This is to certify that the

thesis entitled

SOME RESULTS ON SEPARABILITY
AND PURE INSEPARABILITY
FOR ALGEBRAS OVER COMMUTATIVE RINGS

presented by

Linda Lee Deneen

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Edward C. Ingraham
Major professor

Date October 28, 1980

SOME RESULTS ON SEPARABILITY
AND PURE INSEPARABILITY
FOR ALGEBRAS OVER COMMUTATIVE RINGS

By

Linda Lee Deneen

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1980

ABSTRACT

SOME RESULTS ON SEPARABILITY
AND PURE INSEPARABILITY
FOR ALGEBRAS OVER COMMUTATIVE RINGS

By

Linda Lee Deneen

Let R be a Noetherian inertial coefficient ring and let A be a finitely generated R-algebra (that is, finitely generated as an R-module) with Jacobson radical $J(A)$. Let S be a subalgebra of A with $S + J(A) = A$. We show that for every separable subalgebra T of A there is a unit a of A such that $aTa^{-1} \subseteq S$. Moreover, we show that if $S + I = A$ for a nil ideal I of A, then R can be taken to be an arbitrary commutative ring, and the conjugacy result still holds.

If $A \supseteq S$ are rings, Bogart defined A to be <u>purely inseparable</u> over S if the A-A bimodule map $\mu : A \otimes_S A^\circ \longrightarrow A$ has small kernel. For A a finitely generated R-algebra and S a subalgebra of A, Ingraham defined S to be an <u>inertial subalgebra</u> of A if S is separable over R and $S + J(A) = A$. If A is commutative and $A/J(A)$ is separable, it is shown that S is an inertial subalgebra of A if and only if A is purely inseparable over S and S is separable over R.

If $A/J(A)$ is not separable, the situation is more complicated. We show that if A is a finitely generated algebra over a commutative

semilocal ring R, then there is a finitely generated, faithfully flat (in fact free), commutative R-algebra P such that $(A \otimes_R P)/(B \otimes_R P)$ is P-separable. If B is a subalgebra of A for which $B \otimes_R P$ is an inertial subalgebra of $A \otimes_R P$ for any such P, then extending a definition of Bogart we define B to be a <u>Wedderburn specter</u> for A over R. If A is commutative, we show that B is a specter for A over R if and only if A is purely inseparable over B. We conclude by giving some properties of specters.

ACKNOWLEDGEMENTS

I am deeply grateful to Professor Edward C. Ingraham for his patience and understanding in guiding my work on this thesis. His love of mathematics has been a constant inspiration to me. I am indebted to Professor William C. Brown for his careful reading of this work and for his kind comments and helpful suggestions. I would also like to thank the friends, teachers, and family members whose support over the years has been invaluable. Finally, I greatly appreciate the excellent typing of Barbara Miller.

TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

In this chapter we present some background results which will be used frequently in the remaining chapters of this thesis. The concepts of separability and inertial subalgebras are particularly important.

§1. Notation and General Results

All rings will be assumed to be associative and to possess an identity element 1. All subrings will contain the identity of the overring, and ring homomorphisms will map the identity to the identity.

Suppose A is a ring, R a commutative ring, and θ is a ring homomorphism of R into the center of A. Then θ induces a natural R-module structure on A defined by $r \cdot a = \theta(r)a$ for $r \in R$, $a \in A$, and we say that A is an R-algebra. If A is a commutative ring as well, then we call A a commutative R-algebra. An R-algebra A is said to be finitely generated or projective if it is finitely generated or projective as a module over R. For all rings R we let $J(R)$ denote the Jacobson radical (or radical) of R.

This lemma provides a link between $J(R)$, $J(A)$, and the maximal ideals of R.

Lemma 1.1: [12, Lemma 1.1, p. 78] Let A be a finitely generated R-algebra, and let $\bigcap(mA)$ denote the intersection of the mA as m runs over all maximal ideals of R.

(a) $J(R)\cdot A \subseteq J(A)$.

(b) There exists a positive integer n such that $(J(A))^n \subseteq \bigcap(mA)$.

(c) If A is projective, $J(R)\cdot A \subseteq \bigcap(mA)$.

(d) If A is separable, $J(A) = \bigcap(mA)$.

The next result gives an important connection between the radical of an algebra A and the radical of a subalgebra of A.

Proposition 1.2: [4, Corollary, p. 126] If A is a finitely generated R-algebra and S is its subalgebra, then $J(A)\cap S \subseteq J(S)$.

For a finite-dimensional algebra over a field, the radical of a direct sum of ideals is nice.

Proposition 1.3: [1, Corollary, p.29] Let A be a finite-dimensional R-algebra, where R is a field, and suppose A can be expressed as a direct sum of ideals $A_1, A_2, \ldots, A_n$, say $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Then $J(A) = J(A_1) \oplus J(A_2) \oplus \cdots \oplus J(A_n)$.

The next result is probably proven somewhere in the literature. We sketch a proof here for completeness.

Proposition 1.4: Let A be a finitely generated, commutative algebra over the commutative ring R, and let m be any maximal ideal of R. Then $(J(A))_m \subseteq J(A_m)$, where for any R-module M, M_m denotes the localization of M at m.

Proof: Every ideal of A_m is an extended ideal, and the prime ideals of A_m are in one-to-one correspondence with the prime ideals of A which do not intersect $R - m$. One can show, using the "Going Up Theorem" [2, Theorem 5.11, p. 62] that every maximal ideal of A_m is extended from a maximal ideal of A which does not intersect $R - m$. Thus,

$$J(A_m) = \bigcap_{\substack{Q \text{ maximal in } A \\ Q \cap (R-m) = \emptyset}} (Q_m) \supseteq (\cap Q)_m \supseteq (J(A))_m.$$

□

Next we state the well-known Nakayama's Lemma along with a useful corollary.

Proposition 1.5: (Nakayama's Lemma) [2, Proposition 2.6, p. 21] Let M be a finitely generated R-module and I be an ideal of R contained in $J(R)$. Then $I \cdot M = M$ implies $M = 0$.

Corollary 1.6: [2, Corollary 2.7, p. 22] Let M be a finitely generated R-module, N be a submodule of M, and $I \subseteq J(R)$ be an ideal. If $M = I \cdot M + N$, then $M = N$.

We will also need the following result on localizations.

Proposition 1.7: [7, Proposition 4.4, p. 29] Let M be an R-module such that $M_m = 0$ for every maximal ideal m of R. Then $M = 0$.

§2. The Concept of Separability

The theory of separable field extensions is well known. From the work of Wedderburn, Dickson, Albert, and others during the early 1900's arose a generalization for algebras over fields. If R is a field and A an R-algebra, then A is said to be separable over R if $A \otimes_R F$ is semisimple for every field extension F of R [1, p. 44]. This has since come to be known as classical separability. Albert showed in [1, p. 44] that if A is a field extension of R, then (classical) separability is equivalent to the usual field theoretic definition of separability. He also gave the following useful result.

Theorem 1.8: [1, Theorem 21, p. 44] Let A be a finite-dimensional algebra over the field R. Then A is (classically) separable over R if and only if the center of each simple component of A is a separable field extension of R.

In 1960, Auslander and Goldman [3] extended the definition of separability further to algebras over commutative rings. If A is an R-algebra and if A° denotes the R-algebra opposite to A, then we can form the following short exact sequence of $A \otimes_R A^\circ$-modules:

$$0 \longrightarrow J \longrightarrow A \otimes_R A^\circ \xrightarrow{\mu} A \longrightarrow 0.$$

Here μ is simply "multiplication": $\mu(a \otimes b) = a \cdot b$. μ will be used throughout this thesis to signify the multiplication map, with further notation added to avoid confusion when more than one multiplication map occurs. J is the kernel of μ, and J is the left ideal of $A \otimes_R A^\circ$ generated by all elements of the form $a \otimes 1 - 1 \otimes a$, $a \in A$.

Proposition 1.9: [7, Proposition 1.1, p. 40] The following conditions on an R-algebra A are equivalent:

i) A is projective as a left $A \otimes_R A^\circ$-module under the μ-structure.

ii) $0 \longrightarrow J \longrightarrow A \otimes_R A^\circ \xrightarrow{\mu} A \longrightarrow 0$ splits as a sequence of left $A \otimes_R A^\circ$-modules.

iii) $A \otimes_R A^\circ$ contains an element e such that $\mu(e) = 1$ and $Je = 0$. (e is an idempotent called a separability idempotent for A.)

Definition of Separability: [7, p. 40] An R-algebra A is called separable if it satisfies the equivalent conditions of Proposition 1.9.

If A is separable over R, then from Proposition 1.9 we see that $J = \ker \mu$ is a direct summand of $A \otimes_R A^\circ$; i.e., there is a left ideal H of $A \otimes_R A^\circ$ such that $H \oplus J = A \otimes_R A^\circ$. In contrast we will examine in Chapter 3 a generalization of pure inseparability developed by Sweedler [16] and Bogart [5] in which there is no left ideal H of $A \otimes_R A^\circ$ for which $H + \ker \mu = A \otimes_R A^\circ$.

We list two examples of separable algebras.

Example 1.10: [7, Example II, p. 41] If $M_n(R)$ denotes the $n \times n$ matrices with entries from R, then $M_n(R)$ is separable over R.

Example 1.11: [7, Theorem 2.5, p. 50] If R is a field, then A is separable over R if and only if A is finite-dimensional over R and A is classically separable.

The following properties of separable algebras will be used frequently in Chapters II and III. We list them here for reference.

Property 1.12: [7, p. 45] If $\mathfrak{a}$ is an ideal of R and A is an $R/\mathfrak{a}$-algebra, then A is also an R-algebra. A is R-separable if and only if A is $R/\mathfrak{a}$-separable.

Property 1.13: [7, Proposition 1.11, p. 46] If A is a separable R-algebra and I is an ideal of A, then A/I is a separable R-algebra.

Property 1.14: [7, Corollary 1.7, p. 44] If A is a separable R-algebra and S is any commutative R-algebra, then $A \otimes_R S$ is a separable S-algebra.

Property 1.15: (Transitivity of Separability) [7, Proposition 1.12, p. 46] Let S be a commutative, separable R-algebra, and let A be a separable S-algebra. Then A is a separable R-algebra. Conversely, if A is a separable R-algebra and S is any R-subalgebra of the center of A, then A is separable over S.

Property 1.16: [7, Proposition 1.13, p. 47] Let A_1 be an R_1-algebra and A_2 an R_2-algebra. Then $A_1 \oplus A_2$ is a separable $R_1 \oplus R_2$-algebra if and only if A_1 and A_2 are separable over R_1 and R_2 respectively.

Property 1.17: [15, Theorem 5, p. 5] If A is a finitely generated R-algebra, and if S is a separable subalgebra of A, then S is a finitely generated R-algebra.

§3. Wedderburn Factors and Inertial Subalgebras.

In 1907 Wedderburn proved the famous Wedderburn Principal Theorem in the case where A is a finite-dimensional algebra over a field F of characteristic zero. We state its more general form.

Theorem 1.18: [17] Let A be a finite-dimensional algebra over a field F with $A/J(A)$ separable. Then there exists a (separable) subalgebra S of A such that $S \oplus J(A) = A$.

S is called a Wedderburn factor of A.

In 1951 Azumaya generalized Wedderburn's result to finitely generated algebras over Hensel local rings. Recall that a local ring R with maximal ideal m is called a Hensel local ring if it satisfies Hensel's Lemma; that is, if $f(x) \in R[x]$ is a monic polynomial such that $\overline{f}(x) = g(x)h(x)$ in $R/m[x]$, where $g(x)$ and $h(x)$ are monic and relatively prime, then there are monic polynomials $G(x)$ and $H(x)$ in $R[x]$ with $f(x) = G(x)H(x)$, $\overline{G(x)} = g(x)$, and $\overline{H(x)} = h(x)$.

Theorem 1.19: (Azumaya's Theorem) [4, Theorem 33, p. 145] Let A be a finitely generated algebra over a Hensel local ring R with maximal ideal m, and suppose $A/J(A)$ is separable over R/m. Then there exists a separable subalgebra S of A such that $S + J(A) = A$.

Azumaya called S an inertial subalgebra. In 1965 Ingraham extended this definition to algebras over arbitrary commutative rings. According to [12, Definition 2.1, p. 79] if A is a finitely generated algebra over a commutative ring R, then a subalgebra S of A is called an inertial subalgebra if S is a separable R-algebra such that $S + J(A) = A$. We list two properties of inertial subalgebras.

<u>Property 1.20</u>: [12, Lemma 2.5, p. 80] If $S' \subseteq S$ are two inertial subalgebras of a finitely generated R-algebra A, then $S' = S$.

<u>Property 1.21</u>: [12, Proposition 2.6, p. 80] If A is a commutative, finitely generated R-algebra, then A contains at most one inertial subalgebra.

In [5] Bogart characterized certain subalgebras of a finite-dimensional algebra over a field k which "become" Wedderburn factors upon tensoring up with an appropriate field extension of k. She called these subalgebras Wedderburn specters. We will examine these in Chapter III, where we will generalize some of her results to algebras over commutative rings.

In [12, p. 85] Ingraham defined a commutative ring R to be an <u>inertial coefficient ring</u> if every finitely generated R-algebra A for which $A/J(A)$ is separable contains an inertial subalgebra. By Theorem 1.19 we have that Hensel local rings are inertial coefficient rings. Finite direct sums and homomorphic images of inertial coefficient rings are inertial coefficient rings [12, Proposition 3.2, p. 85, and Corollary 3.4, p. 86]. Also Noetherian Hilbert rings are inertial coefficient rings [13, Corollary 2, p. 553]. (Recall that a commutative ring R is a <u>Hilbert ring</u> if every prime ideal of R is an intersection of maximal ideals of R.)

If A is a finitely generated R-algebra and I is an ideal of A, then we say that we can "lift idempotents from A/I to A" if every idempotent in A/I is the image of an idempotent in A under the natural map from A to A/I. Ingraham has conjectured that if R has the property that idempotents can be lifted from $A/J(A)$ to A for every finitely generated R-algebra A, then R is an inertial coefficient ring. In [14] Kirkman proved the converse of this conjecture.

Theorem 1.22: [14, Theorem 4, p. 221] Let R be an inertial coefficient ring and A a finitely generated R-algebra. Then idempotents can be lifted from $A/J(A)$ to A.

We give one last property of idempotent lifting, due to Greco.

Property 1.23: [10, Corollary 1.3, p. 46] Let R be a commutative ring and A a finitely generated R-algebra with the property that idempotents can be lifted from $A/J(A)$ to A. Let I be an ideal of A with $I \subseteq J(A)$. Then idempotents can be lifted from A/I to A.

CHAPTER II

A THEOREM ON THE LATTICE OF SUBALGEBRAS OF AN ALGEBRA

Let R be a commutative ring and A a finitely generated R-algebra. We are interested in finding conditions under which a maximal separable subalgebra T of A is inertial. It is clear that if S is an inertial subalgebra of A, and if a is a unit of A such that $aTa^{-1} \subseteq S$, then T is inertial. Thus, we are led to look for conditions under which we can conjugate T into S.

We prove a more general result. If T is any R-separable subalgebra of A, and S is a subalgebra of A with the property that $S + J(A) = A$, then under the condition that R is an inertial coefficient ring, T can be conjugated into S. Ford has given an example in [8, Theorem 2.3, p. 43] of a certain class of rings R which are not inertial coefficient rings and for which a finitely generated algebra A over R exists having nonisomorphic inertial subalgebras. In view of this result, we cannot expect to be able to conjugate T into S in general.

The theorem will be proven in six steps. The first two steps reduce to the case where $A/J(A)$ is separable and S is an inertial subalgebra of A. Then we prove the theorem where R is successively a field, a local ring with nilpotent radical, a Noetherian local ring, and finally, a Noetherian ring.

Theorem 2.1: Let R be a Noetherian inertial coefficient ring and A be a finitely generated R-algebra. Let T be a separable subalgebra of A, and let S be a subalgebra of A with the property that $S + J(A) = A$. Then there is a unit a of A such that $aTa^{-1} \subseteq S$.

Proof: Step 1: We first reduce to the case where $A/J(A)$ is separable over R. Let $A_1 = T + J(A)$. By Proposition 1.2, $J(A) \subseteq J(A_1)$, so that $A_1/J(A_1)$ is a homomorphic image of T and hence is separable by Property 1.13. Setting $S_1 = S \cap A_1$, we clearly have $S_1 + J(A) \subseteq A_1$. To show equality, we write an arbitrary element a_1 of $A_1 = T + J(A)$ as $t+n$ where $t \in T$, $n \in J(A)$. Since a_1 also lies in $A = S + J(A)$, we have $t + n = s + n_1$ for $s \in S$, $n_1 \in J(A)$. It follows that $s = t + n - n_1$ is in $S \cap (T+J(A)) = S_1$, so that $a_1 \in S_1 + J(A)$, and we have $S_1 + J(A) = A_1$. If the theorem is true for A_1, then there is a unit $a \in A_1 \subseteq A$ such that $aTa^{-1} \subseteq S_1 \subseteq S$. Thus, it suffices to prove the theorem in the case that $A/J(A)$ is separable over R.

Step 2: We will now reduce to the case where S is separable, hence inertial. By Proposition 1.2, $S \cap J(A) \subseteq J(S)$, but $S/S \cap J(A) \simeq A/J(A)$ is semisimple, so $S \cap J(A) = J(S)$. Since R is an inertial coefficient ring, S contains a separable subalgebra S_1 such that $S_1 + J(S) = S$, and it follows that $S_1 + J(A) = A$. Clearly, if we can conjugate T into S_1, we can conjugate it into S. Therefore, we assume S is an inertial subalgebra.

The remainder of the proof involves the following setting. Let $\overline{A} = A/J(A)$, $\overline{T} = T/(T \cap J(A))$, and $\overline{R} = R/(R \cap J(A))$. Let $f : S \otimes_R T^\circ \longrightarrow \overline{A} \otimes_{\overline{R}} \overline{T}^\circ$ and $g : T \otimes_R T^\circ \longrightarrow \overline{A} \otimes_{\overline{R}} \overline{T}^\circ$ be the natural maps, and let e be a separability idempotent for T with $\overline{e} = g(e)$. Then $\ker f = i[(S \cap J(A)) \otimes_R T^\circ + S \otimes_R (T \cap J(A))^\circ] \subseteq J(S \otimes_R T^\circ)$ [4, Theorem 10, p. 127]. By Theorem 1.22 idempotents can be lifted from $(S \otimes_R T^\circ)/J(S \otimes_R T^\circ)$ to $S \otimes_R T^\circ$, since R is an inertial coefficient ring. Thus, Property 1.23 implies that we can lift idempotents from $\overline{A} \otimes_{\overline{R}} \overline{T}^\circ$ to $S \otimes_R T^\circ$, so let e_1 be an idempotent in $S \otimes_R T^\circ$ such that $f(e_1) = \overline{e}$. The picture looks like this.

$$\begin{tikzcd}
S \otimes_R T^\circ \arrow[dr, "f"] & \\
e_1 \arrow[dr, dashed] & \overline{A} \otimes_{\overline{R}} \overline{T}^\circ \\
T \otimes_R T^\circ \arrow[ur, "g"] & \overline{e} \\
e \arrow[ur, dashed] &
\end{tikzcd}$$

If $\mu : A \otimes_R A^\circ \longrightarrow A$ is the multiplication map, then we will show that $\mu(e_1)$ is the conjugating element we seek. In other words, providing e_1 is an <u>idempotent</u> preimage of $\overline{e}$, we will show that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

<u>Step 3</u>: Let R be a field. The proof of [5, Lemma 2.7, p. 127] gives the existence of a unit α in A of the form $\alpha = 1 + n$ for n in $J(A)$ such that $\alpha T \alpha^{-1} \subseteq S$. If we define $\phi : T \longrightarrow S$ by $\phi(t) = \alpha T \alpha^{-1}$, then the map $\phi \otimes 1 : T \otimes_R T^\circ \longrightarrow S \otimes_R T^\circ$ makes the diagram above commute. Furthermore, $\ker f = i\,[(S \cap J(A)) \otimes_R T^\circ + S \otimes_R (T \cap J(A))^\circ] = 0$,

since S and T are separable over the field R. Therefore, if we let $e_1 = (\phi \otimes 1)(e)$, then e_1 is the unique preimage of $\bar{e}$ in $S \otimes_R T^\circ$, and e_1 is also an idempotent.

Because e is a separability idempotent for T, we have $(1 \otimes t - t \otimes 1) \cdot e = 0$ for every t in T. Applying $\phi \otimes 1$, this becomes $(1 \otimes t - \phi(t) \otimes 1) \cdot e_1 = 0$. Next apply μ, recall that $\phi(t) = \alpha t \alpha^{-1}$, and notice that $\mu(e_1)t - \alpha t \alpha^{-1} \mu(e_1) = 0$. It follows that $\mu(e_1) t \mu(e_1)^{-1} = \alpha t \alpha^{-1}$ is in S, provided $\mu(e_1)$ is invertible. But $\mu(\bar{e}) = 1$, so $\mu(e_1) = 1 + n$ for some n in $J(A)$; consequently, $\mu(e_1)$ is invertible.

<u>Step 4</u>: Suppose (R,m) is a Noetherian local ring with $m^n = 0$ for some positive integer n. We proceed by induction on n. If $n = 1$, then R is a field, and the result follows from Step 3.

Assume the statement is true for $n \leq k$, and consider the case where $n = k + 1$.

Let $\tilde{A} = A/(m^k A)$, $\tilde{R} = R/m^k$, $\tilde{T} = T/(m^k A \cap T)$, and $\tilde{S} = S/(m^k A \cap S)$. Since $m^k A \subseteq J(A)$ by Lemma 1.1, then $J(\tilde{A}) = J(A)/m^k A$. Letting $\tilde{e}_1$ and $\tilde{e}$ be the images of e_1 and e and taking $\tilde{f}$ and $\tilde{g}$ to be the induced maps from f and g, we have the following situation.

$$\begin{tikzcd}
\tilde{S} \otimes_{\tilde{R}} \tilde{T}^\circ \arrow[ddr, "\tilde{f}"] & \\
\tilde{e}_1 \arrow[ddr, dashed] & \\
 & \bar{A} \otimes_{\bar{R}} \bar{T}^\circ \\
 & \bar{e} \\
\tilde{T} \otimes_{\tilde{R}} \tilde{T}^\circ \arrow[uur, "\tilde{g}"] & \\
\tilde{e} \arrow[uur, dashed] &
\end{tikzcd}$$

Both $\tilde{T}$ and $\tilde{S}$ are separable over $\tilde{R}$ by Properties 1.13 and 1.12, $\tilde{e}$ is a separability idempotent for $\tilde{T}$, and $\tilde{S} + J(\tilde{A}) = \tilde{A}$. Then the induction hypothesis gives that $\mu(\tilde{e}_1)\tilde{T}\mu(\tilde{e}_1)^{-1} \subseteq \tilde{S}$. Pulling this inclusion back to A, we have $\mu(e_1)T\mu(e_1)^{-1} \subseteq S + m^kA$.

Now let $C = S + m^kA$ and $T' = \mu(e_1)T\mu(e_1)^{-1}$. S is an inertial subalgebra of C, T' is a separable subalgebra of C, and C is a finitely generated R-algebra, because R is Noetherian. Write $e = \Sigma\gamma_i \otimes \delta_i$, where $\gamma_i \in T$, $\delta_i \in T^\circ$, and let $e' = \Sigma[\mu(e_1)\gamma_i\mu(e_1)^{-1} \otimes \mu(e_1)\delta_i\mu(e_1)^{-1}]$. One easily sees that e' is a separability idempotent for T'. Write $e_1 = \Sigma\alpha_j \otimes \beta_j$ where $\alpha_j \in S$ and $\beta_j \in T^\circ$, let $e_1' = \Sigma\alpha_j \otimes \mu(e_1)\beta_j\mu(e_1)^{-1}$, and notice that e_1' is an idempotent. It is not hard to see that $J(C) = C \cap J(A)$, $C/J(C) = \bar{A}$, and $T'/(T' \cap J(C)) = \bar{T}$. Thus, we have natural maps $f' : S \otimes_R T'^\circ \longrightarrow \bar{A} \otimes_{\bar{R}} \bar{T}^\circ$ and $g' : T' \otimes_R T'^\circ \longrightarrow \bar{A} \otimes_{\bar{R}} \bar{T}^\circ$ with $f'(e_1') = \bar{e} = g'(e')$. We can now use the same argument here for C that we used previously for A to conclude that $\mu(e_1')T'\mu(e_1')^{-1} \subseteq S + m^kC = S + m^k(S + m^kA) = S$. Equivalently, $\mu(e_1')\mu(e_1)T\mu(e_1)^{-1}\mu(e_1')^{-1} \subseteq S$. But $\mu(e_1')\mu(e_1) = (\Sigma\alpha_j\mu(e_1)\beta_j\mu(e_1)^{-1}) \cdot \mu(e_1) = \Sigma\alpha_j\mu(e_1)\beta_j = [\Sigma\alpha_j \otimes \beta_j] \cdot \mu(e_1) = e_1 \cdot \mu(e_1) = \mu(e_1 \cdot e_1) = \mu(e_1)$. Thus we have shown that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

<u>Step 5</u>: Let (R,m) be a Noetherian local ring. Let k be a positive integer, and pass to the factor algebra $\tilde{A} = A/m^kA$ over $\tilde{R} = R/m^k$. Letting $\tilde{T} = T/(m^kA \cap T)$ and $\tilde{S} = S/(m^kA \cap S)$, we have that $\tilde{T}$ is $\tilde{R}$-separable, and $\tilde{S}$ is an $\tilde{R}$-inertial subalgebra of $\tilde{A}$. Taking $\tilde{e}$, $\tilde{e}_1$, $\tilde{f}$, and $\tilde{g}$ to be defined as they were in Step 4, we can again refer to the diagram on page 13. $\tilde{R}$ is a local ring with maximal ideal $\tilde{m} = m/m^k$, and $\tilde{m}^k = 0$, so we can apply the result of Step 4 to get

$\mu(\tilde{e}_1)\tilde{T}\mu(\tilde{e}_1)^{-1} \subseteq \tilde{S}$. Pulling back to A we have $\mu(e_1)T\mu(e_1)^{-1} \subseteq S + m^kA$. This containment holds for every positive integer k, so we can write $\mu(e_1)T\mu(e_1)^{-1} \subseteq \bigcap_{k=1}^{\infty} (S+m^kA)$. But R is a Zariski ring [18, p. 263, 264], so by [18, Theorem 9, p. 262] we have $\bigcap_{k=1}^{\infty} (S+m^kA) = S$, and again we have shown that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

<u>Step 6</u>: Let R be a Noetherian ring and $T' = \mu(e_1)T\mu(e_1)^{-1}$. We will show that $T' \subseteq S$ by showing that $Z = (T'+S)/S$ is the zero module. $Z=0$ if and only if $Z_m = Z \otimes_R R_m = 0$ for every maximal ideal m of R, by Proposition 1.7. By tensoring everything in the diagram on page 12 with R_m over R, we again place ourselves in the setting of Step 5, where we have $T'_m \subseteq S_m$, or equivalently, $Z_m = 0$. We conclude that $Z=0$, and it follows that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

$\square$

<u>Example</u>: Let $R = Z/4Z$, $A = M_2\left(\frac{Z/4Z[x]}{(x^2-2)}\right)$, and $S = M_2(Z/4Z) \subseteq A$. By Example 1.10, S is separable over R. It is not hard to see that $J(A) = A\begin{pmatrix}\bar{x} & 0\\ 0 & \bar{x}\end{pmatrix} + 2A$, so that $S + J(A) = A$, and S is an inertial subalgebra of A. $T_1 = \left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \middle| a,b \in R\right\}$ is a separable subalgebra of A with separability idempotent $\tilde{e} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Then the subalgebra $T = \begin{pmatrix}1 & \bar{x}\\ \bar{x} & 1\end{pmatrix} T_1 \begin{pmatrix}3 & \bar{x}\\ \bar{x} & 3\end{pmatrix} = \left\{\begin{pmatrix}3a+2b & (a+3b)\bar{x}\\ (3a+b)\bar{x} & 2a+3b\end{pmatrix} \middle| a,b \in R\right\}$ is a separable subalgebra of A with separability idempotent $e = \begin{pmatrix}3 & \bar{x}\\ 3\bar{x} & 2\end{pmatrix} \otimes \begin{pmatrix}3 & \bar{x}\\ 3\bar{x} & 2\end{pmatrix} + \begin{pmatrix}2 & 3\bar{x}\\ \bar{x} & 3\end{pmatrix} \otimes \begin{pmatrix}2 & 3\bar{x}\\ \bar{x} & 3\end{pmatrix}$. It is interesting to illustrate the method of proof of Theorem 2.1 with this example, so consider the following diagram.

$$\begin{tikzcd}
S \otimes_R T^\circ \arrow[dr, "f"] & \\
e_1 \arrow[dr, dashed] & \bar{A} \otimes_{\bar{R}} \bar{T}^\circ \\
T \otimes_R T^\circ \arrow[ur, "g"] & \bar{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
e = \begin{pmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{pmatrix} \arrow[ur, dashed] &
\end{tikzcd}$$

We will consider various candidates for e_1.

1. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{pmatrix}$. This element is an idempotent, and $\mu(e_1) = \begin{pmatrix} 3 & \bar{x} \\ \bar{x} & 3 \end{pmatrix}$, so that $\mu(e_1)T\mu(e_1)^{-1} = T_1 \subseteq S$. It is not surprising that $\begin{pmatrix} 3 & \bar{x} \\ \bar{x} & 3 \end{pmatrix}$ arises as $\mu(e_1)$, since it is the inverse of $\begin{pmatrix} 1 & \bar{x} \\ \bar{x} & 1 \end{pmatrix}$, the element used to conjugate T_1 to T in the first place, although we have not shown that all conjugating elements arise as $\mu(e_1)$ for some idempotent e_1.

2. Let $e_1 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{pmatrix}$. This element is also an idempotent, but here $\mu(e_1) = \begin{pmatrix} 3 & 2+\bar{x} \\ \bar{x} & 3 \end{pmatrix}$, and $\mu(e_1)T\mu(e_1)^{-1} = \left\{ \begin{pmatrix} a & 2a+2b \\ 0 & b \end{pmatrix} \middle| \, a,b \in Z/4Z \right\} \subseteq S$. We see from this example that as e_1 varies, both $\mu(e_1)$ and $\mu(e_1)T\mu(e_1)^{-1}$ may vary. The theorem, however, guarantees that as long as e_1 is an idempotent, $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

3. Let $e_1 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 3 & \bar{x} \\ 3\bar{x} & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3\bar{x} \\ \bar{x} & 3 \end{pmatrix}$. This element is not an idempotent, although it is a preimage of $\bar{e}$ in $S \otimes_R T^\circ$. However, this element is simply a scalar multiple of the first candidate for e_1,

so that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$. Thus, some nonidempotent preimages of $\bar{e}$ may satisfy $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

4. Let $e_1 = \begin{pmatrix}3 & 2\\ 2 & 2\end{pmatrix} \otimes \begin{pmatrix}3 & \bar{x}\\ 3\bar{x} & 2\end{pmatrix} + \begin{pmatrix}2 & 2\\ 2 & 1\end{pmatrix} \otimes \begin{pmatrix}2 & 3\bar{x}\\ \bar{x} & 3\end{pmatrix}$. This preimage of $\bar{e}$ is not an idempotent. $\mu(e_1) = \begin{pmatrix}1 & 2+\bar{x}\\ 2+3\bar{x} & 3\end{pmatrix}$, and

$$\mu(e_1)T\mu(e_1)^{-1} = \left\{\begin{pmatrix}a & 2a+2b+2a\bar{x}+2b\bar{x}\\ 2a+2b+2a\bar{x}+2b\bar{x} & b\end{pmatrix} \middle| a,b \in R\right\} \not\subseteq S.$$

Hence, not all preimages of $\bar{e}$ have the property that $\mu(e_1)T\mu(e_1)^{-1} \subseteq S$.

□

In the proof of Theorem 2.1, once the reductions of Step 1 and Step 2 are made, the only place we use the fact that R is an inertial coefficient ring is when we wish to lift idempotents. Thus, if we start with the assumption that $A/J(A)$ is separable and S is an inertial subalgebra of A, we have the following corollary.

<u>Corollary 2.2</u>: Let R be a Noetherian ring with the property that for every finitely generated R-algebra idempotents can be lifted from the algebra modulo its radical to the algebra. Let A be a finitely generated R-algebra with $A/J(A)$ separable, and let S be an inertial subalgebra of A. Then there is a unit a in A with $aTa^{-1} \subseteq S$.

Conjugates of inertial subalgebras are inertial subalgebras, so if we are in a setting where inertial subalgebras exist and Theorem 2.1 applies, the following corollary holds.

<u>Corollary 2.3</u>: If R is a Noetherian inertial coefficient ring and A is a finitely generated R-algebra with $A/J(A)$ separable, then every separable subalgebra is contained in an inertial subalgebra, and every maximal separable subalgebra is an inertial subalgebra.

When A is a commutative, finitely generated algebra over a commutative ring R, the situation becomes much simpler.

<u>Proposition 2.4</u>: Let A be a commutative, finitely generated algebra over a commutative ring R. Let S be a subalgebra of A with $S + J(A) = A$. If T is a separable subalgebra of A, then $T \subseteq S$.

<u>Proof</u>: If we consider A as an S-algebra, then S is an S-inertial subalgebra of A. By Property 1.14, $S \otimes_R T$ is an S-separable algebra, and $S \cdot T$ is a homomorphic image of $S \otimes_R T$, so by Property 1.13, $S \cdot T$ is an S-separable subalgebra of A. Furthermore, since $S \subseteq S \cdot T$, $S \cdot T$ is also an S-inertial subalgebra of A. Therefore, by Property 1.21, $S \cdot T = S$, and, consequently, $T \subseteq S$.

□

It would be nice to be able to eliminate some of the restrictions on R in Theorem 2.1. The following result lifts both the Noetherian and the inertial coefficient ring conditions on R, but we are forced to replace the Jacobson radical of A with a nil ideal of A.

<u>Proposition 2.5</u>: Let R be a commutative ring and A be a finitely generated R-algebra. Let I be any nil ideal of A, and let S be an R-subalgebra of A such that $S + I = A$. If T is any separable subalgebra of A, then there exists an element a in A such that $aTa^{-1} \subseteq S$.

Proof: We shall use the technique of selecting a suitable Hilbert subring R_1 of R and an R_1-algebra A_1 which satisfy the conditions of Theorem 2.1. We then lift the result back to A.

By Property 1.17, T is a finitely generated R-algebra, so write $T = Rt_1 + Rt_2 + \dots + Rt_m$. T is R-separable, so there are elements x_i and y_i in T such that $\Sigma x_i \otimes y_i$ is a separability idempotent for T in $T \otimes_R T^\circ$. Thus, we have, for every $j = 1,\dots,m$,

$$(*) \qquad (t_j \otimes 1 - 1 \otimes t_j)(\Sigma x_i \otimes y_i) = 0 \text{ in } T \otimes_R T^\circ.$$

Think of $T \otimes_R T^\circ$ as a free abilian group with subgroup $\mathcal{A}$ of relations factored out, and notice that there is a finite subset M_j of $T \cup R$ such that the elements of $\mathcal{A}$ making $(*)$ zero in $T \otimes_R T^\circ$ are expressible in terms of the elements of M_j.

Let $a_1,\dots,a_n$ generate A as an R-module. Since $S + I = A$, there exist $s_1,\dots,s_n$ in S and $\mu_1,\dots,\mu_n$ in I with $a_i = s_i + \mu_i$ for $i = 1,\dots,n$.

Now set $B = \{1, a_i a_j, s_i, t_i\}$ and $C = \{1, t_i t_j, x_i, y_i\} \cup (\bigcup_j M_j)$. Write each element of the finite set B as an R-linear combination of $a_1,\dots,a_n$, and write each element of the finite set C as an R-linear combination of $t_1,\dots,t_m$. All of this will involve only finitely many coefficients from R. Let R_1 be the Noetherian subring of R generated by this finite set and the "prime" subring P of R. P is a homomorphic image of the Hilbert ring Z, the integers, so P is a Hilbert ring. R_1 is finitely generated as an algebra over P, so R_1 is a Hilbert ring [9, Theorem 2 and Theorem 3, pp. 136-137]. Therefore, R_1 is an inertial coefficient ring [13, Corollary 2, p. 553].

Define $A_1 = R_1a_1 + R_1a_2 + \cdots + R_1a_n$. By construction of R_1 we have $B \subseteq A_1$, so A_1 is a finitely generated R_1-algebra containing the s_i's and the t_i's. Consequently, we can take S_1 to be the R_1-subalgebra of A_1 generated by $s_1,\ldots,s_n$. Next let $T_1 = R_1t_1 + R_1t_2 + \cdots + R_1t_m$, so T_1 is a finitely generated R_1-algebra containing the set C. Furthermore, $\Sigma x_i \otimes y_i$ is an element of $T_1 \otimes_{R_1} T_1^\circ$ satisfying $(*)$ in $T_1 \otimes_{R_1} T_1^\circ$. Consequently, $\Sigma x_i \otimes y_i$ is a separability idempotent for T_1, so T_1 is R_1-separable. Finally, we let $I_1 = I \cap A_1$. Since I is nil, I_1 is nil, and it follows that $I_1 \subseteq J(A_1)$. Recall that $a_i - s_i = \mu_i$ is in I, and since $a_i - s_i$ is also in A_1, then $a_i - s_i = \mu_i$ is in I_1 for $i = 1,\ldots,n$. The relations $a_i = s_i + \mu_i$ imply that $S_1 + I_1 = A_1$, and it follows that $S_1 + J(A) = A_1$.

Now A_1 satisfies all the conditions of Theorem 2.1, so there is a unit a in A_1 such that $aT_1a^{-1} \subseteq S_1$. We next extend back up to A by multiplying by R to get $RA_1 = A$, $RS_1 \subseteq S$, and $RT_1 = T$. Considering a now as an element of A, we have $aTa^{-1} = a(RT_1)a^{-1} = R(aT_1a^{-1}) \subseteq RS_1 \subseteq S$, and we are done.

□

<u>Remark</u>: If R is a Hilbert ring and A is a finitely generated R-algebra, then it is not difficult to show that $J(A)$ is nil. This is one setting in which Proposition 2.5 applies.

CHAPTER III

PURE INSEPARABILITY AND WEDDERBURN SPECTERS

In [16] Sweedler defines an algebra A over a commutative ring R to be <u>purely inseparable over R</u> if the multiplication map $\mu : A \otimes_R A^\circ \longrightarrow A$ gives an $A \otimes_R A^\circ$-projective cover of A. This is equivalent to $\ker \mu$ being a small left ideal of $A \otimes_R A^\circ$; that is, if M is a left ideal of $A \otimes_R A^\circ$ with $M + \ker \mu = A \otimes_R A^\circ$, then $M = A \otimes_R A^\circ$. Furthermore, A is purely inseparable over R if and only if $\ker \mu \subseteq J(A \otimes_R A^\circ)$.

If $A \supseteq S$ are rings, then Bogart [5] extends Sweedler's definition by taking A to be purely inseparable over S if the $A-A$ bimodule map $\mu : A \otimes_S A^\circ \longrightarrow A$ has small kernel. If $C = Z(A) \cap S$, where $Z(A)$ is the center of A, one can consider μ to be an $A \otimes_C A^\circ$-map, and if A is an R-algebra and S is a subalgebra of A, then μ can be considered to be an $A \otimes_R A^\circ$-map. The smallness of $\ker \mu$ is independent of which of the three module structures one uses.

In the case where S is a commutative ring and A is an algebra over S, Bogart's definition reduces to Sweedler's definition. If A and S are fields, Sweedler has shown that his definition is equivalent to the usual definition for purely inseparable field extensions [16, Theorem 12, p. 351].

Example: Let $R = \mathbb{Z}/2\mathbb{Z}(\alpha)$, the field of functions over $\mathbb{Z}/2\mathbb{Z}$, and let $A = R[x]/(x^2+\alpha)$. Then A is a purely inseparable algebra over R, since A is a purely inseparable field extension of R. As an illustration of the definition of a purely inseparable algebra, we will show that $\ker\mu \subseteq J(A \otimes_R A)$. In Chapter I we saw that $\ker\mu$ is generated as an ideal of $A \otimes_R A$ by elements of the form $y \otimes 1 + 1 \otimes y$ for y in A. Moreover, y can be written as $a+bx$, where a and b are in R. Therefore, $y \otimes 1 + 1 \otimes y = (a+bx) \otimes 1 + 1 \otimes (a+bx) = a \otimes 1 + bx \otimes 1 + 1 \otimes a + 1 \otimes bx = b \cdot (x \otimes 1 + 1 \otimes x)$, so $\ker\mu = (A \otimes_R A) \cdot (x \otimes 1 + 1 \otimes x)$. Furthermore, $(x \otimes 1 + 1 \otimes x)^2 = \alpha \otimes 1 + 1 \otimes \alpha = 0$, so that $x \otimes 1 + 1 \otimes x$ is in $J(A \otimes_R A)$, and it follows that $\ker\mu \subseteq J(A \otimes_R A)$.

While Bogart and Sweedler present results in the general setting of these definitions of pure inseparability, most of their work deals with algebras over fields. This chapter extends certain of their results to algebras over commutative rings. We begin with a proposition giving some basic properties of pure inseparability which will be used throughout this chapter.

Proposition 3.1: Let A be a finitely generated algebra over a commutative ring R.

a) Let S be a subalgebra of A such that A is purely inseparable over S, and let I be an ideal of A. Then A/I is purely inseparable over $S/(S \cap I)$.

b) Let A be commutative and S be a subalgebra of A. Then A is purely inseparable over S if and only if A_m is purely inseparable over S_m for every maximal ideal m of R.

c) Let A be commutative, S be a subalgebra of A, and P be a finitely generated, commutative, faithfully flat R-algebra. Then A is purely inseparable over S if and only if $A \otimes_R P$ is purely inseparable over $S \otimes_R P$.

Proof: (a) The proof of this is essentially the same as Bogart's proof of [5, Proposition 2.10, p. 128]. We include it here for completeness.

Let $\overline{A} = A/I$, $\pi : A \longrightarrow \overline{A}$ be the canonical map, and $\overline{S} = S/S \cap I$. Consider the commutative diagram with exact rows given below.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\pi \otimes \pi^{\circ}|_L"] & A \otimes_S A^{\circ} \arrow[r, "\mu"] \arrow[d, "\pi \otimes \pi^{\circ}"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & K \arrow[r] & \overline{A} \otimes_{\overline{S}} \overline{A}^{\circ} \arrow[r, "\overline{\mu}"] & \overline{A} \arrow[r] & 0
\end{tikzcd}$$

Here $\ker \mu = L$ and $\ker \overline{\mu} = K$. Since the diagram commutes, $(\pi \otimes \pi^{\circ})(L) \subseteq K$. To show equality, we pick $\overline{\beta} \in K$, and let $\beta \in (\pi \otimes \pi^{\circ})^{-1}(\overline{\beta}) \subseteq A \otimes_S A^{\circ}$. If $b = \beta - (\mu(\beta) \otimes 1)$, then $\mu(b) = 0$, so $b \in L$. Furthermore, $(\pi \otimes \pi^{\circ})(b) = \overline{\beta}$, since $\pi(\mu(\beta)) = \overline{\mu}(\overline{\beta}) = 0$, so that $(\pi \otimes \pi^{\circ})(L) = K$. If $\overline{M}$ is an $\overline{A}-\overline{A}$ submodule of $\overline{A} \otimes_{\overline{S}} \overline{A}^{\circ}$ with $\overline{M} + K = \overline{A} \otimes_{\overline{S}} \overline{A}^{\circ}$, then for $M = (\pi \otimes \pi^{\circ})^{-1}(\overline{M})$, we have $M + L = A \otimes_S A^{\circ}$. Since L is small, $M = A \otimes_S A^{\circ}$. This implies that $(\pi \otimes \pi^{\circ})(M) = \overline{M} = \overline{A} \otimes_{\overline{S}} \overline{A}^{\circ}$, so K is small.

(b) Let m be any maximal ideal of R. Consider the following commutative diagram.

$$\begin{array}{ccc} A \otimes_S A & \xrightarrow{\mu} & A \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ A_m \otimes_{S_m} A_m & \xrightarrow{\mu_m} & A_m \end{array}$$

Here μ and μ_m are the multiplication maps, ϕ_2 is the natural map from A to A_m, and ϕ_1 is $\phi_2 \otimes \phi_2$. Since $A_m \simeq A \otimes_R R_m$ and $A_m \otimes_{S_m} A_m \simeq (A \otimes_R R_m) \otimes_{S \otimes_R R_m} (A \otimes_R R_m) \simeq (A \otimes_S A) \otimes_R R_m$, we can rewrite the preceding diagram.

$$\begin{array}{ccc} A \otimes_S A & \xrightarrow{\mu} & A \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ A \otimes_S A \otimes_R R_m & \xrightarrow{\mu \otimes 1} & A \otimes_R R_m \end{array}$$

Since R_m is flat over R, then $\ker(\mu \otimes 1) = (\ker \mu) \otimes_R R_m$, and furthermore, $\ker(\mu \otimes 1) \simeq \ker \mu_m$.

($\Leftarrow$) Suppose M is an A-A submodule of $A \otimes_S A$ such that $M + \ker \mu = A \otimes_S A$. Tensoring up with R_m, this becomes $M \otimes_R R_m + (\ker \mu) \otimes_R R_m = A \otimes_S A \otimes_R R_m$. Then $M \otimes_R R_m + \ker(\mu \otimes 1) = A \otimes_S A \otimes_R R_m$, and, since $\ker(\mu \otimes 1)$ is small, we have $M \otimes_R R_m = A \otimes_S A \otimes_R R_m$. This equality holds for every maximal ideal m of R, so by Proposition 1.7, $M = A \otimes_S A$. Therefore, $\ker \mu$ is small, and A is purely inseparable over S.

($\Rightarrow$) Since A is purely inseparable over S, then $\ker \mu \subseteq J(A \otimes_S A)$. Tensoring up with R_m, we have $(\ker \mu) \otimes_R R_m \subseteq J(A \otimes_S A) \otimes_R R_m$. By Proposition 1.4, $J(A \otimes_S A) \otimes_R R_m \subseteq J(A \otimes_S A \otimes_R R_m)$, and we have $\ker(\mu \otimes 1) \subseteq J(A \otimes_S A \otimes_R R_m)$. Consequently, $\ker \mu_m \subseteq J(A_m \otimes_{S_m} A_m)$, and A_m is purely inseparable over S_m.

(c) Consider the following commutative diagram.

$$\begin{array}{ccc} A \otimes_S A & \xrightarrow{\mu} & A \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ (A \otimes_R P) \otimes_{S \otimes_R P} (A \otimes_R P) & \xrightarrow{\tilde{\mu}} & A \otimes_R P \end{array}$$

Here μ and $\tilde{\mu}$ are multiplication maps, ϕ_2 is the natural inclusion, and ϕ_1 is $\phi_2 \otimes \phi_2$. It is not hard to see that $(A \otimes_R P) \otimes_{S \otimes_R P} (A \otimes_R P) \simeq (A \otimes_S A) \otimes_R P$ via $(a_1 \otimes p_1) \otimes (a_2 \otimes p_2) \longrightarrow (a_1 \otimes a_2) \otimes p_1 p_2$. Therefore, we can rewrite the preceding diagram.

$$\begin{array}{ccc} A \otimes_S A & \xrightarrow{\mu} & A \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ (A \otimes_S A) \otimes_R P & \xrightarrow{\mu \otimes 1} & A \otimes_R P \end{array}$$

Since P is flat over R, then $\ker(\mu \otimes 1) = (\ker \mu) \otimes_R P$, and also $\ker(\mu \otimes 1) \simeq \ker \tilde{\mu}$.

($\Leftarrow$) Let M be an A-A submodule of $A \otimes_S A$ such that $M + \ker \mu = A \otimes_S A$. Then $M \otimes_R P + (\ker \mu) \otimes_R P = (A \otimes_S A) \otimes_R P$. Since $(\ker \mu) \otimes_R P \simeq \ker \tilde{\mu}$ is small, then $M \otimes_R P = (A \otimes_S A) \otimes_R P$, and it follows that $M = A \otimes_S A$ because P is faithfully flat over R. Thus, $\ker \mu$ is small, and A is purely inseparable over S.

($\Longrightarrow$) Let $\Omega(R)$ denote the set of maximal ideals of R. $A \otimes_S A$ is a finitely generated R-algebra, so by Lemma 1.1 there is a positive integer n such that $[J(A \otimes_S A)]^n \subseteq \bigcap_{m\in\Omega(R)} m \cdot (A \otimes_S A)$. Then $[J(A \otimes_S A) \otimes_R P]^n = [J(A \otimes_S A)]^n \otimes_R P \subseteq [\bigcap_{m\in\Omega(R)} m \cdot (A \otimes_S A)] \otimes_R P \subseteq$ $\bigcap_{m\in\Omega(R)} m \cdot (A \otimes_S A \otimes_R P) \subseteq J(A \otimes_S A \otimes_R P)$, where the last inclusion follows from Lemma 1.1. Consequently, $J(A \otimes_S A) \otimes_R P \subseteq J(A \otimes_S A \otimes_R P)$, and since $\ker \mu \subseteq J(A \otimes_S A)$, then $(\ker \mu) \otimes_R P \subseteq J(A \otimes_S A) \otimes_R P \subseteq$ $J(A \otimes_S A \otimes_R P)$. Therefore, $\ker \tilde{\mu} \subseteq J((A \otimes_R P) \otimes_{S \otimes_R P} (A \otimes_R P))$, and $A \otimes_R P$ is purely inseparable over $S \otimes_R P$.

$\square$

<u>Example</u>: The conclusion of Proposition 3.1(c) need not be true if P is not finitely generated over R. If $R = Z_{(q)}$ (the integers localized at a prime ideal q), $A = Z_{(q)}[y]/(y^2 - q)$, and $P = Z_{(q)}[x]$, then P is faithfully flat over R because it is free, but P is not finitely generated over R. Let m be the ideal of $A \otimes_R A$ generated as a module over R by the set $\{q \otimes 1, 1 \otimes y, y \otimes 1, y \otimes y\}$. m is maximal because $(A \otimes_R A)/m$ is a field, and in fact $m = J(A \otimes_R A)$ because the square of each generator of m lies in $J(A \otimes_R A)$. Consequently, $A \otimes_R A$ is local, so $\ker \mu \subseteq m = J(A \otimes_R A)$, and A is purely inseparable over R.

$A \otimes_R P = A \otimes_{Z_{(q)}} Z_{(q)}[x] \simeq A[x]$ via $a \otimes f(x) \longrightarrow a \cdot f(x)$ for $a \in A$, $f(x) \in Z_{(q)}[x]$. Therefore, $(A \otimes_R P) \otimes_P (A \otimes_R P) \simeq A[x] \otimes_P A[x] \simeq$ $(A \otimes_R A)[x]$ via $ax^n \otimes bx^m \longrightarrow (a \otimes b)x^{n+m}$ for $a,b \in A$. We have the following commutative diagram.

$$\begin{array}{ccc} (A \otimes_R P) \otimes_P (A \otimes_R P) & \xrightarrow{\mu_P} & A \otimes_R P \\ \downarrow \simeq & & \downarrow \simeq \\ (A \otimes_R A)[x] & \xrightarrow{\mu} & A[x] \end{array}$$

The map μ acts as multiplication on $A \otimes_R A$ and leaves x fixed, and the element $y \otimes 1 - 1 \otimes y$ is in $\ker \mu$. For the polynomial ring $(A \otimes_R A)[x]$, the Jacobson radical is equal to the nilradical. If n is any even integer, then

$$(y \otimes 1 - 1 \otimes y)^n = \left[\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n}\right]\left(q^{n/2} \otimes 1\right) - \left[\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1}\right] q^{\frac{n-2}{2}} (y \otimes y)$$

$\neq 0$. Therefore, $y \otimes 1 - 1 \otimes y \notin J[(A \otimes_R A)[x]]$, so that $\ker \mu \nsubseteq J[(A \otimes_R A)[x]]$, and consequently, $A \otimes_R P$ is not purely inseparable over $R \otimes_R P \simeq P$.

□

The next result shows the relationship between pure inseparability and maximal separable subalgebras.

Proposition 3.2: Let R be a Noetherian ring and A be a finitely generated R-algebra. Let S be a separable subalgebra of A with A purely inseparable over S. If

a) R is local with maximal ideal m, or

b) A is commutative,

then S is a maximal separable subalgebra of A.

Proof: (a) Step 1: Assume $m^k = 0$ for some positive integer k. Let $\overline{R} = R/m$, $\overline{A} = A/mA$, and $\overline{S} = S/(S\cap mA)$. By Proposition 3.1(a), $\overline{A}$ is purely inseparable over $\overline{S}$. $\overline{A}$ is a finite-dimensional algebra over the field $\overline{R}$, so by [5, Proposition 2.1, p. 124] $\overline{S}$ is a maximal separable subalgebra of $\overline{A}$.

Suppose there is a separable subalgebra S' of A with $S \subseteq S'$. Setting $\overline{S}' = S'/(S'\cap mA)$, we have $\overline{S}' = \overline{S}$, since $\overline{S}$ is maximal separable, and it follows that $S' + mA = S + mA$. Consider the R-algebra $B = S' + mA = S + mA$. Clearly $mA \subseteq J(B)$, so both S' and S are inertial subalgebras of B. Since R is Noetherian, B is finitely generated over R, and Property 1.20 yields $S = S'$. Therefore, S is a maximal separable subalgebra of A.

Step 2: Let (R,m) be a Noetherian local ring. Let k be a positive integer, and pass to the factor algebra $\overline{A} = A/m^kA$ over $\overline{R} = R/m^k$ with separable subalgebra $\overline{S} = S/(m^kA\cap S)$. By Proposition 3.1(a), $\overline{A}$ is purely inseparable over $\overline{S}$, so by Step 1, $\overline{S}$ is a maximal separable subalgebra of $\overline{A}$ for each k.

Suppose there is a separable subalgebra S' of A with $S \subseteq S'$. Setting $\overline{S}' = S'/(m^kA\cap S')$, we have $\overline{S} = \overline{S}'$, and it follows that $S' + m^kA = S + m^kA$ for every positive integer k. In particular, $S' \subseteq S + m^kA$ for each k, or $S' \subseteq \bigcap_{k=1}^{\infty} (S + m^kA)$. Then, since R is a Zariski ring, $S' \subseteq \bigcap_{k=1}^{\infty} (S + m^kA) = S$ [18, Theorem 9, p. 262], and therefore, S is a maximal separable subalgebra of A.

(b) Let S' be a separable subalgebra of A with $S \subseteq S'$, and let m be any maximal ideal of R. By Proposition 3.1(b), A_m is purely inseparable over S_m. By Property 1.14, S_m and S'_m are separable over R_m, so by (a), $S_m = S'_m$. This equality holds for every maximal ideal m of R. Applying Proposition 1.7 we have $S = S'$, so that S is a maximal separable subalgebra of A.

$\square$

The next three results, Lemma 3.3, Proposition 3.4, and Proposition 3.5, are the unpublished work of Edward C. Ingraham. I am grateful to him for allowing me to include them here. They provide us with a link between pure inseparability and inertial subalgebras.

Lemma 3.3: Let $A \supseteq S$ be rings, and let $C = Z(A) \cap S$, where $Z(A)$ is the center of A. Then A is purely inseparable over S if and only if for every x in $A \otimes_S A^\circ$ with $\mu(x) = 1$, $(A \otimes_C A^\circ) \cdot x = A \otimes_S A^\circ$.

Proof: ($\Longrightarrow$) Let $x \in A \otimes_S A^\circ$ with $\mu(x) = 1$. Then $\ker \mu + (A \otimes_C A^\circ) \cdot x = A \otimes_S A^\circ$, whence $(A \otimes_C A^\circ) \cdot x = A \otimes_S A^\circ$ by the smallness of $\ker \mu$.

($\Longleftarrow$) Conversely, suppose $\ker \mu + M = A \otimes_S A^\circ$ for some $A \otimes_C A^\circ$-submodule M of $A \otimes_S A^\circ$. We can write $1 \otimes 1 \in A \otimes_S A^\circ$ as $k + x$, for some $k \in \ker \mu$, $x \in M$. Then $\mu(x) = 1$, and by the assumption, $A \otimes_S A^\circ = (A \otimes_C A^\circ) \cdot x \subseteq M$, so $\ker \mu$ is small.

$\square$

Remark: If L is a finitely generated R-module, then $J(R) \cdot L$ is small in L, since $J(R) \cdot L + M = L$ implies $M = L$ by Corollary 1.6.

<u>Proposition 3.4</u>: If A is a ring, S a subring, and $C = Z(A) \cap S$ such that

1. $A = S + N$ for some ideal N in $J(A)$ such that
2. $i(A \otimes_C N^\circ) + i(N \otimes_C A^\circ) \subseteq J(A \otimes_C A^\circ)$,

then A is purely inseparable over S.

<u>Proof</u>: $A \otimes_S A^\circ = i(S \otimes_S S^\circ) + i(N \otimes_S A^\circ) + i(A \otimes_S N^\circ)$. Choose $x \in A \otimes_S A^\circ$ with $\mu(x) = 1$. Then $x = s + n$, where $s \in i(S \otimes_S S^\circ) \simeq S$ and $n \in i(N \otimes_S A^\circ) + i(A \otimes_S N^\circ)$. Now $1 = \mu(x) = \mu(s) + \mu(n) = s + \mu(n)$, and $\mu(n) \in N \subseteq J(A)$, so s is a unit in A. Thus, $A \otimes_S A^\circ = (A \otimes_C A^\circ) \cdot s$, and $(A \otimes_C A^\circ) \cdot s = (A \otimes_C A^\circ)(x - n) = (A \otimes_C A^\circ) \cdot x + (A \otimes_C A^\circ) \cdot n$. Since $n \in i(A \otimes_S N^\circ) + i(N \otimes_S A^\circ)$, $(A \otimes_C A^\circ) \cdot n$ has elements of the form $\Sigma \alpha_i \otimes \beta_i$, where α_i or $\beta_i \in N$, which implies that $(A \otimes_C A^\circ) \cdot n \subseteq [i(A \otimes_C N^\circ) + i(N \otimes_C A^\circ)] \cdot (A \otimes_S A^\circ) \subseteq J(A \otimes_C A^\circ) \cdot (A \otimes_S A^\circ)$. Thus, by the preceding remark, $(A \otimes_C A^\circ) \cdot n$ is small, since $A \otimes_S A^\circ$ is generated by $1 \otimes 1$ over $A \otimes_C A^\circ$. Therefore, $A \otimes_S A^\circ = (A \otimes_C A^\circ) \cdot x$. Then Lemma 3.3 gives that A is purely inseparable over S.

<u>Proposition 3.5</u>: Let A be a finitely generated R-algebra over the commutative ring R, and let S be an inertial subalgebra of A. Then A is purely inseparable over S.

<u>Proof</u>: Let $C = Z(A) \cap S$, and let $N = J(A)$. By [4, Theorem 10, p. 127], $i(A \otimes_C N^\circ) + i(N \otimes_C A^\circ) \subseteq J(A \otimes_C A^\circ)$, since A is finitely generated over C. Apply Proposition 3.4.

$\square$

The next lemma is a generalization of a standard field theory result and also of a result of Sweedler [16] for algebras over fields. Recall that in the category of rings a homomorphism f is called an <u>epimorphism</u> if for any two homomorphisms g and h the equality of $g \circ f$ and $h \circ f$ implies the equality of g and h.

<u>Lemma 3.6</u>: Let R be a commutative ring and A be an R-algebra. If A is both separable and purely inseparable over R, then $R \longrightarrow A$ is an epimorphism. If in addition A is finitely generated over R, then $A = R$.

<u>Proof</u>: A purely inseparable over R means that for $\mu : A \otimes_R A^{\circ} \longrightarrow A$, $\ker \mu$ is small. A separable over R means that $\ker \mu$ is a direct summand of $A \otimes_R A^{\circ}$. Consequently, $\ker \mu = 0$, and by [15, Theorem 1, p. 2] the map $R \longrightarrow A$ is an epimorphism. If A is finitely generated over R, then [15, Corollary 4.2, p. 4] yields $A = R$.

□

<u>Example</u>: If A is not finitely generated over R in Lemma 3.6, we do not necessarily have $A = R$. Q (the rational numbers) is both separable and purely inseparable over Z (the integers), since $Q \otimes_Z Q \simeq Q$, but $Q \neq Z$.

Now we are able to combine several of the preceding results to get the following proposition, which provides a link between pure inseparability, inertial subalgebras, and maximal separable subalgebras of A when $A/J(A)$ is separable.

Proposition 3.7: Let R be a commutative ring and A a finitely generated, commutative R-algebra. If $A/J(A)$ is separable, then the following are equivalent:

a) S is an inertial subalgebra of A.

b) A is purely inseparable over S, and S is separable over R.

If R is a Noetherian inertial coefficient ring, then (a) and (b) are equivalent to

c) S is a maximal separable subalgebra of A.

Proof:

(a) $\Longrightarrow$ (b) is Proposition 3.5.

(b) $\Longrightarrow$ (a) $A/J(A)$ is separable over R, so $A/J(A)$ is separable over $S/[S \cap J(A)]$ by Property 1.15. Since A is purely inseparable over S, then $A/J(A)$ is purely inseparable over $S/[S \cap J(A)]$ by [16, Proposition 6(e), p. 345]. Lemma 3.6 then gives $A/J(A) = S/[S \cap J(A)]$, which implies $A = S + N$. Since S is separable, then S is an inertial subalgebra of A.

Assume now that R is a Noetherian inertial coefficient ring. Then (b) $\Longrightarrow$ (c) is Proposition 3.2.

(c) $\Longrightarrow$ (a) Since R is an inertial coefficient ring, A contains an inertial subalgebra T. By Proposition 2.4, $S \subseteq T$. Since S is maximal separable, $S = T$, so S is an inertial subalgebra of A.

$\square$

Let A be a finite-dimensional algebra over a field K. In [5] Bogart defines a separable subalgebra B of A to be a Wedderburn specter if for any field extension K of k for which $(A \otimes_k K)/J(A \otimes_k K)$ is a separable K-algebra it follows that $B \otimes_k K$ is a Wedderburn factor for $A \otimes_k K$ over K. An algebraic closure $\overline{k}$ of k is always an extension of k for which $(A \otimes_k \overline{k})/J(A \otimes_k \overline{k})$ is $\overline{k}$-separable, so we can always check to see whether B is a specter for A over k by checking to see whether $B \otimes_k \overline{k}$ is a Wedderburn factor of $A \otimes_k \overline{k}$. In addition, Bogart proved that B is a specter for A over k if and only if A is purely inseparable over B, which provides us with an internal characterization of specters.

If we wish to extend this definition of specter to a finitely generated algebra A over a commutative ring R, we need to find an extension P of R for which $(A \otimes_R P)/J(A \otimes_R P)$ is P-separable and then look for a subalgebra B of A for which $B \otimes_R P$ is an inertial subalgebra of $A \otimes_R P$. We know by Proposition 3.7 that $A \otimes_R P$ must be purely inseparable over $B \otimes_R P$, so if we hope to characterize a specter B of A as a separable subalgebra for which A is purely inseparable over B, then it would be useful to find a P which is finitely generated and faithfully flat over R so that Proposition 3.1(c) holds. The next set of results will show that if R is a commutative, semi-local ring, there is always a finitely generated, commutative, free R-algebra P such that $(A \otimes_R P)/J(A \otimes_R P)$ is P-separable. We begin with two lemmas.

Lemma 3.8: Let R be a commutative ring, A a finitely generated R-algebra, and S a finitely generated commutative R-algebra. If $A/J(A)$ is separable over R, then $(A \otimes_R S)/J(A \otimes_R S)$ is separable over S.

Proof: By [4, Theorem 10, p. 127] $i(J(A) \otimes_R S) \subseteq J(A \otimes_R S)$, so $(A \otimes_R S)/J(A \otimes_R S)$ is a homomorphic image of $(A \otimes_R S)/i(J(A) \otimes_R S)$ $\simeq [A/J(A)] \otimes_R S$. By Property 1.14, $[A/J(A)] \otimes_R S$ is S-separable, and it follows from Property 1.13 that $(A \otimes_R S)/J(A \otimes_R S)$ is S-separable.

□

Lemma 3.9: Let R be a commutative ring, S a commutative, finitely generated R-algebra, and A a finitely generated, separable S-algebra. If $S/J(S)$ is R-separable, then $A/J(A)$ is R-separable.

Proof: By Lemma 1.1 $J(S) \cdot A \subseteq J(A)$, so $A/J(A)$ is a homomorphic image of $A/J(S) \cdot A$. But $A/J(S) \cdot A$ is a separable $S/J(S)$-algebra, so by Property 1.15 $A/J(S) \cdot A$ is a separable R-algebra, and it follows that $A/J(A)$ is R-separable.

□

The next two results, Propositions 3.10 and 3.11, show that if A is a finitely generated algebra over a commutative, semilocal ring R, then we can construct a finitely generated, free, commutative R-algebra P with $(A \otimes_R P)/J(A \otimes_R P)$ separable over P. Proposition 3.10 is the special case where R is a field k, and here P is actually a finite field extension of k. Since this result is used to prove Proposition 3.11, the finiteness of P is important, which explains why we cannot use $\overline{k}$, the algebraic closure of k, for P. In Proposition 3.10 P is shown to be a purely inseparable extension of k, but it is not known whether the P constructed in Proposition 3.11 is purely inseparable over R.

The proof of Proposition 3.10 is in five steps. A is successively a finite field extension of k, a finite-dimensional division ring over k, a matrix ring over a division ring which is finite-dimensional over k, a finite-dimensional, semisimple k-algebra, and, finally, an arbitrary, finite-dimensional k-algebra.

Proposition 3.10: Let A be a finite-dimensional algebra over a field k. Then there exists a finite, purely inseparable, field extension P of k such that $(A \otimes_k P)/J(A \otimes_k P)$ is separable over P.

Proof: Step 1: Let A be a finite field extension of k. Then there are elements $a_1, a_2, \ldots, a_n$ such that $A = k(a_1, a_2, \ldots, a_n)$. Let $f_i(x)$ denote the minimal polynomial in $k[x]$ for a_i, and let F be a splitting field for the family $\{f_i(x)\}_{i=1}^{n}$ over k. Then F is a finite normal extension of k containing A, and it follows that there is a purely inseparable field extension P of k contained in F with F separable over P.

If $\mu : P \otimes_k P \longrightarrow P$ is the multiplication map, then $\ker \mu \subseteq J(P \otimes_k P)$, since P is purely inseparable over k. Therefore, $(P \otimes_k P)/J(P \otimes_k P)$ is a homomorphic image of $(P \otimes_k P)/\ker \mu \simeq P$, which is P-separable, and consequently, $(P \otimes_k P)/J(P \otimes_k P)$ is P-separable. Furthermore, since F is separable over P, then $F \otimes_P (P \otimes_k P) \simeq F \otimes_k P$ is separable over $P \otimes_k P$ by Property 1.14. Thus if we take $F \otimes_k P$ to be A, $P \otimes_k P$ to be S and P to be R in Lemma 3.9, we conclude that $(F \otimes_k P)/J(F \otimes_k P)$ is P-separable.

Now $A \otimes_k P \subseteq F \otimes_k P$, and by Proposition 1.2 $J(F \otimes_k P) \cap (A \otimes_k P) \subseteq J(A \otimes_k P)$. Therefore, $(A \otimes_k P)/J(A \otimes_k P)$ is a homomorphic image of $(A \otimes_k P)/[J(F \otimes_k P) \cap (A \otimes_k P)]$, which is isomorphic to a subalgebra of $(F \otimes_k P)/J(F \otimes_k P)$. Since homomorphisms preserve separability, if we can show that all subalgebras of $(F \otimes_k P)/J(F \otimes_k P)$

are P-separable, then it will follow that $(A \otimes_k P)/J(A \otimes_k P)$ is P-separable, and we will be done.

Let B denote the separable P-algebra $(F \otimes_k P)/J(F \otimes_k P)$. B is commutative, semisimple, and satisfies the descending chain condition on ideals, so by the Wedderburn structure theorems it is a direct sum of fields, say $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ where each B_i is a field extension of P. In fact, Theorem 1.8 yields that each B_i is a separable field extension of P. Let C be any P-subalgebra of B, and let G be any field extension of P. Then $C \otimes_P G$ is a subalgebra of $B \otimes_P G \simeq (B_1 \otimes_P G) \oplus (B_2 \otimes_P G) \oplus \cdots \oplus (B_m \otimes_P G)$, and for each i, $B_i \otimes_P G$ has no nonzero nilpotent elements, since $B_i \otimes_P G$ is G-separable. Hence, $B \otimes_P G$ has no nonzero nilpotent elements, and so $J(C \otimes_P G) = 0$. Therefore, C is separable over P, and we are done.

<u>Step 2</u>: Let A be a finite-dimensional division algebra over k, and let C be the center of A. By Step 1 there is a finite, purely inseparable, field extension P of k such that $(C \otimes_k P)/J(C \otimes_k P)$ is P-separable. A is a central simple C-algebra, so by [11, Lemma 4.1.1, p. 90] A is a separable C-algebra, and it follows that $A \otimes_C (C \otimes_k P) \simeq A \otimes_k P$ is separable over $C \otimes_k P$. Then Lemma 3.9 yields that $(A \otimes_k P)/J(A \otimes_k P)$ is separable over P.

<u>Step 3</u>: Let A be a matrix ring over a division ring D, say $A = M_n(D)$, where D is finite-dimensional over k. By Step 2 there exists a finite, purely inseparable, field extension P of k such that $(D \otimes_k P)/J(D \otimes_k P)$ is separable over P. Since $M_n(D) \otimes_k P \simeq M_n(D \otimes_k P)$ and $J(M_n(D \otimes_k P)) \simeq M_n(J(D \otimes_k P))$, we have
$[M_n(D) \otimes_k P]/J[M_n(D) \otimes_k P] \simeq M_n(D \otimes_k P)/M_n(J(D \otimes_k P)) \simeq M_n((D \otimes_k P)/J(D \otimes_k P))$.
If K is any field extension of P, then $[(D \otimes_k P)/J(D \otimes_k P)] \otimes_P K$ is

semisimple, because $(D \otimes_k P)/J(D \otimes_k P)$ is P-separable. It follows that $M_n[((D \otimes_k P)/J(D \otimes_k P)) \otimes_P K] \simeq M_n[(D \otimes_k P)/J(D \otimes_k P)] \otimes_P K$ is semisimple, so $M_n[(D \otimes_k P)/J(D \otimes_k P)] \simeq (M_n(D) \otimes_k P)/J(M_n(D) \otimes_k P)$ is P-separable.

Step 4: Let A be a finite-dimensional semisimple k-algebra. Then A is a direct sum of matrix rings over division rings, say $A \simeq M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_m}(D_m)$, where each D_i is finite-dimensional over k by the Wedderburn structure theorems. By Step 3 we have for each i a finite, purely inseparable, field extension P_i of k such that $[M_{n_i}(D_i) \otimes_k P_i]/J[M_{n_i}(D_i) \otimes_k P_i]$ is P_i-separable. Then $P_i = k(a_{i1}, a_{i2}, \ldots, a_{im_i})$, where each a_{ij} is purely inseparable over k.

Let $f_{ij}(x)$ denote the minimal polynomial for a_{ij} in $k[x]$, and let K be a splitting field for the family $\{f_{ij}(x)\}$ of polynomials. Then K is a finite normal extension of k, so that K contains a maximal purely inseparable field extension P of k. Furthermore, P contains an isomorphic copy of each P_i.

Consider $A \otimes_k P \simeq (M_{n_1}(D_1) \otimes_k P) \oplus \cdots \oplus (M_{n_m}(D_m) \otimes_k P)$. By Proposition 1.3, the radical of a direct sum of algebras over a field is the direct sum of the radicals. Therefore, $(A \otimes_k P)/J(A \otimes_k P) \simeq$ $(M_{n_1}(D_1) \otimes_k P)/J(M_{n_1}(D_1) \otimes_k P) \oplus \cdots \oplus (M_{n_m}(D_m) \otimes_k P)/J(M_{n_m}(D_m) \otimes_k P)$, and, moreover, tensoring up by any field extension of P preserves the semisimplicity of the sum if and only if it preserves the semisimplicity of the factors. Consequently, $(A \otimes_k P)/J(A \otimes_k P)$ will be P-separable if and only if $(M_{n_i}(D_i) \otimes_k P)/J(M_{n_i}(D_i) \otimes_k P)$ is P-separable for each i. But the latter follows by taking $R = P_i$, $S = P$, and $A = M_{n_i}(D_i) \otimes_k P_i$ in Lemma 3.8.

Step 5: Let A be a finite-dimensional k-algebra. Then $A/J(A)$ is semisimple, and by Step 4 there is a finite, purely inseparable, field extension P of k with $[(A/J(A)) \otimes_k P]/J[(A/J(A)) \otimes_k P]$ separable over P. Consider the following commutative diagram.

$$\begin{tikzcd}
A \otimes_k P \arrow[r, "\beta"] \arrow[d, "\alpha"'] & (A \otimes_k P)/J(A \otimes_k P) \arrow[d, "\delta"] \\
(A/J(A)) \otimes_k P \arrow[r, "\gamma"] & [(A/J(A)) \otimes_k P]/J[(A/J(A)) \otimes_k P]
\end{tikzcd}$$

$J(A)$ is nilpotent, so $J(A) \otimes_k P \subseteq J(A \otimes_k P)$, and $\ker \alpha \subseteq \ker \beta = J(A \otimes_k P)$. Therefore, α maps the radical of $A \otimes_k P$ to the radical of $(A/J(A)) \otimes_k P$, or we can write $\alpha(\ker \beta) = \ker \gamma$, so δ is an isomorphism, and it follows that $(A \otimes_k P)/J(A \otimes_k P)$ is P-separable.

□

It is now possible to use Proposition 3.10 to construct P with the desired properties when A is a finitely generated algebra over a commutative semilocal ring R.

Proposition 3.11: Let A be a finitely generated algebra over a commutative semilocal ring R. Then there exists a finitely generated, free, commutative R-algebra P such that $(A \otimes_R P)/J(A \otimes_R P)$ is separable over P.

Proof: If $\overline{R} = R/J(R)$, then $\overline{R} = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where the F_i are fields, since R is semilocal. If $\overline{A} = A/(J(R) \cdot A)$, then $\overline{A} = F_1\overline{A} \oplus \cdots \oplus F_n\overline{A}$. For each i $F_i\overline{A}$ is a finite-dimensional algebra over the field F_i, so by Proposition 3.10 there is a finite, purely inseparable, field extension P_i of F_i such that $(F_i\bar{A} \otimes_{F_i} P_i)/J(F_i\bar{A} \otimes_{F_i} P_i)$ is P_i-separable.

We now construct an algebra P over R for which $(A \otimes_R P)/J(A \otimes_R P)$ is separable over P. Since each P_i is a finite field extension of F_i, write $P_i = F_i(a_{i1}, a_{i2}, \ldots, a_{ik_i})$. Let $F_{11}(x_{11})$ be the minimal polynomial for a_{11} in $F_1[x_{11}]$. There is a natural map of polynomial rings $\phi_{11} : R[x_{11}] \longrightarrow F_1[x_{11}]$, so we can choose $\tilde{f}_{11}(x_{11})$ to be a monic element of $\phi_{11}^{-1}(f_{11}(x_{11}))$. Define $\tilde{P}_{11} = R[x_{11}]/(\tilde{f}_{11}(x_{11}))$, and notice that ϕ_{11} induces a natural map from $\tilde{P}_{11}$ to $F_1(a_{11})$.

Now let $f_{12}(x_{12})$ be the minimal polynomial for a_{12} in $F_1(a_{11})[x_{12}]$. Let $\phi_{12} : \tilde{P}_{11}[x_{12}] \longrightarrow F_1(a_{11})[x_{12}]$ be the natural map, and let $\tilde{f}_{12}(x_{12})$ be a monic element of $\phi_{12}^{-1}(f_{12}(x_{12}))$. Define $\tilde{P}_{12} = \tilde{P}_{11}[x_{12}]/(\tilde{f}_{12}(x_{12}))$, and notice that ϕ_{12} induces a map from $\tilde{P}_{12}$ to $F_1(a_{11}, a_{12})$. Continue this procedure for $j = 3, 4, \ldots, k_1$. At the final stage we have a map from $\tilde{P}_{1k_1}$ to $P_1 = F_1(a_{11}, a_{12}, \ldots, a_{1k_1})$.

Next let $\phi_{21} : \tilde{P}_{1k_1}[x_{21}] \longrightarrow F_2[x_{21}]$ be defined by letting $\phi_{21}(x_{1j}) = 0$ for $j = 1, \ldots, k_1$, $\phi_{21}(x_{21}) = x_{21}$, and $\phi_{21} : R \longrightarrow F_2$ be the natural map. Let $f_{21}(x_{21})$ be the minimal polynomial for a_{21} in $F_2[x_{21}]$, and choose $\tilde{f}_{21}(x_{21})$ to be a monic element of $\phi_{21}^{-1}(f_{21}(x_{21}))$. Define $\tilde{P}_{21} = \tilde{P}_{1k_1}[x_{21}]/(\tilde{f}_{21}(x_{21}))$, and we have an induced map from $\tilde{P}_{21}$ to $F_2(a_{21})$. Continue on in this way, taking

$$\phi_{ij} : \tilde{P}_{ij}[x_{ij}] \longrightarrow F_i(a_{i1}, \ldots, a_{ij-1})[x_{i1}] \quad \text{where} \quad 2 \le j \le k_i$$

or

$$\phi_{i1} : \tilde{P}_{i-1,k_{i-1}}[x_{ij}] \longrightarrow F_i[x_{i1}] \quad \text{where} \quad j = 1.$$

Then let $P = \tilde{P}_{nk_n}$.

P is a commutative, finitely generated R-algebra which is free over R. Thus, we only need to show that $(A \otimes_R P)/J(A \otimes_R P)$ is separable over P.

Let $\overline{P} = P/J(R) \cdot P$. Then $\overline{P} = F_1\overline{P} \oplus F_2\overline{P} \oplus \cdots \oplus F_n\overline{P}$, and for each i, $F_i\overline{P}$ is a finite-dimensional F_i-algebra containing P_i. Therefore, by Lemma 3.8 $(F_i\overline{A} \otimes_{F_i} F_i\overline{P})/J(F_i\overline{A} \otimes_{F_i} F_i\overline{P})$ is $F_i\overline{P}$-separable. We can write $\overline{A} \otimes_{\overline{R}} \overline{P} = (F_1\overline{A} \oplus \cdots \oplus F_n\overline{A}) \otimes_{(F_1 \oplus \cdots \oplus F_n)} (F_1\overline{P} \oplus \cdots \oplus F_n\overline{P})$ $\simeq (F_1\overline{A} \otimes_{F_1} F_1\overline{P}) \oplus \cdots \oplus (F_n\overline{A} \otimes_{F_n} F_n\overline{P})$. Since $F_i\overline{A} \otimes_{F_i} F_i\overline{P}$ is an ideal of $\overline{A} \otimes_{\overline{R}} \overline{P}$, $J(F_i\overline{A} \otimes_{F_i} F_i\overline{P}) = J(\overline{A} \otimes_{\overline{R}} \overline{P}) \cap (F_i\overline{A} \otimes_{F_i} F_i\overline{P})$, so that $J(F_1\overline{A} \otimes_{F_1} F_1\overline{P}) \oplus \cdots \oplus J(F_n\overline{A} \otimes_{F_n} F_n\overline{P}) \subseteq J(\overline{A} \otimes_{\overline{R}} \overline{P})$. However,

$$\frac{F_1\overline{A} \otimes_{F_1} F_1\overline{P}}{J(F_1\overline{A} \otimes_{F_1} F_1\overline{P})} \oplus \cdots \oplus \frac{F_nA \otimes_{F_n} F_nP}{J(F_n\overline{A} \otimes_{F_n} F_n\overline{P})}$$ is a direct sum of semisimple

Artinian rings and is itself semisimple. Thus, $J(\overline{A} \otimes_R \overline{P}) \simeq J(F_1\overline{A} \otimes_{F_1} F_1\overline{P}) \oplus \cdots \oplus J(F_n\overline{A} \otimes_{F_n} F_n\overline{P})$, and

$$\frac{\overline{A} \otimes_{\overline{R}} \overline{P}}{J(\overline{A} \otimes_{\overline{R}} \overline{P})} \simeq \frac{F_1A \otimes_{F_1} F_1P}{J(F_1\overline{A} \otimes_{F_1} F_1\overline{P})} \oplus \cdots \oplus \frac{F_n\overline{A} \otimes_{F_n} F_n\overline{A}}{J(F_n\overline{A} \otimes_{F_n} F_n\overline{A})}.$$ By Property 1.16

$(\overline{A} \otimes_{\overline{R}} \overline{P})/J(\overline{A} \otimes_{\overline{R}} \overline{P})$ is separable over $\overline{P}$.

Consider the following commutative diagram.

$$\begin{array}{ccc} A \otimes_R P & \longrightarrow & (A \otimes_R P)/J(A \otimes_R P) \\ \pi_1 \big\downarrow & & \big\downarrow \pi_2 \\ \overline{A} \otimes_{\overline{R}} \overline{P} & \longrightarrow & (\overline{A} \otimes_{\overline{R}} \overline{P})/J(\overline{A} \otimes_{\overline{R}} \overline{P}) \end{array}$$

Since $A \otimes_R P$ is finitely generated over R, $J(R) \cdot (A \otimes_R P) \subseteq J(A \otimes_R P)$

by Lemma 1.1. However, $J(R)\cdot(A\otimes_R P) = \ker \pi_1$ so $J(\overline{A}\otimes_{\overline{R}}\overline{P}) = \pi_1(J(A\otimes_R P))$, and it follows that π_2 is an isomorphism. Since $(\overline{A}\otimes_{\overline{R}}\overline{P})/J(\overline{A}\otimes_{\overline{R}}\overline{P})$ is P-separable, $(A\otimes_R P)/J(A\otimes_R P)$ is P-separable.

□

We can now extend Bogart's definition of Wedderburn specter.

Definition 3.12: Let A be a finitely generated algebra over a commutative semilocal ring R with separable subalgebra B. B is said to be a Wedderburn specter (or specter) for A over R if $B\otimes_R P$ is an inertial subalgebra of $A\otimes_R P$ for every finitely generated, faithfully flat, commutative R-algebra such that $(A\otimes_R P)/J(A\otimes_R P)$ is P-separable. (Such a P exists by Proposition 3.11.)

Now we establish a connection between pure inseparability and Wedderburn specters in the case where A is commutative.

Proposition 3.13: Let A be a finitely generated commutative algebra over the commutative semilocal ring R, and let B be a separable subalgebra of A. Then B is a specter for A over R if and only if A is purely inseparable over B.

Proof: By Proposition 3.11, there is a finitely generated, free, commutative R-algebra P such that $(A\otimes_R P)/J(A\otimes_R P)$ is separable over P. $B\otimes_R P$ is separable over P by Property 1.14. A is purely inseparable over B if and only if $A\otimes_R P$ is purely inseparable over $B\otimes_R P$ by Proposition 3.1. Furthermore, Proposition 3.7 yields that $A\otimes_R P$ is purely inseparable over $B\otimes_R P$ if and only if $B\otimes_R P$ is an inertial subalgebra of $A\otimes_R P$. Thus, we have shown that A is purely inseparable over B if and only if B is a specter of A.

□

<u>Remark</u>: Bogart has shown that not all maximal separable subalgebras of A are specters of A, where A is a finite-dimensional algebra over a field.

The remaining results give some properties of specters.

<u>Proposition 3.14</u>: Let R be a commutative, semilocal, inertial coefficient ring. Let A be a commutative, finitely generated algebra over R and $\overline{A} = A/J(A)$. Then A has a specter if and only if $\overline{A}$ does.

<u>Proof</u>: Let $\pi : A \longrightarrow \overline{A}$ be the canonical map.

($\Longrightarrow$) Let B be a specter for A, and let $\overline{B} = \pi(B)$. $\overline{B}$ is separable by Property 1.13, and $\overline{A}$ is purely inseparable over $\overline{B}$ by Proposition 3.1. Then Proposition 3.13 gives that $\overline{B}$ is a specter for $\overline{A}$.

($\Longleftarrow$) <u>Step 1</u>: Assume R is local with maximal ideal m, and let $\tilde{B}$ be a specter for $\overline{A}$. Then $\tilde{B}$ is finitely generated over R by Property 1.17, so we can write $\tilde{B} = R\tilde{a}_1 + R\tilde{a}_2 + \cdots + R\tilde{a}_n$. Let a_i be a preimage of $\tilde{a}_i$ in A for each i. Each a_i is integral over R [2, Proposition 5.1, p. 59], so we can construct a finitely generated R-subalgebra B' of A with the property that $\pi(B') = \tilde{B}$ in the following way. Let $f_1(x_1)$ be a monic polynomial for a_1 over R, and take $B_1 = R[x_1]/(f(x_1))$. Continue inductively, so that $B_i = B_{i-1}[x_i]/(f_i(x_i))$, where $f_i(x_i)$ is a monic polynomial for a_i over R. Then map B_n into A by taking x_i to a_i. Let B' be the image of B_n in A under this map. Then B' is finitely generated over R because B_n is, and $\pi(B') = \tilde{B}$.

By Proposition 1.2, $J(A) \cap B' \subseteq J(B')$, and since $\widetilde{B} = B'/(J(A) \cap B')$, then $J(\widetilde{B}) = \pi(J(B')) = J(B')/(J(A) \cap B')$. $\overline{A}$ is a finitely generated $\widetilde{B}$-algebra, so by Lemma 1.1, $J(\widetilde{B}) \cdot \overline{A} \subseteq J(\overline{A}) = 0$. Consequently, $J(\widetilde{B}) = 0$, and it follows that $J(A) \cap B' = J(B')$. Moreover, $B'/J(B') = \widetilde{B}$ is R-separable. Since R is an inertial coefficient ring, B' has an inertial subalgebra B; that is, B is separable, and $B + J(B') = B'$. Notice that $\pi(B) = B/(B \cap J(A)) = B/(B \cap J(A) \cap B') = B/(B \cap J(B')) \simeq (B + J(B'))/J(B') = B'/J(B') = \widetilde{B}$.

Now $m \cdot A \subseteq J(A)$, by Lemma 1.1, so if we let $\underline{A} = A/mA$, we can let $\pi_1 : A \longrightarrow \underline{A}$ and $\pi_2 : \underline{A} \longrightarrow \overline{A}$ be the natural maps, with $\pi = \pi_2 \circ \pi_1$. Furthermore, if $\underline{B} = \pi_1(B)$, then $\pi_2(\underline{B}) = \widetilde{B}$. We have the following commutative diagram, where L, H, and K are the appropriate kernels.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & A \otimes_B A \arrow[r, "\mu"] \arrow[d, "\pi_1 \otimes \pi_1"] & A \arrow[r] \arrow[d, "\pi_1"] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & \underline{A} \otimes_{\underline{B}} \underline{A} \arrow[r] \arrow[d, "\pi_2 \otimes \pi_2"] & \underline{A} \arrow[r] \arrow[d, "\pi_2"] & 0 \\
0 \arrow[r] & K \arrow[r] & \overline{A} \otimes_{\widetilde{B}} \overline{A} \arrow[r] & \overline{A} \arrow[r] & 0
\end{tikzcd}$$

B is our candidate for a specter of A, so by Proposition 3.13, we need to show that $L = \ker \mu$ is small as an $A \otimes_R A$-submodule of $A \otimes_B A$. If $\ker(\overline{\mu} \circ (\pi_2 \otimes \pi_2) \circ (\pi_1 \otimes \pi_1))$ is small, then L will be small. It is easy to see that $\ker(\overline{\mu} \circ (\pi_2 \otimes \pi_2) \circ (\pi_1 \otimes \pi_1))$ is small if $\ker \overline{\mu}$, $\ker(\pi_2 \otimes \pi_2)$, and $\ker(\pi_1 \otimes \pi_1)$ are all small. $\ker \overline{\mu}$ is small by Proposition 3.13, and since $\underline{A}$ is an R/m-algebra, where R/m is a field, by [5, Proposition 2.10, p. 128], $\ker(\pi_2 \otimes \pi_2)$ is small. Thus, we only need to show that $\ker(\pi_1 \otimes \pi_1)$ is small.

It is not hard to see that $\ker(\pi_1 \otimes \pi_1) = m \cdot (A \otimes_B A)$. Suppose there is an $A \otimes_R A$-submodule of $A \otimes_B A$ such that $M + \ker(\pi_1 \otimes \pi_1) = A \otimes_B A$. Then $M + m \cdot A \otimes_B A = A \otimes_B A$. If we view these modules as R-modules, it follows from Corollary 1.6 that $M = A \otimes_B A$. Therefore, $\ker(\pi_1 \otimes \pi_1)$ is small, and we are done.

Step 2: Let R be semilocal, and let $\tilde{B}$ be a specter for $\bar{A}$. By [6, Proposition 2, p. 11], $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where each R_i is local, and, moreover, each R_i is an inertial coefficient ring. Then $A = R_1 A \oplus \cdots \oplus R_n A$, $\bar{A} = R_1 \bar{A} \oplus \cdots \oplus R_n \bar{A}$, $\tilde{B} = R_1 \tilde{B} \oplus \cdots \oplus R_n \tilde{B}$, and $\pi|_{R_i A} : R_i A \longrightarrow R_i \bar{A}$. Furthermore, $R_i \tilde{B} \subseteq R_i A$, and $R_i \tilde{B}$ is R_i-separable by Property 1.16. Proposition 3.1 yields that $R_i \bar{A}$ is purely inseparable over $R_i \tilde{B}$, so apply Step 1 to find an R_i-subalgebra B_i of $R_i A$ which is a specter for $R_i A$. Let $B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, and note that B is separable over R. Thus, we only need to show that A is purely inseparable over B.

We know that $R_i A$ is purely inseparable over B_i, because B_i is a specter for $R_i A$. If we define $\mu_i : R_i A \otimes_{B_i} R_i A \longrightarrow R_i A$ to be the multiplication map, then it is clear that $\mu : A \otimes_B A \longrightarrow A$ can be written as $\mu_1 \oplus \cdots \oplus \mu_n$. We want to show that $\ker(\mu_1 \oplus \cdots \oplus \mu_n)$ is small as an $A \otimes_R A$-module, so let M be an $A \otimes_R A$-submodule of $A \otimes_B A$ with the property that $M + \ker(\mu_1 \oplus \cdots \oplus \mu_n) = A \otimes_B A$. Multiply by R_i to get $R_i M + R_i[\ker(\mu_1 \oplus \cdots \oplus \mu_n)] = R_i \cdot (A \otimes_B A)$, or, equivalently, $R_i M + \ker \mu_i = R_i A \otimes_{S_i} R_i A$. Since $\ker \mu_i$ is small, it follows that $R_i M = R_i A \otimes_{B_i} R_i A$. Then $A \otimes_B A = \bigoplus \sum_{i=1}^{n} R_i A \otimes_{B_i} R_i A = \bigoplus \sum_{i=1}^{n} R_i M = M$. Thus, $\ker(\mu_1 \oplus \cdots \oplus \mu_n)$ is small, and we are done.

□

The next result is a slight modification of Proposition 3.14.

<u>Proposition 3.15</u>: Let R be an inertial coefficient ring and A be a projective, commutative, finitely generated R-algebra. Then A has an R-separable subalgebra B with A purely inseparable over B if and only if $\overline{A} = A/J(A)$ has an R-separable subalgebra $\tilde{B}$ with $\overline{A}$ purely inseparable over $\tilde{B}$. If $\pi : A \longrightarrow \overline{A}$ is the canonical map, then $\pi(B) = \tilde{B}$.

<u>Proof</u>: ($\Longrightarrow$) This is the same as the proof of Proposition 3.14 ($\Longrightarrow$), with $\tilde{B} = \overline{B}$.

($\Longleftarrow$) This is nearly the same as the proof of Proposition 3.14 ($\Longleftarrow$), Step 1. We need to modify that proof by replacing m by $J(R)$ and letting $\underline{A} = A/J(R) \cdot A$. The arguments that $\ker \overline{\mu}$ and $\ker(\pi_1 \otimes \pi_1)$ are small go through as before, so we only need to show that $\ker(\pi_2 \otimes \pi_2)$ is small. $\ker(\pi_2 \otimes \pi_2) = \pi_1(J(A)) \otimes_{\underline{B}} \underline{A} + \underline{A} \otimes_{\underline{B}} \pi_1(J(A))$, and since A is projective over R, $\pi_1(J(A))$ is nilpotent in $\underline{A} \otimes_{\underline{B}} \underline{A}$ by Lemma 1.1. Therefore, $\ker(\pi_2 \otimes \pi_2)$ is nilpotent, so that $\ker(\pi_2 \otimes \pi_2) \subseteq J(\underline{A} \otimes_{\underline{B}} \underline{A})$, and it follows that $\ker(\pi_2 \otimes \pi_2)$ is small.

$\square$

<u>Proposition 3.16</u>: Let A be a finitely generated, commutative algebra over the commutative, semilocal ring R. If B_1 and B_2 are specters for A over R, then $B_1 = B_2$.

<u>Proof</u>: Let P be a finitely generated, free, commutative R-algebra such that $(A \otimes_R P)/J(A \otimes_R P)$ is P-separable. The existence of P is proven in Proposition 3.11. Since B_1 and B_2 are specters for A over R, then $B_1 \otimes_R P$ and $B_2 \otimes_R P$ are inertial subalgebras of

$A \otimes_R P$ over P. Proposition 2.4 yields $B_1 \otimes_R P = B_2 \otimes_R P$.

Now consider the short exact sequence:

$$0 \longrightarrow B_1 \longrightarrow B_1 + B_2 \longrightarrow (B_1 + B_2)/B_1 \longrightarrow 0.$$

Since P is flat over R, the following is exact:

$$0 \longrightarrow B_1 \otimes_R P \longrightarrow (B_1 + B_2) \otimes_R P \longrightarrow ((B_1 + B_2)/B_1) \otimes_R P \longrightarrow 0.$$

This yields $[(B_1 + B_2) \otimes_R P]/(B_1 \otimes_R P) \simeq ((B_1 + B_2)/B_1) \otimes_R P$, or $[B_1 \otimes_R P + B_2 \otimes_R P]/(B_1 \otimes_R P) \simeq ((B_1 + B_2)/B_1) \otimes_R P$. Since $B_1 \otimes_R P = B_2 \otimes_R P$, then $((B_1 + B_2)/B_1) \otimes_R P = 0$, and because P is faithfully flat over R, $(B_1 + B_2)/B_1 = 0$. Therefore, $B_2 \subseteq B_1$, and by symmetry, $B_1 \subseteq B_2$.

□

Proposition 3.17: Let R be a commutative, semilocal, inertial coefficient ring and A be a finitely generated, commutative algebra over R. Then A has a unique specter B.

Proof: The technique is to find a specter for $\overline{A} = A/J(A)$, and then use Proposition 3.14. $\overline{A}$ is an algebra over $\overline{R} = R/J(R) = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where the F_i are fields. Write $\overline{A} = F_1\overline{A} \oplus F_2\overline{A} \oplus \cdots \oplus F_n\overline{A}$, and notice that $F_i\overline{A}$ is a commutative, finite-dimensional F_i-algebra. Bogart has shown in [5, Corollary 2.13, p. 130] that such an algebra has a (unique) specter; call B_i the specter for $F_i\overline{A}$. Let $\overline{B} = B_1 \oplus \cdots \oplus B_n$, and use the argument given in the proof of Proposition 3.14, Step 2, to show that $\overline{B}$ is a specter for $\overline{A}$ over $\overline{R}$. Proposition 3.14 then gives the existence of a specter B for A over R, and B is unique by Proposition 3.16.

□

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Albert, A. A., Structure of Algebras, American Mathematical Society Colloquium Publication, Volume 24 (1961).

2. Atiyah, M. F., and MacDonald, I. G., Introduction to Commutative Algebra, Addison-Wesley (1969).

3. Auslander, M., and Goldman, O., "The Brauer group of a commutative ring," Transactions of the American Mathematical Society 97 (1960), 367-409.

4. Azumaya, G., "On maximally central algebras," Nagoya Mathematical Journal 2 (1951), 119-150.

5. Bogart, J., "Wedderburn specters and the structure of certain associative algebras," Journal of Algebra 50 (1978), 122-134.

6. Brown, W. C., and Ingraham, E. C., "A characterization of semilocal inertial coefficient rings," Proceedings of the American Mathematical Society 26 (1970), 10-14.

7. DeMeyer, F., and Ingraham, E., Separable Algebras Over Commutative Rings, Lecture Notes In Mathematics, No. 181, Springer-Verlag (1971).

8. Ford, N. S., "Inertial subalgebras of central separable algebras," Proceedings of the American Mathematical Society 60 (1976), 39-44.

9. Goldman, O., "Hilbert rings and the Hilbert Nullstellensatz," Mathematische Zeitschrift, Band 54, Heft 2 (1951), 136-140.

10. Greco, S., "Algebras over nonlocal Hensel rings," Journal of Algebra 8 (1968), 45-59.

11. Herstein, I. N., Noncommutative Rings, Carus Mathematical Monographs, No. 15, Mathematical Association of America (1968).

12. Ingraham, E. C., "Inertial subalgebras of algebras over commutative rings," Transactions of the American Mathematical Society 124 (1966), 77-93.

13. Ingraham, E. C., "On the existence and conjugacy of inertial subalgebras," Journal of Algebra 31 (1974), 547-556.

14. Kirkman, E. E., "Inertial coefficient rings and the idempotent lifting property," Proceedings of the American Mathematical Society 61 (1976), 217-222.

15. Sanders, D., "The dominion and separable subalgebras of finitely generated algebras," Proceedings of the American Mathematical Society 48 (1975), 1-7.

16. Sweedler, M. E., "Purely inseparable algebras," Journal of Algebra 35 (1975), 342-355.

17. Wedderburn, J. H. M., "On hypercomplex numbers," Proceedings of the London Mathematical Society, Series 2, v. 6 (1908), 77-117.

18. Zariski, O., and Samuel, P., Commutative Algebra, V. II, vanNostrand (1960).