m, \dots)$, an exterior algebra on odd dimensional generators by Lemma 5.1 in Chapter II. But then X has p -torsion by Lemma 3.3 in Chapter II. This contradicts the hypothesis that X has no p -torsion. Thus ΩX has no p -torsion.

Theorem 4. If X has rank 2, then ΩX has no p -torsion for any p .

Proof. We divide into 2 cases, (i) X is simply connected, and (ii) X is not simply connected.

(i) Suppose that X is simply connected.

If X has rank one, then X has the homotopy type of S^3 or S^7 [15; Theorem 5.2] and ΩS^3 and ΩS^7 are

torsion free. Suppose now that X has rank 2. By Theorem 2 X has no p -torsion for $p \geq 3$. Thus by Theorem 3 above ΩX has no p -torsion for $p \geq 3$. So, if we can show that ΩX has no 2-torsion, then we are done.

If X has no 2-torsion, then by Lemma 3 ΩX has no 2-torsion and the theorem is proved. Suppose that X has 2-torsion. Since X is 1-connected,

$$H^*(X; \mathbb{Z}_2) \cong H^*(G_2; \mathbb{Z}_2),$$

where the cohomology ring $H^*(G_2; \mathbb{Z}_2)$ has one generator in dimension 3 and one generator in dimension 5. We shall show that ΩX has no 2-torsion. Suppose the contrary holds. As reasoned in the proof of Lemma 3, $H_i(\Omega X; \mathbb{Z}_2)$ contains 2-torsion for some positive odd integer i . Let m be the smallest such i . Then $H_m(\Omega X; \mathbb{Z}_2)$ contains an indecomposable element. Since, by Lemma 5.2 in Chapter II,

$$S_m : Q(H_m(\Omega X; \mathbb{Z}_2)) \rightarrow P(H_{m+1}(X; \mathbb{Z}_2))$$

is a monomorphism for $m \neq 2^t(2n)-2$, where $t \geq 1$ and $2n = \text{dimension}$ of some generator of $H^*(X;Z_2)$, we see that $H_{m+1}(X;Z_2)$ contains a primitive element. Since

$$P(H_{m+1}(X;Z_2)) \cong (Q(H^{m+1}(X;Z_2)))^*,$$

we see that $H^{m+1}(X;Z_2)$ has an indecomposable element; hence a generator. But $m+1$ is even, a contradiction to the fact that $H^*(X;Z_2)$ has generators only in dimensions 3 and 5.

- (ii) If X is not simply connected, then consider the universal covering space $\tilde{X}$ of X . Let $(\Omega X)_*$ denote the path connected component of ΩX containing the base point $*$ and let $p : \tilde{X} \rightarrow X$ be the covering projection. Since $\tilde{\Omega X}$ is path connected and $(\Omega X)_*$ is also path connected, the map $\Omega p : \tilde{\Omega X} \rightarrow (\Omega X)_*$ induces a one-one correspondence between the

path component of $\tilde{\Omega X}$ and that of $(\Omega X)_*$. Since $p_{\#} : \pi_1(\tilde{X}) \cong \pi_1(X)$ for $i \geq 2$ and $\pi_i(Y) \cong \pi_{i-1}(\Omega Y)$ for any Y , we have that $(\Omega p)_{\#} : \pi_{i-1}(\tilde{\Omega X}) \cong \pi_{i-1}(\Omega X) = \pi_{i-1}((\Omega X)_*)$. Thus $\tilde{\Omega X}$ has the weak homotopy type of $(\Omega X)_*$ and thus $\tilde{\Omega X}$ has the homotopy type of $(\Omega X)_*$, cf. [41; Chapter 7]. But $\tilde{X}$ is a simply connected finite H-complex of rank 1 or rank 2, so by part (i) above $\tilde{\Omega X}$ is torsion free. Thus $(\Omega X)_*$ is torsion free. The cohomology of ΩX is the direct sum of the cohomology of the path components of ΩX . Since all path components of ΩX have the homotopy type of $(\Omega X)_*$, it follows that ΩX is torsion free.

Corollary 5. If X is 1-connected and is of rank 2, then X has no higher p -torsion for any prime p .

Proof. By Theorem 4 above ΩX is torsion free. By Lemma 5.3 in Chapter 2, X has no higher p -torsion for any prime p .

Corollary 6. If X is of rank 2 and has no 2-torsion, then X has 3-torsion if and only if $H^2(X; \mathbb{Z})$ has an element of order 3.

Proof. If X has no 2-torsion, then the type of X is $(1,1), (1,3), (1,7), (3,3), (3,5), (3,7)$, or $(7,7)$. Suppose X has 3-torsion. Then $H^*(X; \mathbb{Q})$ has a generator in dimension $2m \cdot 3^k - 1$, where $0 < k < \infty$ and $2m$ is the smallest integer for which $H^{2m}(X; \mathbb{Z})$ has 3-torsion. But $2m$ cannot be greater than or equal to 4 because if $2m \geq 4$, then we have $2m \cdot 3^k - 1 \geq 11$, a contradiction. Thus $2m = 2$.

Remark. As in the remark following the proof of Theorem 1, $\text{PSU}(3)$ has no 2-torsion but has 3-torsion and $H^2(\text{PSU}(3); \mathbb{Z}) \cong \mathbb{Z}_3$.

Theorem 7. (i) If X has rank less than or equal to 5 and if X has no 2-torsion, then X has no p -torsion for $p \geq 7$.

(ii) If X is 1-connected, has rank less than 5, and has no 2-torsion, then X has no p -torsion for $p \geq 5$.

Proof.

(i) Let X be of rank less than or equal to 5 and have no 2-torsion. Then by Lemma 2.3 in Chapter II, the type of X is a union of sets taken from $(1), (3), (7), (3,5), (3,7), (3,5,7), (3,7,11), (3,5,7,9), (3,7,11,15), (3,5,7,9,11),$ or $(3,7,11,15,19).$

Suppose X has p -torsion for some $p \geq 7$. If s is the smallest integer for which $H^s(X;Z)$ has p -torsion, then by Lemma 3.2 in Chapter II, $s = 2m$ and, since $p \neq 2$, then $H_s(X;Z_p)$ has a primitive element. From Lemma 1.1 in Chapter II we have that $P(H_{2m}(X;Z_p)) \cong (Q(H^{2m}(X;Z_p)))^*$, since $H^*(X;Z_p)$ and $H_*(X;Z_p)$ are dual to each other. Thus $H^{2m}(X;Z_p)$ has an indecomposable element; hence, a generator. By Lemma 3.1 in Chapter II, $H^*(X;Q)$ has a generator in dimension $2mp^k - 1$, for some k , $0 < k < \infty$. Let $p=7$. If $2m=2$ and $k=1$, then $2mp^k - 1 = 2 \cdot 7 - 1 = 13$, a contradiction to the possible types of X . If

$2m=2$, $k \geq 2$, then $2mp^k-1 \geq 2 \cdot 7^2-1 = 97$, again a contradiction. If $2m=4$ and $k \geq 1$, then $2mp^k-1 \geq 4 \cdot 7-1 = 27$. Thus in any case X has no 7-torsion. Let $p \geq 11$. If $2m \geq 2$ and $k \geq 1$, then $2mp^k-1 \geq 2 \cdot 11-1 = 21$, which is a contradiction to the possible types of X . Thus X has no p -torsion for $p \geq 7$.

- (ii) Let X be of rank less than or equal to 4 and have no 2-torsion. Then the type of X is a union of sets taken from $(1), (3), (7), (3,5), (3,7), (3,5,7), (3,7,11), (3,5,7,9)$, or $(3,7,11,15)$. By the same argument used in (i) above, we have that $H^*(X;Q)$ has a generator in dimension $2mp^k-1$, where $0 < k < \infty$ and $2m$ is the smallest integer for which $H^{2m}(X;Z)$ has p -torsion for some $p \geq 5$. Since X is 1-connected, $\pi_1(X) = \pi_2(X) = 0$ by Lemma 4.3 in Chapter II; hence $H^1(X) = H^2(X) = 0$. Thus $2m \geq 4$, $k \geq 1$, and $p \geq 5$; so $2mp^k-1 \geq 4 \cdot 5-1 = 19$. This contradicts the possible types of X .

Remark (i). If X is a connected finite H-complex and if X is also homotopy commutative, then X has no 2-torsion [14; Corollary 8.7]. If X is a connected finite H-complex both homotopy commutative and homotopy associative, then X is torsion free [14; Theorem 8.10]. If G is a compact Lie group with trivial center, it follows that G does not admit any homotopy commutative and homotopy associative multiplication.

Remark (ii). For all the known compact connected Lie groups G we have that if G has torsion, then it has 2-torsion. On the other hand, there are finite H-complexes that have 3-torsion or 5-torsion but not 2-torsion [35; § 8]; however, their ranks are not less than or equal to 5.

Theorem 8. If X is 1-connected, associative, and has rank less than or equal to 5, then X has no p -torsion for $p \geq 7$; furthermore, ΩX has no p -torsion for $p \geq 7$.

Proof. Let X be a 1-connected, associative, finite H-complex of rank less than or equal to 5. Then by Lemma 2.5 in Chapter II the type of X is $(3), (3, 3), (3, 5), (3, 7), (3, 11), (3, 3, 3), (3, 3, 5), (3, 3, 7), (3, 5, 7), (3, 3, 11), (3, 7, 11), (3, 3, 3, 3), (3, 3, 3, 5), (3, 3, 5, 5), (3, 3, 3, 7), (3, 3, 5, 7),$

$(3,3,7,7), (3,5,7,9), (3,3,3,11), (3,3,5,11), (3,3,7,11),$
 $(3,7,7,11), (3,3,11,11), (3,7,11,15), (3,11,15,23), (3,3,3,3,3),$
 $(3,3,3,3,5), (3,3,3,5,5), (3,3,3,3,7), (3,3,3,5,7), (3,3,5,5,7),$
 $(3,3,3,7,7), (3,3,5,7,7), (3,3,5,7,9), (3,5,5,7,9), (3,3,3,3,11),$
 $(3,3,3,5,11), (3,3,3,7,11), (3,3,5,7,11), (3,3,7,7,11),$
 $(3,5,7,9,11), (3,3,3,11,11), (3,3,7,11,11), (3,3,7,11,15),$
 $(3,5,7,11,15), (3,7,9,11,15), (3,7,11,11,15), (3,7,11,15,19),$
 or $(3,3,11,15,23)$. By using the same argument as in (i) of Theorem 7 above we see that if X has p -torsion for $p \geq 7$ and if $2m$ is the smallest integer for which $H^{2m}(X; \mathbb{Z})$ has p -torsion, then $H^*(X; \mathbb{Q})$ has a generator in dimension $2mp^k - 1$ for some k , $0 < k < \infty$. Let $p=7$. If $2m=2$, $k=1$, then $2mp^k - 1 = 2 \cdot 7 - 1 = 13$, a contradiction to the possible types of X . If $2m=2$, $k \geq 2$, then $2mp^k - 1 \geq 2 \cdot 7^2 - 1 = 97$, again a contradiction. If $2m=4$ and $k \geq 1$, then $2mp^k - 1 \geq 4 \cdot 7 - 1 = 27$. Thus in any case X has no 7-torsion. Let $p=11$. If $2m=2$, $k=1$, then $2mp^k - 1 = 2 \cdot 11 - 1 = 21$, a contradiction. If $2m \geq 2$, $k \geq 2$, then $2mp^k - 1 \geq 2 \cdot 11 - 1 = 21$, again a contradiction. Let $p \geq 13$. If $2m \geq 1$, $k \geq 1$, then $2mp^k - 1 \geq 2 \cdot 13 - 1 = 25$. Thus X has no p -torsion for $p \geq 7$. The last statement of the theorem follows from Lemma 2.

Remark. Let X be 1-connected, have rank less than or equal to 4, and have no 2-torsion. If all generators in $H^*(X; \mathbb{Z}_p)$ for $p \geq 5$ have dimension 3, then $H_*(\Omega X; \mathbb{Z}_p)$ is primitively generated. This is a special case of Hubbuck's result [31; Corollary 1.3].

Proof. If X is 1-connected, has rank ≤ 4 , and has no 2-torsion, then by (ii) of Theorem 7 above X has no p -torsion for $p \geq 5$. Therefore by Lemma 3.3 in Chapter II

$$H^*(X; \mathbb{Z}_p) \cong \wedge(x_1, \dots, x_i), \quad 1 \leq i \leq 4, \quad p \geq 5,$$

an exterior algebra on odd dimensional generators. By hypothesis $\dim x_j = 3$ for $1 \leq j \leq i$. It follows that $H_*(\Omega X; \mathbb{Z}_p) \cong \mathbb{Z}_p[y_1, \dots, y_i]$, $1 \leq i \leq 4$, $p \geq 5$ and that $\dim y_j = 2$ for $1 \leq j \leq i$ by Lemma 5.1 in Chapter II. Since X is 1-connected, X is 2-connected by Lemma 4.3 in Chapter II. Thus ΩX is 1-connected because $\pi_1(\Omega X) = \pi_2(X) = 0$. It follows that $H_1(\Omega X; \mathbb{Z}_p) = 0$. Now, $\dim y_j = 2$ for $1 \leq j \leq i$ and $H_1(\Omega X; \mathbb{Z}_p) = 0$; so if Δ_* is the homomorphism induced by the diagonal map $\Delta : \Omega X \rightarrow \Omega X \times \Omega X$, then

$$\Delta_*(y_j) = y_j \otimes 1 + 1 \otimes y_j \in \sum_{i+j=2} H_i(\Omega X; \mathbb{Z}_p) \otimes H_j(\Omega X; \mathbb{Z}_p),$$

i.e., y_j is primitive. Therefore $H_*(\Omega X; \mathbb{Z}_p)$ is primitively generated.

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