STRONGLY POINTLIKE MAPS AND COLLAPSIBLE COMPLEXES

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ABSTRACT

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by Paul Francis Dierker

Let $\{a_i | i = 1,2,\ldots,s\}$ be the set of barycenters of the simplexes in the simplicial complex L. If $f: |K| \xrightarrow{\text{onto}} |L|$ is a simplicial map with the property that $f^{-1}(a_i)$ is collapsible for all $i = 1, \ldots, s$, we call f a strongly pointlike map.

In this thesis we examine the relation that exists between a complex and its image under a strongly pointlike map. In particular, in Section I we find that a complex and its image under such a map must be of the same simple homotopy type.

Next, collapsible polyhedra are characterized as those polyhedra having a strongly pointlike map onto a 1-simplex. This characterization is then used to derive a few properties of collapsible polyhedra. In Section V the characterization is used to find a class of non-collapsible polyhedra whose product with the unit interval is collapsible.

STRONGLY POINTLIKE MAPS AND COLLAPSIBLE COMPLEXES

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DEDICATION

To my wife.

CONTENTS

SECTION I

INTRODUCTION

Throughout this thesis we will be considering both simplicial complexes and convex linear cell complexes. All complexes (simplicial or convex linear cell) are to be finite and embedded in some Euclidean space.

If K is a complex, $|K|$ will be used to denote the underlying point set of K, that is, the union of all convex linear cells of K. If K is a convex linear cell complex and T a simplicial complex with $|T| = |K|$, T will be called a simplicial division of $|K|$.

B^n will be called a polyhedral n-cell if there is a piecewise linear homeomorphism from the standard n-simplex onto B^n.

If K_1 and K_2 are complexes with $K_1 \supset K_2$ we say that there is an elementary simplicial collapse from K_1 to K_2 if

$$K_1 = K_2 \dot{\cup} (a * \sigma^n) \qquad \text{and}$$
$$K_2 \cap (a * \sigma^n) = a * \dot{\sigma}^n.$$

K_1 simplicially collapses to K_2 (written $|K_1| \searrow_s |K_2|$) if there is a sequence of elementary simplicial collapses from K_1 to K_2. If K_2 consists of a single vertex, K_1 is said to be simplicially collapsible (written $|K_1| \searrow_s 0$). It is well known [3] that if $|K_1| \searrow_s 0$, then K_1 may be collapsed to any of its vertices.

If $|K_1| \supset |K_2|$ we say that there is an elementary collapse from $|K_1|$ to $|K_2|$ if there exist B^n and B^{n-1}, polyhedral n and n-1 balls respectively, with $B^{n-1} \subset \dot{B}^n$, such that

$$|K_1| = |K_2| \cup B^n \quad \text{and}$$
$$|K_2| \cap B^n = B^{n-1}.$$

$|K_1|$ collapses to $|K_2|$ (written $|K_1| \searrow |K_2|$) if there is a sequence of elementary collapses from $|K_1|$ to $|K_2|$. If $|K_2|$ is a point, $|K_1|$ is said to be collapsible (written $|K_1| \searrow 0$). The following well known theorem [5] gives the connection between collapsible and simplicially collapsible polyhedra. If $|K| \searrow 0$ then there is a simplicial division T of K such that $|T| \searrow_s 0$.

Let K and L be complexes. We say that K and L have the same simple homotopy type if there exists a complex P such that $|P| \searrow |K|$ and $|P| \searrow |L|$.

Let L be a subcomplex of the simplicial complex K. The star of L in K is defined as

$$st_K(L) = \cup \{\zeta \in K |\ \zeta \cap |L| \neq \Phi\}$$

and the link of L in K as

$$lk_K(L) = \cup\{\zeta \in K |\ \zeta \subseteq st_K(L);\ \zeta \cap |L| = \Phi\}$$

If σ is a simplex of K the reduced star of σ in K is defined as

$$\widetilde{st}_K(\sigma) = \cup \{\zeta \in K | \sigma < \zeta\}$$

A subcomplex L of a simplicial complex K is full in K if any simplex of K whose vertices belong to L is in L.

In addition we will consistently use $\dot{A}$ to denote the boundary of A and int A to denote the interior of A.

SECTION II

STRONGLY POINTLIKE MAPS AND SIMPLE HOMOTOPY TYPE

Definition 2.1: Let $f: |K| \longrightarrow |L|$ be a simplicial map of the complex K onto the complex L, and let $\{a_i | i = 1,2,\ldots,s\}$ be the set of barycenters of the simplexes of L. If $f^{-1}(a_i)$ is collapsible for all i the map f is said to be strongly pointlike.

In this section we will show that the existence of a strongly pointlike map between two polyhedra implies that the polyhedra are of the same simple homotopy type. This result will follow quickly from [1].

Definition 2.2: If M is a subcomplex of the complex $K \subset E^n \subset E^{n+1}$ and V is a point of $E^{n+1} - E^n$ we define the quotient complex of K with respect to M as

$$K/M = [K - K|st_K(M)] \cup [V * K|lk_K(M)].$$

Lemma 2.1: Let M be a collapsible subcomplex of K and suppose that $|K|st_K(M)| \searrow_s M$. Then $|K/M|$ is of the same simple homotopy type as $|K|$.

Proof: Since a cone collapses simplicially to any subcone, [5] we have that

$$|V * K|st_K(M)| \searrow_s |V * K|lk_K(M)|.$$

Thus $|V * K|st_K(M) \cup [K - K|st_K(M)]| \searrow_s |K/M|$.

Moreover $|K|st_K(M)| \searrow_s |M| \searrow 0$ so $|K|st_K(M)| \searrow 0$.

Thus by lemma 3 of [3]

$$|V * K|st_K(M) \cup [K - K|st_K(M)]| \searrow |K|$$

and so $|K/M|$ and $|K|$ are of the same simple homotopy type.

<u>Definition 2.3</u>: A subcomplex L of a complex K is locally collapsible if for any simplex σ of K, $\widetilde{st}_K(\sigma) \cap |L|$ is collapsible.

The following lemma is the second portion of proposition 2.1 of [1].

<u>Lemma 2.2</u>: Let M be a subcomplex of K such that

(i) M is full in K

(ii) M is locally collapsible in K,

then $|K|st_K(M)| \underset{s}{\searrow} |M|$.

<u>Definition 2.4</u>: The simplicial mapping $\pi: K \longrightarrow K/M$ defined by

$\pi | \overline{|K| - st_K(M)} = id$ and

$\pi(|M|) = v$

is called the projection of K onto K/M.

<u>Theorem 2.3</u>: If $f: |K| \xrightarrow{onto} |L|$ is a strongly point-like map, then $|K|$ and $|L|$ have the same simple homotopy type.

<u>Proof</u>: Order the simplexes of L in the order of increasing dimension, $\zeta_1, \zeta_2, \ldots, \zeta_s$ and let a_i denote the barycenter of ζ_i. Construct the sequence of quotient complexes

$$\begin{array}{ccccccc} K' & \xrightarrow{\pi_1} & K_0/f^{-1}(a_1) & \xrightarrow{\pi_2} & K_1/\pi_1 f^{-1}(a_2) & \xrightarrow{\pi_3} & \cdots \\ \| & & \| & & \| & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

$$\xrightarrow{\pi_s} \begin{array}{c} K_{s-1}/\pi_{s-1} \cdots \pi_1 f^{-1}(a_s) \\ \| \\ K_s \end{array}$$

where L' is the first barycentric subdivision of L and K' is a first derived complex of K chosen so that $f:K' \longrightarrow L'$ is simplicial [1].

From [1] we have that K_s is isomorphic to L', and that $J_i = K_{i-1}|\pi_{i-1}\cdots\pi_1 f^{-1}(a_i)$ is both full and locally collapsible in K_{i-1}. Thus by Lemma 2.2

$$|K_{i-1}|st_{K_{i-1}}(J_i)| \searrow_s |J_i| \; .$$

In addition, since f is strongly pointlike $f^{-1}(a_i) \searrow 0$ for all i. It is an easy calculation to show that $f^{-1}(a_i) \cap st_K(f^{-1}(a_j)) \subset lk_K(f^{-1}(a_j))$ for all $j \neq i$. Thus $\pi_{i-1}, \pi_{i-2}, \ldots, \pi_1$ are all the identity on $f^{-1}(a_i)$ and so $|J_i| = \pi_{i-1}\,\pi_{i-2}\cdots\pi_1 f^{-1}(a_i) \searrow 0$.

Now we may apply Lemma 2.1 to see that $|K_i| = |K_{i-1}/J_i|$ is of the same simple homotopy type as $|K_{i-1}|$.

Thus $|K_s|$ is of the same simple homotopy type as $|K'|$, and so $|K'|$ and $|L'|$ are of the same simple homotopy type.

SECTION III

STRONGLY POINTLIKE MAPS ONTO COLLAPSIBLE COMPLEXES
A CHARACTERIZATION OF COLLAPSIBLE COMPLEXES

In this section we will establish that the preimage of a collapsible polyhedron under a strongly pointlike map is collapsible. In order to establish this it is helpful to prove first that the preimage of a one simplex under a strongly pointlike map is collapsible. Both the more general result and a characterization of collapsible complexes will follow from the consideration of this special case.

The idea for the proof of the special case is quite simple and is indicated schematically below. Let $f: |K| \xrightarrow{\text{onto}} \langle V_0, V_1 \rangle$ be strongly pointlike, b denote the barycenter of $\langle V_0, V_1 \rangle$, and $b \in \langle b_0, b_1 \rangle \subset \text{int}\langle V_0, V_1 \rangle$. Then $|K|$ may be represented pictorially as in Figure 1. By following the path of collapsing $f^{-1}(b)$ to p we will show that K may be collapsed to the complex represented pictorially in Figure 2. This in turn will be collapsed to the one cell $\langle U_0, U_1 \rangle$ which is collapsible.

Remark: Let $f: \sigma^n \xrightarrow{\text{onto}} \langle V_0, V_1 \rangle$ be a simplicial map. Further let b denote the barycenter of $\langle V_0, V_1 \rangle$ and $\langle b_1, b_2 \rangle$ a one cell contained in $\langle V_0, V_1 \rangle$. Then since f is a linear map we have

$f^{-1}(b) = B^{n-1}$, a convex linear (n-1)-cell

$f^{-1}\langle b_0, b_1 \rangle = B^n$, a convex linear n-cell

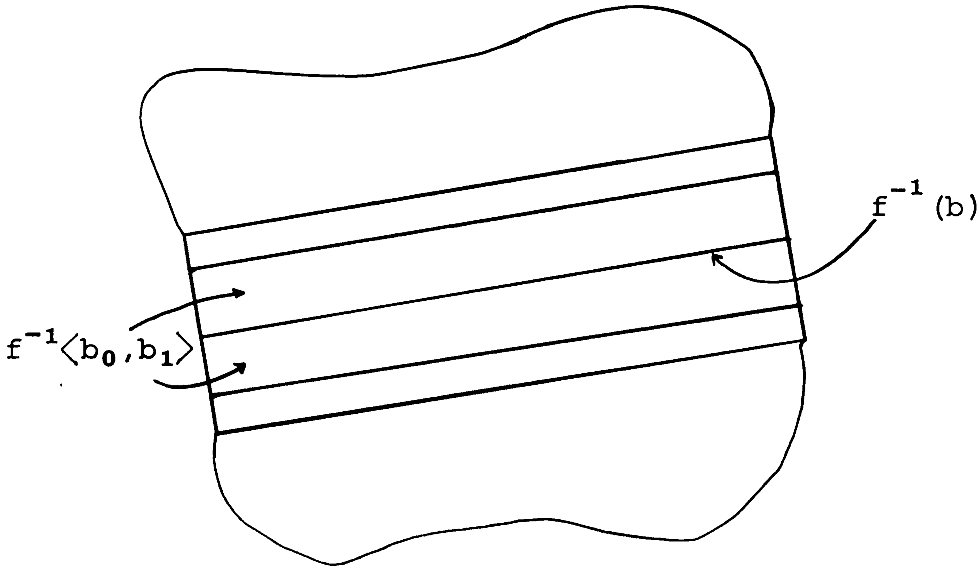

Figure 1.

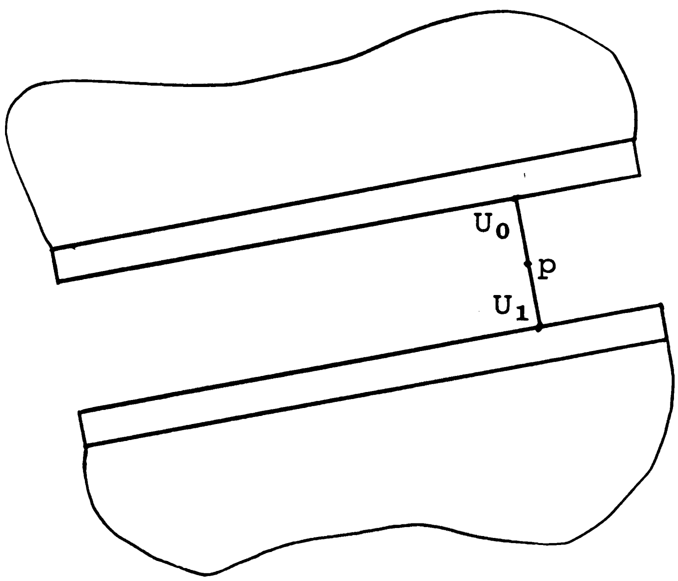

Figure 2.

$$(f^{-1}(b))^{\cdot} = f^{-1}(b) \cap \dot{\sigma}^n$$

$$(f^{-1}\langle b_0,b_1\rangle)^{\cdot} = [f^{-1}(b_0) \cup f^{-1}(b_1)] \cup [f^{-1}\langle b_0,b_1\rangle \cap \dot{\sigma}^n].$$

We shall use these facts throughout the proofs of the following lemmas.

<u>Lemma 3.1</u>: Let $f: |K| \xrightarrow{\text{onto}} \langle V_0,V_1\rangle$ be a simplicial map, b the barycenter of $\langle V_0,V_1\rangle$, and $\langle b_0,b_1\rangle \subset \text{int}\langle V_0,V_1\rangle$. There is a piecewise-linear homeomorphism

$h: f^{-1}(b) \times I \xrightarrow{\text{onto}} f^{-1}\langle b_0,b_1\rangle$ such that

$h(f^{-1}(b) \times \{0\}) = f^{-1}(b_0)$,

$h(f^{-1}(b) \times \{1\}) = f^{-1}(b_1)$, and

$h((f^{-1}(b) \cap \sigma) \times I) = f^{-1}\langle b_0,b_1\rangle \cap \sigma$, $\sigma \in K$.

<u>Proof</u>: The proof will be by induction on k, the number of simplexes $\sigma \in K$ with $\sigma \cap f^{-1}(b) \neq \Phi$. If $k = 1$ and $\sigma \cap f^{-1}(b) \neq \Phi$ then σ must be a one simplex. Otherwise $\sigma \cap f^{-1}(b) \neq \Phi$ implies $\dot{\sigma} \cap f^{-1}(b) \neq \Phi$. Then there is another simplex $\zeta \subset \dot{\sigma}$ such that $\zeta \cap f^{-1}(b) \neq \Phi$ and so $k \neq 1$. Thus if $k = 1$, $f^{-1}(b)$ is a point and $f^{-1}(b) \times I$ a polyhedral one-cell. $f^{-1}\langle b_0,b_1\rangle$ is also a polyhedral one-cell with boundary $f^{-1}(b_0) \cup f^{-1}(b_1)$. The conclusion of the theorem is then obvious for $k = 1$.

Now suppose that the lemma is true for k and let K have $k + 1$ simplexes with $\sigma \cap f^{-1}(b) \neq \Phi$. Let ζ be a principal simplex of K with $\zeta \cap f^{-1}(b) \neq \Phi$. Then $K - \zeta$ is a simplicial complex and $f: |K - \zeta| \xrightarrow{\text{onto}} \langle V_0,V_1\rangle$ a simplicial map. By the induction hypothesis there is a piecewise linear homeomorphism

$$g:[f^{-1}(b) \cap |K - \zeta|] \times I \xrightarrow{\text{onto}} f^{-1}\langle b_0,b_1\rangle \cap |K - \zeta|$$

with

$$g([f^{-1}(b) \cap |K - \zeta|] \times \{0\}) = f^{-1}(b_0) \cap |K - \zeta|$$
$$g([f^{-1}(b) \cap |K - \zeta|] \times \{1\}) = f^{-1}(b_1) \cap |K - \zeta| \quad \text{and}$$
$$g([f^{-1}(b) \cap \sigma] \times I) = f^{-1}\langle b_0,b_1\rangle \cap \sigma, \quad \sigma \in K$$

Note that $\dot{\zeta} \subset |K - \zeta|$ so g is defined on $(f^{-1}(b) \cap \dot{\zeta}) \times I$. Moreover,

$$g[(f^{-1}(b) \cap \dot{\zeta}) \times I] = f^{-1}\langle b_0,b_1\rangle \cap \dot{\zeta} \qquad \text{so}$$
$$g[(f^{-1}(b) \cap \dot{\zeta}) \times \{0\}] = f^{-1}(b_0) \cap \dot{\zeta} \qquad \text{and}$$
$$g[(f^{-1}(b) \cap \dot{\zeta}) \times \{1\}] = f^{-1}(b_1) \cap \dot{\zeta}.$$

$(f^{-1}(b) \cap \dot{\zeta}) \times \{0\}$ and $(f^{-1}(b) \cap \dot{\zeta}) \times \{1\}$ are the boundaries of the polyhedral balls $(f^{-1}(b) \cap \zeta) \times \{0\}$ and $(f^{-1}(b) \cap \zeta) \times \{1\}$ respectively. In addition, $f^{-1}(b_0) \cap \dot{\zeta}$ $f^{-1}(b_1) \cap \dot{\zeta}$ are the boundaries of the polyhedral balls $f^{-1}(b_0) \cap \zeta$ and $f^{-1}(b_1) \cap \zeta$ respectively. By [5] any piecewise linear homeomorphism between the boundaries of two polyhedral balls can be extended to the interiors. Thus we may extend g to $\overline{g}$, a piecewise linear homeomorphism with

$$\overline{g}[(f^{-1}(b) \cap \zeta) \times \{0\}] = f^{-1}(b_0) \cap \zeta \qquad \text{and}$$
$$\overline{g}[(f^{-1}(b) \cap \zeta) \times \{1\}] = f^{-1}(b_1) \cap \zeta \quad .$$

Now $\overline{g}$ is defined on

$$S = [(f^{-1}(b) \cap \zeta) \times \{0\}] \cup [(f^{-1}(b) \cap \zeta) \times \{1\}] \cup [(f^{-1}(b) \cap \dot{\zeta}) \times I] \quad ,$$

the boundary of the polyhedral ball $(f^{-1}(b) \cap \zeta) \times I$, and $\overline{g}(S) = (f^{-1}(b_0) \cap \zeta) \cup (f^{-1}(b_1) \cap \zeta) \cup (f^{-1}\langle b_0,b_1\rangle \cap \dot{\zeta})$, the boundary of the polyhedral ball $f^{-1}\langle b_0,b_1\rangle \cap \zeta$. Thus, as before, we can extend $\overline{g}$ to a piecewise linear homeomorphism h with

$$h[(f^{-1}(b) \cap \zeta) \times I] = f^{-1}\langle b_0, b_1\rangle \cap \zeta.$$

Note that since h is an extension of $\overline{g}$

$$h[(f^{-1}(b) \cap \zeta) \times \{0\}] = f^{-1}(b_0) \cap \zeta \qquad \text{and}$$

$$h[(f^{-1}(b) \cap \zeta) \times \{1\}] = f^{-1}(b_1) \cap \zeta.$$

$$h:[f^{-1}(b) \cap |K|] \times I \xrightarrow{\text{onto}} f^{-1}\langle b_0, b_1\rangle \cap |K|$$

so h is the desired homeomorphism.

<u>Lemma 3.2</u>: Let $f:K \longrightarrow \langle V_0, V_1\rangle$ be a simplicial map and $\langle b_0, b_1\rangle \subset \text{int}\langle V_0, V_1\rangle$, then $f^{-1}\langle V_0, b_0\rangle \searrow f^{-1}(V_0)$ and $f^{-1}\langle b_1, V_1\rangle \searrow f^{-1}(V_1)$.

<u>Proof</u>: We will show that $f^{-1}\langle V_0, b_0\rangle \searrow f^{-1}(V_0)$ by induction on k, the number of simplexes of K with $f^{-1}(b_0) \cap \sigma \neq \Phi$.

If $k = 1$, then as in the previous lemma σ is a one simplex and $f^{-1}\langle V_0, b_0\rangle \cap \sigma$ a polyhedral one cell with a free vertex $f^{-1}(b_0)$. Thus

$$f^{-1}\langle V_0, b_0\rangle \searrow f^{-1}\langle V_0, b_0\rangle - [\text{int}(f^{-1}\langle V_0, b_0\rangle \cup \sigma) \cup f^{-1}(b_0)],$$

and since $k = 1$,

$$f^{-1}\langle V_0, b_0\rangle - [\text{int}(f^{-1}\langle V_0, b_0\rangle \cap \sigma) \cup f^{-1}(b_0)] = f^{-1}(V_0).$$

Now assume that the lemma is true for k and suppose that there are $k+1$ simplexes in K with $\sigma \cap f^{-1}(b_0) \neq \Phi$. Let ζ^n be a principal simplex of K with $\zeta^n \cap f^{-1}(b_0) \neq \Phi$. $f^{-1}\langle V_0, b_0\rangle \cap \zeta^n$ is a convex linear n-cell with a face $f^{-1}(b_0) \cap \zeta^n$. We now show that we can collapse $f^{-1}\langle V_0, b_0\rangle \cap \zeta^n$ across $f^{-1}(b_0) \cap \zeta^n$.

Since ζ^n is a principal simplex of K, $|K-\zeta^n| \cap \zeta^n = \dot{\zeta}^n$ so

$$(f^{-1}\langle v_0, b_0\rangle \cap |K-\zeta^n|) \cap (f^{-1}\langle v_0, b_0\rangle \cap \zeta^n) = f^{-1}\langle v_0, b_0\rangle \cap \dot{\zeta}^n .$$

Moreover $f^{-1}\langle v_0, b_0\rangle \cap \dot{\zeta}^n$ is a polyhedral (n-1) cell since

$$f^{-1}\langle v_0, b_0\rangle \cap \dot{\zeta}^n = \overline{[(f^{-1}\langle v_0, b_0\rangle \cap \dot{\zeta}^n) \cup (f^{-1}(b_0) \cap \zeta^n)] - (f^{-1}(b_0) \cap \zeta^n)}$$

$$= \overline{(f^{-1}\langle v_0, b_0\rangle \cap \zeta^n)^{\cdot} - (f^{-1}(b_0) \cap \zeta^n)} ,$$

the closed complement of a polyhedral (n-1) cell in a polyhedral (n-1) sphere.

Thus we may collapse as follows

$$f^{-1}\langle v_0, b_0\rangle \searrow f^{-1}\langle v_0, b_0\rangle \cap |K - \zeta|$$

and employ the induction hypothesis to find

$$f^{-1}\langle v_0, b_0\rangle \cap |K - \zeta| \searrow f^{-1}(v_0).$$

The proof that $f^{-1}\langle b_1, v_1\rangle \searrow f^{-1}(v_1)$ is analogous.

<u>Lemma 3.3</u>: If $P \searrow 0$ then $P \times I \searrow (P\times\{0\}) \cup (p\times I) \cup (P\times\{1\})$ where $p \in P$.

<u>Proof</u>: Let $P = P_0 \searrow P_1 \searrow \cdots \searrow P_i \searrow P_{i+1} \searrow \cdots \searrow P_k \searrow p$ be a sequence of elementary collapses. That is,

$$P_i = P_{i+1} \cup B_i^n \quad \text{and}$$

$$B_i^n \cap P_{i+1} = B_i^{n-1} \quad \text{where } B_i^n \text{ and } B_i^{n-1}$$

are polyhedral n and $n-1$ balls respectively with $B_i^{n-1} \subset \dot{B}_i^n$. Thus $B_i^n \times I$ is a polyhedral (n+1) ball and by [5], $(B_i^{n-1} \times I) \cup (B_i^n \times \{0\}) \cup (B_i^n \times \{1\}) \subset (B_i^n \times I)^{\cdot}$ is a polyhedral n-ball since $B_i^{n-1} \times I$, $B_i^n \times \{0\}$, and

$B_i^n \times \{1\}$ are all polyhedral n-balls and $(B_i^n \times \{0\}) \cap (B_i^{n-1} \times I)$, and $(B_i^n \times \{1\}) \cap (B_i^{n-1} \times I)$ are polyhedral (n-1) balls. Moreover

$$P_i \times I = [(P_{i+1} \times I) \cup (B_i^n \times \{0\}) \cup (B_i^n \times \{1\})] \cup (B_i^n \times I)$$

and

$$[(P_{i+1} \times I) \cup (B_i^n \times \{0\}) \cup (B_i^n \times \{1\})] \cap (B_i^n \times I)$$
$$= (B_i^{n-1} \times I) \cup (B_i^n \times \{0\}) \cup (B_i^n \times \{1\})$$

so

$$P_i \times I \searrow (P_{i+1} \times I) \cup (B_i^n \times \{0\}) \cup (B_i^n \times \{1\}) \quad \text{for}$$

all i. Then

$$P \times I \searrow (P_1 \times I) \cup (B_0 \times \{0\}) \cup (B_0 \times \{1\})$$
$$\searrow (P_2 \times I) \cup [(B_0 \cup B_1) \times \{0\}] \cup [(B_0 \cup B_1) \times \{1\}]$$
$$\cdots$$
$$\searrow (P_k \times I) \cup [\bigcup_{j=1}^{k-1} B_j \times \{0\}] \cup [\bigcup_{j=1}^{k-1} B_j \times \{1\}]$$
$$\searrow (p \times I) \cup (P \times \{0\}) \cup (P \times \{1\})$$

<u>Theorem 3.4</u>: If $f: |K| \xrightarrow{\text{onto}} \langle V_0, V_1 \rangle$ is a strongly pointlike map, then $|K| \searrow 0$.

<u>Proof</u>: If $\langle b_0, b_1 \rangle \subset \text{int} \langle V_0, V_1 \rangle$, then

$$|K| = f^{-1}\langle V_0, b_0 \rangle \cup f^{-1}\langle b_0, b_1 \rangle \cup f^{-1}\langle b_1, V_1 \rangle.$$

By Lemma 3.1 there is a piecewise linear homeomorphism $h: f^{-1}(b) \times I \xrightarrow{\text{onto}} f^{-1}\langle b_0, b_1 \rangle$ such that $h(f^{-1}(b) \times \{0\}) = f^{-1}(b_0)$ and $h(f^{-1}(b) \times \{1\}) = f^{-1}(b_1)$. By Lemma 3.3 and the hypothesis that $f^{-1}(b) \searrow 0$ we have

$$f^{-1}(b) \times I \searrow (f^{-1}(b) \times \{0\}) \cup (p \times I) \cup (f^{-1}(b) \times \{1\}), \; p \in f^{-1}(b).$$

Let

$$f^{-1}(b)\times I \searrow P_1 \searrow \dots \searrow P_k \searrow (f^{-1}(b)\times\{0\})\cup(p\times I)\cup(f^{-1}(b)\times\{1\})$$

be a sequence of elementary collapses. Then since h is a piecewise linear homeomorphism

$$f^{-1}\langle b_0,b_1\rangle \searrow h(P_1) \searrow \dots \searrow h(P_k) \searrow f^{-1}(b_0)\cup h(p\times I)\cup f^{-1}(b_1)$$

is also a sequence of elementary collapses. Thus $f^{-1}\langle b_0,b_1\rangle \searrow f^{-1}(b_0) \cup h(p\times I) \cup f^{-1}(b_1)$. Since $f^{-1}\langle V_0,b_0\rangle$, $f^{-1}\langle b_0,b_1\rangle$ and $f^{-1}\langle b_1,V_1\rangle$ intersect only on $f^{-1}(b_0)$ and $f^{-1}(b_1)$, we have

$K \searrow f^{-1}\langle V_0,b_0\rangle \cup h(p\times I) \cup f^{-1}\langle b_1,V_1\rangle$. Note also that $h(p\times I) \cap f^{-1}\langle V_0,b_0\rangle = h(p\times\{0\})$ and $h(p\times I) \cap f^{-1}\langle b_1,V_1\rangle = h(p\times\{1\})$. From Lemma 3.2 we have $f^{-1}\langle V_0,b_0\rangle \searrow f^{-1}\langle V_0\rangle$ and by hypothesis $f^{-1}\langle V_0\rangle \searrow 0$ so $f^{-1}\langle V_0,b_0\rangle \searrow 0$. By [5] there is a simplicial subdivision T of $f^{-1}\langle V_0,b_0\rangle$ such that $T \searrow_s 0$ and $h(p\times\{0\})$ is a vertex of T. (Star from $h(p\times\{0\})$ if necessary.) Then we may collapse T simplicially to $h(p\times\{0\})$. Similarly $f^{-1}\langle b_1,V_1\rangle \searrow h(p\times\{1\})$. Thus

$$K \searrow f^{-1}\langle V_0,b_0\rangle \cup h(p\times I) \cup f^{-1}\langle b_1,V_1\rangle \searrow h(p\times I) \searrow 0$$

and the theorem is proved.

In order to prove a similar theorem in the general case where the range is collapsible, and not necessarily a one simplex, we need the following two trivial lemmas.

<u>Lemma 3.5</u>: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be n-dimensional convex linear cell complexes. Suppose there exists a map $g: \mathcal{C}_1 \xrightarrow[1-1]{onto} \mathcal{C}_2$ such that

(a) $g(A \cap B) = g(A) \cap g(B)$ for all $A, B \in \mathcal{C}_1$

(b) $\dim g(A) = \dim A$ for all $A \in \mathcal{C}_1$.

Then there is a piecewise linear homeomorphism $h: |\mathcal{C}_1| \xrightarrow{onto} |\mathcal{C}_2|$ such that $h: |A| \xrightarrow{onto} |g(A)|$ for all $A \in \mathcal{C}_1$.

<u>Proof:</u> The proof will be by induction on k, the cardinality of $\mathcal{C}_1$. If $k = 1$ the result is clear. (In this case $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$ are single points.) Now suppose that the result is true for k and let $\mathcal{C}_1$ have cardinality $k+1$. Let B^n be a convex linear n-cell in $\mathcal{C}_1$. Then $\mathcal{C}_1 - B^n$ is a convex linear cell complex since B^n is principal in $\mathcal{C}_1$. $\mathcal{C}_2 - g(B^n)$ is also a cell complex since by (b) $\dim g(B^n) = \dim B^n = n$, and so $g(B^n)$ is a principal cell in $\mathcal{C}_2$. By the induction hypothesis there is a piecewise linear homeomorphism $h: |\mathcal{C}_1 - B^n| \xrightarrow{onto} |\mathcal{C}_2 - g(B^n)|$ so that if $A \in \mathcal{C}_1 - B^n$ $h: g(A) \xrightarrow{onto} A$.

We will now extend h to $\operatorname{int} B^n$. Note that since $\mathcal{C}_1$ is a cell complex $\dot{B}^n \subset |\mathcal{C}_1 - B^n|$ and $\dot{B}^n = \bigcup_{C \in \mathcal{C}_1 - B^n} (C \cap B^n)$.

Thus
$$g[\dot{B}^n] = g\Big(\bigcup_{C \in \mathcal{C}_1 - B^n} (C \cap B^n)\Big)$$
$$= \bigcup_{C \in \mathcal{C}_1 - B^n} g(C \cap B^n)$$
$$= \bigcup_{C \in \mathcal{C}_1 - B^n} [g(C) \cap g(B^n)]$$
$$= (g(B^n))^{\cdot} .$$

Thus h: $\dot{B}^n \xrightarrow{\text{onto}} (g(B^n))^{\bullet}$ and we may extend h "cone-wise" [5] to a piecewise linear homeomorphism of B^n onto $g(B^n)$. This is the desired homeomorphism.

<u>Lemma 3.6</u>: Let $f\colon |K| \longrightarrow \sigma^n$ be a simplicial map. If b denotes the barycenter of σ^n and $x \in \text{int}(\sigma^n)$, then $f^{-1}(x)$ is piecewise linearly homeomorphic to $f^{-1}(b)$.

<u>Proof</u>: $f^{-1}(x)$ and $f^{-1}(b)$ can be considered as convex linear cell complexes, the convex linear cells being given by $f^{-1}(x) \cap \zeta$ and $f^{-1}(b) \cap \zeta$ for $\zeta \in K$. We will show that there is a one to one map g from the convex linear cells of $f^{-1}(b)$ onto those of $f^{-1}(x)$ such that

$$g(A \cap B) = g(A) \cap g(B) \qquad \text{and}$$

$$\dim g(A) = \dim A.$$

The desired result will then follow from the previous lemma.

Let $\sigma \in K$ and suppose that $\sigma \cap f^{-1}(b) \neq \Phi$. Since f is a simplicial map we must have $f(\sigma) = \sigma^n$ and so $\sigma \cap f^{-1}(x) \neq \Phi$. In the same way $\sigma \cap f^{-1}(x) \neq \Phi$ implies $\sigma \cap f^{-1}(b) \neq \Phi$. Thus if $f^{-1}(b) \cap \sigma^k \neq \Phi$, $f^{-1}(b) \cap \sigma^k = B^{k-n}$ and $f^{-1}(x) \cap \sigma^k = C^{k-n}$ where B^{k-n} and C^{k-n} are convex linear cells of dimension k-n.

Now define the mapping g as follows:

$$g[f^{-1}(b) \cap \sigma^k] = f^{-1}(x) \cap \sigma^k .$$

From the above discussion we have

$$\dim g\,[f^{-1}(b) \cap \sigma^k] = \dim\,[f^{-1}(b) \cap \sigma^k].$$

Moreover,

$$g[(f^{-1}(b) \cap \sigma^{k_1}) \cap (f^{-1}(b) \cap \sigma^{k_2})] = g[f^{-1}(b) \cap (\sigma^{k_1} \cap \sigma^{k_2})]$$
$$= f^{-1}(x) \cap (\sigma^{k_1} \cap \sigma^{k_2})$$
$$= (f^{-1}(x) \cap \sigma^{k_1}) \cap (f^{-1}(x) \cap \sigma^{k_2})$$
$$= g[f^{-1}(b) \cap \sigma^{k_1}] \cap g[f^{-1}(b) \cap \sigma^{k_2}].$$

g is clearly one to one and onto, and so the proof is complete.

We now have immediately

<u>Corollary 3.7</u>: Let $f: |K| \xrightarrow{\text{onto}} |L|$ be a simplicial map. f is strongly pointlike if and only if $x \in L$ implies $f^{-1}(x)$ is collapsible.

<u>Definition 3.1</u>: Let $|K| \searrow_s |K_1|$ be an elementary simplicial collapse, (i.e. $K = K_1 \cup (a * \sigma^n)$ and $K_1 \cap (a * \sigma^n) = a * \dot{\sigma}^n$), and let K' be the subdivision of K obtained by starring from b, the barycenter of σ^n. The simplicial map $f: |K'| \xrightarrow{\text{onto}} |K_1|$ defined by

$$f(b) = a \qquad \text{and}$$
$$f(V) = V \quad \text{for all vertices } V \in K',\ V \neq b$$

will be called the simplicial map associated with the collapse $|K| \searrow_s |K_1|$.

<u>Theorem 3.8</u>: If $f: |K| \xrightarrow{\text{onto}} |L|$ is a strongly pointlike map and $|L| \searrow 0$, then $|K| \searrow 0$.

<u>Proof</u>: Since $|L| \searrow 0$ there is a triangulation L^* of $|L|$ such that $|L^*| \searrow_s 0$. The proof will be by induction on p, the number of simplexes in L^*.

If $p = 1$ the result is trivial for then L^* consists of a single vertex.

Now suppose that the theorem is true for all $p < k$ and let L^* be a simplicial complex with cardinality k. By hypothesis $|L^*| \searrow_s 0$ so there is a sequence of elementary simplicial collapses

$|L^*| = |L_0| \searrow_s |L_1| \searrow_s \cdots \searrow_s 0$ where $L^* = L_1 \cup (a * \sigma^n)$ and $L_1 \cap (a * \sigma^n) = a * \dot{\sigma}^n$. Note that $|L_1| \searrow_s 0$ and L_1 has cardinality $k-2$. Let $g: |L_0'| \longrightarrow |L_1|$ be the simplicial map associated with the simplicial collapse $|L_0| \searrow_s |L_1|$. Subdivide K to K' so that the map $f: |K'| \longrightarrow |L_0'|$ is simplicial. We will now show that the composition $gf: |K'| \longrightarrow |L_1|$ is strongly pointlike and use the induction hypothesis to conclude that $|K| = |K'| \searrow 0$.

(i) If $x \in |L_1| - a * \dot{\sigma}^n$, then $g^{-1}(x) = x$ so $f^{-1}g^{-1}(x) = f^{-1}(x)$ which is collapsible by hypothesis.

(ii) If $x \in a * \dot{\sigma}^n$, $g^{-1}(x)$ is a line segment in $a * \sigma^n$. Triangulate $f^{-1}(g^{-1}(x))$ so that $f: f^{-1}(g^{-1}(x)) \xrightarrow{\text{onto}} g^{-1}(x) = \sigma^1$ is a simplicial map. Then since f is strongly pointlike we may apply Theorem 3.4 to find that $f^{-1}(g^{-1}(x)) \searrow 0$.

Thus $gf: |K'| \xrightarrow{\text{onto}} L_1$ is strongly pointlike and the conclusion follows from the induction hypothesis.

The following corollary and lemma will aid in the characterization of collapsible complexes.

Corollary 3.9: If the simplicial maps f and g are strongly pointlike where $f: |K| \xrightarrow{onto} |L|$ and $g: |L| \xrightarrow{onto} |P|$, then the composition gf is strongly pointlike.

Proof: Let $x \in |P|$. $g^{-1}(x)$ is collapsible since g is strongly pointlike. Moreover $f: f^{-1}g^{-1}(x) \xrightarrow{onto} g^{-1}(x)$ so by the above theorem $f^{-1}g^{-1}(x) \searrow 0$ and gf is a strongly pointlike map.

Lemma 3.10: If $|K| \searrow_s |K_1|$ is an elementary simplicial collapse, the associated simplicial map $f: |K'| \xrightarrow{onto} K_1$ is strongly pointlike.

Proof: If $x \in |K_1| - (a * \dot{\sigma}^n)$, $f^{-1}(x) = x$. If $x \in a * \dot{\sigma}^n$, $f^{-1}(x)$ is a polyhedral one cell. Thus in any case $f^{-1}(x) \searrow 0$.

Theorem 3.11: If $|K| \searrow_s |L|$, there is a strongly pointlike map $f: |K^*| \xrightarrow{onto} |L|$ where K^* is a subdivision of K.

Proof: Since $|K| \searrow_s |L|$ there is a sequence of elementary simplicial collapses from K to L.

$$|K| = |K_0| \searrow_s |K_1| \searrow_s \cdots \searrow_s |K_p| \searrow_s |L|.$$

Let $f_i: |K_i^1| \xrightarrow{onto} |K_{i+1}|$ be the simplicial map associated with the elementary simplicial collapse $|K_i| \searrow_s |K_{i+1}|$. $f_p: |K_p^1| \xrightarrow{onto} |L|$ is a strongly pointlike map by Lemma 3.10. Now subdivide K_{p-1}^1 to K_{p-1}^2 so that the map $f_{p-1}: |K_{p-1}^2| \xrightarrow{onto} |K_p^1|$ is simplicial.

Continue in this manner to arrive at the sequence of strongly pointlike maps

$$|K^*| = |K_0^{p+1}| \xrightarrow{f_0} |K_1^p| \xrightarrow{f_1} |K_2^{p-1}| \longrightarrow \cdots$$

$$\longrightarrow |K_{p-2}^3| \xrightarrow{f_{p-2}} |K_{p-1}^2| \xrightarrow{f_{p-1}} |K_p^1| \xrightarrow{f_p} |L|.$$

By Corollary 3.9 $f = f_p f_{p-1} \ldots f_1 f_0$ is a strongly pointlike map.

We are now able to state as a corollary the converse to Theorem 3.4.

Corollary 3.12: If $|K| \searrow 0$, then there exists a strongly pointlike map $f: |K^*| \xrightarrow{\text{onto}} \sigma^1$ where K^* is a subdivision of K.

Proof: Just note that if K is collapsible to a point the second last collapse takes K to a one-simplex.

Remark: In view of Theorem 3.4 and Corollary 3.12 we have characterized collapsible polyhedra as those having a strongly pointlike map onto a one-simplex.

SECTION IV

FURTHER RESULTS CONCERNING COLLAPSIBLE COMPLEXES

In this section we will use the results of section III to obtain necessary and sufficient conditions for a complex to be collapsible. Further, we will drive some properties of a collapsible complex embedded in a combinatorial manifold.

The following two lemmas, along with the characterization of collapsible complexes of the previous section, will lead quickly to necessary and sufficient conditions that a complex be collapsible.

<u>Lemma 4.1</u>: If $f: |K| \xrightarrow{\text{onto}} \langle V_0, V_1 \rangle$ is a simplicial map and b is the barycenter of $\langle V_0, V_1 \rangle$, then there is a subdivision K' of K such that

(i) K' is a first derived complex of K,

(ii) $f: |K'| \xrightarrow{\text{onto}} \langle V_0, b \rangle \cup \langle b, V_1 \rangle$ is a simplicial map,

and (iii) if $\sigma \in K' | f^{-1}(b)$, then $f(\widetilde{st}_{K'}(\sigma)) = \langle V_0, b \rangle \cup \langle b, V_1 \rangle$.

<u>Proof</u>: We will star the simplexes of K in the order of decreasing dimension, $\sigma_1, \sigma_2, \ldots, \sigma_s$. If $f: \sigma_i \longrightarrow V_j$ for $j = 0,1$, we will star σ_i from b_i, the barycenter of σ_i. If $f: \sigma_i \xrightarrow{\text{onto}} \langle V_0, V_1 \rangle$, we will star σ_i from a point $b_i \in f^{-1}(b) \cap \text{int}\, \sigma_i$. Such a process yields a first derived complex of K. Call this K'.

By [2] $\sigma \in K'$ if an only if σ is of the form $\sigma = \langle b_{i_0}, b_{i_1}, \ldots, b_{i_n} \rangle$ where b_{i_k} is the point from which σ_{i_k} is starred and $\sigma_{i_0} > \sigma_{i_1} > \ldots > \sigma_{i_n}$.

We first show that $f\colon |K'| \longrightarrow \langle V_0,b\rangle \cup \langle b,V_1\rangle$ is a simplicial map. If $f(\sigma_{i_0}) = V_j$, $j = 0,1$, then $f(\sigma_{i_k}) = V_j$ for $k = 0,1,\ldots,n$, and $f(b_{i_k}) = V_j$ for all k. Thus $f(\sigma) = V_j$.

If $f(\sigma_{i_0}) = \langle V_0,V_1\rangle$, then either $f(\sigma_{i_k}) = \langle V_0,V_1\rangle$ for all $k = 0,1,\ldots,n$, or there is a least integer $N < n$ such that $f(\sigma_{i_k}) = V_j$ for a fixed $j = 0,1$ and all $k > N$.

In the first case $f(b_{i_k}) = b$ for all k so $f(\sigma) = b$.

In the second case $f(\sigma_{i_k}) = \langle V_0,V_1\rangle$ for all $k \leq N$, and $f(\sigma_{i_k}) = V_j$ for $k > N$. Then $f(b_{i_k}) = V_j$ for all $k > N$, and $f(b_{i_k}) = b$ for all $k \leq N$. Thus $f(\sigma) = \langle b,V_j\rangle$. In any case f maps simplexes onto simplexes and f is a simplicial map.

We now verify conclusion (iii). If $f(\sigma) = b$, then $\sigma = \langle b_{i_0}, \ldots, b_{i_n}\rangle$, $b_{i_k} \in \sigma_{i_k}$ and $\sigma_{i_0} > \sigma_{i_1} > \cdots > \sigma_{i_n}$. By definition $f(\sigma_{i_k}) = \langle V_0,V_1\rangle$ for all k. In particular $f(\sigma_{i_n}) = \langle V_0,V_1\rangle$, so σ_{i_n} has vertices u_0 and u_1 with $f(u_0) = V_0$, $f(u_1) = V_1$. Then by [2] the sequences

$$\sigma_{i_0} > \sigma_{i_1} > \cdots > \sigma_{i_n} > u_0 \qquad \text{and}$$

$$\sigma_{i_0} > \sigma_{i_1} > \cdots > \sigma_{i_n} > u_1$$

determine simplexes ζ_0 and ζ_1 of K',

$$\zeta_0 = \langle b_{i_0}, b_{i_1}, \ldots, b_{i_n}, u_0\rangle \qquad \text{and}$$

$$\zeta_1 = \langle b_{i_0}, b_{i_1}, \ldots, b_{i_n}, u_1\rangle.$$

Note that $f(\zeta_0) = \langle v_0, b\rangle$ and $f(\zeta_1) = \langle b, v_1\rangle$. Then, since $\sigma < \zeta_0$ and $\sigma < \zeta_1$, we have that $\zeta_0 \cup \zeta_1 \subset \widetilde{st}_{K'}(\sigma)$ and $f(\widetilde{st}_{K'}(\sigma)) = \langle v_0, b\rangle \cup \langle b, v_1\rangle$.

<u>Lemma 4.2</u>: If $f: |K| \xrightarrow{\text{onto}} \langle v_0, v_1\rangle$ is a simplicial map and b is the barycenter of $\langle v_0, v_1\rangle$, then $f^{-1}(b) = lk_{K'}(f^{-1}(v_0))$ where K' is the first derived subdivision of K described in Lemma 4.1.

<u>Proof</u>: If $\sigma \in K'|f^{-1}(b)$, then $f(\sigma) = b$ so $\sigma \cap f^{-1}(v_0) = \Phi$. By conclusion (iii) of Lemma 4.1 $f(\widetilde{st}(\sigma)) = \langle v_0, b\rangle \cup \langle b, v_1\rangle$. Thus $\sigma < \zeta$ where $f(\zeta) = \langle v_0, b\rangle$ so $\zeta \cap f^{-1}(v_0) \neq \Phi$, and $\sigma \in K'|lk_{K'}(f^{-1}(v_0))$.

On the other hand, if $\sigma \in K'|lk_{K'}(f^{-1}(v_0))$, then $\sigma \cap f^{-1}(v_0) = \Phi$ and $\sigma < \zeta \in K'$ where $\zeta \cap f^{-1}(v_0) \neq \Phi$. Since $\zeta \cap f^{-1}(v_0) \neq \Phi$, $f(\zeta) = \langle v_0, b\rangle$ and $f(\sigma) \subset \langle v_0, b\rangle$. Moreover, $\sigma \cap f^{-1}(v_0) = \Phi$ so $f(\sigma) \cap \langle v_0\rangle = \Phi$ and $f(\sigma) = b$. Thus $\sigma \in f^{-1}(b)$.

<u>Remark</u>: Clearly it follows in the same way that $f^{-1}(b) = lk_{K'}(f^{-1}(v_1))$.

<u>Theorem 4.3</u>: Let $|K| \searrow 0$. Then there is a simplicial subdivision K^* of $|K|$ and subcomplexes A, B of K^* with

(i) A, B full in K^*, $A \neq \Phi$, $B \neq \Phi$,

(ii) $A \cap B = \Phi$,

(iii) $A \cup B \supset (K^*)^0$ (the zero skeleton of K^*),

(iv) $|A| \searrow 0$, $|B| \searrow 0$, and

(v) $lk_{K'}(A) = lk_{K'}(B) \searrow 0$ where K' is the first derived subdivision of K^*.

<u>Proof</u>: Since $|K| \searrow 0$ we may employ Corollary 3.12 to find a simplicial subdivision K^* of $|K|$ and a strongly pointlike map $f: |K^*| \xrightarrow{\text{onto}} \langle v_0, v_1 \rangle$.

Let $A = f^{-1}(v_0)$, $B = f^{-1}(v_1)$. Since f is a simplicial map, we note A and B are full in K^*, and since f is onto, $A \neq \Phi$, $B \neq \Phi$. Conclusions (ii) and (iii) are immediate, and (iv) follows from the fact that f is strongly pointlike. Finally, by Lemma 4.2,

$$\mathrm{lk}_{K'}(A) = \mathrm{lk}_{K'}(B) = f^{-1}(b) \searrow 0.$$

Thus conclusion (v) follows.

We will now prove a type of converse to Theorem 4.3.

<u>Theorem 4.4</u>: Let K be a simplicial complex and A, B subcomplexes of K with

(i) A, B full in K,

(ii) $A \cap B = \Phi$,

(iii) $A \cup B \supset K^0$, (the zero skeleton of K),

(iv) $|A| \searrow 0$, $|B| \searrow 0$, and

(v) $\mathrm{lk}_{K'}(A) \searrow 0$ where K' is a first derived subdivision of K.

Then $|K| \searrow 0$.

<u>Proof</u>: Let $f: |K| \xrightarrow{\text{onto}} \langle v_0, v_1 \rangle$ be the simplicial map defined by $f(A) = v_0$, $f(B) = v_1$. Since A and B are full in K, $f^{-1}(v_0) = |A|$ and $f^{-1}(v_1) = |B|$. Thus, by hypothesis, $f^{-1}(v_0) \searrow 0$ and $f^{-1}(v_1) \searrow 0$.

By Lemma 4.2 $f^{-1}(b) = \mathrm{lk}_{K'}(A)$ so $f^{-1}(b) \searrow 0$, and f is a strongly pointlike map. Then by Theorem 3.4 $|K| \searrow 0$.

Remark: Clearly, by symmetry, condition (iii) above can be replaced by $lk_{K'}(B) \searrow 0$.

The next application of the results of Section III involves collapsible polyhedra embedded in a combinatorial manifold.

Theorem 4.5: Let $|K|$ be a polyhedral n-complex in $|M|$, a combinatorial m-manifold. If $|K|$ is collapsible, there exist two polyhedral m-cells N_0 and N_1 in $|M|$ such that

(i) if $x \in |K|$ we can choose N_0 and N_1 so that $x \in \text{int } N_0$,

(ii) $N_0 \cup N_1 \supset |K|$,

(iii) $N_0 \cap |K| \neq \Phi$, $N_1 \cap |K| \neq \Phi$,

(iv) $\dot{N}_0 \cap |K| \searrow 0$, $\dot{N}_1 \cap |K| \searrow 0$,

(v) $N_0 \cap |K| \searrow 0$, $N_1 \cap |K| \searrow 0$, and

(vi) $\dim (\dot{N}_i \cap |K|) \leq n-1$, $i = 0,1$.

Proof: We will suppose that K is a subcomplex of M. If not, there exist subdivisions K_1 and M_1 of K and M respectively such that $K_1 \subset M_1$. Further, since $|K_1| \searrow 0$ there exists a subdivision K_2 of K_1 such that $|K_2| \underset{s}{\searrow} 0$. Extend the subdivision K_2 of K_1 to a subdivision M_2 of M_1. Moreover, if $x \in |K_2|$, we will assume that x is a vertex of K_2. If not, we simply star M_2 from x and call the resulting triangulations M_3 and K_3. Note that since we arrived at K_3 by starring K_2 and $|K_2| \underset{s}{\searrow} 0$, $|K_3| \underset{s}{\searrow} 0$, see [5]. In addition we will suppose that K_3

is full in M_3. If not, we have only to let M_3' denote the first derived subdivision of M_3 and K_3' the corresponding subdivision of K_3. Then $|K_3'| \searrow_s 0$ since it was obtained from K_3 by starring. In order to save notation we will call K_3', K and M_3', M. In summation we now have

(a) $K \subset M$,

(b) x is a vertex of K,

(c) K is full in M, and

(d) $|K| \searrow_s 0$.

Since $|K| \searrow_s 0$, there exists a strongly pointlike map $f\colon |K| \xrightarrow{\text{onto}} \langle V_0, V_1\rangle$. Since x is a vertex of K, either $f(x) = V_0$ or $f(x) = V_1$. Assume that $f(x) = V_0$. Star $\langle V_0, V_1\rangle$ from its barycenter, b, and subdivide K to K_f' so that

$$f\colon\ K_f' \xrightarrow{\text{onto}} \langle V_0, b\rangle \cup \langle b, V_1\rangle$$

is a simplicial map, and K_f' is a first derived complex of K (see Lemma 4.1). Extend K_f' to a first derived subdivision of M, say M'.

Now consider the first derived neighborhood of $f^{-1}(V_i)$,

$$N(f^{-1}(V_i), M') = st_{M'}(f^{-1}(V_i)) \quad , \quad i = 0,1.$$

By [5] these neighborhoods are regular neighborhoods, (i.e. $N(f^{-1}(V_i), M')$ is an m-manifold such that $N(f^{-1}(V_i), M') \searrow f^{-1}(V_i)$), since $f^{-1}(V_i)$ is full in M ($f^{-1}(V_i)$ full in K, K full in M) and M' is a first derived subdivision of M.

Thus since $f^{-1}(V_i) \searrow_s 0$, $N(f^{-1}(V_i),M')$ is an m-ball, and $\dot{N}(f^{-1}(V_i),M') = lk_{M'}(f^{-1}(V_i))$ is an m-1 sphere for $i = 0,1$, (see [5]). Let N_i denote $N(f^{-1}(V_i),M')$.

We next note that

$$\dot{N}_i \cap |K| = f^{-1}(b) \quad , \quad i = 0,1. \tag{1}$$

We will prove this for $i = 0$. (The proof for $i = 1$ is analogous.)

First suppose that $y \in \dot{N}_0 \cap |K|$. Then $y \in \text{int}\ \sigma$ where $\sigma \in K_f'$ and $\sigma \in \dot{N}_0$. Since $\sigma \in \dot{N}_0$, $\sigma \cap f^{-1}(V_0) = \phi$. Thus we must have one of the following three possibilities, $f(\sigma) = b$, $f(\sigma) = V_1$, or $f(\sigma) = \langle b,V_1\rangle$. Suppose that $f(\sigma) = V_1$, or $f(\sigma) = \langle b_1,V_1\rangle$, and $\sigma < \zeta$. Then since f is a simplicial map, $f(\zeta) = V_1$ or $f(\zeta) = \langle b,V_1\rangle$. In either case $\zeta \cap f^{-1}(V_0) = \phi$ so $\zeta \not\subset st_{M'}(f^{-1}(V_0))$ and $\sigma \not\subset lk_{M'}(f^{-1}(V_0))$. That is, $\sigma \not\subset \dot{N}_0$, a contradiction. Thus we must have $f(\sigma) = b$ so $f(y) = b$ and $y \in f^{-1}(b)$.

On the other hand, if $y \in f^{-1}(b)$ then $y \in \sigma \in K_f'$ and $f(\sigma) = b$. Since $f(\sigma) = b$, $\sigma \cap f^{-1}(V_0) = \phi$. But $\sigma \subset \zeta \in K$ where $f(\zeta) = \langle V_0,V_1\rangle$ so σ is the face of some $\sigma' \in K_f'$ where $f(\sigma') = \langle V_0,b\rangle$. Thus $f^{-1}(V_0) \cap \sigma' \neq \phi$, and $\sigma \subset lk_{M'}(f^{-1}(V_0))$.

Finally we will show that $N_0 \cap |K| = f^{-1}(\langle V_0,b\rangle)$ and

$$N_1 \cap |K| = f^{-1}(\langle b,V_1\rangle). \tag{2}$$

Let $y \in N_0 \cap |K|$. Then $y \in N_0$ implies $y \in \text{int}\ \sigma$, $\sigma \in K_f'$ where $\sigma \cap f^{-1}(V_0) \neq \phi$. Thus $f(\sigma) = \langle V_0\rangle$ or $f(\sigma) = \langle V_0,b\rangle$. In either case $y \in \sigma \subset f^{-1}(\langle V_0,b\rangle)$. On the other hand, if $y \in f^{-1}(\langle V_0,b\rangle)$, then $y \in \sigma \in K_f'$ and either $f(\sigma) = \langle V_0\rangle$, $f(\sigma) = \langle V_0,b\rangle$, or $f(\sigma) = b$. In the first two cases $\sigma \cap f^{-1}(V_0) \neq \Phi$. Thus $\sigma \subset st_{M'}(f^{-1}(V_0))$ and $\sigma \subset N_0$. If $f(\sigma) = b$, $\sigma \subset f^{-1}(b)$ and, by (1), $\sigma \subset \mathring{N}_0$. The proof that $N_1 \cap |K| = f^{-1}\langle b, V_1\rangle$ is analogous.

We are now in a position to verify that conclusions (i) through (vi) follow.

(i) follows immediately from the construction since $x \in f^{-1}(V_0) \subset \text{int } N_0$.

(ii) follows since $|K| \subset f^{-1}(\langle V_0, b\rangle) \cup f^{-1}(\langle b, V_1\rangle) = N_0 \cup N_1$.

(iii) follows since $f: |K| \longrightarrow \langle V_0, V_1\rangle$ was onto.

(iv) follows from (1) above. That is, $\mathring{N}_i \cap |K| = f^{-1}(b)$, and $f^{-1}(b)$ is collapsible since f is strongly pointlike.

(v) follows from (2). That is, $N_0 \cap |K| = f^{-1}(\langle V_0, b\rangle)$ and $N_1 \cap |K| = f^{-1}(\langle b, V_1\rangle)$, and the fact that f is strongly pointlike.

(vi) follows from the fact that f is a simplicial map $f: |K| \xrightarrow{\text{onto}} \langle V_0, V_1\rangle$. Thus if $\dim f^{-1}(b) = \dim K$, some $\sigma^n \in K$ would be mapped to b in contradiction to the fact that f is a simplicial map with range $\langle V_0, V_1\rangle$.

SECTION V

CONTRACTIBLE COMPLEXES WHOSE PRODUCT WITH THE UNIT INTERVAL IS COLLAPSIBLE

The dunce hat D is obtained from the 2-simplex $\langle a,b,c\rangle$ by identifying all three sides $\langle a,b\rangle = \langle a,c\rangle = \langle b,c\rangle$. D is of interest since it is one of the simplest contractible polyhedra which is not collapsible (there is no free face from which to start the collapsing). However, it is well known, see [4], that $D \times I \searrow 0$. This fact leads to the following conjecture.

<u>Conjecture 1</u>: If K is a contractible 2-complex, then $K \times I \searrow 0$.

This conjecture is of particular interest since it implies the three dimensional Poincare' Conjecture [4].

<u>Definition 5.1</u>: Let M be a compact polyhedral manifold with boundary. Define a spine K of M to be a subpolyhedron such that $M \searrow K$.

By [4] we may assume that if K is a spine of M, then

(i) $K \subset \text{int } M$, and

(ii) $\dim K < \dim M$.

The proof that Conjecture 1 implies the Poincare' Conjecture depends upon the following proposition (see, for example, [4]).

<u>Proposition 5.1</u>: Let M^3 be a 3-manifold with a 2-sphere boundary and spine K^2. Then if $K^2 \times I \searrow 0$, M^3

is a 3-ball.

In this section we will apply the results of Section III to find a class of complexes K with the property that $|K \times I| \searrow 0$.

<u>Lemma 5.2</u>: Let K be a complex with a subcomplex L. Then $|K \times I| \searrow |(K \times \{0\}) \cup (L \times I)|$.

<u>Proof</u>: Order the simplexes ζ with $\text{int}\ \zeta \subset |K| - |L|$ in the order of decreasing dimension, $\zeta_1, \zeta_2, \ldots, \zeta_s$. First,

$$|K \times I| \searrow |K \times \{0\}| \cup [(|K| - \text{int}\ \zeta_1) \times I]$$

by collapsing the polyhedral ball $\zeta_1 \times I$ across its free face $\zeta_1 \times \{1\}$.

In general,

$$|K \times \{0\}| \cup [(|K| - \bigcup_{i=1}^{j-1} \text{int}\ \zeta_i) \times I] \searrow |K \times \{0\}| \cup [(|K| - \bigcup_{i=1}^{j} \text{int}\ \zeta_i) \times I]$$

by collapsing $\zeta_j \times I$ across $\zeta_j \times \{1\}$. Note that $\zeta_j \times \{1\}$ is a free face of $\zeta_j \times I$ since if $\zeta_j \times \{1\} < \zeta_k \times I$, for $j \neq k$, then $\zeta_j \times \{1\} < \zeta_k \times \{1\}$ and so $\zeta_j < \zeta_k$. Since the simplexes were ordered in the order of decreasing dimension, $k < j$, so

$$\zeta_k \times \{1\} \not\subset |K \times \{0\}| \cup [(|K| - \bigcup_{i=1}^{j-1} \text{int}\ \zeta_i) \times I].$$

Thus

$$|K \times I| \searrow |K \times \{0\}| \cup [(|K| - \bigcup_{i=1}^{s} \text{int}\ \zeta_i) \times I] = |K \times \{0\}| \cup |L \times I| .$$

<u>Theorem 5.3</u>: Let K be a complex with a subcomplex L such that

(i) $|L| \searrow 0$,

(ii) $|L|$ separates $|K|$, $|K| - |L| = A \cup B$, and

(iii) $A \cup |L| \searrow 0$, $B \cup |L| \searrow 0$,

then $|K \times I| \searrow 0$.

<u>Proof</u>: On applying the previous lemma twice we see that

$$
\begin{aligned}
|K \times I| &= ((A \cup |L|) \times I) \cup ((B \cup |L|) \times I) \\
&\searrow (A \times \{0\}) \cup (|L| \times I) \cup ((B \cup |L|) \times I) \\
&\searrow (A \times \{0\}) \cup (|L| \times I) \cup (B \times \{1\}) = M.
\end{aligned}
$$

Now triangulate M by starring the convex linear cells of $L \times I$, $\zeta \times I$, from vertices of $(A \cup |L|) \times \{0\}$ and $(B \cup |L|) \times \{1\}$.

Note that

(i) $(A \cup |L|) \times \{0\}$ is full in M,

(ii) $(B \cup |L|) \times \{1\}$ is full in M, and

(iii) $[(A \cup |L|) \times \{0\}] \cup [(B \cup |L|) \times \{1\}] \supset M^0$, the zero skeleton of M.

Now define a simplicial map $f\colon M \longrightarrow \langle V_0, V_1 \rangle$ by

$$f((A \cup |L|) \times \{0\}) = V_0,$$

$$f((B \cup |L|) \times \{1\}) = V_1,$$

and linearly on $|L| \times I$.

Note that since $(A \cup |L|) \times \{0\}$ and $(B \cup |L|) \times \{1\}$ are full in M,

$$f^{-1}(V_0) = (A \cup |L|) \times \{0\}, \text{ and}$$

$$f^{-1}(V_1) = (B \cup |L|) \times \{1\}.$$

Then since by hypothesis $A \cup |L| \searrow 0$ and $B \cup |L| \searrow 0$, we have $f^{-1}(V_0) \searrow 0$ and $f^{-1}(V_1) \searrow 0$. Moreover, since $f(|L| \times \{0\}) = V_0$ and $f(|L| \times \{1\}) = V_1$, and f is defined linearly on $|L| \times I$, we have

$$f^{-1}\left(\frac{1}{2} V_0 + \frac{1}{2} V_1\right) = |L| \times \left\{\frac{1}{2}\right\}.$$

Since $|L| \times \{\frac{1}{2}\}$ is piecewise linearly homeomorphic to $|L|$, $f^{-1}(\frac{1}{2} V_0 + \frac{1}{2} V_1) \searrow 0$.

Thus f is a strongly pointlike map, and so $M \searrow 0$. Therefore, $K \times I \searrow M \searrow 0$.

<u>Example 1</u>: In Figure 3 we picture a 2-dimensional polyhedron, K, (the house with two rooms), due to R. H. Bing.

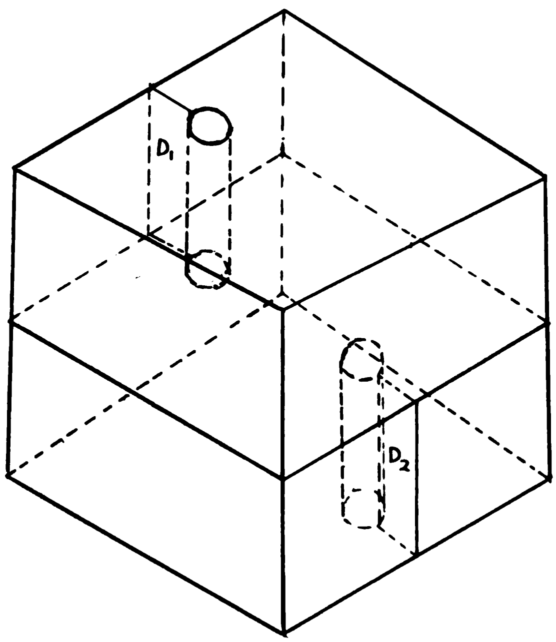

Figure 3.

Although K is not collapsible, since it has no free faces, an application of Theorem 5.3 will show that $K \times I \searrow 0$. Pass a plane P through K so that P contains the rectangular disks D_1 and D_2 of Figure 3. This plane separates K into two components A and B. Figure 4 pictures $A \cup (P \cap K)$. $B \cup (P \cap K)$ is similar.

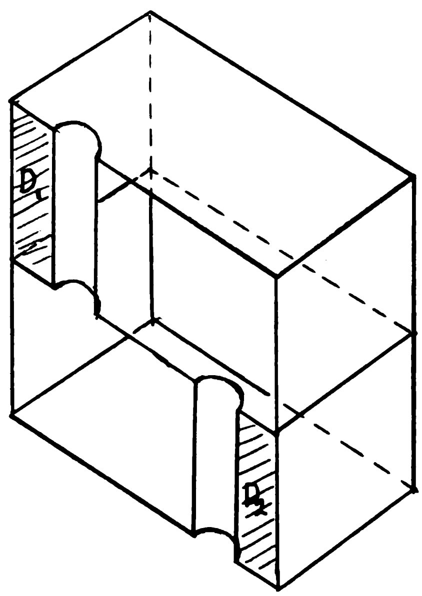

Figure 4.

Clearly $L = P \cap K \searrow 0$, $A \cup L \searrow 0$, and $B \cup L \searrow 0$. Thus, by Theorem 5.3, $K \times I \searrow 0$.

Example 2: Let $\langle V_0, V_1, V_2 \rangle$ be a 2-simplex and let V_1' and V_2' be two interior points of $\langle V_0, V_1, V_2 \rangle$ not co-linear with V_0. Let K_i, $i = 1,2$, indicate two copies of the space obtained by identifying the intervals $\langle V_0, V_1 \rangle = \langle V_0, V_1' \rangle$, $\langle V_0, V_2 \rangle = \langle V_0, V_2' \rangle$, and let L_i

denote the image of $\langle V_1, V_2 \rangle$ in K_i. The polyhedron K_i is pictured in Figure 5.

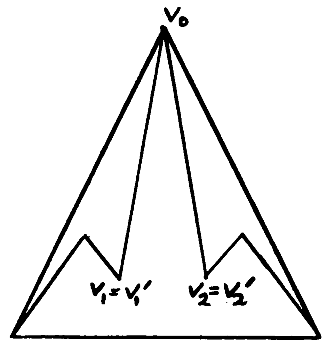

Figure 5.

Let $K = K_1 \cup K_2$ where the union identifies corresponding points of L_1 and L_2.

Note that although K is not collapsible, $L_1 \searrow 0$ and L_1 separates K. If we let $K - L_1 = A \cup B$, then $A \cup L_1 = K_1 \searrow 0$ and $B \cup L_1 = K_2 \searrow 0$. Thus, by Theorem 5.3, $K \times I \searrow 0$.

Using the usual terminology we will call a counterexample to the 3-dimensional Poincare' Conjecture a fake 3-sphere (if such exists). If we triangulate this fake sphere and remove the interior of a 3-simplex, the resulting manifold with boundary will be called a fake 3-ball. Note that a fake 3-ball has a 2-sphere boundary. We may now prove the following theorem.

<u>Theorem 5.4</u>: If K is a spine of a fake 3-ball, and T a tree in K which separates K into components S_1 and S_2, then either $S_1 \cup T$ or $S_2 \cup T$ is not collapsible.

Proof: Let M^3 be a fake 3-ball with K as a spine. We may assume that T is a subcomplex in a triangulation of K. If $S_1 \cup T$ and $S_2 \cup T$ are collapsible we may apply Theorem 5.3 to find that $K \times I \searrow 0$. Then by Proposition 5.1 M^3 is a 3-ball.

In the remainder of this section we will consider an additional method of collapsing $K \times I$ where K is a contractible polyhedron.

Theorem 5.5: If $K_1 \searrow 0$ and $K_1 \searrow K$ by an elementary collapse, then $K \times I \searrow 0$.

Proof: Let

$$K_1 = K \cup B^n, \quad \text{and}$$
$$B^n \cap K = B^{n-1} \subset \dot{B}^n$$

where B^n and B^{n-1} are polyhedral n and n-1 balls respectively.

By Lemma 5.2

$$K \times I \searrow (K \times \{0\}) \cup (B^{n-1} \times I).$$

But $(K \times \{0\}) \cup (B^{n-1} \times I)$ is piecewise linearly homeomorphic to K_1. Let h denote the natural piecewise linear homeomorphism $h: K \times \{0\} \longrightarrow K$. Then h is defined on the n-1 cell $B^{n-1} \times \{0\} \subset (B^{n-1} \times I)^{\cdot}$ onto $B^{n-1} \subset \dot{B}^n$. Thus by [5] h can be extended to a piecewise linear homeomorphism

$$g: (K \times \{0\}) \cup (B^{n-1} \times I) \xrightarrow{\text{onto}} K \cup B^n = K_1.$$

Thus, since $K_1 \searrow 0$, $(K \times \{0\}) \cup (B^{n-1} \times I) \searrow 0$, and we have

$$K \times I \searrow (K \times \{0\}) \cup (B^{n-1} \times I) \searrow 0.$$

<u>Example 3</u>: Consider the dunce hat D. It is well known that $D \times I \searrow 0$, [4]. However, the following application of Theorem 5.5 seems to be a somewhat easier way to prove that $D \times I \searrow 0$.

In Figure 6 we picture a two simplex, two of whose sides have been identified. The identification of a generator of the cone with its base, as indicated by the numbering of vertices in Figure 6, yields the dunce hat.

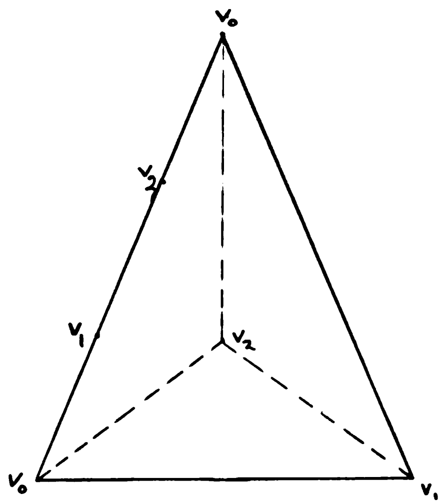

Figure 6.

Now expand D to the complex K indicated in Figure 7. K is simply $D \cup B^3$ where B^3 is the tetrahedron with vertices V_0, V_1, V_3, V_4. Note that $K \searrow D$ since we may collapse B^3 across the 2-simplex $\langle V_1, V_3, V_4 \rangle$. Moreover $K \searrow 0$ as is indicated in Figure 8. The first

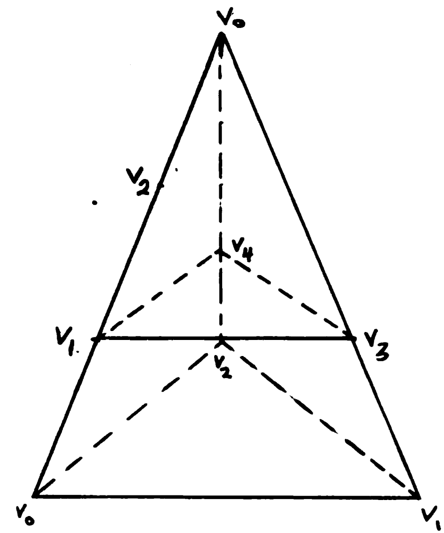

Figure 7.

collapse pictured in Figure 8 is the collapse of B^3 across the 2-simplex $\langle V_0, V_1, V_3 \rangle$. In the second we collapse the 2-cell $\langle V_0, V_1, V_4 \rangle \cup \langle V_0, V_3, V_4 \rangle$ across the 1-cell $\langle V_0, V_3 \rangle$. The third collapse collapses the 2-cell with vertices V_0, V_2, V_4, V_1 across the one cell $\langle V_0, V_2 \rangle$. Finally we collapse the 2-cell with vertices V_1, V_2, V_4, V_3 across the one cell $\langle V_4, V_2 \rangle \cup \langle V_2, V_1 \rangle$. The resultant complex pictured in Figure 8 is a disk which is clearly collapsible. Thus by Theorem 5.5 $D \times I \searrow 0$.

<u>Corollary 5.6</u>: If $K \searrow 0$ and $K \searrow L$, then there exists an integer p such that $L \times I^p \searrow 0$.

<u>Proof</u>: Let $K \searrow K_1 \searrow K_2 \searrow \cdots \searrow K_p = L$ be a sequence of elementary collapses. The proof will be by induction on p.

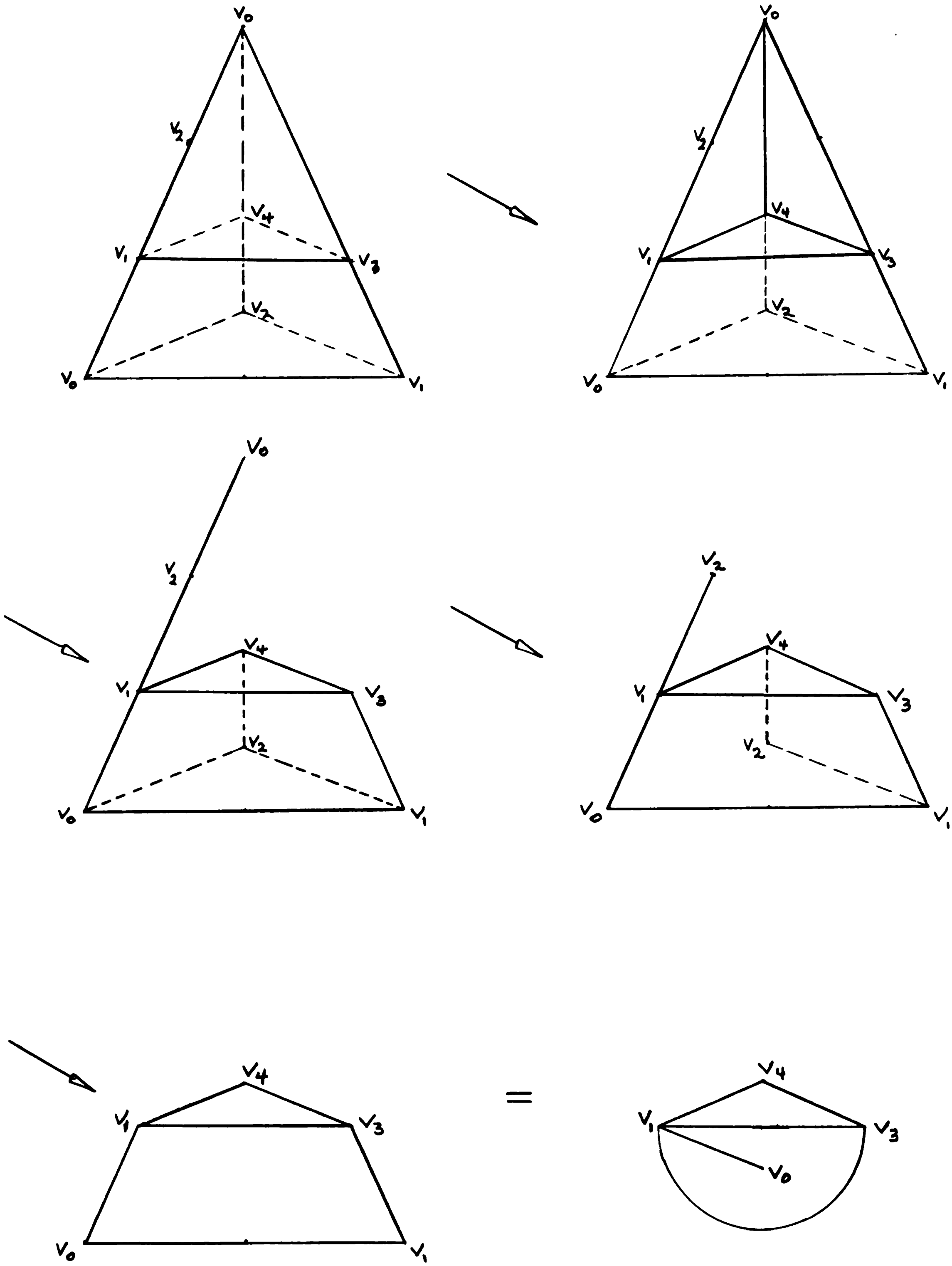

Figure 8.

If $p = 1$ this is just Theorem 5.5.

Now suppose $K_{p-1} \searrow K_p$ and $K_{p-1} \times I^{p-1} \searrow 0$.

Let $K_{p-1} = K_p \cup B^n$ and

$K_p \cap B^n = B^{n-1} \subset \dot{B}^n$ where B^n and B^{n-1} are polyhedral balls. Then since

$$K_{p-1} \times I^{p-1} = (K_p \times I^{p-1}) \cup (B^n \times I^{p-1}) \quad \text{and}$$

$$(K_p \times I^{p-1}) \cap (B^n \times I^{p-1}) = B^{n-1} \times I^{p-1} \subset (B^n \times I^{p-1})^{\cdot}$$

we have

$$K_{p-1} \times I^{p-1} \searrow K_p \times I^{p-1} \quad \text{by an elementary collapse.}$$

Now apply Theorem 5.5 and we find that $K_p \times I^p \searrow 0$.

<u>Corollary 5.7</u>: If K is a homotopically trivial polyhedron, then there is an integer p such that $K \times I^p \searrow 0$.

<u>Proof:</u> By [3] K is of the same simple homotopy type as a point. Thus there exists a complex L such that $L \searrow 0$ and $L \searrow K$. We may now apply Corollary 5.6 to get the desired result.

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