ROTATORY INERTIA EFFECTS OF ATTACHED MASSES ON THE VIBRATION FREQUENCIES OF BEAMS AND PLATES

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ABSTRACT

ROTATORY INERTIA EFFECTS OF ATTACHED MASSES ON THE VIBRATION FREQUENCIES OF BEAMS AND PLATES

by Salil Kumar Das

This investigation concerns itself with the effect of rotatory inertia of attached masses on the vibration frequencies of beams and plates. The equations which are derived, are quite general and can be used for any number of attached masses. The usual assumptions of Hooke's Law, isotropy of material and small deflection theory are assumed in deriving the general equations.

In the case of plates, rotatory inertia and shear deformation of the plate are neglected, whereas, for the beam, only part of the shear is neglected. In the latter case, the resulting equation is compared with Timoshenko's reduced equation and is shown to give the same result.

Solutions were obtained with the help of digital computer and an accuracy of about five places was realized. The tables of values, given in this work, contain only the frequencies, the mode shapes being omitted because of prohibitive amount of space required to tabulate them.

As may be expected, the solutions obtained are approximate, rather than exact. In order to verify these results, a series of experiments were performed and the calculated frequencies compares with the measured ones. The agreement is very encouraging and seems to be quite adequate for most practical purposes.

In Chapter V, a method is developed, that can be used for many problems which involve concentrated masses. A few examples are worked out and results are compared with values from other chapters. The agreement seems to be quite good.

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By

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NOMENCLATURE

Plate:

D	Plate rigidity defined by $\frac{Eh^3(x, y)}{12(1-\nu^2)}$.
U	Work done. Subscripts with U refer to work done by particular type of forces.
(x, y)	Coordinates on plate surface.
E	Modulus of Elasticity.
$h(x, y)$	Thickness of plate at any point (x, y).
ν	Poisson's ratio.
$w(x, y, t)$	Deflection of plate in z direction.
t	Time.
ρ	Mass density of plate material, assumed to be constant.
q	Distributed load per unit area.
V	Potential energy.
T	Kinetic energy.
$W(x, y)$	Plate deflection defined by Equation (5).
p	Circular frequency, radians per sec.
G	Shear Modulus.
$\phi(t)$	Function of time defined by Equation (5).
M_k	Mass of load placed at x_k, y_k.
I_{kn}	Moment of inertia of a mass about n axis, placed at x_k, y_k; $n = x$ or y.
a	Length of the sides of a square plate.
α	Ratio of weight of attached mass to that of the plate.
β	A non-dimensional inertia parameter.

Beam:

E	Modulus of Elasticity.
U	Work done. Subscripts with U refer to work done by particular forces.
y	Deflection of the center line of beam.
L	Length of beam between supports.
x	Distance along center line of beam from left hand support.
ρ	Mass density of beam material, assumed to be constant.
I_b	Moment of inertia of beam sections about plane of bending.
X_m	Normal function of the corresponding uniform beam for the m^{th} mode.
A_n	Coefficients for series expansion of y.
p	Circular frequency, radians per sec.
V	Potential energy of the vibrating beam.
A	Beam cross section.
I_k	Mass moment of inertia of k^{th} mass about plane of bending.
k'	Shape factor for cross-section of beam.
M_k	Mass of k^{th} mass.
m_b	Mass of beam.
α_k	$\frac{M_k}{m_b}$, mass ratio.
β_k	Inertia parameter. This equals to $\frac{R_k}{L}$ for an attached mass in the shape of a disc whose radius is R_k. For other shapes of load, $\beta_k^2 = \frac{4I_k}{\alpha_k m_b L^2}$.
λ	$\frac{\pi^4 EI_b}{2p^2 m_b L^3}$; a frequency parameter.
$(k_n L)_{12}$	kL value for the n^{th} mode, obtained from a 12 terms expansion of y.
kL	$\sqrt[4]{\frac{m_b p^2 L^3}{EI_{bo}}}$; I_{bo} is moment of inertia of beam at $x = 0$.

CHAPTER I

INTRODUCTION

Inertia is an inherent property of matter in motion. When a body oscillates about a point or line, the inertia in question is called rotatory inertia. If a mass is attached to a beam or plate, it is well-known that the natural frequency of the system is reduced due to translatory inertia of the mass. If the rotatory inertia of the mass is also taken into account, keeping the mass constant, the frequency is further reduced. In this investigation, a study has been made to determine the effect of rotatory inertia of attached masses on the frequencies of vibration of beams and plates. In the general theory, rotatory inertia and shear deformation of the plate are neglected, but for the beam, only part of the shear deformation is neglected. The assumptions of Hooke's Law, isotropy of material and small deflection theory, are made in the derivation of the fundamental equations. These equations can be applied to any type of variable thickness beams or plates with an arbitrary number of masses attached to the systems at different points.

In order to investigate this effect of rotatory inertia, several approaches may be taken. In the case of a beam, having one or two masses, the classical approach may be used. For this case, each mass is replaced by the equivalent shear force and bending moment and these, in turn, are employed to give the required boundary conditions for the differential equations. The number of simultaneous differential equations that arise from this procedure is one more than the number of attached masses. As may be expected, this method becomes quite laborious when the number of attached masses is more than two, and also, it is difficult to generalize it to an arbitrary number of masses, because the boundary conditions that will have to be satisfied for each differential equation are influenced by the boundary conditions of the system, as well as the locations of the masses.

To avoid this difficulty, d'Alembert's principle together with the principle of virtual work is used to find the governing partial differential equation. This approach leads to a single differential equation.

Since the primary interest here is in free vibrations, a harmonic oscillation is assumed, with the result that the governing partial differential equation is reduced to an ordinary differential equation. The displacement function is then expanded in terms of the normal functions of the corresponding uniform beam. The number of terms taken in the expansion will depend on the accuracy required of the lower mode frequencies, as well as the number of modes under investigation. Taking a finite number of terms of this expansion, a system of linear algebraic equations are obtained. Finding a solution other than the trivial one demands the vanishing of the determinant of the coefficients of these equations and this gives the frequency equation. Actually, the set of algebraic equations generate a pair of symmetric matrices, the order of each of which is the same as the number of terms taken in the expansion of the displacement function. These, in turn, can be solved by the usual matrix methods. In the present investigation, the digital computer was used to solve these matrices. The eigenvalues of the matrices gave the frequency functions and the mode shapes were obtained from the eigenvectors.

For the case of a beam or plate without any attached masses, this method generates two symmetric matrices, one from the elasticity terms and the other from the translatory motion, rotatory inertia and shear deformation of the system. Once these matrices are known, the addition of masses to the system add certain terms on the latter matrix. As these terms are functions of the points of application of the loads, there are no integrals involved and as such the computation of these values is quite simple.

In Chapter V, an approximate method is derived for finding the frequencies of a system loaded with concentrated masses, when the unloaded frequencies of the original system are known. This method has been introduced previously by D. Young (11)* and applied for the fundamental mode only. The present method differs from that presented in (11) in the sense that there is no trial and error solution necessary and also it is applicable for higher modes. Some results by this method are compared with results from other chapters and agreement is found to be very good.

Historical background:

The earliest work done on beams with rotatory inertia of load apart from Rayleigh's work, seems to be that of R. M. Davies (1-4). In his papers, Davies considered a uniform cantilever beam with a load at the free end. The effects of shear deformation and rotatory inertia of the beam are included in (4). R. H. Scanlan (5, 6) introduced the effect of rotatory inertia of loads by the usual method for lumped systems, obtaining a matrix in terms of displacements and angles. The same work was further investigated by H. E. Fettis (7) who obtained a variation of about 46% on the second mode of a wing when rotatory inertia of the engine was included. R. F. S. Hearmon and E. H. Adams (8) employed Rayleigh's approximation to the case of a loaded vertical strip to include the effect of rotatory inertia of the load. References (9) through (17) deal with concentrated and distributed masses on beams,

*Numbers in parentheses refer to the Bibliography at the end.

neglecting rotatory inertia of the loads. So far, no general solution seems to have been presented for loaded beams and plates considering the rotatory inertia of the loads except in (5, 6). It is to be noted that in (5, 6) the matrix is of higher order due to the use of slope functions as separate unknowns. Also, it should be pointed out that the accuracy of the results in lumped mass systems depend on the manner in which the mass distribution is assumed, as shown by J. P. Ellington (29). This is not necessary in the present method.

Plate

Considerable work has been done on vibration of plates but very few publications were found on rotatory inertia effects of masses on plates. A brief bibliography regarding the effect of engine mass on wing vibration may be found on page 361 of (6). G. B. Warburton (18) gives a detail analysis of vibration of uniform plates together with a long bibliography and B. B. Raju (19) gives a fairly complete bibliography of important publications about variable thickness plates. R. E. Roberson in (20) and (21) analyses the vibration of uniform circular plates with the help of Dirac δ function to represent the attached mass at the center, the former one for a free plate and the latter for a clamped plate. W. F. Z. Lee and E. Saibel (12) consider the case of a simply supported circular plate with a mass at center and obtain the solution in terms of the normal functions of the plate which are Bessel functions. J. Hansen, E. Warlow-Davis and J. Taylor (28) illustrates an interesting way of analyzing experimentally the effects of engine mass on the flexural and torsional vibrations of a wing. This includes the effects of weight of the engine and also its rotatory inertia.

CHAPTER II

GENERAL THEORY

(a) Plate

From classical theory, for a uniform plate,

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = q(x, y, t) \tag{1}$$

If there is no external load $q = 0$ and Equation (1) reduces to

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{2}$$

The general expression of plate vibration, including rotatory inertia and shear deformation of the plate, is given by R. D. Mindlin (22) as

$$\left(\nabla^2 - \frac{\rho}{G} \frac{\partial^2}{\partial t^2}\right)\left(D \nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2}\right) w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{3}$$

This equation may, perhaps, be solved by a direct application of the Ritz method or the equivalent energy equation may be derived and solved. However, when the plate carries attached masses, it is convenient to use energy principles; otherwise, generalization to an arbitrary number of masses is not easily affected. Following R. D. Mindlin (22), the secondary effects due to rotatory inertia and shear deformation of the plate will be neglected in the following discussion.

From (23), the increment of potential energy of a plate element during vibration is given by

$$dV = \frac{Ez^2}{2(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dxdydz; \; -\frac{h}{2} \le z \le \frac{h}{2}$$

Integrating the above equation with respect to z and denoting $\frac{E\{h(x,y)\}^3}{12(1-\nu^2)}$ by D results in

$$V = \frac{1}{2}\left[\iint D\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\} dx\,dy\right] \tag{4}$$

Equation (4) is quite general when deflections are small and lateral dimensions of the plate are large compared to the thickness h(x, y).

Consider $$w = \phi(t)\,W(x,y) \tag{5}$$

Substituting in equation (4)

$$V = \frac{1}{2}\left[\iint D\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right\}\phi^2\,dx\,dy\right] \tag{6}$$

Considering harmonic oscillations, assume

$$\phi(t) = \text{Sin}\ pt \ \text{and}\ W(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m(x)\,Y_n(y) \tag{7}$$

where X and Y are the normal functions of the corresponding uniform beams in the respective directions and m and n are corresponding mode numbers. Substitution of Equation (7) into Equation (6) gives

$$V = \frac{1}{2}\left[\iint D\left\{\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2}Y_n\right)^2 + \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2}\right)^2 + 2\nu\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2}Y_n\right)\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2}\right) + 2(1-\nu)\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}\frac{dY_n}{dy}\right)^2\right\}\text{Sin}^2 pt\,dx\,dy\right] \tag{8}$$

If a virtual displacement is taken in the form

$$\delta W(x, y) = \delta A_{ij} X_i Y_j \tag{9}$$

the virtual work done by the elasticity forces becomes

$$\begin{aligned}
\delta U_e &= - \frac{\partial V}{\partial A_{ij}} \delta A_{ij} \\
&= - \Big[\iint D \Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2 X_m}{dx^2} Y_n \Big) \Big(\frac{d^2 X_i}{dx^2} Y_j \Big) \\
&\quad + \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m \frac{d^2 Y_n}{dy^2} \Big) \Big(X_i \frac{d^2 Y_j}{dy^2} \Big) \\
&\quad + \nu \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2 X_m}{dx^2} Y_n \Big) \Big(X_i \frac{d^2 Y_j}{dy^2} \Big) \\
&\quad + \nu \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m \frac{d^2 Y_n}{dy^2} \Big) \Big(\frac{d^2 X_i}{dx^2} Y_j \Big) \\
&\quad + 2(1-\nu) \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{dX_m}{dx} \frac{dY_n}{dy} \Big) \Big(\frac{dX_i}{dx} \frac{dY_j}{dy} \Big) \Big\} \operatorname{Sin}^2 pt \, dx \, dy \Big] \delta A_{ij}
\end{aligned} \tag{10}$$

The inertia force of an element dx dy of the plate is

$$- \rho h \frac{\partial^2 w}{\partial t^2} dx\, dy = [\rho p^2 h W(x, y) dx\, dy] \operatorname{Sin} pt.$$

Therefore the virtual work done by the inertia force of the entire plate is

$$\delta U_i = [\rho p^2 \iint h \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n \Big) X_i Y_j \, \delta A_{ij} dx\, dy] \operatorname{Sin}^2 pt \tag{11}$$

When the plate is free from attached masses, the total virtual work done by these forces equals zero, from which the natural frequency equation is found to be*

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*This same equation has been derived by Ritz method in Appendix B, for the purpose of verification.

$$\iint D \Big[\Big\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2X_m}{dx^2} Y_n \Big)\Big(\frac{d^2X_i}{dx^2} Y_j \Big) + \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m \frac{d^2Y_n}{dy^2} \Big)$$

$$\Big(X_i \frac{d^2Y_j}{dy^2} \Big) \Big\} + \nu \Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2X_m}{dx^2} Y_n \Big) \Big(X_i \frac{d^2Y_j}{dy^2} \Big)$$

$$+ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m \frac{d^2Y_n}{dy^2} \Big) \Big(\frac{d^2X_i}{dx^2} Y_j \Big) \Big\}$$

$$+ 2(1-\nu) \Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{dX_m}{dx} \frac{dY_n}{dy} \Big) \Big(\frac{dX_i}{dx} \frac{dY_j}{dy} \Big) \Big\} \Big] \, dx \, dy$$

$$= p^2 \rho \iint h \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n \Big) (X_i Y_j) \, dx \, dy \qquad (12)$$

It may be mentioned that when the plate is of uniform thickness, D and h are constants and as such, Equation (12) can be evaluated quite easily. However, it is preferred to leave the equation in its present form to achieve conditions of generality in subsequent developments.

Let there be k number of masses attached to the plate at points (x_1, y_1), (x_2, y_2), - - - - - , (x_k, y_k), having masses M_1, M_2, - - - -, M_k and moments of inertia (I_{1x}, I_{1y}), (I_{2x}, I_{2y}), - - - - -, (I_{kx}, I_{ky}) where the first subscript k denotes the position of the masses and the second subscript x or y denotes the axis about which the moments of inertia are calculated. The total virtual work done by the masses, due to translatory motion, is given by

$$\delta U_t = - \sum_{k=1}^{k} \Big[M_k \frac{\partial^2 w}{\partial t^2} \; \delta w \Big]_{x=x_k,\, y=y_k}$$

$$= p^2 \sum_{k=1}^{k} \Big\{ \Big[M_k \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n \Big] [X_i Y_j] \; \delta A_{ij} \sin^2 pt \Big\}_{x=x_k,\, y=y_k} \qquad (13)$$

To include the effect of rotatory inertia, it should be noted that the masses will have components of rotation about both x and y directions and the net effect will be rotation about some intermediate axis. It is shown in

Appendix A that the virtual work done by two torques in the x and y directions, on two virtual angles $\delta(\frac{\partial w}{\partial y})$ and $\delta(\frac{\partial w}{\partial x})$ respectively, is $T_x\delta(\frac{\partial w}{\partial y}) - T_y\delta(\frac{\partial w}{\partial x})$. For k number of masses, the virtual work due to rotatory inertia of the masses is given by

$$\begin{aligned}
\delta U_r &= \sum_{k=1}^{k} [T_{kx}\,\delta(\frac{\partial w}{\partial y}) - T_{ky}\delta(\frac{\partial w}{\partial x})] \\
&= -\sum_{k=1}^{k}[I_{kx}\frac{\partial^2}{\partial t^2}(\frac{\partial w}{\partial y})\,\delta\,(\frac{\partial w}{\partial y}) + I_{ky}\frac{\partial^2}{\partial t^2}(\frac{\partial w}{\partial x})\,\delta\,(\frac{\partial w}{\partial x})]_{x=x_k, y=y_k} \\
&= p^2\sum_{k=1}^{k}[(I_{kx}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{dY_n}{dy})\,(X_i\,\frac{dY_j}{dy})\,\delta\,A_{ij} \\
&\quad + (I_{ky}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}\,Y_n)\,(\frac{dX_i}{dx}\,Y_j)\,\delta\,A_{ij}]\,\mathrm{Sin}^2\,pt_{x=x_k, y=y_k}
\end{aligned} \tag{14}$$

Adding all the virtual works from Equations (10), (11), (13) and (14), and equating them to zero, the final form of the frequency equation is found to be

$$\begin{aligned}
&\frac{E}{12(1-\nu^2)}\iint h^3[(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2}Y_n)\,(\frac{d^2X_i}{dx^2}\,Y_j) \\
&\qquad + (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2})\,(X_i\,\frac{d^2Y_j}{dy^2}) \\
&\quad + \nu\,\{(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2}\,Y_n)\,(X_i\,\frac{d^2Y_j}{dy^2}) \\
&\qquad + (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2})\,(\frac{d^2X_i}{dx^2}\,Y_j)\} \\
&\quad +2(1-\nu)(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}\,\frac{dY_n}{dy})\,(\frac{dX_i}{dx}\,\frac{dY_j}{dy})]\,dx\,dy \\
&\quad = p^2[\rho\iint h\,(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_mY_n)\,(X_iY_j)\,dx\,dy \\
&\qquad + \sum_{k=1}^{k}\{M_k(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_mY_n)(X_iY_j)\}_{x=x_k, y=y_k} \\
&\qquad + \sum_{k=1}^{k}\{I_{kx}(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{dY_n}{dy})(X_i\,\frac{dY_j}{dy}) \\
&\qquad + I_{ky}\,(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}\,Y_n)\,(\frac{dX_i}{dx}\,Y_j)\}_{x=x_k, y=y_k}]
\end{aligned} \tag{15}$$

where D has been replaced by $\frac{Eh^3}{12(1-\nu^2)}$. This is the general frequency equation for a variable section plate with any number of masses attached to the plate at arbitrary points.

(b) Beam

In this part, as with the plate, the usual assumptions are made regarding Hooke's Law, isotropy and small deflection theory. The general equation contains the effects due to rotatory inertia of the beam and also some part due to shear deformation. These secondary effects are included as in Timoshenko beam theory excepting a few second order terms including the fourth-order time derivative function which are neglected. The validity of the resulting equation, for a uniform beam, is compared in Appendix D with that of (23) and they are found to be exactly the same. The strain energy of bending of the bar at any instant is

$$V = \frac{E}{2} \int_0^L I_b(x)\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \tag{16}$$

Let $y = \phi(t)\ Y(x)$

Considering harmonic oscillations, assume

$$\phi(t) = \text{Sin } pt \quad \text{and} \quad Y(x) = \sum_{n=1}^{\infty} A_n X_n$$

where X_n are the normal functions of the corresponding uniform beam. Thus

$$y = \left(\sum_{n=1}^{\infty} A_n X_n\right) \text{Sin } pt \tag{17}$$

Substitution of Equation (17) into Equation (16) gives

$$V = \frac{E}{2} \left\{ \int_0^L I_b\left(\sum_{n=1}^{\infty} A_n \frac{d^2 X_n}{dx^2}\right)^2 dx \right\} \text{Sin}^2\, pt$$

Taking a variation δA_i in one of the coefficients A_i of y, the virtual work done by elasticity forces during the virtual displacement is given by

$$\delta U_e = -\frac{\partial V}{\partial A_i}\,\delta A_i = -E\left\{\int_0^L I_b\left(\sum_{n=1}^{\infty} A_n \frac{d^2X_n}{dx^2}\right)\frac{d^2X_i}{dx^2}\,dx\right\}\mathrm{Sin}^2\,pt\;\delta A_i \qquad (18)$$

The inertia force of an element dx of the beam at any instant is $-\rho A\frac{\partial^2 y}{\partial t^2}\,dx$. So the total work done by the inertia force on a virtual displacement δy (= $X_i\,\delta A_i$ Sin pt) is

$$\delta U_i = -\int_0^L \left(\rho A\frac{\partial^2 y}{\partial t^2}\,dx\right)(X_i\,\delta A_i)\,\mathrm{Sin}\,pt$$

$$= \rho\,p^2\left\{\int_0^L \left(A\sum_{n=1}^{\infty} A_nX_n\right)X_i\,dx\right\}\mathrm{Sin}^2\,pt\;\delta A_i \qquad (19)$$

From Equations (18) and (19), the frequency equation for a variable thickness beam, neglecting rotatory inertia and shear deformation of the beam is given by

$$p^2\,\rho\int_0^L A\left(\sum_{n=1}^{\infty} A_nX_n\right)X_i\,dx = E\int_0^L I_b\left(\sum_{n=1}^{\infty} A_n\frac{d^2X_n}{dx^2}\right)\frac{d^2X_i}{dx^2}\,dx \qquad (20)$$

Let there be k number of masses attached to the beam at points x_1, x_2, - - - -, x_k, having masses M_1, M_2 - - - -, M_k and moments of inertia I_1, I_2, - - - I_k. The virtual work done by the masses during translatory motion is

$$\delta U_t = -\sum_{k=1}^{k}\left[M_k\frac{\partial^2 y}{\partial t^2}(\delta y)\right]_{x=x_k}$$

$$= p^2\sum_{k=1}^{k}\left[M_k\left(\sum_{n=1}^{\infty} A_nX_n\right)X_i\right]_{x=x_k}\mathrm{Sin}^2\,pt\;\delta A_i \qquad (21)$$

The virtual work done by the rotatory inertia forces of the masses is *

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*See Appendix C

$$\delta U_r = - \sum_{k=1}^{k} [(I_k \frac{\partial^3 y}{\partial x \partial t^2}) \; \delta \; (\frac{\partial y}{\partial x})]_{x=x_k}$$

$$= [p^2 \sum_{k=1}^{k} \{ \; I_k \, (\sum_{n=1} \; A_n \frac{dX_n}{dx}) \frac{dX_i}{dx} \}_{x=x_k} \;] \; \text{Sin}^2 \, pt \;\; \delta A_i \quad (22)$$

Rotatory inertia torque of an element dx of the beam is*

$$- \rho \; I_b \frac{\partial^2 \theta}{\partial t^2} \, dx = - \rho \; I_b \frac{\partial^3 y}{\partial x \partial t^2} \, dx \tag{23}$$

Therefore, the virtual work done is

$$\delta U_{ib} = - \rho \int_o^L \; I_b \frac{\partial^3 y}{\partial x \partial t^2} \;\; \delta \; (\frac{\partial y}{\partial x}) \, dx$$

$$= p^2 \rho \, [\int_o^L I_b (\sum_{n=1}^{\infty} A_n \frac{dX_n}{dx}) \frac{dX_i}{dx} \;\; dx \,] \, \text{Sin}^2 \, pt \;\; \delta A_i \tag{24}$$

Shear torque in an element dx of the beam* is

$$- \; \frac{E}{k'G} \frac{\partial}{\partial x} (I_b \; \frac{\partial^2 y}{\partial t^2}) \, dx$$

So the virtual work done is

$$\delta U_{sb} = - \rho \; \frac{E}{k'G} \int_o^L \frac{\partial}{\partial x} (I_b \frac{\partial^2 y}{\partial t^2}) \;\; \delta \; (\frac{\partial y}{\partial x}) \;\; dx$$

$$= p^2 \frac{\rho E}{k'G} [\int_o^L \frac{\partial}{\partial x} \; (I_b \; \sum_{n=1}^{\infty} A_n X_n) \frac{dX_i}{dx} \, dx] \, \text{Sin}^2 \, pt \;\; \delta \; A_i \quad (25)$$

Adding all the virtual work from Equations (18), (19), (21), (22), (24) and (25) and equating it to zero, the frequency equation is obtained as

$$p^2 [\rho \int_o^L A (\sum_{n=1}^{\infty} A_n X_n) X_i dx + \sum_{k=1}^{k} \{ \; M_k (\sum_{n=1}^{\infty} A_n X_n) X_i \; \}_{x=x_k}$$

$$+ \sum_{k=1}^{k} \{ \, I_k \, (\sum_{n=1}^{\infty} \; A_n \frac{dX_n}{dx}) \; \frac{dX_i}{dx} \; \}_{x=x_k} + \rho \int_o^L I_b (\sum_{n=1}^{\infty} A_n \frac{dX_n}{dx}) \frac{dX_i}{dx} \, dx$$

$$+ \rho \frac{E}{k'G} \; \int_o^L \frac{\partial}{\partial x} (I_b \sum_{n=1}^{\infty} A_n X_n) \, \frac{dX_i}{dx} \; dx]$$

$$= E \quad \int_o^L I_b (\sum_{n=1}^{\infty} A_n \frac{d^2 X_n}{dx^2}) \frac{d^2 X_i}{dx^2} \, dx \tag{26}$$

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*See Appendix D

This is the general frequency equation for a variable thickness beam with k number of masses attached to it at arbitrary points. It is used in its complete form to evaluate the frequencies of a beam in Chapter III and verified by experiments in Chapter IV.

(c) Uniform beam

When the beam is of uniform cross section, Equation (26) reduces to

$$p^2\Big[\rho A\int_0^L \sum_{n=1}^{\infty} A_nX_n)X_i dx + \sum_{k=1}^{k}\{ M_k(\sum_{n=1}^{\infty} A_nX_n)X_i\}_{x=x_k} + \sum_{k=1}^{k} \{I_k(\sum_{n=1}^{\infty} A_n\frac{dX_n}{dx})\frac{dX_i}{dx}\}_{x=x_k}$$

$$+ \rho\, I_b \int_0^L(\sum_{n=1}^{\infty} A_n \frac{dX_n}{dx})\frac{dX_i}{dx}\, dx + \rho\, I_b \frac{E}{k'G}\int_0^L(\sum_{n=1}^{\infty} A_n\frac{dX_n}{dx})\frac{dX_i}{dx}\, dx\Big]$$

$$= E\, I_b \left\{ \int_0^L(\sum_{n=1}^{\infty} A_n \frac{d^2X_n}{dx^2})\frac{d^2X_i}{dx^2}\, dx \right\} \qquad (27)$$

When the rotatory inertia and shear deformation of the beam are neglected, Equation(27) reduces to

$$p^2\Big[\rho A\int_0^L (\sum_{n=1}^{\infty} A_nX_n)X_i dx + \sum_{k=1}^{k}\{ M_k(\sum_{n=1}^{\infty} A_nX_n)X_i\}_{x=x_k} + \sum_{k=1}^{k}\{ I_k (\sum_{n=1}^{\infty} A_n\frac{dX_n}{dx})\frac{dX_i}{dx}\}_{x=x_k}\Big]$$

$$= E\, I_b \int_0^L (\sum_{n=1}^{\infty} A_n \frac{d^2X_n}{dx^2})\frac{d^2X_i}{dx^2}\, dx \qquad (28)$$

Equation (28) is used in Chapter III to find the frequencies for three cases of a simply supported beam. The frequency values are tabulated and plotted in terms of a reference parameter. Actually, the expansion of the series in Equation (28) was carried up to 12^{th} modes, even though the values given in the tables are up to 6^{th} modes. This was done with a view to improve the accuracy of the lower modes. The secondary effects of rotatory inertia and shear deformation of the beam are expected to be small because of the length of the beam.

Due to orthogonal property of the normal modes, Equations (27) and (28) will simplify considerably, but that aspect is shown in each case separately.

Also it may be mentioned that Equation (28) can be derived directly from Equation (15) by assuming $Y_n = Y_j = 1$, $\nu = 0$, $I_{kx} = 0$, and $I_{ky} = I_k$.

CHAPTER III

NUMERICAL EXAMPLES

(a) Plate

For this part, a square plate was used, as in Figure 1. The thickness at the middle was 0.125" which decreased gradually to 0.0625" at the edge. This was the same plate as model E of B. B. Raju (19).

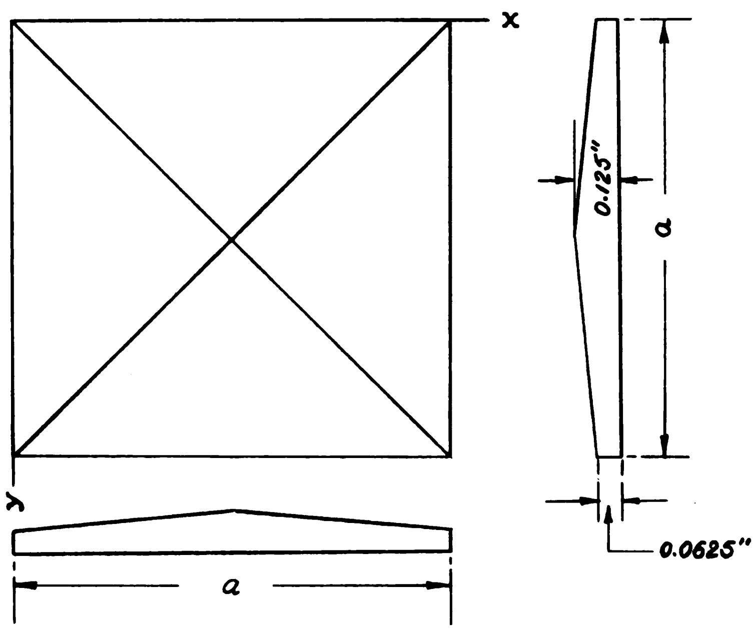


Figure 1. Variable thickness plate.

To facilitate computation, this part of the analysis is divided into two sections. The first section contains the evaluation of the natural frequencies of the plate and section 2 contains that of the plate with mass attached. It may be mentioned that in (19), finite difference method was used to find the natural frequencies of the plate.

(1) Without attached mass. To find the natural frequencies, Equation (12) was used. The plate was assumed to be simply supported. Taking only one term of the series gives,

$$\iint D\Big[\Big\{ \Big(A_{11} \frac{d^2X_1}{dx^2} Y_1\Big) \Big(\frac{d^2X_1}{dx^2} Y_1\Big) + \Big(A_{11}X_1 \frac{d^2Y_1}{dy^2}\Big) \Big(X_1 \frac{d^2Y_1}{dy^2}\Big) \Big\}$$

$$+ \nu \Big\{ \Big(A_{11} \frac{d^2X_1}{dx^2} Y_1\Big) \Big(X_1 \frac{d^2Y_1}{dy^2}\Big) + \Big(A_{11}X_1 \frac{d^2Y_1}{dy^2}\Big) \Big(\frac{d^2X_1}{dx^2} Y_1\Big) \Big\}$$

$$+ 2(1-\nu) \Big\{ \Big(A_{11} \frac{dX_1}{dx} \frac{dY_1}{dy}\Big) \Big(\frac{dX_1}{dx} \frac{dY_1}{dy}\Big) \Big\} \Big] dx\, dy$$

$$= p^2 \rho \iint h \,(A_{11}X_1Y_1)(X_1Y_1)\, dx\, dy \qquad (29)$$

For this case $X_1 = \text{Sin}\,\frac{\pi x}{a}$, $Y_1 = \text{Sin}\,\frac{\pi y}{a}$

Substitution of these values in Equation (29) results in

$$\frac{\pi^4 E}{12a^4(1-\nu^2)} \Big[\iint h^3 \Big\{ 2\,\text{Sin}^2 \frac{\pi x}{a}\, \text{Sin}^2 \frac{\pi y}{a} + 2\,\nu \Big(\text{Sin}^2 \frac{\pi x}{a}\, \text{Sin}^2 \frac{\pi y}{a}\Big)$$

$$+2(1-\nu)\, \text{Cos}^2 \frac{\pi x}{a}\, \text{Cos}^2 \frac{\pi y}{a} \Big\}\, dx\, dy$$

$$= p^2 \rho \Big\{ \iint h\, \text{Sin}^2 \frac{\pi x}{a}\, \text{Sin}^2 \frac{\pi y}{a} \Big\}\, dx\, dy \qquad (30)$$

Let h_0 be the thickness at the center of the plate and assume $\frac{h}{h_0} = \gamma$. Then γ has the following relations:

If $x \le y \le a-x$ $\qquad \gamma(x, y) = \left(\frac{1}{2} + \frac{x}{a}\right)$

If $y \le x \le a-y$ $\qquad \gamma(x, y) = \left(\frac{1}{2} + \frac{y}{a}\right)$

$$0.5 \le \gamma \le 1.0$$

The integrals in Equation (30), as well as others to come, were solved in the digital computer with program EAI-M with 48 divisions between the limits. With this program, an accuracy up to about 7 places has been realized. This program uses quadrature formula (25) Q_{66}, obtained from a $6^{\underline{th}}$ degree polynomial that fits the f(x) values at the seven points indicated, by integrating over the six panels, between x_0 and x_6. Actually, 49 points were used between limits, whereby eight cycles were necessary to cover the complete range. The following values are shown, as representative examples, that were used for Equation (30).

$$\int_0^1 \text{Sin}^2 \frac{\pi y}{a} \int_0^1 \gamma^3 \text{Sin}^2 \frac{\pi x}{a} \, d(\frac{x}{a}) d(\frac{y}{a}) = 0.1304990400$$

$$\int_0^1 \text{Cos}^2 \frac{\pi y}{a} \int_0^1 \gamma^3 \text{Cos}^2 \frac{\pi x}{a} \, d(\frac{x}{a}) d(\frac{y}{a}) = 0.053923941 \qquad (31)$$

$$\int_0^1 \text{Sin}^2 \frac{\pi y}{a} \int_0^1 \gamma \, \text{Sin}^2 \frac{\pi x}{a} \, d(\frac{x}{a}) d(\frac{y}{a}) = 0.198397736$$

In (19), the frequency p is expressed in terms of a reference parameter

$p_0 = \sqrt{\frac{D_0}{\rho h_0 a^4}}$ where

$$D_0 = \frac{E h_0^3}{12(1-\nu^2)}$$

With the help of Equation (31) and taking $\nu = \frac{1}{3}$ the first approximation to the fundamental frequency is found to be

$$p^2/p_0^2 = 2.1164356\pi^4$$

$$\text{or} \quad p/p_0 = 14.3582752$$

The experimental value for this case is 13.78, a variation of only 4.19 p.c. Next, nine terms of the series in Equation (12) were taken for i = 1, 2, 3 and j = 1, 2, 3. To get the non-trivial solution, the determinant of the coefficients must vanish and this gives a determinant of the form $|[A] - \lambda\ [B]| = 0$ where $\lambda = \frac{\pi^4 p_0^2}{p^2}$, [A] is the matrix of the elements on the right hand side of Equation (12) and [B] is the matrix of the remaining part. The two matrices are shown in Table 2. Table 1 contains the frequency values obtained through M-5 program of MISTIC.

Table 1. Natural frequencies of plate.

Mode	p/p_0 from 6 terms	p/p_0 from 9 terms	Extrapolated p/p_0	p/p_0 from (19)
1	13.794	13.738	13.693	13.568
2	35.065	34.731	34.464	33.195
3	35.065	34.731	34.464	33.195
4	55.234	55.234	55.234	52.279
5	73.935	71.665	69.849	66.121
6	70.479	70.479	70.479	68.001
7		88.909		83.339
8		88.909		
9		124.850		

(2) With attached mass. For this part, a mass was assumed to be located at $x = (2/3)a$ and $y = (2/3)a$. This point was chosen with a view to get higher rotatory inertia effects for all modes under consideration. Equation (15) is the governing equation. Assuming the load to be fixed perpendicular to the neutral surface of the plate, $I_x = I_y$, the rotatory inertias about x and y axes.

Let $I_x = I_y = \dfrac{\alpha m_p \beta^2 a^2}{4}$ where $\alpha = \dfrac{M}{m_p}$, β is a non-dimensional inertia parameter*, m_p is the mass of the plate and M is the total attached mass. In this case $m_p = \dfrac{2t_0 a^2}{3} \rho$.

*This notation came from the beam analysis, where $\beta = R/L$, R being the radius of a disc fitted on the beam and L is the length of the beam.

Table 2. Matrices for natural frequencies of plate.

[A]

.1983977	0	0	0	-.0158542	-.0158542	0	0	.0079615
0	.1825435	0	0	0	0	0	-.0078927	0
0	0	.1825435	0	0	0	-.0078927	0	0
0	0	0	.1746508	0	0	0	0	0
-.0158542	0	0	0	.1807845	.0079615	0	0	-.0094989
-.0158542	0	0	0	.0079615	.1807845	0	0	-.0094989
0	0	-.0078927	0	0	0	.1712856	0	0
0	-.0078927	0	0	0	0	0	.1712856	0
.0079615	0	0	0	-.0094989	-.0094989	0	0	.1702535

[B]

.4198960	0	0	0	-.4370515	-.4370515	0	0	.5234754
0	2.3041048	0	0	0	0	0	-.8143869	0
0	0	2.3041049	0	0	0	-.8143869	0	0
0	0	0	5.4698681	0	0	0	0	0
-.4370515	0	0	0	9.6833840	.8705078	0	0	-2.8962648
-.4370515	0	0	0	.8705078	9.6833840	0	0	-2.8962648
0	0	-.8143869	0	0	0	13.8975675	0	0
0	-.8143869	0	0	0	0	0	13.8975675	0
.5234754	0	0	0	-2.8962648	-2.8962648	0	0	27.0160076

Substitution of the above values in Equation (15) gives

$$\frac{Eh_0^2}{12\,\rho p^2(1-\nu^2)}\ [\int_0^1\int_0^1 \gamma^3 \{ (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2} Y_n)(\frac{d^2X_i}{dx^2}Y_j)$$

$$+ (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2})(X_i\frac{d^2Y_j}{dy^2})$$

$$+ \nu(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{d^2X_m}{dx^2}Y_n)(X_i\frac{d^2Y_j}{dy^2}) + \nu(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{d^2Y_n}{dy^2})(\frac{d^2X_i}{dx^2}Y_j)$$

$$+2(1-\nu)(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}\frac{dY_n}{dy})(\frac{dX_i}{dx}\frac{dY_j}{dy})\}\ d(\frac{x}{a})d(\frac{y}{a})$$

$$= [\int_0^1\int_0^1 \gamma (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_mY_n)(X_iY_j)d(\frac{x}{a})d(\frac{y}{a})$$

$$+\frac{2}{3}\{ \alpha (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_mY_n)(X_iY_j)$$

$$+\frac{\alpha\beta^2a^2}{4}(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_m\frac{dY_n}{dy})(X_i\frac{dY_j}{dy})$$

$$+\frac{\alpha\beta^2a^2}{4}(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{dX_m}{dx}Y_n)(\frac{dX_i}{dx}Y_j)\}_{x=\frac{2a}{3},\,y=\frac{2a}{3}} \qquad (32)$$

Assuming $X_m = \text{Sin}\,\frac{m\pi x}{a}$, $Y_n = \text{Sin}\,\frac{n\pi y}{a}$, expanding the series to any number of terms and equating the determinant of the coefficients to zero, a matrix of the form $|[A] - \lambda\ [B]| = 0$ is obtained. This, in turn, was solved by M-5 program of MISTIC. For this case, a 9 terms expansion was used. It is to be noted that the 9 term expansion is quite small for a problem of this type. But as each integral of Equation (32) used to take a long time (about $4\frac{1}{2}$ minutes of machine time) it was decided to be satisfied with 9x9 only. An extrapolation (h^2) is used to improve the accuracy of the results.

In Equation (32), the first term of the right hand side and the complete left hand side has already been evaluated in section 1. To find the other values, the following values of the parameters α and β were used.

α = 1.0 and 2.5

β = 0.1, 0.2, 0.3 and 0.4

Tables 3 and 4 contain the p/p_0 values, some of which are compared with the experimental values in Table 16.

(b) Cantilever beam

In this part of the investigation, a cantilever beam, as in Figure 2, was used. The complete analysis is divided into three sections. Sections (1) and (2) contain the evaluation of the natural frequencies of the beam, without and with corrections for rotatory inertia and shear deformation of the beam respectively. To get reasonable variations due to these correction terms, the length of the beam was purposely made

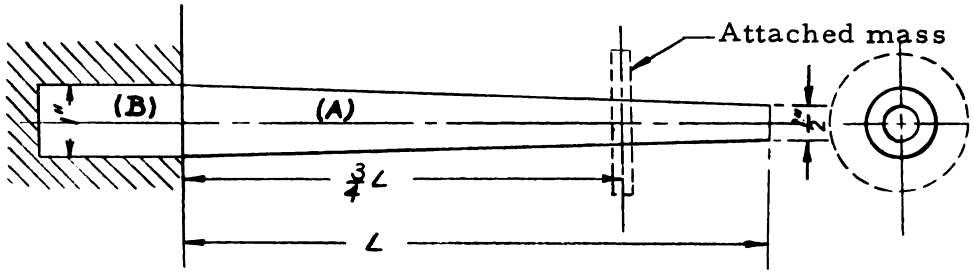


Figure 2. Variable thickness cantilever beam.

short (20"). The results are shown in Table 5. In section (3) a mass is assumed to be attached at a point three-fourths the length of the beam away from the fixed end. Tables 6, 7 and 8 show the values of the frequencies for different values of α and β. In Figure 3 the above values are plotted to show the effect of β on the frequencies. Numbers on the right give the corresponding mode of vibration.

(1) Natural frequency without rotatory inertia and shear deformation. For this case, Equation (26) reduces to

Table 3. Frequencies of plate with attached mass.

	Mode	6 terms expansion	9 terms expansion	Extra-polated
$\alpha = 1.0$, $\beta = 0.1$	1	7.39	7.35	7.32
	2	23.08	22.68	22.36
	3	32.25	32.25	32.25
	4	47.55	46.55	45.75
	5	61.98	53.20	46.18
$\alpha = 1.0$, $\beta = 0.2$	1	7.36	7.33	7.31
	2	21.08	19.88	18.92
	3	25.00	24.27	23.69
	4	38.44	34.99	32.23
	5	48.38	42.56	37.90
$\alpha = 1.0$, $\beta = 0.3$	1	7.32	7.29	7.27
	2	17.96	16.03	14.49
	3	18.85	17.69	16.76
	4	33.25	30.79	28.82
	5	45.95	41.30	37.58
$\alpha = 1.0$, $\beta = 0.4$	1	7.25	7.22	7.20
	2	14.83	12.97	11.48
	3	15.03	13.67	12.58
	4	31.26	29.43	27.97
	5	45.15	40.76	37.25

Table 4. Frequencies of plate with attached mass

	Mode	6 terms $\exp \frac{n}{}$	9 terms $\exp \frac{n}{}$	Extra polated
$\alpha = 2.5,\ \beta = 0.1$	1	5.05	5.02	5.00
	2	21.28	20.58	20.02
	3	28.20	27.83	27.53
	4	42.09	38.90	36.35
	5	50.74	45.39	41.11
$\alpha = 2.5,\ \beta = 0.2$	1	5.03	5.00	4.98
	2	16.82	15.03	13.60
	3	18.09	16.92	15.98
	4	32.59	30.06	28.04
	5	45.82	41.23	37.56
$\alpha = 2.5,\ \beta = 0.3$	1	5.00	4.98	4.96
	2	12.70	10.87	9.41
	3	12.72	11.67	10.83
	4	30.12	28.43	27.08
	5	44.80	40.47	37.01
$\alpha = 2.5,\ \beta = 0.4$	1	4.95	4.93	4.91
	2	9.38	8.44	7.69
	3	10.04	8.84	7.88
	4	29.39	27.93	26.76
	5	44.56	40.33	36.95

$$p^2 \rho \int_o^L A \left(\sum_{n=1}^{\infty} A_n X_n\right) X_i dx = E \int_o^L I_b \left(\sum_{n=1}^{\infty} A_n \frac{d^2X_n}{dx^2}\right) \frac{d^2X_i}{dx^2}\, dx \qquad (33)$$

Here,

$$D = \tfrac{1}{2}\left(2 - \frac{x}{L}\right),\ A = \frac{\pi}{16}\left(2 - \frac{x}{L}\right)^2,\ I_b = \frac{\pi}{1024}\left(2 - \frac{x}{L}\right)^4$$

Now let

$$X_i = \phi_i,\ \frac{d^pX_i}{dx^p} = k_i^p \frac{d^p}{d(k_ix)^p}(\phi_i),\ \frac{x}{L} = z \text{ and } \lambda_v = \frac{E}{64\,\rho p^2 L^4}$$

With these notations, Equation (33) becomes

$$\int_o^1 (2-z)^2 \left(\sum_{n=1}^{\infty} A_n\, \phi_n\right) \phi_i dz = \lambda_v \int_o^1 (2-z)^4 \left\{ \sum_{n=1}^{\infty} A_n (k_nL)^2\, \phi_n''\right\} (k_iL)^2 \phi_i'' dz \qquad (34)$$

The primes on the ϕs indicate differentiation with respect to $(k_n x)$. Expanding the above series, any number of modes may be evaluated. For convenience, only nine terms were taken. It is interesting to see the effect on convergence of the number of terms taken of the series in Equation (34). Taking only one term gives

$$\int_o^1 (2-z)^2 \phi_1^2 dz - \lambda_v (k_1L)^4 \int_o^1 (2-z)^4 (\phi_1'')^2 dz = 0$$

Here $(k_1L)^4 = 12.3623643$

$$(k_1L)^4 \int_o^1 (2-z)^4 (\phi_1'')^2 dz = 137.188301$$

and $\int_o^1 (2-z)^2 \phi_1^2 dz = 1.44843126$

from which $\lambda_v = 1.05579794 \times 10^{-2}$

and $p_1 = 1.21652099 \sqrt{\dfrac{E}{\rho L^4}}$

This is a variation of 5.2 p.c from the result obtained from nine terms expansion. By taking two terms of the series in Equation (34), the result was $p_1 = 1.1574033 \sqrt{\frac{E}{\rho L^4}}$ and the variation is reduced to 0.09 p.c. In Table 5 the frequency values are given in terms of a reference paramater $p_r = 0.87900 \sqrt{\frac{E}{\rho L^4}}$, which is the fundamental frequency of a one inch diameter uniform cantilever beam.

(2) Natural frequency with rotatory inertia and shear deformation. In this part,it was assumed that $\frac{E}{G} = \frac{8}{3}$ and $k' = 0.847$, as given in (27). Values of k' for other cross sections may be obtained from the same paper. With these, Equation (26) reduces to

$$\int_o^1 (2-z)^2 (\sum_{n=1}^{\infty} A_n \phi_n)\phi_i dz + \frac{0.06481823101}{L^2} \int_o^1 (2-z)^4 \{ \sum_{n=1}^{\infty} A_n (k_n L)\phi_n' \} (k_i L)\phi_i dz$$

$$- \frac{0.196772924}{L^2} \int_o^1 (2-z)^3 (\sum_{m=1}^{\infty} A_n \phi_n)(k_i L)\phi_i' dz$$

$$= \lambda_v \int_o^1 (2-z)^4 \{ \sum_{n=1}^{\infty} A_n (k_n L)^2 \phi_n'' \} (k_i L)^2 \phi_i'' dz \qquad (35)$$

From Equation (35) it is clear that the effect of rotatory inertia and shear deformation is dependent on the length of the beam, as is well-known. In case of simple supports, $k_n L = n\pi$ and as such, the length term in the denominator of the second term on the left is replaced by the corresponding wave length term, as in (23).

The second and third terms on the left are the only additional ones and substituting these terms, the following values of p/p_r, as given in Table 5, were obtained.

(3) Frequency with mass, rotatory inertia and shear deformation. In this section, the same beam as in Figure 2 is used with a mass at $x = \frac{3L}{4}$ (arbitrary). For this case Equation (26) becomes

Table 5. **Natural frequencies of cantilever beam.**

Mode	p/p_r Without Correction			p/p_r With Correction			
	p/p_r from 6 terms	p/p_r from 9 terms	Extra-polated	p/p_r from 6 terms	p/p_r from 9 terms	Extra-polated	Variation in p.c
1	1.31549	1.31548	1.31547	1.31508	1.31507	1.31506	0.031
2	5.55997	5.55930	5.55876	5.53911	5.53841	5.53785	0.378
3	13.82067	13.81150	13.80416	13.67616	13.66585	13.65760	1.073
4	26.20372	26.08430	25.98876	25.67889	25.54609	25.43985	2.158
5	43.19072	42.32409	41.63079	41.77829	40.93599	40.26215	3.399
6	70.89293	62.63953	56.03681	67.33397	59.69240	53.57914	4.587

$$p^2[\rho\int_o^L \frac{\pi}{16}(2-\frac{x}{L})^2(\sum_{n=1}^{\infty} A_n\phi_n)\phi_i dx + \{M(\sum_{n=1}^{\infty} A_n\phi_n)\phi_i$$

$$+\frac{I}{L^2}(\sum_{n=1}^{\infty} A_n(k_nL)\phi_n')(k_iL)\phi_i'\}_{\frac{3L}{4}}$$

$$+\frac{\pi\rho}{1024L^2}\{1+\frac{E}{k'G}\}\int_o^L(2-\frac{x}{L})^4(\sum_{n=1}^{\infty} A_n(k_nL)\phi_n')(k_iL)\phi_i'dx$$

$$-\frac{\pi}{265L^2}\frac{\rho E}{k'G}\int_o^L(2-\frac{x}{L})^3(\sum_{n=1}^{\infty} A_n\phi_n)(k_iL)\phi_i'dx$$

$$= \frac{\pi E}{1024L^2}\int_o^L(2-\frac{x}{L})^4(\sum_{n=1}^{\infty} A_n(k_nL)^2\phi_n'')(k_iL)^2\phi_i''dx \quad (36)$$

For the beam in question, m_b = mass of beam = $\frac{7\pi\rho L}{48}$

Let $\lambda_v = \frac{E}{64\rho p^2L^4}$; $\alpha = \frac{M}{m_b}$; $I = \frac{\alpha m_b L^2}{4}\beta^2$ and $\frac{x}{L} = z$

With these notations, Equation (36) reduces to

$$[\int_o^1(2-z)^2(\sum_{n=1}^{\infty} A_n\phi_n)\phi_i dz + \frac{7}{3}\{\alpha(\sum_{n=1}^{\infty} A_n\phi_n)\phi_i + \frac{\alpha\beta^2}{4}(\sum_{n=1}^{\infty} A_n(k_nL)\phi_n')(k_iL)\phi_i'\}_{\frac{3L}{4}}$$

$$+\frac{0.06481823101}{L^2}\int_o^1(2-z)^4\{\sum_{n=1}^{\infty} A_n(k_nL)\phi_n'\}(k_iL)\phi_i'dz$$

$$-\frac{0.196772924}{L^2}\int_o^1(2-z)^3(\sum_{n=1}^{\infty} A_n\phi_n)(k_iL)\phi_i'dz]$$

$$-\lambda_v[\int_o^1(2-z)^4\{\sum_{n=1}^{\infty} A_n(k_nL)^2\phi_n''\}(k_iL)^2\phi_i''dz] = 0 \quad (37)$$

It is seen that Equation (35) is same as Equation (37) except that two more terms are added to the latter equation. Equation (37) was solved for the following values of α and β.

α = 0.5, 1.0, 2.5, 5.0, 7.5 and 10.0.

β = 0.4 and 0.5.

Also, only nine terms of the series in Equation (37) were taken. In Figure 3 the ordinate represents the ratio of the loaded frequency to that of the unloaded one. It is seen that the second mode is the one most effected by change of β. The fifth mode values were not plotted primarily because it was not possible to verify these values experimentally, and secondly, there seemed to be some small error in the values. This was concluded from the fact that the values were larger for $\alpha = 7.5$ than $\alpha = 5.0$ and again decreasing for $\alpha = 10.0$, which does not seem logical. The variation is very small and can be attributed to accumulation error from the digital computer.

(c) Uniform beam

In this section, three cases of a simply supported beam are investigated. The expansion of the series in Equation (28) was carried out to the twelfth mode. But the values given in the tables are up to sixth mode only. Rotatory inertia and shear deformation of the beam are neglected in this section.

The three cases investigated are:

(1) A mass at mid-point of the beam.

(2) Two equal masses at quarter points from the ends.

(3) One mass at quarter point from one end.

(1) <u>A mass at mid-point of the beam (Figure 4a).</u> For this case, Equation (28) is used with $x_k = \frac{L}{2}$ and $X_i = \operatorname{Sin}\frac{i\pi x}{L}$.

Since

$$\int_o^L X_i X_j dx = \frac{L}{2} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

and

$$\int_o^L \frac{d^2X_i}{dx^2}\frac{d^2X_j}{dx^2}\,dx = \frac{i^4\pi^4}{2L^3} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

Table 6. Frequencies of cantilever beam with attached mass. .

Parameter	Mode	p/p_r from 6 terms expansion	p/p_r from 9 terms expansion	Extra-polated
$\alpha = 0.5$ $\beta = 0.4$	1	0.793	0.793	0.793
	2	3.066	3.032	3.005
	3	10.568	10.469	10.390
	4	14.750	13.846	13.123
	5	28.539	26.928	25.639
$\alpha = 0.5$ $\beta = 0.5$	1	0.764	0.763	0.762
	2	2.663	2.628	2.600
	3	10.525	10.370	10.246
	4	14.312	13.492	12.836
	5	28.426	26.872	25.629
$\alpha = 1.0$ $\beta = 0.4$	1	0.619	0.619	0.619
	2	2.345	2.309	2.280
	3	10.102	9.974	9.872
	4	14.015	13.210	12.566
	5	28.186	26.519	25.185
$\alpha = 1.0$ $\beta = 0.5$	1	0.591	0.591	0.591
	2	2.012	1.981	1.956
	3	10.066	9.932	9.825
	4	13.789	12.970	12.315
	5	28.115	26.476	25.165

Table 7. Frequencies of cantilever beam with attached mass.

Parameters	Mode	p/p_r from 6 terms expansion	p/p_r from 9 terms expansion	Extra-polated
$\alpha = 2.5$ $\beta = 0.4$	1	0.420	0.420	0.420
	2	1.565	1.539	1.518
	3	9.727	9.593	9.486
	4	13.549	12.751	12.113
	5	27.917	26.250	24.916
$\alpha = 2.5$ $\beta = 0.5$	1	0.398	0.398	0.398
	2	1.334	1.310	1.291
	3	9.727	9.579	9.461
	4	13.450	12.673	12.051
	5	27.889	26.165	24.786
$\alpha = 5.0$ $\beta = 0.4$	1	0.305	0.305	0.305
	2	1.127	1.106	1.089
	3	9.586	9.480	9.395
	4	13.365	12.602	11.992
	5	27.847	26.109	24.719
$\alpha = 5.0$ $\beta = 0.5$	1	0.288	0.288	0.288
	2	0.959	0.942	0.928
	3	9.586	9.466	9.370
	4	13.323	12.560	11.950
	5	27.832	26.024	24.578

Table 8. Frequencies of cantilever beam with attached mass.

Parameters	Mode	p/p_r from 6 terms expansion	p/p_r from 9 terms expansion	Extra-polated
$\alpha = 7.5$ $\beta = 0.4$	1	0.251	0.251	0.251
	2	0.928	0.910	0.896
	3	9.558	9.420	9.310
	4	13.309	12.560	11.961
	5	27.850	26.137	24.767
$\alpha = 7.5$ $\beta = 0.5$	1	0.237	0.237	0.237
	2	0.788	0.774	0.763
	3	9.551	9.416	9.308
	4	13.280	12.518	11.908
	5	27.848	26.066	24.640
$\alpha = 10.0$ $\beta = 0.4$	1	0.219	0.218	0.217
	2	0.805	0.791	0.780
	3	9.522	9.409	9.319
	4	13.280	12.518	11.908
	5	27.830	26.066	24.655
$\alpha = 10.0$ $\beta = 0.5$	1	0.206	0.206	0.206
	2	0.685	0.673	0.663
	3	9.522	9.409	9.319
	4	13.252	12.489	11.879
	5	27.804	26.024	24.600

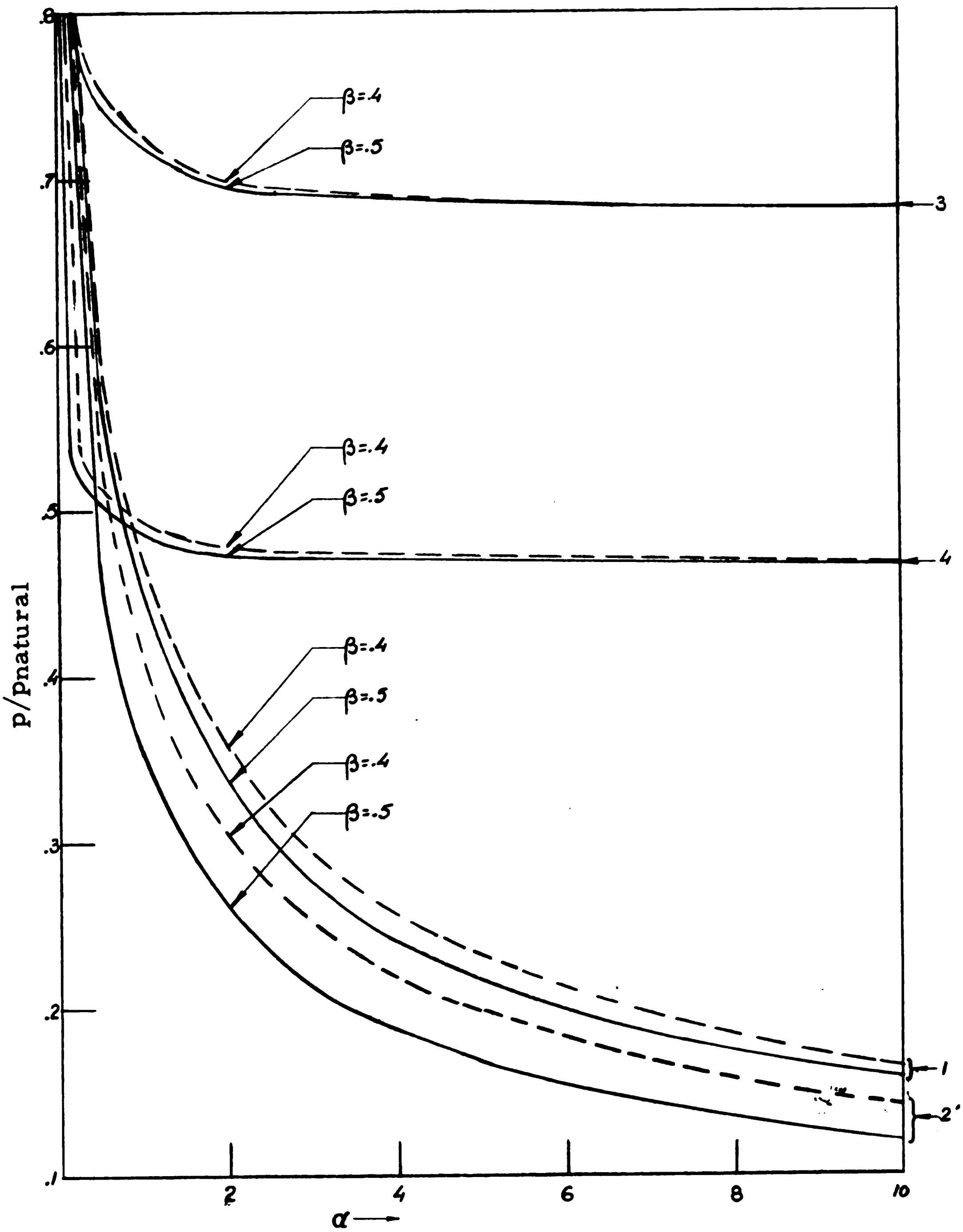


Figure 3. Frequency distribution for cantilever beam.

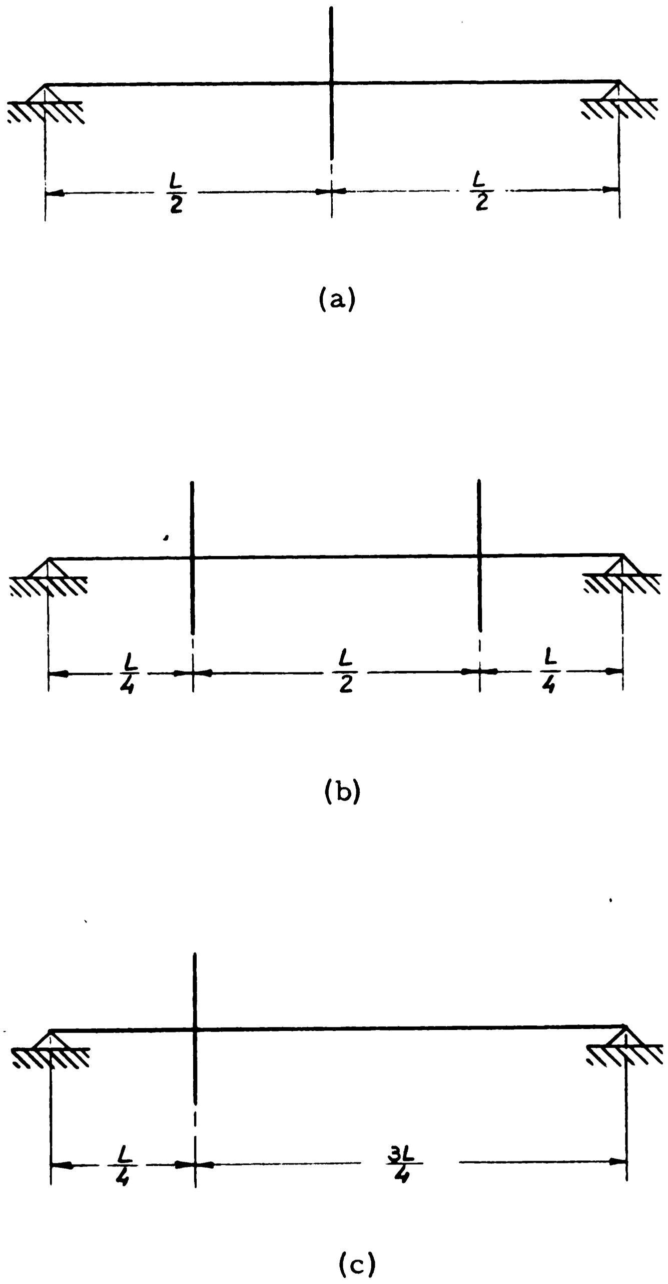


Figure 4. Mass arrangements for simply supported beam.

Equation (28) reduces to

$$p^2[\rho A(\frac{A_i L}{2}) + M(\sum_{n=1}^{\infty} A_n \operatorname{Sin}\frac{n\pi}{2}) \operatorname{Sin}\frac{i\pi}{2} + I(\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \operatorname{Cos}\frac{n\pi}{2}) \frac{i\pi}{L} \operatorname{Cos}\frac{i\pi}{2}]$$

$$= EI_b A_i \frac{i^4\pi^4}{2L^3} \qquad (38)$$

Let $\alpha = \frac{M}{m_b}$, $I = \frac{\alpha m_b L^2}{4} \beta^2$ and $\lambda = \frac{EI_b \pi^4}{2m_b p^2 L^3}$ (39)

With these notations, Equation (38) reduces to

$$[\frac{A_i}{2} + \alpha(\sum_{n=1}^{\infty} A_n \operatorname{Sin}\frac{n\pi}{2}) \operatorname{Sin}\frac{i\pi}{2} + \frac{\pi^2\alpha\beta^2}{4}(\sum_{n=1}^{\infty} A_n\, n \operatorname{Cos}\frac{n\pi}{2})\, i \operatorname{Cos}\frac{i\pi}{2}]$$

$$= \lambda i^4 A_i \qquad (40)$$

For each value of i in Equation (40), the corresponding vibration mode of the beam is obtained. The following values of α and β were used in Equation (40).

α = 0.5, 1.0, 2.5 and 5.0

β = 0, 0.1, 0.2, 0.3, 0.4 and 0.5.

In Figure 5 the ratios of kL values for the loaded beam to that of the unloaded beam are plotted against β values for only three values of α. The function kL for the loaded beam is defined by the following relation

$$kL = \sqrt[4]{\frac{m_b p^2 L^3}{EI_b}} = \pi \sqrt[4]{\frac{1}{2\lambda}} \qquad (41)$$

where λ are the eigenvalues of Equation (40). It should be noticed in Equation (40) and Figure 5, that when the beam is vibrating in odd modes e.g. i = 1, 3, 5 etc., the mass does not rotate and inertia terms drop out of Equation (40). For even modes, the mass only rotates and does not move up and down. As such, the terms with α in Equation (40) drop out.

From (23), for an uniform beam,

$$k^4 = \frac{p^2}{a_1^2} = \frac{p^2 m_b}{EI_b L}, \text{ where } a_1^2 = \frac{EI_b}{\rho A}$$

and this is the same relation as in Equation (41). For a simply supported uniform beam $k_n L = n\pi$. As such, each kL values obtained from Equation (40) were divided by $n\pi$ to plot Figure 5.

To check the convergence of the series in Equation (40), one term of the series is taken with i = 1. This gives

$$\lambda_1 = (\tfrac{1}{2} + \alpha)$$

or

$$k_1 L = \pi \sqrt[4]{\frac{1}{1 + 2\alpha}}$$

The following are the values of $k_1 L$ for different α.

α	0	0.5	1.0	2.5	5.0	7.5	10.0
$k_1 L$	π	0.840897π	0.759836π	0.638943π	0.549100π	0.5π	0.467138π
$(k_1 L)_{12}$	π	0.840125π	0.758601π	0.637330π	0.547455π	0.498408π	0.465604π
p.c variation	0	0.092	0.163	0.253	0.300	0.319	0.329

As may be seen, the error gradually increases with increasing α because with higher α, the mode shape of the beam deviates more and more from a sine curve which is assumed here in this case. This was verified from the eigenvectors.

In this problem, it is seen that the odd and even modes separate out. So, to get two terms of the series in Equation (40), it is necessary to use n = 1, 3 and i = 1, 3. With these, the determinant takes the form

$$\begin{vmatrix} [(\tfrac{1}{2} + \alpha) - \lambda] & (-\alpha) \\ (-\alpha) & [(\tfrac{1}{2} + \alpha) - 81\lambda] \end{vmatrix} = 0$$

The following are the values obtained from this determinant.

α	k_1L	p.c variation from $(k_1L)_{12}$	k_3L	$(k_3L)_{12}$	p.c variation
0.5	0.840243π	0.014	2.71291π	2.69809π	0.549
1.0	0.758790π	0.025	2.64397π	2.62353π	0.779
2.5	0.637576π	0.039	2.58369π	2.55836π	0.990
5.0	0.547706π	0.046	2.55870π	2.53104π	1.078
7.5	0.498651π	0.049	2.54967π	2.52166π	1.111
10.0	0.465838π	0.050	2.54502π	2.51665π	1.127

As may be seen, the series converges very rapidly and even one or two terms of the series give fairly accurate results for this problem.

Next consider the even modes. In this case Equation (40) becomes

$$\left[\frac{A_i}{2} + \frac{\pi^2\alpha\beta^2}{4}\left(\sum_{n=1}^{\infty} A_n n \operatorname{Cos}\frac{n\pi}{2}\right) i \operatorname{Cos}\frac{i\pi}{2}\right] = \lambda\, i^4 A_i \qquad (42)$$

Taking one term of equation (42) gives

$$\left[\frac{1}{2} + \pi^2\alpha\beta^2\right] = 16\lambda \qquad (43)$$

In this equation, the frequency depends on both α and β. This is because the rotatory inertia I of the mass depends on both (refer to Equation (39)). Taking α = 1.0, β = 0.1 and 0.5, the following values are obtained.

$\alpha = 1.0$

β	0.1	0.5
k_2L	1.91193π	1.28138π
$(k_2L)_{12}$	1.90493π	1.18667π
p.c variation	0.367	7.981

It may be seen from the above values that rotatory inertia changes the mode shape considerably from Sin $\frac{2\pi x}{L}$ which is assumed by taking only one term. This is the reason for the high difference of 7.981% for $\beta = 0.5$. From this, one can conclude that for high rotatory inertia, the convergence of the series is slow and needs more terms of the series. This situation was encountered during the experiment with plates [Chapter IV, Part (a)].

Taking two even terms of the series in Equation (42) the frequency equation is obtained as

$$\begin{vmatrix} (\frac{1}{2} + \pi^2 \alpha\beta^2 - 16\lambda) & (-2\pi^2\alpha\beta^2) \\ (-2\pi^2\alpha\beta^2) & (\frac{1}{2} + 4\pi^2\alpha\beta^2 - 256\lambda) \end{vmatrix} = 0 \qquad (44)$$

For $\alpha = 1.0$, $\beta = 0.1$ and 0.5, the following values were obtained for k_2L and k_4L.

$\alpha = 1.0$

β	0.1.	0.5
k_2L	1.90839π	1.22950π
p.c variation from $(k_2L)_{12}$	0.182	3.609
k_4L	3.53084π	2.89061π
$(k_4L)_{12}$	3.37625π	2.64210π
p.c variation	4.579	9.41

The variation of k_2L for $\alpha = 1.0$ and $\beta = 0.5$ from $(k_2L)_{12}$ is now reduced to 3.609%. This shows that the series is reasonably convergent.

Next, the series in Equation (40) were expanded to twelve terms and the roots were obtained through the digital computer. Table 9 gives the kL values up to sixth mode. For this particular problem, the even modes were most effected by β. The convergence was found to be quite rapid and as such extrapolation was not deemed necessary.

Table 9. Frequencies of simply supported uniform beam with mass at center $(\frac{\text{loaded } kL}{\pi})$

Parameters \ Mode	1	2	3	4	5	6
$\alpha = 0.5$ $\beta = 0$	0.8401	2.0	2.6981	4.0	4.6395	6.0
$\alpha = 0.5$ $\beta = 0.1$	0.8401	1.9514	2.6981	3.6286	4.6395	5.1607
$\alpha = 0.5$ $\beta = 0.2$	0.8401	1.8203	2.6981	3.1046	4.6395	4.8164
$\alpha = 0.5$ $\beta = 0.3$	0.8401	1.6569	2.6981	2.8567	4.6395	4.7429
$\alpha = 0.5$ $\beta = 0.4$	0.8401	1.5058	2.6980	2.7466	4.6392	4.7169
$\alpha = 0.5$ $\beta = 0.5$	0.8401	1.3799	2.6980	2.6920	4.6392	4.7057
$\alpha = 1.0$ $\beta = 0$	0.7586	2.0	2.6235	4.0	4.5829	6.0
$\alpha = 1.0$ $\beta = 0.1$	0.7586	1.9049	2.6235	3.3763	4.5829	4.9442
$\alpha = 1.0$ $\beta = 0.2$	0.7586	1.6848	2.6235	2.8859	4.5829	4.7503
$\alpha = 1.0$ $\beta = 0.3$	0.7586	1.4729	2.6236	2.7300	4.5820	4.7137
$\alpha = 1.0$ $\beta = 0.4$	0.7586	1.3100	2.6236	2.6702	4.5820	4.7002
$\alpha = 1.0$ $\beta = 0.5$	0.7586	1.1867	2.6236	2.6421	4.5820	4.6947
$\alpha = 2.5$ $\beta = 0$	0.6373	2.0	2.5584	4.0	4.5400	6.0
$\alpha = 2.5$ $\beta = 0.1$	0.6373	1.7825	2.5584	3.0260	4.5400	4.7900
$\alpha = 2.5$ $\beta = 0.2$	0.6373	1.4432	2.5583	2.7166	4.5407	4.7113
$\alpha = 2.5$ $\beta = 0.3$	0.6373	1.2153	2.5584	2.6477	4.5407	4.6939
$\alpha = 2.5$ $\beta = 0.4$	0.6373	1.0642	2.5583	2.6230	4.5407	4.6916
$\alpha = 2.5$ $\beta = 0.5$	0.6373	0.9568	2.5583	2.6117	4.5413	4.6900
$\alpha = 5.0$ $\beta = 0$	0.5475	2.0	2.5314	4.0	4.5241	6.0
$\alpha = 5.0$ $\beta = 0.1$	0.5475	1.6308	2.5314	2.8326	4.5241	4.7372
$\alpha = 5.0$ $\beta = 0.2$	0.5475	1.2477	2.5315	2.6546	4.5243	4.6978
$\alpha = 5.0$ $\beta = 0.3$	0.5475	1.0350	2.5315	2.6196	4.5243	4.6908
$\alpha = 5.0$ $\beta = 0.4$	0.5475	0.9013	2.5312	2.6068	4.5165	4.7018
$\alpha = 5.0$ $\beta = 0.5$	0.5475	0.8083	2.5312	2.6018	4.5165	4.6861

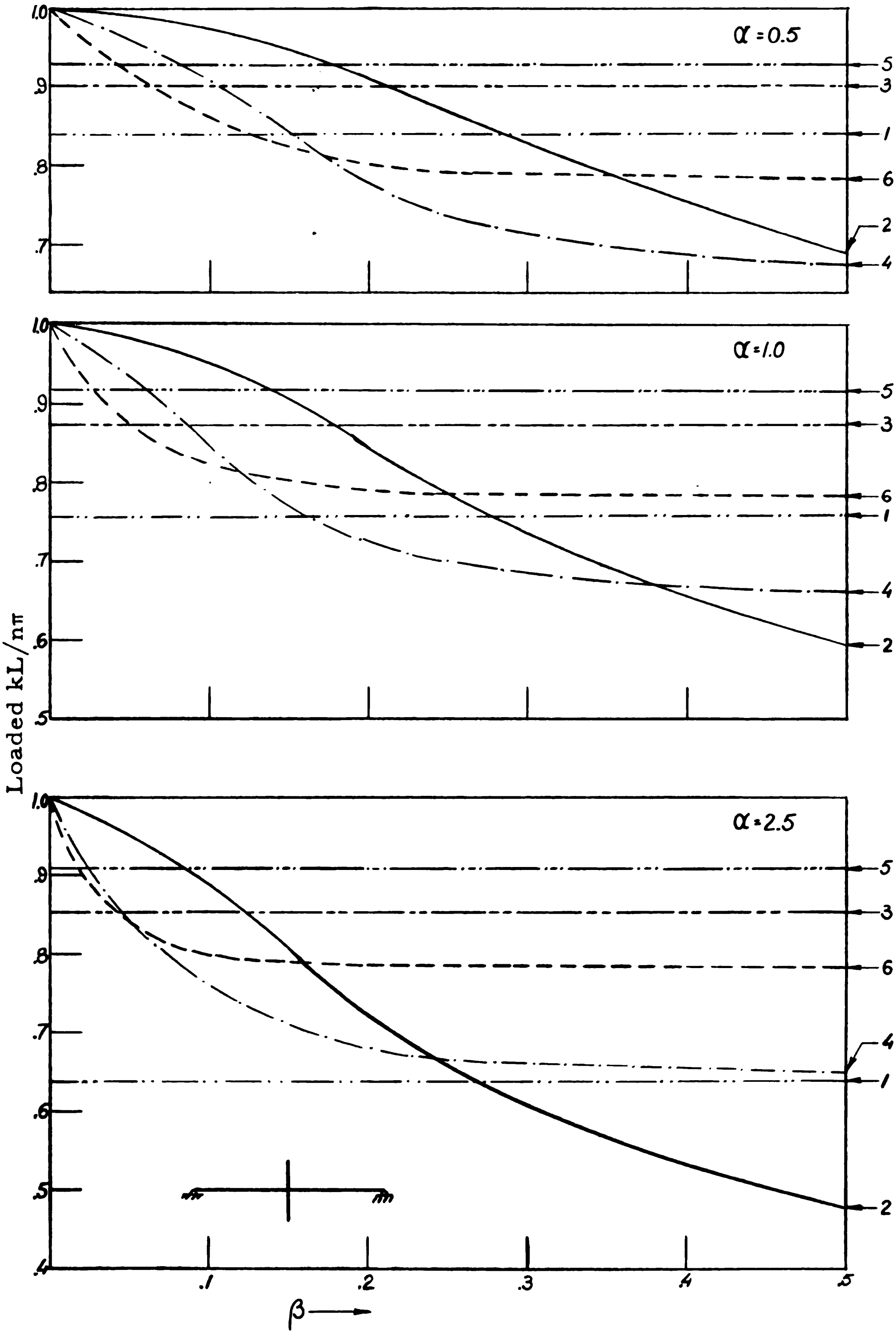


Figure 5. Frequency distribution for simply supported beam; mass at center.

(2) Two equal masses at quarter points from the ends (Figure 4b). For convenience of calculations, the two masses are assumed to be equal in all respects. From Equation (28), the frequency equation is given by

$$\left[\frac{A_i}{2} + \alpha \left\{ \left(\sum_{n=1}^{\infty} A_n \operatorname{Sin}\frac{n\pi}{4}\right) \operatorname{Sin}\frac{i\pi}{4} + \left(\sum_{n=1}^{\infty} A_n \operatorname{Sin}\frac{3n\pi}{4}\right) \operatorname{Sin}\frac{3i\pi}{4} \right\}\right.$$

$$\left. + \frac{\pi^2 \alpha\beta^2}{4} \left\{ \left(\sum_{n=1}^{\infty} A_n\, n \operatorname{Cos}\frac{n\pi}{4}\right)\left(i \operatorname{Cos}\frac{i\pi}{4}\right) + \left(\sum_{n=1}^{\infty} A_n\, n \operatorname{Cos}\frac{3n\pi}{4}\right)\left(i \operatorname{Cos}\frac{3i\pi}{4}\right) \right\}\right]$$

$$= \lambda\, i^4 A_i \tag{45}$$

Comparable to the last section, when i = 4, 8 etc. the α terms drop out and when i = 2, 6, 10 etc., the β terms drop out. In the remaining modes both terms are present.

Taking only one term of the series in Equation (45), with n = i = 1, the first mode frequency equation is obtained as

$$\left[\frac{1}{2} + \alpha + \frac{\pi^2 \alpha\beta^2}{4} \right] = \lambda \tag{46}$$

Assuming α = 1.0, β = 0.1 and 0.5, the following values are obtained for k_1L.

α = 1.0

β	0.1	0.5
k_1L	0.756743π	0.697140π
$(k_1L)_{12}$	0.755667π	0.695177π
p.c variation	0.142	0.282

In a similar manner, Equation (45) was solved with twelve terms of the series. Table 10 gives the $\frac{kL}{\pi}$ values for different values of α and β. The graphs of Figure 6 shows the variation of kL due to variation of β for three values of α. These values are given only up to sixth mode.

Table 10. Frequencies of simply supported uniform beam with two equal masses at quarter points.

Parameters \ Mode		1	2	3	4	5	6
α = 0.5	β = 0	0.8401	1.5173	2.6981	4.00	4.6395	5.2584
α = 0.5	β = 0.1	0.8376	1.5173	2.5803	3.3935	4.1267	5.2584
α = 0.5	β = 0.2	0.8303	1.5173	2.2980	2.6640	3.6266	5.2583
α = 0.5	β = 0.3	0.8186	1.5173	2.0317	2.2254	3.4611	5.2583
α = 0.5	β = 0.4	0.8032	1.5173	1.8386	1.9433	3.3983	5.2576
α = 0.5	β = 0.5	0.7848	1.5173	1.7028	1.7448	3.3692	5.2576
α = 1.0	β = 0	0.7586	1.3345	2.6235	4.0	4.5829	5.1551
α = 1.0	β = 0.1	0.7557	1.3345	2.4113	3.0488	3.8112	5.1549
α = 1.0	β = 0.2	0.7471	1.3345	2.0225	2.2866	3.4196	5.1549
α = 1.0	β = 0.3	0.7334	1.3345	1.7468	1.8891	3.3249	5.1539
α = 1.0	β = 0.4	0.7157	1.3345	1.5714	1.6429	3.2915	5.1539
α = 1.0	β = 0.5	0.6952	1.3345	1.4562	1.4723	3.2756	5.1539
α = 2.5	β = 0	0.6373	1.0951	2.5584	4.0	4.5401	5.0779
α = 2.5	β = 0.1	0.6343	1.0951	2.1293	2.5400	3.4724	5.0779
α = 2.5	β = 0.2	0.6255	1.0951	1.6590	1.8418	3.2650	5.0770
α = 2.5	β = 0.3	0.6116	1.0951	1.4084	1.5110	3.2239	5.0770
α = 2.5	β = 0.4	0.5939	1.0951	1.2635	1.3108	3.2095	5.0770
α = 2.5	β = 0.5	0.5738	1.0951	1.1728	1.1733	3.2017	5.0840
α = 5.0	β = 0	0.5475	0.9313	2.5314	4.0	4.5242	5.0490
α = 5.0	β = 0.1	0.5446	0.9313	1.8782	2.1717	3.3177	5.0507
α = 5.0	β = 0.2	0.5365	0.9313	1.4105	1.5554	3.2063	5.0507
α = 5.0	β = 0.3	0.5237	0.9313	1.1899	1.2730	3.1855	5.0507
α = 5.0	β = 0.4	0.5074	0.9313	1.0667	1.1034	3.1767	5.0384
α = 5.0	β = 0.5	0.4892	0.9313	0.9911	0.9873	3.1723	5.0384

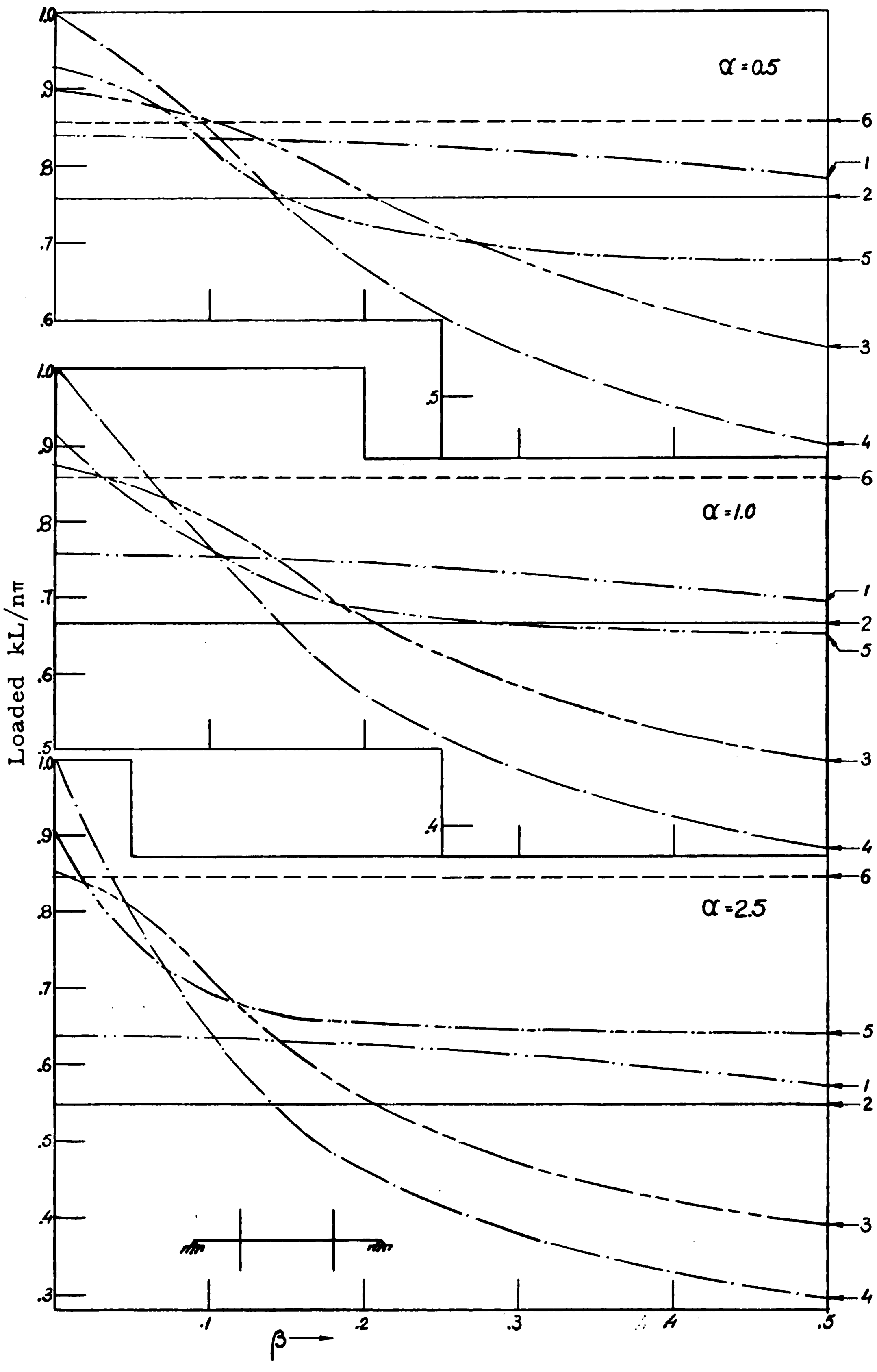


Figure 6. Frequency distribution for simply supported beam; two masses at quarter points.

(3) One mass at quarter point from one end (Figure 4c). For this case, Equation (28) reduces to

$$\left[\frac{A_i}{2} + \alpha \left(\sum_{n=1}^{\infty} A_n \operatorname{Sin}\frac{n\pi}{4}\right) \operatorname{Sin}\frac{i\pi}{4} + \frac{\pi^2\alpha\beta^2}{4}\left(\sum_{n=1}^{\infty} A_n n \operatorname{Cos}\frac{n\pi}{4}\right) i \operatorname{Cos}\frac{i\pi}{4}\right] = \lambda i^4 A_i \quad (47)$$

Taking only one term of the series in Equation (47),

$$\left[\frac{1}{2} + \frac{\alpha}{2} + \frac{\pi^2\alpha\beta^2}{4}\right] = \lambda \quad (48)$$

With $\alpha = 1.0$, $\beta = 0.1$ and 0.5, the following values are obtained for k_1L.

$\alpha = 1.0$

β	0.1	0.5
k_1L	0.838323π	0.786240π
$(k_1L)_{12}$	0.830865π	0.779235π
p.c. variation	0.8976	0.8990

Even with one term, the results are within one percent of the twelve term series.

It is interesting to note that for $\alpha = 1.0$ and $\beta = 0$, k_1L is 0.833165π whereas, for $\alpha = 1.0$ and $\beta = 0.5$, k_1L is 0.779235π, a variation of 6.472 p.c. The combined effect of $\alpha = 1.0$ and $\beta = 0.5$ is 22.077 p.c from the unloaded frequency of the beam.

Next, a twelve term series expansion of Equation (47) was taken and the values of $\frac{kL}{\pi}$ that were obtained are given in Table 11. The graphs of these values are shown in Figure 7.

Table 11. Frequencies of simply supported uniform beam with mass at quarter point.

Parameters \ Mode	1	2	3	4	5	6
$\alpha = 0.5$ $\beta = 0$	0.8998	1.7570	2.8739	4.0	4.7524	5.6075
$\alpha = 0.5$ $\beta = 0.1$	0.8981	1.7554	2.7915	3.6309	4.5726	5.5543
$\alpha = 0.5$ $\beta = 0.2$	0.8929	1.7499	2.4945	3.2539	4.4796	5.4501
$\alpha = 0.5$ $\beta = 0.3$	0.8843	1.7373	2.1855	3.1605	4.4569	5.4096
$\alpha = 0.5$ $\beta = 0.4$	0.8727	1.7089	1.9851	3.1316	4.4476	5.3912
$\alpha = 0.5$ $\beta = 0.5$	0.8582	1.6530	1.8814	3.1192	4.4458	5.3834
$\alpha = 1.0$ $\beta = 0$	0.8332	1.6819	2.8495	4.0	4.7006	5.5749
$\alpha = 1.0$ $\beta = 0.1$	0.8309	1.6767	2.6749	3.4098	4.4857	5.4458
$\alpha = 1.0$ $\beta = 0.2$	0.8240	1.6570	2.2278	3.1483	4.4345	5.3535
$\alpha = 1.0$ $\beta = 0.3$	0.8128	1.6077	1.9514	3.1050	4.4257	5.3289
$\alpha = 1.0$ $\beta = 0.4$	0.7977	1.5215	1.8358	3.0910	4.4193	5.3230
$\alpha = 1.0$ $\beta = 0.5$	0.7792	1.4312	1.7941	3.0845	4.4176	5.3142
$\alpha = 2.5$ $\beta = 0$	0.7180	1.6114	2.8301	4.0	4.6594	5.5530
$\alpha = 2.5$ $\beta = 0.1$	0.7154	1.5919	2.4013	3.1862	4.4242	5.3329
$\alpha = 2.5$ $\beta = 0.2$	0.7076	1.5082	1.9232	3.0827	4.4033	5.2841
$\alpha = 2.5$ $\beta = 0.3$	0.6950	1.3636	1.7866	3.0664	4.4004	5.2743
$\alpha = 2.5$ $\beta = 0.4$	0.6779	1.2408	1.7491	3.0615	4.3999	5.2729
$\alpha = 2.5$ $\beta = 0.5$	0.6576	1.1554	1.7348	3.0588	4.3959	5.2729
$\alpha = 5.0$ $\beta = 0$	0.6238	1.5811	2.8225	4.0	4.6435	5.5444
$\alpha = 5.0$ $\beta = 0.1$	0.6213	1.5327	2.1423	3.1074	4.4028	5.2865
$\alpha = 5.0$ $\beta = 0.2$	0.6138	1.3474	1.7948	3.0607	4.3909	5.2603
$\alpha = 5.0$ $\beta = 0.3$	0.6016	1.1666	1.7356	3.0530	4.3914	5.2562
$\alpha = 5.0$ $\beta = 0.4$	0.5852	1.0504	1.7191	3.0495	4.3903	5.2077
$\alpha = 5.0$ $\beta = 0.5$	0.5658	0.9766	1.7119	3.0486	4.3792	5.1690

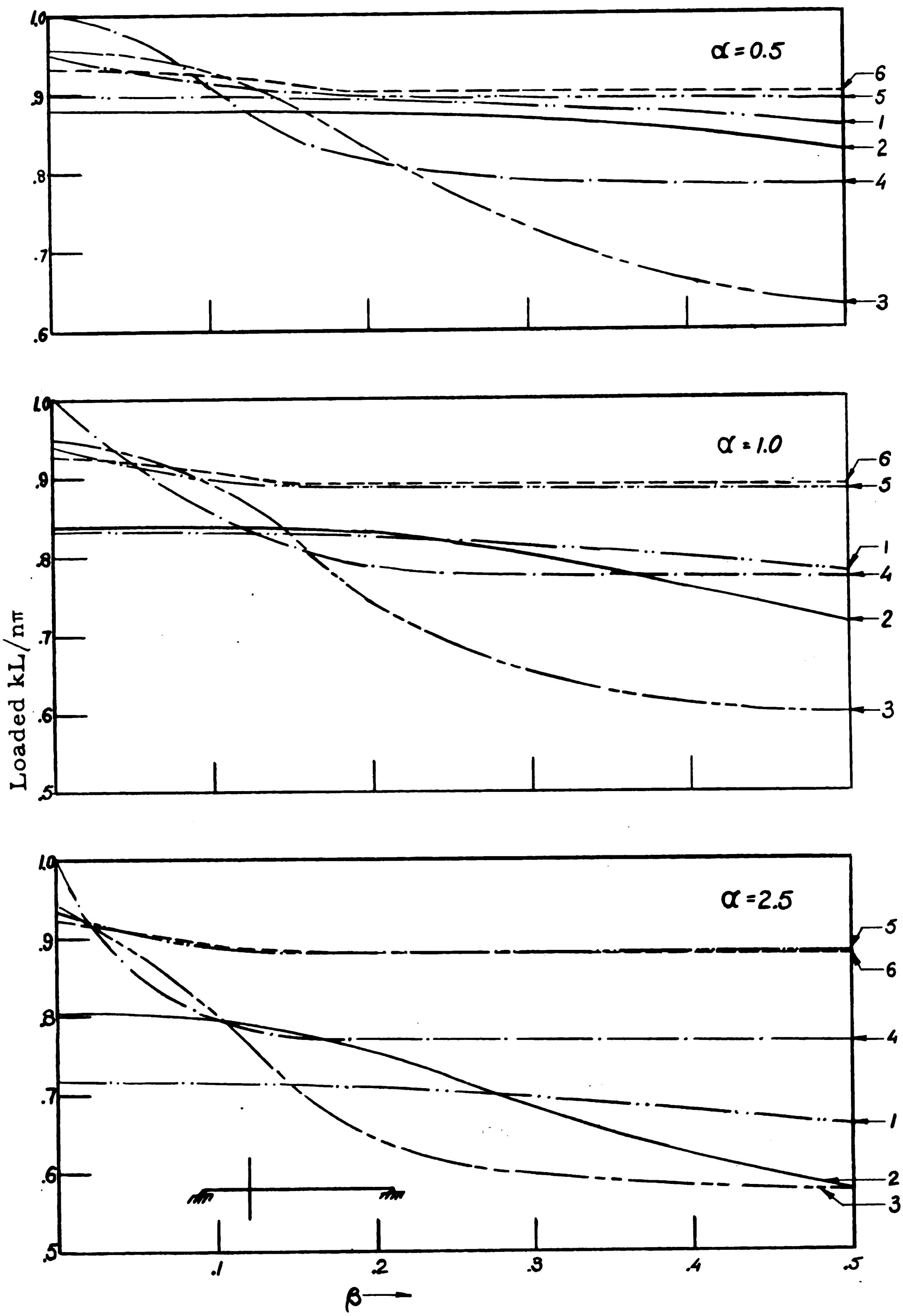


Figure 7. Frequency distribution for simply supported beam; mass at quarter point.

CHAPTER IV

EXPERIMENTAL RESULTS

As mentioned earlier, three cases were investigated experimentally. They are:

(a) Plate: This was a square plate, 9"x9", made of aluminum. The thickness at the middle of the plate was 0.125" which gradually decreased to 0.0625" at the edges. The mass was fixed at $x = \frac{2a}{3}$, $y = \frac{2a}{3}$.

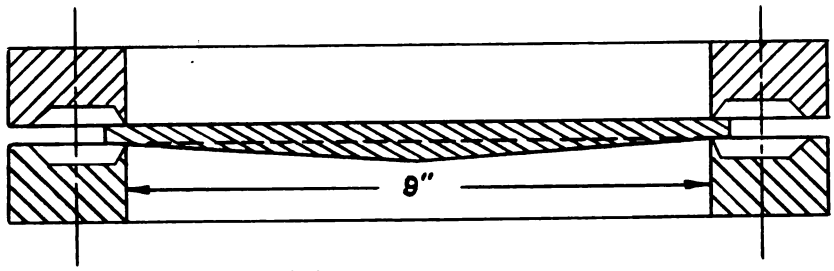
(b) Cantilever beam: The beam was made of aluminum and 20" long. The diameter at the fixed end was 1" which gradually decreased to $\frac{1}{2}$" at the free end. The mass was fixed at $x = \frac{3L}{4}$.

(c) Uniform beam: This was a steel bar, 1" in diameter and 35" long between supports. Two cases of this setup were investigated:

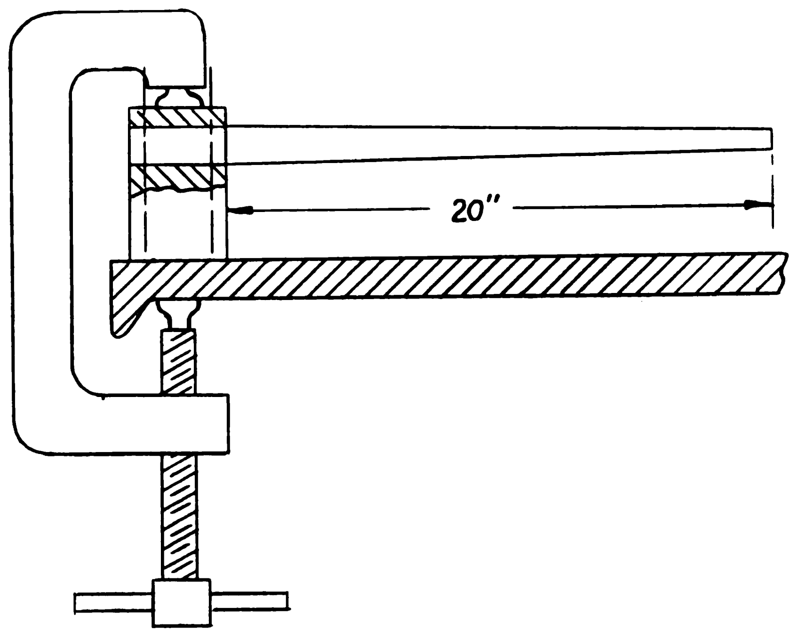
(1) The mass was attached at the mid-point of the beam.

(2) The mass was attached at quarter point.

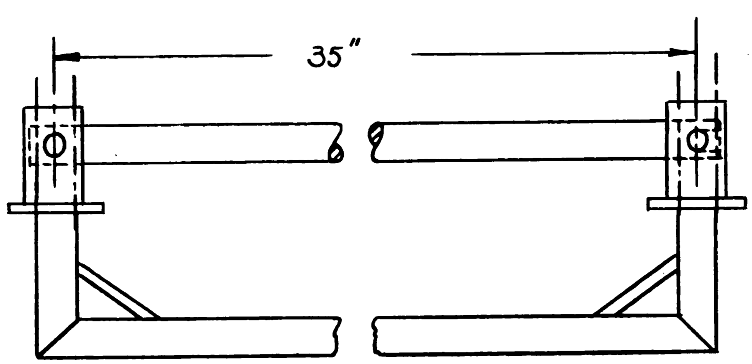
The different types of supports, used in the experiments, are shown schematically in Figure 8. Figure 8a is for the plate which rests on the support along four edges. By spreading some special fine grained sand along the edges and vibrating the plate by pulsed air, it was brought to resonance. The side bolts were then gradually tightened until the sand along the edges just stopped vibrating. This gave the condition for simple supports where the edges should not move but can rotate. Due to finite width of the supports, there was friction and this raised the frequencies of the plate.



(a) Plate



(b) Cantilever beam



(c) Simple support

Figure 8. Schematic diagrams of supports.

Figure 8b shows the support arrangement for the cantilever beam. The beam was fixed on a bearing and the whole assembly was fixed on a heavy steel table with a C-clamp. It was found from the natural frequencies of the beam that the support was weak. But as no other support arrangement could be built near the compressor, this was taken to serve the purpose.

The support for the uniform beam was made from three channel iron sections with stiffening ribs, as shown in Figure 8c. Two steel plates were then welded on the two vertical channels and the bearings were bolted to these two plates. Two 1/2" holes were drilled through the beam at the ends and these were honed to fit two ground 1/2" pins. The two pins were then fitted on the bearings with screws. To remove middle plane forces along the beam, one end of one of the holes on the beam was cut out. The whole assembly was then clamped onto the table with two C-clamps.

The systems were vibrated with pulsed air, supplied from a compressor at about 90 psi. The setup used was the same as used by B. B. Raju (19). To find resonance of the beams, a vibration pickup, instead of SR-4 gages, was used. This had the advantages of higher signal strength and also it could be moved to any point along the beam. A SR-4 gage was used in the case of the plate.

To vary the rotatory inertia of the attached masses, keeping the mass constant for each value of α, it was necessary to fix clamps on the bars with extensions attached to the clamps. The masses were slid on these extensions. As only some part of the attached masses could be slided, it was necessary to compute the lengths at which the sliding masses had to be fixed to get the required amount of rotatory inertia. The computations are shown in the following pages.

Plate

In order to attach masses on the plate, the plate was drilled at $x = \frac{2a}{3}$ $y = \frac{2a}{3}$. A 1/2" diameter magnesium rod was chosen and it was threaded throughout its whole length. This rod and other masses were fixed at desired locations with the help of six nuts. Following are the data for this setup.

Weight of aluminum plate = $m_{p}g = \frac{2\, t_0 a^2}{3}(\rho g) = 0.675\#$

Weight of threaded magnesium rod = 0.0625#

Length of threaded magnesium rod = $5\frac{23}{32}$".

Therefore effective diameter of magnesium rod = 0.463" (at $(\rho g) = .065\#$)

Weight of six nuts = 0.375#

Therefore, weight of each nut = .0625#

Height of each nut = 0.482".

With this setup, only two values of α were used. They were 1.0 and 2.5. Following are the data.

Weight of rod and six nuts = 0.4375#

$\alpha = 1.0$; extra weight = 0.2375#

This weight was made from a steel plate, having a 1/2" diameter hole, with dimensions $2\frac{1}{8}$" x 2" x $\frac{3}{16}$".

$\alpha = 2.5$; extra weight = 1.25 #.

This was also made out of a steel plate with a $\frac{1}{2}$" hole at the middle, having dimensions $3\frac{51}{64}$" x $1\frac{7}{8}$" x $\frac{1}{2}$". The weight of this plate was 1#. So, this plate, together with the first plate and a washer (0.0125# - height 0.1"), the total weight was 1.25#. It was found that if a single weight of 1.25# is made, $\beta = 0.2$ is not possible to obtain.

Let I_r = required moment of inertia;

$$= \frac{\alpha m_p a^2}{4}\beta^2$$

$$= \frac{13.65}{g}\alpha\beta^2$$

From Figure 9, rotatory inertia of two nuts $= \dfrac{0.01315}{g}$

rotatory inertia of rod $= \dfrac{0.171}{g}$

Total rotatory inertia of Figure 9 $= \dfrac{0.18415}{g}$

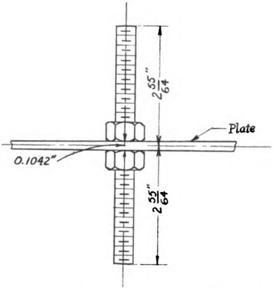


Figure 9. Plate with rod attached

$\underline{\alpha = 1.0}$ ($\beta = 0.1$ and 0.2 are not possible)

$\beta = 0.3$ $$I_r = \frac{1.227}{g} = \frac{0.2765}{g} + \frac{0.3625}{g} L^2$$

So, L = 1.64" (Figure 10a)

Similarly, for $\beta = 0.4$, L = 2.3"

$\underline{\alpha = 2.5}$ $$I_r = \frac{34.15}{g} \beta^2$$

$\beta = 0.2$ $$I_r = \frac{1.366}{g} = \frac{1.1799}{g} + \frac{0.0625 L^2}{g}$$

So, L = 1.728" (Figure 10b)

$\beta = 0.3$ $$I_r = \frac{3.074}{g} = \frac{0.375 L^2}{g} + \frac{0.0161 L}{g} + \frac{1.06081}{g}$$

So, L = 2.29" (Figure 10c)

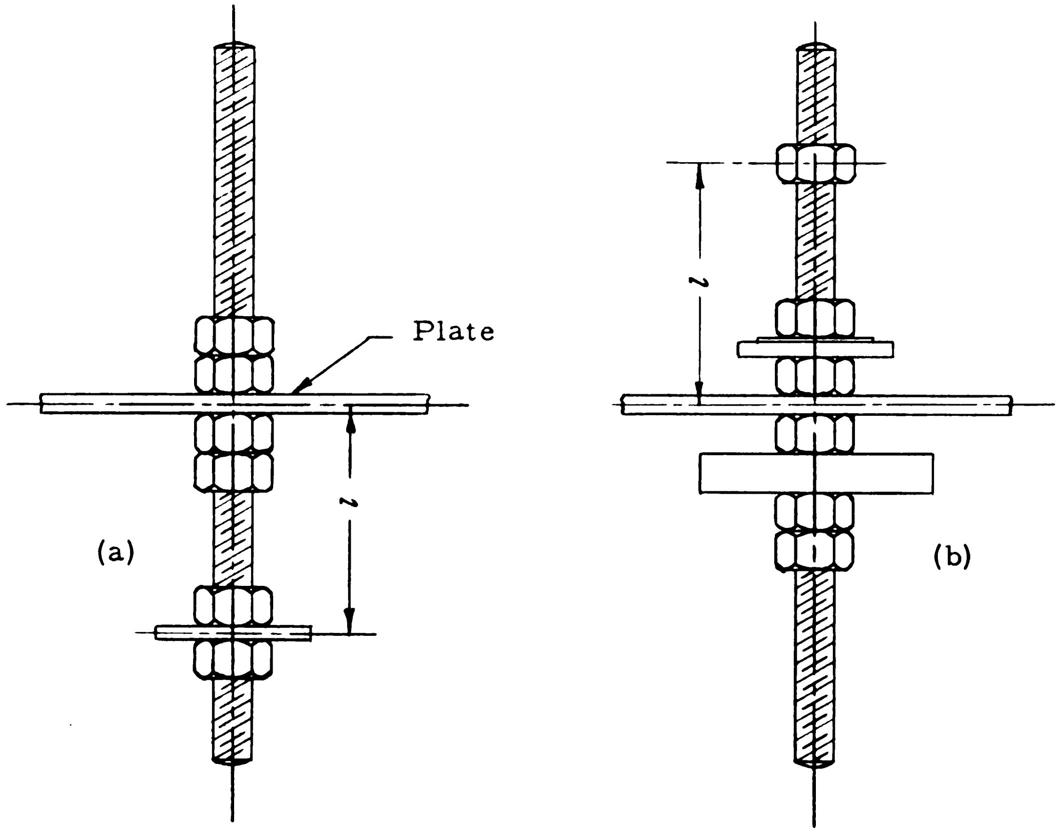
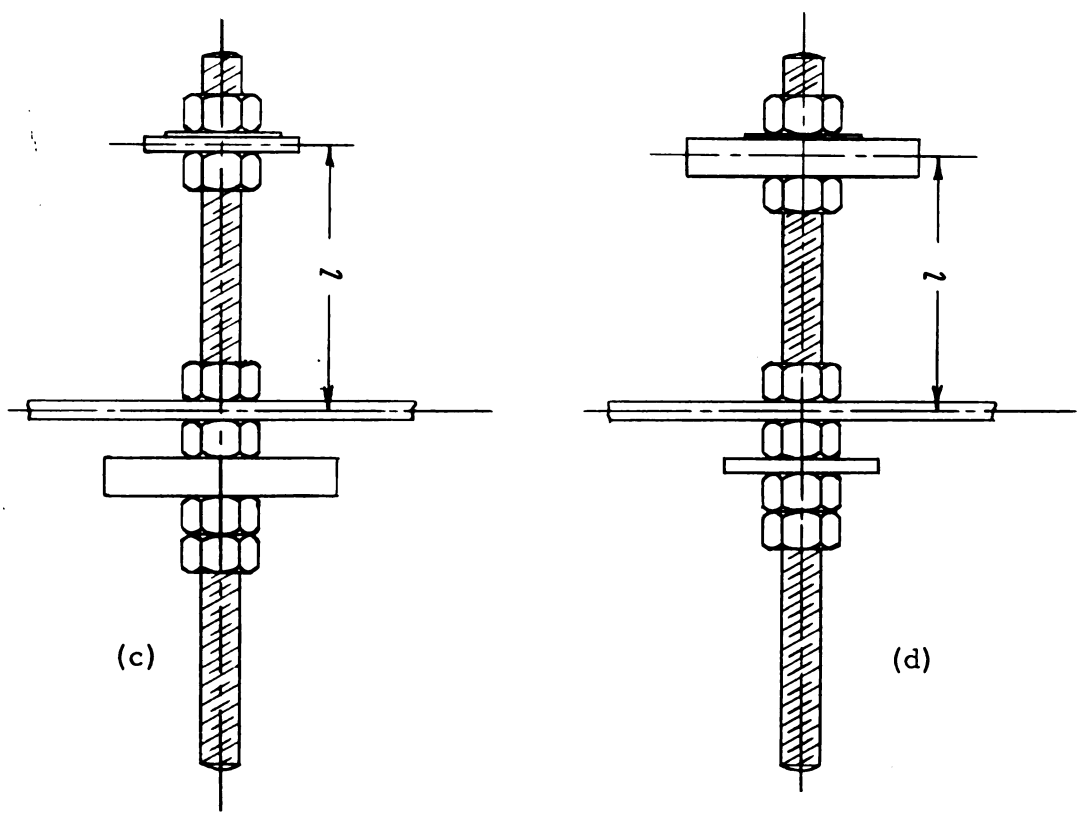


Figure 10. Masses on plate.



$$\beta = 0.4 \qquad I_r = \frac{5.46}{g} = \frac{0.5248}{g} + \frac{0.02\,L}{g} + \frac{1.1375\,L^2}{g}$$

So, L = 2.08" (Figure 10d).

Table 12 gives the natural frequencies of this plate and Table 13 gives those of the loaded plate. Since frequencies in cycles per second gives a better feel of the phenomenon, they are also tabulated.

Table 12. Experimental frequencies of plate without attached mass.

Mode	Theoretical Frequency	Experimental Frequency	Theoretical p/p_r	Experimental p/p_r
1	206.5	209	13.693	13.78
2	518	514	34.464	34.2
3	518	514	34.464	34.2
4	831	824	55.234	54.8
5	1051.5	1026	69.849	68.2

Note: Since the plate is square and symmetrical, the second and third modes are usually accepted as the same. But due to unsymmetric placing of the attached mass, these two modes are not same for the loaded plate. As such they are assumed here as different.

Table 13. Experimental frequencies of plate with attached mass.

α	β	Mode	Theoretical Frequency	Experimental Frequency
1.0	0.3	1	109.3	113
		2	218	214
		3	261.5	256
		4	434	427
		5	564	560
1.0	0.4	1	108.1	112
		2	172.7	159
		3	189.3	186
		4	421	416
		5	560	553
2.5	0.2	1	74.9	76
		2	204.5	---
		3	240.1	---
		4	422	414
		5	563.5	561
2.5	0.3	1	74.6	75
		2	141.9	---
		3	163.7	---
		4	407.5	398
		5	557	552
2.5	0.4	1	73.7	74
		2	113.9	---
		3	118.5	---
		4	403	387
		5	556	547

Note: For $\alpha = 2.5$ and $\beta = 0.2, 0.3, 0.4$, it was not possible to get any good response for the second and third modes, even with the full pressure on. There were some responses near about the theoretical values but the presence of subharmonics of higher modes made it very difficult to locate these values accurately and as such they are not filled in. For the value corresponding to the second mode of $\alpha = 1$ and $\beta = 0.4$, it was noticed that the load vibration was very violent. The high variation between theoretical and experimental value seems to be due to slow convergence of the series. Higher number of terms in the series apparently must be taken to improve these values.

(b) Cantilever beam

At the earlier stages, a steel bar was chosen for this part of the experiment. But while checking the natural frequencies of the unloaded bar, it was found that the support was too weak for this purpose (Figure 8). Consequently, an aluminum bar had to be used. The different weights that were made to suit the different values of α, were no longer found adequate when the beam material was changed. Instead of making the weights again, it was decided to draw curves of theoretical values of frequencies vs. α for each β and compare the experimental frequencies for those α values that were available from all the weights. Only three α values were available and these are 3.64, 6.18 and 7.915. The following calculations are made on this basis. The data for this setup are given below.

Weight of clamp	$= 2\frac{7}{16}$ # (Figure 11)
Weight of beam	$= 1\frac{3}{16}$ # (Figure 2)
Weight of clamping piece	$= \frac{11}{32}$ # (Figure 13a)
Weight of first mass	$= \frac{21}{32}$ # (Figure 13b)
Weight of second mass	$= 3\frac{1}{16}$ # (Figure 13c)
Weight of third mass	$= 4\frac{11}{16}$ # (Figure 13d)

Referring to Figure 2,

Weight of (A) + (B) = $1\frac{3}{16}$ #. So weight of (A) = 0.945#.

$$I_r = \frac{\alpha m_p L^2}{4}\beta^2 = \frac{94.5\alpha\beta^2}{g} \qquad (49)$$

Clamp : steel (Figure 11)

$$\text{Rotatory inertia of clamp} = \rho\left[\int_{\frac{5}{16}}^{1\frac{1}{2}} \frac{5}{8}\,\pi r^3 dr + \int_{1\frac{1}{2}}^{9\frac{1}{2}} \frac{x^2 dx}{2}\right]$$

$$= \frac{40.6}{g} \qquad (50)$$

Let I_x = Moment of inertia of any mass about an axis x, through its center of gravity.

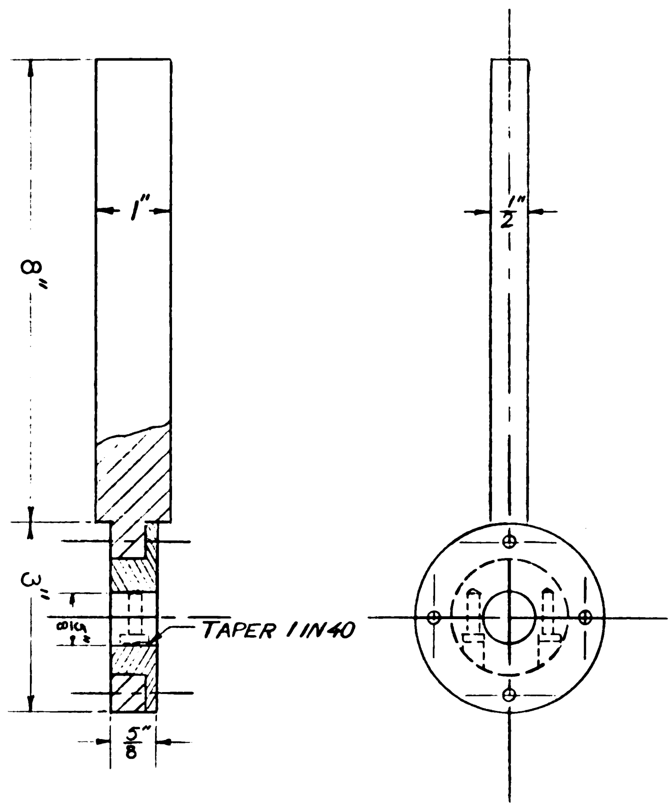


Figure 11. Clamp for cantilever beam.

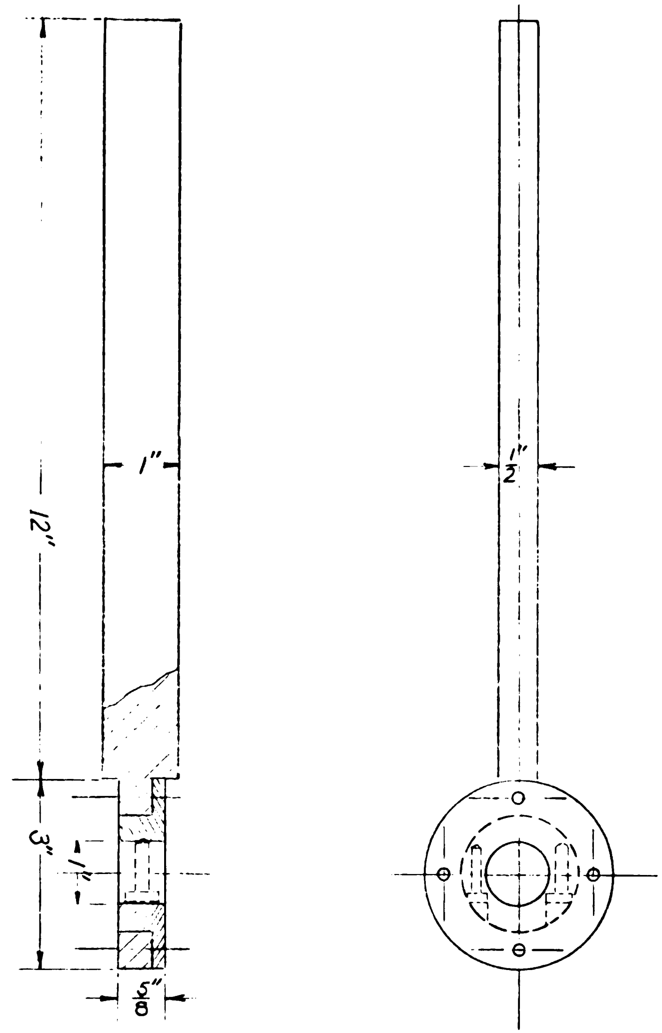


Figure 12. Clamp for simply supported beam.

In the following, I_x of the clamping piece (Figure 13a) will be neglected.

First mass: Figure 13b and clamping piece 13a

Total weight on bar = $3\frac{7}{16}$ #

So, $\alpha = 3.64$

From Equation (49), $I_r = \frac{343.75}{g} \beta^2$

Comparing with Equation (50), $\beta = 0.1$, 0.2 and 0.3 are not possible.

$$I_x \text{ of first mass } = \frac{0.722}{g} \tag{51}$$

For $\beta = 0.4$ $\quad I_r = \frac{55.0}{g} = \frac{40.6}{g} + \frac{0.722}{g} + \frac{L^2}{g}$

So, $L = 3.70''$

Similarly for $\beta = 0.5$ $\quad L = 6.685''$.

Second mass: Figure 13c and clamping piece 13a

Total weight on bar = 5.8431#

So, $\alpha = 6.18$, $I_r = \frac{584.31}{g} \beta^2$

$$I_x \text{ of second mass } = \frac{0.575}{g} \tag{52}$$

Inertia at distance $L = \frac{0.575}{g} + \frac{49}{16g} (L + 0.23)^2$

$$\text{So, total inertia } = \frac{41.3368}{g} + \frac{1.408L}{g} + \frac{3.4065L^2}{g} \tag{53}$$

With these, the following values are obtained.

For $\beta = 0.4$ $\quad L = 3.72''$

For $\beta = 0.5$ $\quad L = 5.335''$

Third mass: Figure 13d and clamping piece 13a.

Total weight on bar = 7.469#

So $\alpha = 7.915$, $I_r = \frac{746.9}{g} \beta^2$

$$I_x \text{ of third mass } = \frac{1.54}{g} \tag{54}$$

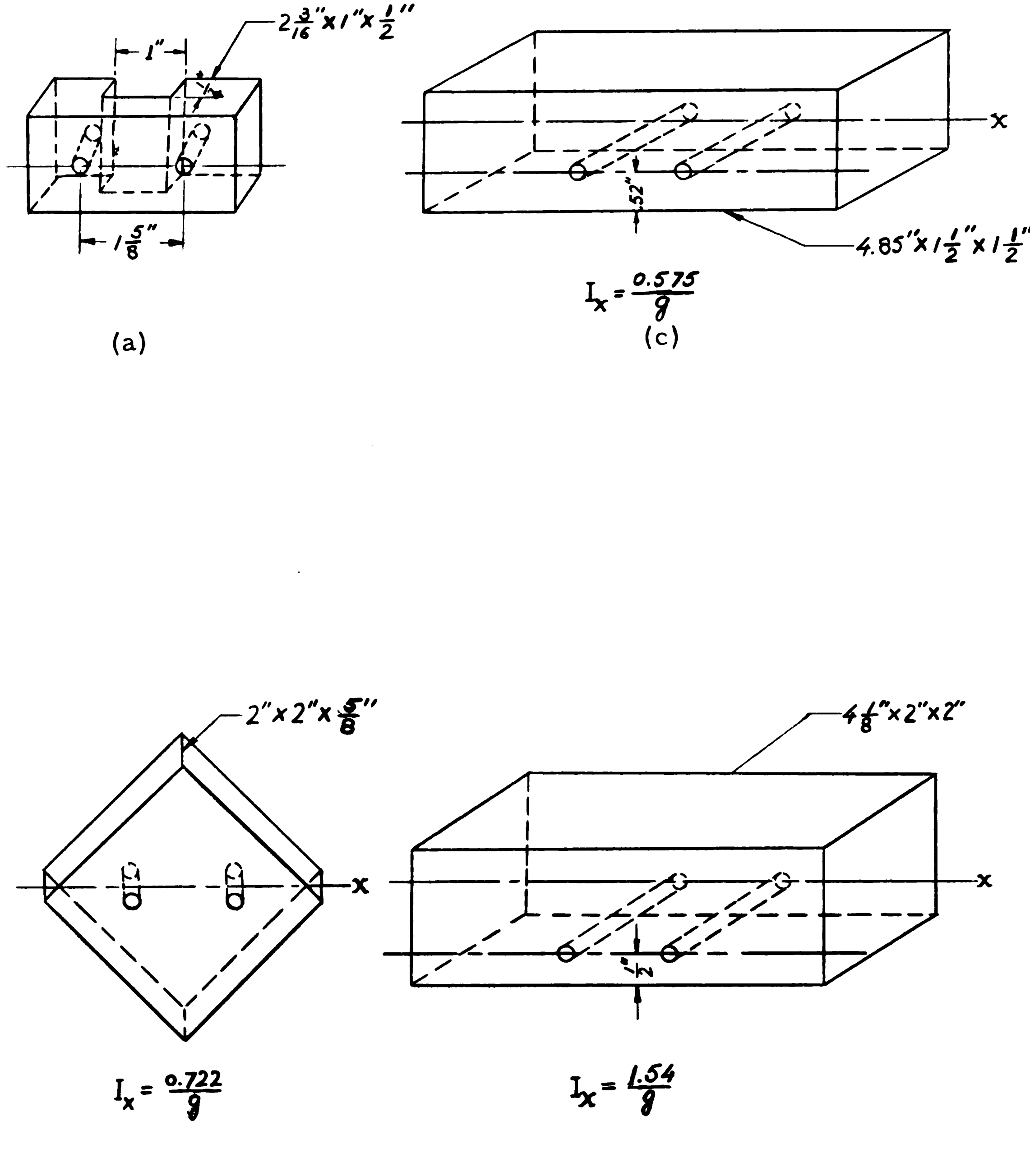


Figure 13. Masses for cantilever beam.

Inertia at distance $L = \frac{1.54}{g} + \frac{75}{16g}(L + 0.5)^2$

$$\text{So, total inertia} = \frac{43.312}{g} + \frac{4.6875L}{g} + \frac{5.0315L^2}{g} \qquad (55)$$

Therefore, for $\beta = 0.4$ $\quad L = 3.45''$

for $\beta = 0.5$ $\quad L = 4.89''$

Table 14 contains the experimental values obtained and these are compared with the theoretical values obtained from Figure 3.

(c) Uniform beam

The same clamp and weights were used for both the settings, the first one being for the masses at the center and the second one for that at quarter point away from one end. Following are the data for this part.

Beam length between supports = 35"

Diameter of beam (uniform) = 1"

Material - Carbon steel drill rod (SAE1096)

Young's Modulus = 30.5×10^6 psi

Density of beam material = 0.283# in^3.

Beam weight = $7\frac{13}{15}$#

Weight of clamp = $2\frac{15}{16}$ #

Weight of clamping piece = $\frac{11}{32}$#

Weight of first mass (α = 0.5) = $\frac{21}{32}$#

Weight of second mass (α = 1.0) = $4\frac{11}{16}$#

Weight of third mass (α = 2.5) = $16\frac{19}{32}$#

Weight of fourth mass (α = 5.0) = $36\frac{5}{8}$#

The following calculations are made on this basis. The same clamping piece as in Figure 13a was used.

Table 14. Experimental frequencies of cantilever beam with and without attached mass.

α	β	Mode	Theoretical Frequency	Experimental Frequency
0	0	1	92.8	91
		2	390.8	385
		3	963.8	947
		4	1795,2	---
3.64	0.4	1	24.9	25
		2	89.75	89
		3	664	657
		4	858	857
	0.5	1	23.45	22.7
		2	77	77
		3	664	648
		4	850	846
6.18	0.4	1	19.5	20
		2	70.6	71
		3	663	658
		4	855	850
	0.5	1	18.3	17
		2	59.75	60
		3	663	655
		4	845	839
7.915	0.4	1	17.3	17
		2	61.5	58
		3	660	651
		4	850	841
	0.5	1	15.63	14.7
		2	52.8	52
		3	660	650
		4	846	838

Clamp : steel (Figure 12)

$$\text{Rotatory inertia of clamp} = \rho\left[\int_{\frac{1}{2}}^{1\frac{1}{2}} \tfrac{5}{8} \quad \pi r^3 dr + \int_{1\frac{1}{2}}^{13\frac{1}{2}} \frac{x^2 dx}{2} \right]$$

$$= \frac{116.4}{g} \tag{56}$$

$$I_r = \frac{\alpha m_b L^2}{4} \beta^2 = \frac{2400}{g} \alpha\beta^2 \tag{57}$$

$\alpha = 0.5$; From Equation (57) $\quad I_r = \dfrac{1200}{g} \beta^2$

From Equation (51) $\quad I_x = \dfrac{0.722}{g}$

$$\text{So, } I_r = \frac{1200}{g} \beta^2 = \frac{116.4}{g} + \frac{0.722}{g} + \frac{L^2}{g}$$

For $\beta = 0.4$ $\qquad L = 8.9''$

For $\beta = 0.5$ $\qquad L = 13.7''$ (not possible).

$\alpha = 1.0$; From Equation (57) $\quad I_r = \dfrac{2400}{g} \beta^2$

From Equation (54) $\quad I_x = \dfrac{1.54}{g}$

$$\text{So, } I_r = \frac{2400}{g} \beta^2 = \frac{116.4}{g} + \frac{1.54}{g} + \frac{11}{32g} L^2 + \frac{75}{16g} (L + 0.5)^2$$

From this, for $\beta = 0.3$ $\quad L = 4.05''$

for $\beta = 0.4$ $\quad L = 6.90''$

for $\beta = 0.5$ $\quad L = 9.46''$

$\alpha = 2.5$; From Equation (57) $\quad I_r = \dfrac{6000}{g} \beta^2$

Referring to Figure 14a,

$I_{x_1} = \dfrac{2.795}{g}$; I_{x_2} = negligible ; I_{x_3} = negligible

$m_1 = \dfrac{14.92}{g}$; $m_2 = \dfrac{1.565}{g}$; $m_3 = \dfrac{0.115}{g}$

$$I_r = \frac{6000}{g} \beta^2 = \frac{147.065}{g} + \frac{39.172}{g} L + \frac{16.94}{g} L^2$$

So, for $\beta = 0.3$ $\quad L = 3.87''$

for $\beta = 0.4$ $\quad L = 5.97''$

for $\beta = 0.5$ $\quad L = 8.06''$

$\alpha = 5.0$; From Equation (57), $I_r = \frac{12000}{g}\beta^2$

Referring to Figure 14b,

$$I_{x_1} = \frac{35.5}{g} \; ; \quad I_{x_2} = \frac{0.426}{g} \; ; \quad I_{x_3} = \frac{0.404}{g}$$

$$m_1 = \frac{26.7}{g} \; ; \quad m_2 = \frac{5.11}{g} \; ; \quad m_3 = \frac{4.84}{g}$$

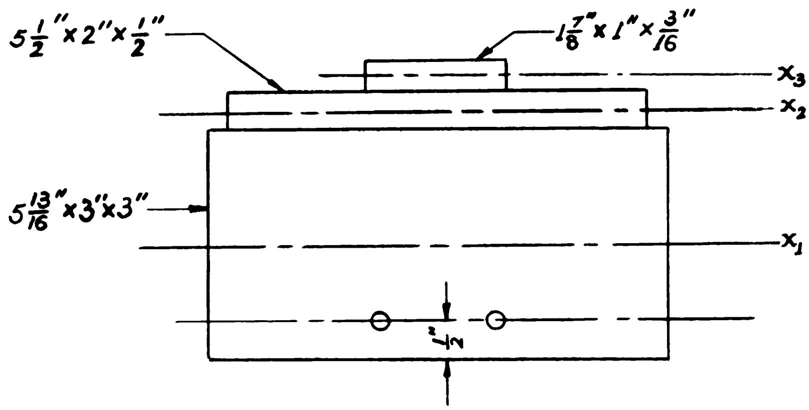
$$I_r = \frac{12000}{g}\beta^2 = \frac{243.93}{g} + \frac{59.43}{g}L + \frac{36.99}{g}L^2$$

From this, for $\beta = 0.3$ $\quad L = 4.08''$

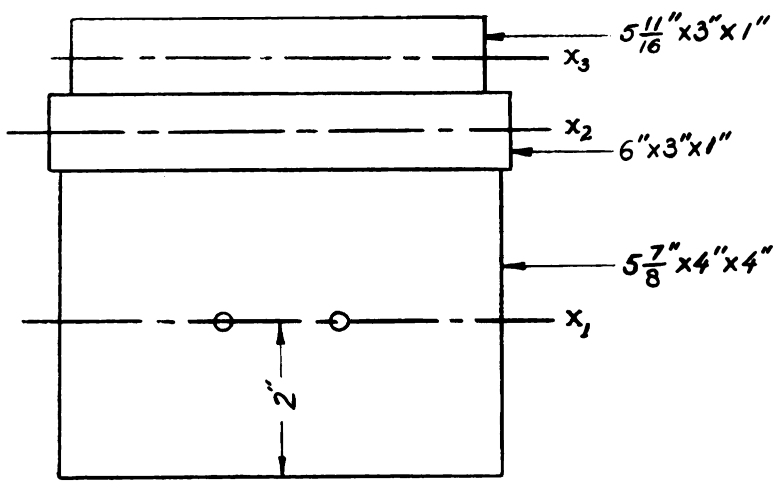
for $\beta = 0.4$ $\quad L = 6.06''$

for $\beta = 0.5$ $\quad L = 7.975''$.

Table 15 contains the frequency values for the system with the mass at the center and Table 16 contains those with the mass at quarter point.



(a)



(b)

Figure 14. Masses for simply supported beam.

Table 15. Experimental frequencies of uniform simply supported beam with mass at the center.

α	β	Mode	Theoretical Frequency	Experimental Frequency
0.5	0.4	1	45.2	46.6
		2	145	145
		3	466	473
		4	483	484
1.0	0.3	1	36.8	37
		2	139	139.5
		3	440	437
		4	477	478
	0.4	1	36.8	37
		2	110	111
		3	440	437
		4	457	456
	0.5	1	36.8	37
		2	90.2	92
		3	440	437
		4	447	446
2.5	0.3	1	26	27
		2	94.5	94
		3	418	421
		4	448	447
	0.4	1	26	27
		2	72.4	71
		3	418	421
		4	440	439
	0.5	1	26	27
		2	58.6	57
		3	418	421
		4	436.5	433
5.0	0.3	1	19.2	19
		2	68.6	68
		3	411	414
		4	439	437

Continued

Table 15 - Continued

α	β	Mode	Theoretical Frequency	Experimental Frequency
5.0	0.4	1	19.2	19
		2	52	52
		3	411	414
		4	436	434
	0.5	1	19.2	19
		2	41.8	41
		3	411	414
		4	433.5	430

Table 16. Experimental frequencies of uniform simply supported beam with mass at quarter point.

α	β	Mode	Theoretical Frequency	Experimental Frequency
0.5	0.4	1	48.75	49
		2	187	188
		3	252	249
		4	627.5	619
1.0	0.3	1	42.3	43
		2	165	167
		3	244	251
		4	617	619
	0.4	1	40.7	42
		2	147.2	146
		3	216	214
		4	612	618
	0.5	1	38.9	35
		2	131	134
		3	206	208
		4	609	615
2.5	0.3	1	30.9	30
		2	119	117
		3	204.6	197
		4	602	594
	0.4	1	29.4	28
		2	98.6	100
		3	195.8	198
		4	600	593
	0.5	1	27.65	27
		2	85.5	84
		3	192.8	188
		4	599	591
5.0	0.3	1	23.15	22
		2	87.2	83
		3	193	191
		4	597	590

Table 16 - Continued

α	β	Mode	Theoretical Frequency	Experimental Frequency
5.0	0.4	1	21.9	21
		2	70.75	72
		3	189	188
		4	596	560
	0.5	1	20.5	19
		2	61	64
		3	187.5	185
		4	550	555

Discussion of Experimental Results

The following remarks seem appropriate before the final results are discussed. In the case of the simply supported uniform beam with a mass at mid-point, the first mode was quite simple to find. The amplitude became fairly large and the scope trace was almost perfectly sinusoidal. The higher modes required judgment on the part of the investigator. As the frequency was increased, the scope trace did not remain sinusoidal until the next higher mode was reached. It was observed that the amplitude of the next higher mode may become smaller than that shown just before the resonant frequency was reached. The only way to obtain the resonant frequency was by noting the trace which should be sinusoidal.

In the case of the same beam with mass at the quarter point, the maximum amplitude was seen to be at the fourth mode. The remarks made for the previous case apply to this case also. Subharmonics of the fourth mode presented considerable difficulty which had to be reduced by the use of a filter. One important aspect noticed in this experiment was that if the nozzle was placed at any arbitrary point of the beam, the resonant frequency was higher than the theoretical values. When the nozzle was placed at the point where the amplitude was maximum, the resonant frequency was minimum and gave best agreement to the theoretical value. This point of maximum amplitude was very close to the value obtained from the eigenvectors.

The results obtained experimentally agreed fairly well with the theoretical values. There are several reasons for discrepancies as listed below.

(a) Slow convergence of the series: This seemed to be the main reason for the cases of variable section beam and plate. This is obvious from the results of the higher modes. Also, in some cases, particularly the second mode of the loaded plate, the load was observed to vibrate considerably in the plane of the plate. This effects the rotatory inertia of the attached mass. The maximum variation of experimental values from the theoretical values are most evident in this mode.

(b) The reasons given in (19) applies to both plate and beams. These are (i) determination of exact resonance, (ii) errors in the reading instruments, (iii) actual model differing from the theoretical model, (iv) inaccuracies in the physical constants, (v) support conditions, (vi) vibration of support, (vii) damping in the material, and (viii) effect of air mass. The effects of (ix) large amplitudes, and (x) shear and rotatory inertia of the system are applicable to plate only, because a vibration pick-up, used in the case of beam, reduced the amplitude and shear and rotatory inertia effects were also taken care of.

(c) Rotatory inertia I_x of the clamping piece. In the calculation of length to produce certain amount of rotatory inertia, this I_x part was neglected. The effect of this is quite small and should be less than a fraction of a percent.

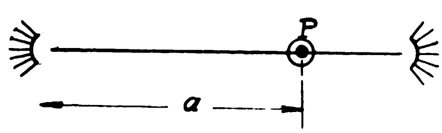
(d) Slackness in tightening bolts. When the masses were fixed on the clamps, they were fastened with two bolts to the clamping piece. It was seen that if these bolts are loose, the frequency was higher because the load had the freedom to remain at the same place instead of moving with the clamp. The contribution due to this appears negligible because the bolts were checked occasionally.

(e) Variation of α values. As may be seen (page 58) for the uniform beam, the weights that were used to produce particular values of α were a little off. The maximum variation was less than one percent and as such the variation in frequencies should be less than one percent.

CHAPTER V

REDUCED MASS SYSTEM

From a paper by D. Young (11) it was seen that the effect of a concentrated mass on a beam can be approximated by reducing the mass of the beam at the position of the concentrated mass in such a way that the natural frequency of the beam without concentrated mass is same as a single mass placed at the position of the concentrated mass and assuming the intertia of the entire beam to be zero. It is stated in the above paper that this method is valid only for the fundamental mode. It is the purpose of this chapter to show that the above method can be used for higher modes also. And the same method may be used, with sufficient accuracy, for plates as well. For this method, the natural frequencies of the unloaded beam or plate are needed, which can be derived by standard formulas and also the stiffness coefficients of the beam or plate at the positions of the concentrated masses. These can be derived or in complicated cases, they may be obtained by measurement. The procedure for beam may be described as follows. The same applies for plate as well.



Let there be a concentrated mass at point P, at a distance "a" from the left support (figure at left). The beam may have any type of supports at the ends. If the first mode frequency of the loaded beam is required, then the mass of the beam is replaced at P by a reduced mass m_r, where m_r is unknown. Let the spring stiffness (force per unit deflection) of the beam at P be k. Then from the elementary formula for a spring mass system, it is known that natural frequency is given by,

$$\omega^2 = \frac{k}{m_r} \tag{58}$$

where ω is in radians per sec.

In order to use this method, ω must be known. From Equation(58),

$$m_r = \frac{k}{\omega^2} \tag{59}$$

As k depends on the position P, even for the same beam, m_r depends on P also. After m_r is obtained, this mass is superposed on the concentrated mass M at P. Then the frequency of the beam with the concentrated mass is given by

$$p^2 = \frac{k}{(M + m_r)} = \frac{k}{(M + \frac{k}{\omega^2})} \tag{60}$$

Example: Simply supported uniform beam with a mass at the center (Figure 15a). From elementary theory

$$y\Big]_{max} = \frac{PL^3}{48EI}$$

Therefore, k = spring stiffness at center

$$= \frac{48EI}{L^3}$$

For this beam, $\omega_1^2 = \dfrac{\pi^4 EI}{m_b L^3}$

So, from Equation(59), $m_r = 0.492767\ m_b$.

Let a concentrated mass M be placed at the center and assume $\alpha = \dfrac{M}{m_b}$

From Equation(60),

$$p_1^2 = \frac{48EI}{m_b L^3(\alpha + 0.492767 m_b)} \tag{61}$$

Table 17 shows the values of p_1 for six different values of α and these values are compared with the values obtained from Equation(40), with $\beta = 0$.

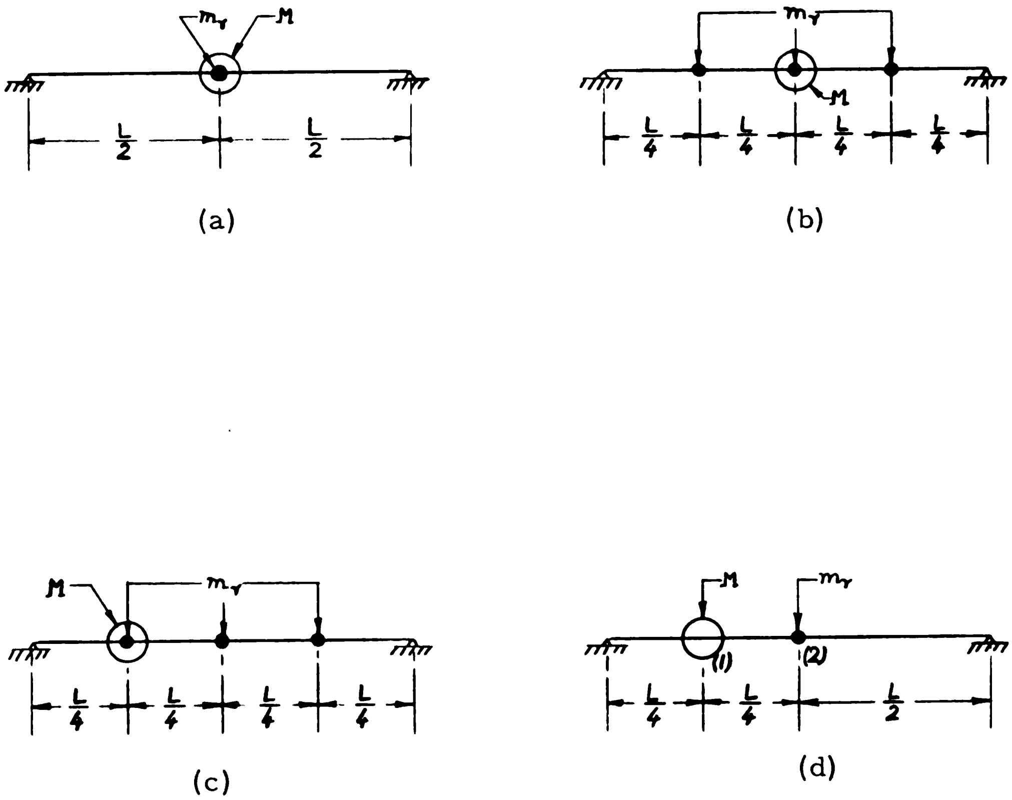


Figure 15. Mass arrangements for reduced mass system.

Table 17. First mode frequencies for a simply supported beam with a concentrated mass at the center, obtained by reduced mass method.

a	p_1 from Equation (61) $\times \sqrt{\frac{EI}{mL^3}}$	p_1 from Equation (40) $\times \sqrt{\frac{EI}{mL^3}}$	Variation in p.c.
0.5	6.9534	6.9661	-0.182
1.0	5.6705	5.6797	-0.162
2.5	4.0048	4.0089	-0.102
5.0	2.9561	2.9580	-0.064
7.5	2.4506	2.4517	-0.044
10.0	2.1388	2.1396	-0.037

If there are n masses or the n^{th} mode is required, then the beam mass should be replaced by n masses. If there are concentrated masses, then these reduced masses should be placed on the concentrated masses, in order to keep the order of the matrix to a minimum. But if only the n^{th} mode is required with lesser number of concentrated masses, then they may be placed anywhere. However, if placed on equal intervals, the computation becomes a little simpler.

Example: As an example, the previous example is taken and the third mode is evaluated. (The second mode is same as an unloaded beam, since the rotatory inertia of the mass is neglected.) For this case, the beam mass is divided into three equal masses m_r at quarter points and mid-point, as shown in Figure 15b. To find the frequency, the influence coefficient method is used. Denoting by δ_{nf} the deflection at n produced by an unit force at f, the following relations may be obtained from any strength of materials book (see Reference 30).

$$\delta_{11} = \delta_{33} = 18c$$

$$\delta_{12} = \delta_{21} = \delta_{32} = \delta_{23} = 22c$$

$$\delta_{13} = \delta_{31} = 14c$$

$$\delta_{22} = 32c$$

where $c = \dfrac{L^3}{1536EI}$

Considering A_1, A_2 and A_3 as the amplitudes of the three locations and assuming harmonic oscillations, the following three equations are obtained.

$$A_1 = m_r p^2 c[18A_1 + 22A_2 + 14A_3]$$

$$A_2 = m_r p^2 c[22A_1 + 32A_2 + 22A_3]$$

$$A_3 = m_r p^2 c[14A_1 + 22A_2 + 18A_3]$$

Solving these equations give,

$$\lambda_1 = 63.1126984, \; \lambda_2 = 4.0, \; \lambda_3 = 0.8873016$$

where $\lambda = \dfrac{1}{m_r p^2 c}$.

Since the highest frequency is needed,

$$p_3^2 = \frac{1}{m_r \lambda_3 c} = \frac{(k_3 L)^3 EI}{m_b L^3}$$

from which, $m_r = 0.2193994 m_b$.

Adding mass M of the concentrated mass to the middle reduced mass, gives

$$m_1 = m_r$$
$$= 0.2193994 m_b$$
$$m_2 = m_r + M$$
$$= m_b(\alpha + 0.2193994)$$
$$m_3 = m_r$$
$$= 0.2193994 m_b$$

With these three new masses, the frequency equations become,

$$\lambda A_1 = 3.9491892A_1 + (22\alpha + 4.8267868)A_2 + 3.0715916A_3$$

$$\lambda A_2 = 4.8267868A_1 + (32\alpha + 7.0207808)A_2 + 4.8267868A_3 \qquad (62)$$

$$\lambda A_3 = 3.0715916A_1 + (22\alpha + 4.8267868)A_2 + 3.9491892A_3$$

The solution of these equations, with $\alpha = 0.5$, give,

$\lambda_1 = 29.744403$; $\lambda_2 = 0.8775976$, $\lambda_3 = 0.2971586$
from which $p_3^2 = 5168.957 \dfrac{EI}{m_b L^3}$.

It is interesting to note that the result obtained from Equation (40) was $p_3^2 = 5162.10 \dfrac{EI}{m_b L^3}$, a variation of only 0.102%. Considering p_1^2, it is found that $p_1^2 = 51.640 \dfrac{EI}{m_b L^3}$, whereas, the value obtained from Equation (40) was $48.526 \dfrac{EI}{m_b L^3}$. This variation is 6.4%, which is fairly high. From this it is concluded that this method is accurate only for the particular mode for which the unloaded frequency is matched.

Table 18 compares the results obtained from both the methods for different values of α.

Table 18. Third mode frequencies for a simply supported beam with a concentrated mass at the center, obtained by reduced mass method.

α	p_3 from Equation (62) $\times \sqrt{\dfrac{EI}{m_b L^3}}$	p_3 from Equation (40) $\times \sqrt{\dfrac{EI}{m_b L^3}}$	Variation in p.c.
0.5	71.895	71.848	0.065
1.0	68.459	67.932	0.776
2.5	65.627	64.599	1.591
5.0	64.497	63.244	1.981
7.5	64.095	62.758	2.130
10.0	63.888	62.509	2.206

Next, the third mode of an unsymmetrical case is investigated, as in Figure 15c. As in the previous case,

$$m_r = 0.2193994\, m_b$$

Adding the concentrated mass M to the left m_r, the frequency equations are found to be

$$\begin{aligned} \lambda A_1 &= (18\alpha + 3.9491892)A_1 + 4.8267868A_2 + 3.0715916A_3 \\ \lambda A_2 &= (22\alpha + 4.8267868)A_1 + 7.0207808A_2 + 4.8267868A_3 \\ \lambda A_3 &= (14\alpha + 3.0715916)A_1 + 4.8267868A_2 + 3.9491892A_3 \end{aligned} \tag{63}$$

Substituting different values of α, the following values of frequencies are obtained, as shown in Table 19, and these are compared with the values obtained from Equation (47).

Table 19. Third mode frequencies for a simply supported uniform beam with a concentrated mass at quarter point, obtained by reduced mass method.

α	p_3 from Equation (63) $\times \sqrt{\frac{EI}{m_b L^3}}$	p_3 from Equation (47) $\times \sqrt{\frac{EI}{m_b L^3}}$	Variation p.c.
0.5	81.689	81.516	0.212
1.0	80.558	80.139	0.523
2.5	79.699	79.049	0.822
5.0	79.374	78.627	0.950
7.5	79.261	78.480	0.995
10.0	79.203	78.404	1.019

This method is helpful in finding the frequencies not only for masses placed at the positions of the reduced masses as explained above, but also can be used for any other positions. In this case, the order of the matrix to be solved increases. This is illustrated by an example below.

Example: Let there be a mass at quarter point of the beam as shown in Figure 15d, and assume that the first mode m_r is placed at the middle of the beam. For this case,

$$m_r = 0.492767 m_b.$$

To find the first mode frequency, assume two lumped masses, m_r at middle of the beam and M at quarter point. Assuming stations (1) and (2), as shown in Figure 15d,

$$\begin{aligned} \lambda A_1 &= 18 M A_1 + 22 m_r A_2 \\ \lambda A_2 &= 22 M A_1 + 32 m_r A_2 \end{aligned} \qquad (64)$$

As mentioned earlier, the order of the matrix has increased, in this case, to 2. If there were n masses and the first mode frequency is desired, then the order of the matrix will be (n + 1). Solving Equations (64) for different values of α, two frequencies are obtained for each α, the lower one being the first mode frequency. This way the values shown in Table 20 were obtained and they are compared with the corresponding values from Equation (47).

The same procedure may be followed for plates also. The natural frequency and spring stiffness of the plate may be measured or calculated, and from this the reduced mass value is obtained. Adding this reduced mass onto the concentrated mass, the loaded frequency can be easily evaluated.

Table 20. First mode frequencies of a simply supported beam with a concentrated mass at quarter point, obtained by reduced mass method.

α	p_1 from Equation (64) $\times \sqrt{\frac{EI}{m_b L^3}}$	p_1 from Equation (47) $\times \sqrt{\frac{EI}{m_b L^3}}$	Variation in p.c.
0.5	8.031	7.991	0.501
1.0	6.889	6.851	0.555
2.5	5.109	5.087	0.432
5.0	3.851	3.840	0.286
7.5	3.216	3.210	0.187
10.0	2.818	2.814	0.142

CHAPTER VI

SUMMARY AND CONCLUSIONS

Summary

The effect of rotatory inertia of attached masses on the vibration frequencies of beams and plates is analyzed by the method of normal mode superposition; this method is also known as the method of undetermined coefficients. General equations are derived for variable thickness beams and plates with arbitrary number of attached masses. In the numerical examples, only one or two masses are used and the resulting eigen value problem is solved by the use of digital computer. In the experimental part, only one mass was used. The frequency values were obtained experimentally with a pulsed-air vibrator. The theoretical and experimental values are compared and the variations between the two are discussed.

In the latter part of this work, a method is developed by which the frequencies of systems, loaded with concentrated masses, can be predicted by the knowledge of the unloaded frequencies of the system. The results from this method are compared with the results from other sections.

Conclusions

The normal mode superposition method seems to be very well suited for problems concerning vibrations of beams and plates. For uniform systems (beams or plates), the integrals can be evaluated quite easily but for systems with variable thicknesses, the use of the digital computer is essential.

The accuracy of the results depends mainly on the number of terms taken in the series expansion of the displacement function. It is apparent that when a load is added to a uniform beam, a certain number of terms of the series are needed for certain accuracy. The number of terms needed for the same accuracy will increase if rotatory inertia of the load is taken into account. If in addition to this, the beam happens to be of variable thickness or the uniform beam is replaced by a uniform plate, a still higher number of terms in the series expansion will be necessary. As may be seen, for a plate of variable thickness with attached mass, the number of terms required will be very high. Since extreme accuracy is not the primary object of this investigation, only nine terms were taken. The extrapolation formula used seems to improve the values reasonably well, at least for the higher modes. However, there is some reservation in the mind of the author at using this formula. A point of uncertainty is the way the shear deformation term is introduced. The transformation of the terms of the differential equation to energy forms was necessitated by the requirement of introduction of the loads and rotatory inertia of the loads. Since the shear deformation term introduces very little correction, especially for the lower modes, the effect of error in the assumption for shear deformation will not effect the final results much. Actually it seems to improve the results as expected. Further study in this area seems to be in order.

When this work was started, it was assumed that normal functions are the best functions in terms of which the deflections may be represented. But as seen from the wedge problem in Appendix E, it may be concluded to be erroneous. Also, normal functions are difficult to handle, particularly for clamped and free edge conditions. This suggests the necessity of investigating other functions, mainly polynomials, which can be used for problems of this type. This will reduce the amount of time required for numerical calculations.

From the curves of frequency vs β with α as parameter, it looks, as if, with higher β, the frequency remains constant. When β is high, however, it means that physically the mass or masses do not rotate. This may be used as a means of application of bending moment or other constraints for further investigation. The same conclusions can be drawn for higher α. In this case, the translatory motion of the mass is reduced.

The results presented here from theory seem to compare quite well with experimental values. To improve the values, two aspects need considerable attention. The most important factor is to use higher number of terms in the theoretical calculations. Even with the best extrapolation formulae, it is not always possible to get very good results. The second part that needs attention is the support conditions in the experimental set-up. The simple support for the beam was fairly good with careful lubrication of the pins and occasional checking of the support screws. But the cantilever support was definitely weak as shown by the results of natural frequencies. The worst case of support was found in the case of plate. Some other design of support seems essential for better verification of the theory.

In the reduced mass method, some of the results seem very good whereas some are not so good. Even though the variations are within 2.5 p.c., this may perhaps be improved. The only reason that can be offered for discrepancies is error in numerical calculations. Further investigation in this line seems advisable.

The following are a few of the items that can be suggested for future investigations. They are given separately for beams, plates, etc.

Beams

(1) Variation of cross-sections other than assumed here. If stepped beam is used, the integrations will contain limit points other than from 0 to 1.

(2) Include the neglected terms for shear deformation. Also a better relation for shear deformation may, perhaps, be derived. One helpful suggestion is to use the normal functions given by T. C. Huang (31).

(3) The nature of constraints that can be incorporated by proper selections of α and β quantities, as mentioned earlier in this chapter.

(4) Addition of work done by W. F. Z. Lee and E. Saibel (12) to the present work, which opens up a whole variety of problems that can be solved quite easily. This may include the cases of continuous beams, sprung masses, elastic foundation, etc.

(5) The effect of stretching of the center line of the beam due to addition of load.

(6) Study of visco-elastic beams.

(7) Use of finite difference for this type of problems.

(8) Forced vibration.

Plates

(1) Inclusion of shear and rotatory inertia of plate. Procedure outlined in (32) may be helpful in this respect.

(2) Use of other end conditions than simple supports.

(3) Continuous plates.

(4) Use of finite difference.

(5) Effect of stretching of the middle plane due to application of loads.

(6) Study of viscoelastic plates.

(7) Forced vibration.

Reduced mass

(1) To include rotatory inertia effects of loads.

(2) Approximate non-linear behaviour of beams or plates, with load attached. Since the method reduces the whole system to a spring mass system, this study seems possible.

In the experimental part, subharmonics and ultraharmonics were observed in almost all cases. One important difference between the two was noticed. Assuming p to be the main frequency, the subharmonics of p had the frequency of p, as measured from the oscilloscope trace, but ultraharmonics had the frequency of the ultraharmonic itself. And this created a little confusion at the early part of the experiment.

Comparing with the natural frequency results of B. B. Raju (19) it is observed that the finite difference results gave a lower bound whereas, this method gave an upper bound to the actual frequency values. The explanation for the latter case seems to follow from Rayleigh's principle but the reason for the lower bound in the former case is not clear. All the same, further investigation should be carried out to establish the validity of this observation.

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APPENDICES

APPENDIX A

WORK DONE BY ROTATORY INERTIA OF ATTACHED MASSES ON PLATE

To include the effect of rotatory inertia of the masses, it should be borne in mind that the masses will, in general, rotate about both x axis as well as y axis and the net effect will be rotation about some other intermediate axis in the x, y plane. To appreciate this physically, consider a right handed Cartesian coordinate system and let a body of mass m be located at point P at a distance L below the x, y plane i.e. in z direction. When the plate vibrates, there will be inertia forces generated due to inertia of the mass and this in turn will produce a torque T_n. Let its components be T_x and T_y in the x and y directions respectively.

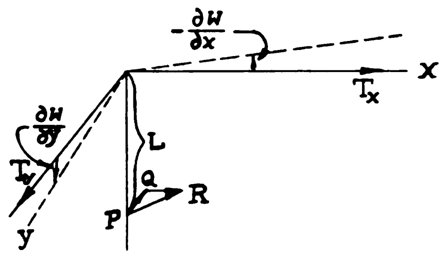


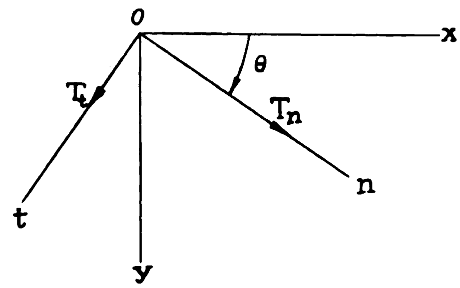
Figure (a)

From elementary theory,

$$T_n = - I_t \frac{\partial^2 \theta_n}{\partial t^2}$$

From Figure (a), it is clear that due to bending in the x direction, T_y will produce work given by $- T_y \frac{\partial w}{\partial x}$ and due to bending in the y direction, T_x will produce work equal to $T_x \frac{\partial w}{\partial y}$. In Figure (a), the point P moves to Q due to $\frac{\partial w}{\partial y}$ and then from Q to R due to $- \frac{\partial w}{\partial x}$. Since both motions are present at the same time and simple harmonic motion is assumed, the mass will move parallel to PR. It is parallel, because the point P will also be moving up and down. Since the force field, generated by the inertia force, is conservative (no damping assumed), the total work done by the torques T_y and T_x in moving thru' $- \frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ is same as the work done by the torque T_n in moving the point P thru' PR.

(An alternative proof of this is given in the next paragraph.*) This gives a simple relation for the work done by the inertia forces as $T_x(\frac{\partial w}{\partial y}) - T_y(\frac{\partial w}{\partial x})$. In this analysis, it was assumed that T_x and T_y are constant and $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ as small which is valid for virtual work principle.



(Figure (b))

For a mathematical proof, the following may be considered.

Referring to Figure (b), let there be two torques T_n and T_t acting in two normal directions at point 0. If two variations are given in the n and t directions i.e. $\delta(\frac{\partial w}{\partial n})$ and $\delta(\frac{\partial w}{\partial t})$, then the work done by the torques T_n and T_t is

$$U_r = T_n \delta(\frac{\partial w}{\partial t}) - T_t \delta(\frac{\partial w}{\partial n})$$

But $$T_n = T_x \text{ Cos } \theta + T_y \text{ Sin } \theta$$

$$T_t = - T_x \text{ Sin } \theta + T_y \text{ Cos } \theta$$

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} \text{ Cos } \theta + \frac{\partial w}{\partial y} \text{ Sin } \theta$$

$$\frac{\partial w}{\partial t} = - \frac{\partial w}{\partial x} \text{ Sin } \theta + \frac{\partial w}{\partial y} \text{ Cos } \theta$$

Substituting these in the above expression for U_r and simplifying, the same result is obtained e.g.,

$$U_r = T_x \delta(\frac{\partial w}{\partial y}) - T_y \delta(\frac{\partial w}{\partial x})$$

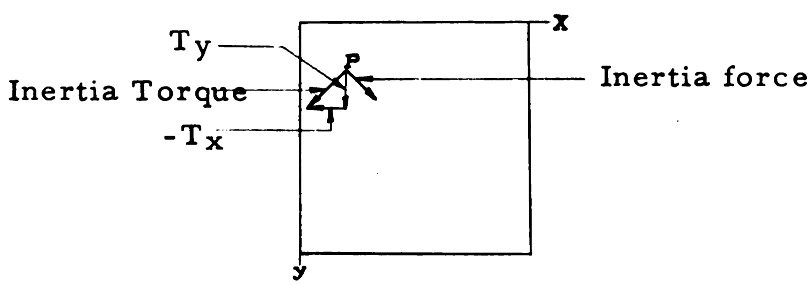


Figure (c)

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*The writer is indebted to Dr. W. A. Bradley, Professor of Applied Mechanics Department, M.S.U., for this proof.

As regards the signs of T_x and T_y, consider Figure (c). Assume that a mass is attached at a certain distance below point P. When the plate vibrates, for positive w, the inertia force will be directed as shown in Figure (c). This force will create a torque as indicated. The components of this torque in the x and y directions have opposite signs. From this it is apparent that both the terms in Equation (14) have negative signs.

APPENDIX B

VERIFICATION OF PLATE EQUATION BY RITZ METHOD

The Ritz method is used in this section to evaluate Equation (12). In this case, the expression for potential energy remains the same as in Equation (4). The kinetic energy of the vibrating plate is given by

$$T = \frac{\rho}{2} \iint h \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy$$

$$= \frac{\rho}{2} p^2 \iint h \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n\right)^2 \text{Cos}^2\, pt\, dx\, dy$$

Let $\rho \iint h \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n\right)^2 \text{Cos}^2 pt\, dx\, dy = Q$ (B-1)

Then $T = p^2 \frac{Q}{2}$

Equating the maximum values of V and T,

$$p^2 = \frac{2\, V_{max}}{Q_{max}} \tag{B-2}$$

Applying Ritz method,

$$\frac{\partial p^2}{\partial A_{ij}} = 0$$

or $$2\, Q_{max} \frac{\partial}{\partial A_{ij}} V_{max} - 2\, V_{max} \frac{\partial}{\partial A_{ij}} Q_{max} = 0 \quad (\text{Since } Q_{max} \neq 0)$$

(B-3)

But from Equation B-2, $V_{max} = \frac{p^2}{2} Q_{max}$.

Substituting in Equation (B-3),

$$2 \frac{\partial}{\partial A_{ij}} V_{max} - p^2 \frac{\partial}{\partial A_{ij}} Q_{max} = 0$$

or $$\frac{\partial}{\partial A_{ij}} \left[2\, V_{max} - p^2\, Q_{max}\right] = 0 \tag{B-4}$$

Substitution of V_{max} from Equation (8) and Q_{max} from Equation (B-1), into Equation B-4 and performing the indicated differentiation, results in

$$\iint D \Big[\Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2X_m}{dx^2} Y_n\Big) \Big(\frac{d^2X_i}{dx^2} Y_j\Big)$$

$$+ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2Y_n}{dx^2}\Big) \Big(X_i \frac{d^2Y_j}{dy^2}\Big)\Big\}$$

$$+ \nu \Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{d^2X_m}{dx^2} Y_n\Big) \Big(X_i \frac{d^2Y_j}{dy^2}\Big)$$

$$+ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m \frac{d^2Y_n}{dy^2}\Big) \Big(\frac{d^2X_i}{dx^2} Y_j\Big) \Big\}$$

$$+ 2(1-\nu) \Big\{ \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{dX_m}{dx} \frac{dY_n}{dy}\Big) \Big(\frac{dX_i}{dx} \frac{dY_j}{dy}\Big) \Big\} \Big] dx\, dy$$

$$= p^2 \rho \iint h \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m Y_n\Big) (X_i Y_j)\, dxdy$$

which is the same as Equation(12).

APPENDIX C

WORK DONE BY ROTATORY INERTIA OF ATTACHED MASSES ON BEAM

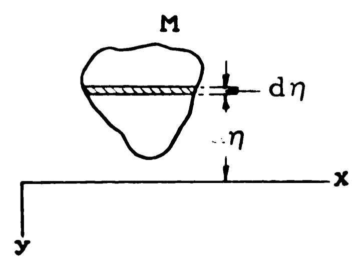

Figure (a)

To find the virtual work done by the rotatory inertia of the masses, an arbitrary mass M is taken as shown in Figure (a). Considering an element of the mass at a distance η from the x axis gives

$$d(\text{Torque}) = d\,(\text{Volume})\rho x(-\text{Acceleration}) \times \eta$$

where ρ = mass density.

If the mass rotates through an angle $\theta = \frac{\partial y}{\partial x}$ about the z axis, then

$$d(\text{Torque}) = -(dV)\rho(\ddot{\theta}\ \eta)\ \eta$$

$$= -\rho \frac{\partial^3 y}{\partial x \partial t^2}\ dV\ \eta^2$$

$\therefore$ virtual work done $= -\rho \frac{\partial^3 y}{\partial x \partial t^2}\ dV\ \eta^2\ \delta\left(\frac{\partial y}{\partial x}\right)$

$$\text{Total virtual work done} = -\int_V \rho \frac{\partial^3 y}{\partial x \partial t^2}\ \eta^2\ \delta\left(\frac{\partial y}{\partial x}\right) dV.$$

Considering only rigid mass and uniform density, it is found that ρ, $\frac{\partial^3 y}{\partial x \partial t^2}$ and $\delta\left(\frac{\partial y}{\partial x}\right)$ is independent of the integration variable. As such, the total work done is given by

$$-\rho \frac{\partial^3 y}{\partial x \partial t^2}\ \delta\left(\frac{\partial y}{\partial x}\right) \int_V \eta^2\ dV$$

$$= -I \frac{\partial^3 y}{\partial x \partial t^2}\ \delta\left(\frac{\partial y}{\partial x}\right)$$

where $I = \rho \int_V \eta^2\ dV$ is the mass moment of inertia about x axis.

APPENDIX D

WORK DONE BY ROTATORY INERTIA AND SHEAR DEFORMATION IN BEAM

Rotatory Inertia

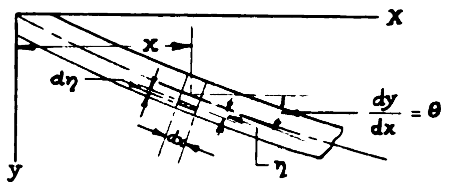

Figure (a)

To find the virtual work done by the rotatory inertia of the beam, an elemental volume dV at a distance η from the bent neutral line of the beam is taken (Figure (a))

$$d(\text{mass}) = \rho dV = \rho b d\eta dx \quad \text{where } b = \text{width of beam at } x \text{ and } \eta$$

$$\therefore \ d(\text{force}) = -b\rho(\ddot{\theta}\eta)d\eta\, dx$$

$$\therefore \ d(\text{torque}) = -b\rho\ddot{\theta}\,\eta^2 d\eta dx.$$

$$\therefore \ \text{Torque} = -\rho \int_A (\ddot{\theta}\ \eta^2 b d\eta)dx = -\rho\ddot{\theta} I_b dx \qquad \text{(D-1)}$$

$\therefore$ the virtual work done by the section dx is

$$d(\text{work}) = -\rho\ddot{\theta}dx \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} b\eta^2 d\eta\, \delta\left(\frac{\partial y}{\partial x}\right)\right]$$

$$= -\rho\ddot{\theta}\, dx\, I_b\, \delta\left(\frac{\partial y}{\partial x}\right) = -\rho I_b \frac{\partial^3 y}{\partial x \partial t^2}\, dx\, \delta\left(\frac{\partial y}{\partial x}\right)$$

$$\therefore \quad \text{total virtual work done} = -\rho \int_0^L I_b \ddot{\theta} dx \delta\left(\frac{\partial y}{\partial x}\right)$$

$$= -\rho \int_0^L I_b \frac{\partial^3 y}{\partial x \partial t^2}\ \delta\left(\frac{\partial y}{\partial x}\right) dx$$

Shear

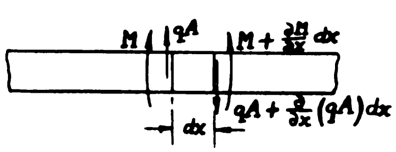

Figure (b)

Considering an element as in Figure (b) it is seen that in order to include the effects of rotatory inertia and shear deformation of the beam, the following relations must be satisfied.

$$-\frac{\partial M}{\partial x} + \frac{\partial}{\partial x}(q\,A) = I_b \rho \frac{\partial^2 \Psi}{\partial t^2}$$

$$\frac{\partial}{\partial x}(q\,A) = A\rho \frac{\partial^2 y}{\partial t^2} \qquad \text{(D-2)}$$

where M = Bending moment across a section = $- EI_b \frac{\partial \Psi}{\partial x}$

q = Average shear = $k'G\beta$ = $k'G(\frac{\partial y}{\partial x} - \Psi)$

Ψ = Slope of center line without shear

β = Shear angle at the center = $\frac{\partial y}{\partial x} - \Psi$

Substituting the respective values in Equations (D-2) gives

$$E\frac{\partial}{\partial x}(I_b \frac{\partial \Psi}{\partial x}) + A\rho \frac{\partial^2 y}{\partial t^2} = I_b \rho \frac{\partial^2 \Psi}{\partial t^2}$$

$$Ak'G(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \Psi}{\partial x}) + k'G\beta \frac{\partial A}{\partial x} = A\rho \frac{\partial^2 y}{\partial t^2} \qquad \text{(D-3)}$$

For a uniform beam, Ψ can be eliminated between these two equations and a single equation may be obtained for y. But when the beam is of variable cross-section, it is difficult to eliminate Ψ. In the case of the cantilever beam under investigation in which the shear deformation has been included

$$A = \frac{\pi}{16}(2 - \frac{x}{L})^2 \quad , \quad \frac{\partial A}{\partial x} = -\frac{\pi}{8L}(2 - \frac{x}{L})$$

$$I = \frac{\pi}{1024}(2 - \frac{x}{L})^4 \quad , \quad \frac{\partial I}{\partial x} = -\frac{\pi}{256L}(2 - \frac{x}{L})^3 \qquad \text{(D-4)}$$

From Equation (D-4) it is seen that when L = 20"

$$(\frac{\partial A}{\partial x})_{max} = -0.0393$$

$$(\frac{\partial I}{\partial x})_{max} = -0.00491 \qquad \text{(D-5)}$$

Since this investigation concerns itself only with modes up to the fourth, the shear angle β at the center may be assumed to be small. As such, the term $k'G\beta \frac{\partial A}{\partial x}$ in the second equation of (D-3) may be neglected in comparison to the first term. This gives

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 y}{\partial x^2} - \frac{\rho}{k'G} \frac{\partial^2 y}{\partial t^2} \tag{D-6}$$

Differentiating the first of Equation (D-3) with respect to x and substituting $\frac{\partial \psi}{\partial x}$ from Equation (D-6), the final form of the equation reduces to

$$E \frac{\partial^2}{\partial x^2}\left(I_b \frac{\partial^2 y}{\partial x^2}\right) + A\rho \frac{\partial^2 y}{\partial t^2} - \rho \frac{\partial}{\partial x}\left(I_b \frac{\partial^3 y}{\partial x \partial t^2}\right) - E \frac{\partial^2}{\partial x^2}\left(\frac{I_b \rho}{k'G} \frac{\partial^2 y}{\partial t^2}\right)$$
$$+ \rho \frac{\partial I_b}{\partial x} \frac{\partial^2 \beta}{\partial t^2} - \frac{I_b \rho}{k'G} \frac{\partial^4 y}{\partial t^4} = 0 \tag{D-7}$$

It is shown in (23) that the last term is of second order compared to the third and fourth terms. Also, for the lower modes, it is reasonable to assume the fifth term to be small since $\frac{\partial I_b}{\partial x}$ is small and $\frac{\partial^2 \beta}{\partial t^2}$ should not be too large. Neglecting these two terms, Equation (D-7) reduces to

$$E \frac{\partial^2}{\partial x^2}\left(I_b \frac{\partial^2 y}{\partial x^2}\right) + A\rho \frac{\partial^2 y}{\partial t^2} - \rho \frac{\partial}{\partial x}\left(I_b \frac{\partial^3 y}{\partial x \partial t^2}\right) - \frac{E\rho}{k'G} \frac{\partial^2}{\partial x^2}\left(I_b \frac{\partial^2 y}{\partial t^2}\right) = 0 \tag{D-8}$$

In Equation (D-8) the third term is the first spatial derivative of the rotatory inertia torque, as may be seen from Equation (D-1). The first term also can be shown to be the first spatial derivative of the elasticity torque (this term is used here to keep the same notation) as follows.

$$\delta U_e = \int_0^L (\text{Torque})\{\delta(\text{angle})\} dx$$
$$= E \int_0^L \frac{\partial}{\partial x}\left(I_b \frac{\partial^2 y}{\partial x^2}\right) \delta\left(\frac{\partial y}{\partial x}\right) dx$$
$$= E \int_0^L \frac{\partial}{\partial x}\left(I_b \frac{\partial^2 y}{\partial x^2}\right) \frac{\partial}{\partial x}(\delta y)\, dx$$

Integrating by parts,

$$\delta U_e = E[I_b \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x}(\delta y)]_o^L - E\int_o^L I_b(\frac{\partial^2 y}{\partial x^2}) \frac{\partial^2}{\partial x^2}(\delta y)\, dx$$

For any standard end conditions, the first term is zero, because

for a fixed end $\frac{\partial}{\partial x}(\delta y) = 0$

for a simple support $\frac{\partial^2 y}{\partial x^2} = 0$

for a free end $\frac{\partial^2 y}{\partial x^2} = 0$

Therefore

$$\delta U_e = - E\int_o^L I_b(\frac{\partial^2 y}{\partial x^2}) \frac{\partial^2}{\partial x^2}(\delta y)\, dx$$

$$= - E\{\int_o^L I_b(\sum_{n=1}^{\infty} A_n \frac{d^2X_n}{dx^2}) \frac{d^2X_i}{dx^2}\, dx\}\, Sin^2 pt\ \delta A_i \qquad \text{(D-9)}$$

where the series expansion of y and δy are substituted from Chapter II, Part (b). As may be seen, Equation (D-9) is the same as Equation (18). It is difficult to show that the second term is the spatial derivative of the inertia torque for a variable cross-section beam. This appears to be due to the neglected term in the second equation of Equations (D-3). An alternate way of defining the shear angle β might remove this difficulty. In the case of a uniform beam, A is constant. Expanding y in terms of an infinite series of normal functions of the beam, it can be shown easily that this represents the inertia torque. This follows easily from the relations of the type $\frac{d^4X_n}{dx^4} = k_n^4 X_n$. From these analogies, it is reasonable to assume that the fourth term is the spatial derivative of the shear torque. With this assumption, the work done by the shear deformation is given by

$$\delta U_{sb} = - \frac{E\rho}{k'G}\int_o^L \frac{\partial}{\partial x}(I_b \frac{\partial^2 y}{\partial t^2})\ \delta\ (\frac{\partial y}{\partial x})\, dx \qquad \text{(D-10)}$$

Since no example was found for the shear correction on a variable thickness beam, the case of an uniform simply supported beam is taken to check the validity of the above assumptions. This case is treated in (23). For this case, Equation(26) reduces to

$$p^2[\rho A \int_o^L (\sum_{n=1}^{\infty} A_n X_n) X_i dx + \rho I_b \int_o^L (\sum_{n=1}^{\infty} A_n \frac{dX_n}{dx}) \frac{dX_i}{dx} dx$$

$$+ \rho I_b \frac{E}{k'G} \int_o^L (\sum_{n=1}^{\infty} A_n \frac{dX_n}{dx}) \frac{dX_i}{dx} dx$$

$$= EI_b \{ \int_o^L (\sum_{n=1} A_n \frac{d^2X_n}{dx^2}) \frac{dX_i^2}{dx^2} dx \} \qquad \text{(D-11)}$$

But $X_n = \text{Sin} \frac{n\pi x}{L}$,

$$\int_o^L X_i X_j dx = \frac{L}{2} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

$$\int_o^L \frac{dX_i}{dx} \frac{dX_j}{dx} dx = \frac{i^2\pi^2}{2L} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

and

$$\int_o^L \frac{d^2X_i}{dx^2} \frac{d^2X_j}{dx^2} dx = \frac{i^4\pi^4}{2L^3} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

Substituting these in Equation (D-11), results in

$$p^2[\rho A(A_i \frac{L}{2}) + \rho I_b(A_i \frac{i^2\pi^2}{2L}) + \rho I_b \frac{E}{k'G}(A_i \frac{i^2\pi^2}{2L})] = EI_b(A_i \frac{i^4\pi^4}{2L^3}) \qquad \text{(D-12)}$$

Assuming ρAL = mass of the beam = m_b

$$\lambda = \frac{L}{i}$$

$$a^2 = \frac{EI_b L}{m_b}$$

and $I_b \rho L = \frac{I_b m_b}{A} = \gamma^2 m_b$

and substituting in Equation (D-12) gives

$$p^2 = \frac{\pi^4 a^2}{\lambda^4} \frac{1}{[1 + \frac{\pi^2\gamma^2}{\lambda^2} + \frac{E}{k'G}\frac{\pi^2}{\lambda^2}\gamma^2]}$$

Considering the last two terms as small compared to unity, the denominator can be expanded in a binomial series. Neglecting all higher order terms, it is found that

$$p = \frac{\pi^2 a}{\lambda^2} [1 - \frac{1}{2}\frac{\pi^2\gamma^2}{\lambda^2}(1 + \frac{E}{k'G})] \qquad \text{(D-13)}$$

which is the same as Equation (140) of (23).

APPENDIX E

VIBRATION OF A WEDGE

This problem was investigated with a view toward checking the convergence of the series of normal functions for a uniform cantilever beam. The problem was investigated by G. R. Kirchoff (24) who obtained an exact solution, neglecting rotatory inertia and shear deformation of the wedge.

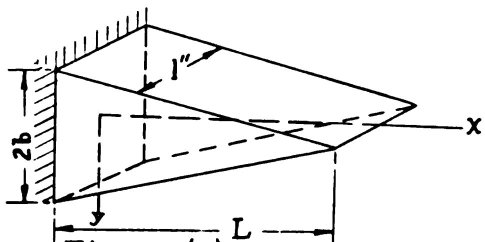

Figure (a)

For this case, Equation (20) represents the frequency equation.

From Figure (a)

$$A = \frac{2b(L-x)}{L}, \quad I_b = \frac{2b^3}{3}\left(1 - \frac{x}{L}\right)^3$$

$$X_n = \text{Cos h } k_n x - \text{Cos } k_n x - \alpha_n(\text{Sin h } k_n x - \text{Sin } k_n x)$$

The values of $k_n x$, α_n were obtained from (26).

Let $\frac{d^p X_n}{dx^p} = k_n{}^p \frac{d^p}{d(k_n x)^p} \phi_n$

With these notations, Equation (20) reduces to

$$p^2\rho \int_0^L A \left(\sum_{n=1}^{\infty} A_n \phi_n\right)\phi_i dx = E \int_0^L I_b \left(\sum_{n=1}^{\infty} A_n k_n^2 \phi_n''\right) k_i^2 \phi_i'' dx \quad \text{(E-1)}$$

where prime represents differentiation with respect to $(k_n x)$.

Assuming only one term of the series in Equation E-1,

$$\frac{2k_1^4 b^2}{3} \int_0^L \left(1 - \frac{x}{L}\right)^3 (\phi_1'')^2 dx = \frac{2p^2\rho}{E} \int_0^L \left(1 - \frac{x}{L}\right)(\phi_1)^2 dx \quad \text{(E-2)}$$

The following are the values of these integrals:

$$\int_o^1 (1 - \frac{x}{L}) (\phi_1)^2 \, d(\frac{x}{L}) = 0.19346191$$

$$\int_o^1 (1 - \frac{x}{L})^2 (\phi_1'')^2 d(\frac{x}{L}) = 0.5791407$$

Substitution in Equation E-2 gives

$$p = 6.0833815 \frac{b}{L^2} \sqrt{\frac{E}{3\rho}}$$

which is a variation of about 14.4 p.c. to that of (24). It may be recalled that by assuming another series and applying Ritz method, the value obtained in (23) by one term approximation was 5.48. This is a variation of only 3.1 p.c. This shows that the use of functions, other than normal functions, may sometimes be profitable but this needs judgment on the part of the investigator. As the normal functions are standard functions, they can be used more effectively, if suitable tables can be prepared for different types of integrals of these functions.

To check the convergence, two terms of the series in Equation E-1 were taken. The value obtained for the first mode frequency was $\frac{5.434991}{2\pi} \frac{b}{L^3} \sqrt{\frac{E}{3\rho}}$, a variation of only 2.26 p.c.

The problem was further investigated with eight terms of the series and the following results were obtained. The exact frequency value for the fundamental mode is $\frac{5.315}{2\pi} \frac{b}{L^3} \sqrt{\frac{E}{3\rho}}$

First mode frequency $\frac{5.3165}{2\pi} \frac{b}{L^3} \sqrt{\frac{E}{3\rho}}$

Second mode frequency $\frac{15.317}{2\pi} \frac{b}{L^3} \sqrt{\frac{E}{3\rho}}$

Third mode frequency $\frac{31.371}{2\pi} \frac{b}{L^3} \sqrt{\frac{E}{3\rho}}$

The exact solution results for the remaining two modes were not available but they are listed here as reference.

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