SOME THEOREMS ON EXTENDING AUTOMORPHISMS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY

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1966

This is to certify that the

thesis entitled

"Some Theorems on
Extending Automorphisms"

presented by

Franklin D. Demana

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Major professor

Date March 14, 1966

O-169

ABSTRACT

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One method to gain some information about the automorphism group, $A(G)$, of a group G is to consider classes of subgroups of G on which the automorphisms act as permutations. Two basic problems must be contended with in any investigation of this kind. First, given a subgroup A of G and an automorphism α of A, does there exist an automorphism γ of G such that $\gamma|A = \alpha$? Secondly, given a normal subgroup A of G, α an automorphism of A, β an automorphism of G/A, does there exist an automorphism γ of G such that $\gamma|A = \alpha$ and γ induces β on G/A ? In chapter one we consider these problems and now list some of the results we have obtained. This list is not meant to be complete but to give examples of the type of results obtained.

(1) Let $G = AB$, $A \triangleleft G$, $\alpha \in A(A)$, and $\beta \in A(B)$. Then necessary and sufficient conditions that there exist $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|B = \beta$ are that

(i) $\alpha|A\cap B = \beta|A\cap B$;

(ii) $\pi_b\alpha = \alpha\pi_{b^\beta}$ on A.

(2) Let $G = AB$, $A \triangleleft G$, $A\cap B = (1)$, and $\alpha \in A(A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ are that there exist $\beta \in A(B)$ and a function f from B to A satisfying

(i) $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1, b_2 \in B$;

(ii) $\pi_b\alpha = \alpha\pi_{f(b)b^\beta}$ on A.

(3) Let G be the relative holomorph of A by K with $Z(A) = (1)$. Then a necessary and sufficient condition that $\alpha \in A(A)$ can be extended to G is that there exists an automorphism Θ of K such that $\alpha^{-1}\beta\alpha \equiv \beta^\Theta \pmod{I(A)}$ for all $\beta \in K$.

(4) Let $A \triangleleft G$, $G = \bigcup_1^n Ab_i$, $G/A = \{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n\}$, $\alpha \in A(A)$, and $\beta \in A(G/A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|G/A = \beta$ are that

(i) there exists a function f from G/A to A such that

$f(\bar{b}_i\bar{b}_j) = a_{ij}^{-\alpha} f(\bar{b}_i)f(\bar{b}_j)^{b_{i*}^{-1}} a_{i*j*}$ where $b_ib_j = a_{ij}b_k$ and $\bar{b}_i^{\beta} = \bar{b}_{i*}$;

(ii) $\pi_{b_i}\alpha = \alpha\pi_{f(\bar{b}_i)b_{i*}}$ on A.

(5) Let $G = AB$, $A \triangleleft G$, $A \cap B = (1)$, and $\gamma \in A(G)$ such that $\gamma|A = \alpha \in A(A)$. If $B^\gamma = x^{-1}Bx$ for some $x \in A$ then the function f, see (2) above, from B to A induced by γ is $f(b) = x^{-1}x^{b^{-\beta}}$ where $b^\gamma = f(b)b^\beta$ and $\beta \in A(B)$.

It would be nice to have a result like (2) above in case A is supplemented in G i.e., there exists a subgroup B of G such that $G = AB$ and $A \cap B \neq (1)$. Although this is not possible we are able to obtain something in this direction by looking at a restricted class of supplements. Next we turn our attention to lifting power and central automorphisms. If A is a normal subgroup of $G = AB$ we define

$\overset{*}{\text{Hom}}(A,Z(A)) = \{f \in \text{Hom}(A,Z(A)) \mid f(a) \neq a^{-1} \text{ for } a \neq 1\}$

$\overset{**}{\text{Hom}}(A,Z(A)) = \{f \in \overset{*}{\text{Hom}}(A,Z(A)) \mid f(b^{-1}ab) = b^{-1}f(a)b\}.$

Then we show that

(6) Let $G = AB$ with $A \lhd G$. If $f \in \overset{**}{\text{Hom}}(A,Z(A))$ and $A \cap B \leq \ker f$ then the central automorphism α_f, $a^{\alpha_f} = f(a)a$, of A associated with f can be extended to G.

(7) Let $G = AB$ with $A \lhd G$. Then $\gamma \in B(A,G) = \{\gamma \in A(G) \mid A^{\gamma}=A\}$ is central if and only if there exist f and g such that

(i) $f \in \overset{*}{\text{Hom}}(A,Z(G) \cap A)$, $g \in \overset{*}{\text{Hom}}(B,Z(G))$, and $f(a)g(b) \neq (ab)^{-1}$ for all $ab \neq 1$;

(ii) $f(b^{-1}ab) = f(a)$ for all $a \in A$ and $b \in B$;

(iii) $f(b) = g(b)$ for all $b \in A \cap B$.

Other results of this type are obtained for power automorphisms.

Now let $B(A,G)$ denote those automorphisms of G which leave A invariant and $C(A,G)$ those automorphisms which fix A elementwise. Then we obtain the following results:

(8) Let $G = AB$, $A \lhd G$, and $A \cap B = (1)$. If $|A| \neq 1$ or 2 then A has a nontrivial automorphism that can be lifted to G.

Let $G = AB$ with $A \lhd G$. We define $B^* = \{\pi_b \in A(A) \mid b \in B\}$. Then

(9) If $G = AB$, $A \lhd G$, $C_B(A) = (1)$, $B^* \trianglelefteq A(A)$, and $A \cap B$ is a characteristic subgroup of A then every automorphism of A can be extended to G i.e., $B(A,G)/C(A,G) \simeq A(A)$.

(10) Let A be a normal abelian subgroup of G. Then $|C(A,G)| = 1$ if and only if G is abelian, $[G:A] = 2$, and $|A|$ is odd.

(10) tells us when an automorphism of A has a unique extension to G since $|C(A,G)|$ is the number of ways an automorphism of A can be extended to G.

In chapter two we turn our attention to the following situation. Let A be a complemented subgroup of G and denote by Ω the set of all complements of A in G. If $D(A,G)$ denotes those elements of $B(A,G)$ which fix every complement of A then $B(A,G)/D(A,G) = X$ is a permutation group on Ω. Our goal was to characterize when all complements of A are conjugate in terms of a permutation condition on (X,Ω). Although we were unable to do this we did obtain some results in this direction.

(11) If (X,Ω) is a primitive permutation group then all complements of A are either conjugate or normal.

(12) If (X,Ω) is a primitive permutation group then either A is characteristically simple or if H is a characteristic subgroup of A then $H\,N_A(B) = \begin{cases} N_A(B) \\ A \end{cases}$.

(13) If (X,Ω) is 3/2-fold transitive then all complements of A are either conjugate or normal provided any one of the following conditions hold:

(i) $(|A|, |B/B'|) = 1$ where B' is the derived group of B;

(ii) if there exists a normal subgroup H of G such that $H \leq A$ and $H \cap N_A(B) = (1)$;

(iii) if $N_A(B)$ is a Hall subgroup of A with a normal complement;

(iv) if $Z(A) \cap N_A(B) = Z(N_A(B))$ and there exists at least one $x \in A - N_A(B)$ such that $(|x|, |N_A(B)|) = 1$;

(v) if $A/N_A(B)$ is not nilpotent.

We also obtained some results in case (X, Ω) is a sharply doubly transitive permutation group.

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1966

PREFACE

The author is deeply indebted to his major professor, Dr. J. E. Adney for his patience and helpful guidance in the preparation of this thesis. The many discussions we had together during the research were of immeasurable value to me.

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INTRODUCTION

When studying automorphisms of a group G some very natural questions arise. Can we start with a subgroup A and by "extending" its automorphisms obtain every automorphism of G? Of course the answer to this question is no if A is not characteristic in G. If A is characteristic then the answer is yes but how to do this is another problem. One might be tempted to think that if A is characteristic then every automorphism of A ought to be extendable to G. The example in 1.1 shows this conjecture to be false. Since not every automorphism of a subgroup can be "lifted" to G we begin by finding necessary and sufficient conditions under which an automorphism can be "extended". First we clear up the notion of "extending" or "lifting" automorphisms.

If A is a subgroup of G and α an automorphism of A then α is "extendable" or "liftable" if and only if there is an automorphism γ of G such that $\gamma|A = \alpha$. Also if $G = AB$ and $\tau \in \mathrm{Hom}(B,G)$ then the pair α, τ is "extendable" or "liftable" if and only if there is an automorphism γ of G such that $\gamma|A = \alpha$ and $\gamma|B = \tau$. Finally, if α is an automorphism of the normal subgroup A and β an automorphism of G/A then the pair α, β is "extendable" or "liftable" if and only if there is an automorphism γ of G such that $\gamma|A = \alpha$ and γ induces β on G/A.

In chapter one we obtain necessary and sufficient conditions in each of the cases mentioned above. We are also concerned with the problem of lifting certain types of automorphisms, namely, power automorphisms and central automorphisms. Let $B(A,G)$ denote the set of automorphisms of G which leave A invariant and $C(A,G)$ the set of automorphisms of G which fix A elementwise. Then $C(A,G)$ is normal in $B(A,G)$ and $B(A,G)/C(A,G)$ is isomorphic to a subgroup of the automorphism group of A. In fact, $|B(A,G)/C(A,G)|$ is the number of automorphisms of A that can be extended to G. We try to find out something about the structure of $B(A,G)/C(A,G)$ and also find conditions when every automorphism or no automorphism can be extended to G. Finally, under very special hypothesis on G, we count the number of ways a particular automorphism of A can be lifted to G and when an automorphism of A can be paired with a unique automorphism of a complement of A and lifted to G.

In Chapter Two we restrict our attention to a complemented subgroup A of a group G. Then we define Ω to be the set of all complements of A in G. The set $D(A,G)$ of all automorphisms of G which fix every complement forms a normal subgroup of the group $B(A,G)$. We then see that the factor group $X = B(A,G)/D(A,G)$ is a permutation group acting on the set Ω. Naturally one would like to know necessary and sufficient conditions that will guarantee when all complements are conjugate. Although we are not able to do this at this time we do obtain several interesting results in this direction. What we do is impose the conditions primitive, 3/2-fold transitive, and sharply 2-fold transitive on X and investigate the complements in these cases. If (X,Ω) is primitive then we show that all complements are either normal or conjugate. By placing certain additional restrictions on G we get the same result in case (X, Ω) is 3/2-fold transitive. We also try to find out something about G, A, and X under these hypotheses. The results are not complete and many interesting questions remain unanswered.

Our main interest was in finite groups and, in fact, all groups considered in Chapter Two and 1.4 of Chapter One are assumed finite. However, in 1.2 and 1.3 of Chapter One the results hold as stated for infinite groups unless otherwise stated.

The reader is asked to consult the index of notation for identification of symbolic notations of groups, sets and relations.

CHAPTER I

1.1 Introduction

Let A be a normal subgroup of the group G. The two main problems considered in this chapter are lifting automorphisms of A to G and pairing an automorphism of A with an automorphism of G/A and lifting to G. In 1.2 we give sets of necessary and sufficient conditions solving these problems under various hypotheses on G. The question of lifting certain types of automorphisms, namely power automorphisms and central automorphisms, is considered in 1.3. Finally, in 1.4, we investigate the group B(A,G) to some extent.

It is difficult to make very general statements since the problem considered is quite complex. To illustrate this we first show that given any odd prime p and any positive integer n greater than 1 there exists a group G with characteristic subgroup A such that $G = AB$, $A \cap B = (1)$, $|B| = 2$, and $\frac{|A(A)|}{|A(G)|} = \frac{p^{n-2}(p^n-1)}{p-1}$. The latter implies that for each automorphism of A that can be extended to G there are at least $\frac{p^{n-2}(p^n-1)}{p-1} - 1$ automorphisms of A that cannot be extended to G.

Let A be an elementary abelian p-group of order p^n generated by $a_1, \ldots, a_n$. Let $B = \langle b \rangle$ where $b^2 = 1$, $ba_1b = a_1^{-1}$, and $ba_ib = a_i$ for $i = 2,3, \ldots, n$. Now A is a normal p-Sylow subgroup of $G = AB$ so clearly characteristic. The center of G is generated by $a_2, \ldots, a_n$ so any automorphism of G must leave $\langle a_2, \ldots, a_n \rangle$ invariant. Let α be any automorphism of $\langle a_2, \ldots, a_n \rangle$ and r and s any two integers such that $0 \leq r \leq p-1$ and $0 < s \leq p-1$. Then the mapping γ of G defined by $a_1^\gamma = a_1^s$, $a_i^\gamma = a_i^\alpha$ $(i = 2, \ldots, n)$, and $b^\gamma = a_1^r b$ is an automorphism of G and one can easily check that these are all the automorphisms of G. It is well known [3, pg. 86] that the order of the automorphism group of $<a_2, \ldots, a_n>$ is $(p^{n-1}-1)(p^{n-1}-p) \ldots (p^{n-1}-p^{n-2})$. Therefore, we have $|A(G)| = p(p-1)(p^{n-1}-1)(p^{n-1}-p) \ldots (p^{n-1}-p^{n-2})$ so that

$$\frac{|A(A)|}{|A(G)|} = \frac{(p^n-1)(p^n-p) \ldots (p^n-p^{n-1})}{p(p-1)(p^{n-1}-1)(p^{n-1}-p) \ldots (p^{n-1}-p^{n-2})} = \frac{p^{n-2}(p^n-1)}{p-1} .$$

1.2 Necessary and Sufficient Conditions for Extending Automorphisms

Definition 1.1: Let A be a subgroup of G. If B is a proper subgroup of G such that G = AB then B is said to be a supplement of A in G. Moreover if $A \cap B = (1)$ then we say that B is a complement of A in G. We refer to A as being supplemented or complemented in G.

Let A be a normal subgroup of G which is not contained in the Frattini subgroup of G. It has been shown [2] that A possesses a supplement, say B, in this case. The first two results we obtain are generalizations of those appearing in [5].

Theorem 1.2: Let $G = AB$, $A \lhd G$, $\alpha \in A(A)$, and $\beta \in A(B)$. Then necessary and sufficient conditions that there exist $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|B = \beta$ are that

(1) $\alpha|A \cap B = \beta|A \cap B$;

(2) $\pi_b \alpha = \alpha \pi_b \beta$ on A.

The proof of this theorem follows immediately from the more general result:

Theorem 1.3: Let A, A_1, B, B_1 be subgroups of G with A and A_1 normal in G, A and B isomorphic to A_1 and B_1 respectively under σ and τ, and $G = AB = A_1B_1$. Then necessary and sufficient conditions for the existence of an automorphism γ of G such that $\gamma|A = \sigma$ and $\gamma|B = \tau$ are that

(1) $\sigma|A\cap B = \tau|A\cap B$;

(2) $\pi_b\sigma = \sigma\pi_{b^\tau}$ on A.

Proof: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \sigma$ and $\gamma|B = \tau$. Then clearly $\sigma|A\cap B = \tau|A\cap B$. If $a \in A$ and $b \in B$ then $(ab)^\gamma = a^\gamma b^\gamma = a^\sigma b^\tau = b^\tau b^{-\tau} a^\sigma b^\tau = b^\tau a^{\sigma b^\tau}$. We can also write $ab = bb^{-1}ab$ so that $(ab)^\gamma = (bb^{-1}ab)^\gamma = b^\gamma(b^{-1}ab)^\gamma = b^\tau(b^{-1}ab)^\sigma = b^\tau a^{b\sigma}$. Hence we have $b^\tau a^{\sigma b^\tau} = b^\tau a^{b\sigma}$ or $a^{\sigma b^\tau} = a^{b\sigma}$ for all $a \in A$. Therefore, $\pi_b\sigma = \sigma\pi_{b^\tau}$.

(b) Sufficiency

Now $G = AB = A_1B_1$ with $A \overset{\sigma}{\simeq} A_1$, $B \overset{\tau}{\simeq} B_1$, $\sigma|A\cap B = \tau|A\cap B$, and $\pi_b\sigma = \sigma\pi_{b^\tau}$. Let ab $(a \in A,\ b \in B)$ be any element of G and define γ by $(ab)^\gamma = a^\sigma b^\tau$. If $a_1b_1 = a_2b_2$ with $a_i \in A$, $b_i \in B$ for $i = 1,2$ then $a_2^{-1}a_1 = b_2b_1^{-1} \in A\cap B$ and since $\sigma|A\cap B = \tau|A\cap B$ we have

$$(a_2^{-1}a_1)^\sigma = (b_2b_1^{-1})^\tau$$

$$(a_2^\sigma)^{-1}a_1^\sigma = b_2^\tau(b_1^\tau)^{-1}$$

$$a_1^\sigma b_1^\tau = a_2^\sigma b_2^\tau$$

$$(a_1b_1)^\gamma = (a_2b_2)^\gamma$$

Hence γ is well defined.

If $ab \in \ker\gamma$ i.e., if $(ab)^\gamma = a^\sigma b^\tau = 1$ then $a^\sigma = (b^\tau)^{-1} \in A_1\cap B_1$ and since $\sigma^{-1}|A_1\cap B_1 = \tau^{-1}|A_1\cap B_1$ we have $(a^\sigma)^{\sigma^{-1}} = [(b^\tau)^{-1}]^{\tau^{-1}}$ or $a = b^{-1}$. Thus $ab = 1$ and $\ker\gamma = (1)$. So γ is 1-1 and clearly γ is onto since $G = A_1B_1$.

Now let $g_1 = a_1b_1$ and $g_2 = a_2b_2$ be any two elements of G where $a_i \in A$ and $b_i \in B$ for $i = 1,2$. Then $g_1g_2 = a_1(b_1a_2b_1^{-1})b_1b_2 = a_1a_2^{b_1^{-1}}b_1b_2$ so that $(g_1g_2)^\gamma = a_1^\sigma a_2^{b_1^{-1}\sigma} b_1^\tau b_2^\tau$. But $\pi_b\sigma = \sigma\pi_{b^\tau}$ so we have

$$\begin{aligned}(g_1g_2)^\gamma &= a_1^\sigma a_2^{b_1^{-1}\sigma} b_1^\tau b_2^\tau \\ &= a_1^\sigma a_2^{\sigma b_1^{-\tau}} b_1^\tau b_2^\tau \\ &= a_1^\sigma b_1^\tau a_2^\sigma b_1^{-\tau} b_1^\tau b_2^\tau \\ &= a_1^\sigma b_1^\tau a_2^\sigma b_2^\tau \\ &= (a_1b_1)^\gamma (a_2b_2)^\gamma \\ &= g_1^\gamma g_2^\gamma\end{aligned}$$

Therefore γ is an automorphism of G such that $\gamma|A = \sigma$ and $\gamma|B = \tau$.

Using this result we can obtain a set of necessary and sufficient conditions for lifting an automorphism of a normal subgroup which is not contained in the Frattini subgroup.

<u>Corollary 1.4</u>: Let $G = AB$, $A \lhd G$, and $\alpha \in A(A)$. Then necessary and sufficient conditions that there exist $\gamma \in A(G)$ such that $\gamma|A = \alpha$ are that

(1) there exists a subgroup B_1 of G isomorphic, say under τ, to B and $G = AB_1$;

(2) $\alpha|A \cap B = \tau|A \cap B$;

(3) $\pi_b\alpha = \alpha\pi_{b^\tau}$ on A.

Proof: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$. (1) is clear if we set $B_1 = B^\gamma$ and $\tau = \gamma|B$. (2) and (3) follow immediately from Theorem 1.3.

(b) Sufficiency

From the sufficiency of Theorem 1.3 with $A_1 = A$ and $\sigma = \alpha$ there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$.

Now if we impose the stronger condition that A be complemented in G then we can refine the conditions somewhat.

Theorem 1.5: Let $G = AB$, $A \triangleleft G$, $A \cap B = (1)$, and $\alpha \in A(A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ are that there exist $\beta \in A(B)$ and a function f from B to A satisfying

(1) $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1, b_2 \in B$;

(2) $\pi_b\alpha = \alpha\pi_{f(b)b^\beta}$ on A.

Proof: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$. Since $A \cap B = (1)$ each $g \in G$ has a unique representation as $g = ab$ with $a \in A$ and $b \in B$. Hence we can write $b^\gamma = f(b)b^*$ with $f(b) \in A$ and $b^* \in B$ where $f(b)$ and b^* are uniquely determined by b. Thus f is a function from B to A. Define β by $b^\beta = b^*$. β is well defined since b^* is uniquely determined by b. If $b_1, b_2 \in B$ then $(b_1b_2)^\gamma = f(b_1b_2)(b_1b_2)^\beta$. But $\gamma \in A(G)$ so we also have.

$$(b_1b_2)^\gamma = b_1^\gamma b_2^\gamma$$
$$= f(b_1)b_1^\beta\ f(b_2)b_2^\beta$$
$$= f(b_1)b_1^\beta\ f(b_2)(b_1^\beta)^{-1}b_1^\beta b_2^\beta$$
$$= f(b_1)\ f(b_2)^{(b_1^\beta)^{-1}}\ b_1^\beta b_2^\beta$$

Hence $f(b_1b_2)(b_1b_2)^\beta = f(b_1)\ f(b_2)^{(b_1^\beta)^{-1}}\ b_1^\beta b_2^\beta$ and since $A \cap B = (1)$ it follows that $f(b_1b_2) = f(b_1)\ f(b_2)^{(b_1^\beta)^{-1}}$ and $(b_1b_2)^\beta = b_1^\beta b_2^\beta$. Thus f has the desired property and β is an endomorphism of B. Let $b \in \ker \beta$ i.e., $b^\beta = 1$. Then we have $[f(b)^{-\gamma^{-1}}b]^\gamma = f(b)^{-1}\ f(b)b^\beta = b^\beta = 1$ and since $\gamma \in A(G)$ we must have $f(b)^{-\gamma^{-1}}b = 1$. Now $b = f(b)^{\gamma^{-1}} \in A$ since $\gamma|A \in A(A)$ so $b \in A \cap B = (1)$. So $\ker \beta = (1)$ and β is 1-1. Clearly β is onto so $\beta \in A(B)$. Define τ by $b^\tau = b^\gamma = f(b)b^\beta$. Then, by the necessary part of Theorem 1.3, we have $\pi_b\alpha = \alpha\pi_{b^\tau} = \alpha\pi_{f(b)b^\beta}$.

(b) Sufficiency

Let f be a function from B to A and $\beta \in A(B)$ satisfying (1) and (2) of the theorem. Define τ by $b^\tau = f(b)b^\beta$. Since f is a function and β an automorphism τ is well defined. Let b_1, b_2 be any two elements of B and we have

$$(b_1b_2)^\tau = f(b_1b_2)(b_1b_2)^\beta$$
$$= f(b_1)\ f(b_2)^{b_1^{-\beta}}\ b_1^\beta b_2^\beta$$
$$= f(b_1)b_1^\beta\ f(b_2)b_1^{-\beta}b_1^\beta b_2^\beta$$
$$= f(b_1)b_1^\beta\ f(b_2)b_2^\beta$$
$$= b_1^\tau\ b_2^\tau\ .$$

Thus τ is a homomorphism from B into G. Let $b \in \ker \tau$ then $b^\tau = f(b)b^\beta = 1$ so that $b^\beta = f(b)^{-1} \in A$. Since $b^\beta \in B$ and $A \cap B = (1)$ we must have $b^\beta = 1$. Hence $b = 1$ since $\beta \in A(B)$. Therefore $\ker \tau = (1)$ and τ is an isomorphism. Set $B_1 = B^\tau$ then clearly $G = AB_1$ and since $\pi_b\alpha = \alpha\pi_{f(b)b^\beta} = \alpha\pi_{b^\tau}$ we have by the sufficiency part of Corllary 1.4 that there exists a $\gamma \in A(G)$ such that $\gamma|A = \alpha$.

The function f in the above theorem has two other properties which will be useful later so we will establish them now.

Lemma 1.6: For the function f in Theorem 1.5 we have

(i) $f(1) = 1$

(ii) $f(b^{-1}) = b^{-\beta} f(b)^{-1} b^\beta$ for all $b \in B$.

Proof: In $f(b_1b_2) = f(b_1)\, f(b_2)^{b_1^{-\beta}}$ set $b_1 = b_2 = 1$ and we get $f(1) = f(1)f(1)$ or $f(1) = 1$. Now set $b_1 = b^{-1}$ and $b_2 = b$ and we get

$$f(b^{-1}b) = f(b^{-1})\, f(b)^{b^\beta}$$

$$f(1) = f(b^{-1})b^{-\beta} f(b)b^\beta.$$

Since $f(1) = 1$ we have $f(b^{-1}) = b^{-\beta} f(b)^{-1}b^\beta$.

We notice that this function f need not be a homomorphism as will be illustrated in the following example.

Example 1: $G = \langle a,b \rangle$, $A = \langle a \rangle$, $B = \langle b \rangle$, $a^n = b^2 = 1$, and $ba = a^{-1}b$. Define the mapping γ by $a^\gamma = a^i$, $b^\gamma = a^jb$ where $(i,n) = 1$ and $0 \leq j \leq n-1$. $\gamma \in A(G)$ and the function f from B to A induced by γ maps b to a^j. Since $|a^j|$ need not be 2, f need not be a homomorphism.

If we add to the hypothesis of Theorem 1.5 the condition that each automorphism of A induced by an element of B is an inner automorphism of A then we can replace condition (2) of the theorem by one of different form. Under this hypothesis we can write $\pi_b = \pi_{g(b)}$ with $g(b) \in A$ for each $b \in B$ and obtain:

Theorem 1.7: Let $G = AB$, $A \triangleleft G$, $A \cap B = (1)$ and assume for each $b \in B$ there is a $g(b) \in A$ such that $\pi_b = \pi_{g(b)}$ on A. Then necessary and sufficient conditions that $\alpha \in A(A)$ can be extended to G is that there exists a function f from B to A and a $\beta \in A(B)$ such that

(1) $f(b_1 b_2) = f(b_1)\, f(b_2)^{g(b_1^{-\beta})}$ for all $b_1, b_2 \in B$;

(2) $f(b) \equiv g(b)^\alpha g(b^\beta)^{-1} \pmod{Z(A)}$ for all $b \in B$.

Proof: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma | A = \alpha$. Then, by Theorem 1.5, there exists a $\beta \in A(B)$ and a function f from B to A such that $f(b_1 b_2) = f(b_1)\, f(b_2)^{b_1^{-\beta}}$ and $\pi_b \alpha = \alpha \pi_{f(b) b^\beta}$. Since $\pi_b = \pi_{g(b)}$ on A we can write $f(b_1 b_2) = f(b_1)\, f(b_2)^{g(b_1^{-\beta})}$. If $x \in A$ then it is easy to show that $\pi_x \alpha = \alpha \pi_{x^\alpha}$ so we have $\pi_{g(b)} \alpha = \alpha \pi_{g(b)^\alpha}$. Thus

$$\pi_b \alpha = \alpha \pi_{f(b) b^\beta}$$

$$\pi_{g(b)} \alpha = \alpha \pi_{f(b)\, b^\beta}$$

$$\alpha \pi_{g(b)^\alpha} = \alpha \pi_{f(b)} \pi_{g(b^\beta)}$$

$$\pi_{g(b)^\alpha} = \pi_{f(b)} \pi_{g(b^\beta)} .$$

Hence $\pi_{f(b)} = \pi_{g(b)^\alpha g(b^\beta)^{-1}}$ on A so that $f(b) \equiv g(b)^\alpha g(b^\beta)^{-1} \pmod{Z(A)}$.

(b) Sufficiency

Let α, β, and f be given satisfying (1) and (2). Since $\pi_{b_1^{-\beta}} = \pi_{g(b_1^{-\beta})}$ we can rewrite (1) as $f(b_1b_2) = f(b_1)\, f(b_2)^{b_1^{-\beta}}$ for all b_1, $b_2 \in B$. Now $\alpha\pi_{f(b)b^\beta} = \alpha\pi_{f(b)}\pi_{b^\beta} = \alpha\pi_{f(b)}\pi_{g(b^\beta)}$ and since $f(b) \equiv g(b)^\alpha g(b^\beta)^{-1} \pmod{Z(A)}$ we have $\pi_{f(b)} = \pi_{g(b)^\alpha g(b^\beta)^{-1}}$ on A. Thus $\alpha\pi_{f(b)b^\beta} = \alpha\pi_{f(b)}\pi_{g(b^\beta)} = \alpha\pi_{g(b)^\alpha g(b^\beta)^{-1}}\pi_{g(b^\beta)} = \alpha\pi_{g(b)^\alpha}$ and since $g(b) \in A$ we have $\alpha\pi_{g(b)^\alpha} = \pi_{g(b)}\alpha$. Hence $\alpha\pi_{f(b)b^\beta} = \pi_{g(b)}\alpha = \pi_b\alpha$. Therefore, by the sufficiency part of Theorem 1.5, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$.

<u>Definition 1.8</u>: Let $G = AK$, $A \cap K = (1)$, $K \leq A(A)$ with defining relations $\alpha^{-1}a\alpha = a^\alpha$ for all $a \in A$ and $\alpha \in K$. Then G is called the <u>relative holomorph</u> of A by K. If $K = A(A)$ then G is called the <u>holomorph</u> of A.

Next we consider some of the previous theorem if we restrict G to be a relative holomorph of A. If we try to apply Theorem 1.2 to G i.e., try to extend $\alpha \in A(A)$, $\Theta \in A(K)$ to G then the condition of the theorem becomes $\beta\alpha = \alpha\beta^\Theta$ for all $\beta \in K$ since $\pi_\beta = \beta$ on A. Therefore if $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|K = \Theta$ then $\gamma = \pi_\alpha$, the inner automorphism of the holomorph of A induced by the element α, where $\alpha \in N_{A(A)}(K)$. However, if we apply Theorem 1.5 to G with the added condition $Z(A) = (1)$ we have a little more success for we are able to obtain the following result:

Theorem 1.9: Let G be the relative holomorph of A by K with $Z(A) = (1)$. Then a necessary and sufficient condition that $\alpha \in A(A)$ can be extended to G is that there exists an automorphism Θ of K such that $\alpha^{-1}\beta\alpha \equiv \beta^{\Theta}$ (mod $I(A)$) for all $\beta \in K$.

Proof: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$. Then, by the necessary part of Theorem 1.5, there exists a $\Theta \in A(K)$ and a function f from K to A such that $f(\beta_1\beta_2) = f(\beta_1)\ f(\beta_2)^{\beta_1^{-\Theta}}$ for all $\beta_1, \beta_2 \in K$ and $\pi_\beta\alpha = \alpha\pi_{f(\beta)}\beta^{\Theta}$. Now $\pi_\beta = \beta$ on A so the latter condition becomes $\beta\alpha = \alpha\pi_{f(\beta)}\beta^{\Theta}$ or $\alpha^{-1}\beta\alpha = \pi_{f(\beta)}\beta^{\Theta}$. Hence $\alpha^{-1}\beta\alpha \equiv \beta^{\Theta}$ (mod $I(A)$) for all $\beta \in K$.

(b) Sufficiency

Let $\alpha \in A(A)$, $\Theta \in A(K)$ such that $\alpha^{-1}\beta\alpha \equiv \beta^{\Theta}$ (mod $I(A)$) for all $\beta \in K$. Then we can write $\alpha^{-1}\beta\alpha = \pi_{f(\beta)}\beta^{\Theta}$ where $f(\beta) \in A$. First we show that f is a function from K to A. If $f(\beta)$ is not unique then we could have $f(\beta) = a_1$ or $f(\beta) = a_2$ where $\alpha^{-1}\beta\alpha = \pi_{a_1}\beta^{\Theta}$ and $\alpha^{-1}\beta\alpha = \pi_{a_2}\beta^{\Theta}$. Hence $\pi_{a_1} = \pi_{a_2}$ on A. Thus $a_1a_2^{-1} \in Z(A)$ and since $Z(A) = (1)$ we have $a_1 = a_2$. Therefore, f is a function from K to A. Let β_1, β_2 be any two elements of K and we have

$$\begin{aligned}\pi_{f(\beta_1\beta_2)}(\beta_1\beta_2)^{\Theta} &= \alpha^{-1}\beta_1\beta_2\alpha \\ &= \alpha^{-1}\beta_1\alpha\alpha^{-1}\beta_2\alpha \\ &= \pi_{f(\beta_1)}\beta_1^{\Theta}\pi_{f(\beta_2)}\beta_2^{\Theta} \\ &= \pi_{f(\beta_1)}\pi_{f(\beta_2)^{\beta_1^{-\Theta}}}\ \beta_1^{\Theta}\beta_2^{\Theta}\end{aligned}$$

since $\beta_1^{\Theta}\pi_{f(\beta_2)} = \pi_{f(\beta_2)^{\beta_1^{-\Theta}}}\ \beta_1^{\Theta}$. Now $\Theta \in A(K)$ so we have $\pi_{f(\beta_1\beta_2)} = \pi_{f(\beta_1)}\pi_{f(\beta_2)^{\beta_1^{-\Theta}}}$ and since $Z(A) = (1)$ it follows that

$f(\beta_1\beta_2) = f(\beta_1)\ f(\beta_2)^{\beta_1^{-\Theta}}$. Since $\pi_\beta = \beta$ on A we can rewrite $\alpha^{-1}\beta\alpha = \pi_{f(\beta)}\beta^{\Theta}$ as $\alpha^{-1}\pi_\beta\alpha = \pi_{f(\beta)}\pi_{\beta^\Theta}$. Thus $\pi_\beta\alpha = \alpha\pi_{f(\beta)\beta^\Theta}$. Summarizing we have $\alpha \in A(A)$, $\Theta \in A(K)$, and a function f from K to A such that $f(\beta_1\beta_2) = f(\beta_1)\ f(\beta_2)^{\beta_1^{-\Theta}}$ for all $\beta_1,\beta_2 \in K$ and $\pi_\beta\alpha = \alpha\pi_{f(\beta)\beta^\Theta}$ on A. Therefore, by the sufficiency part of Theorem 1.5, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$.

Now we consider a relative holomorph of A by a cyclic subgroup of A(A). However, we do not assume that Z(A) = (1) in the result that follows.

<u>Theorem 1.10</u>: Let $G = A\langle\beta\rangle$, $\beta \in A(A)$, and $\beta^{-1}a\beta = a^\beta$ for all $a \in A$. Then necessary and sufficient conditions that $\alpha \in A(A)$ can be extended to G are that

(1) there exists $a \in A$ such that $\alpha^{-1}\beta\alpha = \pi_a\beta^k$ where $(k,|\beta|) = 1$;

(2) $a^{1+\beta^{-k}+\beta^{-2k}+\cdots+\beta^{-(r-1)k}} = 1$ where $r = |\beta|$.

<u>Proof</u>: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$. Then, by the necessary part of Theorem 1.5, there exists a $\Theta \in A(\langle\beta\rangle)$ and a function f from $\langle\beta\rangle$ to A such that $f(\beta_1\beta_2) = f(\beta_1)\ f(\beta_2)^{\beta_1^{-\Theta}}$ for all $\beta_1,\beta_2 \in \langle\beta\rangle$ and $\pi_{\beta^t}\alpha = \alpha\pi_{f(\beta^t)(\beta^t)^\Theta}$ on A. Since $\Theta \in A(\langle\beta\rangle)$ we have $\beta^\Theta = \beta^k$ where $(k,|\beta|) = 1$. Let $r = |\beta|$ and we have

$$1 = 1^\gamma = (\beta^r)^\gamma = (\beta^\gamma)^r$$

$$= (f(\beta)\beta^k)^r$$

$$= f(\beta)^{1+\beta^{-k}+\beta^{-2k}+\cdots+\beta^{-(r-1)k}}\beta^{rk}$$

$$= f(\beta)^{1+\beta^{-k}+\beta^{-2k}+\cdots+\beta^{-(r-1)k}}$$

Thus we have (2) with $a = f(\beta)$. Now set $t = 1$ in $\pi_{\beta^t}\alpha = \alpha\pi_{f(\beta^t)(\beta^t)^\theta}$ and we get $\pi_\beta\alpha = \alpha\pi_{f(\beta)\beta^\theta}$. But $\pi_\beta = \beta$ on A so we have $\beta\alpha = \alpha\pi_a\beta^\theta$ or $\alpha^{-1}\beta\alpha = \pi_a\beta^k$.

(b) Sufficiency

Let $\alpha \in A(A)$, $a \in A$, and k be given satisfying (1) and (2) of the theorem. Define Θ by $\beta^\theta = \beta^k$ then clearly $\Theta \in A(\langle\beta\rangle)$ since $(k,|\beta|) = 1$. Let $f(\beta) = a$ and define $f(\beta^i) = f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(i-1)k}}$ for all $i \geq 2$. Now $f(\beta^s\beta^t) = f(\beta^{s+t}) = f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(s+t-1)k}}$ and

$$\begin{aligned} f(\beta^s)\, f(\beta^t)^{\beta^{-sk}} &= f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(s-1)k}} \left(f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(t-1)k}}\right)^{\beta^{-sk}} \\ &= f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(s+t-1)k}} \\ &= f(\beta^s\beta^t). \end{aligned}$$

Hence we have shown that $f(\beta_1\beta_2) = f(\beta_1)\, f(\beta_2)^{\beta_1^{-\theta}}$ for all $\beta_1,\beta_2 \in \langle\beta\rangle$. Since $\alpha^{-1}\beta\alpha = \pi_{f(\beta)}\beta^k$ we have

$$\begin{aligned} (\alpha^{-1}\beta\alpha)^i &= (\pi_{f(\beta)}\beta^k)^i \\ \alpha^{-1}\beta^i\alpha &= \pi_{f(\beta)^{1+\beta^{-k}+\cdots+\beta^{-(i-1)k}}}\ \beta^{ik} \\ \alpha^{-1}\beta^i\alpha &= \pi_{f(\beta^i)}\ \beta^{ik}. \end{aligned}$$

Now $\pi_\beta = \beta$ on A so we can rewrite the above equation as $\alpha^{-1}\pi_{\beta^i}\alpha = \pi_{f(\beta^i)}\pi_{\beta^{ik}}$ or $\pi_{\beta^i}\alpha = \alpha\pi_{f(\beta^i)(\beta^i)^\theta}$. Thus we have $\alpha \in A(A)$,

$\Theta \in A(<\beta>)$, and a function f from $<\beta>$ to A satisfying $f(\beta_1\beta_2) = f(\beta_1)\, f(\beta_2)^{\beta_1^{-\Theta}}$ for all $\beta_1, \beta_2 \in <\beta>$ and $\pi_{\beta^i}\alpha = \alpha\pi_{f(\beta^i)(\beta^i)^\Theta}$ on A. Therefore, by the sufficiency part of Theorem 1.5, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$.

Now we turn our attention to the following question. Let A be a normal subgroup of G, $\alpha \in A(A)$, and $\beta \in A(G/A)$. Under what conditions can we put α and β together to get an automorphism of G i.e., under what conditions does there exist a $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and γ induces the automorphism β on G/A? We will use the notation $\gamma|G/A = \beta$ to mean that γ induces the automorphism β on G/A. First we consider the special case when A is supplemented in G and notice that we have essentially answered this question in Corollary 1.4. In the following theorem G = AB and we must insist that coset representatives be chosen from B so that all the statements made are meaningful.

<u>Theorem 1.11</u>: Let G = AB, $A \triangleleft G$, $\alpha \in A(A)$, and $\beta \in A(G/A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|G/A = \beta$ are that

(1) there exists a subgroup B_1 of G such that $G = AB_1$, $B \cong B_1$ under τ, and τ induces β on G/A;

(2) $\alpha|A \cap B = \tau|A \cap B$;

(3) $\pi_b\alpha = \alpha\pi_{b^\tau}$ on A.

<u>Proof</u>: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|G/A = \beta$. Set $B_1 = B^\gamma$ and

$\tau = \gamma|B$. Clearly $\gamma|G/A = \tau|G/A = \beta$ where we mean of course that coset representatives are chosen from B. Now (1), (2), and (3) follow from Corollary 1.4.

(b) Sufficiency

Let α, β, τ, and B_1 be given satisfying (1), (2), and (3) of the theorem. Then, by Corollary 1.4, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|B = \tau$. Thus $\gamma|G/A = \tau|G/A = \beta$ and we have the result.

In the next result we only assume that A is a normal subgroup of G. Let a set of coset representatives of A in G be fixed so that we can write $G = Ab_1 \cup Ab_2 \cup \cdots \cup Ab_n$ where $b_1 = 1$. We denote the elements of G/A by $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n$. If b_i and b_j are any two coset representatives then we can write $b_i b_j = a_{ij} b_k$ where $a_{ij} \in A$ and the set of a_{ij} constitute a factor set. For more information concerning factor sets the reader can consult [3, pg. 218]. Now if $\beta \in A(G/A)$ we denote the image of $\bar{b}_i$ under β by $\bar{b}_{i^*}$ where i^* is some positive integer from 1 to n. Then with this notation we have the following result.

<u>Theorem 1.12</u>: Let $A \triangleleft G$, $G = \bigcup_{i=1}^{n} Ab_i$, $G/A = \{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n\}$, $\alpha \in A(A)$, and $\beta \in A(G/A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|G/A = \beta$ are that

(1) there exists a function f from G/A to A such that $f(\bar{b}_i \bar{b}_j) = a_{ij}^{-\alpha} f(\bar{b}_i)\, f(\bar{b}_j)^{b_{i^*}^{-1}}\, a_{i^* j^*}$;

(2) $\pi_{b_i}\alpha = \alpha\pi_{f(\overline{b}_i)b_{i^*}}$ on A.

<u>Proof</u>: (a) Necessity

Let $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|G/A = \beta$. We can write $b_i^\gamma = f(\overline{b}_i)b_{i^*}$ where $f(\overline{b}_i) \in A$ and $\overline{b}_i^\beta = \overline{b}_{i^*}$ since $\gamma|G/A = \beta$. Thus f is a function from G/A to A since each $g \in G$ has a unique representation of the form $g = ab_i$ where $a \in A$. Let b_i and b_j be any two coset representatives then $b_ib_j = a_{ij}b_k$ and $\overline{b}_i\overline{b}_j = \overline{b}_k$. Since $\gamma \in A(G)$ we have $(b_ib_j)^\gamma = b_i^\gamma b_j^\gamma$ so that

$$(a_{ij}b_k)^\gamma = b_i^\gamma b_j^\gamma$$

$$a_{ij}^\alpha f(\overline{b}_k)b_{k^*} = f(\overline{b}_i)b_{i^*}f(\overline{b}_j)b_{j^*}$$

$$a_{ij}^\alpha f(\overline{b}_i\overline{b}_j)b_{k^*} = f(\overline{b}_i)b_{i^*}f(\overline{b}_j)b_{i^*}^{-1}b_{i^*}b_{j^*}$$

$$a_{ij}^\alpha f(\overline{b}_i\overline{b}_j)b_{k^*} = f(\overline{b}_i)f(\overline{b}_j)^{b_{i^*}^{-1}}a_{i^*j^*}b_{k^*}$$

Hence $f(\overline{b}_i\overline{b}_j) = a_{ij}^{-\alpha}f(\overline{b}_i)f(\overline{b}_j)^{b_{i^*}^{-1}}a_{i^*j^*}$ which establishes (1). Let a be any element of A and b_i any coset representative and we can write $b_ia = b_iab_i^{-1}b_i = a^{b_i^{-1}}b_i$. Hence

$$(b_ia)^\gamma = (a^{b_i^{-1}}b_i)^\gamma$$

$$f(\overline{b}_i)b_{i^*}a^\alpha = a^{b_i^{-1}\alpha}f(\overline{b}_i)b_{i^*}$$

$$f(\overline{b}_i)b_{i^*}a^\alpha b_{i^*}^{-1}b_{i^*} = a^{b_i^{-1}\alpha}f(\overline{b}_i)b_{i^*}$$

$$f(\bar{b}_i)a^{\alpha b_{i^*}^{-1}} = a^{b_i^{-1}\alpha} f(\bar{b}_i)$$

$$a^{\alpha b_{i^*}^{-1} f(\bar{b}_i)^{-1}} = a^{b_i^{-1}\alpha}.$$

Therefore, $\pi_{b_i^{-1}}\alpha = \alpha\pi_{(f(\bar{b}_i)b_{i^*})^{-1}}$ or $\pi_{b_i}\alpha = \alpha\pi_{f(\bar{b}_i)b_{i^*}}$.

(b) Sufficiency

Let $\alpha \in A(A)$, $\beta \in A(G/A)$, and f be a function from G/A to A which satisfies (1) and (2) of the theorem. Define γ by $(ab_i)^\gamma = a^\alpha f(\bar{b}_i)b_{i^*}$. Since each element of G is uniquely represented in the form ab_i with $a \in A$ and $1 \leq i \leq n$ γ is well defined. Let $g_1 = a_1b_i$ and $g_2 = a_2b_j$ be any two elements of G and then we have $g_1g_2 = a_1b_ia_2b_j = a_1b_ia_2b_i^{-1}b_ib_j = a_1a_2^{b_i^{-1}} a_{ij}b_k$ where $\bar{b}_i\bar{b}_j = \bar{b}_k$. Now using (1) and (2) at the appropriate steps we have

$$\begin{aligned}
(g_1g_2)^\gamma &= a_1^\alpha a_2^{b_i^{-1}\alpha} a_{ij}^\alpha f(\bar{b}_k)b_{k^*} \\
&= a_1^\alpha a_2^{b_i^{-1}\alpha} a_{ij}^\alpha f(\bar{b}_i\bar{b}_j)b_{k^*} \\
&= a_1^\alpha a_2^{b_i^{-1}\alpha} a_{ij}^\alpha a_{ij}^{-\alpha} f(\bar{b}_i)f(\bar{b}_j)^{b_{i^*}^{-1}} a_{i^*j^*}b_{k^*} \\
&= a_1^\alpha a_2^{b_i^{-1}\alpha} f(\bar{b}_i)f(\bar{b}_j)^{b_{i^*}^{-1}} b_{i^*}b_{j^*} \\
&= a_1^\alpha a_2^{\alpha(f(\bar{b}_i)b_{i^*})^{-1}} f(\bar{b}_i)b_{i^*}f(\bar{b}_j)b_{i^*}^{-1}b_{i^*}b_{j^*} \\
&= a_1^\alpha f(\bar{b}_i)b_{i^*}a_2^\alpha b_{i^*}^{-1} f(\bar{b}_i)^{-1}f(\bar{b}_i)b_{i^*}f(\bar{b}_j)b_{j^*} \\
&= a_1^\alpha f(\bar{b}_i)b_{i^*}a_2^\alpha f(\bar{b}_j)b_{j^*}
\end{aligned}$$

$= (a_1b_i)^\gamma(a_2b_j)^\gamma$

$= g_1^\gamma\ g_2^\gamma.$

Thus γ is an endomorphism of G. Now let $ab_i \in \ker\gamma$ i.e., $(ab_i)^\gamma = a^\alpha f(\bar{b}_i)b_{i^*} = 1$. Then $b_{i^*} = b_1 = 1$ and since $\beta \in A(G/A)$ and $\bar{b}_i^\beta = \bar{b}_{i^*}$ we must have $b_i = b_1 = 1$. If in (1) we set $\bar{b}_i = \bar{b}_j = \bar{b}_1$ we get $f(\bar{b}_1) = f(\bar{b}_1)f(\bar{b}_1)$ or $f(\bar{b}_1) = 1$ since $a_{11} = 1$. Hence $a^\alpha = 1$ and since $\alpha \in A(A)$ we have $a = 1$. Therefore $\ker\gamma = (1)$ and we have shown that γ is 1-1. Finally let $g = ab_i$ be any element of G. Since $\beta \in A(G/A)$ there exists a b_j such that $\bar{b}_j^\beta = \bar{b}_i$. Hence $\bar{b}_j^\beta = \bar{b}_{j^*} = \bar{b}_i$ or $b_{j^*} = b_i$. Since $\alpha \in A(A)$ there exists $a_1 \in A$ such that $a_1^\alpha = af(\bar{b}_j)^{-1}$. Therefore $(a_1b_j)^\gamma = a_1^\alpha\ f(\bar{b}_j)b_{j^*} = a\ f(\bar{b}_j)^{-1}\ f(\bar{b}_j)b_i = ab_i$ and γ is onto. We have shown that $\gamma \in A(G)$ and clearly $\gamma|A = \alpha$. Further $(Ab_i)^\gamma = Ab_i^\gamma = A\ f(\bar{b}_i)b_{i^*} = Ab_{i^*} = (Ab_i)^\beta$ so that $\gamma|G/A = \beta$.

We can observe that nowhere in the proof of the above theorem did we need the fact that G/A was finite. It is easily verified that this theorem is true even if G/A is infinite.

The conditions obtained in the above theorem leave something to be desired. However, we must remember that the hypothesis includes the case that A is contained in the Frattini subgroup of G. From [3, pg. 156] we have that the Frattini subgroup consists of the non-generators of G. Thus it is not surprising that the conditions should be as they are.

We can obtain a corollary to the above result in the following way. Suppose $A \lhd G$ and $\alpha \in A(A)$. Then necessary and sufficient conditions that there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ are that there exists $\beta \in A(G/A)$ and a function f from G/A to A which satisfy (1) and (2) of Theorem 1.12.

Now we will illustrate Theorem 1.12 with two examples. In the first f will turn out to be a homomorphism. In general f need not be a homomorphism, in fact, it may be less well behaved than the function which appears in Theorem 1.5. This will be pointed out by the second example given below.

Example 2: $G = \langle a,c\rangle$, $a^8 = c^2 = 1$, $cac = a^5$. Let $A = Z(G) = \Phi(G) = \langle a^2\rangle$ and as coset representatives we choose $b_1 = 1$, $b_2 = a$, $b_3 = c$, and $b_4 = ac$. The mapping γ defined by $a^\gamma = ac$, $c^\gamma = a^4c$ is an automorphism of G. If $\alpha = \gamma|A$ and $\beta = \gamma|G/A$ then we have

$$\alpha : a^2 \to a^6 \text{ and } \beta : \begin{matrix} \langle a^2\rangle a \to \langle a^2\rangle ac \\ \langle a^2\rangle c \to \langle a^2\rangle c \end{matrix} .$$

Direct computation gives $f(\bar{b}_1) = 1$, $f(\bar{b}_2) = 1$, $f(\bar{b}_3) = a^4$, $f(\bar{b}_4) = a^4$, and

$a_{11} = 1$	$a_{21} = 1$	$a_{31} = 1$	$a_{41} = 1$
$a_{12} = 1$	$a_{22} = a^2$	$a_{32} = a^4$	$a_{42} = a^6$
$a_{13} = 1$	$a_{23} = 1$	$a_{33} = 1$	$a_{43} = 1$
$a_{14} = 1$	$a_{24} = a^2$	$a_{34} = a^4$	$a_{44} = a^6$.

Since $A = Z(G)$ condition (1) of Theorem 1.12 reduces to $f(\bar{b}_i\bar{b}_j) = f(\bar{b}_i)f(\bar{b}_j)a_{ij}^{-\alpha}a_{i^*j^*}$. One can verify from the information given above that $a_{ij}^{\alpha} = a_{i^*j^*}$ where $1^* = 1$, $2^* = 4$, $3^* = 3$, and $4^* = 2$. Thus f is a homomorphism.

<u>Example 3:</u> $G = \langle a,c \rangle$, $a^{27} = c^3 = 1$, $cac^{-1} = a^{19}$. Let $A = \Phi(G) = \langle a^3 \rangle$ and as coset representatives we choose $b_1 = 1$, $b_2 = a$, $b_3 = a^2$, $b_4 = c$, $b_5 = c^2$, $b_6 = ac$, $b_7 = a^2c$, $b_8 = ac^2$, and $b_9 = a^2c^2$. The mapping γ defined by

$$\gamma : \begin{array}{l} a \to a^4 c \\ c \to a^9 c \end{array}$$

is an automorphism of G. If $\alpha = \gamma|A$ and $\beta = \gamma|G/A$ then we have

$$\alpha : a^3 \to a^{12} \text{ and } \beta : \begin{array}{l} \langle a^3 \rangle a \to \langle a^3 \rangle ac \\ \langle a^3 \rangle c \to \langle a^3 \rangle c \end{array} .$$

Now $a_{22} = 1$ so $a_{22}^{\alpha} = 1$. Since $(\overline{b}_2)^{\beta} = \overline{b}_6$ we have $2^* = 6$. But $a_{66} = a^{18}$ so that $a_{22}^{\alpha} \neq a_{2^*2^*} = a_{66}$ contrary to the previous example. Now $f(\overline{b}_2\overline{b}_2) = f(\overline{b}_3) = a^{24}$ and $f(\overline{b}_2) = a^3$ so that $f(\overline{b}_2\overline{b}_2) \neq f(\overline{b}_2)f(\overline{b}_2)$ and f is not a homomorphism.

We observe from example 2 that if $A \leq Z(G)$ then condition (1) of Theorem 1.12 reduces to $f(\overline{b}_i\overline{b}_j) = f(\overline{b}_i)f(\overline{b}_j)a_{ij}^{-\alpha}\, a_{i^*j^*}$. Then f is a homomorphism iff $a_{ij}^{\alpha} = a_{i^*j^*}$.

If, in Theorem 1.12, we impose the further restriction that $G = AB$ and $A \cap B = (1)$ then we can choose the coset representatives as elements of B. From [3, pg. 221] we know that the factor set is trivial in this case i.e., $a_{ij} = 1$ for all i and j so that (1) reduces to $f(\overline{b}_i\overline{b}_j) = f(\overline{b}_i)f(\overline{b}_j)^{b_{i^*}^{-1}}$. Since $G/A \cong B$ we can consider f as a function from B to A with the property that $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\tau_\beta}}$ for all $b_1,b_2 \in B$ where τ_β is the automorphism of B induced by β. So (2) reduces to $\pi_b\alpha = \alpha\pi_{f(b)b^{\tau_\beta}}$. Thus this function is the same as the function which occurs in Theorem 1.5. We notice that f is almost a crossed homomorphism [4, pg. 105] of B to A and, in fact, would be if we insisted that A be abelian. In the next two results we try to find out when f behaves like a principal crossed homomorphism [4, pg. 106].

Theorem 1.13: Let $G = AB$, $A \triangleleft G$, $A \cap B = (1)$, $\alpha \in A(A)$, $\beta \in A(G/A)$, and τ_β the automorphism of B induced by β. If there exists an $x \in A$ such that $\pi_b \alpha = \alpha \pi_{x^{-1} b^{\tau_\beta} x}$ then the pair α, β can be extended to an automorphism γ of G. Moreover, $B^\gamma = x^{-1}Bx$ and the function f from B to A induced by γ acts like a principal crossed homomorphism.

Proof: For each $b \in B$ we define $f(b) \in A$ by $f(b) = x^{-1}x^{b^{-\tau_\beta}}$. Clearly f is a function from B to A and if b_1, b_2 are any two elements of B we have

$$\begin{aligned} f(b_1b_2) &= x^{-1}x^{(b_1b_2)^{-\tau_\beta}} \\ &= x^{-1}x^{b_2^{-\tau_\beta} b_1^{-\tau_\beta}} \\ &= x^{-1}x^{b_1^{-\tau_\beta}}(x^{-1})^{b_1^{-\tau_\beta}} x^{b_2^{-\tau_\beta} b_1^{-\tau_\beta}} \\ &= (x^{-1}x^{b_1^{-\tau_\beta}})(x^{-1}x^{b_2^{-\tau_\beta}})^{b_1^{-\tau_\beta}} \\ &= f(b_1)f(b_2)^{b_1^{-\tau_\beta}} . \end{aligned}$$

Since $f(b)b^{\tau_\beta} = x^{-1}x^{b^{-\tau_\beta}} b^{\tau_\beta} = x^{-1}b^{\tau_\beta} x b^{-\tau_\beta} b^{\tau_\beta} = x^{-1}b^{\tau_\beta} x$ we have $\pi_{f(b)b^{\tau_\beta}} = \pi_{x^{-1}b^{\tau_\beta}x} = \alpha^{-1}\pi_b \alpha$ or $\pi_b \alpha = \alpha \pi_{f(b)b^{\tau_\beta}}$. Summarizing we have $\alpha \in A(A)$, $\tau_\beta \in A(B)$, and a function f from B to A such that $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\tau_\beta}}$ for all $b_1, b_2 \in B$ and $\pi_b \alpha = \alpha \pi_{f(b)b^{\tau_\beta}}$. Therefore, by the sufficiency part of Theorem 1.5, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $b^{\gamma} = f(b)b^{\tau\beta}$. Now, as was shown previously, $b^{\gamma} = f(b)b^{\tau\beta} = x^{-1}b^{\tau\beta}x$ so that $B^{\gamma} = x^{-1}Bx$. Finally, $(Ab)^{\gamma} = Ab^{\gamma} = Af(b)b^{\tau\beta} = Ab^{\tau\beta} = (Ab)^{\beta}$ so that $\gamma|G/A = \beta$.

Theorem 1.14: Let $G = AB$, $A \lhd G$, $A \cap B = (1)$, and $\gamma \in A(G)$ such that $\gamma|A = \alpha \in A(A)$. If $B^{\gamma} = x^{-1}Bx$ for some $x \in A$ then the function f from B to A induced by γ is $f(b) = x^{-1}x^{b^{-\beta}}$ where $b^{\gamma} = f(b)b^{\beta}$ with $\beta \in A(B)$.

Proof: By the necessary part of Theorem 1.5 $b^{\gamma} = f(b)b^{\beta}$ where $\beta \in A(B)$. Since $B^{\gamma} = x^{-1}Bx$ we can write $b^{\gamma} = x^{-1}\bar{b}x$ for each $b \in B$ where $\bar{b} \in B$. Now $b^{\gamma} = x^{-1}\bar{b}x = x^{-1}x^{\bar{b}^{-1}}\bar{b}$ so we have $f(b)b^{\beta} = x^{-1}x^{\bar{b}^{-1}}\bar{b}$ and since $A \cap B = (1)$ it follows that $f(b) = x^{-1}x^{\bar{b}^{-1}}$ and $\bar{b} = b^{\beta}$. Hence $f(b) = x^{-1}x^{b^{-\beta}}$.

If A is complemented in G and we also assume that all complements of A are conjugate (this could be achieved by insisting $(|A|, |B|) = 1$ for example) then we can obtain the following corollary to Theorem 1.14.

Corollary 1.15: Let $G = AB$, $A \lhd G$, $A \cap B = (1)$, and assume all complements of A are conjugate. If $\gamma \in B(A,G)$ then $b^{\gamma} = f(b)b^{\beta}$ where $\beta \in A(B)$ and $f(b) = x^{-1}x^{b^{-\beta}}$ for some $x \in A$.

Proof: By the necessary part of Theorem 1.5 $b^{\gamma} = f(b)b^{\beta}$ where $\beta \in A(B)$. Since $\gamma \in B(A,G)$ we have $G^{\gamma} = (AB)^{\gamma} = A^{\gamma}B^{\gamma} = AB^{\gamma}$ so that B^{γ} is a complement of A. Thus there exists an $x \in A$ such that $B^{\gamma} = x^{-1}Bx$. The result now follows from Theorem 1.14.

One might be tempted to try to obtain a result like Theorem 1.5 in case A is supplemented in G. However, it is not possible to relax the condition $A \cap B = (1)$ as will be seen by the following example.

Example 4: $G = \langle a,b \rangle$, $a^4 = b^4 = 1$, $a^2 = b^2$, and $ba = a^{-1}b$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ then $A \cap B = \langle a^2 \rangle$. Let γ be any automorphism of G which leaves A invariant. We can write $y^\gamma = f(y)y^\beta$ for any $y \in B$ where $f(y) \in A$ and $y^\beta \in B$. However, $1 = 1^\gamma = b^2b^2 = a^2b^2$ so we could choose $f(1) = a^2$ so that f violates (1) of Lemma 1.6.

If $G = AB$, $A \lhd G$, and $\gamma \in B(A,G)$ then we can write $b^\gamma = f(b)b^\beta$ with $f(b) \in A$ and $b^\beta \in B$. Since $ab = bb^{-1}ab$ we have $(ab)^\gamma = (bb^{-1}ab)^\gamma$ and exactly as in the proof of Theorem 1.3 we can show that $\pi_b\alpha = \alpha\pi_{f(b)b^\beta}$ on A for all $b \in B$ where $\alpha = \gamma|A$. If $b \in A \cap B$ then we have $b^\alpha = f(b)b^\beta$. However, in general, we cannot decide if $\beta \in A(B)$ or if f behaves like the function in Theorem 1.5. If we examine the first part of the proof of Theorem 1.5 we find that the key step in concluding that f and β satisfy (1) and (2) was the application of $A \cap B = (1)$ to the equation $f(b_1b_2)(b_1b_2)^\beta = f(b_1)f(b_2)^{(b_1^\beta)^{-1}}b_1^\beta b_2^\beta$. But if we know that either $\beta \in A(B)$ or $f(b_1b_2) = f(b_1)f(b_2)^{(b_1^\beta)^{-1}}$ then both must hold. Hence we have:

Theorem 1.16: Let $G = AB$, $A \lhd G$, and $\gamma \in B(A,G)$. Then $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1,b_2 \in B$ iff $\beta \in A(B)$ where $b^\gamma = f(b)b^\beta$ with $f(b) \in A$ and $b^\beta \in B$.

Throughout the remainder of this section G will be a finite group. If B_1 is a supplement of A in G such that $|B_1| = |B|$ and $B_1 = \{f(b)b^{\beta} | \beta \in A(B)$ or equivalently $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1, b_2 \in B\}$ then we say that B_1 is related to B and write $B_1 \sim B$. We call f an associating function and β an associating automorphism. With this notation we have:

Theorem 1.17: Let $G = AB$ with $A \lhd G$. Then $x^{-1}Bx \sim B$ for all $x \in A$. Moreover we can choose an associating function f which behaves like a principal crossed homomorphism and an associating automorphism $\beta = 1_B$.

Proof: We can write $x^{-1}bx = x^{-1}bxb^{-1}b = x^{-1}x^{b^{-1}}b$. For each $b \in B$ define $f(b) = x^{-1}x^{b^{-1}}$. Then $x^{-1}Bx = \{f(b)b | f(b) = x^{-1}x^{b^{-1}}$ for all $b \in B\}$. Hence $x^{-1}Bx \sim B$ with the desired properties.

Summarizing the remarks preceeding Theorem 1.16 we have:

Theorem 1.18: Let $G = AB$ with $A \lhd G$. If $\gamma \in B(A,G)$, $b^{\gamma} = f(b)b^{\beta}$ with $f(b) \in A$ and $b^{\beta} \in B$, and $\gamma | A = \alpha$ then

(1) $b^{\alpha} = f(b)b^{\beta}$ for all $b \in A \cap B$;

(2) $\pi_b \alpha = \alpha \pi_{f(b)b^{\beta}}$ on A.

Next we obtain a partial converse of the above theorem.

Theorem 1.19: Let $G = AB$, $A \lhd G$, and $B_1 \sim B$ with associating function f and associating automorphism β. Then $\alpha \in A(A)$ can be extended to G if

(1) $b^{\alpha} = f(b)b^{\beta}$ for all $b \in A \cap B$;

(2) $\pi_b \alpha = \alpha \pi_{f(b)b^{\beta}}$ on A.

Proof: Define τ by $b^{\tau} = f(b)b^{\beta}$. Since f is a function and β an automorphism τ is well defined. Let b_1, b_2 be any two elements of B and we have

$$\begin{aligned}(b_1b_2)^{\tau} &= f(b_1b_2)(b_1b_2)^{\beta} \\ &= f(b_1)f(b_2)^{b_1^{-\beta}} b_1^{\beta}b_2^{\beta} \\ &= f(b_1)b_1^{\beta}f(b_2)b_1^{-\beta}b_1^{\beta}b_2^{\beta} \\ &= f(b_1)b_1^{\beta}\, f(b_2)b_2^{\beta} \\ &= b_1^{\tau}\, b_2^{\tau}\ .\end{aligned}$$

Thus τ is a homomorphism from B to B_1. Clearly τ is onto and since G is finite τ is an isomorphism. From (1) we have $b^{\alpha} = b^{\tau}$ for all $b \in A \cap B$. From (2) we have $\pi_b \alpha = \alpha \pi_{b^{\tau}}$ on A. Therefore, from the sufficiency part of Theorem 1.3, there exists $\gamma \in A(G)$ such that $\gamma | A = \alpha$.

1.3 Power and Central Automorphisms

In the first three results of this section we consider the problem of extending a power automorphism of A to G. A power automorphism maps each $a \in A$ onto a^k where k is some fixed positive integer.

Theorem 1.20: Let G = AB, $A \triangleleft G$, and $A \cap B = (1)$. Then every power automorphism of A can be extended to G.

Proof: Let α be any power automorphism of A i.e., $a^{\alpha} = a^{k}$ for all $a \in A$ where k is some fixed positive integer. We show that the pair α, 1_B can be lifted to G. By Theorem 1.2, we need only show that $\pi_b\alpha = \alpha\pi_b$ on A. Now if a is any element of A we have $a^{b\alpha} = (b^{-1}ab)^{\alpha} = (b^{-1}ab)^{k} = b^{-1}a^{k}b = b^{-1}a^{\alpha}b = a^{\alpha b}$ so that $\pi_b\alpha = \alpha\pi_b$ on A.

The above result tells us that when A is complemented then every power automorphism can be lifted. Now if $\gamma \in B(A,G)$ such that $\gamma|A$ is a power automorphism then what is its effect on the rest of G? This question is answered by the next theorem when A is supplemented in G.

Theorem 1.21: Let G = AB with $A \lhd G$. If $\gamma \in B(A,G)$ such that $\gamma|A = \alpha$ is a power automorphism of A then $b^{\gamma} = g(b)b$ where g is a function from B to $C_G(A)$ and $g(b_1b_2) = g(b_1)g(b_2)^{b_1^{-1}}$ for all $b_1,b_2 \in B$.

Proof: By Theorem 1.3 we know that $\pi_b\alpha = \alpha\pi_{b^{\gamma}}$ on A. Let $a^{\alpha} = a^{k}$ where k is some fixed positive integer. Now

$$
\begin{aligned}
a^{b\alpha} &= a^{\alpha b^{\gamma}} \\
(b^{-1}ab)^{\alpha} &= b^{-\gamma}a^{\alpha}b^{\gamma} \\
(b^{-1}ab)^{k} &= b^{-\gamma}a^{k}b^{\gamma} \\
b^{-1}a^{k}b &= b^{-\gamma}a^{k}b^{\gamma}
\end{aligned}
$$

so that $b^{\gamma}b^{-1} \in C_G(A)$. Set $b^{\gamma}b^{-1} = g(b)$ with $g(b) \in C_G(A)$ then clearly g is a function since $\gamma \in A(G)$. Since $\gamma \in A(G)$, $(b_1b_2)^{\gamma} = b_1^{\gamma}b_2^{\gamma}$ for all $b_1,b_2 \in B$. Hence

$$
\begin{aligned}
g(b_1b_2)b_1b_2 &= g(b_1)b_1g(b_2)b_2 \\
g(b_1b_2)b_1b_2 &= g(b_1)b_1g(b_2)b_1^{-1}b_1b_2 \\
g(b_1b_2) &= g(b_1)g(b_2)^{b_1^{-1}}.
\end{aligned}
$$

Under certain conditions we can take such a function as described in the above theorem, pair it with a power automorphism of A and extend to G.

Theorem 1.22: Let $G = AB$, $A \triangleleft G$, and $A \cap B = (1)$. Let $\alpha \in A(A)$ be a power automorphism and g a function from B to $C_G(A)$ such that $g(b_1b_2) = g(b_1)g(b_2)^{b_1^{-1}}$ for all $b_1, b_2 \in B$ and $g(b) \not\equiv b^{-1} \pmod{(A)}$ for $b \neq 1$. Then the pair α, τ, where $b^\tau = g(b)b$, can be extended to $\gamma \in A(G)$ provided G is finite.

Proof: We will show that we have all the conditions of Theorem 1.3 satisfied. (1) is clear. Since $g(b) \in C_G(A)$ we have $\pi_{b^\tau} = \pi_{g(b)b} = \pi_b$ on A. α is a power automorphism of A so that $\pi_b\alpha = \alpha\pi_b$. Hence $\alpha\pi_{b^\tau} = \alpha\pi_b = \pi_b\alpha$ so we have (2). Now τ is well defined since g is a function and if b_1, b_2 are any two elements of B we have

$$\begin{aligned}(b_1b_2)^\tau &= g(b_1b_2)b_1b_2\\ &= g(b_1)g(b_2)^{b_1^{-1}} b_1b_2\\ &= g(b_1)b_1g(b_2)b_1^{-1}b_1b_2\\ &= g(b_1)b_1g(b_2)b_2\\ &= b_1^\tau b_2^\tau .\end{aligned}$$

Thus τ is a homomorphism. Let $b \in \ker \tau$ i.e., $b^\tau = g(b)b = 1$. Then $g(b) = b^{-1}$ so clearly $g(b) \equiv b^{-1} \pmod{(A)}$. Thus $b = 1$ and $\ker \tau = (1)$. Since G is finite τ is an isomorphism. Finally we must show that $G = AB_1$ where $B_1 = B^\tau$. Assume otherwise. Then we must have $A \cap B_1 \neq (1)$ since G is finite. So there exists a $b \in B$ and an $a \in A$ such that

$g(b)b = a$ where $a \neq 1$. Then $g(b) = ab^{-1}$ so that $g(b) \equiv b^{-1} \pmod{(A)}$. Hence $b = 1$ and from $g(b_1b_2) = g(b_1)g(b_2)^{b_1^{-1}}$ one can easily show that $g(1) = 1$. Therefore, $a = 1$ which is a contradiction. Hence we must have $G = AB_1$.

The remaining results in this section have to do with lifting central automorphisms of A to G. γ is said to be a central automorphism of the group H if $h^{\gamma} = z_h h$ where $z_h \in Z(H)$ for all $h \in H$. One can show that the mapping h to z_h is a homomorphism of H into $Z(H)$. Thus if α is a central automorphism of A we can write $a^{\alpha} = f(a)a$ for all $a \in A$ where $f \in \text{Hom}(A,Z(A))$. Clearly $f(a) \neq a^{-1}$ if $a \neq 1$. We define $\overset{*}{\text{Hom}}(A,Z(A)) = \{f \in \text{Hom}(A,Z(A)) | f(a) \neq a^{-1} \text{ for } a \neq 1\}$. It was shown in [1] that there is a 1-1 correspondence between the central automorphisms of A and the elements of $\overset{*}{\text{Hom}}(A,Z(A))$. Let A be supplemented by B in G and define $\overset{**}{\text{Hom}}(A,Z(A)) = \{f \in \overset{*}{\text{Hom}}(A,Z(A)) | f(b^{-1}ab) = b^{-1}f(a)b \text{ for all } a \in A \text{ and } b \in B\}$. Then we have

Theorem 1.23: Let $G = AB$ with $A \lhd G$. If $f \in \overset{**}{\text{Hom}}(A,Z(A))$ and $A \cap B \leq \ker f$ then the central automorphism α_f, $a^{\alpha_f} = f(a)a$, of A associated with f can be extended to $\gamma \in A(G)$ such that $\gamma|B = 1_B$. Conversely, if $\gamma \in B(A,G)$ is such that $\gamma|A = \alpha_f$ is central and $\gamma|B = 1_B$ then $f \in \overset{**}{\text{Hom}}(A,Z(A))$ and $A \cap B \leq \ker f$.

Proof: (a) Let $f \in \overset{**}{\text{Hom}}(A,Z(A))$ such that $A \cap B \leq \ker f$. Then α_f, defined by $a^{\alpha_f} = f(a)a$, is a central automorphism of A. We will show that $\alpha_f|A \cap B = 1_B|A \cap B$ and $\pi_b\alpha_f = \alpha_f\pi_b$ so that, by Theorem 1.2, the pair $\alpha_f, 1_B$ can be extended to G. Now if $b \in A \cap B$ then $b^{\alpha_f} = f(b)b = b$ since $b \in \ker f$. Hence $\alpha_f|A \cap B = 1_B|A \cap B$. Let a be any element of A and we have $a^{b\alpha_f} = (b^{-1}ab)^{\alpha_f} = f(b^{-1}ab)b^{-1}ab =$ $b^{-1}f(a)bb^{-1}ab = b^{-1}f(a)ab = b^{-1}a^{\alpha_f}b = a^{\alpha_f b}$ so that $\pi_b\alpha_f = \alpha_f\pi_b$.

(b) Let $\gamma \in B(A,G)$ such that $\gamma|A = \alpha_f$ is central and $\gamma|B = 1_B$. Then, by Theorem 1.2, we must have $\pi_b\alpha_f = \alpha_f\pi_b$ on A and $\alpha_f|A \cap B = 1_B|A \cap B$. Let $b \in A \cap B$ then we have $b = b^{1B} = b^{\alpha_f} = f(b)b$ so that $f(b) = 1$ and $A \cap B \leq \ker f$. Let a be any element of A and we have

$$a^{b\alpha_f} = a^{\alpha_f b}$$

$$(b^{-1}ab)^{\alpha_f} = (f(a)a)^b$$

$$f(b^{-1}ab)b^{-1}ab = b^{-1}f(a)ab$$

$$f(b^{-1}ab)b^{-1}ab = b^{-1}f(a)bb^{-1}ab$$

$$f(b^{-1}ab) = b^{-1}f(a)b.$$

Therefore, $f \in \overset{**}{\mathrm{Hom}}(A,Z(A))$.

If we add to the above theorem the further restriction $A \cap B = (1)$ then the condition $A \cap B \leq \ker f$ is trivally satisfied so we can rephrase the theorem as follows:

<u>Theorem 1.24</u>: Let $G = AB$, $A \lhd G$, and $A \cap B = (1)$. If $f \in \overset{**}{\mathrm{Hom}}(A,Z(A))$ then the central automorphism α_f of A associated with f can be extended to $\gamma \in A(G)$ such that $\gamma|B = 1_B$ and conversely.

What conditions must we put on $\alpha \in A(A)$ so that α can be extended to a central automorphism of G? This question is answered in the next result under the hypothesis that A is supplemented in G.

<u>Theorem 1.25</u>: Let $G = AB$ with $A \lhd G$. Then $\gamma \in B(A,G)$ is central iff there exist homomorphisms f and g such that

(1) $f \in \overset{*}{\mathrm{Hom}}\ (A, Z(G) \cap A)$, $g \in \overset{*}{\mathrm{Hom}}\ (B, Z(G))$, and $f(a)g(b) \neq (ab)^{-1}$ for all $ab \neq 1$;

(2) $f(b^{-1}ab) = f(a)$ for all $a \in A$ and $b \in B$;

(3) $f(b) = g(b)$ for all $b \in A \cap B$.

Moreover $a^{\alpha} = f(a)a$, $b^{\tau} = g(b)b$ where $\alpha = \gamma|A$ and $\tau = \gamma|B$.

Proof: (a) Necessity

Let $\gamma \in B(A,G)$ such that γ is a central automorphism of G. Then we can write $x^{\gamma} = h(x)x$ for all $x \in G$ where $h \in \overset{*}{\mathrm{Hom}}\ (G, Z(G))$. Define f and g by $f = h|A$ and $g = h|B$. Clearly $g \in \overset{*}{\mathrm{Hom}}\ (B, Z(G))$. Since $\gamma \in B(A,G)$ $a^{\gamma} = f(a)a \in A$ so we must have $f(a) \in A$. Thus $f(a) \in Z(G) \cap A$ and $f \in \overset{*}{\mathrm{Hom}}\ (A, A \cap Z(G))$. Since $\gamma \in A(G)$, $(ab)^{\gamma} = h(ab)ab \neq 1$ if $ab \neq 1$ so $h(ab) = f(a)g(b) \neq (ab)^{-1}$ if $ab \neq 1$. Since $f = h|A$ we have $f(b^{-1}ab) = h(b^{-1}ab) = h(b)^{-1}h(a)h(b) = h(a) = f(a)$ since $h(a) \in Z(G)$. (3) is obvious in view of the way f and g are defined.

(b) Sufficiency

Now suppose f and g are given satisfying (1), (2), and (3) of the theorem. Define h by $h(ab) = f(a)g(b)$ and γ by $(ab)^{\gamma} = a^{\alpha}b^{\tau}$ where $a^{\alpha} = f(a)a$ and $b^{\tau} = g(b)b$. Then we have $(ab)^{\gamma} = f(a)ag(b)b = f(a)g(b)ab = h(ab)ab$ where $h(ab) \in Z(G)$. If we show that $h \in \overset{*}{\mathrm{Hom}}\ (G, Z(G))$ we will have the desired result. Now if $a_1b_1 = a_2b_2$ with $a_i \in A$ and $b_i \in B$ for $i = 1,2$ then $a_2^{-1}a_1 = b_2b_1^{-1} \in A \cap B$ so by (3) we have

$$f(a_2^{-1}a_1) = g(b_2b_1^{-1})$$

$$f(a_2)^{-1}f(a_1) = g(b_2)g(b_1)^{-1}$$

$$f(a_1)g(b_1) = f(a_2)g(b_2)$$

$$h(a_1b_1) = h(a_2b_2).$$

Thus h is well defined. Let a_1b_1 and a_2b_2 be any two elements of G where $a_i \in A$ and $b_i \in B$ for i = 1,2. Then $a_1b_1a_2b_2 = a_1(b_1a_2b_1^{-1})b_1b_2$ so that using (2) at the appropriate step we have

$$\begin{aligned} h(a_1b_1a_2b_2) &= h(a_1(b_1a_2b_1^{-1})b_1b_2) \\ &= f(a_1(b_1a_2b_1^{-1}))g(b_1b_2) \\ &= f(a_1)f(b_1a_2b_1^{-1})g(b_1)g(b_2) \\ &= f(a_1)f(a_2)g(b_1)g(b_2) \\ &= f(a_1)g(b_1)f(a_2)g(b_2) \\ &= h(a_1b_1)h(a_2b_2). \end{aligned}$$

Hence $h \in \text{Hom}\,(G,Z(G))$. Finally if $ab \neq 1$ then, by (1), we have $h(ab) = f(a)g(b) \neq (ab)^{-1}$ so that $h \in \check{\text{Hom}}\,(G,Z(G))$.

The above result tells us little of what happens to B. If we further hypothesize A abelian and $A \cap B = (1)$ then the result can be sharpened. Under these conditions it is easy to show that $Z(G) = A_1B_1$ where $A_1 = C_A(B)$ and $B_1 = C_B(A) \cap Z(B)$. With this notation we have:

<u>Theorem 1.26</u>: Let $G = AB$, $A \triangleleft G$, A abelian, and $A \cap B = (1)$. Then necessary and sufficient conditions that $\gamma \in B(A,G)$ is central are that there exists a homomorphism f from B to A_1 and a $\beta \in A(B)$ such that

(1) $\alpha(=\gamma|A)$ and β are multiplier automorphisms with multipliers from A_1 and B_1 respectively;

(2) $\pi_b\alpha = \alpha\pi_b$ on A.

Proof: (a) Necessity

Let $\gamma \in B(A,G)$ be central. Then we can write $g^{\gamma} = h(g)g$ where $h(g) \in Z(G)$ for all $g \in G$. By Theorem 1.5 there exists a function f from B to A and a $\beta \in A(B)$ such that $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1,b_2 \in B$ and $\pi_b\alpha = \alpha\pi_{f(b)b^{\beta}}$. Now $b^{\gamma} = f(b)b^{\beta}$ so we have $b^{\gamma} = f(b)b^{\beta} = h(b)b$ and by the remarks preceding this theorem we can write $h(b) = a_1b_1$ with $a_1 \in A_1$ and $b_1 \in B_1$ since $h(b) \in Z(G)$. Since $A \cap B = (1)$ it follows that $f(b) = a_1$ and $b^{\beta} = b_1b$. Therefore f is a function from B to A_1 and β is a multiplier automorphism with multipliers from B_1. Since $A_1 = C_A(B)$ we have $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}} = f(b_1)f(b_2)$ so that f is a homomorphism from B to A_1. Let $a \in A$ then since $\gamma \in B(A,G)$, $a^{\gamma} = h(a)a \in A$ so that $h(a) \in A$. Thus $h(a) \in A \cap Z(G) \leq A_1$ so that $\alpha = \gamma|A$ is a multiplier automorphism with multipliers from A_1. Now $\pi_b\alpha = \alpha\pi_{f(b)b^{\beta}} = \alpha\pi_{f(b)b_1b} = \alpha\pi_b$ since $f(b) \in A$ and $b_1 \in C_B(A)$. Thus $_b\alpha = \alpha\pi_b$.

(b) Sufficiency

Let α, β, and f be given satisfying (1), (2), and (3) of the theorem. By (2) we can write $b^{\beta} = b_1b$ with $b_1 \in B_1$. Thus $\alpha\pi_{f(b)b^{\beta}} = \alpha\pi_{f(b)b_1b} = \alpha\pi_b$ since $f(b) \in A$ and $b_1 \in B_1 \leq C_B(A)$. But, by (3), $\alpha\pi_b = \pi_b\alpha$ so that $\alpha\pi_{f(b)b^{\beta}} = \pi_b\alpha$. Hence, by the sufficiency part of Theorem 1.5, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $b^{\gamma} = f(b)b^{\beta}$. By (2) we can write $a^{\alpha} = a_1a$ with $a_1 \in A_1$. Therefore $(ab)^{\gamma} = a^{\alpha}b^{\gamma} = a_1af(b)b^{\beta} = a_1af(b)b_1b = a_1f(b)b_1ab$ and since $a_1f(b)b_1 \in A_1B_1 = Z(G)$ we have that γ is a central automorphism of G.

1.4 The Group B(A,G)

Recall that $B(A,G) = \{\gamma \in A(G) | \gamma|A = \alpha \in A(A)\}$. We define $C(A,G) = \{\gamma \in B(A,G) | \gamma|A = 1_A\}$. Then we have

Theorem 1.27: $C(A,G) \lhd B(A,G)$ and $[B(A,G):C(A,G)]$ = the number of automorphisms of A that can be lifted to G.

Proof: Let $\gamma \in B(A,G)$ and $\bar{\gamma} \in C(A,G)$. If a is any element of A then we have $a^{\gamma^{-1}\bar{\gamma}\gamma} = (a^{\gamma^{-1}})^{\bar{\gamma}\gamma} = (a^{\gamma^{-1}})^{\gamma} = a^{1_A}$ since $a^{\gamma^{-1}} \in A$. Thus $\gamma^{-1}\bar{\gamma}\gamma|A = 1_A$ and $\gamma^{-1}\bar{\gamma}\gamma \in C(A,G)$. Hence $C(A,G) \lhd B(A,G)$.

If $\gamma_1,\gamma_2 \in B(A,G)$ and $\gamma_1|A = \gamma_2|A$ then $\gamma_2^{-1}\gamma_1|A = 1_A$ so that $\gamma_2^{-1}\gamma_1 \in C(A,G)$. Thus $\gamma_1 C(A,G) = \gamma_2 C(A,G)$ and we have the second assertion of the theorem.

The next result tells us when no automorphism of A different from 1_A can be lifted to G under the hypothesis that A is complemented in G. In view of the above theorem this also tells us when $B(A,G) = C(A,G)$.

Theorem 1.28: Let $G = AB$, $A \lhd G$, and $A \cap B = (1)$. If $|A| \neq 2$ then A has a nontrivial automorphism that can be lifted to G i.e., $|B(A,G)/C(A,G)| \neq 1$.

Proof: Suppose that $B(A,G) = C(A,G)$. Then since $\pi_g \in B(A,G)$ for all $g \in G$ we must have $a^g = a$. Thus $A \leq Z(G)$ so $B \lhd G$ and $G = A \times B$. Let α be any element of $A(A)$. Since $\pi_b = 1_A$ for all $b \in B$ we have $\pi_b\alpha = \alpha\pi_b$. Hence, by Theorem 1.2, the pair $\alpha, 1_B$ can be extended to G. Therefore, we must have $\alpha = 1_A$ i.e., $|A(A)| = 1$ and since A is abelian it follows that $|A| = 2$. This is a contradiction so the theorem is true.

The next theorem gives us a set of sufficient conditions under which every automorphism of A can be extended to G. But first we prove a lemma which is of some interest itself. Let B be a supplement of A in G and define B^* to be the subgroup of $A(A)$ consisting of those automorphisms of A induced by the elements of B i.e., $B^* = \{\pi_b | A | b \in B\}$. Then we have:

<u>Lemma 1.29</u>: Let $G = AB$, $A \lhd G$, and $C_B(A) = 1$. If $\alpha \in N_{A(A)}(B^*)$ and $A \cap B$ is α-invariant then there exists a unique $\beta \in A(B)$ such that the pair α, β can be lifted to an automorphism of G.

<u>Proof</u>: Let b_1 and b_2 be any two elements of B. If $\pi_{b_1} = \pi_{b_2}$ on A then $b_2^{-1}b_1 \in C_B(A)$ and since $C_B(A) = 1$ it follows that $b_1 = b_2$. We will use this fact several times in the proof. From the above remarks it follows that $B \simeq B^*$. Since $\alpha \in N_{A(A)}(B^*)$ and $B \simeq B^*$ we have for each $b \in B$ a unique $\bar{b} \in B$ such that $\alpha^{-1}\pi_b\alpha = \pi_{\bar{b}}$. Define β by $b^\beta = \bar{b}$. Clearly β is well defined. Let $b \in \ker \beta$ i.e., $b^\beta = 1$. Then $\alpha^{-1}\pi_b\alpha = \pi_1 = 1_A$ so that $\pi_b = 1_A$. Hence $b \in C_B(A) = 1$ and $\ker \beta = (1)$. β is onto since α induces an automorphism of B^* and $B^* \simeq B$. Now let b_1 and b_2 be any two elements of B and we have

$$\pi_{(b_1b_2)^\beta} = \alpha^{-1}\pi_{b_1b_2}\alpha = \alpha^{-1}\pi_{b_1}\pi_{b_2}\alpha = \alpha^{-1}\pi_{b_1}\alpha\alpha^{-1}\pi_{b_2}\alpha = \pi_{b_1^\beta}\pi_{b_2^\beta} = \pi_{b_1^\beta b_2^\beta}.$$

Therefore, since $(b_1b_2)^\beta \in B$ and $b_1^\beta b_2^\beta \in B$ we have that $(b_1b_2)^\beta = b_1^\beta b_2^\beta$. So $\beta \in A(B)$.

If $x \in A$ then it is easy to show that $\alpha^{-1}\pi_x\alpha = \pi_{x^\alpha}$. Thus if $b \in A \cap B$ then $\pi_{b^\beta} = \alpha^{-1}\pi_b\alpha = \pi_{b^\alpha}$ since $b \in A$. Since $A \cap B$ is α-invariant, $b^\alpha \in A \cap B \leq B$. Therefore $\pi_{b^\beta} = \pi_{b^\alpha}$ implies that $b^\beta = b^\alpha$. Hence $\alpha|A \cap B = \beta|A \cap B$. Summarizing we have $\alpha \in A(A)$ and $\beta \in A(B)$ such that $\alpha|A \cap B = \beta|A \cap B$ and $\pi_b\alpha = \alpha\pi_{b^\beta}$. Therefore, by Theorem 1.2, there exists $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|B = \beta$.

Finally suppose that the pairs α,β_1 and α,β_2 can both be extended to G. Then, by Theorem 1.2, we have $\pi_b\alpha = \alpha\pi_{b^{\beta_1}}$ and $\pi_b\alpha = \alpha\pi_{b^{\beta_2}}$. Hence $\pi_{b^{\beta_1}} = \alpha^{-1}\pi_b\alpha = \pi_{b^{\beta_2}}$ so that $b^{\beta_1} = b^{\beta_2}$. Since b is an arbitrary element of B we must have $\beta_1 = \beta_2$.

If we add to the hypothesis of the above lemma $B^* \trianglelefteq A(A)$ and $A \cap B$ characteristic in A then we can say that each $\alpha \in A(A)$ can be paired with a unique $\beta \in A(B)$ and extended to G since $N_{A(A)}(B^*) = A(A)$ and $A \cap B$ is α-invariant for each $\alpha \in A(A)$. The next result gives us a set of sufficient conditions under which every automorphism of A can be extended to G that was referred to above Lemma 1.29.

<u>Theorem 1.30</u>: Let $G = AB$, $A \triangleleft G$, $C_B(A) = (1)$, $B^* \trianglelefteq A(A)$, and assume that $A \cap B$ is a characteristic subgroup of A. Then $B(A,G)/C(A,G) \simeq A(A)$.

<u>Proof</u>: Clearly $B(A,G)/C(A,G) \simeq$ to a subgroup of $A(A)$ under the correspondence $\gamma C(A,G) \iff \gamma|A$. By the remarks following Lemma 1.29 each automorphism of A can be extended to G so, by Theorem 1.27, $[B(A,G):C(A,G)] = |A(A)|$. Thus $B(A,G)/C(A,G) \simeq A(A)$.

If an automorphism of a subgroup can be extended to the whole group then it can usually be done in many ways. In fact Theorem 1.27 tells us that it can be extended $|C(A,G)|$ ways. In our next result we try to find out when an automorphism of a normal abelian subgroup has a unique extension to G. This, of course, is equivalent to asking when is $|C(A,G)| = 1$. First we prove the following:

<u>Lemma 1.31</u>: If $A \triangleleft G$ and $C(A,G) = (1_G)$ then $C_G(A) = Z(G)$.

<u>Proof</u>: Clearly $Z(G) \leq C_G(A)$. Let $g \in C_G(A)$ then $\pi_g \in C(A,G) = (1_G)$ so $x^g = x$ for all $x \in G$. Hence $g \in Z(G)$ and we have $C_G(A) \leq Z(G)$. Thus $C_G(A) = Z(G)$.

<u>Theorem 1.32</u>: Let A be a normal abelian subgroup of G. Then $C(A,G) = (1_G)$ iff G is abelian, $[G:A] = 2$, and $|A|$ is odd.

<u>Proof</u>: (a) Necessity

Suppose $C(A,G) = (1_G)$. Then, by Lemma 1.31, $C_G(A) = Z(G)$. Since A is abelian we have $A \leq C_G(A) = Z(G)$. Hence if $g \in G$ then $\pi_g \in C(A,G)$ so $x^g = x$ for all $x \in G$. Thus $g \in Z(G)$ and since g was arbitrary we have that G is abelian. Now we will show that G must split over A.

We can write $G = S_{p_1} \times \cdots \times S_{p_r}$, $A = A_{p_1} \times \cdots \times A_{p_r}$, and $G/A = \overline{S}_{p_1} \times \cdots \times \overline{S}_{p_r}$ where S_{p_i} is a p_i-Sylow subgroup of G, $A_{p_i} = A \cap S_{p_i}$, and $\overline{S}_{p_i} \simeq S_{p_i}/A_{p_i}$. We will assume that G does not split over A and arrive at a contradiction by constructing a nontrivial automorphism γ of G such that $\gamma|A = 1_A$ and $\gamma|G/A = 1_{G/A}$. By Theorem 1.12 we need only construct a nontrivial homomorphism f from G/A to A to accomplish this. If $(|A|,[G:A]) = 1$ then G splits over A so we may assume that for some i, $A_{p_i} \neq (1)$ and $\overline{S}_{p_i} \neq (\overline{1})$. If A_{p_i} is a direct factor of S_{p_i} for each i then G splits over A so we may assume for some i that A_{p_i} is not a direct factor of S_{p_i}. There is no loss in generality if we assume i = 1. Define $f|\overline{S}_{p_i} = 1$ for i > 1. Now S_{p_1} is a direct product of cyclic subgroups, say $S_{p_1} = C_1 \times \cdots \times C_s$ where C_i is generated by x_i. Then we can write $A_{p_1} = A_1 \times \cdots \times A_s$ where $A_i = A_{p_1} \cap C_i$. Since A_{p_1} is not a direct factor of S_{p_1} there exists at least one A_i such that $1 \lneq A_i \lneq C_i$, say A_1. Define $f|\overline{C}_i = 1$ for $i > 1$. Let $|x_1| = p_1^t$ and $A_1 = \langle x_1^{p_1^u} \rangle$ where we may assume $0 < u < t$. The only cosets of G/A on which f has not been defined are of the form $x_1^k A$. Define $f(x_1^k A) = (x_1^{p_1^u})^k$. Clearly f maps G/A into A_1. Now any two elements $\overline{g}_1, \overline{g}_2$ of G/A can be written in the form $\overline{g}_1 = x_1^{k_1} yA$, $\overline{g}_2 = x_1^{k_2} zA$ where y and z belong to $C_2 \times \cdots \times C_s \times S_{p_2} \times \cdots \times S_{p_r}$. Hence $f(\overline{g}_1) = (x_1^{p_1^u})^{k_1}$, $f(\overline{g}_2) = (x_1^{p_1^u})^{k_2}$, and $f(\overline{g}_1\overline{g}_2) = (x_1^{p_1^u})^{(k_1+k_2)}$ since $\overline{g}_1\overline{g}_2 = x_i^{(k_1+k_2)} yzA$. Therefore f is a homomorphism from G/A to A and $f(x_1 A) = x_1^{p_1^u} \neq 1$ so f is nontrivial and we have arrived at a contradiction. Thus we may assume that G splits over A.

Let G = AB, $A \cap B = (1)$, and $\beta \in A(B)$. Since $\pi_b = 1_A$ for all $b \in B$ we have $\pi_b 1_A = 1_A \pi_{b^\beta}$ so that, by Theorem 1.3, the pair $1_A, \beta$ can be extended to $\gamma \in A(G)$. Now $\gamma \in C(A,G) = (1_G)$ so we must have $\beta = 1_B$ i.e., $|A(B)| = 1$. Since B is abelian we must have $|B| = 2$. Let $B = \langle b \rangle$ where $b^2 = 1$. If there exists an $x \in A$ such that $x^2 = 1$ then, using Theorem 1.3 as before, we can show that the pair $1_A, \tau$ where $b^\tau = xb$ can be extended to $\gamma \in A(G)$. But $\gamma \in C(A,G) = (1_G)$ so we must have $\gamma = 1_G$. Hence $b^\tau = b$ i.e., x = 1 and |A| is odd.

(b) Sufficiency

Let G be an abelian group such that $G = AB$, $A \cap B = (1)$, $|B| = 2$, and $|A|$ odd. Let $B = \langle b \rangle$ where $b^2 = 1$. If $\gamma \in C(A,G)$ then b^γ is an element of the coset Ab since $G = A \cup Ab$. Hence there exists an $x \in A$ such that $b^\gamma = xb$. But $|b| = 2$ so that $|xb| = 2$. Hence $(xb)^2 = x^2b^2 = x^2 = 1$ which contradicts the fact that $|A|$ is odd unless $x = 1$. Therefore, $\gamma = 1_G$ and we have shown that $C(A,G) = (1_G)$.

Now we try to find out something about the structure of $B(A,G)/C(A,G)$. We know that $B(A,G)/C(A,G)$ is isomorphic to a subgroup of $A(A)$ and also that $N_G(A)/C_G(A)$ is isomorphic to a subgroup of $A(A)$. The next result tells us that $B(A,G)/C(A,G)$ contains a subgroup isomorphic to $N_G(A)/C_G(A)$ and we make no assumptions about A.

<u>Theorem 1.33</u>: $B(A,G)/C(A,G)$ contains a subgroup isomorphic to $N_G(A)/C_G(A)$.

<u>Proof</u>: We define $\Theta: N_G(A)/C_G(A) \longrightarrow B(A,G)/C(A,G)$ by $(gC_G(A))^\Theta = \pi_g C(A,G)$. Since $g \in N_G(A)$, $\pi_g \in B(A,G)$. If $g_1C_G(A) = g_2C_G(A)$ then $g_2^{-1}g_1 \in C_G(A)$ so that $\pi_{g_2^{-1}g_1} = \pi_{g_2^{-1}}\pi_{g_1} \in C(A,G)$. Therefore $\pi_{g_1} C(A,G) = \pi_{g_2} C(A,G)$ and Θ is well defined. Let $g_1C_G(A)$ and $g_2C_G(A)$ be any two elements of $N_G(A)/C_G(A)$. Then we have

$$\begin{aligned}(g_1C_G(A)g_2C_G(A))^\Theta &= (g_1g_2C_G(A))^\Theta \\ &= \pi_{g_1g_2}C(A,G) \\ &= \pi_{g_1}\pi_{g_2}C(A,G) \\ &= \pi_{g_1}C(A,G)\pi_{g_2}C(A,G)\end{aligned}$$

$$= (g_1C_G(A))^\Theta(g_2C_G(A))^\Theta$$

so that Θ is a homomorphism. Let $gC_G(A) \in \ker \Theta$ i.e., $(gC_G(A))^\Theta = C(A,G)$. Then $\pi_g \in C(A,G)$ so $a^g = a$ for all $a \in A$. Hence $g \in C_G(A)$ and $\ker \Theta = (1)$. Thus Θ is an isomorphism.

In the remarks preceeding Lemma 1.31 we mentioned that the number of ways an automorphism of A can be lifted to G equals $|C(A,G)|$. In the next result we calculate $|C(A,G)|$ under very special hypothesis on G.

<u>Theorem 1.34</u>: Let $G = AB$, $A \triangleleft G$, $A \cap B = (1)$, $C_B(A) = (1)$, A abelian and assume that all complements of A are conjugate. Then $|C(A,G)| = [A:C_A(B)]$.

<u>Proof</u>: Let $\alpha \in N_{A(A)}(B^*)$. By Lemma 1.29 there exists a unique $\beta \in A(B)$ such that the pair α,β can be extended to G. Then, by Theorem 1.2, $\pi_b\alpha = \alpha\pi_{b^\beta}$ on A. Let $x \in A$ and define τ_β by $b^{\tau_\beta} = x^{-1}b^\beta x$. Since A is abelian we have $\pi_{x^{-1}b^\beta x} = \pi_{b^\beta} = \alpha^{-1}\pi_b\alpha$ on A. Therefore, $\pi_b\alpha = \alpha\pi_{b^{\tau_\beta}}$ and, by Theorem 1.3, the pair α,τ_β can be extended to $\gamma \in A(G)$ such that $\gamma|A = \alpha$ and $\gamma|B = \tau_\beta$.

Now let $\gamma \in B(A,G)$ such that $\gamma|A = \alpha$. Then, by Corollary 1.15, $b^\gamma = f(b)b^\Theta$ where $\Theta \in A(B)$ and $f(b) = x^{-1}x^{b^{-\Theta}}$ for some $x \in A$. Thus $b^\gamma = x^{-1}x^{b^{-\Theta}}b^\Theta = x^{-1}b^\Theta x b^{-\Theta}b^\Theta = x^{-1}b^\Theta x$ so that, by Theorem 1.3, we must have $\pi_b\alpha = \alpha\pi_{b^\gamma} = \alpha\pi_{x^{-1}b^\Theta x}$. Since $x \in A$ and A is abelian this reduces to $\pi_b\alpha = \alpha\pi_{b^\Theta}$. Therefore, by the sufficiency part of Theorem 1.2, the pair α,Θ can be extended to G. But β is unique so we must have $\Theta = \beta$. We have shown that if $\gamma \in B(A,G)$ such that $\gamma|A = \alpha$ then $b^{\gamma} = x^{-1}b^{\beta}x$ for some $x \in A$. Finally if $x,y \in A$ and $x^{-1}b^{\beta}x = y^{-1}b^{\beta}y$ then $yx^{-1} \in C_A(B)$. Hence the number of ways α can be extended to G is the number of distinct cosets of $C_A(B)$ in A i.e., $[A:C_A(B)]$. Therefore, $|C(A,G)| = [A:C_A(B)]$.

If we add $B^* \trianglelefteq A(A)$ to the hypothesis of the above theorem then, by Theorem 1.30, $B(A,G)/C(A,G) \simeq A(A)$ and it follows that $|B(A,G)| = |C(A,G)|\ |A(A)| = [A:C_A(B)]|A(A)|$.

CHAPTER II

2.1 Introduction

Throughout this chapter A will be a complemented subgroup of G and at first we do not insist that A be normal in G. We will construct a permutation group on the complements of A in G and investigate what happens to G under various hypotheses on this permutation group. For the sake of completeness several definitions relevant to the theory of permutation groups will be included. One question we have in mind is what condition on this permutation group will insure that all complements of A in G are conjugate.

Denote by Ω the set of all complements of A in G. If $\gamma \in B(A,G)$ and B is any complement of A in G then it follows that $G = G^{\gamma} = (AB)^{\gamma} = A^{\gamma}B^{\gamma} = AB^{\gamma}$ so that B^{γ} is a complement of A in G. Thus the elements of $B(A,G)$ can be considered as permutations of the set Ω. $B(A,G)$ need not be a permutation group on Ω since different elements of $B(A,G)$ may induce the same permutation on Ω. However, we can obtain a permutation group on Ω in the following way. Let $D(A,G)$ denote those elements of $B(A,G)$ which fix every element of Ω i.e., $\gamma \in D(A,G)$ if and only if $B^{\gamma} = B$ for all $B \in \Omega$. Then we have:

<u>Lemma 2.1</u>: $D(A,G) \triangleleft B(A,G)$

<u>Proof:</u> Let $\gamma \in B(A,G)$ and $\Theta \in D(A,G)$. If B is any complement of A in G then we have $B^{\gamma\Theta\gamma^{-1}} = (B^{\gamma})^{\Theta\gamma^{-1}} = (B^{\gamma})^{\gamma^{-1}} = B$ since $B^{\gamma} \in \Omega$. Hence $\gamma\Theta\gamma^{-1} \in D(A,G)$ and $D(A,G)$ is normal in $B(A,G)$.

Now $X = B(A,G)/D(A,G)$ can be considered as permutations on Ω by defining $B^{\bar{\gamma}} = B^{\gamma}$ where $\bar{\gamma} = \gamma D(A,G)$, $\gamma \in B(A,G)$. If $\bar{\gamma}_1, \bar{\gamma}_2 \in X$ induce the same permutation on Ω i.e., if $B^{\gamma_1} = B^{\gamma_2}$ for all $B \in \Omega$ then $B^{\gamma_1\gamma_2^{-1}} = B$ for all $B \in \Omega$ so that $\gamma_1\gamma_2^{-1} \in D(A,G)$. Thus $\bar{\gamma}_1 = \bar{\gamma}_2$. For $\gamma \in B(A,G)$ we denote by $\bar{\gamma}$ the coset of X which contains γ . At this point we give an example that will be referred to later.

Example 1: $G = < a,b,c >$, $a^5 = b^2 = c^4 = 1$, $bab = a^{-1}$, $bc = cb$, $c^{-1}ac = a^2$. Let $A = < a,b >$ then the complements of A are:

$B_0 = \langle c\rangle = \{1,c,c^2,c^3\}$ $\qquad B_5 = \langle bc\rangle = \{1,bc,c^2,bc^3\}$

$B_1 = \langle ac\rangle = \{1,ac,a^4c^2,a^3C^3\}$ $\qquad B_6 = \langle abc\rangle = \{1,abc,a^3c^2,a^2bc^3\}$

$B_2 = \langle a^2c\rangle = \{1,a^2c,a^3c^2,ac^3\}$ $\qquad B_7 = \langle a^2bc\rangle = \{1,a^2bc,ac^2,a^4bc^3\}$

$B_3 = \langle a^3c\rangle = \{1,a^3c,a^2c^2,a^4c^3\}$ $\qquad B_8 = \langle a^3bc\rangle = \{1,a^3bc,a^4c^2,abc^3\}$

$B_4 = \langle a^4c\rangle = \{1,a^4c,ac^2,a^2c^3\}$ $\qquad B_9 = \langle a^4bc\rangle = \{1,a^4bc,a^2c^2,a^3bc^3\}$

Now the automorphisms of G are:

$a \to a^i$	$a \to a^i$	$a \to a^i$	$a \to a^i$	$a \to a^i$
$b \to b$	$b \to a^4b$	$b \to a^3b$	$b \to a^2b$	$b \to ab$
$c \to c$	$c \to ac$	$c \to a^2c$	$c \to a^3c$	$c \to a^4c$
$a \to a^i$	$a \to a^i$	$a \to a^i$	$a \to a^i$	$a \to a^i$
$b \to b$	$b \to a^3b$	$b \to a^4b$	$b \to ab$	$b \to a^2b$
$c \to bc^3$	$c \to a^2bc^3$	$c \to abc^3$	$c \to a^4bc^3$	$c \to a^3bc^3$

where $i = 1,2,3,4$. From the above it is clear that A is characteristic in G so $B(A,G) = A(G)$. The only automorphisms which fix B_0 are $\begin{matrix} a \to a^i \\ b \to b \\ c \to c \end{matrix}$ for $i = 1,2,3,4$. Of these only the identity fixes

B_1 so clearly $D(A,G) = (1)$. Thus $X \simeq A(G)$. Now $a^{-i}B_0a^i = \langle a^{2^i}c \rangle$ for $i = 1, 2, 3, 4$ so B_0, B_1, B_2, B_3, B_4 form a conjugate class. Similarly $a^{-i}B_5a^i = \langle a^{2^i}bc \rangle$ for $i = 1,2,3,4$ and B_5, B_6, B_7, B_8, B_9 form a conjugate class. To show that (X,Ω) is a transitive permutation group we need only show the existence of an automorphism which sends B_0 to B_5. Now $\begin{matrix} a \to a \\ b \to b \\ c \to bc^3 \end{matrix}$ sends B_0 to B_5 so (X,Ω) is transitive.

If all complements of A in G are conjugate then (X,Ω) is a transitive permutation group; in fact, the subgroup $I(G)D(A,G)/D(A,G)$ is transitive on Ω. However, from the above example we can see that transitivity of (X,Ω) is not sufficient to insure that all complements of A are conjugate. In the next example we point out the fact that (X,Ω) need not be transitive.

<u>Example 2</u>: $G = \langle x,y,z \rangle$, $x^5 = y^2 = z^2 = 1$, $yxy = x^{-1}$, $yz = zy$, $zxz = x^{-1}$. Let $A = \langle x,y \rangle$ then the complements of A are: $B_0 = \langle z \rangle$, $B_1 = \langle xz \rangle$, $B_2 = \langle x^2z \rangle$, $B_3 = \langle x^3z \rangle$, $B_4 = \langle x^4z \rangle$, $B_5 = \langle yz \rangle$. Now $x^{-i}B_0x^i = \langle x^{3^i}z \rangle$ for $i = 1,2,3,4$ so that B_0, B_1, B_2, B_3, B_4 form a conjugate class. It is easily verified that $Z(G) = B_5$ so no automorphism of G can map B_5 to any other complement. Consequently (X,Ω) is not transitive. We may observe that this group is a subgroup of the group in example one where $x = a$, $y = b$, and $z = c^2$.

Our objective in this study is to find a permutation condition that will characterize when all complements of A in G are conjugate but at present we have not found such a condition. However, we do have several interesting results in this direction.

2.2 The Group (X, Ω)

If $\Delta = \{B_1, B_2, \cdots, B_r\}$ is a subset of Ω and $\bar{\gamma} = \gamma D(A,G) \in X$ Then by $\Delta^{\bar{\gamma}}$ we mean $\Delta^{\bar{\gamma}} = \{B_1^{\gamma}, B_2^{\gamma}, \cdots, B_r^{\gamma}\}$.

Definition 2.2: If (X, Ω) is transitive then a subset Δ of Ω is called a block if and only if

(1) $1 < |\Delta| < |\Omega|$

(2) $\Delta^{\bar{\gamma}} \cap \Delta = \begin{cases} \emptyset \\ \Delta \end{cases}$ for each $\bar{\gamma} \in X$.

Definition 2.3: The transitive permutation group (X, Ω) is said to be primitive if and only if it has no blocks.

Now we can obtain the following result:

Theorem 2.4: If (X, Ω) is a primitive permutation group then either

(1) all complements of A are normal

or (2) all complements of A are conjugate.

Proof: Let $B \in \Omega$. If $B \lhd G$ then every complement of A is normal since (X, Ω) is transitive. Suppose B is not normal in G and consider the set $\Delta = \{B^{\bar{\pi}_x} = x^{-1}Bx \mid x \in A\}$. Suppose $\Delta^{\bar{\gamma}} \cap \Delta \neq \emptyset$ for some $\bar{\gamma} \in X$ i.e., there exist x and y in A such that $(x^{-1}Bx)^{\bar{\gamma}} = y^{-1}By$. Now $(x^{-1}Bx)^{\bar{\gamma}} = (x^{-1}Bx)^{\gamma} = (x^{\gamma})^{-1}B^{\gamma}x^{\gamma}$ where $\bar{\gamma} = \gamma D(A,G)$. Since $\gamma \in B(A,G)$ we have $x^{\gamma} \in A$ so that $B^{\gamma} = x^{\gamma}y^{-1}Byx^{-\gamma} \in \Delta$. Thus $\Delta^{\bar{\gamma}} = \Delta$. If $1 < |\Delta| < |\Omega|$ then Δ is a block of (X, Ω) so since (X, Ω) is primitive we must have either $|\Delta| = 1$ or $|\Delta| = |\Omega|$. $|\Delta| = 1$ implies that $B \lhd G$ so we must have $|\Delta| = |\Omega|$. Therefore, all complements of A in G appear in Δ i.e., all complements of A in G are conjugate.

To this point we have not hypothesized A normal in G. However, if A is normal in G then from the above result we see that all complements of A in G are conjugate if (X,Ω) is a primitive permutation group and A is not a direct factor of G. That these conditions are not necessary is pointed out by the following example.

<u>Example 3</u>: $G = \langle a,b \rangle$, $a^{15} = b^2 = 1$, $bab = a^{-1}$. Let $A = \langle a \rangle$ then the complements of A are: $B_j = \langle a^j b \rangle$, $j = 0,1,2,\cdots,14$. Now A is normal in G and $(|A|, [G:A]) = 1$ so that A is characteristic. Hence $B(A,G) = A(G)$. The automorphisms of G are:

$$\gamma_{i,j}: \begin{array}{l} a \to a^i \\ b \to a^j b \end{array}$$

where $(i,15) = 1$ and $j = 0,1,2,\cdots,14$. The only automorphisms which fix B_0 are of the form $\gamma_{i,0}$. Of these only the identity fixes B_1 so clearly $D(A,G) = (1)$. Hence $X \simeq A(G)$. Now $a^s B_0 a^{-s} = \langle a^{2s} b \rangle$ so all complements are conjugate and (X,Ω) is transitive. However, (X,Ω) is not primitive for we will show that $\Delta = \{B_0, B_5, B_{10}\}$ is a block of order 3. Let $B_r = \langle a^r B \rangle \in \Delta$ i.e., $r = 0,5,10$ then $B_r^{\gamma_{i,j}} = \langle a^{ri+j} b \rangle$. If 5 does not divide j then 5 does not divide $ri + j$ and $B_r^{\gamma_{i,j}} \notin \Delta$. Hence $\Delta^{\gamma_{i,j}} \cap \Delta = \emptyset$ in this case. If $j = 0,5,10$ then $ri + j$ will take on the values $0,5,10 \mod 15$ as r takes on the values $0,5,10$. Hence $\Delta^{\gamma_{i,j}} \cap \Delta = \Delta$. Therefore, Δ is a block and (X,Ω) is imprimitive.

In the next result we obtain some information about the structure of A in case (X,Ω) is primitive.

Theorem 2.5: If (X_1, Ω) is primitive then either

(1) A is characteristically simple

or (2) if H is a characteristic subgroup of A then $HN_A(B) = \begin{cases} N_A(B) \\ A \end{cases}$

where B is any element of Ω .

Proof: Suppose A is not characteristically simple and let H be a characteristic subgroup of A . By Theorem 2.4 all complements of A are normal or all complements of A are conjugate. If $B \in \Omega$ and $B \lhd G$ then $H\, N_A(B) = H\, A = A$. Thus we can assume that B is not normal in G . Form the set $\Delta = \{B^{\bar{\pi}x} = x^{-1}Bx \mid x \in H\}$ and let $\bar{\gamma} \in X$. If $\Delta^{\bar{\gamma}} \cap \Delta \neq \emptyset$ i.e., if there exist x and y in H such that $(x^{-1}Bx)^{\bar{\gamma}} = y^{-1}By$ then $B^{\gamma} = x^{\gamma}y^{-1}Byx^{-\gamma}$ where $\bar{\gamma} = \gamma D(A,G)$. Since $\gamma|A \in A(A)$ and H is characteristic in A we must have $x^{\gamma} \in H$. Thus $B^{\gamma} \in \Delta$. If $1 < |\Delta| < |\Omega|$ then Δ is a block so since (X, Ω) is primitive we must have either $|\Delta| = 1$ or $|\Delta| = |\Omega|$. If $|\Delta| = 1$ then $H \leq N_A(B)$. If $|\Delta| = |\Omega|$ then all complements of A appear in Δ and it follows that $|\Omega| = [H : H \cap N_A(B)]$. Since all complements of A are conjugate we also have $|\Omega| = [A : N_A(B)]$. Thus

$$|N_A(B)H| = \frac{|N_A(B)|\ |H|}{|H \cap N_A(B)|} = |N_A(B)|\ |\Omega| = |N_A(B)|[A : N_A(B)] = |A|$$

so that $N_A(B)H = A$.

We can observe that we did not use the full power of H being characteristic in A but just invariant under X . So we have the following as a corollary to the proof of Theorem 2.5.

Corollary 2.6: If (X, Ω) is primitive and H a subgroup of A

invariant under X then $HN_A(B) = \begin{cases} N_A(B) \\ A \end{cases}$.

From this point on we will assume that A is a normal complemented subgroup of G . The fixed group of an element $B \in \Omega$, denoted by X_B, is the set of $\bar{\gamma}$ in X such that $B^{\bar{\gamma}} = B$. It is well known that [7, pg 15] in a primitive permutation group the fixed group of a point is maximal. Thus X_B is maximal in X and we can obtain the following result where $\overline{I(A)} = I(A)D(A,G)/D(A,G)$.

Theorem 2.7: If (X, Ω) is a primitive permutation group and A is not a direct factor of G then $X = X_B\overline{I(A)}$.

Proof: If $\bar{\pi}_a \in X_B$ for all $a \in A$ i.e., if $a^{-1}Ba = B$ for all $a \in A$ then $B \lhd G$ and A is a direct factor of G since $A \lhd G$. Hence there exists $x \in A$ such that $\bar{\pi}_x \notin X_B$. Thus the group generated by X_B and π_x is X since X_B is maximal in X . Let $\bar{\Theta}_1, \bar{\Theta}_2 \in X_B$ where $\bar{\Theta}_i = \Theta_i D(A,G)$ for $i = 1,2$. One can easily show that $\Theta_1\pi_x\Theta_2 = \Theta_1\Theta_2\pi_{x^{\Theta_2}}$ so that we have $\bar{\Theta}_1\bar{\pi}_x\bar{\Theta}_2 = \bar{\Theta}_1\bar{\Theta}_2\bar{\pi}_{x^{\Theta_2}}$.

Since $x \in A$, $x^{\Theta_2} \in A$ and it follows that the group generated by X_B and $\bar{\pi}_x$ is $X_B\overline{I(A)}$.

We can replace the hypothesis A not a direct factor of G by the condition $(|A|,|B|) = 1$ in the above theorem for if $B \lhd G$ then B is characteristic since $(|A|,|B|) = 1$. But (X, Ω) is transitive so that $|\Omega| = 1$. Thus we have:

Theorem 2.8: If (X, Ω) is a non-trivial primitive permutation group and $(|A|,|B|) = 1$ then $X = X_B\overline{I(A)}$.

Definition 2.9: The permutation group (X, Ω) is said to be 3/2-fold transitive if and only if (X, Ω) is transitive and the orbits of $(X_B, \Omega - \{B\})$ have equal length.

Definition 2.10: A subgroup N of the permutation group (X, Ω) is said to be semiregular on Ω if and only if $N_B = (1)$ for all $B \in \Omega$.

Example 1 shows that transitivity of (X, Ω) will not insure that all complements of A are conjugate. Also, from example 3, we see that primitivity of (X, Ω) is too strong. So the condition seems to rest somewhere between transitivity and primitivity. Now we will impose the condition 3/2-fold transitivity on (X, Ω) which is weaker than primitivity. We will obtain a theorem like Theorem 2.4 under additional hypothesis on G. But first we need.

Lemma 2.11: If (X, Ω) is transitive and $\overline{I(G)}$ is semiregular on Ω then

(a) $N_G(B_1) = N_G(B) = N_A(B)B$ for any two complements B_1, B in Ω ;

(b) Every complement of A is contained in $N_G(B)$ for any $B \in \Omega$;

(c) $N_A(B)$ and $N_G(B)$ are normal subgroups of G ;

(d) $N_A(B)$ and $N_G(B)$ are invariant under $B(A,G)$;

(e) If $\gamma \in B(A,G)$ and $b^\gamma = f(b)b^\beta$ with f and β as given in Theorem 1.5 then $f \in \mathrm{Hom}(B, Z(N_A(B)))$;

(f) If $B \in \Omega$ then $B \cap B_1$ is normal in B for any B_1 in Ω ;

(g) For any $B \in \Omega$ we have $B' \leq C_B(A) \leq \bigcap_{x \in A} x^{-1}Bx$ where B' is the derived group of B.

Proof: (a) Let g be any element of $N_G(B_1)$. Then $\overline{\pi}_g \in \overline{I(G)}_{B_1} = (1)$ so that $\pi_g \in D(A,G)$. Thus $g \in N_G(B)$ and we have $N_G(B_1) \leq N_G(B)$. Similarly $N_G(B) \leq N_G(B_1)$ so $N_G(B) = N_G(B_1)$.

(b) From (a) we have $N_G(B_1) = N_G(B)$ and since $B_1 \leq N_G(B_1)$ we have $B_1 \leq N_G(B)$ for any $B_1 \in \Omega$.

(c) For $g \in G$ we have $g^{-1}N_G(B)g = N_G(g^{-1}Bg)$ and, by (a), $N_G(g^{-1}Bg) = N_G(B)$ so $g^{-1}N_G(B)g = N_G(B)$. Now $g^{-1}N_A(B)g \leq A$ since $A \lhd G$ and $g^{-1}N_A(B)g \leq N_G(B)$ since $N_G(B) \lhd G$. Thus $g^{-1}N_A(B)g \leq A \cap N_G(B) = N_A(B)$.

(d) Let $\gamma \in B(A,G)$ then $(N_G(B))^\gamma = N_G(B^\gamma)$ and $N_G(B^\gamma) = N_G(B)$ by (a). Thus $(N_G(B))^\gamma = N_G(B)$. Now $(N_A(B))^\gamma \leq A$ since $A^\gamma = A$ and $(N_A(B))^\gamma \leq N_G(B)$ since $(N_G(B))^\gamma = N_G(B)$. Hence $(N_A(B))^\gamma \leq A \cap N_G(B) = N_A(B)$.

(e) Let $B \in \Omega$ and $\gamma \in B(A,G)$. Then for any $b \in B$ we have $b^\gamma = f(b)b^\beta$ where $\beta \in A(B)$ and f is a function from B to A satisfying $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ for all $b_1, b_2 \in B$. Also, from Theorem 1.5, we know that $\pi_b\alpha = \alpha\pi_{f(b)b^\beta}$ where $\alpha = \gamma|A$. By (b) $B^\gamma \leq N_G(B) = N_A(B)B$ so $f(b) \in N_A(B)$ for all $b \in B$. Since $G = AB$, $A \cap B = (1)$, and $A \lhd G$ we know that $N_A(B) = C_A(B)$. Thus $f(b_1b_2) = f(b_1)f(b_2)^{b_1^{-\beta}}$ reduces to $f(b_1b_2) = f(b_1)f(b_2)$. By (d), $\gamma \in B(N_A(B), N_G(B))$. Therefore, by Theorem 1.5, $\pi_b\alpha = \alpha\pi_{f(b)b^\beta}$ on $N_A(B)$. But on $N_A(B) = C_A(B)$ we have $\pi_b = 1_{N_A(B)}$ so this equation reduces to $\pi_{f(b)} = 1_{N_A(B)}$ or $f(b) \in Z(N_A(B))$. Hence $f \in \mathrm{Hom}(B, Z(N_A(B)))$.

(f) Let B, B_1 be any two elements of Ω. Since (X, Ω) is transitive there exists $\gamma \in B(A,G)$ such that $B^\gamma = B_1$. Therefore, by

Theorem 1.5, we can write $B_1 = B^{\gamma} = \{f(b)b^{\beta} | b \in B\}$ with f and β as given in the theorem. By (e) f is a homomorphism and we will show that $B \cap B_1 = (\ker f)^{\beta}$. Let $b \in B \cap B_1$ then there exists $b_1 \in B$ such that $b = f(b_1)b_1^{\beta}$. Since $A \cap B = (1)$ and $f(b_1) \in A$ we must have $f(b_1) = 1$. Thus $b_1 \in \ker f$ and $b_1^{\beta} = b$. This implies $B \cap B_1 \leq (\ker f)^{\beta}$ and it is easily shown that $(\ker f)^{\beta} \leq B \cap B_1$. Thus $B \cap B_1 = (\ker f)^{\beta}$. Since $\ker f \lhd B$ we have $B \cap B_1 \lhd B$.

(g) Let $x \in A$ and $b \in B$. Then $x^{-1}bx = (x^{-1}bxb^{-1})b$. Set $f(b) = x^{-1}bxb^{-1}$ then $x^{-1}Bx = \{f(b)b | b \in B\}$ and, by (e), f is a homomorphism from B to $Z(N_A(B))$. Therefore, for any $b_1, b_2 \in B$ we must have $f(b_1b_2) = f(b_2b_1)$. Thus

$$\begin{aligned} x^{-1}b_1b_2xb_2^{-1}b_1^{-1} &= x^{-1}b_2b_1xb_1^{-1}b_2^{-1} \\ b_1b_2xb_2^{-1}b_1^{-1} &= b_2b_1xb_1^{-1}b_2^{-1} \\ (b_1^{-1}b_2^{-1}b_1b_2)x &= x(b_1^{-1}b_2^{-1}b_1b_2) . \end{aligned}$$

So $b_1^{-1}b_2^{-1}b_1b_2 \in C_B(A)$ and $B' \leq C_B(A)$. Clearly $C_B(A) \leq \bigcap_{x \in A} x^{-1}Bx$ so we have the desired result.

We may observe that the transitivity of (X, Ω) was used only in part (f).

If Δ is a subset of Ω then by X_{Δ} we mean the set of all $\bar{\gamma} \in X$ such that $B^{\bar{\gamma}} = B$ for all $B \in \Delta$.

<u>Definition 2.12</u>: The transitive permutation group (X, Ω) is said to be <u>Frobenius</u> if and only if $X_B \neq (1)$ and $X_{\Delta} = (1)$ whenever $|\Delta| = 2$.

If (X, Ω) is a Frobenius group then from [7, pg. 11] we know that the elements of X which fix no element of Ω together with the identity form a nilpotent characteristic subgroup of X. For the sake of reference we will state two results which appear on pages 25 and 32 respectively of [7]:

<u>Theorem 2.13</u>: If (X, Ω) is 3/2-fold transitive then either (X, Ω) is primitive or (X, Ω) is a Frobenius group.

<u>Theorem 2.14</u>: If (X, Ω) is 3/2-fold transitive and $N \lhd X$ then either N is transitive on Ω or N is semiregular on Ω.

Now we are in a position to obtain the theorem referred to earlier.

<u>Theorem 2.15</u>: If (X, Ω) is 3/2-fold transitive then all complements of A in G are either normal or conjugate provided any one of the following conditions hold:

(a) $(|A|, |B/B'|) = 1$ where B' is the derived group of B;

(b) If there exists a normal subgroup H of G contained in A such that $H \cap N_A(B) = (1)$;

(c) If $N_A(B)$ is a Hall subgroup of A with a normal complement;

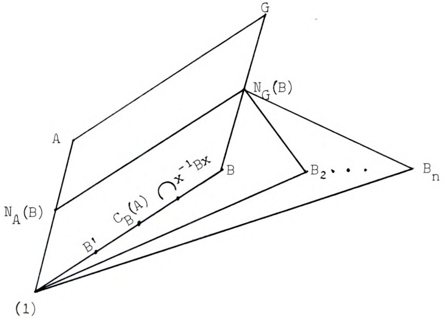
(d) If $Z(A) \cap N_A(B) = Z(N_A(B))$ and there exists at least one element of $A - N_A(B)$ whose order is relatively prime to $|N_A(B)|$;

(e) If $A/N_A(B)$ is not nilpotent.

<u>Proof</u>: Assume (X, Ω) is 3/2-fold transitive and the complements of A are not all normal or conjugate. By Theorem 2.13 either (X, Ω) is primitive or (X, Ω) is a Frobenius group. If (X, Ω) is primitive then, by Theorem 2.4, all complements are normal or

conjugate. So we may assume that (X, Ω) is a Frobenius group. Now $\overline{I(G)} \lhd X$ so, by Theorem 2.14, either $\overline{I(G)}$ is transitive on Ω or semiregular on Ω. If $\overline{I(G)}$ is transitive on Ω then all complements are conjugate so we may assume that $\overline{I(G)}$ is semiregular on Ω. Next we show that if $B \in \Omega$ then $(1) \neq N_A(B) \neq A$. If $N_A(B) = A$ Then $B \lhd G$ and every complement is normal as (X, Ω) is transitive. Thus $N_A(B) \neq A$. If $N_A(B) = (1)$ then $N_G(B) = N_A(B)B = B$. $\overline{I(G)}$ is semiregular so, by Lemma 2.11(c), $N_G(B) \lhd G$. This is a contradiction since $B = N_G(B)$ is not normal in G. Hence $N_A(B) \neq (1)$.

Thus (X, Ω) is a Frobenius group, $\overline{I(G)}$ is semiregular, $(1) \neq N_A(B) \neq A$, and all complements of A are not normal or conjugate. If $B_1 = B, B_2, \cdots, B_n$ are the elements of Ω then, applying Lemma 2.11, we have the following structure on G:



Now we will impose the conditions (a) through (e).

(a) Let $B \in \Omega$. By Lemma 2.11(e) if $B_1 = \{f(b)b^{\beta} | b \in B\}$ is any complement of A we know that $f \in \mathrm{Hom}\,(B, Z(N_A(B)))$. Since the image of B under f is abelian we may consider f as an element of $\mathrm{Hom}\,(B/B', Z(N_A(B)))$ in the obvious way. But $|f(b)|$ must divide $|b|$ and since $(|A|, |B/B'|) = 1$ we must have $f(b) = 1$ for all $b \in B$. Thus $B_1 = B$ so that $\Omega = \{B\}$. Hence B is the only complement so must be normal.

(b) Let $H \lhd G$, $H \leq A$, and $H \cap N_A(B) = (1)$. If $h \in H$ and b is any element of B then $h^{-1}bhb^{-1} \in H$ since $H \lhd G$. Now $b \in N_G(B)$ and, by Lemma 2.11(c), $N_G(B) \lhd G$ so that $h^{-1}bhb^{-1} \in N_G(B)$. Hence $h^{-1}bhb^{-1} \in H \cap N_G(B) = H \cap N_A(B) = (1)$ so $bh = hb$ for all $b \in B$. Thus $h \in N_A(B)$, a contradiction, and the result holds in this case.

(c) Let $A = N_A(B)D$ where $D \cap N_A(B) = (1)$ and $D \lhd A$. Since $N_A(B)$ is a Hall subgroup of A we have $(|N_A(B)|, [A : N_A(B)]) = 1$. Thus $(|D|, [A : D]) = 1$ and since $D \lhd A$ we must have D characteristic in A. Since $A \lhd G$ and D is a characteristic subgroup of A we have that $D \lhd G$. Then $D \lhd G$, $D \leq A$, and $D \cap N_A(B) = (1)$ and we can apply (b).

(d) Let $x \in A - N_A(B)$ such that $(|x|, |N_A(B)|) = 1$. Since $x \notin N_A(B) = C_A(B)$ there is at least one element of B, say b, such that $x^{-1}bxb^{-1} \neq 1$. But, as in the proof of Lemma 2.11(g), the function f defined by $f(y) = x^{-1}yxy^{-1}$ for all $y \in B$ is a homomorphism from B to $Z(N_A(B))$. Thus $x^{-1}bxb^{-1} \in Z(N_A(B))$, say $x^{-1}bxb^{-1} = n$, and $n \neq 1$. Since $Z(A) \cap N_A(B) = Z(N_A(B))$ we have $n \in Z(A)$. Let $|x| = d$ then from $bxb^{-1} = xn$ we have

$$(bxb^{-1})^d = (xn)^d$$

$$bx^d b^{-1} = x^d n^d$$

$$1 = n^d.$$

However, $(d, |N_A(B)|) = 1$ so $n^d \neq 1$ and we have a contradiction. Thus the theorem is true in this case.

(e) Since (X, Ω) is a Frobenius group we have, from the remarks preceeding Theorem 2.13, that the elements of X which fix no element of Ω together with the identity form a nilpotent characteristic subgroup, say Y, of X. By Lemma 2.11(c), $N_A(B)$ and $N_G(B)$ are normal in G. We show that each $g \in G - N_G(B)$ fixes no element of Ω so that $G/N_G(B)$ is isomorphic to a subgroup of Y. Now if π_g fixes $B_1 \in \Omega$ then $\overline{\pi}_g \in \overline{I(G)}_{B_1} = (1)$ so that $\pi_g \in D(A,G)$. Hence π_g fixes B i.e., $g \in N_G(B)$, a contradiction. Therefore, $A/N_A(B) \simeq G/N_G(B)$ and since $G/N_G(B)$ is isomorphic to a subgroup of Y we have $A/N_A(B)$ is isomorphic to a subgroup of Y. But Y is nilpotent so $A/N_A(B)$ is nilpotent which is a contradiction. Thus the theorem is true in this case.

Whether the conditions in the above theorem are necessary is an open question at this time. We do not have an example of a group G in which (X, Ω) is 3/2-fold transitive and the complements of A are neither conjugate nor normal.

Finally we conclude this chapter by imposing a condition on (X, Ω) which is considerably stronger than primitivity.

Definition 2.16: The permutation group (X, Ω) is said to be 2-fold transitive if and only if X_B is transitive on the set $\Omega - \{B\}$.

From [7, pg 20] every 2-fold transitive group is primitive so this condition is stronger than primitivity.

Definition 2.17: (X, Ω) is said to be sharply 2-fold transitive if and only if

(1) (X, Ω) is 2-fold transitive;

(2) $|X_\Delta| = 1$ whenever $|\Delta| = 2$.

Now we will assume that (X, Ω) is sharply 2-fold transitive. This is a stronger condition than primitivity so all previous results hold true in this case.

Theorem 2.18: If (X, Ω) is sharply 2-fold transitive and $(|A|, |B|) = 1$ then X_B is fixed-point-free on the set $A - N_A(B)$. Moreover $|X_B| > 1$.

Proof: From the remarks preceeding Theorem 2.8 we note that B cannot be normal in G. Hence $A - N_A(B) \neq \emptyset$. Let $a \in A - N_A(B)$ and $\bar{\gamma} \in X_B$ such that $\bar{\gamma} \notin D(A,G)$. This can be done since (X, Ω) is 2-fold transitive. Since $|X_{\{B_1, a^{-1}Ba\}}| = 1$ and $\bar{\gamma}$ is not the identity we must have $(a^{-1}Ba)^{\bar{\gamma}} \neq a^{-1}Ba$. Let $\bar{\gamma} = \gamma D(A,G)$, $\gamma \in B(A,G)$ and we have $(a^{-1}Ba)^{\bar{\gamma}} = (a^{-1}Ba)^{\gamma} = (a^{\gamma})^{-1}B^{\gamma}a^{\gamma} = (a^{\gamma})^{-1}Ba^{\gamma}$ so that $a^{\gamma}a^{-1} \notin N_G(B)$. In particular, $a^{\gamma}a^{-1} \neq 1$ so $\bar{\gamma}$ is fixed-point-free on $A - N_A(B)$.

Now we will show that $|X_B| > 1$ by showing that there exists a $b \in B$ such that $\pi_b \notin D(A,G)$. If $\pi_b \in D(A,G)$ for all $b \in B$ then for any $a \in A - N_A(B)$ and all $b \in B$ we must have $b^{-1}(a^{-1}Ba)b = a^{-1}Ba$. Thus $aba^{-1} \in N_G(B)$. Since $A \triangleleft G$ and $A \cap B = 1$ we have $N_G(B) = N_A(B)B$ and $N_A(B) = C_A(B)$. Since $(|A|, |B|) = 1$, $|aba^{-1}| = |b|$, and $N_A(B) = C_A(B)$ we must have

$aba^{-1} \in B$. Therefore, $a \in N_A(B)$ which is a contradiction.

In the proof of the above theorem we did not use the condition $(|A|, |B|) = 1$ to show that X_B was fixed-point-free on $A - N_A(B)$. However, without this hypothesis it is possible that $B \triangleleft G$ so that the set $A - N_A(B)$ is empty. As a consequence of the above theorem we have:

Theorem 2.19: If (X, Ω) is sharply 2-fold transitive, $(|A|, |B|) = 1$, and A is not an elementary abelian p-group then $(1) \subsetneq N_A(B) \subsetneq A$.

Proof: As in the proof of Theorem 2.10 we must have $N_A(B) \subsetneq A$. By Theorem 2.10 $|X_B| > 1$ and X_B is fixed-point-free on $A - N_A(B)$. If $N_A(B) = (1)$ then X_B contains a fixed-point-free automorphism of A of prime order. Thompson [6] has shown that if a group has a fixed-point-free automorphism of prime order then the group is nilpotent. So A is nilpotent. Now (X, Ω) is primitive and $N_A(B) = 1$ so, by Theorem 2.5, A must be characteristically simple. But A nilpotent and characteristically simple implies that A must be an elementary abelian p-group which is a contradiction. Therefore, $(1) \subsetneq N_A(B) \subsetneq A$.

INDEX OF NOTATION

I. Relations:

$\leq$	Is a subgroup of
$\lneq$	Is a proper subgroup of
$\triangleleft$	Is a normal subgroup of
$\simeq$	Is isomorphic to
$\equiv$	$x \equiv y \bmod A$ means $xy^{-1} \in A$
$\in$	Is an element of
$\notin$	Is not an element of

II. Operations:

G^{θ}	The image of the group G under the mapping θ
g^{θ}	The image of the element g under the mapping θ
g^{x}	$x^{-1}gx$
$\gamma \vert A$	The restriction of the mapping γ to the set A
π_x	The automorphism sending g to $x^{-1}gx$
$g^{\theta_1+\theta_2}$	$g^{\theta_1}g^{\theta_2}$
1_A	The identity automorphism of A
G/H	Factor Group
$\times$	Direct product of groups
$[G:H]$	Index of H in G
$\langle \; \rangle$	Subgroup generated by
$\{ \; \}$	Set whose members are
$\{x \vert P\}$	Set of all x such that P is true
$\vert G \vert$	Number of elements in G

$|g|$ Order of the element g

III. Groups and Sets:

$Hom(G,H)$	The group of all homomorphisms from G to H
$\ker \gamma$	The set of all g such that $g^{\gamma} = 1$
$A(G)$	The automorphism group of G
$B(A,G)$	The set of all $\gamma \in A(G)$ such that $A^{\gamma} = A$
$Z(G)$	The center of G
$C_G(H)$	The centralizer of H in G
$N_G(H)$	The normalizer of H in G
$\Phi(G)$	The Frattini subgroup of G
$I(G)$	The inner automorphisms of G
$\emptyset$	The empty set
$A - B$	The set of all x in A and not in B

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