GENERALIZED SYLOW TOWER GROUPS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
JAMES BERNARD DERR
1968

JAMES BERNARD DERR

Ph. D.

GENERALIZED SYLOW TOWER GROUPS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
JAMES BERNARD DERR
1968

This is to certify that the

thesis entitled

Generalized Sylow Tower Groups

presented by

James Bernard Derr

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Major professor

Date September 4, 1968

O-169

can t

In th

if, a

permu

Sylow

anoth

Sylow

S is

S sa

rela

grou

Sylo

call

grou

and

dire

stru

foll

ABSTRACT

GENERALIZED SYLOW TOWER GROUPS

By

James B. Derr

Both finite solvable groups and finite nilpotent groups can be characterized in terms of their Sylow structures. In this vein, P. Hall showed that a finite group is solvable if, and only if the group has a complete set of pairwise permutable Sylow subgroups. And it is well known that the Sylow subgroups of a finite nilpotent group centralize one another.

The aim here is to study the structure of groups whose Sylow subgroups satisfy a normalizer condition (N). If S is a collection of subgroups of a finite group, we say S satisfies (N) if, for every pair of subgroups of relatively prime orders in S, at least one of the subgroups normalizes the other.

We first consider groups having a complete set of Sylow subgroups which satisfies (N). These groups are called generalized Sylow tower groups (GSTG). Sylow tower groups are GSTG's and GSTG's are solvable. All subgroups and factor groups of GSTG's are again GSTG's and the direct product of GSTG's having the same normalizing structure is a GSTG. The main result on GSTG's is the following.

G i

divi

is a

Syl

grou

N-gr

a GS

G is

ties

the c

are n

condi

group

tivel

chara

group

nilpo

Sylow

of K

Theorem: If G is a GSTG, the nilpotent length of G is less than or equal to the number of distinct prime divisors of the order of G and, if equality holds, G is a STG.

We next consider condition (N) for the set of all Sylow subgroups of a group. If the set of all Sylow subgroups of a group satisfies (N), the group is called an N-group. An N-group is necessarily a GSTG and, if G is a GSTG whose order is divisible by at most 3 primes, then G is either an N-group or a STG. The inheritance properties of N-groups are identical with those of GSTG's and the classes of N-groups with similar normalizing structures are non-saturated formations.

B. Huppert has studied groups with a permutability condition (V) on the set of all Sylow subgroups. A group satisfies (V) if any two Sylow subgroups of relatively prime orders permute as subgroups. We can then characterize N-groups as follows.

Theorem: Let G satisfy (V). Then G is an N-group if, and only if

(1) G is a partially complemented extension of a nilpotent group H by a nilpotent group K, and

(2) if p and q are distinct primes, then the Sylow p-subgroup of H normalizes the Sylow q-subgroup of K or the Sylow q-subgroup of H normalizes the Sylow

p-su

of a

grou

the

exten

K wh

eithe

group.

cates

class

N

I

of N-g

whose

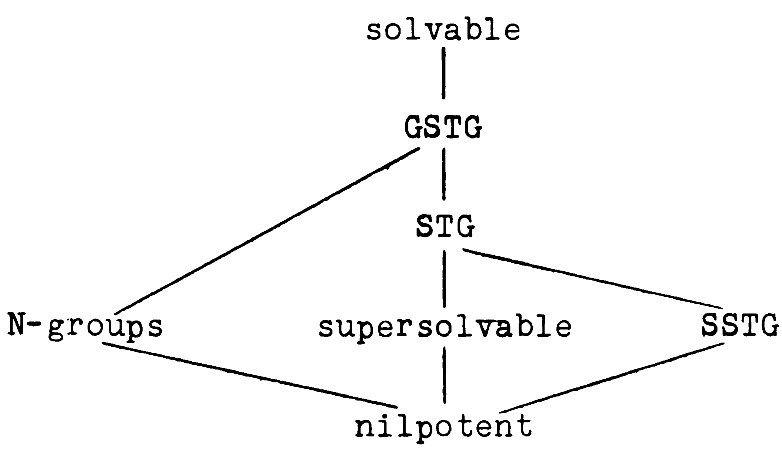
invari

p-subgroup of K.

We briefly consider condition (N) for Sylow systems of a group. A solvable group is a strongly Sylow towered group (SSTG) if some Sylow system satisfies (N). We give the following characterization of these groups.

<u>Theorem</u>: G is a SSTG if, and only if G is an extension of a nilpotent group H by a nilpotent group K where H and K have relatively prime orders and either H or K is a p-group.

The relative positions of various classes of solvable groups are shown by the following diagram. A line indicates that the lower class of groups lies in the higher class.



In order to give a decomposition of GSTG's in terms of N-groups, we examine the invariant series of a group whose factor groups are N-groups. These series are called invariant N-series.

James B. Derr

A GSTG G has a unique descending invariant N-series, whose length is denoted by $m(G)$. $m(G)$ is the minimal length of an invariant N-series of G and hence gives a measure of the deviation of the GSTG from an N-group. A GSTG G also has a unique ascending invariant N-series, whose length is denoted by $e(G)$. We show by example that $e(G)$ need not equal $m(G)$.

GENERALIZED SYLOW TOWER GROUPS

By

James Bernard Derr

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1968

this

and

ment

tion

L. S

probl

654037
3-18-69

PREFACE

I want to thank Professor W. E. Deskins for suggesting this problem and for his guidance throughout the preparation and writing of the dissertation. His patience and encouragement have served as a source of inspiration for the completion of this work.

I should also like to thank Professors J. Adney and L. Sonneborn for stimulating discussions concerning the problem.

TABLE OF CONTENTS

desc

show

the

subg

be t

Hupp

Sylo

norm

If s

satis

tivel

other

group

set o

group

The co

the se

called

satisf

necess

I

our co

work o

INTRODUCTION

Some well-known classes of finite groups can be described in terms of their Sylow structure. P. Hall [6] showed that finite solvable groups are characterized by the existence of a complete set of permutable Sylow subgroups. And finite nilpotent groups are known [13] to be the direct product of their Sylow subgroups. Recently Huppert [9,10] has characterized groups in which any two Sylow subgroups of different orders permute as groups.

The aim here is to study groups which satisfy a normalizer condition (N) on a collection of subgroups. If S is a collection of subgroups of a group, we say S satisfies (N) if, for every pair of subgroups of relatively prime orders in S, at least one normalizes the other. In chapter I we take a complete set of Sylow subgroups for our collection of subgroups. If some complete set of Sylow subgroups of a group satisfies (N), the group is called a generalized Sylow tower group (GSTG). The collection of subgroups considered in chapter II is the set of all Sylow subgroups of a group. A group G is called an N-group if the set of all Sylow subgroups of G satisfies (N). It is clear that every N-group is necessarily a GSTG.

In chapter III we take a Sylow system of a group for our collection of subgroups. This choice is based on the work of P. Hall [6] on solvable groups. A group G is

call
sys
meas
N-gr
with
is a
N-se
hype
the
grou
ascer
N-ser
lower
N-ser

called a strongly Sylow towered group (SSTG) if some Sylow system of G satisfies (N). In chapter IV we try to measure the deviation of an arbitrary GSTG from being an N-group. To accomplish this, we consider the normal series with N-factor groups of a GSTG. Among these series there is a unique descending invariant series, called the lower N-series of the group. This series is similar to the hypercommutator series of a solvable group. The length of the lower N-series of a GSTG gives the deviation of the group from being an N-group. There is also a unique ascending invariant series of a GSTG, called the upper N-series of G. An example shows that the length of the lower N-series may be shorter than the length of the upper N-series.

of a

the

(N)

group

the S

such

main

is at

of

then

const

high

inclu

group

(m,n)

CHAPTER I

Solvable groups are characterized [6] by the existence of a complete set of permutable Sylow subgroups. We replace the permutability requirement by a normalizer condition (N) and examine groups with a complete set of Sylow subgroups which satisfies (N). This class of groups includes the Sylow tower groups (STG) among others and so we call such groups generalized Sylow tower groups (GSTG). The main result shows that the nilpotent length of a GSTG G is at most $\pi(G)$, the number of distinct prime divisors of $|G|$. And if the nilpotent length of G equals $\pi(G)$, then G is a STG. We conclude the section with a construction process which yields GSTG's of arbitrarily high nilpotent length.

For the sake of completeness and easy reference, we include some definitions and basic theorems for solvable groups.

<u>Theorem 1.1</u> [4]

If G is a solvable group of order mn, where $(m,n) = 1$, then

i.) G has at least one subgroup of order m

ii.) any two subgroups of G of order m are conjugate

iii.) any subgroup of G whose order divides m belongs to some subgroup of order m.

G i

a Ha

c(H)

will

Where

Sylow

$\mathcal{S}$ =

subgr

Sylow

The s

all (

of (

We sa

$T^g \in$

has a

Definition 1.2

A subgroup H of G is called a Hall subgroup of G if $(|H|,[G:H]) = 1$. If π is a set of primes, then a Hall subgroup H of G is a Hall π-subgroup of G if $c(H) \subseteq \pi$. For a set of primes π and a group G, G_π will denote a Hall π-subgroup of G.

Definition 1.3

Let G be a finite group of order $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \cdots, p_r$ are distinct primes. If S_i is a Sylow p_i-subgroup of G $(i = 1,2,\cdots,r)$, then a set $\mathcal{S} = \{S_1, S_2, \cdots, S_r\}$ is called a complete set of Sylow subgroups of G. A complete set of pairwise permutable Sylow subgroups of G is called a Sylow basis of G. The set of 2^r Hall subgroups of G formed by taking all (group theoretic) products of members of a Sylow basis of G is called a Sylow system of G.

Definition 1.4

Let $\mathcal{S}$ and $\mathcal{T}$ be Sylow systems of a group G. We say $\mathcal{S}$ and $\mathcal{T}$ are conjugate if for some $g \in G$, $T^g \in \mathcal{S}$ for all $T \in \mathcal{T}$. We denote this by $\mathcal{T}^g = \mathcal{S}$.

Theorem 1.5 [5]

A finite group G is solvable if, and only if G has a Sylow system.

Theorem 1.6 [6]

All Sylow systems of a solvable group G are conjugate.

let

syst

in

is th

grou

of di

its Sy

Sylow

of

(N)

<u>Theorem 1.7</u> [6]

Let H be a subgroup of the solvable group G and let $\mathcal{T}$ be a Sylow system of H. Then there is a Sylow system $\mathcal{S}$ of G such that $\mathcal{T} = \{S \cap H \mid S \in \mathcal{S}\}$.

<u>Definition 1.8</u>

If $\mathcal{S}$ is any Sylow system of G, the <u>system normalizer</u>, $N_G(\mathcal{S})$, associated with $\mathcal{S}$ is the set of elements in G which normalize every member of $\mathcal{S}$:

$$N_G(\mathcal{S}) = \{g \in G \mid S^g = S, \quad \text{for all} \quad S \in \mathcal{S}\}.$$

One can easily show that the system normalizer $N_G(\mathcal{S})$ is the intersection of the normalizers of the Sylow subgroups which belong to $\mathcal{S}$.

<u>Theorem 1.9</u> [7]

The index of a system normalizer in G is the number of distinct Sylow systems of G.

<u>Theorem 1.10</u> [7]

A system normalizer $N_G(\mathcal{S})$ is the direct product of its Sylow subgroups N_i, where $N_i = S_i \cap N_G(S_i')$, S_i = Sylow p_i-subgroup of $\mathcal{S}$ and S_i' = Sylow p_i-complement of $\mathcal{S}$.

We are now ready to introduce the normalizer condition (N) which is used throughout this work.

<u>Definition 1.11</u>

Let $\mathcal{S}$ be a collection of subgroups of a group G.

We say <u>$\mathcal{S}$ satisfies (N)</u> if, for any two subgroups of relatively prime orders belonging to $\mathcal{S}$, at least one of the subgroups normalizes the other.

<u>Definition 1.12</u>

If G has a complete set of Sylow subgroups which satisfies (N), G is called a <u>generalized Sylow tower group</u> (GSTG).

<u>Proposition 1.13</u>

A GSTG G is necessarily solvable.

<u>Proof</u>:

Let $\mathcal{S} = \{S_1, S_2, \cdots, S_r\}$ be a complete set of Sylow subgroups of G which satisfies (N). Then $S_iS_j = S_jS_i$ for $1 \leq i \leq j \leq r$ and $\mathcal{S}$ is a Sylow basis for G. By Theorem 1.5, G is solvable.

$\square$

<u>Definition 1.14</u>

A finite group G is called a <u>Sylow tower group</u> (STG) if every nontrivial factor group of G has a nontrivial, normal Sylow subgroup.

Equivalently a group G is a Sylow tower group if G has an ascending series of normal subgroups S_i with $S_0 = 1$, $S_k = G$ and $^{S_i}/S_{i-1}$ = the normal Sylow p_i-subgroup of $^{G}/S_{i-1}$. This series is called a <u>Sylow tower</u> of G.

<u>Theorem 1.15</u>

If G is a STG, then G is a GSTG.

only

norm

Let

G h

compl

By i

(T_i)

norma

set c

G is

satis

s

groups

is a S

any ot

$\mathcal{T}$ =

$S_i \leq N$

(N).

Proof: (Induction on $\pi(G)$).

If $\pi(G) = 1$, G is a p-group and the assertion is trivial. If $\pi(G) = 2$ and G is a STG, then G has a normal Sylow subgroup and G is clearly a GSTG.

Now assume the theorem holds whenever $\pi(G) \leq k$ and let G be a STG with $\pi(G) = k + 1$. Since G is a STG, G has a normal Sylow subgroup S. Let S' be any complement of S in G. S' is a STG and $\pi(S') = k$. By induction, S' has a complete set of Sylow subgroups $\{T_1, T_2, \cdots, T_k\}$ which satisfies (N). Since S is a normal subgroup of G, $\{S, T_1, T_2, \cdots, T_k\}$ is a complete set of Sylow subgroups of G which satisfies (N) and so G is a GSTG.

$\square$

Proposition 1.16

If G is a GSTG, then every Sylow basis of G satisfies (N).

Proof:

Since G is a GSTG, some complete set of Sylow subgroups of G, $\mathcal{S} = \{S_1, \cdots, S_k\}$, satisfies (N). Then $\mathcal{S}$ is a Sylow basis of G which satisfies (N). If $\mathcal{T}$ is any other Sylow basis of G, then by theorem 1.6 $\mathcal{T} = \mathcal{S}^g = \{S_1{}^g, \cdots, S_k{}^g\}$ for some $g \in G$. Since $S_i \leq N_G(S_j)$ implies $S_i{}^g \leq N_G(S_j{}^g)$, $\mathcal{T} = \mathcal{S}^g$ satisifes (N).

$\square$

of a

of

of G

then

subgr

K = S

of $\mathcal{S}$

group

gate

$\{S_{i_1}^{g},$

H wh

is a

than

Hall s

1970.

More generally, whenever a collection $\mathcal{S}$ of subgroups of a group G satisfies (N), every conjugate $\mathcal{S}^g$ $(g \in G)$ of $\mathcal{S}$ satisfies (N).

We now study the inheritence properties of the class of GSTG's.

<u>Lemma 1.17</u>

If G is a GSTG and H is a Hall subgroup of G, then H is a GSTG.

<u>Proof</u>:

Let $\mathcal{S} = \{S_1, \cdots, S_k\}$ be a complete set of Sylow subgroups of G which satisfies (N). Let $K = S_{i_1} \cdots S_{i_m}$ be the product of those Sylow subgroups of $\mathcal{S}$ whose orders divide $|H|$. Then K is a Hall subgroup of G and $|K| = |H|$. By theorem 1.1 K is conjugate to H and hence $H = K^g$ for some $g \in G$. Then $\{S_{i_1}^g, \cdots \cdots, S_{i_m}^g\}$ is a complete set of Sylow subgroups of H which satisfies (N). $\square$

<u>Theorem 1.18</u>

If G is a GSTG and H is a subgroup of G, then H is a GSTG.

<u>Proof</u>: (Induction on $|G|$)

Assume the theorem holds for all groups of order less than $|G|$. If $\pi(H) \subsetneq \pi(G)$, then H lies in a proper Hall subgroup $\overline{H}$ of G. By the preceding lemma $\overline{H}$ is a GSTG. Induction then implies that H is a GSTG.

H, t

G s

H ∩

then

of

subg

(N)

(N).

of G

H, a

of G

I

$\mathscr{A}$ =

of G

$\mathscr{A}^{0}$ =

of G^{0}

S_j,

S

and G

Now let $\pi(H) = \pi(G)$. If $\mathcal{H}$ is any Sylow system of H, then, by theorem 1.7, there is a Sylow system $\mathcal{S}$ of G such that $\mathcal{H} = \{H \cap S \mid S \in \mathcal{S}\}$. Since $\pi(H) = \pi(G)$, $H \cap S$ is nontrivial for every $S \in \mathcal{S}$. In particular then, if $\{S_1, \cdots, S_k\}$ is the set of all Sylow subgroups of $\mathcal{S}$, $\{H \cap S_1, \cdots, H \cap S_k\}$ is a complete set of Sylow subgroups of H. By theorem 1.16 $\{S_1, \cdots, S_k\}$ satisfies (N) and consequently $\{H \cap S_1, \cdots, H \cap S_k\}$ satisfies (N). Then H is a GSTG. □

Lemma 1.19 [13,p.134]

Let G be a finite group and let σ be a homomorphism of G onto G^σ. Then P is a Sylow p-subgroup of G^σ if, and only if $P = G_p^\sigma$ for some Sylow p-subgroup G_p of G.

Theorem 1.20

The homomorphic image of a GSTG G is again a GSTG.

Proof:

Let σ be a homomorphism of G onto G^σ and let $\mathcal{S} = \{S_1, \cdots, S_k\}$ be a complete set of Sylow subgroups of G which satisfies (N). By lemma 1.19, $\mathcal{S}^\sigma = \{S_1^\sigma, \cdots, S_k^\sigma\}$ is a complete set of Sylow subgroups of G^σ. Since S_i^σ normalizes S_j^σ whenever S_i normalizes S_j, $\mathcal{S}^\sigma$ satisfies (N) and G^σ is a GSTG. □

Suppose H is a normal subgroup of G and both H and G/H are GSTG's. Can we conclude that G is a GSTG?

The

for

norm

S_4

Sylow

of pr

S_3 x

and

of a

ily a

follo

Sylow

$\bar{\delta}$ =

of Sy

The following examples show that we cannot answer "yes" even for simple cases.

Example 1.

A_4 is a GSTG, since the Sylow 2-subgroup of A_4 is normal. And $^{S_4}/A_4$ is a cyclic group of order 2. Yet S_4 is not a GSTG since $\pi(S_4) = 2$ and S_4 has no normal Sylow subgroup. Hence a GSTG extended by a cyclic group of prime order is not necessarily a GSTG.

Example 2.

$S_3 \times A_4$ is not a GSTG since $\pi(S_3 \times A_4) = 2$ and $S_3 \times A_4$ has no normal Sylow subgroup. Yet both $C_3 \triangleleft S_3$ and $^{S_3 \times A_4}/C_3$ are GSTG's. This shows that an extension of a cyclic group of prime order by a GSTG is not necessarily a GSTG.

Relative to this question however, we can show the following.

Theorem 1.21

G is a GSTG if, and only if $^G/Z(G)$ is a GSTG.

Proof:

Because of theorem 1.20, we need only show the necessity.

Choose $\mathcal{S} = \{G_1, \cdots \cdots, G_k\}$ to be a complete set of Sylow subgroups of G such that

$\bar{\mathcal{S}} = \left\{ {}^{G_1Z(G)}/_{Z(G)}, \cdots \cdots, {}^{G_kZ(G)}/_{Z(G)} \right\}$ is a complete set of Sylow subgroups of $^G/Z(G)$ which satisfies (N).

Supp

$1 \leq$

$x \in$

G_j^x

p_j^{-s}

G_jZ

sati

if, a

centr

By th

Since

and t

induc

Suppose ${}^{G_iZ(G)}/_{Z(G)}$ normalizes ${}^{G_jZ(G)}/_{Z(G)}$ for $1 \leq i, j \leq k$. Then $(G_jZ(G))^x = G_j \cdot Z(G)$ for all $x \in G_i$. Since $G_j \lhd G_j \cdot Z(G)$ and $G_j^x \leq (G_j \cdot Z(G))^x = G_j \cdot Z(G)$, $G_j^x = G_j$, the unique Sylow p_j-subgroup of $G_j \cdot Z(G)$. Hence, ${}^{G_iZ(G)}/_{Z(G)}$ normalizes ${}^{G_jZ(G)}/_{Z(G)}$ implies $G_i \leq N_G(G_j)$ and it follows that $\mathscr{A}$ satisfies (N). □

<u>Corollary 1.22</u>

Let $Z_\infty(G)$ be the hypercenter of G. G is a GSTG if, and only if ${}^{G}/_{Z_\infty(G)}$ is a GSTG.

<u>**Proof**</u>: Theorem 1.20 shows the sufficiency.

Let $1 \lhd Z(G) = Z_1 \lhd \cdots \lhd Z_k = Z_\infty(G)$ be the upper central series of G defined inductively by ${}^{Z_{i+1}}/_{Z_i} = Z({}^{G}/_{Z_i})$. By the isomorphism theorems

$$\frac{G}{Z_k} \cong \frac{\frac{G}{Z_{k-1}}}{\frac{Z_k}{Z_{k-1}}} = \frac{\frac{G}{Z_{k-1}}}{Z({}^{G}/_{Z_{k-1}})} .$$

Since ${}^{G}/_{Z_k}$ is a GSTG, $\dfrac{\frac{G}{Z_{k-1}}}{Z({}^{G}/_{Z_{k-1}})}$ is a GSTG and theorem 1.21 implies ${}^{G}/_{Z_{k-1}}$ is a GSTG. Proceeding inductively shows at the k^{th} step that G is a GSTG. □

G.

g-s

we s

p

and

of

H a

prim

or

group

is

K a

We now consider the direct product of GSTG's.

Definition 1.23

Let $\mathcal{S}$ be a collection of Sylow subgroups of a group G. If all Sylow p-subgroups of $\mathcal{S}$ normalize all Sylow q-subgroups of $\mathcal{S}$, where p and q are distinct primes, we say p normalizes q relative to $\mathcal{S}$ and write $p \xrightarrow[\mathcal{S}]{} q$.

Definition 1.24

Let H and K be GSTG's and let $\mathcal{H} = \{H_1, \cdots, H_s\}$ and $\mathcal{K} = \{K_1, \cdots, K_t\}$ be complete sets of Sylow subgroups of H and K (respectively) which satisfy (N). We say H and K are (N)-similar if for any pair of distinct primes $p, q \in c(H) \cap c(K)$, either

i) $p \xrightarrow[\mathcal{H}]{} q$ and $p \xrightarrow[\mathcal{K}]{} q$

or ii) $q \xrightarrow[\mathcal{H}]{} p$ and $q \xrightarrow[\mathcal{K}]{} p$.

Since any two complete sets of Sylow subgroups of a group which satisfy (N) are conjugate, definition 1.24 is independent of the choice of the sets $\mathcal{H}$ and $\mathcal{K}$.

The proofs of theorems 1.18 and 1.20 show the following.

Proposition 1.25

If H and K are subgroups of a GSTG, then H and K are (N)-similar GSTG's.

Proposition 1.26

If H and K are normal subgroups of a GSTG G, the factor groups G/H and G/K are (N)-similar GSTG's.

We can now describe the direct product of GSTG's.

Theorem 1.27

$H \times K$ is a GSTG if, and only if H and K are (N)-similar GSTG's.

Proof:

Proposition 1.25 shows the sufficiency. To prove the necessity, let H and K be (N)-similar GSTG's and let $\mathcal{H} = \{H_r \mid r \in c(H)\}$ and $\mathcal{K} = \{K_r \mid r \in c(K)\}$ be complete sets of Sylow subgroups of H and K which satisfy (N). The set $\mathfrak{D} = \{H_r \times K_r \mid H_r = 1$ if $r \notin c(H)$, $K_r = 1$ if $r \notin c(K)$; $r \in c(H) \cup c(K)\}$ is a complete set of Sylow subgroups of $H \times K$. Let p and q be distinct prime divisors of $|H \times K|$. Since H and K are (N)-similar we may assume $p \xrightarrow[\mathcal{H}]{} q$ and $p \xrightarrow[\mathcal{K}]{} q$. Then $H_p \leq N_H(H_q)$, $K_p \leq N_K(K_q)$ and consequently $H_p \times K_p$ normalizes $H_q \times K_q$. Since p and q are arbitrary, $\mathfrak{D}$ satisfies (N) and $H \times K$ is a GSTG.

□

The previous result can be extended to central products.

Definition 1.28

A group $G = H \cdot K$ with $H \cap K \leq Z(G)$ and $H \leq C_G(K)$ is called the central product of H and K.

Theorem 1.29

The central product of H and K is a GSTG if, and only if H and K are (N)-similar GSTG's.

Proof: Theorem 1.25 shows the sufficiency.

To verify the necessity, let H and K be (N)-similar GSTG's. Since $H \leq C_G(K)$, $M = H \cap K$ is normal in $G = H \cdot K$. Then H/M and K/M are (N)-similar GSTG's and hence $G/M = H/M \times K/M$ is a GSTG. Therefore $\frac{G}{Z(G)} \cong \frac{G}{M}\Big/\frac{Z(G)}{M}$ is a GSTG and theorem 1.21 shows G is a GSTG.

□

The class of all (N)-similar GSTG's may be shown to be a formation in the sense of Gaschütz [3].

Lemma 1.30

If H and K are normal subgroups of G, then $G/H \cap K$ is isomorphic to a subgroup of $G/H \times G/K$.

Proof:

Define the mapping $t : G \to G/H \times G/K$ by $t(g) = gH \times gK$, for all $g \in G$. t is a homomorphism into $G/H \times G/K$ whose kernel is $H \cap K$.

□

Proposition 1.31

Let H and K be normal subgroups of G. If G/H and G/K are (N)-similar GSTG's, then $G/H \cap K$ is a GSTG.

Proof:

If G/H and G/K are (N)-similar GSTG's, then $G/H \times G/K$ is a GSTG. Since $G/H \cap K$ is isomorphic to a subgroup of $G/H \times G/K$, $G/H \cap K$ is a GSTG. $\square$

Definition 1.32

A formation F is a collection of finite solvable groups satisfying

i) $1 \in F$

ii) $G \in F$ implies every homomorphic image of G belongs to F

iii) N_1 and N_2 normal subgroups of G with G/N_1, $G/N_2 \in F$ implies $G/N_1 \cap N_2 \in F$.

The formation is saturated if $G/\Phi(G) \in F$ implies $G \in F$.

Let $*$ be a relation on the set of all primes. A GSTG G is compatible with $*$ if, for some complete set of Sylow subgroups $\mathcal{S}$ of G, $p * q$ implies the Sylow p-subgroup of $\mathcal{S}$ normalizes the Sylow q-subgroup of $\mathcal{S}$. Then if H and K are GSTG's compatible with a relation $*$, H and K are (N)-similar. Consequently, theorems 1.26, 1.27 and 1.31 show that the set of all GSTG's compatible with a relation $*$ is a formation. It is not known if the formation is saturated.

The following definitions and theorems are needed for the main result of this section and for chapter IV.

<u>Definition 1.34</u>

Let G be a solvable group.

i) The <u>Fitting subgroup</u> of G, denoted $F(G)$, is the maximal normal nilpotent subgroup of G. It is well known [13, p.166] that $F(G)$ is the product of all normal nilpotent subgroups of G.

ii) The <u>Fitting series</u> of G is defined inductively by $F_o = 1$, ${}^{F_{i+1}}/_{F_i} = F({}^{G}/_{F_i})$, $F_\ell = G$.

iii) The length of the Fitting series of G, denoted $\ell(G)$, is the <u>nilpotent length</u> of G.

iv) A normal series with nilpotent factors is called a <u>nilpotent series</u>.

We combine several results of Spencer [12] in the next statement.

<u>Proposition 1.35</u>

Let G be a solvable group. Then

i) The length of any nilpotent series of G is greater than or equal to $\ell(G)$.

ii) For a normal subgroup K of G, $\ell({}^{G}/_{K}) \leq \ell(G)$.

iii) For H a subgroup of G, $\ell(H) \leq \ell(G)$.

iv) For normal subgroups H and K of G,

$\ell({}^{G}/_{H \cap K}) \leq \max \{\ell({}^{G}/_{H}), \ell({}^{G}/_{K})\}$.

v) $\ell({}^{G}/_{\Phi(G)}) = \ell(G)$.

We are now able to state the main result of chapter I.

Theorem 1.36

If G is a GSTG, then $\ell(G) \leq \pi(G)$.

Proof: (Induction on $|G|$).

If $\pi(G) \leq 2$, then G has a normal Sylow subgroup and surely $\ell(G) \leq \pi(G)$.

Let G be a GSTG with $\pi(G) = k \geq 3$ and take $\mathcal{S}$ to be a complete set of Sylow subgroups which satisfies (N).

Case 1. G has a normal Sylow subgroup, T (say).

Since G is a GSTG, the factor group $\overline{G} = {}^{G}/_{T}$ is a GSTG. $\pi(\overline{G}) = k - 1$ and by the induction assumption $\ell(\overline{G}) \leq \pi(\overline{G}) = k - 1$. Hence $\overline{G}$ has a nilpotent series

$$\overline{1} = {}^{T}/_{T} \triangleleft {}^{H_1}/_{T} \triangleleft \cdots \triangleleft {}^{H_{k-1}}/_{T} = {}^{G}/_{T}.$$

Then $1 \triangleleft T \triangleleft H_1 \triangleleft \cdots \triangleleft H_{K-1} = G$ is a nilpotent series of G of length k and theorem 1.35 (i) implies $\ell(G) \leq k$.

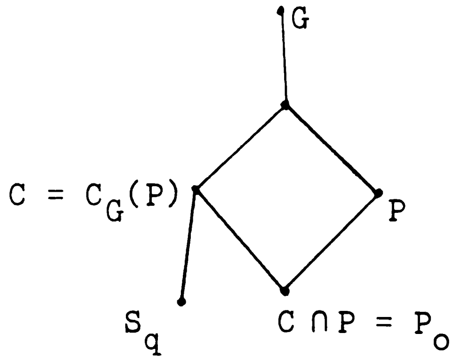
Case 2. No Sylow subgroup of G is normal.

Since G is a GSTG, G is solvable and has a minimal normal subgroup M, which has prime power order. Let $|M| = p^m$ and let P denote the maximal normal p-subgroup of G. If S_p is the Sylow p-subgroup of G belonging to $\mathcal{S}$, then S_p contains P.

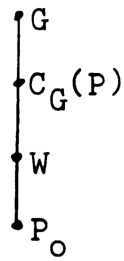
By assumption $S_p \ntriangleleft G$. Then, for some prime $q \neq p$, the Sylow q-subgroup S_q of $\mathcal{S}$ fails to normalize S_p.

Since $\mathcal{S}$ satisfies (N), this means S_p normalizes S_q. Then $S_q \leq C_G(P) = C$, because P normalizes S_q, $P \lhd G$ and $(|P|, |S_q|) = 1$. Notice that, since $P \lhd G$, $C = C_G(P) \trianglelefteq G$.

Now let $P_o = C \cap P$ denote the maximal normal p-subgroup of C. Since P is a p-group, $Z(P) \neq 1$ and $1 \neq Z(P) \leq P_o$. Since $S_q \leq C$, C/P_o is a nontrivial solvable group and therefore has a minimal normal subgroup $V/P_o \neq \bar{1}$. Suppose $|V/P_o|$ is a power of the prime r.



Now $r \neq p$, for $r = p$ implies that V is a normal p-subgroup of C properly containing P_o - contradicting the maximality of P_o. Let W/P_o denote the maximal normal r-subgroup of C/P_o. Then W/P_o is characteristic in $C/P_o \lhd G/P_o$ and hence normal in G/P_o.



By the isomorphism theorems, $W \lhd G$.

Since W centralizes P and W/P_o is an r-group, $W = W_o \times P_o$ where W_o is the Sylow r-subgroup of W. Since the normal Sylow r-subgroup W_o of W is characteristic in $W \lhd G$, W_o is normal in G.

Now consider the factor groups G/P and G/W_o. By the induction hypothesis, $\ell(G/P) \leq \pi(G/P) = k$ and $\ell(G/W_o) \leq \pi(G/W_o) = k$. Since $P \cap W_o = 1$, Proposition 1.34 iv) shows that $\ell(G) = \ell(G/P \cap W_o) \leq k$ and the theorem is proved. $\square$

Lemma 1.37

Let G be a STG with $\ell(G) = \pi(G)$. If S and T are Sylow subgroups of G having relatively prime orders, S does not centralize T.

Proof: (by minimal counterexample)

Assume that G is a STG of minimal order which satisfies the hypothesis of the theorem but fails to satisfy the conclusion. Then $\ell(G) = \pi(G) = n \geq 3$ and G has a Sylow tower

$$1 \triangleleft S_1 \triangleleft S_2S_1 \triangleleft \cdots \cdots \triangleleft S_n \cdots S_2S_1 = G,$$

where S_i is a Sylow p_i-subgroup of G $(i = 1,2,\cdots,n)$.

The factor group G/S_1 is a STG with $\pi(G/S_1) = n-1$. Since G/S_1 is a STG and $\ell(G) = n$, $\ell(G/S_1) = n-1 = \pi(G/S_1)$. Consequently, by the minimality of G, no Sylow subgroups of G/S_1 having relatively prime orders centralize one another. But, since we assumed G does not satisfy the conclusion of the theorem, some Sylow p_j-subgroup T of G centralizes some Sylow p_k-subgroup V of G. Then the Sylow p_j-subgroup TS_1/S_1 of G/S_1 centralizes the Sylow

p_k-subgroup ${}^{VS_1}/_{S_1}$ of ${}^{G}/_{S_1}$.

This is impossible unless ${}^{TS_1}/_{S_1} = \bar{1}$ or ${}^{VS_1}/_{S_1} = \bar{1}$. Consequently $j = 1$ or $k = 1$ and we may assume that the Sylow p_k-subgroup V of G centralizes S_1. Since $S_1 \lhd G$ and $S_k = V^g$ for some $g \in G$, S_k centralizes S_1.

Since $S_1 \lhd G$, $C_G(S_1)$ is a normal subgroup of G containing S_k. Let m be the smallest integer, $2 \leq m \leq n$, for which $p_m \,\big|\, |C_G(S_1)|$ and put $R = S_m \cdots S_2S_1 \cap C_G(S_1)$. R is a normal subgroup of G and by the choice of m, p_1 and p_m are the only prime divisors of $|R|$. Since $R \leq C_G(S_1)$, any Sylow p_m-subgroup R_m of R centralizes the normal Sylow p_1-subgroup $R_1 = S_1 \cap C_G(S_1)$. Then $R = R_1 \times R_m$ and R_m is a characteristic subgroup of the normal subgroup R of G and hence $R_m \lhd G$.

Suppose R_m is a Sylow subgroup of G. Then $R_mS_1 = R_m \times S_1$ is a normal nilpotent subgroup of G and $1 \lhd S_1R_m \lhd S_2S_1R_m \lhd \cdots \lhd S_{m-1}\cdots S_2S_1R_m \lhd S_{m+1}S_{m-1}\cdots S_2S_1R_m \lhd \cdots \lhd G$ is a nilpotent series of G having length $n-1$. Then proposition 1.35 i) implies $\ell(G) \leq n - 1$, which is a contradiction. Therefore R_m is not a Sylow subgroup of G and consequently we are assured that ${}^{S_kR_m}/_{R_m}$ is a nontrivial Sylow p_k-subgroup of ${}^{G}/_{R_m}$. Since $S_k \leq C_G(S_1)$, ${}^{S_kR_m}/_{R_m}$ centralizes ${}^{S_1R_m}/_{R_m}$. Then $\pi({}^{G}/_{R_m}) = n$ and the minimality of G imply $\ell({}^{G}/_{R_m}) < n$. Since $\ell({}^{G}/_{S_1}) < n$

and $R_m \cap S_1 = 1$, proposition 1.35 iv) implies that $\ell(G) = \ell(G/_{R_m \cap S_1}) < n$ - which is a contradiction. $\square$

Theorem 1.38

If G is a GSTG and $\ell(G) = \pi(G)$, then G is a STG.

Proof: (Induction on $|G|$).

If $\pi(G) \leq 2$, G has a normal Sylow subgroup and clearly G is a STG.

Let G be a GSTG with $\ell(G) = \pi(G) = n \geq 3$. We distinguish two cases.

Case 1. G has a normal Sylow subgroup K.

$G/_K$ is a GSTG and, by theorem 1.36, $\ell(G/_K) \leq \pi(G/_K) = n-1$. Since $\ell(G) = n$, we must have $\ell(G/_K) = n-1$. Then the induction assumption implies that $G/_K$ is a STG. Let $G/_K$ have Sylow tower $\overline{1} = K/_K \lhd S_1K/_K \lhd \cdots \cdots \lhd S_{n-1}\cdots S_1K/_K = G/_K$, where S_i is a Sylow p_i-subgroup of G. Then

$$1 \lhd K \lhd S_1K \lhd \cdot \quad \cdot \quad \cdot \lhd S_{n-1}\cdots S_1K = G$$

is a Sylow tower of G and G is a STG.

Case 2. No Sylow subgroup of G is normal.

Then, as in the proof of theorem 1.36, there exist normal prime power order subgroups M_1 and M_2 of G with $(|M_1|, |M_2|) = 1$. Consider the factor groups $G/_{M_1}$ and $G/_{M_2}$ of G. If $\ell(G/_{M_1}) < n$ and $\ell(G/_{M_2}) < n$, proposition 1.35 iv) implies $\ell(G) < n$, a contradiction. Therefore we can assume $\ell(G/M_1) = n$. Since M_1 cannot be a normal Sylow subgroup of G, $\pi(G/M_1) = n = \ell(G/M_1)$ and the induction assumption implies G/M_1 is a STG.

Let $\overline{1} = M_1/M_1 \lhd S_1M_1/M_1 \lhd \cdot \cdot \cdot \lhd S_n \cdots S_1M_1/M_1 = G/M_1$ be a Sylow tower of G/M_1, where S_i is a Sylow p_i-subgroup of G $(i = 1,2,\cdots,n)$. Furthermore, choose S_n to be the Sylow p_n-subgroup of a Sylow basis $\mathcal{S}$ of G which satisfies (N) and contains S_1.

Suppose $M_1 \leq S_k$, for some $k < n$. Then $H = S_k \cdots S_1M_1 = S_k \cdots S_1$ is a normal Hall subgroup of G and, since $\ell(G) = n$, $\ell(H) = \pi(H) = k$. By the induction assumption H is then a STG and consequently has a normal Sylow subgroup W. Since H is a normal Hall subgroup of G, W is then a normal Sylow subgroup of G - a contradiction. Consequently, since a normal p-subgroup of a group lies in every Sylow p-subgroup of the group, we must have $M_1 \leq S_n$.

Since S_1 and S_n were chosen to belong to a Sylow basis $\mathcal{S}$ of G which satisfies (N), either S_1 normalizes S_n or S_n normalizes S_1. Suppose first that $S_n \leq N_G(S_1)$. Then M_1 normalizes S_1 and $S_1M_1 = S_1 \times M_1$, since $[S_1,M_1] \leq S_1 \cap M_1 = 1$. Consequently S_1 is a characteristic subgroup of $S_1M_1 \lhd G$ and hence S_1 is a normal Sylow p_1-subgroup of G - a contradiction. Now suppose

$S_1 \leq N_G(S_n)$. Since $S_1M_1 \lhd G$, we then have that the Sylow p_1-subgroup S_1M_1/M_1 of G/M_1 centralizes the Sylow p_n-subgroup $S_nM_1/M_1 = S_n/M_1$ of G/M_1. Consequently, since G/M_1 is a STG, $\ell(G/M_1) \leq n-1$ - which is a contradiction. Therefore this case cannot occur and the theorem is proved.

□

The remainder of the chapter is devoted to showing that GSTG's of arbitrarily high nilpotent length do exist. The construction process given utilizes wreath products, which we now describe.

Let A and B be finite groups and let $A^{|B|}$ denote the direct product of $|B|$ copies of A. Suffix the $|B|$ direct factors of $A^{|B|}$ by the elements of B and construct a group $W = A \text{ wr } B$, which is the extension of $A^{|B|}$ by B. To complete the definition of W, we specify the automorphisms induced on $A^{|B|}$ by the elements $b \in B$. If $a \in A_{b_i}$, define $b^{-1}ab$ as the element corresponding to a in $A_{b_i b}$. This determines $b^{-1}nb$ for any element $n \in A^{|B|}$. The group $W = A \text{ wr } B$ is called the <u>wreath product of A by B</u> and its order is $|A|^{|B|} \cdot |B|$.

<u>Proposition 1.39</u>

If A and B are GSTG's of relatively prime orders, $A \text{ wr } B$ is a GSTG.

Proof:

Let $\mathfrak{A} = \{A_1,\cdot\cdot,A_k\}$ and $\mathfrak{B} = \{B_1,\cdot\cdot,B_\ell\}$ be Sylow bases of A and B (respectively) which satisfy (N). Let $A_i^{|B|}$ denote the subgroup of $A^{|B|}$ which is the direct product of $|B|$ copies of A_i $(i = 1,\cdot\cdot,k)$. Then $A_i^{|B|}$ is a Sylow subgroup of $A^{|B|}$ and every element of B normalizes $A_i^{|B|}$ $(i = 1,\cdot\cdot,k)$. Furthermore, $A_i \leq N_A(A_j)$ implies $A_i^{|B|}$ normalizes $A_j^{|B|}$. Therefore $\mathcal{W} = \{A_1^{|B|},\cdot\cdot,A_k^{|B|},B_1,\cdot\cdot,B_\ell\}$ is a Sylow basis of $W = A \text{ wr } B$ which satisfies (N). $\square$

Lemma 1.40

Let A and B be finite groups and let $K \trianglelefteq A$. If $W = A \text{ wr } B$, then $W/K^{|B|} \cong A/K \text{ wr } B$.

Proof:

$K^{|B|}$ is normal in $W = A^{|B|} \cdot B$, since $K^{|B|} \trianglelefteq A^{|B|}$ and by the definition of multiplication in W, B normalizes $K^{|B|}$. We will show that the mapping

$\varphi : b\cdot a_1\cdot\cdot\cdot a_{|B|} \to b\cdot a_1K\cdot\cdot a_{|B|}K$ for $b \in B$, $a_i \in A_{b_i}$

is a homomorphism from W to $A/K \text{ wr } B$ with kernel $K^{|B|}$ and image $A/K \text{ wr } B$.

Clearly the mapping is well-defined and has image $A/K \text{ wr } B$. Let $c,d \in B$ and $n_i,g_i \in A_{b_i} \leq A^{|B|}$ for $i = 1,\cdot\cdot,|B|$.

(1) φ is a homomorphism.

$\varphi\,[(c\ g_1\cdot\cdot g_{|B|})\cdot(d\ n_1\cdot\cdot n_{|B|})] =$

$= \varphi\,[cd\ (g_1\cdot\cdot g_{|B|})^d(n_1\cdot\cdot n_{|B|})] =$

$= \varphi\,[cd\ (h_1 n_1\cdot\cdot h_{|B|} n_{|B|})],\ h_t = g_s{}^d \in A_{b_t};\ s,\ t = 1,2,\cdot\cdot|B|$

$= cd\ (h_1 n_1 K\cdot\cdot h_{|B|} n_{|B|} K)$

$= cd\ (h_1 K\cdot\cdot h_{|B|} K)(n_1 K\cdot\cdot n_{|B|} K)$

$= cd\ (g_1 K\cdot\cdot g_{|B|} K)^d(n_1 K\cdot\cdot n_{|B|} K)$

$= [c(g_1 K\cdot\cdot g_{|B|} K)][d(n_1 K\cdot\cdot n_{|B|} K)]$

$= \varphi\ (cg_1\cdot\cdot g_{|B|})\ \cdot\ \varphi\ (d\ n_1\cdot\cdot n_{|B|}).$

(2) $\operatorname{Ker}\ \varphi = K^{|B|}$.

$\varphi\ (c\ g_1\cdot\cdot g_{|B|}) = \bar{1} = K^{|B|} \Leftrightarrow c\ g_1 K\cdot\cdot g_{|B|} K = K^{|B|}$

$\Leftrightarrow c = 1$ and $g_i \in K_{b_i}\quad i = 1,\cdot\cdot,|B|$

$\Leftrightarrow c\ g_1\cdot\cdot g_{|B|} \in K^{|B|}.$ $\square$

<u>Theorem 1.41</u> [13, p.167]

If G is a finite group, then $\operatorname{Fit}(G) = \otimes \prod_{p\,|\,|G|} K(p)$,

where $K(p)$ is the intersection of all Sylow p-subgroups of G and $\operatorname{Fit}(G)$ denotes the Fitting subgroup of G.

As a corollary of theorem 1.41, we have the following.

<u>Lemma 1.42</u>

If $G = G_1 \times\cdot\cdot\times G_k$ is the direct product of finite groups, then $\operatorname{Fit}(G) = \operatorname{Fit}(G_1) \times\cdot\cdot\times \operatorname{Fit}(G_k)$.

<u>Proposition 1.43</u>

If G is a finite group of nilpotent length k and

$C = \langle c \rangle$ is a cyclic group of order ℓ, $(|G|, \ell) = 1$, then $G \text{ wr } C$ has nilpotent length $k + 1$.

Proof:

(1) $\mathrm{Fit}\,(G \text{ wr } C) = (\mathrm{Fit}(G))^{|C|}$.

By lemma 1.42 $\mathrm{Fit}(G^{|C|}) = (\mathrm{Fit}(G))^{|C|}$ and consequently $(\mathrm{Fit}(G))^{|C|}$ is a normal nilpotent subgroup of $G \text{ wr } C$. Hence $(\mathrm{Fit}(G))^{|C|} \leq \mathrm{Fit}(G \text{ wr } c)$.

Suppose $(\mathrm{Fit}(G))^{|C|} < \mathrm{Fit}\,(G \text{ wr } C)$ and let $d \cdot n \in \mathrm{Fit}(G \text{ wr } c) \setminus (\mathrm{Fit}(G))^{|C|}$, where $d \in C$ and $n \in G^{|C|}$. Then $d = c^i$ for $1 \leq i \lneq \ell$.

Since $(dn)^t = d^t \; n^{d^{t-1}} \cdots n^d n$ for any integer t, $d \cdot n$ has order $j \cdot \ell'$ where $j \,\big|\, |G|^{|C|}$ and $1 \neq \ell' \,|\, \ell$. Then $(dn)^j$ has order ℓ' and $(dn)^j \in \mathrm{Fit}(G \text{ wr } C)$. Since $G \text{ wr } C$ is solvable, there exists $x \in G \text{ wr } C$ such that $(dn)^j \in C^x$ and therefore $(dn)^j = (c^w)^x$ for some w, $1 \leq w \lneq \ell$. Then $c^w \in \mathrm{Fit}(G \text{ wr } C)$ and $\mathrm{Fit}(G \text{ wr } C)$ has a nontrivial Hall subgroup H whose order divides $|C|$. Then $H \leq C$ and hence $H \leq Z(G \text{ wr } C)$ - since C is cyclic and $(|H|, |G|) = 1$. This is impossible and therefore we must have $(\mathrm{Fit}(G))^{|C|} = \mathrm{Fit}(G \text{ wr } C)$.

Now use induction on $\ell(G)$ to complete the argument. If $\ell(G) = 1$, then surely $\ell(G \text{ wr } C) \leq 2$. If $\ell(G \text{ wr } C) = 1$, then $G \text{ wr } C = G \times C$, which is nonsense. Therefore $\ell(G \text{ wr } C) = 2$ in this case.

Let $\ell(G) = m$ and put $F = \mathrm{Fit}(G)$. Then $\ell(G/F) = m - 1$

and by induction $\ell(G/F \text{ wr } C) = m$. Applying lemmas 1.40 and 1.42 we have $G/F \text{ wr } C \cong {}^{G \text{ wr } C}/_{F|C|} \cong {}^{G \text{ wr } C}/_{\mathrm{Fit}(G \text{ wr } C)}$.

Therefore $\ell({}^{G \text{ wr } C}/_{\mathrm{Fit}(G \text{ wr } C)}) = m$ and consequently $\ell(G \text{ wr } C) = m + 1$ - and the proof is completed. □

Let $C_1, C_2, \cdot\cdot, C_n$ denote cyclic groups whose orders are relatively prime in pairs. Then, by the previous results, we see that the repeated wreath product $W = ((\cdot(C_1 \text{ wr } C_2)\text{wr}\cdot\cdot)\text{wr } C_n$ is a GSTG of nilpotent length n. Hence if k and n are any positive integers and $k \geq n$, we can construct a GSTG G with $\ell(G) = n$ and $\pi(G) = k$.

Chapter II

In this chapter we consider condition (N) for the collection of all Sylow subgroups of a group. If the set of all Sylow subgroups of G satisfies (N), we call G an N-group. Clearly every N-group is a GSTG. The inheritance properties of N-groups are identical with those of GSTG's and certain subclasses of N-groups are (non-saturated) formations.

Using some results of Huppert [10], it is shown that the nilpotent length of an N-group is at most 2. A characterization of N-groups as a special type of product of nilpotent groups is then obtained. The remaining results describe the Sylow structure of N-groups or the structure of groups which just fail being N-groups.

We begin by formally defining an N-group.

<u>Definition 2.1</u>

Let $\mathrm{Syl}(G)$ denote the set of all Sylow subgroups of G. A finite group G is an <u>N-group</u> if $\mathrm{Syl}(G)$ satisfies (N).

Several observations follow immediately. If G is an N-group, then every complete set of Sylow subgroups of G satisfies (N). In particular, every N-group is a GSTG and consequently is solvable. And since a finite nilpotent group is the direct product of its Sylow subgroups, a finite nilpotent group is clearly an N-group. Furthermore, every group which is a nilpotent group extended by a p-group is an N-group. S_3 is an example of such a group. The next two examples further describe the position of N-groups relative to other classes of solvable groups.

Example 1.

The following is an N-group which is not a STG. Let

$$H = \otimes \pi_{i=1}^{5} < a_i \,|\, a_i^2 = 1 > \times \otimes \pi_{j=1}^{2} < b_j \,|\, b_j^3 = 1 > \times \otimes \pi_{k=1}^{3} < c_k \,|\, c_k^5 = 1 >$$

and $k = <x,y,z> \leq Aut(H)$, where

$$a_i^x = a_{i+1},\ b_j^x = b_j,\ c_k^x = c_k \qquad (a_6 = a_1)$$

$$b_j^y = b_{j+1},\ a_i^y = a_i,\ c_k^y = c_k \qquad (b_3 = b_1)$$

$$c_k^z = c_{k+1},\ a_i^z = a_i,\ b_j^z = b_j \qquad (c_4 = c_1).$$

Define G to be the split extension of H by K - $G = [H]\cdot K$. Then G is an N-group whose Sylow structure is described by

$$\begin{tikzcd} & 5 \arrow[dl] & \\ 2 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

. Since G has no normal Sylow subgroup, G is not a STG.

Example 2.

A supersolvable group which is not an N-group. Let G be the split extension of $C_7 = < x \,|\, x^7 = 1 >$ by $Aut(C_7) = < y \,|\, x^y = x^3 > \cong C_6$, and put $G_2 = <y^3>$, $G_3 = <y^2>$. G is supersolvable since $1 \lhd C_7 \lhd C_7G_3 \lhd G$ is an invariant series of G with cyclic factors. And G is not an N-group because $G_3 \nleq N_G(G_2^x)$ and $G_2^x \nleq N_G(G_3)$.

Proposition 2.2

If G is a GSTG and $\pi(G) \leq 3$, then G is either a

STG or an N-group.

Proof:

If $\pi(G) \leq 2$, then G has a normal Sylow subgroup and G is a STG and an N-group. Suppose $\pi(G) = 3$ and let $\{G_p, G_q, G_r\}$ be a Sylow basis for G which satisfies (N). If G has a normal Sylow subgroup, then G is a STG. Otherwise we may assume $G_p \leq N_G(G_q)$, $G_q \leq N_G(G_r)$ and $G_r \leq N_G(G_p)$. We will show that every conjugate of G_p in G normalizes G_q. Consider G_p^g, for any $g \in G$. Since $G = G_pG_rG_q$, $g = xyz$ for some $x \in G_p$, $y \in G_r$, $z \in G_q$. Then $G_p^g = G_p^{xyz} = G_p^{yz} = G_p^z$, since $y \in G_r \leq N_G(G_p)$. Since G_p normalizes G_q and $z \in G_q$, $G_p^g = G_p^z$ normalizes $G_q (= G_q^z)$. Therefore every Sylow p-subgroup of G normalizes every Sylow q-subgroup of G. The argument may be repeated to show $G_q^g \leq N_G(G_r)$ and $G_r^g \leq N_G(G_p)$, for all $g \in G$. $\square$

Proposition 2.3

Let G be an N-group and let some Sylow p-subgroup of G normalize some Sylow q-subgroup of G, where $p \neq q$. Then every Sylow p-subgroup of G normalizes every Sylow q-subgroup of G.

Proof:

Let $P \in \mathrm{Syl}\, p(G)$, $Q \in \mathrm{Syl}\, q(G)$ and suppose P normalizes Q. Since all Sylow subgroups of G of the same order are conjugate in G, it is sufficient to show that P^x normalizes Q^y, for all $x,y \in G$.

Since every complete set of Sylow subgroups of G forms a Sylow basis, P^x and Q^y belong to some basis $\mathcal{T}$ of G. Likewise P and Q belong to some Sylow basis $\mathcal{S}$ of G. By theorem 1.6 $\mathcal{S}$ and $\mathcal{T}$ are conjugate in G and so there exists $g \in G$ such that $P^x = P^g$, $Q^y = Q^g$. Since P normalizes Q, P^g normalizes Q^g and we are done. □

We now examine the inheritance properties of N-groups.

Lemma 2.4

Let H be a subgroup of G. If P is a Sylow p-subgroup of H, then $P = H \cap \overline{P}$ for some Sylow p-subgroup $\overline{P}$ of G.

Proof:

By theorem 1.1 P lies in some Sylow p-subgroup $\overline{P}$ of G. Then $P \leq H \cap \overline{P}$. Since $H \cap \overline{P}$ is a p-subgroup of H which contains a Sylow p-subgroup P of H, we must have $P = H \cap \overline{P}$. □

Proposition 2.5

If G is an N-group and $H \leq G$, then H is an N-group.

Proof:

Let $P \in \mathrm{Syl}\, p(H)$, $Q \in \mathrm{Syl}\, q(H)$ where $p \neq q$. By lemma 2.4 there exist subgroups $\overline{P} \in \mathrm{Syl}\, p(G)$ and $\overline{Q} \in \mathrm{Syl}\, q(G)$ such that $P = H \cap \overline{P}$ and $Q = H \cap \overline{Q}$. Since G is an N-group we may assume $\overline{Q}$ normalizes $\overline{P}$. Then $Q = H \cap \overline{Q}$ normalizes $P = H \cap \overline{P}$. □

Proposition 2.6

The homomorphic image of an N-group G is again an N-group.

Proof:

Let σ be a homomorphism of G onto G^{σ}. Let $P \in \mathrm{Syl}\,p(G^{\sigma})$ and $Q \in \mathrm{Syl}\,q(G^{\sigma})$ where $p \nmid q$. By lemma 1.19 there exist Sylow p- and q-subgroups G_p and G_q of G such that $P = G_p{}^{\sigma}$ and $Q = G_q{}^{\sigma}$. Since G is an N-group we may assume G_q normalizes G_p. Then $G_q{}^{\sigma}$ normalizes $G_p{}^{\sigma}$ and the assertion follows. $\square$

The next definition is crucial for the study of direct products of N-groups.

Definition 2.7

The N-groups H and K are said to be similar if for any pair of distinct primes $p, q \in c(H) \cap c(K)$, either

i) $p \xrightarrow[\mathrm{Syl}(H)]{} q$ and $p \xrightarrow[\mathrm{Syl}(K)]{} q$

or ii) $q \xrightarrow[\mathrm{Syl}(H)]{} p$ and $q \xrightarrow[\mathrm{Syl}(K)]{} p$.

The proofs of propositions 2.5 and 2.6 have established the next results.

Proposition 2.8

If H and K are any subgroups of an N-group, then H and K are similar N-groups.

Proposition 2.9

If H and K are any normal subgroups of an N-group G, then G/H and G/K are similar N-groups.

The next theorem is the analogue of theorem 1.27 for N-groups.

Proposition 2.10

$H \times K$ is an N-group if, and only if H and K are similar N-groups.

Proof:

The sufficiency follows from Proposition 2.8.

To show the necessity, let H and K be similar N-groups. Let p and q be distinct primes and suppose $p \xrightarrow[\mathrm{Syl}(H)]{} q$ and $p \xrightarrow[\mathrm{Syl}(K)]{} q$. If $P \in \mathrm{Syl}\, p\,(H \times K)$ and $Q \in \mathrm{Syl}\, q\,(H \times K)$, then $P = H_p \times K_p$ and $Q = H_q \times K_q$ for some $H_p \in \mathrm{Syl}\, p\,(H)$, $H_q \in \mathrm{Syl}\, q\,(H)$, $K_p \in \mathrm{Syl}\, p\,(K)$ and $K_q \in \mathrm{Syl}\, q\,(K)$. Consequently $P = H_p \times K_p$ normalizes $Q = H_q \times K_q$ and the assertion follows. □

The groups S_4 and $S_3 \times A_4$, which were discussed in the first chapter, show that an N-group extended by a cyclic group need not be an N-group and that a cyclic group extended by an N-group need not be an N-group.

Using the same argument as for GSTG's [theorem 1.21], we can prove that a group G is an N-group if, and only if $G/_{Z(G)}$ is an N-group. Consequently proposition 2.10 can be extended to central products using the argument for theorem 1.29. Similarly the proof of proposition 1.31 carries over to the present case and establishes the next result.

<u>Proposition 2.11</u>

Let H and K be normal subgroups of G. If $G/_H$ and $G/_K$ are similar N-groups, then $G/_{H \cap K}$ is an N-group.

We are now in a position to see what collections of N-groups are formations.

<u>Definition 2.12</u>

Let $*$ be a relation on the set of all primes. An N-group G is <u>compatible with $*$</u> if $p * q$ implies that every Sylow p-subgroup of G normalizes every Sylow q-subgroup of G.

Since any two N-groups compatible with a relation $*$ are similar, it follows from propositions 2.6 and 2.11 that the set of all N-groups compatible with a given relation $*$ is a formation. The next example shows these formations are not necessarily saturated.

Example: Let S be the nonabelian group of order 7^3 and exponent 7 defined by $S = \langle x,y,z \mid x^7 = y^7 = z^7 = 1, y^z = yx, x^z = x, x^y = x \rangle$. S is generated by y and z and $S' = \langle x \rangle$. Let D be the group of automorphisms of S defined by $D = \langle d \mid d^6 = 1, y^{d^2} = y^2, z^{d^2} = z, y^{d^3} = y, z^{d^3} = z^6 \rangle$. Take G to be the split extension of S by D. Then $\Phi(G) = \langle x \rangle = S'$ and

$G/_{\Phi(G)} = \langle y,d^2,x\rangle/_{\langle x\rangle} \times \langle z,d^3,x\rangle/_{\langle x\rangle}$. Since $G/_{\Phi(G)}$ is the direct product of similar N-groups, $G/_{\Phi(G)}$ is an

N-group. Yet G is not an N-group because $\langle d^2\rangle^x \not\trianglelefteq N_G(\langle d^3\rangle)$ and $\langle d^3\rangle \not\trianglelefteq N_G(\langle d^2\rangle^x)$.

Huppert [10] has studied groups which satisfy (v): every pair of Sylow subgroups of different orders permute as subgroups. He proves the following result.

Theorem 2.13

If G satisfies (v), then $G/\Phi(G)$ is isomorphic to a subgroup of a direct product of groups of orders $p_i^{a_i} q_i^{b_i}$, where p_i and q_i are primes (not necessarily distinct).

As a consequence of this we have the following.

Proposition 2.14

If G is an N-group, then $\ell(G) \leq 2$.

Proof:

Since an N-group clearly satisfies (v), theorem 2.13 shows $G/\Phi(G) \cong H \subseteq \otimes \prod_{i=1}^{n} T_i$, where $|T_i| = p_i^{a_i} q_i^{b_i}$. For each integer k, $1 \leq k \leq n$, define the projection π_k of H into T_k by $\pi_k(t_1 \cdots t_k \cdots t_n) = t_k$, where $t_i \in T_i$. Since H is isomorphic to a factor group of G, H is an N-group. Consequently the homomorphic image of H under π_k, denoted $Im(\pi_k)$, is an N-group for $k = 1,\cdots,n$. Since $Im(\pi_k) \leq T_k$, at most two primes divide $|Im(\pi_k)|$ and so $\ell(Im(\pi_k)) \leq 2$. Since $H \leq \otimes \prod_{k=1}^{n} Im(\pi_k)$ proposition 1.35 implies

$\ell(H) \leq \ell(\otimes \prod_{k=1}^{n} \mathrm{Im}\,(\pi_k)) \leq \max_{1 \leq k \leq n} \{\ell(\mathrm{Im}(\pi_k))\} \leq 2$. Consequently $\ell(G) \leq 2$. □

We now examine N-groups having nilpotent length exactly 2 more closely.

Definition 2.15

Let G be a group with normal subgroup H. We say K is a partial complement of H in G if $G = H \cdot K$ and K is a proper subgroup of G. The subgroup $H \cap K$ is normal in K, but is not necessarily the identity.

Theorem 2.16 [8]

If G/H is any nilpotent factor of G, then some system normalizer $N_G(\mathcal{S})$ of G is a partial complement of H in G.

Let the hypercommutator of G be the intersection of all normal subgroups K of G such that G/K is nilpotent. $G/H_\infty(G)$ is nilpotent and $H_\infty(G)$ is the minimal (normal) subgroup with this property. If $\ell(G) = 2$ and G is an A-group (i.e. all Sylow subgroups of G are abelian), Taunt [14] shows that any system normalizer of G is a complement of $H_\infty(G)$.

We can characterize N-groups as a special product of nilpotent groups.

Theorem 2.17

Let G satisfy (v). Then G is an N-group if, and only if

(1) G is a partially complemented extension of a nilpotent group H by a nilpotent group K

and (2) if p and q are distinct primes, then H_p normalizes K_q or H_q normalizes K_p.

Proof:

Let G be an N-group of nilpotent length 2. By theorem 2.16, $G = H_\infty(G) \cdot N_G(\mathcal{S})$ for some system normalizer $N_G(\mathcal{S})$ of G. Since $\ell(G) = 2$, $H_\infty(G)$ is nilpotent and, by theorem 1.10, $N_G(\mathcal{S})$ is always nilpotent. Consequently (1) holds.

To verify (2), let p and q be any distinct prime divisors of $(|H_\infty(G)|, |{}^G/_{H_\infty(G)}|)$. Put $H = H_\infty(G)$, $K = N_G(\mathcal{S})$ and consider the subgroup $L_1 = H_p(K_pK_q)$ of G. L_1 is an N-group of order p^aq^b and consequently K_q normalizes H_pK_p or H_pK_p normalizes K_q. In the latter case H_p normalizes K_q. In the first case, we may assume $K_q \leq N_G(H_pK_p)$ and $H_pK_p \nleq N_G(k_q)$ - for otherwise we are done. Then, since L_1 is similar to G, H_qK_q must normalize H_pK_p.

Now consider the subgroup $L_2 = H_q(K_qK_p)$ of G. L_2 is an N-group of order p^aq^b and so K_p normalizes H_qK_q or H_qK_q normalizes K_p. In the latter case, $H_q \leq N_G(K_p)$ and we are done. Otherwise we may assume K_p normalizes H_qK_q and $H_qK_q \nleq N_G(K_p)$. Since L_2 is similar to G, we must have H_pK_p normalizes H_qK_q.

Therefore H_pK_p and H_qK_q normalize one another and, since they have relatively prime orders, $[H_pK_p, H_qK_q] \leq H_pK_p \cap H_qK_q = 1$. This establishes the sufficiency.

We now show the necessity. Suppose G satisfies $(\vee)$ and conditions (1) and (2) hold. We proceed by induction on $\pi(G)$.

If G is a p-group, the theorem is trivial. Suppose $|G| = p^a q^b$ and let $G = H \cdot K = (H_pH_q)(K_pK_q)$. By (2) we may assume H_p normalizes K_q. Since K is nilpotent, K_p normalizes K_q. And since H is a nilpotent normal subgroup of G, $H_q \triangleleft G$. Then H_pK_p normalizes H_qK_q and G is an N-group.

Suppose $\pi(G) = k \geq 3$ and let p and q be any distinct prime divisors of $|G|$. Let G_p and G_q be any Sylow p- and q-subgroups of G (respectively). By Sylow arguments we then have

$$G_p = (H_pK_p)^x = H_pK_p{}^x \quad \text{for some} \quad x \in G$$

$$G_q = (H_qK_q)^y = H_qK_q{}^y \quad \text{for some} \quad y \in G.$$

Since G satisfies $(\vee)$, $L = (H_pK_p)(H_qK_q{}^{yx^{-1}})$ is a subgroup of G, for all $x,y \in G$. L is a proper subgroup of G which satisfies the induction hypothesis and so L is an N-group. Hence H_pK_p normalizes $H_qK_q{}^{yx^{-1}}$ or $H_qK_q{}^{yx^{-1}}$ normalizes H_pK_p. Then $G_p = H_pK_p{}^x$ normalizes $H_qK_q{}^y = G_q$

or $G_q = H_qK_q^{\;y}$ normalizes $H_pK_p^{\;x} = G_p$. Since G_p and G_q were chosen to be arbitrary Sylow subgroups of G, we have shown that G is an N-group. $\square$

Owing to Taunt's result on A-groups, we see that condition (1) of theorem 2.17 can be improved to complements (in place of partial complements) for N-groups with abelian Sylow subgroups. It is not known if this is generally true for N-groups. The following examples show the conditions of theorem 2.17 cannot be relaxed.

Example 1.

Let G be the split extension of C_7 by $\mathrm{Aut}\,(C_7) \cong C_6$. We already know G is not an N-group. G satisfies (1) and (2), but fails to satisfy (v).

Example 2.

Let $W = \otimes \prod_{i=1}^{3} < a_i | a_i^{\,2} = 1 > \times \otimes \prod_{j=1}^{2} < b_j | b_j^{\,3} = 1 >$

and let $H = < x, v | a_i^{\,x} = a_{i+1},\ b_j^{\,x} = b_j,\ b_j^{\,v} = b_{j+1},$ $a_i^{\,v} = a_i,\ a_4 = a_1,\ b_3 = b_1 > \leq \mathrm{Aut}\,(W)$. Put $G = [W]\cdot H$, the split extension of W by H. G satisfies (v) and condition (1) but is not an N-group - since $\pi(G) = 2$ and G has no normal Sylow subgroup.

In view of theorems 2.16 and 2.17, we try to describe the system normalizers of an N-group.

Proposition 2.18

Let G be an N-group. If $\mathcal{S} = \{S_1, \cdot\cdot, S_k\}$ is a Sylow basis for G, then the system normalizer associated with $\mathcal{S}$ is $N_G(\mathcal{S}) = \otimes \prod_{k=1}^{n} N_k$, where $N_k = C_{S_k}(T_1 \cdot \cdot \cdot T_\ell)$ for $T_1, \cdot \cdot \cdot, T_\ell$ those members of $\mathcal{S} \setminus \{S_k\}$ which normalize S_k.

Proof:

By theorem 1.10, $N_G(\mathcal{S}) = \otimes \prod_{i=1}^{n} L_i$, where $L_i = S_i \cap N_G(S_i')$, $S_i' = \prod_{j \neq i} S_j$.

Let $x \in L_k = S_k \cap N_G(S_k')$ and let $y \in T_1 \cdots T_\ell$. Then $[y,x] \in S_k \cap S_k' = 1$ and so $x \in N_K = C_{S_k}(T_1 \cdot\cdot T_\ell)$.

Conversely, let $w \in C_{S_k}(T_1 \cdot\cdot T_\ell) = N_k$. Then $w \in S_k$ and by our choice of $T_1, \cdots, T_\ell$, $w \in N_G(S_k')$ - since S_k normalizes all members of $\mathcal{S} \setminus \{T_1, \cdots, T_\ell\}$, $\square$

The order of a system normalizer can also be computed using theorem 1.9 : $[G : N_G(\mathcal{S})]$ = the number of distinct Sylow bases of G. Since the number of Sylow bases for an N-group G is the number n of complete sets of Sylow subgroups of G, we see $n = \prod_{k=1}^{\pi(G)} [G : N_G(S_k)]$ and $|N_G(\mathcal{S})| = {}^{|G|}/_{n}$.

Although we gave an example to show that G need not be an N-group whenever $G/_{\Phi(G)}$ is an N-group, we can make some progress in this direction.

Proposition 2.19

If $|G| = p^a q^b$ and $G/_{\Phi(G)}$ is an N-group, then G is an N-group.

Proof:

We may assume that $G/\Phi(G)$ has normal Sylow p-subgroup $G_p\Phi(G)/\Phi(G)$. Let $W = N_G(G_p)$. If $W = G$, then $G_p \lhd G$ and we are done. Otherwise W lies in some maximal subgroup S of G. Let $x \in G \setminus S$. Since $G_p\Phi(G)/\Phi(G) \lhd G/\Phi(G)$, we have $G_p^x\Phi(G)/\Phi(G) = (G_p\Phi(G)/\Phi(G))^x = G_p\Phi(G)/\Phi(G) \leq S/\Phi(G)$. Hence $G_p^x \leq G_p^x\Phi(G) \leq S$ and there exists $s \in S$ with $G_p^s = G_p^x$. Since $G_p^{xs^{-1}} = G_p$, $xs^{-1} \in W = N_G(G_p)$ and $(xs^{-1})s = x \in S$ - a contradiction. □

Proposition 2.20

If $G/\Phi(G)$ is an N-group and all maximal (proper) subgroups of G are N-groups, then G is an N-group.

Proof:

Let $G_p \in \mathrm{Syl}\, p(G)$ and $G_q \in \mathrm{Syl}\, q(G)$ for $p \neq q$. Since $G/\Phi(G)$ is an N-group, we may assume $G_q\Phi(G)/\Phi(G)$ normalizes $G_p\Phi(G)/\Phi(G)$. Let $W = N_G(G_p)$. If $W = G$, then $G_p \lhd G$ and we are done. Otherwise W lies in some maximal subgroup S of G. Since $G_q\Phi(G)/\Phi(G)$ normalizes $G_p\Phi(G)/\Phi(G)$, we see $G_p^x\Phi(G)/\Phi(G) = (G_p\Phi(G)/\Phi(G))^x$ $= G_p\Phi(G)/\Phi(G) \leq S/\Phi(G)$, for all $x \in G_q$. Hence $G_p^x \leq S$

and there exists $s \in S$ with $G_p^{\,x} = G_p^{\,s}$. Then $xs^{-1} \in W \leq S$ and $x \in S$. Hence $G_q \leq S$. Since S is an N-group by assumption and $< G_p, G_q > \leq S$, either G_p normalizes G_q or G_q normalizes G_p. □

We mention a result of Rose [11] for classes of groups closed under homomorphic images (i.e. Q-closed).

<u>Theorem 2.21</u>

Let $\mathcal{C}$ be any Q-closed class of groups. Suppose G is a finite solvable but non-nilpotent group in which every abnormal maximal subgroup is a $\mathcal{C}$-group. Then G has a normal subgroup W of prime power order such that G/W is a $\mathcal{C}$-group.

Since N-groups form a Q-closed class, the theorem applies if we let $\mathcal{C}$ = class of all N-groups.

Chapter III

The idea of a Sylow system is basic to P. Hall's investigation of solvable groups. In this section we consider condition (N) for Sylow systems of a group. If some Sylow system of G satisfies (N), we call G a strongly Sylow towered group (SSTG). The inheritance properties of SSTG's are derived and we give a characterization of these groups.

We begin with a formal definition.

Definition 3.1

Let G be a solvable group. If some Sylow system $\mathcal{S}$ of G satisfies (N), G is called a strongly Sylow towered group (SSTG).

It follows immediately that a SSTG is necessarily a GSTG. The next examples show that SSTG's are unrelated to N-groups.

Example 1.

Let $G = [C_7]\cdot C_6$, the split extension of C_7 by its automorphism group. Write $Aut(C_7) = C_6$ as the product of its Sylow subgroups, $C_2 \times C_3$. G is a SSTG since $\mathcal{S} = \{1, C_7, C_3, C_2, C_7C_2, C_7C_3, C_2C_3, G\}$ is a Sylow system of G which satisfies (N). By a previous remark, G is not an N-group.

Example 2.

We will show [Proposition 3.4] that a SSTG is necessarily a STG. Therefore the example of an N-group which is not a STG, given in chapter II, is an N-group which is not a SSTG.

Proposition 3.2

If G is a SSTG, then every Sylow system of G satisfies (N).

Proof:

This follows at once from the fact that all Sylow systems of a group are conjugate [Theorem 1.6]. $\square$

Proposition 3.3

If G is a SSTG, then every subgroup H of G is a SSTG.

Proof:

Since G is solvable, the subgroup H is solvable. Let $\mathcal{H}$ be any Sylow system of H. By theorem 1.7 there is a Sylow system $\mathcal{S}$ of G such that $\mathcal{H} = \{H \cap S \mid S \in \mathcal{S}\}$. Since $\mathcal{S}$ satisfies (N), $\mathcal{H}$ satisfies (N). $\square$

Proposition 3.4

If G is a SSTG, then G is a STG.

Proof: (Induction on $\pi(G)$)

If $\pi(G) \leq 2$, then G has a normal subgroup and clearly G is a STG. Suppose $\pi(G) = n \geq 3$ and let $\mathcal{S}$ be a Sylow system of G. Let T and W be complementary (proper) subgroups of G belonging to $\mathcal{S}$ - i.e. $G = T \cdot W$ and $T \cap W = 1$. Since $\mathcal{S}$ satisfies (N), we may assume T normalizes W. Then $W \triangleleft G = T \cdot W$.

By induction, both T and W are STG's. Let

$$1 \triangleleft T_1 \triangleleft \cdots \quad \cdots \triangleleft T_k \cdots T_1 = T$$

and $$1 \triangleleft W_1 \triangleleft \cdots \quad \cdots \triangleleft W_\ell \cdots W_1 = W$$

be Sylow towers of T and W. Then

$$1 \triangleleft W_1 \triangleleft \cdots \quad \triangleleft W_\ell \cdots W_1 = W \triangleleft T_1 W \triangleleft \cdots \quad \cdots \triangleleft T_k \cdots T_1 W = G$$

is a Sylow tower of G and G is a STG. □

Lemma 3.5

Let G be a finite solvable group and σ a homomorphism of G onto G^σ. Then L is a Hall π-subgroup of G^σ if, and only if $L = H^\sigma$, for some Hall π-subgroup H of G.

Proof:

Let V be any Hall π-subgroup of G. Then V^σ is a Hall π-subgroup of G^σ, since $[G^\sigma : V^\sigma] \,\big|\, [G : V]$ and $|V^\sigma| \,\big|\, |V|$. Since G is solvable, G^σ is solvable and theorem 1.1 implies V^σ and L are conjugate in G^σ. Therefore $L = (V^\sigma)^{x^\sigma} = (V^x)^\sigma$, some $x \in G$. □

Proposition 3.6

If G is a SSTG and σ is a homomorphism of G onto G^σ, then G^σ is a SSTG.

Proof:

Let $\mathcal{S}$ be a Sylow system of G. Then $\mathcal{S}^\sigma = \{S^\sigma \mid S \in \mathcal{S}\}$ is, by the lemma, a Sylow system of G^σ. And S^σ normalizes T^σ whenever S normalizes T $(S,T \in \mathcal{S})$. Therefore $\mathcal{S}^\sigma$ satisfies (N) and G^σ is a SSTG. □

In order to study direct products of SSTG's, we need to define similarity for SSTG's.

Definition 3.7

Let H and K be SSTG's with Sylow systems $\mathcal{H} = \{H_\pi \mid \pi \subseteq c(H)\}$ and $\mathcal{K} = \{K_\pi \mid \pi \subseteq c(K)\}$. Let Σ and Δ be arbitrary disjoint sets of primes. If either

(i) H_Σ normalizes H_Δ and K_Σ normalizes K_Δ or

(ii) H_Δ normalizes H_Σ and K_Δ normalizes K_Σ, we say H and K are similar SSTG's.

It is easy to check [Propositions 3.3 and 3.6] that all subgroups and factor groups of a SSTG are similar SSTG's. And by adapting the argument used for the direct product of GSTG's [theorem 1.27] to the present situation, we have the following.

Proposition 3.8

$H \times K$ is a SSTG if, and only if H and K are similar SSTG's.

SSTG's can be characterized as follows.

Theorem 3.9

G is a SSTG if, and only if G is a split extension of a nilpotent group A by a nilpotent group B with $(|A|, |B|) = 1$ and either A or B a p-group.

Proof:

Let $\mathcal{S}$ be any Sylow system of G. Let S and T be any distinct non-normal Sylow subgroups of $\mathcal{S}$ and let S' and T' be (respectively) the complements of S and T

in $\mathcal{S}$. Since $\mathcal{S}$ satisfies (N) and $S \ntrianglelefteq G$, $T \ntrianglelefteq G$, we must have $S' \trianglelefteq G$ and $T' \trianglelefteq G$. Furthermore, we may assume S normalizes T. Now $S \leq T' \triangleleft G$ implies $[S,T] \leq T' \cap T = 1$ and hence, any distinct non-normal Sylow subgroups of $\mathcal{S}$ centralize one another. If we let A be the product of all the normal Sylow subgroups of G and B be the product of the non-normal Sylow subgroups of G in $\mathcal{S}$, then $G = [A]\cdot B$ and $(|A|,|B|) = 1$.

We now show that either A or B is a p-group. Assume this is not the case and suppose $\pi(A) = k \geq 2$, $\pi(B) = \ell \geq 2$. Let $N_1,\cdots,N_k$ denote the normal Sylow subgroups of G and let $S_1,\cdots,S_\ell$ denote the non-normal Sylow subgroups of G belonging to $\mathcal{S}$. Without loss of generality we may assume that N_1 does not normalize S_1. If some normal Sylow subgroup N_i $(i \geq 2)$ fails to normalize some S_j $(j \geq 2)$, consider the subgroups N_1S_j and N_iS_1. Now N_1S_j and N_iS_1 belong to $\mathcal{S}$ and $(|N_1S_j|,|N_iS_1|) = 1$. Hence N_1S_j normalizes N_iS_1 or N_iS_1 normalizes N_1S_j. Then $[N_1,S_1] \leq N_1 \cap N_iS_1 = 1$ or $[N_i,S_j] \leq N_i \cap N_1S_j = 1$ - a contradiction. Therefore we see N_i normalizes S_j whenever $i,j \geq 2$. Consequently N_1 must not normalize S_j for $j = 2,\cdots,\ell$. In particular, N_1 does not normalize S_2.

Now suppose N_i normalizes S_1 whenever $i \geq 2$. Then N_i normalizes every non-normal Sylow subgroup S_j

and $H = \prod_{i=2}^{k} N_i \cdot \prod_{j=1}^{\ell} S_j$ is a nilpotent subgroup of G.
Therefore $G = [N_1] \cdot H$ - which is a contradiction. Hence, we may assume that N_2 fails to normalize S_1.

Consider the subgroups N_1S_1 and N_2S_2 of G. N_1S_1 and N_2S_2 belong to $\mathcal{S}$ and have relatively prime orders. Therefore N_1S_1 normalizes N_2S_2 or N_2S_2 normalizes N_1S_1. Then $[N_1,S_2] \leq N_1 \cap N_2S_2 = 1$ or $[N_2,S_1] \leq N_2 \cap N_1S_1 = 1$ - which is a contradiction. This establishes the sufficiency.

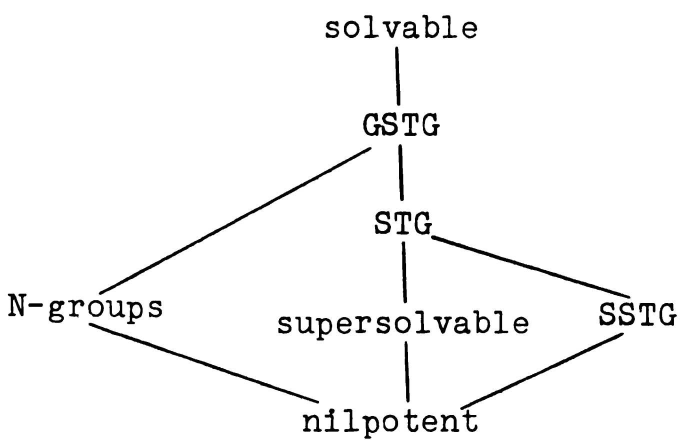
Let G be the split extension of a p-group A by a nilpotent group B where $(p,|B|) = 1$. Let $B_1,\cdots,B_n$ denote the Sylow subgroups of B. Then the Sylow system of G generated by the Sylow basis $\{A,B_1,\cdots,B_n\}$ clearly satisfies (N) and G is a SSTG.

Now let G be the split extension of a nilpotent group A by a p-group B with $(|A|,p) = 1$. Let $\{A_1,\cdots,A_k\}$ be the complete set of Sylow subgroups of A. Then the Sylow system of G generated by the Sylow basis $\{A_1,\cdots,A_k,B\}$ of G clearly satisfies (N) and G is a SSTG. $\square$

We now give an example of a supersolvable group which is not a SSTG. Let G be the split extension of $C_3 \times C_{11}$ by its automorphism group $C_2 \times C_{10}$. Then $1 \triangleleft C_3 \triangleleft C_3 \times C_{11} \triangleleft C_2\,(C_3 \times C_{11}) \triangleleft G$ is a cyclic invariant series of G, and G is supersolvable. By the preceeding theorem, G is not a SSTG.

The group $G = C_3 \text{ wr } C_2$ is an example of a SSTG which is not supersolvable.

The following diagram shows the relative positions of various classes of solvable groups. A line indicates the class at the lower end lies in the class at the upper end. All containments are proper.



Chapter IV

If G is a solvable group, the Fitting length of G is a measure of the nilpotency of G. We ask if there is a measure of the deviation of a GSTG from an N-group. This leads to an examination of invariant series whose factor groups are N-groups. Such series will be called <u>invariant N-series</u>.

A GSTG G has a unique descending invariant N-series, called the lower N-series for G. The length of the lower N-series, denoted by $m(G)$, is the minimal length of an invariant N-series of G. A GSTG G also has a unique ascending invariant N-series, called the upper N-series for G. The length of this ascending series is denoted by $e(G)$. The nilpotent length of G is greater than or equal to $e(G)$.

An invariant N-series $1 \lhd L_1 \lhd \cdots \lhd L_k = G$ with L_{i+1}/L_i a maximal normal N-subgroup of G/L_i $(0 \leq i \leq k-1)$ is called a <u>maximal invariant N-series</u> of G. If k is the length of a maximal invariant N-series of G, then $m(G) \leq k \leq e(G)$. We show by example that k can be less than $e(G)$, but it is not known if k must equal $m(G)$.

We begin with the following observation.

<u>Lemma 4.1</u>

Let H and K be (N)-similar GSTG's. If H and K

are N-groups, they are similar N-groups.

Proof:

Let p and q be distinct primes. Since H and K are (N)-similar GSTG's, we may assume that some Sylow p-subgroup of H normalizes some Sylow q-subgroup of H and, some Sylow p-subgroup of K normalizes some Sylow q-subgroup of K. If H and K are N-groups, proposition 2.3 then implies that every Sylow p-subgroup of H normalizes every Sylow q-subgroup of H and, every Sylow p-subgroup of K normalizes every Sylow q-subgroup of K. $\square$

Our study of ascending invariant N-series of a GSTG is motivated by Baer's work [1] on supersolubly immersed subgroups.

Definition 4.2

A subgroup H of a group G is N-embedded in G if for every pair of distinct primes (p,q), either $G_q \leq N_G(H_p)$ or $H_p \leq N_G(G_q)$ for all $H_p \in \mathrm{Syl}\, p\,(H)$, $G_q \in \mathrm{Syl}\, q\,(G)$.

A normal p-subgroup of a group is an example of an N-embedded subgroup. An N-embedded subgroup need not be normal however. For example, a Sylow 2-subgroup of S_3 is an N-embedded subgroup which is not a normal subgroup.

Proposition 4.3

If H is N-embedded in G, then H is an N-group.

<u>Proof</u>:

Let H_p and H_q be arbitrary Sylow subgroups of H corresponding to distinct primes p and q. Let G_q be a Sylow q-subgroup of G containing H_q. Then $G_q \leq N_G(H_p)$ or $H_p \leq N_G(G_q)$. Consequently we see H_q normalizes H_p or H_p normalizes $G_q \cap H = H_q$. □

<u>Proposition 4.4</u>

If H and K are N-embedded in G, then $H \cap K$ is N-embedded in G.

<u>Proof</u>:

Let p and q be distinct primes and consider $(H \cap K)_p \in \mathrm{Syl}\, p\,(H \cap K)$ and $G_q \in \mathrm{Syl}\, q\,(G)$. By lemma 2.4 $(H \cap K)_p = H_p \cap K_p$ for some $H_p \in \mathrm{Syl}\, p\,(H)$, $K_p \in \mathrm{Syl}\, p\,(K)$. Now if $H_p \leq N_G(G_q)$ or $K_p \leq N_G(G_q)$, then $(H \cap K)_p = H_p \cap K_p$ normalizes G_q. Otherwise G_q normalizes both H_p and K_p. Then G_q normalizes $H_p \cap K_p = (H \cap K)_p$. □

<u>Proposition 4.5</u>

If K is N-embedded in G and H is a Hall subgroup of K, then H is N-embedded in G.

<u>Proof</u>:

This follows at once since Sylow subgroups of H are also Sylow subgroups of K.

□

An arbitrary subgroup of an N-embedded subgroup is not necessarily an N-embedded subgroup. For example, let G be the split extension of $A \times B$ by C, where $A = \langle a \mid a^3 = 1 \rangle$, $B = \langle b \mid b^3 = 1 \rangle$ and $C = \langle c \mid a^c = b,\ b^c = a \rangle \subseteq \mathrm{Aut}\,(A \times B)$. Then $A \times B$ is N-embedded in G but neither A nor B is N-embedded in G.

<u>Proposition 4.6</u>

If H is N-embedded in G and $K \leq H$, $K \trianglelefteq G$, then K is N-embedded in G.

<u>Proof</u>:

Let p and q be distinct primes and $K_p \in \mathrm{Syl}\,p\,(K)$, $G_q \in \mathrm{Syl}\,q\,(G)$. Since $K \leq H$, $K_p = K \cap H_p$ for some $H_p \in \mathrm{Syl}\,p\,(H)$. If $H_p \leq N_G(G_q)$, then $K_p = K \cap H_p$ normalizes G_q. Otherwise $G_q \leq N_G(H_p)$ and, since $K \trianglelefteq G$, G_q normalizes $K \cap H_p = K_p$. $\square$

<u>Proposition 4.7</u>

If H is an N-embedded (normal) subgroup of G and σ is a homomorphism of G onto G^σ, then H^σ is an N-embedded (normal) subgroup of G^σ.

<u>Proof</u>:

Let $p \nmid q$ and let $P \in \mathrm{Syl}\,p\,(H^\sigma)$, $Q \in \mathrm{Syl}\,q\,(G^\sigma)$. By [13, p.134], $P = (H_p)^\sigma$ and $Q = (G_q)^\sigma$ for some $H_p \in \mathrm{Syl}\,p\,(H)$, $G_q \in \mathrm{Syl}\,q\,(G)$. Since H is N-embedded in G, either $H_p \leq N_G(G_q)$ or $G_q \leq N_G(H_p)$. Then P

normalizes Q or Q normalizes P. □

In particular, every conjugate of an N-embedded subgroup is an N-embedded subgroup.

All the previous results hold for arbitrary finite groups. From this point on, we restrict our attention to GSTG's.

Proposition 4.8

Let G be a GSTG with N-embedded subgroup H. Then every Sylow p-subgroup of H normalizes every Sylow q-subgroup of G or every Sylow q-subgroup of G normalizes every Sylow p-subgroup of H, for p and q distinct primes.

Proof:

Let $p \neq q$ and consider $H_p \in \mathrm{Syl}\, p(H)$, $G_q \in \mathrm{Syl}\, q(G)$. Since H is N-embedded in G, $H_p G_q^x$ is a subgroup of G, for all $x \in G$. Since G is a GSTG, all the subgroups $H_p G_q^x$ are (N)-similar. Therefore, either G_q^x normalizes H_p or H_p normalizes G_q^x, for all $x \in G$. □

Theorem 4.9

Let H be an N-embedded (normal) subgroup of G and let K be an N-embedded normal subgroup of G. Then the product $H \cdot K$ is an N-embedded (normal) subgroup of G.

Proof:

Let $p \neq q$ and consider $(H \cdot K)_p \in \mathrm{Syl}\, p(HK)$ and $G_q \in \mathrm{Syl}\, q(G)$.

We first show $(H\cdot K)_p = H_p^{\ x}K_p^{\ x}$ for some $H_p \in \mathrm{Syl}\, p(H)$, $K_p \in \mathrm{Syl}\, p(K)$ and $x \in H\cdot K$. Let P be any Sylow p-subgroup of H. Then $P \leq G_p$ for some $G_p \in \mathrm{Syl}\, p(G)$ and $P = H \cap G_p$. Since $K \trianglelefteq G$, $K \cap G_p = K_p$ is a Sylow p-subgroup of K and $K_p \trianglelefteq G_p$. Then $P\cdot K_p$ is a p-subgroup of $H\cdot K$ and $|P\cdot K_p| = \dfrac{|P|\cdot|K_p|}{|P \cap K_p|}$, the order of a Sylow subgroup of $H\cdot K$. Therefore $(H\cdot K)_p = (P\cdot K_p)^x = P^xK_p^{\ x}$ for some $x \in H\cdot K$.

Since G is a GSTG, all subgroups of G are (N)-similar GSTG's and hence $p \xrightarrow[\mathscr{S}]{} q$ or $q \xrightarrow[\mathscr{S}]{} p$, where $\mathscr{S}$ is a Sylow basis of any subgroup of G.

Suppose first that $p \xrightarrow[\mathscr{S}]{} q$. Since H is N-embedded in G, every conjugate H^x of H is N-embedded in G. Then $H_p^{\ x}G_q$ is a subgroup of G and consequently $H_p^{\ x}$ normalizes G_q $(x \in G)$. Likewise $K_p^{\ x}G_q$ is a subgroup of G and $K_p^{\ x}$ normalizes G_q $(x \in G)$. Then $H_p^{\ x}K_p^{\ x} = (HK)_p$ normalizes G_q.

Now suppose $q \xrightarrow[\mathscr{S}]{} p$. Since $H_p^{\ x}G_q$ and $K_p^{\ x}G_q$ are subgroups of G, G_q normalizes both $H_p^{\ x}$ and $K_p^{\ x}$ $(x \in G)$. Then G_q normalizes $H_p^{\ x}K_p^{\ x} = (HK)_p$ and the theorem is proved. □

<u>Corollary 4.10</u>

The product of all normal N-embedded subgroups of G is a characteristic N-embedded subgroup of G. We denote this maximal normal N-embedded subgroup of G by $E(G)$.

Corollary 4.11

The characteristic subgroup $E(G)$ of G is the intersection of all maximal N-embedded subgroups of G.

Proof:

Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal N-embedded subgroups of G. By theorem 4.4 the intersection $\bigcap_{\lambda \in \Lambda} M_\lambda$ is an N-embedded subgroup of G. And, by theorem 4.7, $\bigcap_{\lambda \in \Lambda} M_\lambda$ is a characteristic subgroup of G. Consequently $\bigcap_{\lambda \in \Lambda} M_\lambda \leq E(G)$.

By the preceding theorem we know $M_\lambda \cdot E(G)$ is an N-embedded subgroup of G, for each $\lambda \in \Lambda$. Since M_λ is a maximal N-embedded subgroup, $E(G) \leq M_\lambda$ for each $\lambda \in \Lambda$ and $E(G) \leq \bigcap_{\lambda \in \Lambda} M_\lambda$. □

Using the maximal normal N-embedded subgroup $E(G)$ of G we can define a unique ascending invariant N-series of the GSTG G. Before proceeding to this, we examine the subgroup $E(G)$ in more detail.

Proposition 4.12

If G is a GSTG, then $Fit(G) \leq E(G)$.

Proof:

The Sylow subgroups of $Fit(G)$ are normal p-subgroups of G. Since a normal p-subgroup is N-embedded, the assertion is proved. □

Since S_3 is an N-group which is not nilpotent, $Fit(S_3) \lneqq S_3 = E(S_3)$. Therefore we cannot hope for equality in proposition 4.12.

Theorem 4.13

Let K be a normal N-subgroup of G. If H is a normal N-embedded subgroup of G, then the product $H \cdot K$ is an N-group.

Proof:

Let $p \neq q$ and consider $(HK)_p \in Syl_p(HK)$ and $(HK)_q \in Syl\, q\,(HK)$. Let $(HK)_p \leq G_p$ and $(HK)_q \leq G_q$ where $G_p \in Syl\, p\,(G)$, $G_q \in Syl\, q\,(G)$. Since H and K are normal in G, we have the following relations:

$H \cap G_p = H_p \in Syl\, p\,(H)$, $H \cap G_q = H_q \in Syl\, q\,(H)$,

$K \cap G_p = K_p \in Syl\, p\,(K)$, $K \cap G_q = K_q \in Syl\, q\,(K)$ and

$(HK)_p = HK \cap G_p = H_pK_p$, $(HK)_q = HK \cap G_q = H_qK_q$.

By lemma 4.1 we may assume that $p \xrightarrow[\mathcal{S}]{} q$ for a Sylow basis $\mathcal{S}$ of any N-subgroup of G. Since H is N-embedded in G, H_pG_q is an N-subgroup of G and hence $H_p \leq N_G(G_q)$. Since $HK \trianglelefteq G$, this implies that H_p normalizes $HK \cap G_q = (HK)_q$. H_qG_p is also an N-subgroup of G and so $G_p \leq H_G(H_q)$. Therefore $K_p = K \cap G_p$ normalizes H_q. Since K is an N-subgroup of G, K_p also normalizes K_q and hence K_p normalizes the product $H_qK_q = (HK)_q$. Therefore both H_p and K_p normalize $(HK)_q$ and the assertion follows. $\square$

The following example shows we cannot drop the normality of the subgroup K in theorem 4.13. $G = [C_7]\cdot C_6$ is the product of a normal N-embedded subgroup C_7 and a (non-normal) cyclic subgroup C_6, but G is not an N-group.

Corollary 4.14

A maximal normal N-subgroup of G contains every normal N-embedded subgroup of G.

In particular, the subgroup $E(G)$ lies in every maximal normal N-subgroup of G. The next example shows that $E(G)$ is not necessarily the intersection of all maximal normal N-subgroups of G.

Let $S_3 = \langle a,b \mid a^3 = b^2 = 1,\ a^b = a^2\rangle$, $C_5 = \langle d \mid d^5 = 1\rangle$ and $G = S_3 \text{ wr } C_5$. We show that $E(G)$ is the (normal) Sylow 3-subgroup of G and $S_3^{|C_5|}$ is the only maximal normal N-subgroup of G.

(1) $E(G) = G_3$, the Sylow 3-subgroup of G.

Since G is the split extension of $S_3^{|C_5|}$ by C_5, $S_3^{|C_5|}$ is a maximal normal N-subgroup of G. Then $E(G) \leq S_3^{|C_5|}$ and consequently $5 \nmid |E(G)|$. Furthermore, $G_3 \trianglelefteq G$ implies $G_3 \leq \mathrm{Fit}(G) \leq E(G)$. We will show $2 \nmid |E(G)|$.

Assume to the contrary that $2 \mid |E(G)|$ and let

$t = (y_1, y_2, y_3, y_4, y_5)$ be an element of order 2 in $E(G)$. Conjugating t by the appropriate element of G_3 gives an element $w = (x_1, x_2, x_3, x_4, x_5) \neq 1$ in $E(G)$, where each x_i is b or 1. Since all cyclic permutations of w and all products of these cyclic permutations are elements of $E(G)$, we may assume without loss of generality that $w = (b, b, x_3, x_4, x_5)$. By the normality of $E(G)$ in G, $z = (b^a, b, x_3, x_4, x_5) \in E(G)$. Let z belong to a Sylow 2-subgroup E_2 of $E(G)$. Since E_2 does not normalize C_5, C_5 must normalize E_2. Consequently $z^d \in E_2$ and $zz^d = (b^a x_5, bb^a, x_3 b, x_4 x_3, x_5 x_4) \in E_2$. But $bb^a = a^{-1}$ is a 3-element and hence zz^d is not a 2-element - which is nonsense. Therefore $2 \nmid |E(G)|$.

(2) $S_3^{|C_5|}$ is the only maximal normal N-subgroup of G.

Suppose to the contrary that M is a maximal normal N-subgroup of G different from $S_3^{|C_5|}$. Since $E(G) \leq M$, $G_3 \leq M$. Then $M \neq S_3^{|C_5|}$ implies M contains a 5-element of G. Since $M \trianglelefteq G$, M must then contain every 5-element of G. Let $x = (b,1,1,1,1)$. Then $d^x \in M$ and $d^{-1}d^x = (b,b,1,1,1) \in M$. Since $M \trianglelefteq G$, this implies $t = (b^a,b,1,1,1) \in M$ and $t^d = (1,b^a,b,1,1) \in M$. Let t belong to the Sylow 2-subgroup M_2 of M. Since M is an N-group and $5 \underset{M}{\longrightarrow} 2$, C_5 normalizes M_2. Consequently $t^d \in M_2$ and $tt^d = (b^a, bb^a, b, 1, 1) \in M_2$. But $bb^a = a^{-1}$ is a 3-element and tt^d is therefore not a 2-element -

which is nonsense.

Proposition 4.15

If H and K are (N)-similar GSTG's, then $E(H \times K) = E(H) \times E(K)$.

Proof:

Since $E(H)$ and $E(K)$ are normal N-embedded subgroups of $H \times K$, $E(H) \times E(K) \leq E(H \times K)$.

Let $E_1 = \{h \in H \mid hk \in E(H \times K)$ for some $k \in K\}$ and $E_2 = \{k \in K \mid hk \in E(H \times K)$ for some $h \in H\}$. By the isomorphism theorems $E_1 \cong E_1/_{E_1 \cap K} \cong E_1K/_K = E(H \times K) \cdot K/_K$.

Since $E(H \times K) \cdot K/_K$ is a normal N-embedded subgroup of $H \times K/_K$, E_1 is a normal N-embedded subgroup of H and so $E_1 \leq E(H)$. Similarly $E_2 \leq E(K)$. Therefore, $E(H \times K) \leq E_1 \times E_2 \leq$ $\leq E(H) \times E(K)$. □

Baer [1] has shown that a normal supersolvably immersed subgroup K of a group G satisfies the following properties:

(1) If $K \leq S \leq G$ and $S/_K$ is supersolvable, then S is supersolvable.

(2) The elements of G induce a supersolvable group of automorphisms on K.

The corresponding properties do not hold for normal N-embedded subgroups of a group.

(1) Let $G = [C_7] \cdot C_6$. $E(G) = C_7$ and $G/_{C_7} \cong C_6$ but G is not an N-group.

(2) Let $G = C_5$ wr S_4 and let G_5 be the Sylow 5-subgroup

of G. Since S_4 is not an N-group, the automorphism group induced on G_5 by the elements of G is not an N-group.

Consequently we expect the ascending invariant N-series of a GSTG which arises in connection with the subgroup E(G) to have some shortcomings.

Definition 4.16

If G is a GSTG, the upper N-series of G is the invariant series of G defined inductively by $E_0 = 1$, $E_{i+1}/E_i = E(G/E_i)$. The length of this series, denoted by e(G), is the number of distinct nontrivial terms in the series.

Since the Fitting subgroup of G lies in E(G), the next result is clear.

Proposition 4.17

If G is a GSTG, then $e(G) \leq \ell(G)$, the nilpotent length of G.

We now compare the upper N-series of a GSTG G with the maximal invariant N-series of G.

Theorem 4.18

Let t be the length of a maximal invariant N-series of G. Then $e(G) \geq t$.

<u>Proof</u>:

Let $1 = M_0 \lhd M_1 \lhd \cdots \lhd M_t = G$ be a maximal invariant N-series of G and let

$$1 = E_0 \lhd E_1 \lhd \cdots \lhd E_n = G$$

be the upper N-series of G. Since M_1 is a maximal normal N-subgroup of G, $E_1 = E(G) \leq M_1$. By the isomorphism theorems $E_2M_1/M_1 \cong E_2/E_2 \cap M_1 \cong \dfrac{E_2/E_1}{(E_2 \cap M_1)/E_1}$. Then E_2M_1/M_1 is a normal N-embedded subgroup of G/M_1 and $E_2M_1/M_1 \leq M_2/M_1$. So $E_2 \leq M_2$. Inductively E_kM_{k-1}/M_{k-1} is a normal N-embedded subgroup of G/M_{k-1} and $E_k \leq M_k$. Therefore $e(G) \geq t$. □

We show by example that the length of the upper N-series may be strictly greater than the length of some maximal invariant N-series of a GSTG.

Example: Let $G = (S_3 \text{ wr } C_5) \text{ wr } C_7$.

By our previous remarks we know $\ell(G) = 4$ and it is easy to see that G is a STG with Sylow tower $1 \lhd G_3 \lhd G_{2,3} \lhd G_{2,3,5} \lhd G$. Furthermore, $1 \lhd G_{2,3} \lhd G$ is a maximal invariant N-series of G having length 2.

We will show that $e(G) = 3$. Since $G_{2,3}$ is a maximal normal N-subgroup of G, $E(G) \leq G_{2,3}$. We showed previously that $E(S_3 \text{ wr } C_5) = C_3^{|C_5|}$. Hence, by 4.15, $E(G_{2,3,5}) = (C_3^{|C_5|})^{|C_7|} = G_3$. Then, by proposition 4.6, $E(G) = E(G) \cap G_{2,3,5} \leq E(G_{2,3,5}) = G_3$. And since $G_3 \trianglelefteq G$, $G_3 \leq E(G)$ and we have shown $E(G) = G_3$. Since $G/E(G)$ has nilpotent length 3 and $G_{2,3}/E(G) \leq$ $\leq E(G/E(G))$, it follows that $e(G) = 3$.

In our search for a unique invariant N-series of minimal length we turn to a consideration of descending invariant N-series.

<u>Proposition 4.19</u>

Let M_1 and M_2 be normal subgroups of a GSTG G. If G/M_1 and G/M_2 are N-groups, then $G/M_1 \cap M_2$ is an N-group.

<u>Proof</u>:

By lemma 4.1, G/M_1 and G/M_2 are similar N-groups and consequently $G/M_1 \times G/M_2$ is an N-group. Since $G/M_1 \cap M_2 \lesssim G/M_1 \times G/M_2$, $G/M_1 \cap M_2$ is an N-group. $\square$

<u>Proposition 4.20</u>

Let M be a normal subgroup of G and σ a homomorphism of G onto G^σ. If G/M is an N-group, then $M^\sigma \trianglelefteq G^\sigma$ and G^σ/M^σ is an N-group.

Proof:

By the isomorphism theorems $M^{\sigma} \trianglelefteq G^{\sigma}$. Let P and Q be arbitrary Sylow p- and q-subgroups of G^{σ}/M^{σ}. Then $P = G_p{}^{\sigma}M^{\sigma}/M^{\sigma}$ and $Q = G_q{}^{\sigma}M^{\sigma}/M^{\sigma}$ for some $G_p \in \mathrm{Syl}\,p(G)$, $G_q \in \mathrm{Syl}\,q(G)$. Since G/M is an N-group, G_pM/M normalizes G_qM/M or G_qM/M normalizes G_pM/M. It follows that P normalizes Q or Q normalizes P. $\square$

For a GSTG G, let $M(G)$ denote the intersection of all normal subgroups K such that G/K is an N-group. We then have the following result.

Corollary 4.21

The subgroup $M(G)$ is the minimal normal subgroup K of G such that G/K is an N-group. $M(G)$ is a characteristic subgroup of G.

Proof:

The first statement follows at once from proposition 4.19. And proposition 4.20 shows $M(G)^{\sigma} = M(G)$ for every automorphism σ of G. $\square$

The subgroup $M(G)$ will be used to define a unique descending invariant N-series of G. First we develop some properties of this subgroup.

Proposition 4.22

If G is a GSTG with subgroup K, $M(K) \leq M(G)$.

Proof:

By the isomorphism theorems

$^{K}/_{K\cap M(G)} \cong {}^{K\cdot M(G)}/_{M(G)} \leq {}^{G}/_{M(G)}$. Then $^{K}/_{K\cap M(G)}$ is an N-group and $M(K) \leq K \cap M(G) \leq M(G)$. $\square$

Let $G = [C_7]\cdot C_6$ and $K = C_7\cdot C_2 \trianglelefteq G$. Then $M(K) = 1$ and $K \cap M(G) = C_7 = M(G)$. Therefore, we cannot hope for equality in proposition 4.22.

Proposition 4.23

Let G be a GSTG with normal subgroup K. If $K \leq L$, $L \leq G$, then $M(^{LK}/_{K}) = {}^{M(L)\cdot K}/_{K}$.

Proof:

By proposition 4.20 $\dfrac{LK}{K}\Big/\dfrac{M(L)K}{K}$ is an N-group and consequently $M(^{LK}/_{K}) \leq {}^{M(L)\cdot K}/_{K}$.

Now let $M(^{LK}/_{K}) = {}^{W}/_{K}$. Hence $\dfrac{LK}{K}\Big/\dfrac{W}{K} \cong {}^{LK}/_{W}$ is an N-group. Since $K \leq W \leq LK$, $^{LK}/_{W} = {}^{LW}/_{W} \cong {}^{L}/_{L\cap W}$ is an N-group. Then $M(L) \leq L \cap W \leq W$ and consequently $^{M(L)K}/_{K} \leq {}^{W}/_{K} = M(^{LK}/_{K})$. $\square$

The next result is mentioned only for completeness.

Proposition 4.24

Let A and B be (N)-similar GSTG's. Then $M(A \times B) = M(A) \times M(B)$.

Proof:

Since ${}^{A\times B}/_{M(A)\times M(B)} = {}^{A}/_{M(A)} \times {}^{B}/_{M(B)}$,
$M(A \times B) \leq M(A) \times M(B)$.

By the isomorphism theorems, ${}^{A}/_{A\cap M(A\times B)} \cong$
$\cong {}^{M(A\times B)\cdot A}/_{M(A\times B)}$, which is an N-group. Hence
$M(A) \leq A \cap M(A \times B) \leq M(A \times B)$. Similarly $M(B) \leq M(A\times B)$
and the assertion follows. □

Definition 4.25

Let G be a GSTG. The lower N-series of G is the invariant series of G defined inductively by $M_o = G$, $M_i = M(M_{i-1})$. The length of this series, denoted by $m(G)$, is the number of distinct nontrivial terms in the series.

Proposition 4.26

If $K \leq G$, then $m(K) \leq m(G)$.

Proof:

Proposition 4.22. □

Proposition 4.27

If $K \lhd G$, then $m({}^{G}/_{K}) \leq m(G)$.

Proof:

Let $G = M_o \rhd M_1 \rhd \cdots \rhd M_\ell = 1$ be the lower N-series for G. Then, by Proposition 4.23,

$$ {}^{G}/_{K} = {}^{M_o}/_{K} \trianglerighteq {}^{M_1K}/_{K} \trianglerighteq \cdots \trianglerighteq {}^{M_\ell K}/_{K} = {}^{K}/_{K} $$

is the lower N-series of ${}^{G}/_{K}$. □

The next result shows that the lower N-series of G is an invariant N-series of G having minimal length.

Theorem 4.28

Let G be a GSTG. Then any invariant N-series of G has at least $m(G)$ distinct nontrivial terms.

Proof:

Let $G = L_o \rhd L_1 \rhd \cdots \cdots \rhd L_k = 1$ be any invariant N-series of G and let $G = M_o \rhd M_1 \rhd \cdots \rhd M_\ell = 1$ be the lower N-series of G.

Since G/L_1 is an N-group, $M_1 \leq L_1$. Next,

$M_1/M_1 \cap L_2 \cong M_1L_2/L_2 \leq L_1/L_2$, which is an N-group. Hence

$M_1/M_1 \cap L_2$ is an N-group and $M_2 \leq M_1 \cap L_2 \leq L_2$. Proceeding inductively, $M_i/M_i \cap L_{i+1} \cong L_i/L_{i+1}$ implies that

$M_{i+1} \leq M_i \cap L_{i+1} \leq L_{i+1}$. It follows that $k \geq \ell = m(G)$.

$\square$

The preceding argument has established the following relation between the terms of the lower N-series and the terms of the upper N-series.

Corollary 4.29

Let $1 = E_o \lhd E_1 \lhd \cdots \lhd E_n = G$ be the upper N-series of G and let $G = M_o \rhd M_1 \rhd \cdots \rhd M_\ell = 1$ be the lower N-series of G. Then $L_k \leq E_{n-k}$, $0 \leq k \leq \ell$.

We have shown that the length t of any maximal invariant N-series of a GSTG G satisfies the relation $m(G) \leq t \leq e(G)$. An example was given to show that t need not be $e(G)$. However, it is not known if t must equal $m(G)$.

There is a refinement theorem for N-series which should be mentioned, although we have not been able to use it.

<u>Theorem 4.30</u>

Let $1 = H_o \lhd H_1 \lhd \cdots \lhd H_\ell = G$ and $1 = K_o \lhd K_1 \lhd \cdots \lhd K_t = G$ be invariant N-series of G. If $\ell \geq t$, then

i) $1 \trianglelefteq H_1 \cap K_1 \trianglelefteq \cdots \trianglelefteq H_t \cap K_t \lhd H_{t+1} \trianglelefteq \cdots \trianglelefteq H_\ell = G$ is an invariant N-series of G, and

ii) $G \trianglerighteq K_{t-1} \cap H_{\ell-1} \trianglerighteq \cdots \trianglerighteq K_1 \cap H_{\ell-(t-1)} \trianglerighteq 1$ is an invariant N-series of G.

<u>**Proof**</u>:

i) We show $H_{i+1} \cap K_{i+1} / H_i \cap K_i$ is an N-group for $0 \leq i \leq \ell - 1$. $H_{i+1} \cap K_{i+1} / H_{i+1} \cap K_i \cong K_i(H_{i+1} \cap K_{i+1}) / K_i \leq$ $\leq K_{i+1}/K_i$, which is an N-group. Similarly $H_{i+1} \cap K_{i+1} / H_i \cap K_{i+1} \cong H_{i+1}/H_i$, which is an N-group. Then

$$H_{i+1} \cap K_{i+1} / H_i \cap K_i \;=\; H_{i+1} \cap K_{i+1} / (H_{i+1} \cap K_i) \cap (H_i \cap K_{i+1})$$ is an

N-group.

ii) Consider consecutive terms $K_a \cap H_b$ and $K_{a+1} \cap H_{b+1}$ of the series. By the isomorphism theorems

$$K_{a+1} \cap H_{b+1} / K_{a+1} \cap H_b \;\cong\; H_{b+1} / H_b$$ and

$$K_{a+1} \cap H_{b+1} / K_a \cap H_{b+1} \;\cong\; K_{a+1} / K_a .$$ Then $K_{a+1} \cap H_{b+1} / K_{a+1} \cap H_b$

and $K_{a+1} \cap H_{b+1} / K_a \cap H_{b+1}$ are N-groups. Therefore

$K_{a+1} \cap H_{b+1} / K_a \cap H_b$ is an N-group. $\square$

Although I have not been able to show that the length of a maximal invariant N-series of a GSTG G is $m(G)$, I strongly suspect that this is the case. If so, GSTG's might be described in terms of their maximal invariant N-series. This possibility will be considered in the near future.

INDEX OF NOTATIONS

I. Relations:

$\subseteq$	Is a subset of
$\leq$	Is a subgroup of
$\nleq$	Is not a subgroup of
$<$	Is a proper subgroup of ($\lneq$ for emphasis)
$\trianglelefteq$	Is a normal subgroup of
$\ntrianglelefteq$	Is not a normal subgroup of
$\triangleleft$	Is a proper normal subgroup of
$\Diamond$	Is a characteristic subgroup of
$\in$	Is an element of
$\notin$	Is not an element of

II. Operations:

$\setminus$	Set difference
$\langle\ \rangle$	Subgroup generated by
$\times$	Direct product of groups
$\otimes \prod_{i=1}^{n} G_i$	$G_1 \times \cdots \times G_n$
$\prod_{i=1}^{n} G_i$	$\langle G_1, \cdots, G_n \rangle$
$\lvert G \rvert$	The number of elements in G
$c(G)$	Set of prime divisors of $\lvert G \rvert$
$\pi(G)$	The number of prime divisors of $\lvert G \rvert$
$\ell(G)$	The Fitting (nilpotent) length of G
$e(G)$	Defined on page 61

$m(G)$	Defined on page 66
$[G:H]$	Index of H in G
G/H	Factor group
G^x	$x^{-1}\,G\,x$
a^x	$x^{-1}\,a\,x$
$[x,y]$	$x^{-1}\,y^{-1}\,x\,y$
$[H,K]$	Subgroup generated by all $[h,k]$; $h \in H$, $k \in K$

III. <u>Other</u>:

$Z(G)$	Center of G
$Z_\infty(G)$	Hypercenter of G
$H_\infty(G)$	Hypercommutator of G
$F(G)$	Fitting subgroup of G (also denoted $Fit(G)$)
$\Phi(G)$	Frattini subgroup of G
G'	Derived group of G
$E(G)$	Defined on page 56
$M(G)$	Defined on page 64
$Aut(G)$	Automorphism group of G
S_n	Symmetric group of degree n
A_n	Alternating group of degree n
$Syl\ (G)$	Set of all Sylow subgroups of G
$Syl\ p(G)$	Set of all Sylow p-subgroups of G.
$[H]\cdot K$	Split extension of H by K
H wr K	Wreath product of H by K

BIBLIOGRAPHY

1. R. Baer, Supersoluble immersion, Canadian Journal of Mathematics, 11(1959), pp. 353-369.

2. ________, Sylowturmgruppen II, Mathematische Zeitschrift, 92(1966), pp. 256-268.

3. W. Gaschütz, Zur theorie der endlichen auflösbaren Gruppen, Mathematische Zeitschrift, 80(1963), pp. 300-305.

4. P. Hall, A note on soluble groups, Journal of the London Mathematical Society, 3(1928), pp. 98-105.

5. ________, A characteristic property of soluble groups, Journal of the London Mathematical Society, 12(1937), pp. 198-200.

6. ________, On the Sylow systems of a soluble group, Proceedings of the London Mathematical Society, 43(1937), pp. 316-323.

7. ________, On the system normalizers of a soluble group, Proceedings of the London Mathematical Society, 43(1937), pp. 507-528.

8. ________, The construction of soluble groups, Journal fur die reine und angewandte Mathematik, 182 (1940), pp. 206-214.

9. B. Huppert, Zur Sylowstruktur auflösbarer Gruppen, Arkiv der Mathematik, 12(1961), pp. 161-169.

10. __________, Zur Sylowstruktur auflösbarer Gruppen II, Arkiv der Mathematik, 15(1964), pp. 251-257.

11. J. S. Rose, Finite groups with prescribed Sylow tower subgroups, Proceedings of the London Mathematical Society, 16(1966), pp. 577-589.

12. A. E. Spencer, Maximal chains in solvable groups, Ph.D. thesis (1967).

13. W. R. Scott, Group theory, (Englewood Cliffs, New Jersey: Prentice Hall, 1964).

14. D. R. Taunt, On A-groups, Proceedings of the Cambridge Philosophical Society, 45(1949), pp. 24-42.