KOTHE FAMILIES IN VECTOR LATTICES

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
CHARLES GOOD DENLINGER
1971

This is to certify that the

thesis entitled

KÖTHE FAMILIES IN VECTOR LATTICES

presented by

Charles Good Denlinger

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

John Masterson
Major professor

Date February 22, 1971

O-7639

ABSTRACT

KÖTHE FAMILIES IN VECTOR LATTICES

By

Charles Good Denlinger

In 1934 Köthe and Toeplitz introduced the notion of the α-dual λ^{x} of a vector space λ of real sequences:

$$\lambda^{x} = \{\{y_n\}: \sum|x_n y_n| < \infty, \quad \{x_n\} \varepsilon \lambda\},$$

and defined a "perfect" sequence space λ to be one such that $\lambda = \lambda^{xx}$. In many important cases, the α-dual and the Banach dual of a sequence space coincide. In any case, the ideas suggested by the notion of α-duality have stimulated extensive investigations into the topological and order properties of real sequence spaces. This theory, with its various generalizations, has become known as the theory of Köthe spaces.

In the present thesis we proceed from the original notion of Köthe sequence spaces, along lines of generalization different from those heretofore undertaken. We consider families $[x_i, I]$ of elements x_i in a vector lattice E, indexed by an arbitrary set I. We let $\omega_I(E)$ denote the collection of all such families, and consider certain vector sublattices of $\omega_I(E)$, analogous to the familiar subspaces ϕ, c_0, c, and ℓ_p $(1 \leq p \leq \infty)$ of the space ω of all real sequences. The order properties of these vector sublattices are studied.

Charles Good Denlinger

The notion of an "order summable" family $[x_i, I]$ is of fundamental importance in the development of this thesis, and appears not to have been exploited before, in the manner in which it is used here. Consider the collection $\nabla(I)$ of all finite subsets of I, partially ordered by inclusion, as a directed set. If we let $\{\tau_J\}$ denote the net of sums $\tau_J = \sum_{i\varepsilon J}|x_i|$ $(J \varepsilon \nabla(I))$, then a family $[x_i, I]$ is said to be order summable if the net $\{\tau_J\}$ is order bounded in E; equivalently, $\{\tau_J\}$ is order convergent in the Dedekind completion $\hat{E}$ of E. The collection of all such $[x_i, I]$ generalizes the sequence space ℓ_1, and is denoted $\ell_I^1(E)$.

This concept of summability leads to a generalization of Köthe's α-dual. We embed E in its universal completion $E^{\#}$ and choose a weak order unit 1 in $E^{\#}$. By the work of Vulikh and others we know that there is a unique multiplication operation in $E^{\#}$, relative to 1. We use this multiplication to define the "X-dual" of $\lambda_I(E)$ $\omega_I(E)$ as the collection $\lambda_I^X(E)$ of all $[y_i, I]$ in $\omega_I(E)$ such $[x_iy_i, I]$ is an order summable family of elements of $\hat{E}$ for every $[x_i, I]$ in $\lambda_I(E)$. In addition, we use this multiplication operation in $E^{\#}$ to define the vector lattice $\ell_I^p(E, 1)$ $(1 < p < \infty)$ as the collection of all $[x_i, I]$ in $\omega_I(E)$ such that $[x_i^p, I] \varepsilon \ell_I^1(\hat{E})$. Analogues of the Hölder and Minkowski inequalities are established, and attention is directed to conditions which will guarantee $\phi_I(E) \subseteq \ell_I^p(E, 1) \subseteq \ell_I^q(E, 1) \subseteq m_I(E)$, if $1 \leq p < q < \infty$, and $[\ell_I^p(E, 1)]^X = \ell_I^q(E, 1)$, if $\frac{1}{p} + \frac{1}{q} = 1$. Corresponding to each $y \varepsilon \lambda_I^X(E)$, the mapping y^* from $\lambda_I(E)$ into $\hat{E}$ defined by $y^*(x) = \sum_{i\varepsilon I} x_iy_i$ is shown to be a

positive, order-continuous linear mapping.

Throughout the thesis considerable attention is given to the case in which E is a normed vector lattice. We define ℓ-norms on $\ell_I^1(E)$ and p-norms on $\ell_I^p(E, 1)$, which behave like the usual norms on ℓ_1 and ℓ_p, respectively. Conditions under which these norms are essentially unique are studied. These considerations involve the problem of extending a monotone norm from E to its Dedekind completion $\hat{E}$. Semi-continuity and continuity of the norm on E are relevant.

Another dual, the y-dual of a subset $\lambda_I(E) \subseteq \omega_I(E)$ is defined and discussed briefly in the final section of the thesis. It is similar to the x-dual, but is independent of the choice of unit 1 in $E^{\#}$.

An interesting result of a different nature is obtained at the end of Chapter I. Using an approach to Banach limits for bounded sequences of real numbers, developed in a paper by Simons, we are able to define, and prove an existence theorem for, Banach order limits of order bounded sequences in vector lattices. Several criteria for almost order convergence are developed.

KÖTHE FAMILIES IN VECTOR LATTICES

By

Charles Good Denlinger

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1971

To Gloria and Joey

ACKNOWLEDGMENTS

I wish to thank Professor John J. Masterson for introducing me to the theory of vector lattices, for stimulating me to pursue research in this area, and for his patient guidance along the way. I also wish to thank Professor George W. Crofts for several enlightening conversations. In fact, the original question which led me to define and consider "order summable families" was raised by Professor Masterson in a seminar on nuclear spaces, led by Professor Crofts.

I also wish to express my indebtedness to those many people - family, friends, colleagues, and teachers - whose individual and collective influences have led me to undertake an academic career, with its accompanying responsibilities.

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INTRODUCTION

A <u>vector lattice</u> is a vector space E over the real scalar field $\mathbb{R}$, equipped with a partial order $\leq$ (which is reflexive, anti-symmetric, and transitive) such that

(1) $x \leq y$ implies $x + z \leq y + z$ for all z in E,

(2) $x \geq 0$ implies $\alpha x \geq 0$ for all real $\alpha \geq 0$, and

(3) E is a lattice under $\leq$.

Thus every two-element set $\{x,y\}$ in E has a supremum, denoted $\sup\{x,y\}$ or $x \vee y$, and an infimum, denoted $\inf\{x,y\}$ or $x \wedge y$.

Since its inception around the year 1930, the theory of vector lattices has greatly enriched the field of functional analysis. A brief account of its origins and early contributors may be found in the forward of Vulikh [18] and in the introduction to Zaanen [21]. For expositions of some of the major results of the theory of vector lattices, the reader is referred to Vulikh [18], Luxemburg and Zaanen [6], and Nakano ([7] and [8]).

Vector lattices have been called by various other names; for example, "linear lattices" or "semi-ordered linear spaces" in the works of Nakano, "Riesz spaces" in the works of Bourbaki, Vulikh, Luxemburg and Zaanen, and "K-lineals" in the works of various Russians.

The basic definitions and fundamental properties of vector lattices may be found scattered throughout the literature; for example, see Namioka [9], Peressini [10] or Vulikh [18]. As a convenience to the reader a few of these basic facts will be given here. The set $E^+ = \{x \in E : x \geq 0\}$ is called the positive cone of E, and has the properties $E^+ + E^+ \subseteq E^+$, $\lambda E^+ \subseteq E^+$ for all $\lambda \in R^+$, and $E^+ \cap (-E^+) = \{0\}$. If $x \in E$ we let $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ \vee x^-$; then $x = x^+ - x^-$, $|x| = x^+ + x^-$ and $x^+ \wedge x^- = 0$. If $|x| \wedge |y| = 0$ we say that x and y are disjoint and write $x \perp y$.

Vector lattices are necessarily distributive lattices. If x, y, and z are arbitrary elements of a vector lattice E, the following relations hold:

(4) $-(x \vee y) = (-x) \wedge (-y)$, $-(x \wedge y) = (-x) \vee (-y)$;

(5) $(x \vee y)^+ = x^+ \vee y^+$, $(x \wedge y)^+ = x^+ \wedge y^+$;

(6) $(x \vee y)^- = x^- \vee y^-$, $(x \wedge y)^- = x^- \wedge y^-$;

(7) if $\alpha \in R^+$, then $\alpha(x \vee y) = \alpha x \vee \alpha y$ and $\alpha(x \wedge y) = \alpha x \wedge \alpha y$;

(8) $(x \vee y) + z = (x+z) \vee (y+z)$, $(x \wedge y) + z = (x+z) \wedge (y+z)$;

(9) $x + y = (x \vee y) + (x \wedge y)$, $|x-y| = (x \vee y) - (x \wedge y)$;

(10) $x \vee y = (x-y)^+ + y$, $x \wedge y = x - (x-y)^+$;

(11) $(x+y)^+ \leq x^+ + y^+$, $(x+y)^- \leq x^- + y^-$;

(12) $|x+y| \leq |x| + |y|$, $||x| - |y|| \leq |x-y|$;

(13) $|(x \vee z) - (y \vee z)| \leq |x-y|$, $|(x \wedge z) - (y \wedge z)| \leq |x-y|$;

(14) if $x, y, z \geq 0$, then $(x+y) \wedge z \leq (x \wedge z) + (y \wedge z)$;

(15) $x \leq y$ if and only if $x^+ \leq y^+$ and $x^- \geq y^-$;

(16) $|x| \leq y$ if and only if $-y \leq x \leq y$;

(17) $x \perp y$ if and only if $|x| \vee |y| = |x| + |y|$;

(18) if $x = y - z$ for $y, z \geq 0$, then $x^+ \leq y$ and $x^- \leq z$; if in addition $y \perp z$, then $x^+ = y$ and $x^- = z$.

Proofs for these well-known relations may be found in the literature cited above.

Given vector lattices E and F, a linear isomorphism ψ: E $\to$ F such that $x \geq 0$ if and only if $\psi(x) \geq 0$ is called a vector lattice isomorphism; we then write ψ: E $\simeq$ F. If ψ: E $\to$ F is a vector lattice isomorphism and $A \subseteq E$, then sup A exists in E if and only if sup $\psi(A)$ exists in F; in that case, $\psi(\sup A) = \sup \psi(A)$. The analogous statement for infima also holds. Unless otherwise specified, the term "isomorphism" will, in this thesis, mean "vector lattice isomorphism".

A linear subspace M of a vector lattics E is said to be a vector sublattice of E if, for each x, y in M, the supremum of x and y in E is also in M. It then follows from (4) above that the infimum of x and y in E is also in M.

A (lattice) ideal of a vector lattice E is a linear subspace $I \subseteq E$ such that $y \in I$ whenever $|y| \leq |x|$ for some $x \in I$. An ideal is necessarily a vector sublattice. A vector sublattice M of E is said to be order closed if it contains sup A, for every subset A of M having a supremum in E. An order closed ideal is sometimes called a band.

A vector lattice E is said to be Archimedean if every $x \in E^+$ satisfies the equation $\inf\{\frac{1}{n}x: n \in N\} = 0$; equivalently, if $x, y \in E^+$ such that $\lambda x \leq y$ for all $\lambda > 0$, then $x = 0$. It is well-known (see Vulikh [18], p. 75) that any n-dimensional Archimedean vector lattice is isomorphic to R^n. Every vector sublattice of an Archimedean vector lattice is Archimedean. Any vector lattice of real-valued functions, with the usual pointwise-defined linear operations and

order, is Archimedean.

A vector lattice E is Dedekind complete (resp. Dedekind σ-complete) if every subset (resp. countable subset) of E which is bounded above in E has a supremum in E. Either of these conditions is sufficient to imply that E is Archimedean. Every ideal in a Dedekind complete (resp. σ-complete) vector lattice is also Dedekind complete (resp. σ-complete). A Dedekind complete vector lattice in which every set of pairwise disjoint elements is bounded is said to be universally complete. Any order-closed vector sublattice of a universally complete vector lattice is also universally complete.

A vector lattice E is order separable if every subset A of E^+ having a supremum in E contains a countable subset with the same supremum. Any ideal in an order separable vector lattice is again order separable.

An element 1 in a vector lattice E is called a weak order unit if $x \wedge 1 > 0$ whenever $x > 0$ in E, and is called a strong order unit if for every $x \in E^+$ there exists a scalar $\lambda > 0$ such that $\lambda x \leq 1$.

Given vector lattices E and F, E is said to be order-dense in F if every $f \geq 0$ in F^+ satisfies the condition $f = \sup\{e \in E: 0 \leq e \leq f\}$; E is said to be quasi-order-dense if for each $f > 0$ in F there exists an $e \in E$ such that $0 < e \leq f$. It is well-known that in an Archimedean space F, order-denseness and quasi-order-denseness are equivalent.

We list here a few of the common vector lattices. The space ω of all real sequences, with its usual (coordinate-wise defined) linear operations and order relation, is an example of a universally complete vector lattice. Real Euclidean n-space R^n is isomorphic to an order-closed ideal of ω, and hence R^n is also universally complete. The familiar sequence spaces ℓ_p $(1 \leq p \leq \infty)$, c_0 (the space of all zero-convergent sequences), and ϕ (the space of finite sequences) are ideals in ω; hence ℓ_p $(1 \leq p \leq \infty)$, c_0 and ϕ are Dedekind complete vector lattices. The space ℓ_∞ of bounded sequences is often denoted m. The vector sublattice c, consisting of all convergent real sequences, is not an ideal in ω, and is not Dedekind complete. The vector lattice F(X) of all real-valued functions on an abstract set X, with linear operations and order defined pointwise, is universally complete. A universally complete vector lattice of key interest in current research is the space $\mathfrak{M}(X, \Omega, \mu)$ of equivalence classes of almost-everywhere finite-real-valued, μ-measurable functions on the measure space (X, Ω, μ), again with linear operations and order defined pointwise. The spaces $L_p(X, \Omega, \mu)$ $(1 \leq p \leq \infty)$ are ideals in $\mathfrak{M}(X, \Omega, \mu)$, and hence are Dedekind complete vector lattices. Finally, the space C(X) of continuous real-valued functions defined on a compact Hausdorff space is a vector lattice which need not be Dedekind complete.

If $\{x_\alpha\}$ is a monotone increasing (resp. decreasing) net in a vector lattice E, with supremum (resp. infimum) x_0 in E, then we write $x_\alpha \uparrow x_0$ (resp. $x_\alpha \downarrow x_0$). A net $\{x_\alpha\}$ in E is said to <u>order converge</u> to x_0 in E if it is order bounded and if there exists a net $\{y_\alpha\}$ in E^+ such that $|x_\alpha - x| \leq y_\alpha \downarrow 0$. If $\{x_\alpha\}$ order converges to x_0, we write $x_\alpha \xrightarrow{o} x_0$, or $x_0 = \text{o-}\lim_\alpha x_\alpha$.

If $\{x_\alpha\}$ order converges then it has a unique o-limit, and every subnet of $\{x_\alpha\}$ o-converges to the same o-limit. Order convergence preserves sums, scalar products, suprema and infima; more precisely, if $x_\alpha \xrightarrow{o} x$ and $y_\alpha \xrightarrow{o} y$, then $x_\alpha + y_\alpha \xrightarrow{o} x + y$, $\lambda x_\alpha \xrightarrow{o} \lambda x$ $(\lambda \in R)$, $x_\alpha \vee y_\alpha \xrightarrow{o} x \vee y$, and $x_\alpha \wedge y_\alpha \xrightarrow{o} x \wedge y$. Given a bounded net $\{x_\alpha\}$ in a Dedekind complete space, we define $\overline{\lim_\alpha}\, x_\alpha = \inf_\alpha \sup_{\beta \geq \alpha} x_\beta$ and $\underline{\lim_\alpha}\, x_\alpha = \sup_\alpha \inf_{\beta \geq \alpha} x_\beta$; then $x_\alpha \xrightarrow{o} x$ if and only if $\overline{\lim_\alpha}\, x_\alpha = x = \underline{\lim_\alpha}\, x_\alpha$.

It is easily seen that if I is an ideal in a vector lattice E and $\{x_\alpha\}$ is an order bounded net in I, then $x_\alpha \xrightarrow{o} x$ in E if and only if $x_\alpha \xrightarrow{o} x$ in I.

A sequence $\{x_n\}$ in a vector lattice E is said to <u>order *-converge</u> to an element $x \in E$ (denoted $x_n \xrightarrow{*} x$) if every subsequence of $\{x_n\}$ has a subsequence that order-converges to x. The properties of order convergence described in the preceding two paragraphs hold for order *-convergence as well.

If we consider the vector lattice $\mathfrak{M}$ of equivalence classes of finite (a.e.) Lebesgue-measurable functions on $[0,1]$, with the usual order, then order convergence is equivalent to convergence almost everywhere and order *-convergence is equivalent to convergence in measure.

A net $\{x_\alpha\}$ in a vector lattice E is said to be <u>y-uniformly convergent</u> to an element $x \in E$ (for a given $y \in E$) if for every positive real number δ, there exists α_0 such that $|x_\alpha - x| \leq \delta y$ for all $\alpha \geq \alpha_0$. A net $\{x_\alpha\}$ in E is said to be <u>uniformly convergent</u> if it is y-uniformly convergent for some $y \in E$; in symbols, $x_\alpha \xrightarrow{u} x$. In an Archimedean vector lattice, uniform convergence implies order convergence.

Many of the vector lattices studied in functional analysis are also normed spaces. If $\|.\|$ is a norm on a vector lattice E such that $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, then $\|.\|$ is called a <u>monotone norm</u>, and $(E, \leq, \|.\|)$ is said to be a <u>normed vector lattice</u>. A <u>Banach lattice</u> is a norm-complete normed vector lattice; if E is a Banach lattice, E^+ is necessarily norm-closed. An <u>M-space</u> is a Banach lattice in which $x, y \geq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$; an <u>L-space</u> is a Banach lattice in which $x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$. In his famous paper [3] Kakutani proved that every L-space is isomorphic and isometric to $L_1(X, \Omega, \mu)$ for some locally compact Hausdorff space X and some positive Radon measure μ defined on X. In another famous paper [4] Kakutani proved that every M-space is isomorphic and isometric to a vector sublattice of C(X) for some compact Hausdorff space X. Vulikh ([18], theorem VII.7.1) has shown that every L-space is Dedekind complete and has the property that $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$.

In the present thesis it is shown that the theory of (real) Köthe sequence spaces (see Köthe [5], § 30) can be generalized in a meaningful way to spaces of families of elements in a vector lattice. Given a vector lattice E and a nonempty set I, the notation $[x_i, I]$ will denote a family of elements x_i of E, indexed by the set I. This is a generalization of the notion of a sequence $[x_n, N]$ in E. Accordingly, the question arises as to whether it is possible and fruitful to define spaces of families in E analogous to the familiar sequence spaces ω, ϕ, c_0, c, and ℓ_p $(1 \leq p \leq \infty)$, using only the theory

of vector lattices. It is this question which gave rise to the author's investigations, culminating in this thesis.

The basic spaces $\omega_I(E)$, $\phi_I(E)$ and $m_I(E)$ generalizing ω, ϕ and m, respectively, are introduced in the beginning of Chapter I. A large portion of Chapter I is devoted to an investigation of the order properties of these, as well as of more general, vector sublattices $\lambda_I(E)$ of $\omega_I(E)$. Order convergence and order *-convergence in $\lambda_I(E)$ are related to order convergence in E. Possession by E of properties such as the Archimedean property, Dedekind completeness, universal completeness and order separability is related to the possession by $\lambda_I(E)$ of the same properties. Dedekind and universal completions are discussed in this context.

Chapter I ends with an application, somewhat unrelated to the remaining material in the thesis. A paper [16] of S. Simons is extended to yield a definition of, and an existence theorem for, "Banach o-limits" in vector lattices. Several criteria for "almost o-convergence" are developed.

Chapter II introduces and develops a theory of order summable families in a vector lattice E. Of primary interest are families $[x_i, I]$ for which there exists an element $u \in E$ such that for every finite subset $J \subseteq I$, $\sum_{i \in J} |x_i| \leq u$. The collection of all such families is a generalization of ℓ_1, and will be denoted $\ell^1{}_I(E)$. For each $[x_i, I]$ in $\ell^1{}_I(E)$ there exists a "sum" $\sum_{i \in I} |x_i|$ in $\hat{E}$, the Dedekind completion of E. This sum is seen to be an order continuous function of $[x_i, I]$. Several other types of summable families are considered.

In Chapter III representation theory, universal completion, and multiplication in Archimedean vector lattices are used to generalize the theory of ℓ_p spaces and Köthe dual sequence spaces to the context of Archimedean vector lattices. For $1 \leq p < \infty$, the space $\ell^p{}_I(E,1)$ is defined as $\{[x_i, I] \in \omega_I(E): \sum_{i \in I} |x_i|^p \leq u$, for some $u \in E\}$, where the multiplication is performed in $E^{\#}$, the universal completion of E, relative to a fixed weak order unit 1 in $E^{\#}$. Analogues of the Hölder and Minkowski inequalities are established, and some attention is directed to conditions which will guarantee $\phi_I(E) \subseteq \ell^p{}_I(E,1) \subseteq \ell^q{}_I(E,1) \subseteq m_I(E)$ if $1 \leq p < q < \infty$.

Given a weak order unit 1 in $E^{\#}$, the Kothe X-dual of a vector sublattice $\lambda_I(E)$ of $\omega_I(E)$ is the ideal $\lambda_I^X(E) = \{[y_i, I] \in \omega_I(E): [x_i y_i, I] \in \ell^1{}_I(\hat{E}), \forall [x_i, I] \in \lambda_I(E)\}$. Attention is given to conditions which will guarantee the duality relationships among the aforementioned spaces which one is led to expect from their analogous real sequence spaces. Corresponding to each $y \in \lambda_I^X(E)$, the mapping y^* from $\lambda_I(E)$ into E is defined $y^*(x) = \sum_{i \in I} x_i y_i$; y^* is shown to be a positive order-continuous linear mapping.

Another type of dual, the Köthe y-dual of a vector sublattice $\lambda_I(E)$ of $\omega_I(E)$, is introduced and discussed briefly in the final section of Chapter III. Its chief advantage lies in its independence of the choice of weak unit 1 in $E^{\#}$.

Throughout the thesis considerable attention is given to the case in which $(E, \leq, \|.\|)$ is a normed vector lattice. Norm-bounded and order-bounded families $[x_i, I]$ are discussed and related in Chapter I. If $(E, \leq, \|.\|)$ is a normed vector lattice, a monotone norm $\|.\|_\ell$ on $\ell^1{}_I(E)$ such that (1) $x_i = 0$ for all but finitely many $i \in I$ implies

$\|[x_i, I]\|_\ell = \|\sum_{i\varepsilon I}|x_i|\|$, and (2) $\sum_{i\varepsilon J}|x_i| \leq y$ for every finite subset $J \subseteq I$ implies $\|[x_i, I]\|_\ell \leq \|y\|$, is called an ℓ-norm on $\ell^1{}_I(E)$. In Chapter II it is shown that for every normed vector lattice E there exists a minimal and a maximal ℓ-norm on $\ell^1{}_I(E)$. Conditions under which $\ell^1{}_I(E)$ has a unique ℓ-norm are considered. These considerations involve the problem of extending a monotone norm from E to $\hat{E}$, as recently studied by Solov'ev [17] and Reichard [12]. Semi-continuity and continuity of the norm $\|.\|$ on E are utilized in this study. In Chapter III this line of study continues, with the definition and discussion of p-norms and p-additive p-norms on $\ell^p{}_I(E, 1)$.

CHAPTER I

GENERAL FAMILIES IN A VECTOR LATTICE

Section 1. The fundamental spaces $\omega_I(E)$, $m_I(E)$, and $\phi_I(E)$.

We begin with a vector lattice E and an arbitrary nonempty index set I. The notation $[x_i, I]$ will be used to denote a family of elements x_i in E indexed by the set I; strictly speaking, $[x_i, I]$ could be regarded as a mapping from I into E. Proceeding in analogy with sequence spaces, we define the spaces

$$\omega_I(E) = \{[x_i, I]: x_i \in E \ \forall i \in I\},$$

$$m_I(E) = \{[x_i, I] \in \omega_I(E): \exists\, u \in E \text{ such that } |x_i| \leq u \ \forall i \in I\},$$

and

$$\phi_I(E) = \{[x_i, I] \in \omega_I(E): x_i = 0 \text{ for all but finitely many } i\}.$$

The elements of $m_I(E)$ will be called <u>(order)-bounded families</u>, and the elements of $\phi_I(E)$, <u>finite families</u>.

The proof of the following proposition is straightforward and easy; it will be omitted.

<u>(1.1) Proposition</u>. If E is a vector lattice and I is an arbitrary index set, then $\omega_I(E)$ is a vector lattice, relative to the definitions

$$[x_i, I] + [y_i, I] = [x_i + y_i, I],$$

$$\lambda[x_i, I] = [\lambda x_i, I] \qquad (\lambda \in R),$$

$$[x_i, I] \leq [y_i, I] \Leftrightarrow x_i \leq y_i \ \forall i \in I.$$

In the vector lattice $\omega_I(E)$ it turns out that

$$[x_i, I] \vee [y_i, I] = [x_i \vee y_i, I] \text{ and}$$

$$[x_i, I] \wedge [y_i, I] = [x_i \wedge y_i, I].$$

Moreover, $m_I(E)$ and $\phi_I(E)$ are ideals in $\omega_I(E)$.

Note that in case E= $\mathbb{R}$ and I = $\mathbb{N}$ (the set of natural numbers), the spaces $\omega_I(E)$, $m_I(E)$ and $\phi_I(E)$ coincide with the usual sequence spaces ω, m and ϕ, respectively. This case also serves to show that $m_I(E)$ is not necessarily a band in $\omega_I(E)$. The set $\{(1, 2, \ldots, n, 0, 0, \ldots): n = 1, 2, \ldots\}$ is a subset of $m_{\mathbb{N}}(\mathbb{R})$ having supremum $(1, 2, \ldots, n, n+1, \ldots)$ in $\omega_{\mathbb{N}}(\mathbb{R})$, but this supremum does not belong to $m_{\mathbb{N}}(\mathbb{R})$. Similarly, consideration of the set $\{e_n: n = 1, 2, \ldots\}$ of sequences $e_n = (0, 0, \ldots, 1, 0, \ldots)$, whose n^{th} term is 1, shows that $\phi_I(E)$ is not necessarily a band in $m_I(E)$ or $\omega_I(E)$.

For any vector lattice E and any $i_0 \in I$, it is evident that the set

$$\alpha_{I,i_0}(E) = \{[x_i, I] \in \omega_I(E): x_i = 0 \ \forall\, i \neq i_0\}$$

is a vector sublattice of $\omega_I(E)$ isomorphic to E.

The proofs of the following two propositions are straightforward, and not very instructive for our purposes; they will thus be omitted.

<u>(1.2) Proposition</u>. If E and F are isomorphic vector lattices, then there exists an isomorphism $\psi: \omega_I(E) \rightarrow \omega_I(F)$ whose restrictions to $m_I(E)$ and $\phi_I(E)$, respectively, are isomorphisms onto $m_I(F)$ and $\phi_I(F)$.

<u>(1.3) Proposition</u>. If E is a vector lattice, and I and J are

two sets of the same cardinality, then there exists an isomorphism $\psi: \omega_I(E) \to \omega_J(E)$ whose restrictions to $m_I(E)$ and $\phi_I(E)$, respectively, are isomorphisms onto $m_J(E)$ and $\phi_J(E)$.

The converse of (1.2) is false. As a counterexample, consider the spaces $\omega_N(R)$ and $\omega_N(R^2)$. The map $[(x_{n1}, x_{n2}), N] \to [x_{(n,i)}, N\times\{1, 2\}]$ is clearly an isomorphism from $\omega_N(R^2)$ onto $\omega_{N\times\{1,2\}}(R)$, and by (1.3) the latter space is isomorphic to $\omega_N(R)$. Thus $\omega_N(R) \simeq \omega_N(R^2)$. However, $R \neq R^2$.

Suppose E is a vector lattice on which there is defined a norm $\|.\|$. We call a family $[x_i, I]$ in $\omega_I(E)$ <u>norm-bounded</u> if there is a positive real number M such that $\|x_i\| \leq M$ for all $i \in I$. In general, there is no relationship between the order-boundedness and norm-boundedness of a family $[x_i, I]$, as the following examples will demonstrate.

Consider the vector lattice ϕ of finite real sequences, with its usual order and "sup" norm. For each $n \in N$ we use the customary notation e_n to denote the "n^{th} basis unit vector" $e_n = (0, 0, \ldots, 1, 0, \ldots)$ whose k^{th} term is 1 if $k = n$, 0 if $k \neq n$. In $\omega_N(\phi)$ the family $[e_n, N]$ is norm-bounded but not order-bounded.

Let $BV_0[0,1]$ denote the vector space of all real-valued functions f of bounded variation on the unit interval $[0,1]$such that $f(0) = 0$. The set $\{f \in BV_0[0,1]: f(t) \geq 0 \ \ \forall\, 0 \leq t \leq 1\}$ is a positive cone, inducing a partial order under which $BV_0[0,1]$ is a vector lattice. We take the total variation as norm; i.e. $\|f\| = T_0^1(f)$. Then the family $[f_n,N]$ defined by the following formula,

$$f_n(t) = \begin{cases} \sin\frac{1}{t} & \text{if } \frac{1}{n} \leq t \leq 1, \\ 0 & \text{if } 0 \leq t < \frac{1}{n}, \end{cases}$$

cannot be norm-bounded. However $|f_n(t)| \leq 2$ for every $n \in \mathbb{N}$ and $0 \leq t \leq 1$; thus $[f_n, \mathbb{N}]$ is order bounded. Thus order-bounded sets need not be norm-bounded.

If E is a vector lattice on which there is defined a norm $\|.\|$, and I is a nonempty set, we let $N_I(E)$ denote the collection of all norm-bounded families of elements of E;

$N_I(E) = \{[x_i, I] \in \omega_I(E)\colon \exists\, M \in \mathbb{R}^+ \ni \|x_i\| \leq M \quad \forall\, i \in I\}$.

Now $N_I(E)$ is a vector subspace of $\omega_I(E)$. Moreover, if we define $\|[x_i, I]\|_m = \sup_{i\in I} \|x_i\|$ for each $[x_i, I]$ in $N_I(E)$, then $N_I(E)$ is easily seen to be a normed space. With the ordering induced from $\omega_I(E)$, $N_I(E)$ is a partially ordered vector space, but need not be a vector lattice.

To see that $N_I(E)$ need not be a vector lattice, consider again the vector space $E = BV_0[0,1]$. This time let the positive cone be $\{f \in BV_0[0,1]\colon f \text{ is non-decreasing}\}$ and let the norm be $\|f\| = \sup\{|f(t)|\colon 0 \leq t \leq 1\}$. We first show that for this ordering, f^+ is the positive variation of the function f:

$$f^+(t) = P_0^t(f) = \sup\{\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]^+\colon n \in \mathbb{N} \text{ and }$$
$$0 = x_1 < x_2 < \ldots < x_n = 1\}.$$

Now $0 \leq s \leq t \leq 1$ implies $P_0^t(f) - P_0^s(f) = P_s^t(f) \geq f(t) - f(s)$ which implies $P_0^t(f) - f(t) \geq P_0^s(f) - f(s)$. Thus $P_0^t(f) - f(t)$ is a non-decreasing function of t for $0 \leq t \leq 1$; i.e. $P_0^{(\,)}(f) \geq f$. Also,

$P_0^{()}(f) \geq 0$ since $P_0^{()}(f)$ is non-decreasing. Now suppose $g \geq f$ and $g \geq 0$; that is, g and $g-f$ are both non-decreasing on $[0,1]$. Then for any $0 \leq s < t \leq 1$ and any partition $s = t_0 < t_1 < \ldots < t_n = t$ of the interval $[s,t]$, we have for each $1 \leq i \leq n$,

$$[g(t_i) - f(t_i)] - [g(t_{i-1}) - f(t_{i-1})] \geq 0,$$

so that $g(t_i) - g(t_{i-1}) \geq f(t_i) - f(t_{i-1})$. We also have $g(t_i) - g(t_{i-1}) \geq 0$. Therefore

$$g(t_i) - g(t_{i-1}) \geq [f(t_i) - f(t_{i-1})]^+,$$

and hence $g(t) - g(s) = P_s^t(g) \geq P_s^t(f) = P_0^t(f) - P_0^s(f)$. We thus have $g(t) - P_0^t(f) \geq g(s) - P_0^s(f)$. From this we see that $g - P_0^{()}(f)$ is an increasing function on $[0,1]$; i.e. $g \geq P_0^{()}(f)$. Therefore, $f^+(t) = P_0^t(f)$.

By this result, we conclude further that $|f|$ is the total variation function of f: $|f|(t) = T_0^t(f)$, for $0 \leq t \leq 1$.

Now consider the family $[f_n, N] \in \omega_N(E)$ $(E = BV_0[0,1])$, given by

$$f_n(t) = \begin{cases} \sin\frac{1}{t} & \text{if } \frac{1}{n} \leq t \leq 1, \\ 0 & \text{if } 0 \leq t < \frac{1}{n}. \end{cases}$$

Then $[f_n, N] \in N_I(E)$ since $\|f_n\| \leq 1$, but $|[f_n, N]| = [|f_n|, N]$ $\notin N_I(E)$ since $\||f_n|\| \to \infty$ as $n \to \infty$. Therefore, $N_I(BV_0[0,1])$ is not a vector lattice.

(1.4) Proposition. If $(E,\leq)$ is a vector lattice on which there is defined a norm $\|.\|$, and if I is an arbitrary set, then

(a) $(E,\leq,\|.\|)$ is norm-complete if and only if $(N_I(E),\leq,\|.\|_m)$ is norm-complete;

(b) $\|.\|$ is monotone on E if and only if $(N_I(E),\leq,\|.\|_m)$ is a normed vector lattice;

(c) $(E, \leq, \|.\|)$ is an M-space if and only if $(N_I(E), \leq, \|.\|_m)$ is an M-space.

Proof. (a) Suppose $(E, \|.\|)$ is norm-complete, and let $\{[x_i^{(n)}, I]\}$ be a Cauchy sequence in $N_I(E)$. Then for each i in I, $\{x_i^{(n)}\}$ is a Cauchy sequence in E; hence it norm-converges to some element x_i in E. Consider the family $[x_i, I]$. Let $\delta > 0$. Since $\exists\ M > 0$ such that $n, m \geq M$ implies $\|[x_i^{(m)} - x_i^{(n)}, I]\|_m < \delta$, we also have $\|[x_i^{(n)}, I]\|_m < \|[x_i^{(M)}, I]\|_m + \delta$ whenever $n \geq M$. For each i in I, we then have

$$\|x_i^{(n)}\| \leq \|x_i^{(M)}\| + \delta;$$

thus letting $n \to \infty$ we have

$$\|x_i\| = \lim_n \|x_i^{(n)}\| \leq \|x_i^{(M)}\| + \delta.$$

Therefore $[x_i, I] \in N_I(E)$. It remains to show that $\{[x_i^{(n)}, I]\}$ $\|\cdot\|_m$ - converges to $[x_i, I]$.

Let $\delta > 0$ and choose M as above. Then $m \geq M$ implies $\|x_i - x_i^{(m)}\| = \lim_{n\to\infty} \|x_i^{(n)} - x_i^{(m)}\| < \delta$ for all i in I. Hence, $m \geq M$ implies $\|[x_i^{(m)}, I] - [x_i, I]\|_m < \delta$, which establishes the desired convergence.

To prove the converse, suppose that $(N_I(E), \|\cdot\|_m)$ is norm-complete, and let $\{x_n\}$ be a Cauchy sequence in E. The sequence of constant families $[x_i^{(n)}, I]$ given by $x_i^{(n)} = x_n$ is then a Cauchy sequence in $N_I(E)$, and thus must converge to an element $[x_i, I]$ in $N_I(E)$. Thus for every $\delta > 0$ there exists $M > 0$ such that $n \geq M$ implies $\|[x_i^{(n)}, I] - [x_i, I]\|_m \leq \delta$; hence, $\|x_n - x_i\| = \|x_i^{(n)} - x_i\| \leq \delta$ $(i \in I)$. Therefore, $\{x_n\}$ is norm-convergent in E, proving that E is norm-complete.

(b) Suppose that $\|.\|$ is monotone on E. Let $[x_i, I] \varepsilon N_I(E)$; then there is some u in R such that $\|x_i\| \leq u$ for all $i \varepsilon I$. Hence, $\||x_i|\| = \|x_i\| \leq u$ for all $i \varepsilon I$, so that $|[x_i, I]| = [|x_i|, I] \varepsilon N_I(E)$. Therefore $N_I(E)$ is a vector lattice. We shall next show that $\|.\|_m$ is monotone on $N_I(E)$. If $|[x_i, I]| \leq |[y_i, I]|$ in $N_I(E)$, i.e., $[|x_i|, I] \leq [|y_i|, I]$, then $|x_i| \leq |y_i|$ for all $i \varepsilon I$; thus $\|x_i\| \leq \|y_i\|$ for all $i \varepsilon I$, which implies $\|[x_i, I]\|_m \leq \|[y_i, I]\|_m$.

The converse is easily seen by the fact that $N_I(E)$ contains constant families.

(c) If $(E, \leq, \|.\|)$ is an M-space, then for $[x_i, I]$, $[y_i, I]$ in $N_I(E)^+$, $\|[x_i, I] \vee [y_i, I]\|_m = \|[x_i \vee y_i, I]\|_m = \sup \{\|x_i \vee y_i\| : i \varepsilon I\}$ $= \sup \{\|x_i\| \vee \|y_i\| : i \varepsilon I\} \leqq (\sup_{i\varepsilon I} \|x_i\|) \vee (\sup_{i\varepsilon I} \|y_i\|) = \|[x_i, I]\|_m \vee \|[y_i, I]\|_m$, which, together with (a), implies that $(N_I(E), \leq, \|\cdot\|_m)$ is an M-space. The converse is easily seen, since $N_I(E)$ contains constant families.

(1.5) Observations. If $(E, \leq, \|.\|)$ is a normed vector lattice, then $m_I(E)$ and $N_I(E)$ are normed vector lattices and are ideals in $\omega_I(E)$. The proof of (c) above shows that if E is a normed vector lattice, then $\|.\|_m$ preserves suprema on $m_I(E)^+$ if and only if $\|.\|$ preserves suprema on E^+.

If $(E, \leq, \|.\|)$ is a normed vector lattice in which every norm-bounded set is order-bounded, then the norm $\|.\|$ is said to be monotone bounded. Note that in such a case, $m_I(E) = N_I(E)$.

Section 2. Cartesian products and direct sums.

It is well-known that the Cartesian product $\prod_\alpha E_\alpha$ of an arbitrary collection $\{E_\alpha: \alpha \in \Gamma\}$ of vector spaces, i.e., the set of all maps $f: \Gamma \to \bigcup_\alpha E_\alpha$ such that for all $\alpha \in \Gamma$, $f(\alpha) \in E_\alpha$, is a vector space under the operations

$$(f + g)(\alpha) = f(\alpha) + g(\alpha),$$

$$(\lambda f)(\alpha) = \lambda(f(\alpha)).$$

The subspace consisting of all f such that $f(\alpha) = 0$ for all but finitely many α is called the direct sum of the spaces E_α and is denoted $\bigoplus_\alpha E_\alpha$. If each of the spaces E_α is a vector lattice, then $\prod_\alpha E_\alpha$ is also a vector lattice, under the ordering:

$$f \leq g \text{ if and only if } f(\alpha) \leq g(\alpha) \text{ for all } \alpha \in \Gamma.$$

The positive cone in $\prod_\alpha E_\alpha$ is thus $\prod_\alpha (E_\alpha^+)$. Moreover, $\bigoplus_\alpha E_\alpha$ is an ideal in $\prod_\alpha E_\alpha$, with positive cone $\bigoplus_\alpha (E_\alpha{}^+)$.

If $E_\alpha = E$ for all $\alpha \in \Gamma$, then the space $\prod_\alpha E_\alpha$ may be denoted $\prod_\alpha E$ or E^Γ.

(2.1) Proposition. Under the ordering described above, $\omega_I(E) \simeq \prod_{i \in I} E$ $(= E^I)$ and $\phi_I(E) \simeq \bigoplus_{i \in I} E$.

(2.2) Proposition. If $\{E_\alpha: \alpha \in \Gamma\}$ is a collection of vector lattices and I is an arbitrary set, then

$$\omega_I(\prod_{\alpha \in \Gamma} E_\alpha) \simeq \prod_{\alpha \in \Gamma} \omega_I (E_\alpha).$$

Moreover, if $E_\alpha = E$ for all α in Γ, then both of the above spaces are isomorphic to $\omega_{I \times \Gamma}(E)$.

Proof. Let $[f_i, I] \in \omega_I(\prod_{\alpha \in \Gamma} E_\alpha)$. Then for each $i \in I$, f_i is a mapping from Γ to $\bigcup_\alpha E_\alpha$ such that $f_i(\alpha) \in E_\alpha$ for all $\alpha \in \Gamma$. Let

$\psi([f_i, I])$ be the map $F\colon \Gamma \to \bigcup_\alpha \omega_I(E_\alpha)$ such that $F(\alpha) = [f_i(\alpha), I]$ for all $\alpha \in \Gamma$. That is,

$$\psi([f_i, I])\,(\alpha) = [f_i(\alpha), I],$$

and hence, $\psi\colon \omega_I(\prod_\alpha E_\alpha) \to \prod_\alpha \omega_I(E_\alpha)$. It is a routine matter to verify that ψ is a (vector lattice) isomorphism.

In case $E_\alpha = E$ for all $\alpha \in \Gamma$, we use the map $\pi\colon \omega_I(\prod_\alpha E_\alpha) \to \omega_{I\times\Gamma}(E)$ given by the formula

$$\pi([f_i, I])\,(i, \alpha) = f_i(\alpha).$$

Again, it is routine to verify that π is an isomorphism.

(2.3) <u>Remarks</u>. If $\{E_\alpha : \alpha \in \Gamma\}$ is a collection of vector lattices and I is an arbitrary set, then

$$m_I(\prod_{\alpha\in\Gamma} E_\alpha) \simeq \prod_{\alpha\in\Gamma} m_I(E_\alpha).$$

The restriction of the map ψ given in (2.2) establishes the desired isomorphism.

The second assertion of (2.2) does not extend to the spaces $m_I(.)$; in general, $m_{\Gamma\times I}(E)$ is not necessarily isomorphic to $\prod_{\alpha\in\Gamma} m_I(E)$. For example,

$$m_N(\prod_{n\in N} R) \neq m_{N\times N}(R).$$

This may be seen by noting that $m_{N\times N}(R)$ has a strong order unit (any positive constant map on $N\times N$), while $m_N(\prod_{n\in N} R)$ does not have a strong order unit (since $\prod_n R$ does not).

Section 3. Order properties in subspaces of $\omega_I(E)$.

In this section we investigate relationships between various order properties which may be possessed by the vector lattice E, and those possessed by $\omega_I(E)$ and its vector sublattices.

(3.1) Proposition. If E is an Archimedean vector lattice, then any vector sublattice of $\omega_I(E)$ is Archimedean.

Proof. Since a vector sublattice of an Archimedean vector lattice is Archimedean, it suffices to consider $\omega_I(E)$. Suppose E is Archimedean. If $[x_i, I], [y_i, I] \in \omega_I(E)^+$ with $\lambda[x_i, I] \leq [y_i, I]$ for all $\lambda \geq 0$, then $\lambda x_i \leq y_i$ for all $\lambda \geq 0$ and $i \in I$; hence, $x_i = 0$ for all $i \in I$, so that $[x_i, I] = 0$. Therefore, $\omega_I(E)$ is Archimedean.

(3.2) Definition. A vector sublattice $\omega_I(E)$ will be said to have the regular supremum property if, for arbitrary $U \subseteq \lambda_I(E)$, an element $[z_i, I]$ in $\lambda_I(E)$ is the supremum of U only if, for every $i_0 \in I$, $z_{i_0} = \sup \{x_{i_0} : [x_i, I] \in U\}$.

Note that the phrase "only if" may be replaced by "if and only if", since the addition of "if" merely adds a condition which is trivially satisfied by every vector sublattice of $\omega_I(E)$.

(3.3) Proposition. If E is a vector lattice, every ideal $\lambda_I(E)$ in $\omega_I(E)$ has the regular supremum property.

Proof. First we show that $\omega_I(E)$ itself has the regular supremum property. Let $U \subseteq \omega_I(E)$ and $[z_i, I] = \sup U$. Let $i_0 \in I$. Then z_{i_0} is an upper bound in E for $\{x_{i_0} : [x_i, I] \in U\}$. Suppose y is another upper bound for this set. Then the family $[y_i, I]$ in $\omega_I(E)$ given by

$$y_i = \begin{cases} y \text{ if } i = i_0, \\ z_i \text{ if } i \neq i_0, \end{cases}$$

is an upper bound in $\omega_I(E)$ for U. Thus $[y_i, I] \geq [z_i, I]$; in particular, $y = y_{i_0} \geq z_{i_0}$. Therefore $z_{i_0} = \sup \{x_{i_0} : [x_i, I] \in U\}$.

Now let $\lambda_I(E)$ by any ideal in $\omega_I(E)$, and let $U \subseteq \lambda_I(E)$ with $[z_i, I] = \sup U$ in $\lambda_I(E)$. Then $[z_i, I] = \sup U$ in $\omega_I(E)$, since $\lambda_I(E)$ is an ideal. Thus $z_{i_0} = \sup \{x_{i_0} : [x_i, I] \in U\}$, as shown above. Therefore, $\lambda_I(E)$ has the regular supremum property.

(3.4) <u>Proposition.</u> If E is a vector lattice, every vector sublattice $\lambda_I(E)$ of $\omega_I(E)$ containing $\phi_I(E)$ has the regular supremum property.

<u>Proof.</u> Suppose $\phi_I(E) \subseteq \lambda_I(E)$, and let U be a subset of $\lambda_I(E)$ having supremum $[z_i, I]$ in $\lambda_I(E)$. Let $i_0 \in I$. Then z_{i_0} is an upper bound for $\{x_{i_0} : [x_i, I] \in U\}$. Suppose y is another upper bound; then $\frac{1}{2}(y + z_{i_0})$ is still another upper bound, since $y + z_{i_0} \geq x_{i_0} + x_{i_0}$ for all $[x_i, I] \in U$. Define the family $[y_i, I]$ by

$$y_i = \begin{cases} y - z_{i_0} \text{ if } i = i_0, \\ 0 \text{ if } i \neq i_0. \end{cases}$$

Then $[y_i, I] \in \phi_I(E) \subseteq \lambda_I(E)$, so that the family $[(\frac{1}{2}y_i + z_i), I] = \frac{1}{2}[y_i, I] + [z_i, I]$ belongs to $\lambda_I(E)$ and is an upper bound for U. Thus $[(\frac{1}{2}y_i + z_i), I] \geq [z_i, I]$; in particular, $\frac{1}{2}y + \frac{1}{2}z_{i_0} \geq z_{i_0}$, which implies $y \geq z_{i_0}$. Therefore, $z_{i_0} = \sup \{x_{i_0} : [x_i, I] \in U\}$.

To find examples of vector sublattices $\lambda_I(E)$ which do not have the regular supremum property, one must thus go beyond the familiar sequence spaces. Consider E = R, and I = [0,1], the closed unit interval. As a vector sublattice of $\omega_I(R)$, the space $C[0,1]$ does not have the regular supremum property. Indeed, the family $U = \{f_n : n \in N\}$ given by $f_n(x) = x^{\frac{1}{n}}$ $(0 \leq x \leq 1)$, has the constant function 1 as its supremum in $C[0,1]$; but $\sup\{f_n(0): n \in N\} = 0$.

In view of the fact that $\omega_I(E)$ has the regular supremum property, we see that a vector sublattice $\lambda_I(E)$ has the regular supremum property if and only if any subset U having a supremum in $\lambda_I(E)$ has the same supremum in $\omega_I(E)$.

(3.5) Proposition. Let E be a vector lattice, let $\lambda_I(E)$ be an ideal in $\omega_I(E)$, and let $\{[x_i^{(\alpha)}, I]\}_{\alpha\in\Gamma}$ be a bounded net in $\lambda_I(E)$. Then $[x_i^{(\alpha)}, I] \xrightarrow{o} [x_i, I]$ in $\lambda_I(E)$ if and only if $x_i^{(\alpha)} \xrightarrow{o} x_i$ (in E) for all $i \in I$.

Proof. First note that each net $\{x_i^{(\alpha)}\}$ is order-bounded in E.

Suppose $[x_i^{(\alpha)}, I] \xrightarrow{o} [x_i, I]$ in E. Then there is a net $\{[y_i^{(\alpha)}, I]\}$ in $\lambda_I(E)$ such that $|[x_i^{(\alpha)}, I] - [x_i, I]| \leq [y_i^{(\alpha)}, I] \downarrow 0$. That is, $[|x_i^{(\alpha)} - x_i|, I] \leq [y_i^{(\alpha)}, I] \downarrow 0$. Thus for each $i \in I$, $|x_i^{(\alpha)} - x_i| \leq y_i^{(\alpha)} \downarrow 0$, using the regular supremum property. Hence, $x_i^{(\alpha)} \xrightarrow{o} x_i$ in E for each $i \in I$.

Now suppose that for each $i \in I$, $x_i^{(\alpha)} \xrightarrow{o} x_i$. Then for each i, there exists a net $\{y_i^{(\alpha)}\}$ in E with $|x_i^{(\alpha)} - x_i| \leq y_i^{(\alpha)} \downarrow 0$. Then $[y_i^{(\alpha)}, I] \downarrow 0$ in $\omega_I(E)$, and $|[x_i^{(\alpha)}, I] - [x_i, I]| \leq [y_i^{(\alpha)}, I]$. Therefore, $[x_i^{(\alpha)}, I] \xrightarrow{o} [x_i, I]$ in $\omega_I(E)$, as well as in the ideal $\lambda_I(E)$.

(3.6) Proposition. Let E be a vector lattice, let $\lambda_N(E)$ be an ideal in $\omega_N(E)$, and let $\{[x_n^{(k)}, N]\}$ be a bounded sequence in $\lambda_N(E)$.

Then $[x_n^{(k)}, N] \xrightarrow{*} [x_n, N]$ in $\lambda_N(E)$ if and only if $x_n^{(k)} \xrightarrow{*} x_n$ in E for all $n \in N$.

Proof. Suppose $[x_n^{(k)}, N] \xrightarrow{*} [x_n, N]$ in $\lambda_N(E)$. Let $n \in N$, and consider any subsequence $\{x_n^{k(i)}\}_{i=1}^{\infty}$ of $\{x_n^{(k)}\}$. By $*$-convergence, $\{[x_n^{k(i)}, N]\}_{i=1}^{\infty}$ has a subsequence $\{[x_n^{k'(i)}, N]\}_{i=1}^{\infty}$ order converging to $[x_n, N]$ in $\lambda_N(E)$. Then $\{x_n^{k(i)}\}_{i=1}^{\infty}$ has the subsequence $\{x_n^{k'(i)}\}_{i=1}^{\infty}$ order converging in E to x_n, by (3.5). Therefore $x_n^{(k)} \xrightarrow{*} x_n$ in E.

Now suppose that for every $n \in N$, $x_n^{(k)} \xrightarrow{*} x_n$. Let $\{[x_m^{k(i)}, N]\}_{i=1}^{\infty}$ be an arbitrary subsequence of $\{[x_n^{(k)}, N]\}$. By $*$-convergence there exists a subsequence $\{k_1(i)\}_{i=1}^{\infty}$ of $\{k(i)\}_{i=1}^{\infty}$ such that $x_1^{k_1(i)} \xrightarrow{o} x_1$. Continuing inductively, for every n = 2,3,... there exists a subsequence $\{k_n(i)\}_{i=1}^{\infty}$ of $\{k_{n-1}(i)\}_{i=1}^{\infty}$ such that $x_n^{k_n(i)} \xrightarrow{o} x_n$. For each $n \in N$ we thus have a sequence $\{y_n^{(i)}\}_{i=1}^{\infty}$ in E such that $|x_n^{k_n(i)} - x_n| \leq y_n^{(i)} \downarrow 0$.

Let $n \in N$. Then for $j \geq n$, $k_j(j)$ is a term of the sequence $\{k_n(i)\}$; i.e., $k_j(j) = k_n(i)$ for some $i \geq j$. Then for $j \geq n$, $\left|x_n^{k_j(j)} - x_n\right| = \left|x_n^{k_n(i)} - x_n\right| \leq y_n^{(i)} \leq y_i^{(n)} \downarrow 0$. That is, for each $n \in N$, $x_n^{k_i(i)} \xrightarrow{o} x_n$. Therefore, by (3.5) $[x_n^{k_i(i)}, N] \xrightarrow{o} [x_n, N]$ in $\lambda_N(E)$. But this means that every subsequence of $\{[x_n^{(k)}, N]\}$ has in turn a subsequence order converging in $\lambda_N(E)$ to $[x_n, N]$. Therefore, $[x_n^{(k)}, N] \xrightarrow{*} [x_n, N]$ in $\lambda_N(E)$.

It may be of some interest to note that the proofs given for the "only if" parts of (3.5) and (3.6) remain valid under the less

restrictive hypothesis that $\lambda_I(E)$ (resp. $\lambda_N(E)$) is a vector sublattice of $\omega_I(E)$ with the regular supremum property.

(3.7) Definition. Given a vector lattice E, we define $\overline{\phi}_I(E) = \{[x_i, I] \in \omega_I(E): x_i = 0$ for all but countably many $i \in I\}$.

We note that $\overline{\phi}_I(E)$ is an ideal in $\omega_I(E)$.

(3.8) Proposition. Let E be a nontrivial vector lattice.

(a) If $\lambda_I(E)$ is an ideal in $\overline{\phi}_I(E)$ containing $\alpha_{I,i_0}(E)$ for some $i_0 \in I$, then E is order separable if and only if $\lambda_I(E)$ is order separable.

(b) $\omega_I(E)$ is order separable if and only if E is order separable and I is a countable set.

Proof. (a) Since ideals in order separable spaces are order separable, it suffices to prove that $\overline{\phi}_I(E)$ is order separable whenever E is. Thus suppose that E is order separable, and let A be a subset of $\overline{\phi}_I(E)^+$ having supremum $[z_i, I]$ in $\overline{\phi}_I(E)$. Then $z_i = 0$ for all but countably many i, say $i = i_1, i_2, \ldots$. Since for each j, $z_j = \sup \{x_j: [x_i, I] \in A\}$, we know that for each $[x_i, I]$ in A, $x_i = 0$ except for $i = i_1, i_2, \ldots$. Thus by order separability of E, for each $n = 1,2,\ldots$, there is a countable set $\{[x_i^{(n,j)}, I]: j = 1,2,\ldots\}$ in A such that

$$z_{i_n} = \sup \{x_{i_n}^{(n,j)}: j = 1,2,\ldots\}.$$

Then $z_{i_n} \leq \sup \{x_{i_n}^{(h,j)}: h = 1,2,\ldots; j = 1,2,\ldots\} \leq \sup \{x_{i_n}: [x_i, I] \in A\} = z_{i_n}$, since $\overline{\phi}_I(E)$ has the regular supremum property. Therefore,

$$[z_i, I] = \sup \{[x_i^{(h,j)}, I]: h = 1,2,\ldots; j = 1,2,\ldots\},$$

and we see that A has a countable subset with the same supremum.

(b) In view of (a) it suffices to prove that if I is uncountable, then $\omega_I(E)$ is not order separable. Suppose I is uncountable. Pick any $e > 0$ in E, and let $[e, I]$ denote the constant family $[x_i, I]$ where $x_i = e$ for all $i \in I$. For each $i_0 \in I$ define $[e(i_0), I]$ to be the family $[y_i, I]$, where $y_i = 0$ if $i \neq i_0$, and $y_i = e$ if $i = i_0$. Let $A = \{[e(i_0), I]: i_0 \in I\}$. It is clear that sup $A = [e, I]$ and that A has no countable subset whose supremum is $[e, I]$. Therefore $\omega_I(E)$ is not order separable.

(3.9) Proposition. Let E be a vector lattice. Then E has a weak order unit if and only if $\omega_I(E)$ has a weak order unit. E has a strong order unit if and only if $m_I(E)$ has a strong order unit.

Proof. (a) If 1 is a weak order unit in E, a weak order unit $[e_i, I]$ in $\omega_I(E)$ may be obtained by defining $e_i = 1$ for all $i \in I$. Conversely, if $[z_i, I]$ is a weak order unit in $\omega_I(E)$ we may pick $i_0 \in I$ and let $e = z_{i_0}$; then e is easily seen to be a weak order unit in E.

(b) Suppose 1 is a strong order unit in E, and again define $e_i = 1$ for all $i \in I$. Then $[e_i, I] \in m_I(E)$. If $[y_i, I] \in m_I(E)$ then there exists $y \in E^+$ such that $|y_i| \leq y$ for all $i \in I$. There exists $\lambda > 0$ in R such that $\lambda y \leq 1$; hence, $|\lambda[y_i, I]| \leq 1$. Therefore, $[e_i, I]$ is a strong order unit in $m_I(E)$.

On the other hand, if $[z_i, I]$ is a strong order unit in $m_I(E)$, pick $i_0 \in I$ and let $e = z_{i_0}$. Let $x \in E^+$. Defining $x_i = x$ for all $i \in I$, we have $[x_i, I] \in m_I(E)$. Then there exists $\lambda > 0$ in R such that $0 \leq \lambda[x_i, I] \leq [z_i, I]$. In particular, $\lambda x = \lambda x_{i_0} \leq z_{i_0} = e$.

Therefore, e is a strong order unit in E.

(3.10) Proposition. If E is a vector lattice, and $\lambda_I(E)$ is any ideal in $\omega_I(E)$ containing $\alpha_{I,i_0}(E)$ for some $i_0 \in I$, then

(a) E is Dedekind complete if and only if $\lambda_I(E)$ is Dedekind complete;

(b) E is Dedekind σ-complete if and only if $\lambda_I(E)$ is Dedekind σ-complete.

Proof. We prove (a) only; the proof of (b) is analogous. Since Dedekind completeness is inherited by ideals, it is sufficient to prove that $\omega_I(E)$ is Dedekind complete whenever E is.

Suppose E is Dedekind complete, and let $A = \{[x_i^{(\alpha)}, I]: \alpha \in \Gamma\}$ be a subset of $\omega_I(E)$ having an upper bound in $\omega_I(E)$. Then for each $i \in I$, $\{x_i^{(\alpha)}: \alpha \in \Gamma\}$ is bounded above in E; hence it has a supremum z_i in E. By the regular supremum property, $[z_i, I] = \sup A$ in $\omega_I(E)$. Therefore, $\omega_I(E)$ is Dedekind complete.

The condition that $\lambda_I(E)$ be an ideal in $\omega_I(E)$ was included in (3.10) to facilitate the "only if" part of the proof. Consider the example $E = \mathsf{R}$, $\omega_{\mathsf{N}}(E) = s$, $\lambda_{\mathsf{N}}(E) = c$. Then c is a vector sublattice of $\omega_{\mathsf{N}}(E)$ which contains $\alpha_{\mathsf{N},i_0}(E)$ for every $i_0 \in \mathsf{N}$, but E is Dedekind complete while c is not Dedekind complete. Notice that c even has the regular supremum property by (3.4).

The condition that $\lambda_I(E)$ contain $\alpha_{I,i_0}(E)$ for some $i_0 \in I$ was included in (3.10) to facilitate the "if" part of the proof. If we leave out this condition, we can easily find a counterexample to (a). Take $E = c$, $I = \{1\}$, and $\lambda_I(c) = c_0$. Then $\lambda_I(E)$ is Dedekind complete,

while E is not.

(3.11) Proposition. If E is a vector lattice, and $\lambda_I(E)$ is any order closed ideal of $\omega_I(E)$ containing $\alpha_{I,i_0}(E)$ for some $i_0 \in I$, then E is universally complete if and only if $\lambda_I(E)$ is universally complete.

Proof. Since universal completeness is inherited by order closed ideals, it suffices to prove that $\omega_I(E)$ is universally complete whenever E is.

Suppose E is universally complete. Then $\omega_I(E)$ is Dedekind complete by (3.10). Let $\{[x_i^{(\alpha)}, I]: \alpha \in \Gamma\}$ be a set of pairwise disjoint positive elements of $\omega_I(E)$. Then for every $i \in I$, $\{x_i^{(\alpha)}: \alpha \in \Gamma\}$ is a set of pairwise disjoint elements of E^+, so this set must have an upper bound u_i in E:

$$u_i \geq x_i^{(\alpha)} \quad (\text{for all } \alpha \in \Gamma).$$

But then $[u_i, I] \geq [x_i^{(\alpha)}, I]$ for all $\alpha \in \Gamma$. Therefore $\omega_I(E)$ is universally complete.

(3.12) Proposition. Suppose E is a vector sublattice of a vector lattice F. Let $\lambda_I(E)$ denote one of the spaces $\omega_I(E)$, $\phi_I(E)$, $\alpha_{I,i_0}(E)$ or $m_I(E)$, and let $\lambda_I(F)$ denote the corresponding space for F. Then, considering $\lambda_I(E)$ as a vector sublattice of $\lambda_I(F)$,

(a) if E is an ideal in F, then $\lambda_I(E)$ is an ideal in $\lambda_I(F)$;

(b) if E is order dense in F, then $\lambda_I(E)$ is order dense in $\lambda_I(F)$;

(c) if E is quasi order dense in F, then $\lambda_I(E)$ is quasi order dense in $\lambda_I(F)$;

(d) if E is order closed in F, then $\lambda_I(E)$ is order closed in $\lambda_I(F)$.

Proof. (a) Let $[x_i, I] \in \lambda_I(E)$ and $[y_i, I] \in \omega_I(F)$ such that

$|[y_i, I]| \leq |[x_i,I]|$; i.e., $|y_i| \leq |x_i|$ for all $i \in I$. If E is an ideal in F, then $y_i \in E$ for all $i \in I$, so that $[y_i, I] \in \lambda_I(E)$.

(b) Suppose E is order dense in F, and let $[f_i, I] \geq 0$ in $\lambda_I(F)$. Then for each $i \in I$, $f_i = \sup \{e: 0 \leq e \leq f_i, e \in E\}$. Since $\lambda_I(E)$ has the regular supremum property, we thus have $[f_i, I] = \sup \{[e_i, I]: 0 \leq [e_i, I] \leq [f_i, I], [e_i, I] \in \lambda_I(E)\}$. But this means that $\lambda_I(E)$ is order dense in $\lambda_I(F)$.

(c) The proof of (c) is trivial.

(d) Suppose E is order closed in F. Let $A \subseteq \omega_I(E)$ have supremum $[f_i, I]$ in $\omega_I(F)$. By the regular supremum property in $\omega_I(F)$, $f_i = \sup \{e_i: [e_i, I] \in A\}$ for every i in I. But then $f_i \in E$ for each i. Therefore $[f_i, I] \in \omega_I(E)$. Therefore $\omega_I(E)$ is order closed in $\omega_I(F)$.

Section 4. Dedekind completion and universal completion.

The relationship between the Dedekind completion of E and that of $\omega_I(E)$ is simple and natural. In this section we establish this relationship and its ramifications for certain subspaces of $\omega_I(E)$. Similarly we examine the universal completions of E and $\omega_I(E)$. These results will find application in the next chapter.

Propositions (4.4) and (4.7) are the only original results in this section. Definition (4.1) and proposition (4.2) may be found in Luxemburg and Zaanen [6, section 32]; proposition (4.3) may be found in Nakano [8, §30], Peressini [10, pp. 151-154] or Vulikh [18, pp. 108-113].

(4.1) Definition. Given a vector lattice E, a Dedekind completion

of E is a vector lattice $\hat{E}$ such that

(a) $\hat{E}$ is Dedekind complete,

(b) there exists a one-one linear $\psi: E \to \hat{E}$ such that $\psi(x) \leq \psi(y)$ whenever $x \leq y$,

(c) for every $\hat{x} > 0$ in $\hat{E}$ there exist x, y in E such that $0 < \psi(x) \leq \hat{x} \leq \psi(y)$.

(4.2) Proposition. Let E, $\hat{E}$ and ψ be as defined in (4.1).

(a) The map ψ preserves arbitrary suprema and infima.

(b) Every $\hat{x}$ in $\hat{E}$ satisfies $\sup \{\psi(x): x \in E, \psi(x) \leq \hat{x}\} = \hat{x} = \inf \{\psi(y): y \in E, \psi(y) \geq \hat{x}\}$.

(c) Condition (c) of (4.1) may be replaced by the pair of conditions: (i) for each $0 < \hat{x} \in \hat{E}$ there exists $x \in E$ such that $0 < x \leq \hat{x}$, and (ii) the ideal generated in $\hat{E}$ by E is $\hat{E}$.

(d) If 1 is a weak (resp. strong) order unit in E, then $\psi(1)$ is a weak (resp. strong) order unit in $\hat{E}$.

(4.3) Proposition. A vector lattice E has a Dedekind completion if and only if E is Archimedean. Moreover, any two Dedekind completions of E are isomorphic.

Henceforth we shall regard an Archimedean vector lattice E as already a subspace of its Dedekind completion $\hat{E}$, with the map ψ being merely the inclusion mapping.

(4.4) Proposition. If E is an Archimedean vector lattice, then

(a) $\widehat{\omega_I(E)} \simeq \omega_I(\hat{E})$;

(b) $\widehat{\phi_I(E)} \simeq \phi_I(\hat{E})$;

(c) $\widehat{m_I(E)} \simeq m_I(\hat{E})$.

Proof. Since $E \subseteq \hat{E}$, we may regard $\omega_I(E)$, $\phi_I(E)$ and $m_I(E)$ as vector sublattices of $\omega_I(\hat{E})$, $\phi_I(\hat{E})$ and $m_I(\hat{E})$, respectively.

(a) By (3.10) $\omega_I(\hat{E})$ is Dedekind complete. Now let $0 < [\hat{x}_i, I] \in \omega_I(\hat{E})$. Then for every $i \in I$ such that $\hat{x}_i \neq 0$, there exist x_i, y_i in E such that $0 < x_i \leq \hat{x}_i \leq y_i$. Defining $x_i = y_i = 0$ for those i such that $\hat{x}_i = 0$, we have

$$0 < [x_i, I] \leq [\hat{x}_i, I] \leq [y_i, I].$$

Therefore, $\omega_I(\hat{E}) \simeq \omega_I(E)$ by the uniqueness result of (4.3).

(b) The proof of (a) carries over to this case.

(c) Only a slight modification needs to be made to the proof of (a) to cover this case. If $[\hat{x}_i, I] \in m_I(\hat{E})$, then there exists $\hat{y} \in \hat{E}$ such that $\hat{x}_i < \hat{y}$ for every $i \in I$; but then by (4.1) there exists $y \in E$ such that $\hat{y} < y$. Then take $y_i = y$ for all $i \in I$, and proceed as in the proof of (a).

The notion of universal completion may be found in Nakano [8, §34] or Vulikh [18, pp. 142-144], although Vulikh prefers the term "maximal extension". Both (4.5) and (4.6) are drawn from these sources.

(4.5) Definition. A universal completion of a vector lattice E is a vector lattice $E^{\#}$ such that

(a) $E^{\#}$ is universally complete,

(b) E is isomorphic to an order-dense vector sublattice of $E^{\#}$.

(4.6) Proposition. A vector lattice E has a universal completion if and only if E is Archimedean. Moreover, any two universal completions of E are isomorphic.

In view of proposition (4.6), we shall hereafter, for the sake

of convenience, regard E as a vector sublattice of its universal completion $E^{\#}$. It is not hard to see that $\hat{E}$ is then the ideal generated by E in $E^{\#}$.

(4.7) Proposition. If E is an Archimedean vector lattice, then $\omega_I(E^{\#}) \simeq \omega_I(E)^{\#}$.

Proof. By (3.11) $\omega_I(E^{\#})$ is a universally complete vector lattice containing $\omega_I(E)$, and by (3.12) $\omega_I(E)$ is order dense in $\omega_I(E^{\#})$. Apply the uniqueness part of (4.6).

Section 5. Banach o-limits in Archimedean vector lattices.

In a recent paper by S. Simons [16] we find a novel approach to the theory of ordinary Banach limits for bounded real sequences. This approach may readily be adapted to the context of vector lattices, leading to the establishment of "Banach o-limits" for the space of order bounded sequences in a vector lattice E, i.e., $m_N(E)$. The results (5.4) - (5.12) of this section are thus generalizations of results already known for ordinary Banach limits, as presented in Simons [16] or Goffman and Pedrick [2].

Throughout this section, let E represent an arbitrary Archimedean vector lattice. The elements $[x_n, N]$ of $m_N(E)$ are merely sequences in E, and thus for convenience will often be denoted $\{x_n\}$.

(5.1) Definition. (a) $\sigma: m_N(E) \to m_N(E)$, called the "shift operator," is defined by $\sigma\{x_n\} = \{x_{n+1}\}$.

(b) $L: m_N(E) \to \hat{E}$ is defined by $L\{x_n\} = \overline{\lim_n}\, x_n$ $(= \inf_n \sup_{k \geq n} x_k$, in $\hat{E})$.

(c) $S\colon m_N(E) \to \hat{E}$ is defined by $S\{x_n\} = \sup_n x_n$ (in $\hat{E}$).

(d) $c(E) = \{\{x_n\} \in m_N(E)\colon \overline{\lim_n}\, x_n = \underline{\lim_n}\, x_n$, in $\hat{E}\}$.

Note that $c(E)$ is the set of all sequences in E which order converge in $\hat{E}$; $c(E)$ is a vector sublattice of $m_N(E)$.

(5.2) Definition. A Banach o-limit on $m_N(E)$ is a linear map $g\colon m_N(E) \to \hat{E}$ such that

$$g \circ \sigma \leq g \leq S;$$

i.e., $g \circ \sigma\,(x) \leq g(x) \leq S(x)$ for all $x \in m_N(E)$.

We let $\mathfrak{BL}(E)$ denote the collection of all Banach o-limits on $m_N(E)$.

A mapping $T\colon E \to F$ from E into a vector lattice F is said to be sublinear if

(a) $T(\lambda x) = \lambda T(x)$ for all $x \in E$, $\lambda \in R^+$, and

(b) $T(x + y) \leq T(x) + T(y)$ for all $x,y \in E$.

As an example, observe that the standard argument used for real sequences applies here to show that the map $L\colon m_N(E) \to \hat{E}$ is sublinear. It is easily seen that a sublinear map $T\colon E \to F$ will have the following additional properties:

(c) $T(0) = 0$;

(d) $-T(x) \leq T(-x)$ for all $x \in E$;

(e) if $f\colon E \to F$ is linear and $f(x) \leq T(x)$ for all $x \in E$, then $-T(-x) \leq f(x) \leq T(x)$ for all $x \in E$.

It has been noted (see e.g. [10], p. 78) that the proof of the classical Hahn-Banach theorem remains valid if R is replaced by an arbitrary Dedekind complete vector lattice.

(5.3) *Proposition*. (Hahn-Banach). Let E be a vector space, F be a Dedekind complete vector lattice, and $p: E \to F$ be sublinear. If $f: E_1 \to F$ is a linear map defined on a linear subspace E_1 of E, with $f(x) \leq p(x)$ for all $x \in E_1$, then there exists a linear extension $\overline{f}$ of f to all of E, with $\overline{f}(x) \leq p(x)$ for all $x \in E$.

(5.4) *Proposition*. If $T: E \to F$ is a sublinear mapping from E into a Dedekind complete vector lattice F, then for every $x \in E$,

$$T(x) = \sup \{g(x) \colon g \in L(E, F),\ g \leq T\},$$

where $L(E, F)$ denotes the space of all linear maps from E to F.

Proof. Let $x \in E$. We need only show that $T(x) \leq \sup \{g(x) \colon g \in L(E, F),\ g \leq T\}$. Let $[x]$ denote the linear span of x, and define $f: [x] \to F$ by $f(\alpha x) = \alpha T(x)$. Then f is linear and dominated on $[x]$ by T, since

(i) $\alpha \geq 0$ implies $f(\alpha x) = \alpha T(x) = T(\alpha x)$, and

(ii) $\alpha < 0$ implies $f(\alpha x) = \alpha T(x) = (-\alpha)[-T(x)] \leq (-\alpha)T(-x) = T(\alpha x)$.

Thus by the Hahn-Banach theorem, there exists $g \in L(E, F)$ with $g \leq T$ and $g(x) = T(x)$. Therefore $T(x) \leq \sup \{g(x) \colon g \in L(E, F),\ g \leq T\}$.

(5.5) *Proposition*. If $g \in \mathcal{BL}(E)$, then

(a) $g \circ \sigma = g$,

(b) $g \leq L$,

(c) $\{x_n\} \in c(E)$ implies $g(x)$ equals the order limit of $\{x_n\}$,

(d) $g \geq 0$.

Proof. (a) Let $x \in m_N(E)$. Then $-(g \circ \sigma)(x) = (g \circ \sigma)(-x) \leq g(-x) = -g(x)$ by (5.2) and the linearity of g and σ. Thus $(g \circ \sigma)(x) \geq g(x)$. Combining with (5.2), we have $g \circ \sigma = g$.

(b) Let $x \in m_{\mathbb{N}}(E)$. For each $n \in \mathbb{N}$, $g(x) = (g \circ \sigma)(x) = \ldots$ $= (g \circ \sigma^n)(x) \leq \sup \{x_k : k \geq n\}$ by (5.2). Thus $g(x) \leq \inf_n \sup_{k \geq n} x_k = L(x)$.

(c) Let $x = \{x_n\} \in c(E)$. Then $\underline{\lim}_n x_n = \overline{\lim}_n x_n$ in $\hat{E}$, so $-L(-x) = L(x)$. But since L is a sublinear map satisfying (b), $-L(-x) \leq g(x) \leq L(x)$. Therefore, $g(x) = L(x)$, which equals the order limit of x.

(d) $x \geq 0$ implies $-x \leq 0$; whence, $g(-x) \leq L(-x) = \inf_n \sup_{k \geq n} x_k \leq 0$, so that $g(x) = -g(-x) \geq 0$.

Observe that if $g \in L(E, F)$ satisfies (a) through (d) of (5.5), then $g \in \mathcal{BL}(E)$. Thus our present definition of Banach o-limits coincides with the usual definition of Banach limits (see Goffman and Pedrick [2], section 2.10) in case $E = \mathbb{R}$. As in the case of ordinary Banach limits, we say that a sequence $\{x_n\} \in m_{\mathbb{N}}(E)$ is <u>almost o-convergent</u> to $\hat{x} \in \hat{E}$ if $g\{x_n\} = \hat{x}$ for every $g \in \mathcal{BL}(E)$. For each $e \in E$, the sequence $(e,0,e,0,\ldots)$ is almost o-convergent to $\frac{1}{2}e$, by (5.5).

We now consider the existence of Banach o-limits.

(5.6) <u>Definition</u>. A sublinear map $T: m_{\mathbb{N}}(E) \to \hat{E}$ is said to

(a) <u>generate Banach o-limits</u> if every $g \in L(m_{\mathbb{N}}(E), \hat{E})$ such that $g \leq T$ is a Banach o-limit;

(b) <u>dominate all Banach o-limits</u> if $g \leq T$ for each $g \in \mathcal{BL}(E)$.

Observe that L dominates all Banach o-limits by (5.5); moreover, any sublinear T generating Banach o-limits must satisfy $T \leq L$ by (5.4) and (5.5)

As a consequence of (5.4), it follows that the existence of Banach o-limits can be proved by exhibiting a sublinear T which generates Banach o-limits.

(5.7) <u>Proposition</u>. A sublinear map $T: m_N(E) \to \hat{E}$ generates Banach o-limits if and only if

(a) $T \leq S$, and

(b) $T \circ (\sigma - I) \leq 0$

(where I denotes the identity map on $m_N(E)$).

<u>Proof</u>. First, assume (a) and (b). Thus every $g \in L(m_N(E), \hat{E})$ such that $g \leq T$ must satisfy $g \leq S$ and $g \circ (\sigma - I) \leq 0$. By linearity of g the latter inequality becomes $g \circ \sigma \leq g$. Thus $g \in \mathcal{BL}(E)$.

To prove the converse, suppose T generates Banach o-limits. Let $g \in L(m_N(E), \hat{E})$ with $g \leq T$. Then g is a Banach o-limit, so $g \circ \sigma \leq g \leq S$; hence, $g \leq S$ and $g \circ (\sigma - I) = (g \circ \sigma) - g \leq 0$. Applying (5.4), we have (a) and (b).

As a consequence, we can see that the map L of (5.1) does not generate Banach o-limits, since L does not satisfy (b) of (5.7). Indeed, if $e > 0$ in E, then $L \circ (\sigma - I)(e,0,e,0,\ldots) = L(-e,e,-e,e,\ldots) = e > 0$.

(5.8) <u>Proposition</u>. (Existence of Banach o-limits). The map

$$Q(\{x_n\}) = \overline{\lim_n}\, \frac{1}{n} \sum_{k=1}^{n} x_k$$

is a sublinear map $Q: m_N(E) \to \hat{E}$ which generates Banach o-limits.

<u>Proof</u>. Sublinearity of Q follows from sublinearity of $\overline{\lim}$. Let

$x = \{x_n\} \in m_N(E)$. Then $Q(x) \leq \overline{\lim_n}\, \frac{1}{n} \sum_{k=1}^{n} (S(x)) = S(x)$, and

$$\begin{aligned} Q \circ (\sigma - I)(x) &= Q(\sigma x - x) \\ &= \overline{\lim_n}\, \frac{1}{n} \sum_{k=1}^{n} (x_{k+1} - x_k) \\ &= \overline{\lim_n}\, \frac{1}{n} (x_{n+1} - x_1) \end{aligned}$$

$$\leq \overline{\lim_n} \frac{1}{n} [S(x) - x_1] = 0.$$

Therefore by (5.7), Q generates Banach o-limits.

(5.9) Proposition. If we define the map W: $m_N(E) \to \hat{E}$ by

$$W(x) = \sup \{g(x): g \in \mathcal{BL}(E)\}$$

and let T denote a sublinear mapping from $m_N(E)$ into $\hat{E}$, then

(a) W is sublinear, generates Banach o-limits, and dominates all Banach o-limits;

(b) T generates Banach o-limits if and only if $T \leq W$;

(c) T dominates all Banach o-limits if and only if $W \leq T$;

(d) a sequence $x = \{x_n\}$ is $m_N(E)$ is almost o-convergent if and only if $-W(-x) = W(x)$.

Proof. That W is sublinear is clear. Since each $g \in \mathcal{BL}(E)$ satisfies the conditions of (5.7), so must W. Thus W generates Banach o-limits. That W dominates all Banach o-limits is inherent in its definition. Then (b) follows directly from (a) and (5.4); (c) follows from (a) and the definition of W.

By our remarks following (5.2), we know that for all $g \in \mathcal{BL}(E)$, $-W(-x) \leq g(x) \leq W(x)$. Thus $-W(-x) = W(x)$ implies that x almost o-converges to W(x).

Conversely, suppose x almost o-converges to $\hat{x} \in \hat{E}$. Then $-x$ almost o-converges to $-\hat{x}$, so that $W(-x) = g(-x)$ for arbitrary $g \in \mathcal{BL}(E)$. But then the linearity of g implies $-W(-x) = -g(-x) = g(x) = W(x)$. Therefore, (d) holds.

(5.10) Proposition. If we let $S_N(E)$ denote the collection of all $x = \{x_n\} \in m_N(E)$ such that the sequence $\{s_n\}$ of partial sums

$s_n = \sum_{k=1}^{n} x_k$ is order bounded, then for every $x \in m_N(E)$

$$W(x) = \inf \{S(x + z): z \in S_N(E)\}.$$

Proof. Let $V(x) = \inf \{S(x + z): z \in S_N(E)\}$. By (5.9) it suffices to prove that V is a sublinear mapping which both generates and dominates all Banach o-limits.

Let $x, y \in m_N(E)$. For every $z_1, z_2 \in S_N(E)$ we have $z_1 + z_2 \in S_N(E)$ and $S(x + y + z_1 + z_2) \leq S(x + z_1) + S(y + z_2)$. Thus

$$\inf \{S(x + y + z): z \in S_N(E)\} \leq S(x + z_1) + S(y + z_2);$$

then taking infima, first over z_1 in $S_N(E)$, then over z_2 in $S_N(E)$, we obtain

$$V(x + y) \leq V(x) + V(y).$$

Therefore, V is sublinear.

Now $V \leq S$, since $0 \in S_N(E)$. Let $x \in m_N(E)$. Then $\sigma x - x \in S_N(E)$ and every $z \in S_N(E)$ satisfies $V(z) \leq 0$; hence, $V \circ (\sigma - I)(x) \leq 0$. Therefore, (5.7) implies that V generates Banach o-limits.

Let $g \in \mathcal{BL}(E)$ and $x \in m_N(E)$. Each $z \in S_N(E)$ may be written $z = \sigma y - y$, where $y = \{0, s_1, s_2, \ldots, s_n, \ldots\} \in m_N(E)$ with $s_n = \sum_{i=1}^{n} z_i$. Thus $S(x + z) = S(x + \sigma y - y) \geq g(x + \sigma y - y) = g(x) + (g\circ\sigma)y - g(y) = g(x)$. Taking infimum over all z in $S_N(E)$, we obtain $V(x) \geq g(x)$. Therefore, V dominates all Banach o-limits.

(5.11) Proposition. For every $x = \{x_n\} \in m_N(E)$, $W(x) = \inf \{\overline{\lim_{j\to\infty}} \frac{1}{k} \sum_{i=1}^{k} x_{n_i+j} : k \in N;\ n_1, n_2, \ldots, n_k \in N\}$.

[Equivalently, $W(x) = \inf \{\frac{1}{k} S(\sum_{i=1}^{k} \sigma^{n_i}(x)): k \in N;\ n_1, n_2, \ldots, n_k \in N\}$.]

Proof. Let $p(x) = \inf\{\overline{\lim}_{j\to\infty} \frac{1}{k}\sum_{i=1}^{k} x_{n_i+j} : k \in N;\ n_1, n_2, \ldots, n_k \in N\}$. By (5.9) it suffices to prove that p is a sublinear mapping which both generates and dominates all Banach o-limits.

(a) We first show that p is sublinear. That $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ is clear from the definition of p. To establish subadditivity, let $x = \{x_n\}$ and $y = \{y_n\}$ belong to $m_N(E)$. Let $k, l \in N$ and $n_1, n_2, \ldots, n_k$, $m_1, m_2, \ldots, m_l \in N$. By definition of p,

$$p(x+y) \leq \overline{\lim_{t\to\infty}}\ \frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{n_i+m_j+t} + y_{n_i+m_j+t}),$$

and using the subadditivity of $\overline{\lim}$,

$$p(x+y) \leq \overline{\lim_{t\to\infty}}\ \frac{1}{kl}\sum_{i=1}^{k}\sum_{j=i}^{l} x_{n_i+m_j+t} + \overline{\lim_{t\to\infty}}\ \frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l} y_{n_i+m_j+t}$$

$$\leq \frac{1}{l}\sum_{j=1}^{l}\Big(\overline{\lim_{t\to\infty}}\ \frac{1}{k}\sum_{i=1}^{k} x_{n_i+m_j+t}\Big) + \frac{1}{k}\sum_{i=1}^{k}\Big(\overline{\lim_{t\to\infty}}\ \frac{1}{l}\sum_{j=1}^{l} y_{n_i+m_j+t}\Big).$$

Since this inequality holds for arbitrary $k \in N$ and n_1+m_j, $n_2+m_j, \ldots$, $n_k+m_j \in N$, upon taking the infimum over these variables, we obtain the inequality

$$p(x+y) \leq \frac{1}{l}\sum_{j=1}^{l} p(x) + \frac{1}{k}\sum_{i=1}^{k}\Big(\overline{\lim_{t\to\infty}}\ \frac{1}{l}\sum_{j=1}^{l} y_{n_i+m_j+t}\Big),$$

and of course $\frac{1}{l}\sum_{j=1}^{l} p(x) = p(x)$. Then taking the infimum over $l \in N$ and n_i+m_1, $n_i+m_2, \ldots$, $n_i+m_l \in N$, we obtain

$$p(x+y) \leq p(x) + p(y).$$

(b) Let $g \in \mathcal{BL}(E)$, and let $k, n_1, n_2, \ldots, n_k \in N$. Then by (5.5) $g(x) = \frac{1}{k}\, g(\sum_{i=1}^{k} \sigma^{n_i}(x)) \leq \frac{1}{k}\, S(\sum_{i=1}^{k} \sigma^{n_i}(x))$. Hence, taking the infimum over $k, n_1, \ldots, n_k$, we obtain $g(x) \leq p(x)$. Therefore, p dominates all Banach o-limits.

(c) To show that p generates Banach o-limits we use (5.7).

Clearly $p \leq S$. Let $x \in m_N(E)$; then there is some $u \in E$ such that $|x_n| \leq u$ for all $n \in N$. For arbitrary $k, n_1, n_2, \ldots, n_k \in N$ we have $p(\sigma x - x) \leq \overline{\lim_{j\to\infty}} \frac{1}{k} \sum_{i=1}^{k}(x_{n_i+1+j} - x_{n_i+j})$. In particular, we may take $n_i = i$ for $i = 1, \ldots, k$, and obtain the inequality

$$p(\sigma x - x) \leq \overline{\lim_{j\to\infty}} \frac{1}{k} \sum_{i=1}^{k}(x_{i+1+j} - x_{i+j}) = \overline{\lim_{j\to\infty}} \frac{1}{k} (x_{k+1+j} - x_{1+j}).$$

Thus $p(\sigma x - x) \leq \frac{2}{k}u$ for all $k \in N$. By the Archimedean property in $\hat{E}$, we thus have $p(\sigma x - x) \leq 0$. Therefore by (5.7), p generates Banach o-limits.

(5.12) <u>Proposition</u>. A sequence $\{x_n\} \in m_N(E)$ almost o-converges to $\hat{x} \in \hat{E}$ if and only if

$$\hat{x} = \lim_{p\to\infty} \frac{1}{p} (x_n + x_{n+1} + \ldots + x_{n+p})$$

holds uniformly in n.

This proposition may be proved by a straightforward alteration of the proof for ordinary Banach limits, presented by Goffman and Pedrick [2]. The alteration is required in order to remove the dependence of their proof on the linearity of the order. Proposition (5.11) was proved here by a similar alteration of proofs given in Goffman and Pedrick.

CHAPTER II

SUMMABLE FAMILIES IN A VECTOR LATTICE

Section 6. Some types of summable families.

We continue to let E denote an arbitrary vector lattice and I denote an arbitrary nonempty set. The theory of summable families will require the use of convergent nets in $\omega_I(E)$. The basic notions of order convergence were presented in the introduction.

Let $\nabla(I)$ denote the collection of all finite subsets of I, and partially order $\nabla(I)$ by inclusion: $J_1 \leq J_2$ if $J_1 \subseteq J_2$. Given a family $x = [x_i, I]$ in $\omega_I(E)$, we shall be concerned with the nets $\{\sigma_J(x)\}$ and $\{\tau_J(x)\}$ defined over $\nabla(I)$ by

$$\sigma_J(x) = \sum_{i \varepsilon J} x_i, \text{ and } \tau_J(x) = \sum_{i \varepsilon J} |x_i|.$$

We say that $[x_i, I]$ is <u>order summable</u> (to an element $x \varepsilon E$) if $\sigma_J(x) \xrightarrow{o} x_0$; in symbols,

$$x_0 = \sum_{i \varepsilon I} x_i.$$

We say that $[x_i, I]$ is <u>absolutely order summable</u>, if $\{\tau_J(x)\}$ order converges to some element of E. Thus x is absolutely order summable if and only $|x|$ is order summable. If $\sigma_J(x) \xrightarrow{u} x_0$, then we say that $[x_i, I]$ is <u>uniformly summable</u> to x_0; if $\{\tau_J(x)\}$ converges uniformly to an element of E, we say that $[x_i, I]$ is <u>absolutely uniformly summable</u>.

(6.1) Definition. Given a vector lattice E and an arbitrary nonempty set I, we let $\ell_I^o(E)$ denote the collection of all $[x_i, I]$ in $\omega_I(E)$ which are order summable, and we let $\ell_I^u(E)$ denote the collection of all $[x_i, I]$ in $\omega_I(E)$ which are uniformly summable. Finally, we let

$$\ell_I^1(E) = \{[x_i, I] \in \omega_I(E): \exists u \in E \ni \sum_{i \in J} |x_i| \leq u \ \forall J \in \nabla(I)\}.$$

(6.2) Proposition. $\ell_I^o(E)$ and $\ell_I^u(E)$ are vector subspaces of $\omega_I(E)$; $\ell_I^1(E)$ is an ideal in $\omega_I(E)$.

Proof. It is clear that each of these spaces is closed under scalar multiplication. Let $x = [x_i, I]$, $y = [y_i, I] \in \ell_I^o(E)$. Note that for each $J \in \nabla(I)$, $\sigma_J(x+y) = \sum_{i \in J}(x_i + y_i) = \sum_{i \in J} x_i + \sum_{i \in J} y_i =$ $\sigma_J(x) + \sigma_J(y)$. Since the nets $\{\sigma_J(x)\}$ and $\{\sigma_J(y)\}$ are order convergent, say to x_0 and y_0 respectively, they are order bounded, and there exist nets $\{u_J\}$ and $\{v_J\}$ in E such that

$$|\sigma_J(x) - x_0| \leq u_J \downarrow 0 \text{ and } |\sigma_J(y) - y_0| \leq v_J \downarrow 0.$$

Then $|\sigma_J(x+y) - (x_0+y_0)| \leq |\sigma_J(x) - x_0| + |\sigma_J(y) - y_0| \leq u_J + v_J \downarrow 0$. Thus $x + y \in \ell_I^o(E)$. Therefore ℓ_I^o is closed under addition. The proof that $\ell_I^u(E)$ is closed under addition is similar. Therefore $\ell_I^o(E)$ and $\ell_I^u(E)$ are vector subspaces of $\omega_I(E)$.

The proof that $\ell_I^1(E)$ is an ideal is easy, and will be omitted.

In contrast to the space $\ell_I^o(E)$, the collection of all absolutely order summable families $[x_i, I]$ in $\omega_I(E)$ does not even form a vector subspace of $\omega_I(E)$. As a counterexample, consider E = c, the space of convergent sequence. For each $n \in \mathbb{N}$ we continue to let $e_n = (0,0, \ldots,1,0,\ldots)$ whose k^{th} term is 1 if $k = n$, and 0 if $k \neq n$. We let

$\mathbb{1} = (1,1,\ldots,1,\ldots)$ each of whose terms is 1. The families $x = [e_n, \mathbb{N}]$ and $y = [(-1)^n e_n, \mathbb{N}]$ are each absolutely order summable, with $\sum_{n\varepsilon\mathbb{N}}|e_n| = \sum_{n\varepsilon\mathbb{N}}|(-1)^n e_n| = \mathbb{1}$. On the other hand, given $k \varepsilon \mathbb{N}$, $\sum_{n=1}^{k}|e_n + (-1)^n e_n| = (0,2,\ldots,0,2,0,0,\ldots)$; thus $\tau_J(x + y)$ is not order convergent in c. Hence $x + y$ is not absolutely order summable. Therefore, the collection of all absolutely order summable families $[x_n, \mathbb{N}] \varepsilon \omega_{\mathbb{N}}(c)$ does not form a vector subspace of $\omega_{\mathbb{N}}(c)$.

(6.3) Proposition. If E is an Archimedean vector lattice, then $\ell_I^u(E) \subseteq \ell_I^o(E) \subseteq \ell_I^1(E)$.

Proof. In an Archimedean vector lattice, uniform convergence implies order convergence. Thus $\ell_I^u(E) \subseteq \ell_I^o(E)$.

Let $[x_i, I] \varepsilon \ell_I^o(E)$. Since order convergent nets are bounded, there exists $u \varepsilon E$ such that $|\sum_{i\varepsilon J} x_i| \leq u$ for every $J \varepsilon \nabla(I)$. As shown in the appendix, an Archimedean vector vector lattice E is isomorphic to a vector sublattice of $C_\infty(Q)$, for some extremal compactum Q. For convenience we identify E with its image in $C_\infty(Q)$. Then for every $J \varepsilon \nabla(I)$ and every $t \varepsilon Q$, $|\sum_{i\varepsilon J} x_i(t)| = |\sum_{i\varepsilon J} x_i|(t) \leq u(t)$. Remember that each $x_i(t)$ is a real number. Consider an arbitrary $J \varepsilon \nabla(I)$. For each $t \varepsilon Q$, let $J_t = \{i\varepsilon J\colon x_i(t) \geq 0\}$. Then $\sum_{i\varepsilon J} x_i^+(t) = \sum_{i\varepsilon J_t} x_i(t) \leq u(t)$. Therefore, for every $J \varepsilon \nabla(I)$, $\sum_{i\varepsilon J} x_i^+ \leq u$. Similarly we can show that $\sum_{i\varepsilon J} x_i^- \leq u$. Hence for every $J \varepsilon \nabla(I)$,

$$\sum_{i\varepsilon J}|x_i| = \sum_{i\varepsilon J} x_i^+ + \sum_{i\varepsilon J} x_i^- \leq 2u.$$

Therefore, $[x_i, I] \varepsilon \ell_I^1(E)$. We conclude that $\ell_I^o(E) \subseteq \ell_I^1(E)$.

(6.4) Proposition. (a) If E is a finite dimensional Archimedean vector lattice, then $\ell_I^o(E) = \ell_I^u(E)$.

(b) If E is a Dedekind complete vector lattice, then $\ell_I^o(E) = \ell_I^1(E)$.

Proof. (a) It is well known that any finite dimensional Archimedean vector lattice E, say of dimension n, is isomorphic to R^n. In R^n, order convergence and uniform convergence are equivalent. Thus $\ell_I^o(E) = \ell_I^u(E)$.

(b) Suppose E is Dedekind complete. Let $x = [x_i, I] \in \ell_I^1(E)$. Then $x^+ = [x_i^+, I]$ and $x^- = [x_i^-, I]$ also belong to $\ell_I^1(E)$. Since $\{\tau_J(x^+)\}$ and $\{\tau_J(x^-)\}$ are bounded monotone nets in a Dedekind complete space, they must order converge to some elements $u, v \in E$. Now $\sigma_J(x) = \sigma_J(x^+) - \sigma_J(x^-) = \tau_J(x^+) - \tau_J(x^-)$. Hence, $\{\sigma_J(x)\}$ order converges, which implies that $x \in \ell_I^o(E)$. Therefore, $\ell_I^1(E) \subseteq \ell_I^o(E)$.

A Dedekind complete vector lattice is necessarily Archimedean. Thus by (6.3) we also have $\ell_I^o(E) \subseteq \ell_I^1(E)$. Therefore, $\ell_I^o(E) = \ell_I^1(E)$.

Proposition (6.4) shows that for a Dedekind complete vector lattice E, a family $[x_i, I]$ is order summable if and only if it is absolutely order summable. The same proof would show that in a Dedekind σ-complete vector lattice E, a sequence $[x_n, N]$ is order summable if and only if it is absolutely order summable.

If $[x_i, I]$ is absolutely order summable, then $\sum_{i\in I} |x_i| = \sup \{\sum_{i\in J} |x_i| : J \in \nabla(I)$; that is, $\tau_J(x) \uparrow \sum_{i\in I} |x_i|$.

The proof of the following proposition is straightforward and is omitted.

(6.5) <u>Proposition</u>. Let A and B be disjoint nonempty subsets of I, and let $[x_i, I] \in \omega_I(E)$ such that $\sum_{i\in A} x_i$ and $\sum_{i\in B} x_i$ exist in E. Define the families $[u_i, I]$ and $[v_i, I]$ by

$$u_i = \begin{cases} x_i & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases} \qquad v_i = \begin{cases} x_i & \text{if } i \in B, \\ 0 & \text{if } i \notin B. \end{cases}$$

Then

(a) $\sum_{i\in A} x_i = \sum_{i\in I} u_i$ and $\sum_{i\in B} x_i = \sum_{i\in I} v_i$,

(b) $\sum_{i\in A\cup B} x_i$ exists in E, and is equal to $\sum_{i\in A} x_i + \sum_{i\in B} x_i$.

Section 7. The space $\ell_I^1(E)$.

We begin this section with several examples, obtaining $\ell_N^1(E)$ for several familiar sequence spaces E. Recall the definitions of the sequence spaces ω, ϕ, m, and ℓ_1 given in the introduction.

It is quite easy to see that $\ell_N^1(R)$ coincides with the familiar vector lattice ℓ_1.

If λ is a vector sublattice of the space ω of real sequences, then we may view $\ell_N^1(\lambda)$ as the collection of all infinite matrices (x_{nm}) (with the customary component-wise definitions of the linear operations and order) such that each column of (x_{nm}) satisfies the defining conditions on λ, and each row satisfies $\sum_{m=1}^{\infty} |x_{nm}| = y_n < \infty$, for some sequence $\{y_n\} \in \lambda$.

It is thus clear that

$$\ell_N^1(\omega) = \{(x_{nm}) : \forall n, \sum_{m=1}^{\infty} |x_{nm}| < \infty\},$$

and from this it follows that $\ell_N^1(\omega) \simeq \omega_N(\ell_1)$.

A matrix (x_{nm}) belongs to $\ell_N^1(\phi)$ if and only if (i) for every m,

$x_{nm} = 0$ for all but finitely many n, (ii) for every n there exists $y_n \in R$ such that $\sum_{m=1}^{\infty}|x_{nm}| = y_n$, and (iii) $y_n = 0$ for all but finitely many n. Taken together these conditions yield a simpler description of $\ell_N^1(\phi)$ as the space of all (x_{nm}) such that

(i') there exists n_0 such that $x_{nm} = 0$ for all $n \geq n_0$,

(ii') for every n, $\sum_{m=1}^{\infty}|x_{nm}| < \infty$;

that is, the rows are all ℓ_1-sequences, and from some row on, all the rows are zero. It is thus apparent that

$$\ell_N^1(\phi) \simeq \phi_N(\ell_1).$$

(7.1) Proposition.

(a) $\ell_N^1(m) = \{(x_{nm}): \exists\, u \in R \ni \forall n, \sum_{m=1}^{\infty}|x_{nm}| \leq u\}$;

(b) $\ell_N^1(m)$ contains a proper ideal isomorphic to $m_N(\ell_1)$.

Proof. (a) A matrix (x_{nm}) belongs to $\ell_N^1(m)$ if and only if (i) for every m there exists $u_m \in R$ such that $|x_{nm}| \leq u_m$ for every $n \in N$, (ii) for every n there exists $y_n \in R$ such that $\sum_{m=1}^{\infty}|x_{nm}| = y_n$, and (iii) there exists $u_0 \in R$ such that $y_n \leq u_0$ for every n. Observe that (i) follows from (ii) and (iii). Then (a) is apparent.

(b) It is clear that $m_N(\ell_1)$ is isomorphic to the space $M = \{(x_{nm})$: (i) $\forall n, \sum_{m=1}^{\infty}|x_{nm}| < \infty$, (ii) $\forall m\, \exists u_m \in R \ni |x_{nm}| \leq u_m\, \forall n$, and (iii) $\sum_{m=1}^{\infty} u_m < \infty.\}$, with the pointwise-defined linear operations and order. If $(x_{nm}) \in M$, then for every n, $\sum_{m=1}^{\infty}|x_{nm}| \leq \sum_{m=1}^{\infty} u_m < \infty$; hence $(x_{nm}) \in \ell_N^1(m)$. Therefore, $m_N(\ell_1)$ is isomorphic to $M \subseteq \ell_N^1(m)$. From its definition it is clear that M is an ideal in $\ell_N^1(m)$. Now let I denote the identity matrix $I = (I_{nm})$ with $I_{nm} = 0$ if $n \neq m$, and 1 if $n = m$. Then by (a) $I \in \ell_N^1(m)$. On the other hand $I \notin M$. Hence, M is a proper ideal in $\ell_N^1(m)$.

(7.2) Proposition. $\ell^1_N(\ell_1) = \{(x_{nm}) : \exists\, u \in R \ni \forall\, k,l\ ,\ \sum_{n=1}^{k} \sum_{m=1}^{l} |x_{nm}| \leq u\}$.

Proof. By the remarks at the beginning of this section, a matrix (x_{nm}) belongs to $\ell^1_N(\ell_1)$ if and only if (i) for every m there exists $u_m \in R$ such that $\sum_{n=1}^{\infty} |x_{nm}| = u_m$, (ii) for every n there exists $y_n \in R$ such that $\sum_{m=1}^{\infty} |x_{nm}| = y_n$, and (iii) $\sum_{n=1}^{\infty} y_n < \infty$.

If $(x_{nm}) \in \ell^1_N(\ell_1)$, we define y_n as above and let $u = \sum_{n=1}^{\infty} y_n$. Then for every k,l, $\sum_{n=1}^{k} \sum_{m=1}^{l} |x_{nm}| \leq \sum_{n=1}^{k} \sum_{m=1}^{\infty} |x_{nm}| = \sum_{n=1}^{k} y_n \leq u$.

On the other hand suppose (x_{nm}) is a matrix and $u \in R$ such that for every k, l, $\sum_{n=1}^{k} \sum_{m=1}^{l} |x_{nm}| \leq u$. Then (i) and (ii) clearly hold for (x_{nm}). We also have (iii), since $\sum_{n=1}^{\infty} y_n = \lim_{k\to\infty} \sum_{n=1}^{k} (\sum_{m=1}^{\infty} |x_{nm}|) = \lim_{k\to\infty} \lim_{l\to\infty} \sum_{n=1}^{k} \sum_{m=1}^{l} |x_{nm}| \leq u$. Therefore $(x_{nm}) \in \ell^1_N(\ell_1)$.

As a consequence of (7.2) we see that $\ell^1_N(\ell_1)$ is the space of all (x_{nm}) such that the double series $\sum_{n,m} x_{nm}$ is absolutely convergent.

We turn now to $\ell^1_I(E)$ for a general Archimedean vector lattice E. Recall that we regard E as embedded in its Dedekind completion $\hat{E}$. Consider an arbitrary $[x_i, I]$ in $\ell^1_I(E)$. Since $[x_i, I]$ may also be regarded as an element of $\ell^1_I(\hat{E})$, we know by (6.4) that there exist elements $\hat{x}$, $\hat{y}$ in $\hat{E}$ such that

$$\hat{x} = \sum_{i\in I} x_i \text{ and } \hat{y} = \sum_{i\in I} |x_i|.$$

(7.3) Proposition (order continuity of the sum). Let E be an Archimedean vector lattice, and suppose $\{[x_i^{(\alpha)}, I]\}_{\alpha\in\Gamma}$ is a net in

$\ell^1_I(E)$ such that $[x_i^{(\alpha)}, I] \xrightarrow{o} [y_i, I] \in \ell^1_I(E)$. Then in $\hat{E}$,

$$\hat{x}^{(\alpha)} \xrightarrow{o} \hat{y},$$

where $\hat{x}^{(\alpha)} = \sum_{i\varepsilon I} x_i^{(\alpha)}$ and $\hat{y} = \sum_{i\varepsilon I} \hat{y}_i$.

Proof. (a) First suppose $[x_i^{(\alpha)}, I] \geq 0$ for all $\alpha\varepsilon\Gamma$, and $[x_i^{(\alpha)}, I] \uparrow [y_i, I]$. Then clearly $\hat{x}^{(\alpha)} \uparrow$ and $\sup_\alpha \hat{x}^{(\alpha)} = \sup_\alpha \sup_J \sum_{i\varepsilon J} x_i^{(\alpha)} = \sup_J \sup_\alpha \sum_{i\varepsilon J} x_i^{(\alpha)} = \sup_J (\text{o-}\lim_\alpha \sum_{i\varepsilon J} x_i^{(\alpha)})$ $\sup_J \sum_{i\varepsilon J} y_i = \hat{y}$. Therefore, $\hat{x}^{(\alpha)} \uparrow \hat{y}$.

(b) Now suppose $[x_i^{(\alpha)}, I] \downarrow 0$. Pick an arbitrary $\alpha_0 \varepsilon \Gamma$. Then $[x_i^{(\alpha_0)} - x_i^{(\alpha)}, I]_{\alpha\geq\alpha_0} = [x_i^{(\alpha_0)}, I] - [x_i^{(\alpha)}, I] \uparrow [x_i^{(\alpha_0)}, I]$ in $\ell^1_I(E)^+$. Thus by (a) $\sum_{i\varepsilon I} x_i^{(\alpha_0)} - \sum_{i\varepsilon I} x_i^{(\alpha)} = \sum_{i\varepsilon I}(x_i^{(\alpha_0)} - x_i^{(\alpha)})$ $\uparrow \sum_{i\varepsilon I} x_i^{(\alpha_0)}$ (for $\alpha \geq \alpha_0$). That is, $\hat{x}^{(\alpha_0)} - \hat{x}^{(\alpha)} \uparrow \hat{x}^{(\alpha)}$ $(\alpha \geq \alpha_0)$, and hence by cancellation, $-\hat{x}^{(\alpha)} \uparrow 0$ $(\alpha \geq \alpha_0)$. Therefore, $\hat{x}^{(\alpha)} \downarrow_{(\alpha\geq\alpha_0)} 0$. Since this is true for every $\alpha_0 \varepsilon \Gamma$, we conclude that $\hat{x}^{(\alpha)} \downarrow 0$.

(c) Finally suppose $[x_i^{(\alpha)}, I] \xrightarrow{o} [y_i, I]$. Then there exists a net $\{[u_i^{(\alpha)}, I]\}$ in $\ell^1_I(E)$ such that $[|x_i^{(\alpha)} - y_i|, I] \leq [u_i^{(\alpha)}, I] \downarrow 0$. By (b) $\sum_{i\varepsilon I} u_i^{(\alpha)} \downarrow 0$. Then $|\hat{x}^{(\alpha)} - \hat{y}| = |\sum_{i\varepsilon I} x_i^{(\alpha)} - \sum_{i\varepsilon I} y_i| \leq$ $\sum_{i\varepsilon I}|x_i^{(\alpha)} - y_i| \leq \sum_{i\varepsilon I} u_i^{(\alpha)} \downarrow 0$. Therefore, $\hat{x}^{(\alpha)} \xrightarrow{o} \hat{y}$.

(7.4) Proposition. If $\{E_\alpha : \alpha\varepsilon\Gamma\}$ is a collection of vector lattices and I is an arbitrary nonempty set, then

$$\ell^1_I(\Pi_{\alpha\varepsilon\Gamma} E_\alpha) \simeq \Pi_{\alpha\varepsilon\Gamma}\ell^1_I(E_\alpha).$$

The proof of (7.4) amounts to showing that mapping ψ defined in the proof of (2.), when restricted to $\ell_I^1(\Pi_\alpha E_\alpha)$, gives the desired isomorphism.

Many of the results of section 3 carry over to $\ell_I^1(E)$. In particular, the proof of (2.12) remains valid, establishing the following proposition:

(7.5) Proposition. Given that E is a vector sublattice of F,

(a) if E is an ideal in F, then $\ell_I^1(E)$ is an ideal in $\ell_I^1(F)$;

(b) if E is order dense in F, then $\ell_I^1(E)$ is order dense in $\ell_I^1(F)$;

(c) if E is quasi order dense in F, then $\ell_I^1(E)$ is quasi order dense in $\ell_I^1(F)$;

(d) if E is order closed in F, then $\ell_I^1(E)$ is order closed in $\ell_I^1(F)$.

(7.6) Proposition. If E is an Archimedean vector lattice, then $\ell_I^1(E)$ is order dense in the (Dedekind complete) vector lattice $\ell_I^1(\hat{E})$, but in general $\widehat{\ell_I^1(E)}$ does not coincide with $\ell_I^1(\hat{E})$. Specifically, $\widehat{\ell_I^1(E)}$ is the ideal generated by $\ell_I^1(E)$ in $\ell_I^1(\hat{E})$.

Proof. By (7.5) $\ell_I^1(E)$ is order dense in $\ell_I^1(\hat{E})$. Let λ denote the ideal generated by $\ell_I^1(E)$ in $\ell_I^1(\hat{E})$. Then λ is a Dedekind complete vector lattice in which $\ell_I^1(E)$ is order dense. If $[\hat{x}_i, I] \in \lambda^+$, then by a well-known characterization of the ideal generated by a vector sublattice, there exists some $[x_i, I] \in \ell_I^1(E)$ such that $[\hat{x}_i, I] \leq [x_i, I]$. Therefore, by (4.1) $\lambda = \widehat{\ell_I^1(E)}$.

We now present an example of a vector lattice E such that $\ell_I^1(\hat{E}) \neq \widehat{\ell_I^1(E)}$. Let E denote the vector lattice of all eventually

constant sequences of real numbers. Then $\hat{E} = m$. Let $\{p_k : k \in N\}$ denote the set of prime numbers, with $p_k \neq p_{k'}$ if $k \neq k'$. For each $k \in N$, let $\{x_n^{(k)}\}$ denote the sequence

$$x_n^{(k)} = \begin{cases} 1 \text{ if } p_k \text{ divides } n, \\ 0 \text{ if } p_k \text{ does not divide } n. \end{cases}$$

Then $[\{x_n^{(k)}\}, k \in N] \in \ell_N^1(m) = \ell_N^1(\hat{E})$, since $\forall J \in \nabla(N)$, $\sum_{k\in J} |\{x_n^{(k)}\}| \leq (1,1,1,\ldots,1,\ldots) \in m$. Suppose, on the other hand, that $[\{x_n^{(k)}\}, k\in N] \in \widehat{\ell_N^1(E)}$. Then there exists $[\{y_n^{(k)}\}, k \in N] \in \ell_N^1(E)$ such that $x_n^{(k)} \leq y_n^{(k)}$ for every n, k. Thus $\exists \{y_n\} \in E$ such that $\sum_{k\in N} y_n^{(k)} \leq y_n$ for every n. Now by definition of E, for each k the sequence $\{y_n^{(k)}\}$ is eventually constant; hence its terms are eventually ≥ 1. Let $p \in N$. For each $i = 1,2,\ldots,p$ there exists $n_i \in N \ni n \geq n_i$ implies $y_n^{(i)} \geq 1$. Thus $n \geq \max\{n_1, n_2, \ldots, n_p\}$ implies $\sum_{k=1}^{p} y_n^{(k)} \geq p$; hence, $y_n \geq p$. This result contradicts the eventually constant nature of $\{y_n\}$. Therefore, $[\{x_n^{(k)}\}, k \in N] \notin \widehat{\ell_N^1(E)}$.

Section 8. $\ell_I^1(E)$ as a normed space.

Throughout this section we shall assume that $(E, \leq, \|\cdot\|)$ is a normed vector lattice as defined in the introduction. We shall examine several natural norms induced on $\ell_I^1(E)$ by the norm $\|\cdot\|$ in E.

(8.1) <u>Definition</u>. An <u>ℓ-norm</u> on $\ell_I^1(E)$, relative to $\|\cdot\|$, is any monotone norm $\|\cdot\|_\ell$ on $\ell_I^1(E)$ such that

(a) $[x_i, I] \in \phi_I(E)$ implies $\|[x_i, I]\|_\ell = \|\sum_{i\in I} |x_i|\|$;

(b) if $\sum_{i\in J} |x_i| \leq y \in E$ for all $J \in \nabla(I)$, then $\|[x_i, I]\|_\ell \leq \|y\|$.

For each $[x_i, I] \in \ell^1_I(E)$ we define

$$\|[x_i, I]\|_1 = \sup \{\|\textstyle\sum_{i\varepsilon J}|x_i|\| : J \in \nabla(I)\},$$

$$\|[x_i, I]\|_{1'} = \inf \{\|u\|: \textstyle\sum_{i\varepsilon J}|x_i| \leq u \in E \ \ \forall J \in \nabla(I)\}.$$

Existence of the real numbers $\|[x_i, I]\|_1$ and $\|[x_i, I]\|_{1'}$ follows from the monotonicity of $\|.\|$ on E, the order boundedness of the set $\{\sum_{i\varepsilon J}|x_i| : J \in \nabla(I)\}$, and the completeness of R.

(8.2) <u>Proposition</u>. If $(E, \leq, \|.\|)$ is a normed vector lattice, then

(a) $\|.\|_1$ and $\|.\|_{1'}$ are ℓ-norms on $\ell^1_I(E)$;

(b) any ℓ-norm $\|.\|_\ell$ on $\ell^1_I(E)$ satisfies the inequality

$$\|.\|_1 \leq \|.\|_\ell \leq \|.\|_{1'}.$$

<u>Proof</u>. (a) We first establish the triangle inequality for $\|.\|_1$ and $\|.\|_{1'}$. Let $[x_i, I], [y_i, I] \in \ell^1_I(E)$. Then $\|[x_i,I] + [y_i,I]\|_1$ $= \sup_J \|\sum_{i\varepsilon J}|x_i + y_i|\| \leq \sup_J \|\sum_{i\varepsilon J}|x_i| + \sum_{i\varepsilon J}|y_i|\| \leq \sup_J \Big(\|\sum_{i\varepsilon J}|x_i|\|$ $+ \|\sum_{i\varepsilon J}|y_i|\|\Big) \leq \|[x_i, I]\|_1 + \|[y_i, I]\|_1$. Thus $\|.\|_1$ satisfies the triangle inequality. Let $u,v \in E$ such that for every $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i| \leq u$ and $\sum_{i\varepsilon J}|y_i| \leq v$. Then $\sum_{i\varepsilon J}|x_i + y_i| \leq u + v$ and hence, by monotonicity of $\|.\|$ on E, $\|[x_i, I] + [y_i, I]\|_{1'} \leq \|u + v\| \leq$ $\|u\| + \|v\|$. Taking infima in this inequality, first over all such u and then over all such v, we obtain $\|[x_i, I] + [y_i, I]\|_{1'} \leq \|[x_i, I]\|_{1'}$ $+ \|[y_i, I]\|_{1'}$. Therefore, $\|.\|_1$ and $\|.\|_{1'}$ satisfy the triangle inequality. It is then quite clear that $\|.\|_1$ and $\|.\|_{1'}$ are norms on $\ell^1_I(E)$.

That the norms $\|.\|_1$ and $\|.\|_{1'}$ are monotone is also clear. Now

suppose $[x_i, I] \in \phi_I(E)$. Let $J_0 = \{i \in I: x_i \neq 0\}$. Then by monotonicity of $\|.\|$, $\|[x_i, I]\|_1 = \|\sum_{i\varepsilon J_0}|x_i|\| = \|[x_i, I]\|_{1'}$. That $\|.\|$, and $\|.\|_{1'}$ satisfy (b) of (8.1) is obvious. Thus $\|.\|_1$ and $\|.\|_{1'}$ are ℓ-norms on $\ell^1_I(E)$.

(b) Suppose that $\|.\|_\ell$ is an ℓ-norm on E, and let $[x_i, I] \in \ell^1_I(E)$. For each $J \in \nabla(I)$ define $[x_i(J), I] \in \phi_I(E)$ by

$$x_i(J) = \begin{cases} |x_i| & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

Then $[|x_i|, I] = \sup_J [x_i(J), I]$. As $\ell^1_I(E)$ is a normed vector lattice, $\|[x_i,I]\|_\ell = \|[|x_i|, I]\|_\ell = \|\sup_J[x_i(J), I]\|_\ell \geq \sup_J \|[x_i(J), I]\|_\ell =$ $\sup_J\|\sum_{i\varepsilon J}|x_i|\| = \|[x_i, I]\|_1$. That $\|[x_i, I]\|_\ell \leq \|[x_i, I]\|_{1'}$ is clear from definition (8.1). Therefore, (b) is true.

(8.3) Proposition. The norm $\|.\|$ is additive on E^+ if and only if $\|.\|_1$ is additive on $\ell^1_I(E)^+$.

Proof. Suppose $\|.\|$ is additive on E^+, and let $[x_i, I], [y_i, I] \in \ell^1_I(E)^+$. Then $\|[x_i, I] + [y_i, I]\|_1 = \sup_J \|\sum_{i\varepsilon J}(x_i+y_i)\| = \sup_J\|\sum_{i\varepsilon J} x_i + \sum_{i\varepsilon J} y_i\| = \sup_J (\|\sum_{i\varepsilon J} x_i\| + \|\sum_{I\varepsilon J} y_i\|) = \sup_J \|\sum_{i\varepsilon J} x_i\| + \sup_J \|\sum_{i\varepsilon J} y_i\|$, since "$\sup_J$" here represents the o-limit of a net of real members, and o-limits preserve sums. Thus $\|.\|_1$ is additive on $\ell^1_I(E)^+$.

For the converse, suppose $\|.\|_1$ is additive on $\ell^1_I(E)^+$. Given $x, y \in E^+$, pick arbitrary $i_0 \in I$ and define $[x_i, I]$ and $[y_i, I]$ by

$$x_i = \begin{cases} x & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0, \end{cases} \qquad y_i = \begin{cases} y & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0. \end{cases}$$

Then $\|x + y\| = \|[x_i, I] + [y_i, I]\|_1 = \|[x_i, I]\|_1 + \|[y_i, I]\|_1 =$

$\|x\| + \|y\|$, and hence, $\|.\|$ is additive on E^+.

(8.4) Example of a normed vector lattice $(E, \leq, \|.\|)$ for which the norms $\|.\|_1$ and $\|.\|_{1'}$ are not equivalent. Let $E = m$, with the usual order but with the norm

$$\|\{x_n\}\| = \sup_n \frac{|x_n|}{n} + \overline{\lim_n}|x_n|.$$

Recall from section 7 the representation of the elements of $\ell^1_N(m)$ as matrices. For each $k \in N$ let S_k denote the matrix whose i, j^{th} entry is 0 if $i \neq j$ or if $i = j < k$, and is 1 if $i = j \geq k$:

$$S_k = \begin{array}{l} \\ \\ \\ \\ \left[\begin{matrix} 0 & 0 & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \end{matrix}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ - k^{th} \text{ row} \\ \\ \\ \\ \end{array}.$$

Then $\{S_k\}$ is a sequence in $\ell^1_N(m)$, the n^{th} column of which is a sequence $\{s_n^{(k)}\} \in m$; $S_k = [\{s_n^{(k)}\}, n\varepsilon N]$. By definition $\|S_k\|_1 = \sup_J \|\sum_{n\varepsilon J}|s_n^{(k)}|\| = \frac{1}{k} + 0 = \frac{1}{k}$, while $\|S_k\|_{1'} = \inf\{\|u\|: \forall J \in \nabla(N), \sum_{n\varepsilon J}|s_n^{(k)}| \leq u\} = \frac{1}{k} + 1$. Thus $\|S_k\|_1 \to 0$, while $\|S_k\|_{1'} \to 1$. Therefore, the two norms are not equivalent.

(8.5) Proposition. Let $\|.\|_\ell$ denote an ℓ-norm on $\ell^1_I(E)$.

(a) If $\{[x_i^{(n)}, I]\}_{n=1}^\infty$ $\|.\|_\ell$-converges to an element $[y_i, I]$ of $\ell^1_I(E)$, then for each $i \in I$, $\{x_i^{(n)}\}$ $\|.\|$-converges in E to y_i.

(b) $(\ell^1_I(E), \leq, \|\cdot\|_\ell)$ contains a norm-closed subspace, namely $\alpha_{I,i_0}(E)$, which is isometric to E. [Thus $(\ell^1_I(E), \|.\|_\ell)$ is norm complete only if E is $\|.\|$-complete.]

Proof. (a) Since $\|\cdot\|_1 \leq \|\cdot\|_\ell$, it suffices to prove the result for $\|\cdot\|_1$. If $\|[x_i^{(n)}, I] - [y_i, I]\|_1 \to 0$, then $\sup_J \|\sum_{i\varepsilon J}|x_i^{(n)} - y_i|\| \to 0$, hence $\forall i \in I, \|x_i^{(n)} - y_i\| = \||x_i^{(n)} - y_i|\| \to 0$.

(b) Let $i_0 \in I$. We already know that the map $y \to [y_i, I]$, where $y_{i_0} = y$ and $y_i = 0$ for $i \neq i_0$, is an isomorphism of E onto $\alpha_{I,i_0}(E)$. By definition $\|[y_i, I]\|_\ell = y$. Thus this map is also an isometry. That $\alpha_{I,i_0}(E)$ is norm closed follows from (a).

In many important cases there is only one suitable ℓ-norm on $\ell_I^1(E)$. It is thus of interest to consider conditions under which this will happen. Two concepts which prove to be important in this investigation are semi-continuity of $\|.\|$ on E and uniqueness of extension of $\|.\|$ to the Dedekind completion $\hat{E}$. These concepts are presented more fully in the thesis of R. Reichard [12], and I shall present here only those details which seem relevant or enlightening in the present context.

(8.6) Definition. The norm $\|.\|$ in a normed vector lattice $(E, \leq, \|.\|)$ is said to be

(a) semi-continuous if $0 \leq x_\tau \uparrow x$ implies $\sup_\tau \|x_\tau\| = \|x\|$;

(b) continuous if $x_\tau \downarrow 0$ implies $\inf_\tau \|x_\tau\| = 0$;

(c) sequentially continuous if $x_n \downarrow 0$ implies $\inf_n \|x_n\| = 0$.

Notice that a continuous norm is also semi-continuous, for if $\|.\|$ is continuous and $x_\tau \uparrow x$, then $x - x_\tau \downarrow 0$ and $\|x - x_\tau\| \downarrow 0$; hence, $0 \leq \|x\| - \sup_\tau \|x_\tau\| = \inf_\tau \{\|x\| - \|x_\tau\|\} \leq \inf_\tau \|x - x_\tau\| = 0$, and therefore, $\|x\| = \sup_\tau \|x_\tau\|$.

Also note that in a normed vector lattice with continuous norm, order convergence implies norm convergence. For if $x_\alpha \to x$, then there exists a net $\{y_\alpha\}$ with $|x - x_\alpha| \leq y_\alpha \downarrow 0$. But then $\|x - x_\alpha\| \leq \|y_\alpha\| \downarrow 0$. Therefore, we have in particular,

$$x = \sum_{i\varepsilon I} x_i \text{ implies } x = \|.\|\text{-}\lim_J \sum_{i\varepsilon J} x_i.$$

As noted in the introduction, every L-space has a continuous norm. The spaces L_p and ℓ_p are additional examples of normed vector lattices with continuous norm.

In the space m with the usual ordering, the usual norm $\|x\| = \sup_n |x_n|$ is semi-continuous but not continuous. Semi-continuity may be seen as follows: if $s_\tau = \{x_n^{(\tau)}\} \uparrow \{y_n\} = y$, then $\sup_\tau \|s_\tau\| = \sup_\tau (\sup_n |x_n^{(\tau)}|) = \sup_n \sup_\tau |x_n^{(\tau)}| = \sup_n |y_n| = \|y\|$. That $\|.\|$ is not continuous may be seen by considering $s_k = (0,0,\ldots,1,1,\ldots,1,\ldots)$, all of whose terms are 1 except the first k, which are 0. Clearly $s_k \downarrow 0$, but $\|s_k\| = 1 \ \forall k$.

Consider again the space m with its usual ordering, but this time with the norm $\|x\|_* = \sup_n |x_n| + \overline{\lim_n}\, x_n$. If for each $k\varepsilon N$ we let s_k denote the sequence $(1,1,\ldots,1,0,0,\ldots)$ whose n^{th} component ξ_{kn} is 1 if $n \leq k$ and is 0 if $n > k$, then $s_k \uparrow 1 = (1,1,\ldots,1,1,\ldots)$. But $\|s_k\|_* = 1$ for every k, while $\|1\|_* = 2$. Thus we have an example of a norm $\|.\|_*$ which is not even semi-continuous.

(8.7) <u>Proposition</u>. Suppose $(E, \leq, \|.\|)$ is a Dedekind complete normed vector lattice with semi-continuous $\|.\|$, and $\|.\|_\ell$ is an ℓ-norm on $\ell_I^1(E)$. If $\{[x_i^{(\alpha)}, I]\}_{\alpha\varepsilon\Gamma}$ is a net in $\ell_I^1(E)$ which $\|.\|_\ell$-converges to $[y_i, I] \ \varepsilon \ \ell_I^1(E)$, and if we define (for all $\alpha \ \varepsilon \ \Gamma$) $x^{(\alpha)} = \sum_{i\varepsilon I} |x_i^{(\alpha)}|$

and $y = \sum_{i\varepsilon I}|y_i|$, then $x^{(\alpha)} \xrightarrow{\|\cdot\|} y$. That is,

$$[x_i^{(\alpha)}, I] \xrightarrow{\|\cdot\|_\ell} [y_i, I] \text{ implies } \sum_{i\varepsilon I}|x_i^{(\alpha)}| \xrightarrow{\|\cdot\|} \sum_{i\varepsilon I}|y_i|.$$

Proof. Since $\|\cdot\|_1 \leq \|\cdot\|_\ell$ it suffices to prove the result for $\|\cdot\|_1$. Suppose $\|[x_i^{(\alpha)}, I] - [y_i, I]\|_1 \to 0$; that is, $\sup_J \|\sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|\| \to 0$. For every $\alpha \in \Gamma$, $|x_i^{(\alpha)}| \leq |x_i^{(\alpha)} - y_i| + |y_i|$ $(i \in I)$; hence for each $J \in \nabla(I)$,

$$\begin{aligned}\sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)}| &\leq \sup_J \left(\sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i| + \sum_{i\varepsilon J}|y_i|\right) \\ &\leq \sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i| + \sup_J \sum_{i\varepsilon J}|y_i|.\end{aligned}$$

Interchanging $|x_i^{(\alpha)}|$ and $|y_i|$, we obtain a similar inequality, and combining these two inequalities we obtain

$$\left|\sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)}| - \sup_J \sum_{i\varepsilon J}|y_i|\right| \leq \sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|;$$

that is, $|x^{(\alpha)} - y| \leq \sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|$, from which it follows that

$$\|x^{(\alpha)} - y\| \leq \left\|\sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|\right\|.$$

Now for each $\alpha \in \Gamma$, $\sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i| \uparrow \sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|$; hence, by semi-continuity of $\|\cdot\|$, $\sup_J \|\sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|\| = \|\sup_J \sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|\|$. Therefore, $\|x^{(\alpha)} - y\| \leq \sup_J \|\sum_{i\varepsilon J}|x_i^{(\alpha)} - y_i|\| \to 0$.

One can easily construct an example to show that without the hypothesis of semi-continuity, the conclusion of (8.7) need not hold. Consider $E = m$, with the usual order and with the norm

$$\|\{x_n\}\| = \sup_n \frac{|x_n|}{n} + \overline{\lim_n}|x_n|.$$

Let S_k be defined as in (8.4), let I denote the infinite identity matrix, let I_k denote the $k \times k$ identity matrix, and let V_k denote the infinite matrix all of whose entries are 0 except the first k

entries along the diagonal, which are 1:

$$V_k = \left(\begin{array}{c|c} I_k & O \\ \hline O & O \end{array}\right), \qquad S_k = \left(\begin{array}{c|c} O & O \\ \hline O & I \end{array}\right)$$

Then $\{V_k\}$ is a sequence in $\ell_N^1(m)$ and $\|V_k - I\|_1 = \|S_{k+1}\|_1 = \frac{1}{k+1} \to 0$, as shown in (8.4). Representing V_k and I in the form $V_k = [v_n^{(k)}, n \in N]$ and $I = [e_n, N]$ ($v_n^{(k)}, e_n \in E$) we have $[v_n^{(k)}, n\in N] \xrightarrow{\|\cdot\|_1} [e_n, N]$. On the other hand, for each $k \in N$, $\sum_{n\in N} v_n^{(k)} = (1,1,\ldots,1,0,\ldots)$ whose first k terms are 1, while $\sum_{n\in N} e_n = (1,1,\ldots,1,\ldots)$, all of whose terms are 1. Thus $\left\|\sum_{n\in N} v_n^{(k)} - \sum_{n\in N} e_n\right\| = \|(0,0,\ldots,1,1,\ldots)\|$ $= \frac{1}{k+1} + 1 \not\to 0$. We therefore have the desired counterexample: $[v_n^{(k)}, n\in N] \xrightarrow{\|\cdot\|_1} [e_n, N]$, while $\sum_{n\in N}|v_n^{(k)}| \overset{\|\cdot\|}{\not\to} \sum_{n\in N}|e_n|$.

(8.8) <u>Proposition</u>. If $(E, \leq, \|\cdot\|)$ is a normed vector lattice with semi-continuous $\|\cdot\|$, then, relative to $\|\cdot\|$,

(a) $\|\cdot\|_1$ is semi-continuous on $\ell_I^1(E)$;

(b) $\|\cdot\|_1$ is the <u>only</u> semi-continuous ℓ-norm on $\ell_I^1(E)$;

(c) $x = \sum_{i\in I}|x_i|$ implies $\|[x_i, I]\|_\ell = \|x\|$ for every ℓ-norm on $\ell_I^1(E)$.

<u>Proof</u>. (a) Suppose $0 \leq [x_i^{(\tau)}, I] \uparrow [y_i, I]$ in $\ell_I^1(E)$. By (3.3) for each $i \in I$, $0 \leq x_i^{(\tau)} \uparrow y_i$; hence, for all $J \in \nabla(I)$, $0 \leq \sum_{i\in J} x_i^{(\tau)} \uparrow \sum_{i\in J} y_i$. Then by semi-continuity of $\|\cdot\|$, $\sup_\tau \left\|\sum_{i\in J} x_i^{(\tau)}\right\| = \left\|\sum_{i\in J} y_i\right\|$. We therefore have $\sup_\tau \|[x_i^{(\tau)}, I]\|_1 = \sup_\tau \sup_J \left\|\sum_{i\in J} x_i^{(\tau)}\right\|$ $= \sup_J \sup_\tau \left\|\sum_{i\in J} x_i^{(\tau)}\right\| = \sup_J \left\|\sum_{i\in J} y_i\right\| = \|[y_i, I]\|_1$. Therefore, $\|\cdot\|_1$ is semi-continuous.

(b) Let $\|\cdot\|_\ell$ be a semi-continuous ℓ-norm. Define the net $\{[x_i(J), I]\}_{J\varepsilon\nabla(I)}$ as in (8.2). Since $[|x_i(J)|, I] \uparrow [|x_i|, I] = |[x_i, I]|$, we have by semi-continuity and definition of ℓ-norm that

$$\|[x_i, I]\|_\ell = \|[|x_i|, I]\|_\ell = \sup_J \|[|x_i(J)|, I]\|_\ell$$
$$= \sup_J \| \textstyle\sum_{i\varepsilon J} |x_i| \| = \|[x_i, I]\|_1.$$

Therefore, $\|\cdot\|_\ell = \|\cdot\|_1$.

(c) $\sum_{i\varepsilon J}|x_i| \uparrow x$ over the directed set $\nabla(I)$. Hence, by semi-continuity of $\|\cdot\|$, $\sup_J \|\sum_{i\varepsilon J}|x_i|\| = \|x\|$; i.e., $\|[x_i, I]\|_1 = \|x\|$. On the other hand, $\sum_{i\varepsilon J}|x_i| \leq x \;\forall J \varepsilon \nabla(I)$ implies $\|[x_i, I]\|_\ell \leq \|x\|$. Thus we have $\|\cdot\|_\ell \leq \|\cdot\|_1$, which in combination with (b) of (8.2) yields the desired equality $\|\cdot\|_\ell = \|\cdot\|_1$.

(8.9) <u>Corollary</u>. If $(E, \leq, \|\cdot\|)$ is a Dedekind complete vector lattice with semi-continuous $\|\cdot\|$, then there is one and only one ℓ-norm on $\ell_I^1(E)$, relative to $\|\cdot\|$; namely $\|\cdot\|_1$.

The proof follows directly from (6.4) and (8.8).

(8.10). <u>Example</u>: We shall exhibit here a normed vector lattice $(E, \leq, \|\cdot\|)$ with semi-continuous $\|\cdot\|$, such that not all ℓ-norms on $\ell_I^1(E)$ are semi-continuous. By virtue of (b) in (8.8) we need only produce an example in which $\|\cdot\|_1 \neq \|\cdot\|_{1'}$. Note that in our example of (8.4) we did not have a semi-continuous $\|\cdot\|$.

Let E denote the space of eventually constant sequences of real numbers, with the usual ordering and with the norm

$$\|x\| = \sup_n \delta(n)\, |x_n|,$$

where $\delta(n) = 1$ if n is even, and 2 if n is odd. That $\|\cdot\|$ is a norm

is easy to see; we show only the triangle inequality: $\|x + y\| = \sup_n \delta(n)|x_n+y_n| \leq \sup_n \delta(n)\,(|x_n| + |y_n|) \leq \sup_n \delta(n)\,|x_n| + \sup_n \delta(n)\,|y_n| = \|x\| + \|y\|$. Semi-continuity of $\|.\|$ is also easy to see. Suppose $0 \leq x^{(\tau)} \uparrow y$ in E. Let $x^{(\tau)} = \{x_n^{(\tau)}\}$ and $y = \{y_n\}$; then for each n, $0 \leq x_n^{(\tau)} \uparrow y_n$ by (3.3). Thus $\sup_\tau \|x^{(\tau)}\| = \sup_\tau \sup_n \delta(n)\, x_n^{(\tau)} = \sup_n \delta(n) \sup_\tau x_n^{(\tau)} = \sup_n \delta(n)\, y_n = \|y\|$.

For each $n \in \mathbb{N}$, let e_n denote the sequence $e_n = (0,0,\ldots,1,0,\ldots)$, having n^{th} term 1 and all others zero. Consider $[e_{2n}, \mathbb{N}] \in \ell_I^1(E)$. We have $\|[e_{2n}, \mathbb{N}]\|_1 = \sup_J \|\sum_{n\varepsilon J} e_{2n}\| = 1$, since each e_{2n} has all its odd terms equal to 0. But if $\sum_{i\varepsilon J} e_{2n} \leq u \in E$ for every $J \in \nabla(I)$, then u must have its even terms equal to 1, from some point on; hence $\|u\| \geq 2$. Thus $\|[e_{2n}, \mathbb{N}]\|_{1'} \geq 2$, and therefore

$$\|\cdot\|_1 \neq \|\cdot\|_{1'}.$$

(8.11) Monotone extensions of $\|.\|$ to $\hat{E}$. In this number we present results without proof. For details and references, consult R. Reichard [12] or V. A. Solov'ev [17].

Suppose $(E, \leq, \|.\|)$ is a normed vector lattice. By a (monotone) extension $\|.\|^*$ of $\|.\|$ to the Dedekind completion $\hat{E}$ of E we mean a norm on $\hat{E}$ which agrees with $\|.\|$ on E and is monotone on $\hat{E}$. Then $(\hat{E}, \leq, \|.\|^*)$ is clearly a normed vector lattice. Given an arbitrary $\hat{x} \in \hat{E}$ we define

$$\|\hat{x}\|_\mu = \inf\{\|y\|: |\hat{x}| \leq y \in E\}, \text{ and}$$
$$\rho(\hat{x}) = \sup\{\|z\|: 0 \leq z \leq |\hat{x}|,\ z \in E\}.$$

Clearly every monotone extension $\|.\|^*$ of $\|.\|$ satisfies $\rho(.) \leq \|.\|^* \leq \|.\|_\mu$.

The following facts will be useful in the sequel.

(a) Proposition (Vulikh). $\|.\|_\mu$ is a monotone extension of $\|.\|$ to $\hat{E}$.

(b) Proposition (Nishura - Lozanovski). If $(E, \leq, \|.\|)$ is a Banach lattice, then $(\hat{E}, \leq, \|.\|_\mu)$ is also a Banach lattice.

(c) Note. the map $\rho: \hat{E} \to R^+$ is not generally a norm.

(d) Proposition. If $\|.\|$ is semi-continuous on E, then ρ is a semi-continuous norm on $\hat{E}$, which we denote $\|.\|_\rho$. In fact, $\|.\|_\rho$ is the unique semi-continuous monotone extension of $\|.\|$ to $\hat{E}$.

(e) Proposition (Solov'ev - Reichard). If $\|.\|$ is continuous on E, then $\|.\|_\mu$ is continuous on $\hat{E}$ and is the unique monotone extension of $\|.\|$ to $\hat{E}$.

(f) Proposition (Solov'ev). If $\|.\|$ is sequentially continuous on E, then $\|.\|$ is continuous if and only if E is of countable type (i.e., every bounded subset of pairwise disjoint elements of E is countable).

(g) Proposition (Vulikh [18], Chapt. VII, §6). If $(E, \leq, \|.\|)$ is Dedekind σ-complete, with sequentially continuous $\|.\|$, then

(i) $\|.\|$ is continuous.

(ii) $(E, \leq)$ is Dedekind complete.

(8.12) Proposition. If $(E, \leq, \|.\|)$ is a normed vector lattice, and $[x_i, I] \in \ell^1_I(E)$, with $\hat{x} = \sum_{i\varepsilon I}|x_i|$ in $\hat{E}$, then

(a) $\|[x_i, I]\|_{1'} = \|\hat{x}\|_\mu$;

(b) if $\|.\|$ is semi-continuous, then $\|[x_i, I]\|_1 = \|\hat{x}\|_\rho$.

Proof: (a) $\|[x_i, I]\|_{1'} = \inf\{\|u\|: \sum_{i\varepsilon J}|x_i| \leq u \ \forall J \in \nabla(I)\} =$

$\inf\{\|u\|: \sum_{i\varepsilon I}|x_i| = \hat{x} \leq u \;\varepsilon\; E\} = \|\hat{x}\|_\mu$.

(b) In the semi-continuous case we have, using (8.11) (d),

$$\|[x_i, I]\|_1 = \sup_J \|\sum_{i\varepsilon J}|x_i|\| = \sup_J \|\sum_{i\varepsilon J}|x_i|\|_\rho = \|\sup_J \sum_{i\varepsilon J}|x_i|\|_\rho = \|\hat{x}\|_\rho.$$

(8.13) Corollary. If $\|.\|$ is a continuous norm on E, then there exists one and only one ℓ-norm on $\ell^1_I(E)$, relative to $\|.\|$.

Proof. In the case of a continuous $\|.\|$, we have $\|.\|_\mu = \|.\|_\rho$ on $\hat{E}$ by (8.11) (e). Apply (8.12) and (8.2).

It is clear, in view of (8.12), that if the norm $\|.\|$ is semi-continuous, then any property on $(E, \leq, \|.\|)$ which implies $\rho(.) = \|.\|_\mu$ on $\hat{E}$, will imply that $\ell^1_I(E)$ has a unique ℓ-norm. Such properties were investigated by R. Reichard, in [12]. In particular, the "projection property" in E is one such property.

We next consider the question of continuity of the norm $\|.\|_1$ on $\ell^1_I(E)$ for a given continuous norm $\|.\|$ on E. It is clear that continuity of $\|.\|$ alone does not imply continuity of $\|.\|_1$. For in view of (8.11), continuity of $\|.\|_1$ entails that $\ell^1_I(E)$ be of countable type, which cannot be true if I is an uncountable set.

(8.14) Proposition. If $(E, \leq, \|.\|)$ is a normed vector lattice with continuous $\|.\|$, then $\|.\|_1$ is sequentially continuous on $\ell^1_I(E)$.

Proof. Suppose $\{[x_i^{(n)}, I]\}$ is a sequence in $\ell^1_I(E)$, with $[x_i^{(n)}, I] \downarrow 0$. For each $n \;\varepsilon\; \mathbb{N}$ there exists $\hat{y}^{(n)} \;\varepsilon\; \hat{E}$ such that

$$\hat{y}^{(n)} = \sup_J \sum_{i\varepsilon J} x_i \qquad (J \;\varepsilon\; \nabla(I)),$$

and by (8.13)

$$\|\hat{y}^{(n)}\|_\mu = \|[x_i^{(n)}, I]\|_1.$$

Note that for each n, the net $\{\sum_{i\notin J} x_i^{(n)}\}_{J\in\nabla(I)}$ decreases in $\hat{E}$ to 0, since

$$\inf_J \sum_{i\notin J} x_i^{(n)} = \inf_J \left(\sum_{i\in I} x_i^{(n)} - \sum_{i\in J} x_i^{(n)}\right) \quad \text{(see (6.5))}$$
$$= \hat{y}^{(n)} - \sup_J \sum_{i\in J} x_i^{(n)}$$
$$= 0.$$

Thus by continuity of $\|\cdot\|_\mu$, for each n we have $\inf_J \|\sum_{i\notin J} x_i^{(n)}\| = 0$.

Let $\delta > 0$. Then there exists $J_0 \in \nabla(I)$ such that $\|\sum_{i\notin J_0} x_i^{(1)}\| < \frac{\delta}{2}$. For each $i \in I$, $x_i^{(n)} \downarrow 0$; hence, $\sum_{i\in J_0} x_i^{(n)} \downarrow 0$. By continuity of $\|\cdot\|$, $\|\sum_{i\in J_0} x_i^{(n)}\| \downarrow 0$. Thus there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|\sum_{i\in J_0} x_i^{(n)}\| < \frac{\delta}{2}$.

For every $n \geq n_0$, we thus have $\|\hat{y}^{(n)}\|_\mu = \|\sum_{i\in J_0} x_i^{(n)} + \sum_{i\notin J_0} x_i^{(n)}\| \leq \|\sum_{i\in J_0} x_i^{(n)}\| + \|\sum_{i\notin J_0} x_i^{(n)}\| < \frac{\delta}{2} + \|\sum_{i\notin J_0} x_i^{(1)}\| < \delta$. Therefore, $\inf_n \|[x_i^{(n)}, I]\|_1 = \inf_n \|\hat{y}^{(n)}\|_\mu = 0$.

Given a norm-complete normed vector lattice $(E, \leq, \|\cdot\|)$, one may ask whether $\ell_I^1(E)$ is norm-complete for some ℓ-norm $\|\cdot\|_\ell$. The following two results are partial answers to this question. Recall that the norm $\|\cdot\|$ on the vector lattice E is said to be <u>monotone bounded</u> if every $\|\cdot\|$-bounded subset of E is order bounded. We say that a norm $\|\cdot\|$ on a vector lattice E is <u>monotone complete</u> if, for each sequence $\{x_n\}$ in E such that $0 \leq x_n \uparrow$ and $\|x_n\| \leq M < \infty$ for all $n \in \mathbb{N}$, there exists an element x in E such that $x_n \uparrow x$.

(8.15) <u>Proposition</u>. If $(E, \leq, \|\cdot\|)$ is a Banach lattice with a monotone bounded norm $\|\cdot\|$, then $\ell_I^1(E)$ is $\|\cdot\|_1$-complete.

Proof. Let $\{[x_i^{(n)}, I]\}$ be a $\|.\|_1$-Cauchy sequence in $\ell_I^1(E)$. Then for each $i \in I$, $\{x_i^{(n)}\}$ is a $\|.\|$-Cauchy sequence in E. Thus, by completeness of E, for each $i \in I$ there exists $x_i \in E$ such that $\|x_i^{(n)} - x_i\| \to 0$.

Now consider the family $[x_i, I]$. Let $\delta > 0$. Then there exists $M \in N$ such that $n \geq M$ implies $\|[x_i^{(n)} - x_i^{(M)}, I]\|_1 < \delta$, by the Cauchy property. That is, $n \geq M$ implies $\sup_J \|\sum_{i\in J}|x_i^{(n)} - x_i^{(M)}|\| < \delta$. Observe that for each $J \in \nabla(I)$, $\|\sum_{i\in J}|x_i - x_i^{(M)}|\| = \|\sum_{i\in J}|\lim_n x_i^{(n)} - x_i^{(M)}|\| = \lim_n \|\sum_{i\in J}|x_i^{(n)} - x_i^{(M)}|\| \leq \delta$. Since $\|.\|$ is monotone bounded, and since the set $\{\sum_{i\in J}|x_i - x_i^{(M)}| : J \in \nabla(I)\}$ is norm bounded, this set must also be order bounded. Further,

$$\sum_{i\in J}|x_i| \leq \sum_{i\in J}|x_i - x_i^{(M)}| + \sum_{i\in J}|x_i^{(M)}|,$$

and $[x_i^{(M)}, I] \in \ell_I^1(E)$. Therefore, $[x_i, I] \in \ell_I^1(E)$. Finally, $\|[x_i^{(n)}, I] - [x_i, I]\|_1 \to 0$, since for each $J \in \nabla(I)$, $n \geq M$ implies $\|[x_i - x_i^{(n)}, I]\|_1 \leq \|[x_i - x_i^{(M)}, I]\|_1 + \|[x_i^{(M)} - x_i^{(n)}, I]\|_1$, and it has already been shown that each of the two terms on the right side of this last inequality are less than δ. Therefore, $\|[x_i, I] - [x_i^{(n)}, I]\|_1 \to 0$. Therefore, $(\ell_I^1(E), \|.\|_1)$ is norm complete.

(8.16) Proposition. If $(E, \leq, \|.\|)$ is a normed vector lattice with semi-continuous, monotone complete norm $\|.\|$, then every ℓ-norm $\|.\|_\ell$ on $\ell_N^1(E)$, relative to $\|.\|$, is monotone complete.

Proof. It is clearly sufficient to prove that $\|.\|_1$ is monotone complete. Suppose that $0 \leq [x_n^{(k)}, N] \uparrow$ and for all $k \in N$,

$\|[x_n^{(k)}, N]\|_1 \leq M < \infty$. Then for each n there exists an element x_n of E such that $x_n^{(k)} \uparrow x_n$ by our hypothesis on E. Then $[x_n^{(k)}, N] \uparrow [x_n, N]$ in $\omega_N(E)$. It remains to show that $[x_n, N]$ belongs to $\ell_N^1(E)$.

For each $r \in N$, let $\tau_r = \sum_{n=1}^r |x_n|$ $(= \sum_{n=1}^r x_n)$. Then $0 \leq \tau_r \uparrow$, and for every r, $\|\tau_r\| = \|\sum_{n=1}^r (\text{o-}\lim_K x_n^{(k)})\| = \|\text{o-}\lim_K \sum_{n=1}^r x_n^{(k)}\| = \|\sup_k \sum_{n=1}^r x_n^{(k)}\| = \sup_k \|\sum_{n=1}^r x_n^{(k)}\|$, since $\|.\|$ is semi-continuous. But $\sup_k \|\sum_{n=1}^r x_n^{(k)}\| = \|[x_n^{(k)}, N]\|_1 \leq M$, for every r. Therefore, by the monotone completeness of $\|.\|$ on E, there exists an element $x \in E$ such that $\tau_r \uparrow x$. Therefore, $[x_n, N] \in \ell_N^1(E)$.

CHAPTER III

A KÖTHE-TYPE DUALITY THEORY

Section 9. Multiplication in Archimedean vector lattices.

In Chapter III we consider only Archimedean vector lattices. It will be seen that Archimedean vector lattices all have a general multiplicative structure (however weak) which is sufficient to enable us to formulate a general theory of what we may reasonably call Köthe-family spaces, analogous to real Köthe sequences spaces.

(9.1) Definition. Let E be a Dedekind σ-complete vector lattice with a weak order unit 1. We say that E admits a multiplication operation relative to 1 if, corresponding to certain pairs x, y of elements of E, there exists a "product" xy, which obeys the conditions:

(M1) for every $x \in E$, $x1$ exists and equals x;

(M2) if xy exists, then yx exists and equals xy;

(M3) if xy, $(xy)z$ and yz all exists, then $x(yz)$ exists and equals $(xy)z$;

(M4) if xy and xz exists, then $x(y + z)$ exists and equals $xy + yz$;

(M5) if xy exists and $\alpha \in R$, then $(\alpha x)y$ exists and equals $\alpha(xy)$;

(M6) if $x, y \geq 0$ and xy exists, then $xy \geq 0$;

(M7) if xy exists, $|x'| \leq |x|$ and $|y'| \leq |y|$, then $x'y'$ exists;

(M8) $x \perp y$ if and only if xy exists and equals 0.

(9.2) Observations. For any multiplication operation relative to a weak order unit 1 on a Dedekind σ-complete vector lattice E, the following additional properties hold:

(M9) if $0 \leq x \leq y$, $z \geq 0$ and yz exists, then $0 \leq xz \leq yz$;

(M10) if xy exists, then

$$(xy)^+ = x^+y^+ + x^-y^-,$$
$$(xy)^- = x^+y^- + x^-y^+,$$
$$|xy| = |x|\ |y|;$$

(M11) if 1 is a strong order unit, then E is closed under this multiplication.

In 1940 B. Z. Vulikh ([19] and [20]) demonstrated a method for constructing, within an arbitrary Dedekind complete vector lattice with weak order unit 1, a product xy satisfying (M1) through (M8). Vulikh's method has been made accessible to the English-reading mathematician in the papers ([13] and [14]) of Rice. We give here a brief outline of this approach.

Let E be a Dedekind complete vector lattice with weak order unit 1. Define the collection of "unitary" elements of E by

$$u(E, 1) = \{e \in E: e \wedge (1-e) = 0\}.$$

Every element $x \in E^+$ is the supremum of all the scalar multiples of unitary elements that lie below it. The multiplication is then defined as follows:

(a) for $e, e' \in u(E, 1)$, $ee' = e \wedge e'$;

(b) for $x, y \geq 0$, $xy = \sup \{\alpha\beta ee': 0 \leq \alpha e \leq x,\ 0 \leq \beta e' \leq y,$ $e, e' \in u(E, 1),\ \alpha, \beta \in R^+\}$ if this supremum exists (otherwise xy is not defined);

(c) for $x, y \in E$, $xy = x^+y^+ - x^+y^- - x^-y^+ + x^-y^-$ if all the products on the right side exist (otherwise xy is not defined).

(9.3) Proposition. (Uniqueness of the product; Rice [13], theorem 5.1). In a Dedekind complete vector lattice E with weak order unit $\mathbb{1}$, if $x * y$ denotes another multiplication on E satisfying (M1) - (M8), then $x * y$ exists if and only if xy exists; moreover $x * y = xy$.

(9.4) Proposition. (Rice [13], lemma 5.2 and theorem 5.3): Let $\mathbb{1}$, $\mathbb{1}'$ be two units in a Dedekind complete vector lattice E. Denote the product relative to $\mathbb{1}$ by xy and the product relative to $\mathbb{1}'$ by $x * y$. Then

$$u(E, \mathbb{1}') = \{\mathbb{1}'e\colon e \in u(E, \mathbb{1})\},$$

and for every $x, y \in E$, if xy and $x * y$ both exist, then

$$x * y = (xy) * \mathbb{1}.$$

The development of the representation theory, outlined in the appendix, provided another proof that every Dedekind complete vector lattice admits a multiplication operation, and made possible further elucidation of the nature and properties of such multiplications. Let Q denote an arbitrary quasi-extremal compactum, as defined in the appendix. It can be shown that the usual (pointwise) multiplication of functions is an operation on $C_\infty(Q)$ satisfying (M1) through (M8), relative to the usual $\mathbb{1}$-function: $\mathbb{1}(t) = 1$ for all $t \in Q$. Moreover, $C_\infty(Q)$ is closed under this operation; $C_\infty(Q)$ is a commutative ring with identity. The same remarks hold for $C(Q)$.

Using the representation theory, and this multiplication in $C_\infty(Q)$, we have immediately the following proposition:

(9.5) Proposition. Every Dedekind σ-complete vector lattice E with a weak order unit 1 admits a multiplication operation relative to 1. If, in addition, 1 is a strong order unit, then E is closed under this multiplication.

(9.6) Multiplication in Archimedean vector lattices. If E is an arbitrary Archimedean vector lattice, we may embed E in its universal completion $E^{\#}$, and thus by picking an arbitrary weak order unit 1 in $E^{\#}$ we obtain, as a consequence of (9.5), a "product" defined on certain pairs of elements of E, which has the properties (M2) through (M6) and (M8). Of course, if E already contains a weak order unit 1, then 1 is also a weak order unit for $E^{\#}$; if we take the multiplication relative to this 1, we will then have property (M1) in E.

From (M7) we see that if E is closed under the multiplication induced from $E^{\#}$, then so is $\hat{E}$.

Proposition (9.5) showed how a change of unit elements affects the definition of the product of two elements. The following proposition will show that, given two different universal completions of E (necessarily isomorphic), the unit elements may be chosen in such a way as to induce the same multiplication in E.

(9.7) Proposition. Suppose $E_1^{\#}$ and $E_2^{\#}$ are universally complete vector lattices, and suppose $\psi\colon E_1^{\#} \to E_2^{\#}$ is an isomorphism. Let 1_1 denote a weak order unit in $E_1^{\#}$; then $1_2 = \psi(1_1)$ is a weak order unit

in $E_2^{\#}$. Denote the products in $E_1^{\#}$ and $E_2^{\#}$, relative to 1_1 and 1_2 respectively, by xy and $x*y$, respectively. Then

$$\psi(xy) = \psi(x) * \psi(y).$$

Proof. In view of (M10) it suffices to prove this identity for $x, y \geq 0$. Note that $u(E_2^{\#}, 1_2) = \{f \in E_2^{\#}: f \wedge (1_2 - f) = 0\} = \{\psi(e): e \in E_1^{\#}, e \wedge (1_1 - e) = 0\} = \psi[u(E_1^{\#}, 1_1)]$. Now $xy = \sup \{\alpha\beta(e \wedge e'): 0 \leq \alpha e \leq x, 0 \leq \beta e' \leq y; e, e' \in u(E_1^{\#}, 1_1); \alpha, \beta \in R^+\}$. Since ψ is an isomorphism, we thus have

$$\begin{aligned}
\psi(xy) &= \sup \{\alpha\beta(\psi(e) \wedge \psi(e')): 0 \leq \alpha e \leq x,\ 0 \leq \beta e' \leq y, \text{ etc.}\} \\
&= \sup \{\alpha\beta(\psi(e) * \psi(e')): 0 \leq \alpha(\psi(e)) \leq \psi(x),\ 0 \leq \beta(\psi(e')) \leq \psi(y), \text{ etc.}\} \\
&= \sup \{\alpha\beta(f*f'): 0 \leq \alpha f \leq \psi(x),\ 0 \leq \beta f' \leq \psi(y);\ f, f' \in u(E_2^{\#}, 1_2); \\
&\qquad \alpha, \beta \in R^+\}. \\
&= \psi(x) * \psi(y).
\end{aligned}$$

By theorem 3.1 of Rice [13], and observing that $E \subseteq \hat{E}$, we have the following proposition.

(9.8) Proposition. If E is an Archimedean vector lattice with weak order unit 1, then for every $x \in E^+$ and every positive ingeger n, there is a unique positive n^{th} root of x (denoted $x^{\frac{1}{n}}$) in $\hat{E}$.

Later in this chapter we shall wish to raise elements of our vector lattice to arbitrary positive real powers. If E is an Archimedean vector lattice, $x \in E^+$ and p is a positive real number, the symbol x^p is ambiguous, in the sense that it depends upon the choice of multiplications taken in $E^{\#}$. Nevertheless, corresponding to each representation of $E^{\#}$ as a space $C_\infty(Q)$ and each choice of weak order unit 1 in E, the symbol x^p is well-defined by the representation theory.

(9.9) Proposition. If E is an Archimedean vector lattice closed under the multiplication induced on E by multiplication relative to some unit 1 in $E^{\#}$, then for every $x \in E^+$ and every positive real number p, $x^p \in \hat{E}$.

Proof. Let n denote the least positive integer greater than or equal to p. Representing x as an element of $C_\infty(Q)$, with 1 represented by the function $1(t) = t$, for all $t \in Q$, we see that

$$x^p \leq x^n \vee x^{n-1},$$

since $x(t) < 1$ implies $x^p(t) \leq x^{n-1}(t)$, while $x(t) \geq 1$ implies $x^p(t) \leq x^n(t)$. But E is closed under the multiplication, and $\hat{E}$ is an ideal in $E^{\#}$. Therefore, $x^n \vee x^{n-1} \in E$; hence, $x^p \in \hat{E}$.

The following proposition and corollary may be found in Rice [13] as theorem 10.3 and corollary 10.3.1.

(9.10) Proposition. (order continuity of the product) Let E be a universally complete vector lattice, and let $\{x_\alpha\}$, $\{y_\alpha\}$ be two nets in E indexed by the same directed set. If $x_\alpha \xrightarrow{o} x$ and $y_\alpha \xrightarrow{o} y$ in E, then $x_\alpha y_\alpha \xrightarrow{o} xy$.

(9.11) Corollary. Suppose $\{x_\alpha\}$, $\{y_\alpha\}$ are two nets in a Dedekind complete vector lattice E, indexed by the same directed set. If $x_\alpha \xrightarrow{o} x$, $y_\alpha \xrightarrow{o} y$, $x_\alpha y_\alpha \in E$ for all α, and there exists $z \in E$ such that $|x_\alpha y_\alpha| \leq z$ for all α, then the product xy exists in E and $x_\alpha y_\alpha \xrightarrow{o} xy$.

Many of the spaces encountered in applications of vector lattice theory have a natural multiplication defined on them already, without

reference to their universal completion. Many of these spaces are not Dedekind complete, or do not contain a unit element. For example, the space $C[0,1]$ and the sequence space c are vector lattices which are not Dedekind complete, while the space ϕ of finite sequences does not have a weak order unit. In each of these spaces the pointwise (or coordinatewise) product provides a natural multiplication operation.

In cases such as these it is natural to ask whether the given multiplication coincides with that obtained by embedding the vector lattice in its universal completion and choosing some appropriate weak order unit.

Suppose E is a vector lattice of real-valued functions $f: X \to R$ defined on some abstract set X, with the vector lattice structure induced from $\omega_X(R)$, and with the pointwise multiplication operation. Assume further that for every $\xi \in X$, there exists $f \in E$ such that $f(\xi) \neq 0$. Then $E^{\#} = \omega_X(R)$. Let $\mathbb{1}$ denote the constant function, $\mathbb{1}(\xi) = 1$ for all $\xi \in X$. (We do not assume $\mathbb{1} \in E$.) Then the pointwise multiplication operation on $E^{\#}$ satisfies (M1) through (M8), and hence by (9.3) is the <u>only</u> such operation that can be defined on $E^{\#}$ with $\mathbb{1}$ as unit. Therefore in this case, the multiplication on E <u>does</u> coincide with multiplication induced from $E^{\#}$, relative to this unit $\mathbb{1}$.

Section 10. The spaces $\ell^p_I(E, \mathbb{1})$, $1 \leq p \leq \infty$.

We continue to assume that E is an Archimedean vector lattice, embedded in its universal completion $E^{\#}$. The vector lattice $E^{\#}$ has a weak order unit $\mathbb{1}$, which of course is not unique.

(10.1) <u>Definition</u>. Given an Archimedean vector lattice E, and an arbitrary weak order unit 1 in $E^{\#}$, for each $1 < p < \infty$, we define <u>$\ell_I^p(E, 1)$</u> $= \{[x_i, I] \in \omega_I(E): \exists\, u \in E$ such that $\forall J \in \nabla(I)$, $\sum_{i\in J}|x_i|^p \leq u\}$. Here the powers $|x_i|^p$ are taken in $E^{\#}$, relative to 1. Thus neither $|x_i|^p$ nor $\sum_{i\in J}|x_i|^p$ can be expected to belong to E. But if $[x_i, I] \in \ell_I^p(E, 1)$, then, since $\hat{E}$ is an ideal in $E^{\#}$, all the powers $|x_i|^p$ and all the sums $\sum_{i\in J}|x_i|^p$ belong to $\hat{E}$. Thus

$$\ell_I^p(E, 1) = \{[x_i, I] \in \omega_I(E): [x_i^p, I] \in \ell_I^1(\hat{E})\}.$$

For the sake of uniformity of notation, we define $\ell_I^1(E, 1) = \ell_I^1(E)$, and $\ell_I^\infty(E, 1) = \ell_I^\infty(E) = m_I(E)$.

(10.2) <u>Example</u>. Let E = m; then $E^{\#} = s$. Consider the family $[e_n, N] \in \omega_N(E)$, where $e_n = (0,0,\ldots,1,0,\ldots)$, as defined in the examples following (1.1). If we take $1 = (1,1,\ldots,1,\ldots)$ then $[e_n,N] \in \ell_N^p(E, 1)$ for every $1 \leq p \leq \infty$. On the other hand, suppose we take $1' = (1,1/2,1/3,\ldots)$. Then 1' is a weak order unit in $E^{\#}$. For each $n \in N$, $\frac{1}{n}e_n \in u(E, 1')$. Relative to 1', $(\frac{1}{n}e_n)^p = \frac{1}{n}e_n$, and $|e_n|^p = n^p(\frac{1}{n}e_n)^p = n^{p-1}\, e_n$. Thus relative to 1', $\sum_{n\in\mathbf{N}}|e_n|^p \notin E$ if $p > 1$. Therefore, $[e_n, N] \notin \ell_N^p(E, 1')$ if $1 < p < \infty$. This example shows that $\ell_I^p(E, 1)$ does indeed depend upon the choice of unit taken in $E^{\#}$.

This example exhibits another phenomenon which will be of some significance later. E is closed under multiplication relative to 1, but not under multiplication relative to 1'. Let us denote the product of two elements $x, y \in E^{\#}$, relative to 1', by $x * y$. Then

$1 * 1 = \sup\{\alpha\beta ee': 0 \leq \alpha e \leq 1,\ 0 \leq \beta e' \leq 1,\ e, e' \in u(E, 1')\}$

$\geq \sup\{n^2(\frac{1}{n}e_n) * (\frac{1}{n}e_n): n = 1, 2, \ldots\} = \sup\{ne_n: n = 1, 2, \ldots\}$.

Hence, $1 * 1 \notin m = E$. Thus $E\ (= \hat{E})$ is not closed under multiplication relative to $1'$.

The following proposition shows that definition (10.1) is neither vacuous nor trivial.

(10.3) Proposition. If I is an infinite set, and $1 \leq p < q \leq \infty$, then (a) $\ell_I^p(E, 1) - \phi_I(E)$ is nonempty;

(b) $\ell_I^q(E, 1) - \ell_I^p(E, 1)$ is nonempty;

(c) if $\hat{E}$ is closed under multiplication, then $\phi_I(E) \subseteq \ell_I^p(E, 1)$;

(d) if $1 \in \hat{E}$, then $\ell_I^p(E, 1) \subseteq \ell_I^\infty(E)$;

(e) if $1 \in \hat{E}$ and $\hat{E}$ is closed under multiplication, then

$$\phi_I(E) \subseteq \ell_I^p(E, 1) \subseteq \ell_I^q(E, 1) \subseteq \ell_I^\infty(E).$$

(f) if $\phi_I(E) \subseteq \ell_I^p(E, 1)$ for some $2 \leq p < \infty$, then $\hat{E}$ is closed under multiplication.

Proof. Since I is an infinite set we may assume, without loss of generality, that $N \subseteq I$.

(a) The result is clear if $p = \infty$; hence, we assume $p < \infty$. Pick an arbitrary $u > 0$ in E. Since E is order-dense in $E^\#$, we may pick an $x \in E$ such that $0 < x < u^{1/p}$ and consequently $0 < x^p < u$. Define $[x_i, I] \in \omega_I(E)$ by

$$x_i = \begin{cases} \left(\frac{1}{i}\right)^{1/(2p)} x & \text{if } i \in N, \\ 0 & \text{if } i \notin N. \end{cases}$$

Then $[x_i, I] \in \ell_I^p(E, 1)$ since for every $J \in \nabla(I)$, $\sum_{i\in J}|x_i|^p = \sum_{n\in J\cap N} n^{-2}\, x^p \leq (\sum_{n\in N} n^{-2})\, u \in E$. Clearly, $[x_i\ I] \notin \phi_I(E)$.

(b) First suppose $q < \infty$. Pick an arbitrary $u \in E$ and $x \in E$ with $0 < x^q < u$ as in (a) above. Define $[x_i, I] \in \omega_I(E)$ by

$$x_i = \begin{cases} \left(\frac{1}{i}\right)^{1/p} x & \text{if } i \in N, \\ 0 & \text{if } i \notin N. \end{cases}$$

For every $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i|^q = \sum_{n\varepsilon J_N} n^{-q/p}x^q \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^{q/p}}\right) u \in E$,

since $q > p$. Thus $[x_i, I] \in \ell_I^q(E, 1)$. However, $\sum_{i\varepsilon J}|x_i|^p =$ $\sum_{n\varepsilon J\cap N}|n^{-1}x^p| = \left(\sum_{n\varepsilon J\cap N} n^{-1}\right) x^p$ which, by the Archimedean property in $\hat{E}$, cannot be bounded by an element of E. Thus $[x_i, I] \notin \ell_I^p(E, 1)$.

Now let $q = \infty$. Pick an arbitrary $u > 0$ in E and define

$$y_i = \begin{cases} u & \text{if } i \in N, \\ 0 & \text{if } i \notin N. \end{cases}$$

Then $[y_i, I] \in \ell_I^\infty(E)$, but $[y_i, I] \notin \ell_I^p(E, 1)$ since the sums $\sum_{i\varepsilon J}|y_i|^p$, being integral multiples of u^p, cannot be bounded in $\hat{E}$, again by the Archimedean property.

(c) Suppose $\hat{E}$ is closed under multiplication. If $[x_i, I] \in \phi_I(E)$, let $J_0 = \{i \in I: x_i \neq 0\}$; then $\sum_{i\varepsilon J}|x_i|^p \leq \sum_{i\varepsilon J_0}|x_i|^p$, which belongs to $\hat{E}$ by (9.9). Thus $[x_i, I] \in \ell_I^p(E, 1)$.

(d) Let $1 \in \hat{E}$ and $[x_i, I] \in \ell_I^p(E, 1)$; say $\sum_{i\varepsilon J}|x_i|^p \leq u \in E$, for all $J \in \nabla(I)$. By representing x_i and u as functions in $C_\infty(Q)$, we see that $|x_i|^p \leq u\ \forall i$ implies $|x_i| \leq u \vee 1 \in \hat{E}\ \forall i \in I$. Hence, $[x_i, I] \in \ell_I^\infty(E)$.

(e) Suppose $1 \in \hat{E}$ and $\hat{E}$ is closed under multiplication. Let $[x_i, I] \in \ell_I^p(E, 1)$. In view of (d) we assume $q < \infty$. Then there exist $u, v \in E$ such that for every $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i|^p \leq u$, and for every $i \in I$, $|x_i| \leq v$. By (9.9) $v^{q-p} \in \hat{E}$. We thus have, for any $J \in \nabla(I)$,

$$\sum_{i\varepsilon J}|x_i|^q = \sum_{i\varepsilon J}|x_i|^{q-p}\ |x_i|^p \leq v^{q-p} \sum_{i\varepsilon J}|x_i|^p \leq v^{q-p}\ u \in \hat{E}.$$

Therefore, $[x_i, I] \in \ell_I^q(E, 1)$.

(f) Suppose $\phi_I(E) \subseteq \ell_I^p(E)$ for some $2 \leq p < \infty$. Since $xy = \frac{1}{4}[(x+y)^2 - (x-y)^2]$ we need only show that $\hat{E}$ is closed under squaring. Let $\hat{x} \in \hat{E}$, $\hat{x} \neq 0$; then there exists $x \in E$ such that $|\hat{x}| < x$. Pick an arbitrary $i_0 \in I$ and define

$$x_i = \begin{cases} x & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0. \end{cases}$$

Then by hypothesis, $[x_i, I] \in \ell_I^p(E, 1)$, so there exists $u \in E$ such that $|x_i|^p \leq u$ for all $i \in I$. From the representation theory we see that $x^2 \leq x^p \vee x$. Thus

$$\hat{x}^2 \leq |\hat{x}|^2 \leq x^2 \leq x^p \vee x \leq u \vee x \in E.$$

Thus $\hat{x}^2 \in \hat{E}$, since $\hat{E}$ is an ideal in $E^\#$.

(10.4) Corollary. If 1 is a strong order unit for E or $\hat{E}$, and $1 \leq p < q \leq \infty$, then $\phi_I(E) \subseteq \ell_I^p(E, 1) \subseteq \ell_I^q(E, 1) \subseteq \ell_I^\infty(E)$.

Proof. If 1 is a strong order unit, then by (M7), $\hat{E}$ is closed under multiplication.

(10.5) Examples. It is not hard to find spaces E which satisfy the condition

$$(\dagger)\quad \phi_I(E) \subseteq \ell_I^p(E, 1) \subseteq \ell_I^q(E, 1) \subseteq \ell_I^\infty(E).$$

Consider R, R^n, the space R^I of all real-valued functions on an abstract set I, the space $C(X)$ of all bounded, continuous, real-valued functions on a topological space X, the space m of all bounded sequences, the space c of all convergent sequences, and the space of eventually constant sequences. All these spaces satisfy condition (e) of proposition (10.3); thus they satisfy (†).

On the other hand, many common spaces do not satisfy condition (†). Consider $E = \ell_1$, with termwise multiplication, and take $\mathbb{1}$ to be the usual constant sequence $\mathbb{1} = \{1,1,1,\ldots\}$ in $E^{\#}$ $(= \omega)$. For each $n \in \mathbb{N}$ define the sequence $S_n = \{x_{nk}\}$ with

$$x_{nk} = \begin{cases} \frac{1}{k} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

If $q > 1$, then $[S_n, \mathbb{N}] \in \ell^q_{\mathbb{N}}(E, \mathbb{1})$, since the series $\sum_{k=1}^{\infty} (\frac{1}{k})^q$ converges and hence, $[S^q_n, \mathbb{N}] \in \ell^1_{\mathbb{N}}(E)$. But the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges; hence, there is no $S \in \ell_1$ such that $S \geq S_n$ $\forall n$. Thus $[S_n, \mathbb{N}] \notin \ell^{\infty}_{\mathbb{N}}(E)$. We therefore have in this case,

$$\ell^q_{\mathbb{N}}(E, \mathbb{1}) \not\subseteq \ell^{\infty}_{\mathbb{N}}(E, \mathbb{1}),$$

for all $q > 1$.

Now consider $E = L_1[0,1]$, with the linear operations and order defined pointwise, and with the usual constant function $\mathbb{1}$. Recall that $L_1[0,1] \supseteq L_p[0,1] \supsetneqq L_q[0,1] \supsetneqq L_{\infty}[0,1]$, for $1 \leq p < q < \infty$. Thus there exists a function $f \in E^+$ such that $f \in L_p[0,1] - L_q[0,1]$. Pick any $i_0 \in I$ and define $[f_i, I] \in \omega_I(E)$ by setting

$$f_i = \begin{cases} f & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0. \end{cases}$$

Since $f^p \in E$ but f^q is not bounded above by any element of E, $[f_i, I] \in \ell^p_I(E, \mathbb{1})$ and $[f_i, I] \in \phi_I(E)$, but $f \notin \ell^q_I(E, \mathbb{1})$. For this example, therefore,

$$\phi_I(E) \not\subseteq \ell^q_I(E, \mathbb{1}), \text{ and}$$

$$\ell^p_I(E, \mathbb{1}) \not\subseteq \ell^q_I(E, \mathbb{1}).$$

(10.6) <u>Proposition</u> (order Hölder inequality). If p, q are finite

positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $[x_i y_i, I] \in \ell^1_I(\hat{E})$ whenever $[x_i, I] \in \ell^p_I(E, 1)$ and $[y_i, I] \in \ell^q_I(E, 1)$. More precisely, if for all $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i|^p \leq u$ and $\sum_{i\varepsilon J}|y_i|^q \leq v$, then for all $J \in \nabla(I)$,

$$\sum_{i\varepsilon J}|x_i y_i| \leq u \vee v.$$

Proof. In the representation of $E^{\#}$ as $C_\infty(Q)$, let $\overline{x_i}$ and $\overline{y_i}$ represent x_i and y_i, respectively. We use a well-known lemma, found in Royden [15], page 112, which says that if $\alpha, \beta \in R^+$ and $0 < \lambda < 1$, then
$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta.$$
Thus $|\overline{x_i}(t)\ \overline{y_i}(t)| \leq \frac{1}{p}|\overline{x_i}(t)|^p + \frac{1}{q}|\overline{y_i}(t)|^q$ whenever $\overline{x_i}(t)$ and $\overline{y_i}(t)$ are both finite. In case $|\overline{x_i}(t)| = \infty$ or $|\overline{y_i}(t)| = \infty$, this inequality still holds. Therefore, for each $i \in I$,
$$|x_i y_i| \leq \frac{1}{p}|x_i|^p + \frac{1}{q}|y_i|^q,$$
so that for all $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i y_i| \leq \frac{1}{p}\sum_{i\varepsilon J}|x_i|^p + \frac{1}{q}\sum_{i\varepsilon J}|y_i|^q \leq \frac{1}{p}u + \frac{1}{q}v \leq (\frac{1}{p} + \frac{1}{q})\ (u \vee v) = u \vee v.$

(10.7) Proposition (order Minkowski inequality). For each $p \geq 1$, if $[x_i, I], [y_i, I] \in \ell^p_I(E, 1)$, then $[x_i + y_i, I] \in \ell^p_I(E, 1)$; more precisely, for $J \in \nabla(I)$,
$$\sum_{i\varepsilon J}|x_i + y_i|^p \leq 2^p\ (\ \sum_{i\varepsilon J}|x_i|^p + \sum_{i\varepsilon J}|y_i|^p).$$
Thus $\ell^p_I(E, 1)$ is an ideal in $\omega_I(E)$.

Proof. Note that $|x_i + y_i|^p \leq (|x_i| + |y_i|)^p \leq [2(|x_i| \vee |y_i|)]^p$. In the representation of $E^{\#}$ as $C_\infty(Q)$ let $\overline{x_i}$ and $\overline{y_i}$ represent x_i and y_i, respectively. Then for all $t \in Q$, $(|\overline{x_i}| \vee |\overline{y_i}|)\ (t) = |\overline{x_i}(t)| \vee |\overline{y_i}(t)|$, and since these are extended real numbers, $[(|\overline{x_i}| \vee |\overline{y_i}|)(t)]^p$

$= |\overline{x_i}(t)|^p \vee |\overline{y_i}(t)|^p$. Thus $(|x_i| \vee |y_i|)^p = |x_i|^p \vee |y_i|^p \leq |x_i|^p + |y_i|^p$. Therefore, for all $J \in \nabla(I)$, $\sum_{i\varepsilon J}|x_i + y_i|^p \leq 2^p \sum_{i\varepsilon J}(|x_i| \vee |y_i|)^p \leq 2^p (\sum_{i\varepsilon J}|x_i|^p + \sum_{i\varepsilon J}|y_i|^p)$.

(10.8) <u>Proposition</u>. If $[x_i^{(\alpha)}, I] \xrightarrow{o} [y_i, I]$ in $\ell_I^p(E, 1)$, then $\sum_{i\varepsilon I}|x_i^{(\alpha)}|^p \xrightarrow{o} \sum_{i\varepsilon I}|y_i|^p$ in $\hat{E}$.

<u>Proof</u>. Since $[|x_i^{(\alpha)}|^p, I]$ and $[|y_i|^p, I]$ belong to $\ell_I^1(\hat{E})$, we may apply (7.3) to yield the desired conclusion.

(10.9) <u>Proposition</u>. (a) If E is Dedekind complete, so is $\ell_I^p(E, 1)$.

(b) $\ell_I^p(E, 1)$ is order-dense in $\ell_I^p(\hat{E}, 1)$; $\widehat{\ell_I^p(E, 1)}$ is the ideal generated by $\ell_I^p(E, 1)$ in $\ell_I^p(\hat{E}, 1)$.

<u>Proof</u>. (a) As shown in (3.10), if E is Dedekind complete, then $\omega_I(E)$ is Dedekind complete. By (10.7) $\ell_I^p(E, 1)$ is an ideal in $\omega_I(E)$; hence, it must be Dedekind complete.

(b) The proof of (3.12) (b) remains valid, with $\lambda_I(E) = \ell_I^p(E, 1)$ and $F = \hat{E}$, showing that $\ell_I^p(E, 1)$ is order dense in $\ell_I^p(\hat{E}, 1)$. The ideal generated by $\ell_I^p(E, 1)$ majorizes $\ell_I^p(E, 1)$ in the sense of definition (4.1) (c); hence, by (4.1), it must equal $\widehat{\ell_I^p(E, 1)}$.

There are spaces which satisfy the condition (†) of (10.5), but which fail to satisfy the criteria of (10.3) (e). We shall see as a result of (10.12) that among such spaces are the sequence space ϕ, the space of all real-valued step functions on R, and the space $C^*(R)$ of real-valued continuous functions on R with compact support. We now consider a property which will guarantee condition (†) in a vector lattice.

(10.10) Definition. A subset $\{e_\alpha: \alpha \in \Gamma\}$ of a vector lattice E is said to be a generalized strong order unit for E if

(a) $e_\alpha \wedge e_\beta = 0$ whenever $\alpha \neq \beta$, and

(b) given any $x \in E^+$, there exists $T \subseteq \Gamma$ and $0 < \lambda < 1$ such that $\sup\{e_\alpha: \alpha \in T\} \in E$ and $\lambda x \leq \sup\{e_\alpha: \alpha \in T\}$.

(10.11) Proposition. Let E be an Archimedean vector lattice with a generalized strong order unit $\{e_\alpha: \alpha \in \Gamma\}$. Then in $E^\#$ the element $1 = \sup\{e_\alpha: \alpha \in \Gamma\}$ is a weak order unit, and for any $T \subseteq \Gamma$, $\sup\{e_\alpha: \alpha \in T\} \in u(E^\#, 1)$.

Proof. First note that $1 \in E^\#$ since every set of pairwise disjoint elements of $E^\#$ has a supremum.

Now let $x^\# > 0$ in $E^\#$. There exists an element $x \in E$ such that $0 < x < x^\#$, and there exists a subset $T \subseteq \Gamma$ such that $0 < \lambda x \leq \sup\{e_\alpha: \alpha \in T\} \leq 1$. Hence, $1 \wedge x^\# \geq 1 \wedge x \geq 1 \wedge \lambda x = \lambda x > 0$. Thus 1 is a weak order unit in $E^\#$.

Let $\alpha \in \Gamma$. Then $1 - e_\alpha = (\sum_{\beta\in\Gamma} e_\beta) - e_\alpha = \sum_{\beta\neq\alpha} e_\beta = \sup\{e_\beta: \beta \neq \alpha\}$. Thus $e_\alpha \wedge (1 - e_\alpha) = e_\alpha \wedge (\sup\{e_\beta: \beta \neq \alpha\}) = \sup\{(e_\alpha \wedge e_\beta): \beta \neq \alpha\} = 0$; thus, $e_\alpha \in u(E^\#, 1)$.

Now $u(E^\#, 1)$ is a complete Boolean algebra (see Vulikh [18], theorem IV.2.1). Hence, for every $T \subseteq \Gamma$, $\sup\{e_\alpha: \alpha \in T\} \in u(E^\#, 1)$.

(10.12) Proposition. If E is an Archimedean vector lattice with generalized strong order unit $\{e_\alpha: \alpha \in \Gamma\}$, and if $1 = \sup\{e_\alpha: \alpha \in \Gamma\}$ in $E^\#$, then

(a) $\hat{E}$ is closed under multiplication relative to 1;

(b) $\phi_I(E) \subseteq \ell^p_I(E, 1) \subseteq \ell^q_I(E, 1) \subseteq \ell^\infty_I(E)$, for any $1 \leq p < q \leq \infty$.

Proof. (a) Let $x, y \in \hat{E}$. Then $x \leq \lambda_1 f_1$ and $y \leq \lambda_2 f_2$ for some $f_1 = \sup \{e_\alpha : \alpha \in T_1\} \in E$, $f_2 = \sup \{e_\alpha : \alpha \in T_2\} \in E$, $0 < \lambda_1 < 1$, and $0 < \lambda_2 < 1$. Then $xy \leq (\lambda_1 f_1)(\lambda_2 f_2) = (\lambda_1\lambda_2)(f_1 f_2) = \lambda_1\lambda_2(f_1 \wedge f_2) \in E$, so that $xy \in \hat{E}$.

(b) That $\phi_I(E) \subseteq \ell_I^p(E, 1)$ follows from (10.3)(c). Let $[x_i, I] \in \ell_I^p(E, 1)$. Then there exists $u \in E$ such that $\sum_{i\in J}|x_i|^p \leq u$ for every $J \in \nabla(I)$. There exists $f = \sup \{e_\alpha : \alpha \in T\} \in E$ and $0 < \lambda < 1$ such that $u \leq \lambda f$. Then for all $i \in I$, $|x_i|^p \leq \lambda f$. Note that $f^{1/p} = f$ since $f \in u(E^\#, 1)$. Thus for all $i \in I$, $|x_i| \leq \lambda^{1/p} f \in E$. Therefore, $[x_i, I] \in \ell_I^\infty(E)$. Therefore, $\ell_I^p(E, 1) \subseteq \ell_I^\infty(E)$.

Again let $[x_i, I] \in \ell_I^p(E, 1)$. Then there exist $u, v \in E$ such that for all $J \in \nabla(I)$, $\sum_{i\in J}|x_i|^p \leq u$ and for all $i \in J$, $|x_i| \leq v$. Now $v \leq \lambda 1$ for some $\lambda > 0$. Thus for all $J \in \nabla(I)$, $\sum_{i\in J}|x_i|^q =$
$\sum_{i\in J}|x_i|^{q-p}|x_i|^p \leq v^{q-p}\sum_{i\in J}|x_i|^p \leq (\lambda^{q-p}1)u = \lambda^{q-p}u \in E$.
Therefore, $[x_i, I] \in \ell_I^q(E, 1)$. We have proved $\ell_I^p(E, 1) \subseteq \ell_I^q(E, 1)$.

Section 11. p-norms on the spaces $\ell_I^p(E, 1)$.

Throughout this section we assume that $(E, \leq, \|.\|)$ is a normed vector lattice. It is natural to question whether it is then possible to define a norm on $\ell_I^p(E, 1)$ which behaves like the usual norm $\|x\| = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$ for $x = \{x_n\}$ in the familiar sequence space ℓ_p. In section 9 we saw that there is possibly more than one ℓ-norm on $\ell_I^1(E)$ relative to $\|.\|$; similarly, we shall have to allow for more than one suitable p-norm on $\ell_I^p(E, 1)$, relative to $\|.\|$.

(11.1) Definition: If $(E, \leq, \|\cdot\|)$ is a normed vector lattice, then a p-norm on $\ell_I^p(E, 1)$, relative to $\|\cdot\|$, is any monotone norm $\|\cdot\|_p$ on $\ell_I^p(E, 1)$ such that if $\hat{x} = \sum_{i\varepsilon I}|x_i|^p$ in $\hat{E}$, then for all $u, v \varepsilon E$ such that $u \leq \hat{x} \leq v$,

$$\|u\|^{1/p} \leq \|[x_i, I]\|_p \leq \|v\|^{1/p}.$$

Equivalently, $\rho(\hat{x})^{1/p} \leq \|[x_i, I]\|_p \leq \|\hat{x}\|_\mu^{1/p}$ (see (8.11)).

In view of the last inequality, one would expect the presence of a p-norm on $\ell_I^p(E, 1)$ to be closely related to the presence of a monotone extension $\|\cdot\|_*$ of the norm $\|\cdot\|$ onto all of $\hat{E}$. Indeed it turns out that to each such extension there corresponds, in a natural way, a p-norm $\|\cdot\|_{*_p}$ on $\ell_I^p(E, 1)$. The following definition is motivated by (8.12).

Suppose $\|\cdot\|_*$ is a monotone extension of $\|\cdot\|$ to $\hat{E}$. For each $[x_i, I]$ $\varepsilon\ \ell_I^p(E, 1)$ there exists a unique element

$$\hat{x} = \sum_{i\varepsilon I}|x_i|^p \ \varepsilon\ \hat{E}.$$

If we define, for every $[x_i, I] \ \varepsilon\ \ell_I^p(E, 1)$,

$$\|[x_i, I]\|_{*_p} = \|\hat{x}\|_*^{1/p},$$

then , as we shall show in corollary (11.4), $\|\cdot\|_{*_p}$ is a p-norm on $\ell_I^p(E, 1)$. This norm will be called the "p-norm associated with $\|\cdot\|_*$".

Note that if we define $\|[x_i, I]\|_{*_1} = \|\hat{x}\|_*$, where $\hat{x} = \sum_{i\varepsilon I}|x_i|$, then $\|\cdot\|_{*_1}$ is an ℓ-norm on $\ell_I^1(E)$.

If E is Dedekind complete, then we write $\|\cdot\|_p$ instead of $\|\cdot\|_{*_p}$.

(11.2) Proposition (Hölder's Inequality). Let p, q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$, and suppose $\|\cdot\|_*$ is a monotone extension of $\|\cdot\|$ to $\hat{E}$. If $[x_i, I] \ \varepsilon\ \ell_I^p(E, 1)$ and $[y_i, I] \ \varepsilon\ \ell_I^q(E, 1)$ then

$[x_i y_i, I] \in \ell_I^1(\hat{E})$ and

$$\|[x_i y_i, I]\|_{*_1} \leq \|[x_i, I]\|_{*_p} \|[y_i, I]\|_{*_q}.$$

<u>Proof</u>. That $[x_i y_i, I] \in \ell_I^1(\hat{E})$ was proved in (10.6). Let $x = [x_i, I]$ and $y = [y_i, I]$. Assume $x \neq 0$ and $y \neq 0$, since otherwise the desired inequality is trivially true. Now represent each x_i and y_i as functions $\overline{x_i}, \overline{y_i} \in C_\infty(Q)$. For each $t \in Q$, let $\lambda = \frac{1}{p}$,

$$\alpha = \left|\frac{\overline{x_i}(t)}{\|x\|_{*_p}}\right|^p \text{ and } \beta = \left|\frac{\overline{y_i}(t)}{\|y\|_{*_p}}\right|^p,$$

and apply the lemma mentioned in the proof of (10.6). We thus obtain

$$\frac{|x_i y_i|}{\|x\|_{*_p} \|y\|_{*_q}} \leq \frac{1}{p}\left(\frac{|x_i|}{\|x\|_{*_p}}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|y\|_{*_q}}\right)^q \qquad (\forall\, i \varepsilon I).$$

Then, letting $\hat{x} = \sum_{i\varepsilon I}|x_i|^p$ and $\hat{y} = \sum_{i\varepsilon I}|y_i|^q$, we have

$$\frac{\sum_{i\varepsilon J}|x_i y_i|}{\|x\|_{*_p} \|y\|_{*_q}} \leq \frac{1}{p(\|x\|_{*_p})^p} \sum_{i\varepsilon J}|x_i|^p + \frac{1}{q(\|y\|_{*_q})^q} \sum_{i\varepsilon J}|y_i|^q$$

$$= \frac{1}{p\|\hat{x}\|_*} \sum_{i\varepsilon J}|x_i|^p + \frac{1}{q\|\hat{y}\|_*} \sum_{i\varepsilon J}|y_i|^q$$

$$\leq \frac{\hat{x}}{p\|\hat{x}\|_*} + \frac{\hat{y}}{q\|\hat{y}\|_*}.$$

Thus, taking the norm $\|.\|_*$ of both sides, we obtain

$$\frac{\left\|\sum_{i\varepsilon J}|x_i y_i|\right\|_*}{\|x\|_{*_p} \|y\|_{*_q}} \leq 1,$$

from which the desired inequality follows immediately.

In the proof of the following proposition, the argument is analogous to the usual derivation of Minkowski's inequality from Hölder's inequality (see Goffman and Pedrick [2], pp. 4,5).

(11.3) <u>Proposition</u> (Minkowski's inequality). Suppose $\|\cdot\|_*$ is a monotone extension of $\|\cdot\|$ to $\hat{E}$, and let $1 < p < \infty$. Then for all $x = [x_i, I]$ and $y = [y_i, I]$ in $\ell_I^p(E, 1)$,

$$\|x + y\|_{*_p} \leq \|x\|_{*_p} + \|y\|_{*_p}.$$

<u>Proof</u>. For each $i \in I$, $|x_i + y_i|^p \leq |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$. Note that $[|x_i + y_i|^{p-1}, I] \in \ell_I^q(E, 1)$, since $(|x_i + y_i|^{p-1})^q = |x_i + y_i|^p$. Thus by the order Hölder inequality (10.6), both $[|x_i + y_i|^{p-1}|x_i|, I]$ and $[|x_i + y_i|^{p-1}|y_i|, I]$ belong to $\ell_I^1(\hat{E})$, and in $\hat{E}$ we have

$$\sum_{i\varepsilon I}|x_i + y_i|^p \leq \sum_{i\varepsilon I}\left(|x_i + y_i|^{p-1}|x_i|\right) + \sum_{i\varepsilon I}\left(|x_i + y_i|^{p-1}|y_i|\right).$$

Taking the norm $\|\cdot\|_*$ of both sides, and applying Hölder's inequality (11.2), we obtain the inequality $\left\|\sum_{i\varepsilon I}|x_i + y_i|^p\right\|_* \leq \|[|x_i + y_i|^{p-1}, I]\|_{*_q}\|[x_i, I]\|_{*_p} + \|[|x_i + y_i|^{p-1}, I]\|_{*_q}\|[y_i, I]\|_{*_p}$. That is,

$$\|x + y\|_{*_p}^p \leq \|(x + y)^{p-1}\|_{*_q}(\|x\|_{*_p} + \|y\|_{*_p}),$$

and we thus have

$$\|x + y\|_{*_p}\|x + y\|_{*_p}^{p-1} \leq (\|x\|_{*_p} + \|y\|_{*_p})\|(x + y)^{p-1}\|_{*_q}.$$

Now $\|x + y\|_{*_p}^{p-1} = \left(\left\|\sum_{i\varepsilon I}|x_i + y_i|^p\right\|_*^{1/p}\right)^{p-1} = \left\|\sum_{i\varepsilon I}|x_i + y_i|^p\right\|_*^{1/q}$ $= \left\|\sum_{i\varepsilon I}|x_i + y_i|^{pq-q}\right\|_*^{1/q} = \|(x + y)^{p-1}\|_{*_q}$. Therefore, cancelling this common factor from both sides of the last inequality above, we obtain the Minkowski inequality.

(11.4) <u>Corollary</u>. Given any monotone extension $\|\cdot\|_*$ of $\|\cdot\|$ to $\hat{E}$, and any $p \geq 1$, $\|\cdot\|_{*_p}$ is a p-norm on $\ell_I^p(E, 1)$.

(11.5) Observations. We list here a few facts about p-norms on $\ell_I^p(E, 1)$ which follow rather easily. Suppose $(E, \leq, \|.\|)$ is a normed vector lattice.

(a) If $\sum_{i\in I}|x_i|^p = x \in E$, then $\|[x_i, I]\|_p = \|x\|^{1/p}$, for any p-norm $\|.\|_p$. Hence, if E is Dedekind complete, there exists only one p-norm on $\ell_I^p(E, 1)$, relative to $\|.\|$.

(b) If $\|.\|_*$ and $\|.\|_\#$ are equivalent monotone extensions of $\|.\|$ to $\hat{E}$, then $\|.\|_{*p}$ and $\|.\|_{\#p}$ are equivalent p-norms on $\ell_I^p(E, 1)$.

(c) If $\|.\|_*$ is a semi-continuous monotone extension of $\|.\|$ to $\hat{E}$, then $\|.\|_{*p}$ is semi-continuous on $\ell_I^p(E, 1)$.

(d) Any condition on E which will guarantee that $\rho(.) = \|.\|_\mu$ on $\hat{E}$ (e.g., continuity of $\|.\|$, or the projection property in E) will imply that $\ell_I^p(E, 1)$ has only one p-norm, relative to $\|.\|$.

(e) If $\|.\|$ is a continuous norm on E, then $\|.\|_{\mu p}$ is sequentially continuous on $\ell_I^p(E, 1)$.

(f) If $\|.\|_*$ is a monotone extension of $\|.\|$ to $\hat{E}$, then we may obtain a p-norm $\|.\|_{*p*}$ extending $\|.\|_{*p}$ to $\ell_I^p(\hat{E}, 1)$ by defining

$$\|[\hat{x}_i, I]\|_{*p*} = \left\|\sum_{i\in I}|\hat{x}_i|^p\right\|_*^{1/p}.$$

(11.6) Definition. If $(E, \leq, \|.\|)$ is a normed vector lattice and $1 < p < \infty$, the norm $\|.\|$ is said to be p-additive if $\|x + y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$ in E.

An obvious example of a p-additive norm is the norm $\|\{x_n\}\| = \left(\sum_{n=1}^\infty|x_n|^p\right)^{1/p}$ on ℓ_p.

(11.7) Proposition. Suppose E is Dedekind complete. If $\|.\|$ is additive on E^+, then $\|.\|_p$ is a p-additive p-norm on $\ell_I^p(E, 1)$. If, in

addition, $1 \in E$ and I contains at least two elements, then the converse is true.

Proof. (a) Suppose $\|\cdot\|$ is additive on E^+, and let $[x_i, I]$, $[y_i, I] \in \ell^p_I(E, 1)$ with $[x_i, I] \wedge [y_i, I] = 0$. Then $x_i \wedge y_i$ for all $i \in I$, and

$$\begin{aligned}\|[x_i, I] + [y_i, I]\|_p^p &= \left\|\textstyle\sum_{i\in I}|x_i + y_i|^p\right\| \\ &= \left\|\textstyle\sum_{i\in I}\left(|x_i|^p + |y_i|^p\right)\right\| \quad (\text{since } x_i \wedge y_i = 0) \\ &= \left\|\textstyle\sum_{i\in I}|x_i|^p\right\| + \left\|\textstyle\sum_{i\in I}|y_i|^p\right\| \quad (\text{since } \|\cdot\| \text{ is additive on } E^+) \\ &= \|[x_i, I]\|_p^p + \|[y_i, I]\|_p^p.\end{aligned}$$

(b) Suppose $\|\cdot\|$ is a norm on E such that $\|\cdot\|_p$ is p-additive. Suppose that $1 \in E$ and that there exist $i_0 \neq i_1$ in I. Let $x, y \in E^+$. Observe from the representation of $E^\#$ as a space $C_\infty(Q)$ that $x^{1/p} \leq 1 \vee x \in E$. Hence, $x^{1/p} \in \hat{E} = E$. Similarly, $y^{1/p} \in E$. We thus define $[x_i, I], [y_i, I] \in \ell^p_I(E, 1)$ by setting

$$x_i = \begin{cases} x^{1/p} & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0 \end{cases} \quad \text{and} \quad y_i = \begin{cases} y^{1/p} & \text{if } i = i_1, \\ 0 & \text{if } i \neq i_1. \end{cases}$$

Since $\|\cdot\|_p$ is a p-additive p-norm on $\ell^p_I(E, 1)$ we then have $\|x + y\| = \|(x^{1/p})^p + (y^{1/p})\|^p = \left\|\sum_{i\in I}(x_i + y_i)^p\right\| = \|[x_i + y_i, I]\|_p^p = \|[x_i, I]\|_p^p + \|[y_i, I]\|_p^p = \|(x^{1/p})^p\| + \|(y^{1/p})^p\| = \|x\| + \|y\|$. Therefore, $\|\cdot\|$ is additive on E^+.

(11.8) Proposition (Solov'ev [17], theorem 8). If $\|\cdot\|$ is additive on E^+, then there exists a monotone extension $\|\cdot\|_*$ of $\|\cdot\|$ to $\hat{E}$ which is additive on $\hat{E}^+$.

(11.9) Corollary. $\|\cdot\|_{*_p}$ is then a p-additive p-norm on $\ell_I^p(E, 1)$.

Section 12. Köthe X-dual spaces

Throughout section 12 we continue to assume that E is an Archimedean vector lattice, embedded in its universal completion $E^{\#} \simeq C_{\infty}(Q)$, and that 1 is a fixed weak order unit in $E^{\#}$. We may assume that 1 is represented by the constant function, $1(t) = t$ for all $t \in Q$. We denote the product (relative to 1) of two elements $x, y \in E^{\#}$ by xy.

(12.1) Definition. If $\lambda_I(E)$ is an arbitrary subset of $\omega_I(E)$, we define its Köthe X-dual, relative to 1, to be the set $[\lambda_I(E)]_1^X =$ $\{[y_i, I] \in \omega_I(E): \forall [x_i, I] \in \lambda_I(E), \exists u \in E$ such that $\sum_{i\in J}|x_i y_i| \le u$ $\forall J \in \nabla(I)\}$. Since $\hat{E}$ is an ideal we have

$$[\lambda_I(E)]_1^X = \{[y_i, I] \in \omega_I(E): [x_i y_i, I] \in \ell_I^1(\hat{E}) \ \forall [x_i, I] \in \lambda_I(E)\}.$$

Observe that from (10.6) it follows that if p and q are finite real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_I^p(E, 1) \subseteq [\ell_I^q(E, 1)]_1^X$.

To simplify notation, $[\lambda_I(E)]_1^X$ will often be written $\lambda_I^X(E)$, if the element 1 is understood to be fixed. It must be emphasized that the space $[\lambda_I(E)]_1^X$ depends upon the element 1. In (10.2) we have seen an example of a vector lattice E containing weak order units 1,1' such that $\hat{E}$ is closed under multiplication relative to 1, but not under multiplication relative to 1'. Using proposition (12.5) below, we see that $[\ell_I^1(m)]_1^X \neq [\ell_I^1(m)]_{1'}^X$.

We employ another notational simplification: we often write $\lambda_I^{XX}(E)$ or $[\lambda_I(E)]_1^{XX}$ instead of $[[\lambda_I(E)]_1^X]_1^X$.

(12.2) <u>Proposition</u>. If $\lambda_I(E)$ is an arbitrary subset of $\omega_I(E)$, then $\lambda_I^{\times}(E)$ is an ideal in $\omega_I(E)$.

<u>Proof</u>. Let $[y_i, I]$, $[z_i, I] \in \lambda_I^{\times}(E)$ and $[x_i, I] \in \lambda_I(E)$. There exist $u, v \in E$ such that for every $J \in \nabla(I)$, $\sum_{i\in J}|x_i y_i| \leq u$ and $\sum_{i\in J}|x_i z_i| \leq v$. By (M4) we may use the distributive law, and thus $\sum_{i\in J}|x_i(y_i + z_i)| \leq \sum_{i\in J}|x_i y_i| + \sum_{i\in J}|x_i z_i| \leq u + v$. Therefore, $[x_i, I] + [y_i, I] \in \lambda_I^{\times}(E)$. That $\lambda_I^{\times}(E)$ is closed under scalar multiplication is obvious.

Let $[v_i, I] \in \omega_I(E)$, with $|[v_i, I]| \leq |[y_i, I]|$. Then for all $J \in \nabla(I)$, $\sum_{i\in J}|x_i v_i| = \sum_{i\in J}|x_i||v_i| \leq \sum_{i\in J}|x_i||y_i| = \sum_{i\in J}|x_i y_i| \leq u$. Therefore, $\lambda_I^{\times}(E)$ is an ideal.

(12.3) <u>Definition</u>. A subset $\lambda_I(E)$ of $\omega_I(E)$ is <u>X-perfect</u> (relative to 1) if $\lambda_I(E) = [\lambda_I(E)]_1^{\times\times}$.

(12.4) <u>Proposition</u>. If $\lambda_I(E)$ is an arbitrary subset of $\omega_I(E)$, then

(a) $\mu_I^{\times}(E) \subseteq \lambda_I^{\times}(E)$ whenever $\lambda_I(E) \subseteq \mu_I(E)$;

(b) $\lambda_I(E) \subseteq \lambda_I^{\times\times}(E)$;

(c) $\lambda_I^{\times}(E)$ is X-perfect;

(d) $\lambda_I^{\times\times}(E)$ is the smallest X-perfect subset of $\omega_I(E)$ containing $\lambda_I(E)$;

(e) if $\lambda_I(E)$ is X-perfect, then $\lambda_I(E)$ is an ideal in $\omega_I(E)$;

(f) if $\lambda_I(E) = [\lambda_I(E)]_1^{\times}$, then $\lambda_I(E) \subseteq \ell_I^2(E, 1)$.

The proof of (12.4) follows the standard argument. See Köthe [5], §30.

(12.5) <u>Proposition</u>. If I is an infinite set, the following are equivalent:

(a) $[\phi_I(E)]_1^\times = \omega_I(E)$;

(b) $\phi_I(E) \subseteq [\omega_I(E)]_1^\times$;

(c) $\hat{E}$ is closed under multiplication (relative to 1);

(d) $\ell_I^\infty(E) \subseteq [\ell_I^1(E)]_1^\times$;

(e) $\ell_I^1(E) \subseteq [\ell_I^\infty(E)]_1^\times$.

<u>Proof</u>. We shall prove (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (c). Clearly, (c) $\Rightarrow$ (a). That (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (e) are seen immediately by using (13.3) - (a), (b). That (b) $\Rightarrow$ (c) and (e) $\Rightarrow$ (c) are also seen readily, by noting that either (b) or (e) will imply that $\phi_I(E) \subseteq \ell_I^2(E, 1)$, which, by (10.3) (f), implies (c). Thus it remains only to prove (c) $\Rightarrow$ (d).

Assume (c) and let $[x_i, I] \in \ell_I^\infty(E)$, $[y_i, I] \in \ell_I^1(E)$. There exist $u, w \in E$ such that $|x_i| \leq u$ for all $i \in I$ and $\sum_{i\varepsilon J}|y_i| \leq w$ for all $J \in \nabla(I)$, and hence $\sum_{i\varepsilon J}|x_iy_i| \leq \sum_{i\varepsilon J} u\,|y_i| = u \sum_{i\varepsilon J}|y_i| \leq u\,w \in \hat{E}$. Thus $[x_i, I] \in [\ell_I^1(E)]_1^\times$.

(12.6) <u>Proposition</u>. If $1 \in \hat{E}$, then

(a) $[\omega_I(E)]_1^\times \subseteq m_I(E)$;

(b) $[\ell_I^\infty(E)]_1^\times \subseteq \ell_I^1(E)$.

<u>Proof</u>. (a) If $1 \in \hat{E}$, then by (10.3) - (d), $\ell_I^2(E, 1) \quad m_I(E)$. But $[\omega_I(E)]_1^\times \subseteq \ell_I^2(E, 1)$.

(b) Since $1 \in \hat{E}$, $1 \leq e$ for some $e \in E$. Let $[x_i, I] \in [\ell_I^\infty(E)]_1^\times$. The family $[y_i, I]$ defined by $y_i = e$, for $i \in I$, belongs to $\ell_I^\infty(E)$;

thus there exists $u \in E$ such that $\sum_{i\in J}|x_i y_i| \leq u$ for all $J \in \nabla(I)$. Then for all $J \in \nabla(I)$,

$$\sum_{i\in J}|x_i| \leq \sum_{i\in J}|x_i|e = \sum_{i\in J}|x_i y_i| \leq u.$$

Therefore, $[x_i, I] \in \ell^1_I(E)$.

By (12.5) (e) and (12.6) we have the following result.

(12.7) Corollary. If $1 \in \hat{E}$ and $\hat{E}$ is closed under multiplication relative to 1, then $[\ell^\infty_I(E)]^X_1 = \ell^1_I(E)$. (Thus, $\ell^\infty_I(E)$ is X-perfect, relative to 1.)

(12.8) Proposition. If E has a countable exhausting set (i.e., a set $\{e_n: n = 1,2,\ldots\}$ such that for every $x \in E$, $|x| \leq e_n$ for some $n \in N$), then $[\omega_I(E)]^X_1 \subseteq \phi_I(E)$.

Proof. Let $\{e_n: n = 1,2,\ldots\}$ be a countable exhausting set in E^+. Let $[y_i, I] \in [\omega_I(E)]^X_1$ and $J_0 = \{i\in I: y_i \neq 0\}$. We shall show that J_0 is finite. Suppose to the contrary that J_0 is infinite; without loss of generality, assume $N \subseteq J_0$. By the Archimedean property in $E^\#$, for each $n \in N$ there exists $\lambda_n \in R^+$ with $|\lambda_n y_n^2| \not\leq e_n$. Define $[x_i, I] \in \omega_I(E)$ by

$$x_i = \begin{cases} \lambda_i y_i & \text{if } i \in N, \\ 0 & \text{if } i \notin N. \end{cases}$$

Let $u \in E^+$. Then there exists $n_0 \in N$ such that $u \leq e_{n_0}$. If J is any finite subset of N containing n_0, we have $\sum_{n\in J}|x_n y_n| = \sum_{n\in J}|\lambda_n y_n^2| \not\leq e_{n_0}$; hence, $\sum_{n\in J}|x_n y_n| \not\leq u$. Therefore, $[y_i, I] \notin [\omega_I(E)]^X_1$.

(12.9) Example of a vector lattice E (in this case, Dedekind

complete and with $1 \in E$) such that $[\ell_N^1(E)]_1^X \not\subseteq \ell_N^\infty(E)$. Let $E = \ell_1$, $E^\# = \omega$ with the usual vector operations, order and unit. Let $[e_n, N] \in \omega_N(E)$ be given by $e_n = (0,0,\ldots,1,0,\ldots)$ with 1 in the n^{th} position only. Clearly, $[e_n, N] \notin \ell_N^\infty(E)$. But $[e_n, N] \in [\ell_N^1(E)]_1^X$. For suppose $[x_n, N] \in \ell_N^1(E)$; say $\sum_{i\in J}|x_i| \leq u$ for all $J \in \nabla(N)$. Then $\sum_{i\in J}|x_i e_i| \leq \sum_{i\in J}|x_i| \leq u$ for all $J \in \nabla(N)$. Therefore, $[\ell_N^1(E)]_1^X \not\subseteq \ell_N^\infty(E)$.

(12.10) <u>Proposition</u>. If 1 is a strong order unit for $\hat{E}$, and if $1 < p,q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$[\ell_I^p(E, 1)]_1^X = \ell_I^q(E, 1).$$

Thus each $\ell_I^p(E, 1)$ is X-perfect. The only self X-dual subspace of $\omega_I(E)$ is $\ell_I^2(E, 1)$.

<u>Proof</u>. Note that $\hat{E}$ is closed under multiplication, since $1 \in \hat{E}$. Assume I is an infinite set, since the result is trivial if I is finite. Without loss of generality we may assume $N \subseteq I$.

(a) If $p = \infty$ and $q = 1$, the result follows by (12.7).

(b) Consider $p = 1$ and $q = \infty$. Suppose $[y_i, I] \notin \ell_I^\infty(E)$. Then there exists a sequence $\{i_k\}$ of distinct positive integers such that $|y_{i_k}| \not\leq k^3 1$. Define $[x_i, I] \in \omega_I(E)$ by setting

$$x_i = \begin{cases} \frac{1}{k^2}1 & \text{if } i = i_k, \\ 0 & \text{if } i \notin \{i_k : k \in N\}. \end{cases}$$

Then $[x_i\ I] \in \ell_I^1(E)$ since, for each $J \in \nabla(I)$, $\sum_{i\in J}|x_i| = \sum_{i_k\in J}|x_{i_k}| = \sum_{i_k\in J} k^{-2}1 \leq \left(\sum_{k=1}^\infty k^{-2}\right)1$. However, $\sum_{i\in J}|x_i y_i| = \sum_{i_k\in J} k^{-2}|y_i|$

$\not\leq k\,1$. Thus the sums $\sum_{i\varepsilon J}|x_i y_i|$ $(J \varepsilon \nabla(I))$ are unbounded in E; hence, $[y_i, I] \notin [\ell_I^1(E)]_1^X$. Therefore, $[\ell_I^1(E)]_1^X \subseteq \ell_I^\infty(E)$. Since $\hat{E}$ is closed under multiplication, the converse containment holds by (12.5) (d). Therefore, equality holds.

(c) Consider $1 < p,q < \infty$. Let $[y_i, I] \varepsilon [\ell_I^p(E, 1)]_1^X$. Suppose for contradiction that $[y_i, I] \notin \ell_I^q(E, 1)$. Since $p > 1$, there exists $n_0 \varepsilon N$ such that $n_0(p-1) > 2$. Then there exists a sequence $\{J_k\}$ of pairwise disjoint subsets of N such that for each $k \varepsilon N$,

$$\sum_{i\varepsilon J_k}|y_i|^q = M_k \not\leq k^{n_0+1}\,1.$$

For each $k \varepsilon N$, let $\lambda_k = \sup\{\lambda\colon 0 < \lambda < 1,\ \lambda M_k \leq k^{n_0+1}\,1\}$. Then $\lambda_k M_k \leq k^{n_0+1}\,1$, but $\lambda_k^{1/p} M_k \not\leq k^{n_0+1}\,1$. Define $[x_i, I] \varepsilon \omega_I(E)$ by

$$x_i = \begin{cases} \lambda_k^{1/p}\, k^{-n_0}\, |y_i|^{q-1} & \text{if } i \varepsilon J_k, \\ 0 & \text{if } i \notin \bigcup_{k=1}^\infty J_k. \end{cases}$$

Then $[x_i, I] \varepsilon \ell_I^p(E, 1)$, since for each $k \varepsilon N$, $\sum_{i\varepsilon J_k}|x_i|^p = \sum_{i\varepsilon J_k}\lambda_k\, k^{-n_0 p}|y_i|^{(q-1)p} = \lambda_k\, k^{-n_0 p}\sum_{i\varepsilon J_k}|y_i|^q = k^{-n_0 p}\,\lambda_k M_k \leq k^{-n_0 p}\, k^{n_0+1}\,1 = k^{-n_0(p-1)+1}\,1 \leq 1$ since $-n_0(p-1) + 1 < -2 + 1 = -1$. On the other hand, $\sum_{i\varepsilon J_k}|x_i y_i| = \sum_{i\varepsilon J_k}\lambda_k^{1/p}\, k^{-n_0}|y_i|^q = k^{-n_0}\lambda_k^{1/p}\, M_k \not\leq k^{-n_0}\, k^{n_0+1}\,1 = k1$. Thus there exists no $k \varepsilon N$ such that for all $J \varepsilon \nabla(I)$, $\sum_{i\varepsilon J}|x_i y_i| \leq k\,1$. That is, $[y_i, I] \notin [\ell_I^p(E, 1)]_1^X$. This is a contradiction. Therefore, $[\ell_I^p(E, 1)]_1^X \subseteq \ell_I^q(E, 1)$. Combining this result with (10.6), we see that the desired equality holds.

(d) Finally, suppose $\lambda_I(E) = \lambda_I^X(E)$. By (12.4) (f), $\lambda_I(E) \subseteq$

$\ell^2_I(E, 1)$. Hence, $\lambda_I(E) = \lambda_I^{\times}(E) \supseteq [\ell^2_I(E, 1)]_1^{\times} = \ell^2_I(E, 1)$. Therefore, $\lambda_I(E) = \ell^2_I(E, 1)$.

Section 13. Linear mappings from $\lambda_I(E)$ into $\hat{E}$.

(13.1) Definition. Given vector lattices E and F, a linear mapping $f: E \to F$ is said to be

(a) positive if $x \geq 0$ implies $f(x) \geq 0$;

(b) strictly positive if $x > 0$ implies $f(x) > 0$;

(c) order bounded if f is bounded on each order interval of E;

(d) sequentially order continuous if f is order bounded and if for sequences $\{x_n\}$ in E, $x_n \xrightarrow{o} 0$ implies $f(x_n) \xrightarrow{o} 0$;

(e) order continuous if f is order bounded and if for arbitrary nets $\{x_\alpha\}$ in E, $x_\alpha \xrightarrow{o} 0$ implies $f(x_\alpha) \xrightarrow{o} 0$.

It is clear that strictly positive $\Rightarrow$ positive $\Rightarrow$ order-bounded. Note that any vector lattice isomorphism is both strictly positive and order continuous.

If E and F are arbitrary vector lattices, we let $L^+(E, F)$ denote the family of all positive linear maps from E to F. Then $L^+(E, F)$ is a cone (not necessarily generating) in the vector space $L(E, F)$ of all linear maps from E into F. Let $L^b(E, F)$ denote the family of all order bounded linear maps from E to F; $L^b(E, F)$ is a linear subspace of $L(E, F)$, partially ordered by the cone $L^+(E, F)$.

Suppose that F is Dedekind complete. Then $L^b(E, F)$ is a vector lattice with the order induced by the cone $L^+(E, F)$; in fact, $L^b(E, F)$ is Dedekind complete (see Vulikh [18], theorem VIII. 2.1). When F is

Dedekind complete, a necessary and sufficient condition for an additive map $f\colon E \to F$ to be order-bounded is that $f = g_1 - g_2$, for some $g_1, g_2 \in L^+(E, F)$ (Vulikh [18], theorem VIII. 2.2). Moreover, the spaces $L^o(E, F)$ and $L^{so}(E, F)$ of all order continuous and all sequentially order continuous maps $f \in L(E, F)$, respectively, are order-closed ideals (bands) in $L^b(E, F)$ (Vulikh [18], theorems VIII. 3.3 and VIII. 4.3).

For an arbitrary vector lattice E, any Banach o-limit $g \in \mathcal{BL}(E)$ is an example of a positive (hence order bounded) linear mapping $g\colon m(E) \to \hat{E}$, as shown in (5.5).

(13.2) <u>Example</u>. For the vector lattice $E = m$ there exists <u>no</u> sequentially order continuous $g \in \mathcal{BL}(E)$. To see this let $g \in \mathcal{BL}(E)$. Define $x^{(k)} \in m_{\mathbb{N}}(E)$ by setting $x^{(k)} = \{x_{kn}\}_{n=1}^{\infty}$ where

$$x_{kn} = \begin{cases} 0 \text{ if } 1 \leq n \leq k-1, \\ 1 \text{ if } n \geq k. \end{cases}$$

Then $x^{(k)} \downarrow 0$; but for each k, $g(x^{(k)}) = \lim_n x_n^{(k)} = 1$, since $x^{(k)} \in c(E)$. Therefore, g is not sequentially order continuous.

(13.3) <u>Proposition</u>. If order convergence of sequences in E implies uniform convergence, then every $g \in \mathcal{BL}(E)$ is sequentially order continuous.

<u>Proof</u>. Let $x_n \xrightarrow{o} 0$ in E. Then there exists $u \in E$ so that for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n| \leq \delta u$. Thus $n \geq n_0$ implies $|g(x_n)| \leq g(|x_n|) \leq \delta g(u)$. Therefore, $g(x_n) \xrightarrow{u} 0$ in $\hat{E}$, which implies $g(x_n) \xrightarrow{o} 0$ in $\hat{E}$.

(13.4) <u>Example</u>. The linear map $f: \ell_I^1(E) \to \hat{E}$ given by $f([x_i, I]) = \sum_{i\varepsilon I} x_i$ is order continuous, as shown in (7.3).

(13.5) <u>Proposition</u>. Let $\lambda_I(E)$ be a vector sublattice of $\omega_I(E)$, and $y = [y_i, I] \varepsilon [\lambda_I(E)]_1^\times$. Define the map $y^*: \lambda_I(E) \to \hat{E}$ by $y^*(x) = \sum_{i\varepsilon I} x_i y_i$, for all $x = [x_i, I] \varepsilon \lambda_I(E)$. Then

(a) $y^* \varepsilon L^b(\lambda_I(E), \hat{E})$;

(b) $y^* \geq 0$ if $y \geq 0$;

(c) if $\lambda_I(E)$ is an ideal, then $y^* \geq 0$ if and only if $y_i \geq 0$ for every $i \varepsilon I$ such that $\lambda_I(E)$ contains an element $[x_i, I]$ with $x_i \neq 0$;

(d) if $\lambda_I(E)$ is an ideal, then y^* is order continuous.

<u>Proof</u>. (a) The results of section 6, along with (M4) and (M5), show that y^* is linear. It is clear that $y^* = (y^+)^* - (y^-)^*$, and that both $(y^+)^*$ and $(y^-)^*$ are positive. Therefore, $y^* \varepsilon L^b(\lambda_I(E), \hat{E})$.

(b) If $y \geq 0$, then it is clear from the definition of y^* that $y^* \geq 0$.

(c) Suppose $\lambda_I(E)$ is an ideal and $y^* \geq 0$. Suppose $\lambda_I(E)$ contains $[x_i, I]$ with $x_{i_0} \neq 0$. Then $\lambda_I(E)$ contains the element $[x_i', I]$, where $x_i' = 0$ if $i \neq i_0$ and $x_{i_0}' = |x_{i_0}|$. Since $[x_i', I] > 0$ and $x_{i_0}' y_{i_0} = \sum_{i\varepsilon I} x_i' y_i \geq 0$, we must have $y_{i_0} \geq 0$ (see (M10)).

(d) Suppose $x^{(\alpha)} = [x_i^{(\alpha)}, I] \xrightarrow{o} 0$ in $\lambda_I(E)$. Recalling (3.5), $x_i^{(\alpha)} \xrightarrow{o} 0$ in E (hence in $E^\#$) for all $i \varepsilon I$. Thus for each i, $x_i^{(\alpha)} y_i \xrightarrow{o} 0$ in $E^\#$ (hence in the ideal $\hat{E}$) by (9.10). Applying (3.5) again, $[x_i^{(\alpha)} y_i, I] \xrightarrow{o} 0$ in $\ell_I^1(\hat{E})$. Therefore, by (7.3) $y^*(x^{(\alpha)}) =$

$\sum_{i\varepsilon I} x_i^{(\alpha)} y_i \xrightarrow{o} 0$ in $\hat{E}$. Therefore, y^* is order-continuous.

(13.6) Proposition. Suppose E is an Archimedean vector lattice and 1 is a weak order unit in $E^{\#}$. Then, taking multiplication relative to 1,

(a) if p and q are finite real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then for every $y \varepsilon \ell_I^p(E, 1)$, y^* is a positive, order continuous linear mapping from $\ell_I^q(E, 1)$ into $\hat{E}$;

(b) if E is closed under multiplication, relative to 1, then for every $1 \leq p,q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for every $y \varepsilon \ell_I^p(E, 1)$, y^* is a positive, order continuous linear mapping from $\ell_I^q(E, 1)$ into $\hat{E}$.

Proof. Both (a) and (b) are corollaries of (10.6), (12.5), and (13.5).

Section 14. Convergent families in a vector lattice.

In this section we define and discuss a notion of convergence for families $[x_i, I]$ in $\omega_I(E)$. We shall not assume that I is a directed set, or even a partially ordered set. Thus this notion is not to be confused with that of convergence of a net over I. In addition to abstracting a notion of convergence, this idea leads to an example of a vector sublattice of $\omega_I(E)$ whose Köthe X-dual can

be described in terms of the spaces already discussed.

(14.1) Definition. A family $[x_i, I] \in \omega_I(E)$ is said to be zero-convergent if there is some $y > 0$ in E such that for every $\delta > 0$ there exists $J \in \nabla(I)$ such that

$$|x_i| \leq \delta y \quad \text{whenever } i \notin J.$$

We let $c_I^0(E)$ denote the collection of all zero-convergent $[x_i, I]$ in $\omega_I(E)$.

(14.2) Proposition. (a) $c_I^0(E)$ is an ideal in $\omega_I(E)$, with $\phi_I(E) \subseteq c_I^0(E) \subseteq m_I(E)$;

(b) if I is an infinite set and E is nontrivial, then $c_I^0(E)$ is not a band in $\omega_I(E)$;

(c) if E is Archimedean, Dedekind σ-complete or Dedekind complete, then $c_I^0(E)$ has the same property; and conversely.

Proof. (a) is trivial. Pick $e > 0$ in E. For each $j \in I$, define $[x_i^{(j)}, I]$ by setting $x_i^{(j)} = 0$ if $i \neq j$, and $x_i^{(j)} = e$ if $i = j$. Then $\{[x_i^{(j)}, I]: j \in I\}$ is a subset of $c_I^0(E)$ having supremum $[y_i, I]$ in $\omega_I(E)$, where $y_i = e$ for all $i \in I$; since $[y_i, I] \notin c_I^0(E)$, $c_I^0(E)$ is not a band. Finally, (d) is a consequence of (3.1) and (3.10).

Let E be an Archimedean vector lattice. Consider an arbitrary $[x_i, I] \in m_I(E)$. In $\hat{E}$ construct the net $\{\hat{x}_J: J \in \nabla(I)\}$ indexed by the directed set $\nabla(I)$ with its usual partial order (inclusion), by defining

$$\hat{x}_J = \sup_{i \notin J} |x_i|,$$

the supremum being taken in $\hat{E}$. Observe that $\hat{x}_J \downarrow$. The following

result is then clear.

(14.3) Proposition. If E is an Archimedean vector lattice, a family $[x_i, I]$ in $m_I(E)$ is zero-convergent if and only if $\hat{x}_J \downarrow 0$ uniformly in $\hat{E}$.

(14.4) Example. In $m_R(R)$ define $[x_r, R]$ by

$$x_r = \begin{cases} 1 \text{ if } 0 \leq r \leq 1, \\ 0 \text{ if } |r| > 1. \end{cases}$$

Then $[x_r, R]$ is a family which is not zero-convergent, even though as a net over the directed set R, $x_r \xrightarrow{o} 0$.

(14.5) Proposition. For any vector lattice E and any set I, $\ell_I^u(E) \subseteq c_I^0(E)$.

Proof. We may assume I is infinite, since for finite I, $\ell_I^u(E) = c_I^0(E) = \omega_I(E)$. Let $[x_i, I] \in \ell_I^u(E)$. There exist x, y $\in$ E such that for all $\delta > 0$ there exists $J_\delta \in \nabla(I)$ such that $J \supseteq J_\delta$ implies $|(\sum_{i\in J} x_i) - x| \leq \frac{\delta}{2} y$.

Let $\delta > 0$. For each $i \notin J_\delta$, if we let $J' = J_\delta \cup \{i\}$, then $|x_i| = |\sum_{i\in J'} x_i - \sum_{i\in J_\delta} x_i| \leq |(\sum_{i\in J'} x_i) - x| + |(\sum_{i\in J_\delta} x_i) - x| \leq \delta y$. Therefore $[x_i, I] \in c_I^0(E)$.

(14.6) Definition. A family $[x_i, I]$ in $\omega_I(E)$ is said to be convergent to an element $x \in E$ if $[x_i - x, I]$ is zero-convergent. We denote this by $x_i \xrightarrow[I]{} x$. We let $c_I(E)$ denote the collection of all convergent families $[x_i, I]$.

(14.7) Proposition. If $[x_i, I]$, $[y_i, I] \in c_I(E)$, $\lambda \in R$, and

$x_i \xrightarrow[I]{} x$, $y_i \xrightarrow[I]{} y$, then

(a) $x_i + y_i \xrightarrow[I]{} x + y$;

(b) $\lambda x_i \xrightarrow[I]{} \lambda x$;

(c) $x_i \vee y_i \xrightarrow[I]{} x \vee y$;

(d) $x_i \wedge y_i \xrightarrow[I]{} x \wedge y$.

Proof. There exist $u_1, u_2, u_3 \in E$ such that for all $\delta > 0$, $\lambda \neq 0$, there exist $J_i, J_2, J_3 \in \nabla(I)$ such that $|x_i - x| \leq \delta u_1$, $|y_j - y| \leq \delta u_2$, and $|x_k - x| \leq \frac{\delta}{|\lambda|} u_3$ whenever $i \notin J_1$, $j \notin J_2$, $k \notin J_3$. Let $\delta > 0$, $J = J_1 \cup J_2 \cup J_3$ and $z = 2(u_1 \vee u_2)$. Then for all $i \notin J$, we have

(a) $|(x_i + y_i) - (x + y)| \leq |x_i - x| + |y_i - y| \leq \delta u_1 + \delta u_2 \leq \delta z$. Thus $x_i + y_i \xrightarrow[I]{} x + y$.

(b) $|\lambda x_i - \lambda x| = |\lambda||x_i - x| \leq \delta u_3$. Thus $\lambda x_i \xrightarrow[I]{} \lambda x$.

(c) Note that $(x_i \vee y_i) - (x \vee y) = [x_i - (x \vee y)] \vee [y_i - (x \vee y)] \leq (x_i - x) \vee (y_i - y) \leq \frac{\delta}{2} z$, and similarly, $(x \vee y) - (x_i \vee y_i) \leq (x - x_i) \vee (y - y_i)$. Therefore, $|(x_i \vee y_i) - (x \vee y)| \leq \delta z$. Thus $x_i \vee y_i \xrightarrow[I]{} x \vee y$.

(d) By (b) $-x_i \xrightarrow[I]{} -x$ and $-y \xrightarrow[I]{} -y$. Then by (c) $-x_i \vee -y_i \xrightarrow[I]{} -x \vee -y$; hence, by (b) $-(-x_i \vee -y_i) \xrightarrow[I]{} -(-x \vee -y)$. That is, $x_i \wedge y_i \xrightarrow[I]{} x \wedge y$.

(14.8) Proposition. (a) $c_I^0(E) \subseteq c_I(E) \subseteq m_I(E)$;

(b) $c_I(E)$ is a vector sublattice of $\omega_I(E)$, but can not be an ideal in $\omega_I(E)$ if I is an infinite set and E is nontrivial.

Proof. (a) is easy to see. That $c_I(E)$ is a vector sublattice follows from (14.7). That $c_I(E)$ is not an ideal is seen by the

following argument. Without loss of generality we assume $\mathbb{N} \subseteq I$. Pick any $e > 0$ in E, and consider the family $[x_i, I]$ defined by

$$x_i = \begin{cases} e & \text{if } i = 2n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $0 \leq [x_i, I] \leq [y_i, I] \in c_I(E)$, where $y_i = e$ for all $i \in I$; but $[x_i, I] \notin c_I(E)$ while $[y_i, I] \in c_I(E)$. Therefore, $c_I(E)$ is not an ideal.

(14.9) <u>Proposition</u>. For any Archimedean vector lattice E,

(a) if $\hat{E}$ is closed under multiplication relative to 1, then $\ell^1_I(E) \subseteq [c_I(E)]^X_1 \subseteq [c^0_I(E)]^X_1$ and $c^0_I(E) \subseteq c_I(E) \subseteq [\ell^1_I(E)]^X_1$.

(b) if 1 is a strong order unit for $\hat{E}$, then $[c^0_I(E)]^X_1 = [c_I(E)]^X_1 = \ell^1_I(E)$.

<u>Proof</u>. (a) By (12.4) and (12.5), $c^0_I(E) \subseteq c_I(E) \subseteq m_I(E)$ implies $\ell^1_I(E) = [m_I(E)]^X_1 \subseteq [c_I(E)]^X_1 \subseteq [c^0_I(E)]^X_1$. On the other hand, by (12.5) we also have $m_I(E) \subseteq [\ell^1_I(E)]^X_1$.

(b) Pick an element $e \in E$ such that $1 \leq e$. Then e is also a strong order unit for $\hat{E}$. We assume I is an infinite set, since otherwise $[c^0_I(E)]^X_1 = \ell^1_I(E) = \omega_I(E)$. Without loss of generality we may thus assume $\mathbb{N} \subseteq I$.

Suppose $[x_i, I] \notin \ell^1_I(E)$. Then there exists a sequence $\{J_k\}$ of pairwise disjoint subsets of $\mathbb{N}$ such that for all $k \in \mathbb{N}$, $\sum_{i \in J_k} |x_i| \not\leq k^2 e$. Then defining $[y_i, I] \in \omega_I(E)$ by $y_i = \begin{cases} \frac{1}{k} e & \text{if } i \in J_k, \\ 0 & \text{if } i \notin \bigcup_{k=1}^\infty J_k, \end{cases}$

we have $[y_i, I] \in c_I^0(E)$. But $\sum_{i\in J_k} |x_i y_i| = \sum_{i\in J_k}\left(\frac{1}{k} e|x_i|\right) \geq \frac{1}{k}\sum_{i\in J_k}|x_i|$ $\not\leq k\, e$. Thus $\sum_{i\in J_k}|x_i y_i| \not\leq k\mathbb{1}$; hence, $[x_i, I] \notin [c_I^0(E)]_1^X$. Therefore, $[c_I^0(E)]_1^X \subseteq \ell_I^1(E)$. The desired equality then follows with the aid of (a).

Section 15. Köthe y-dual spaces.

The theory of Köthe y-duality presented here is based on a definition which may not seem as natural as the corresponding definition in the theory of Köthe X-duality. It does, however, have two major advantages; it is independent of the unit $\mathbb{1}$ chosen in $E^\#$, and certain expected duality relationships may be established with apparently weaker hypotheses.

Throughout section 15 we shall again assume that E is an Archimedean vector lattice, embedded in its universal completion $E^\# \simeq C_\infty(Q)$, with $\mathbb{1}$ denoting the constant function $\mathbb{1}(t) = 1\ \forall t \in Q$. If $x, y \in E^\#$, xy will denote the product of x and y, relative to $\mathbb{1}$.

(15.1) Definition. If $\lambda_I(E)$ is an arbitrary subset of $\omega_I(E)$, then we define its Köthe y-dual to be

$$[\lambda_I(E)]^y = \{[y_i, I] \in \omega_I(E) : [x_i y_i, I] \in \ell_I^1(E^\#), \forall [x_i, I] \in \lambda_I(E)\},$$

where the multiplication is taken relative to $\mathbb{1}$. For convenience we shall sometimes write $\lambda_I^y(E)$.

(15.2) Remarks. Using the result (9.4) of Rice, we see that the definition of $[\lambda_I(E)]^y$ is not affected by the choice of the unit $\mathbb{1}$.

For if $1'$ is another unit and multiplication of x and y relative to $1'$ is denoted $x * y$, then by (9.4) $\sum_{i\varepsilon J}|x_i y_i| \leq u$ implies

$$\sum_{i\varepsilon J}|x_i * y_i| = \sum_{i\varepsilon J}|(x_i y_i) * 1| = \left(\sum_{i\varepsilon J}|x_i y_i|\right) * 1 \leq u * 1,$$

and similarly, $\sum_{i\varepsilon J}|x_i * y_i| \leq v$ implies $\sum_{i\varepsilon J}|x_i y_i| \leq v1'$.

Moreover, proposition (9.7) shows that the definition of $[\lambda_I(E)]^y$ is not affected by the choice of universal completions $E^\#$ of E. Thus the Köthe y-dual of $\lambda_I(E)$ is determined intrinsically by $\lambda_I(E)$ itself, and the embedding of E into $E^\#$ is merely instrumental in computing $[\lambda_I(E)]^y$.

We say that a vector sublattice $\lambda_I(E)$ of $\omega_I(E)$ is <u>y-perfect</u> if $\lambda_I(E) = [\lambda_I(E)]^{yy}$.

Corresponding to propositions (12.2) and (12.4) for x-duals, we have the following proposition for y-duals. As the arguments are entirely analogous, we shall omit them.

(15.3) <u>Proposition</u>. Suppose $\lambda_I(E)$ is an arbitrary subset of $\omega_I(E)$. Then

(a) $\lambda_I^y(E)$ is an ideal in $\omega_I(E)$;

(b) $\lambda_I(E) \subseteq \mu_I(E)$ implies $\mu_I^y(E) \subseteq \lambda_I^y(E)$, for any $\mu_I(E) \subseteq \omega_I(E)$;

(c) $\lambda_I(E) \subseteq \lambda_I^{yy}(E)$;

(d) $\lambda_I^y(E) = \lambda_I^{yyy}(E)$ ($\lambda_I^y(E)$ is y-perfect);

(e) $\lambda_I^{yy}(E)$ is the smallest y-perfect vector sublattice of $\omega_I(E)$ containing $\lambda_I(E)$;

(f) if $\lambda_I(E)$ is y-perfect, then $\lambda_I(E)$ is an ideal in $\omega_I(E)$.

(15.4) Proposition. (a) $\phi_I^y(E) = \omega_I(E)$; $\phi_I(E) \subseteq \omega_I^y(E)$;

(b) if $E^\#$ has a countable exhausting set, then $\omega_I^y(E) = \phi_I(E)$;

(c) if $\lambda_I(E)$ is y-perfect, then $\phi_I(E) \subseteq \lambda_I(E)$.

Proof. (a) is trivial. The proof of (b) uses the same line of thought as the proof of (12.9), and uses (a) as well. To see (c), observe that $\phi_I(E) \subseteq \omega_I^y(E) \subseteq \lambda_I^{yy}(E)$, for any $\lambda_I(E)$, since $\lambda_I^y(E) \subseteq \omega_I(E)$.

(15.5) Definition. (a) $\overline{m}_I(E) = \{[x_i, I] \in \omega_I(E)\colon \exists u^\# \in E^\#$ such that $|x_i| \le u^\# \ \forall i \in I\}$. We also write $\overline{\ell}_I^\infty(E) = \overline{m}_I(E)$.

(b) $\overline{\ell}_I^1(E) = \{[x_i, I] \in \omega_I(E)\colon \exists u^\# \in E^\#$ such that $\forall J \in \nabla(I)$, $\sum_{i\in J}|x_i| \le u^\#\}$.

(c) for each $1 < p < \infty$, $\overline{\ell}_I^p(E) = \{[x_i, I] \in \omega_I(E)\colon \exists u^\# \in E^\#$ such that $\forall J \in \nabla(I)$, $\sum_{i\in J}|x_i|^p \le u^\#\}$.

Note that as in (15.2), the definition of $\overline{\ell}_I^p(E)$ is independent of the universal completion $E^\#$ and the unit $\mathbb{1}$ chosen in $E^\#$.

(15.6) Proposition. For any infinite set I and any $1 \le p < q < \infty$, we have $\phi_I(E) \subsetneq \overline{\ell}_I^p(E) \subsetneq \overline{\ell}_I^q(E) \subsetneq \overline{\ell}_I^\infty(E)$.

Proof. The arguments used in proving (10.3) may easily be revised to establish these results. We do not include the resulting proofs.

It is quite easy to extend the order Hölder inequality (10.6) to the spaces $\overline{\ell}_I^p(E)$, $\overline{\ell}_I^q(E)$ for extended real numbers p,q such that $\frac{1}{p} + \frac{1}{q} = 1$; from this it follows that

$$\overline{\ell}_I^{p\,y}(E) \subseteq \overline{\ell}_I^{q}(E),$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. It is also quite easy to extend the order Minkowski inequality, from which it follows that $\overline{\ell}_I^{p}(E)$ is an ideal.

(15.7) Proposition. (a) $[\overline{\ell}_I^{\infty}(E)]^y = \overline{\ell}_I^{1}(E)$.

(b) If $E^{\#}$ has a strong order unit, then for all $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $[\overline{\ell}_I^{p}(E)]^y = \overline{\ell}_I^{q}(E)$.

The proof is completely analogous to (12.7) and 12.10).

Finally, since a norm $\|\cdot\|$ on E cannot, in general, be extended monotonely to $E^{\#}$, we do not consider ℓ-norms or p-norms for the spaces $\overline{\ell}_I^{1}(E)$ or $\overline{\ell}_I^{p}(E)$.

APPENDIX

APPENDIX

REPRESENTATION THEORY AND UNIVERSAL COMPLETION

In this appendix we present, without proofs, a few pertinent results from the "representation theory" of vector lattices, as developed in Vulikh [18]. The power of this theory lies in the result that every Archimedean vector lattice E may be represented by a function space; more precisely, E is isomorphic to an order-dense vector sublattice of a known space $C_\infty(Q)$. Moreover, $C_\infty(Q)$ turns out to be the universal completion of E, and thus the Dedekind completion of E is the ideal generated by E in $C_\infty(Q)$. For proofs and more details concerning these results the reader may consult Chapter V of Vulikh's book [18], which is the most exhaustive presentation of this representation theory available in the English language.

(16.1) Definition. A compactum (compact Hausdorff space) is said to be

(a) totally disconnected if the open-closed sets form a basis for its topology;

(b) extremal if the closure of every open set is open-closed;

(c) quasi-extremal if the closure of every open F_σ-set is open-closed.

By a rather simple point-set argument, one can show that every

quasi-extremal compactum is totally disconnected. However, the subspace $X = \{0,\pm 1,\pm\frac{1}{2},\pm\frac{1}{3}, \ldots,\pm\frac{1}{n},\ldots\}$ of the real line $\mathbb{R}$ provides a counterexample to the converse; X is totally disconnected but not quasi-extremal.

If Q is an extremal or quasi-extremal compactum, then $C_\infty(Q)$ will denote the collection of all continuous, extended real-valued functions f on Q such that $\{x \in Q: |f(x)| = \infty\}$ is nowhere dense in Q. The assumption that Q is extremal or quasi-extremal allows one to show that $C_\infty(Q)$ is a vector lattice, under the usual (pointwise) definition of addition, scalar multiplication, and order. Note that any non-negative constant function serves as a weak unit for $C_\infty(Q)$.

(16.2) Proposition. If Q is an extremal (resp. quasi-extremal) compactum, then $C_\infty(Q)$ is a Dedekind complete (resp. Dedekind σ-complete) vector lattice. Moreover, if Q is extremal, $C_\infty(Q)$ is universally complete.

(16.3) Proposition. If X is a Dedekind complete (resp. Dedekind σ-complete) vector lattice with a weak order unit $\mathbb{1}$, then there exists an extremal (resp. quasi-extremal) compactum Q such that X is isomorphic to an order dense ideal X' of the space $C_\infty(Q)$.

Moreover, the isomorphism can be realized so that C(Q), the set of finite-valued functions f in $C_\infty(Q)$, is a subset of X', and so that the unit $\mathbb{1}$ is mapped onto the function which is identically 1 on Q.

(16.4) Proposition. Every Dedekind complete vector lattice X is isomorphic to an order-dense ideal X' in $C_\infty(Q)$ for some extremal compactum Q, which is unique up to homeomorphism.

Moreover, X is universally complete if and only if $X' = C_\infty(Q)$; i.e., if and only if $X \simeq C_\infty(Q)$.

(16.5) Corollary. A vector lattice E has a universal completion if and only if E is Archimedean; any two universal completions of E are isomorphic.

The proof of (16.5) consists of embedding E isomorphically as a vector sublattice of its Dedekind completion $\hat{E}$, and then embedding $\hat{E}$ in $C_\infty(Q)$ for appropriate Q. In the composite embedding, arbitrary suprema and infima in E are preserved. The algebraic and lattice operations on elements of E correspond to the pointwise operations on the extended-real-valued functions in $C_\infty(Q)$.

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