AN APPLICATION OF THE PROCESS OF REGULARIZATION TO THE ANALYSIS OF DISTRIBUTIONS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
CHARLES RICHARD DIMINNIE
1970

This is to certify that the

thesis entitled

AN APPLICATION OF THE PROCESS OF REGULARIZATION

TO THE ANALYSIS OF DISTRIBUTIONS

presented by

Charles Richard Diminnie

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

C. E. Weil
Major professor

Date July 27, 1972

O-169

ABSTRACT

AN APPLICATION OF THE PROCESS OF REGULARIZATION
TO THE ANALYSIS OF DISTRIBUTIONS

By

Charles Richard Diminnie

This work represents an attempt to apply certain classical techniques of real analysis to the study of distributions. Historically, mathematicians such as Denjoy and Clarkson have employed the sets $E_{\alpha\beta} = \{x: \alpha < f(x) < \beta\}$ to study the behavior of derivatives of functions. In the present case, a similar approach is used to analyze distributions.

Let $\mathcal{D}$ and $\mathcal{D}'$ denote the spaces of test functions and distributions, respectively, as defined by L. Schwartz. Choose $\phi \in \mathcal{D}$ satisfying the following conditions:
1. $\phi(x) \geq 0$ on R; 2. $\int_R \phi(x)dx = 1$; 3. $\phi(x) = \phi(-x)$ for all x; 4. the support of $\phi = [-1,1]$; and 5. $\phi'(x) > 0$ on $]-1,0[$, while $\phi'(x) < 0$ on $]0,1[$. Next, define $\phi_\lambda(x) = \lambda\,\phi(\lambda x)$ for all $x \in R$. Then, the net $\{\phi_\lambda\}$ converges to the Dirac measure δ in $\mathcal{D}'$, which implies that the net $\{T*\phi_\lambda\}$ converges to T in $\mathcal{D}'$ for each distribution T. Hence, each distribution T may be represented by the net of regularizations $\{T*\phi_\lambda\}$. Further, it is demonstrated in Chapter II that the function $\Gamma_T[(x,\lambda)] = T*\phi_\lambda(x)$ is continuous on the space $R_2^+ = \{(x,\lambda): \lambda \geq 1\}$.

These considerations indicate that it might be possible to examine each distribution T by means of the sets $E_{\alpha\beta}^{T,N} = \{(x,\lambda): \alpha < T*\phi_\lambda(x) < \beta, \lambda \geq N\}$ for large values of N. In Chapter IV, it is shown that these sets satisfy the following conditions.

Theorem 4.10. For all real numbers α,β such that $\alpha < \beta$, exactly one of the following cases occurs:

1. $T \leq \alpha$ or $T \geq \beta$.
2. For all N, $m_2(E_{\alpha\beta}^{T,N}) > 0$ but $m_2(E_{\alpha\beta}^{T,N}) \to 0$ as $N \to \infty$.
3. For all N, $m_2(E_{\alpha\beta}^{T,N}) = \infty$.

Theorem 4.10 leads to the definition of a series of classes of distributions.

Definition 5A. A distribution T is in Class 0-S if for all α and β, exactly one of the following is satisfied:

1. $T \leq \alpha$ or $T \geq \beta$.
2. There is a set $E \subseteq R$ and a number N' such that $m_1(E) > 0$ and $(E \times [N,\infty[) \subseteq E_{\alpha\beta}^{T,N}$ for all $N \geq N'$.

Definition 5B. T is in Class 0-W if for all α and β, exactly one of the following is satisfied:

1. $T \leq \alpha$ or $T \geq \beta$.
2. $m_2(E_{\alpha\beta}^{T,N}) = \infty$ for all N.

Definition 5C. T is in Class θ for $\theta > 0$, if for all α and β, exactly one of the following is satisfied:

1. $T \leq \alpha$ or $T \geq \beta$.
2. There exist numbers N', M such that $m_2(E_{\alpha\beta}^{T,N}) \geq M(1/N)^\theta$ for $N \geq N'$.

It is easily proven that Class 0-S $\subseteq$ Class 0-W $\subseteq$ Class θ, for $\theta > 0$. Also, for $\theta_1 < \theta_2$, Class $\theta_1 \subseteq$ Class θ_2.

The major results concerning these classes are given in Chapter V. If f is an ordinary derivative, then f may be used to define a distribution T^f in this way: $\langle T^f, \psi\rangle = P\int_R f(x)\psi(x)dx$ for all $\psi \in \mathcal{D}$, where the notation "$P\int$" denotes Perron integration. Theorem 5.2 states that all distributions T^f, where f is an ordinary derivative, are included in Class 0-S. The main result for the remaining classes is given by Theorem 5.7. The notation $D^{(n)}T$ is used to denote the n^{th} distributional derivative of T.

Theorem 5.7. If $T = D^{(n)}g$, where g is locally bounded, then $T \in$ Class n, (Class 0-W if $n = 0$); if $T = D^{(n)}g$, where g is a locally L^p function for $p \geq 1$, then $T \in$ Class $(n + 1/p)$; if $T = D^{(n)}\mu$ for some measure μ, then $T \in$ Class $(n + 1)$.

Finally, Chapter VI gives examples to illustrate the following distinctions between the θ-classes:

1. Class 0-S $\subsetneq$ Class 0-W.
2. Class 0-W $\subsetneq$ Class θ, for any $\theta > 0$.
3. For any ν satisfying $0 < \nu < 2$,

$$\bigcup_{\theta<\nu} \text{Class } \theta \subsetneq \text{Class } \nu.$$

AN APPLICATION OF THE PROCESS OF REGULARIZATION TO THE ANALYSIS OF DISTRIBUTIONS

By

Charles Richard Diminnie

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1970

G-65566
1-20-71

ACKNOWLEDGMENTS

I would like to thank Professor Clifford E. Weil who proposed the original problem and made valuable suggestions throughout the course of this work. Also, I am grateful to my wife Carol who typed the first draft of this thesis and helped to smooth over the many rough edges contained therein. Finally, I must acknowledge the contribution of my son, David Charles, whose arrival provided the necessary impetus for the completion of this paper.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In the past, there has been a considerable amount of research devoted to the problem of studying functions by means of the sets $E_{\alpha\beta} = \{x: \alpha < f(x) < \beta\}$. One of the notable results in this area was originally formulated by Denjoy in [2] and later refined by Clarkson in [1]. It reads as follows:

Theorem I. If f is the derivative of a continuous function, then for all numbers α and β such that $\alpha < \beta$, we have either $E_{\alpha\beta} = \Phi$ or $m_1(E_{\alpha\beta}) > 0$, where m_1 denotes one-dimensional Lebesgue measure.

Since distributions are usually considered to be "generalized functions", it is only natural to ask whether they can be studied by means of a similar approach. It was in the course of developing this problem that the results of this paper were formulated.

Since it is impossible to define a set similar to $E_{\alpha\beta}$ for a distribution, we use an alternate approach. For any distribution T, there is a family of infinitely differentiable functions $\{\psi_\lambda\}$ such that $\psi_\lambda \to T$ in the sense of distributions. With this in mind, we study T using the sets $\{x: \alpha < \psi_\lambda(x) < \beta\}$ for large values of λ. More specifically, we consider the relationship of the two-dimensional Lebesgue measure of the sets $\{(x,\lambda): \alpha < \psi_\lambda(x) < \beta, \lambda \geq N\}$ to powers of $(1/N)$ as the criterion for defining classes of distributions. The major result obtained by using these classes is that for most distributions, membership in the appropriate class is dependent solely on the character of the primitives of these distributions. Thus, a result which is somewhat analogous to that of Denjoy and Clarkson may be obtained by our method.

In Chapter II, we give a further justification for the approach described above and we discuss the type of family $\{\psi_\lambda\}$ that we will choose to "represent" the distribution T. Chapter III is devoted to the development of a certain collection of distributions which are not usually discussed in the literature. A thorough analysis of the nature of the sets $\{(x,\lambda): \alpha < \psi_\lambda(x) < \beta, \lambda \geq N\}$ is given in Chapter IV. Chapter V describes the classes of distributions defined by means of these sets and gives the main results concerning these classes. Chapter VI consists of examples which lend more weight to the results of Chapter V. Finally, a short discussion of conclusions and open questions is found in Chapter VII.

Before we proceed to the work at hand, it might be advisable to include a brief discussion of the space of distributions and a statement of some of the results which are used in this paper. The development described here is basically that of Laurent Schwartz, with minor adjustments to meet the requirements of our problem. The reader is referred to either [3], [9], or [11] for a more detailed account of the theory of distributions.

We begin by considering $C_C(R)$, the space of continuous, real-valued functions with compact support. Since the usual topology assigned to $C_C(R)$ is somewhat complicated, we will not specify it completely. Instead, we will make note of the definition of convergence in $C_C(R)$.

$\psi_n \to 0$ in $C_C(R)$ if

1. there is a compact set K such that the support of ψ_n is contained in K for all n;
2. $\psi_n(x) \to 0$ uniformly.

If μ is a continuous, real-valued, linear functional on $C_C(R)$, then μ is called a <u>measure</u>. (The relationship between these measures and the usual notion of set functions is given by the famous Reisz representation theorem.) The following are specific examples of measures:

1. The Dirac delta measure δ is defined by

   $<\delta,\psi> = \psi(0)$ for all $\psi \in C_C(R)$.
2. If f is a function which is Lebesgue integrable on each finite interval, then f may be used to define a measure in this way:

   $<f,\psi> = \int_R f(x)\psi(x)dx$ for all $\psi \in C_C(R)$.

Note that it is conventional to identify the function f with the functional it defines and to use f to denote both concepts.

The space of measures forms a first generalization to the notion of function. It includes most functions as well as certain other objects, such as the Dirac measure, which have been (incorrectly) used as functions in classical physics and mathematics. However, for the needs of differential equations and certain other aspects of mathematics, it becomes necessary to enlarge the space of measures. This is accomplished by reducing the set of objects on which the functionals are to be applied. The following reduction of the space $C_c(R)$ leads us to the expansion of the space of measures to the space of distributions.

Let $C_c^\infty(R)$ denote the space of infinitely differentiable, real-valued functions with compact support. We use the symbol $\mathcal{D}$ to describe $C_c^\infty(R)$ endowed with the topology which has the following definition of convergence:

$\psi_n \to 0$ in $\mathcal{D}$ if

1. there is a compact set K such that the support of ψ_n is contained in K for all n;
2. for each integer $k \geq 0$, $\psi_n^{(k)}(x) \to 0$ uniformly, where $\psi_n^{(k)}$ denotes the k^{th} ordinary derivative of ψ_n.

We will frequently refer to the elements of $\mathcal{D}$ as "test functions".

Next, we define the space of distributions, $\mathcal{D}'$, to be the space of continuous, real-valued, linear functionals on $\mathcal{D}$. Since $\mathcal{D}$ forms a topological vector subspace of $C_c(R)$, it is clear that $\mathcal{D}'$ is an enlargement of the space of measures. The usual topology given to $\mathcal{D}'$ is such that $T_n \to 0$ in $\mathcal{D}'$ if $<T_n,\psi> \to 0$ for all $\psi \in \mathcal{D}$.

All measures and locally Lebesgue integrable functions may now be considered as distributions. Again, we will not distinguish between the measure μ and the distribution μ or between the function f and the distribution f.

The concept of distributions has an immediate advantage over that of measures since there is a convenient method for defining differentiation in $\mathcal{D}'$. We define the n^{th} distributional derivative of a distribution T, $D^{(n)}T$, to be that distribution satisfying:

$$(1.1) \qquad <D^{(n)}T,\psi> = (-1)^{(n)}<T,\psi^{(n)}> \quad \text{for all} \quad \psi \in \mathcal{D}.$$

It is clear from the definition of convergence in $\mathcal{D}$, that the linear functional $D^{(n)}T$ is continuous on $\mathcal{D}$ and hence defines a distribution. Further, under the above definition of differentiation, every distribution has derivatives of all orders. If f is a function which is n-times continuously differentiable, then for all $\psi \in \mathcal{D}$, we have

$$<D^{(n)}f,\psi> = (-1)^n<f,\psi^{(n)}> = (-1)^n\int_R f(x)\psi^{(n)}(x)dx.$$

Using integration by parts n-times, we obtain

$$<D^{(n)}f,\psi> = \int_R f^{(n)}(x)\psi(x)dx = <f^{(n)},\psi> \quad \text{for all} \quad \psi \in \mathcal{D}.$$

Thus, the functional $D^{(n)}f$ is the same as the functional defined by $f^{(n)}$. According to our convention, we identify $D^{(n)}f$ and $f^{(n)}$ and say that the n^{th} distributional derivative of f is the same as the n^{th} ordinary derivative of f.

A distribution T is said to be positive, denoted $T \geq 0$, if for every $\psi \in \mathcal{D}$ such that $\psi(x) \geq 0$ on R, we have $\langle T,\psi\rangle \geq 0$. For two distributions S and T, $S \geq T$ if $(S-T) \geq 0$. The following result relates positive distributions to the subspace of measures (c.f. [9], Chapter I, Theorem V.).

Theorem II. If T is a positive distribution, then T is a positive measure.

If $\psi \in \mathcal{D}$, then we may also consider ψ as an element of $\mathcal{D}'$. For any $T \in \mathcal{D}'$, we define the convolution product $T*\psi$ to be the distribution given by the function $T*\psi(x) = \langle T_t, \psi(x-t)\rangle$, where the subscript t on T is used to indicate that the functional T is operating on $\psi(x-t)$ considered as a function of t. It is easily seen that in the case where T is a locally Lebesgue integrable function f, the above notion of convolution agrees with the classical definition. It can be shown that $T*\psi(x)$ is an infinitely differentiable function of x, whose derivatives are given by

$$(1.2) \qquad (T*\psi)^{(n)}(x) = (D^{(n)}T)*\psi(x) = T*\psi^{(n)}(x).$$

The convolution product $T*\psi$ is also called the regularization of T by ψ. The concept of convolution can be generalized to certain other distributions, but the conditions on the distributions involved are somewhat complicated. Since we will make little use of this notion, we will not consider it further.

In [8], Schwartz gives examples of families of test functions $\{\phi_\lambda\}$ which converge to δ in $\mathcal{D}'$. Further, he gives sufficient conditions for a family of test functions to converge to δ in $\mathcal{D}'$ (c.f. [8], Chapter II, Theorem 13). If $\{\phi_\lambda\}$ is a family of test functions which converges to δ in $\mathcal{D}'$, it can be shown that for any distribution T, the family $\{T*\phi_\lambda\}$ converges to T in $\mathcal{D}'$ (c.f. [8], Chapter III, Theorem 7). This justifies our earlier statement that every distribution T is the limit in $\mathcal{D}'$ of a family of infinitely differentiable functions.

CHAPTER II

DISTRIBUTIONS AS LIMITS OF INFINITELY DIFFERENTIABLE FUNCTIONS

As mentioned in Chapter I, we are going to study each distribution T by means of a net of infinitely differentiable functions $\{\psi_\lambda\}$ which converges to T in $\mathcal{D}'$ as $\lambda \to \infty$. We will demonstrate that this is an appropriate method by briefly relating the approach used by Mikusinski in [6] and Temple in [10] to arrive at an alternate definition of the space $\mathcal{D}'$.

Basically, they define the space of distributions as a completion of the set of infinitely differentiable functions. The approach is essentially the same as that used by Cantor in the construction of the real number system from the rationals. To begin, we define a sequence of functions $\{\psi_n\}$ to be <u>regular</u> if

1. each ψ_n is infinitely differentiable;
2. for each $\eta \in \mathcal{D}$, $\int_R \psi_n(x)\eta(x)dx$ converges to a limit, which we will denote by $\langle L,\eta\rangle$;
3. this limit $\langle L,\eta\rangle$ is continuous on $\mathcal{D}$, i.e., $\langle L,\eta_n\rangle \to 0$ whenever $\eta_n \to 0$ in $\mathcal{D}$.

Two regular sequences $\{\psi_n\}$ and $\{\sigma_n\}$ are said to be <u>equivalent</u> if for each $\eta \in \mathcal{D}$, we have

$$\int_R [\psi_n(x) - \sigma_n(x)]\eta(x)dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since this notion is obviously an equivalence relation, it partitions the set of regular sequences into equivalence classes which are designated as generalized functions. Thus, each of these generalized functions may be specified by any of the regular sequences in its class. In particular, the space of generalized functions is a completion of the set of infinitely differentiable functions. Each such function ψ may be considered as the generalized function represented by the sequence $\{\psi_n\}$, where $\psi_n = \psi$ for all n. Operations on generalized functions are defined by means of the corresponding operations on the regular sequences which represent these functions. For example, if g denotes the generalized function represented by the regular sequence $\{\psi_n\}$, then the m^{th} derivative of g, $D^{(m)}g$, is defined to be the generalized function represented by the sequence $\{\psi_n^{(m)}\}$.

The preceding paragraphs present a basic description of the construction of these generalized functions. Rather than complete the development of this theory, we will now address ourselves to the work of reconciling these concepts to the distributions of Schwartz and ultimately to the task of applying these ideas to the problem before us. First of all, it is easy to see that if the generalized function g is represented by the regular sequence $\{\psi_n\}$, it may be associated with the functional T on $\mathcal{B}$ defined by $\langle T,\eta\rangle = \lim_{n\to\infty} \int_R \psi_n(x)\eta(x)dx$. The continuity of T then follows directly from condition 3 of the definition of regular sequence. Thus, the generalized functions of Mikusinski and Temple are included in the space of Schwartzian distributions.

To show the correspondence in the other direction, we will require the services of an auxiliary function $\phi \in \mathcal{D}$ having the following properties:

1. $\phi(x) \geq 0$ for all $x \in R$;
2. $\int_R \phi(x)dx = 1$;
3. the support of $\phi(x) = [-1,1]$;

(2.1) 4. $\phi(x) = \phi(-x)$ for all x;

5. $\phi'(x) > 0$ for $x \in]-1,0[$ and $\phi'(x) < 0$ for $x \in]0,1[$;
6. $\max_{x \in R} \phi(x) = \phi(0)$.

(Actually, 2.1.6 follows from 2.1.3 and 2.1.5, but it will be used so often in this work that we will list it as a separate property.) A particular example of such a function is the approximating function ϕ^* defined by:

$$\phi^*(x) = \begin{cases} 0 & \text{for } |x| \geq 1 \\ C \exp[-1/(1-x^2)] & \text{for } |x| < 1, \end{cases}$$

where C is chosen so that $\int_R \phi^*(x)dx = 1$.

For ϕ satisfying conditions (2.1), we will make the following definitions:

(2.2) 1. For $\lambda > 0$, $\phi_\lambda(t) = \lambda\phi(\lambda t)$.

2. For $\lambda > 0$ and $x \in R$, $\phi_{x,\lambda}(t) = \phi_\lambda(x-t) = \lambda\phi[\lambda(x-t)]$.

It can be shown that the net $\{\phi_\lambda\}$ converges to δ in $\mathcal{D}'$ as $\lambda \to \infty$ and hence, for any distribution T, the net

$\{T*\phi_\lambda\}$ converges to T in $\mathcal{D}'$ as $\lambda \to \infty$ (c.f. [8], Chapter II, Theorem 13 and Chapter III, Theorem 7). In particular, since each of the regularizations $T*\phi_\lambda$ is an infinitely differentiable function, we see that the sequence $\{T*\phi_n\}$, where $\phi_n(t) = n\phi(nt)$, is an appropriate regular sequence to represent T as a generalized function.

Thus, the theory of generalized functions of Mikusinski and Temple is in complete accord with Schwartz' theory of distributions. For our purposes, it is even more important to notice that their theory illustrates that it is entirely natural to study a distribution T by means of the sequence of regularizations $\{T*\phi_n(x)\}$, since they are essentially the same object. Also, we note that all of the methods of Mikusinski and Temple may be applied through the use of regular nets instead of regular sequences. The reasons why we choose to use the regular net $\{T*\phi_\lambda\}$ to represent T will be made more apparent later in this chapter.

The remainder of this chapter will be devoted to displaying the continuity properties of the regularizations $T*\phi_\lambda(x)$ considered as a function of both x and λ. We begin with the following two lemmas.

Lemma 2.1. If $(x_n,\lambda_n) \to (x,\lambda)$ in R_2 (two dimensional Euclidean space) and λ as well as all the λ_n are positive, then ϕ_{x_n,λ_n} converges to $\phi_{x,\lambda}$ in $\mathcal{D}$.

Proof: Since the numbers $|x_n - x|$ and $|1/(\lambda_n) - 1/\lambda|$ are bounded independent of n and the support of each $\phi_{x_n,\lambda_n} =$

$[x_n-1/(\lambda_n), x_n+1/(\lambda_n)]$, it is clear that there is a compact set K such that the support of each ϕ_{x_n,λ_n} is contained in K.

The compactness of K implies that there is a constant C such that $\max_{t\in K} |t-x| \le C$. Then, for any $t \in K$, we obtain

$$|\lambda_n(x_n-t)-\lambda(x-t)| \le \lambda_n|x_n-x| + |\lambda_n-\lambda||x-t| \le \lambda_n|x_n-x| + |\lambda_n-\lambda|C.$$

Therefore, $\lambda_n(x_n-t) \to \lambda(x-t)$ uniformly on K since $x_n \to x$ and $\lambda_n \to \lambda$.

Let $\varepsilon > 0$ be given. By the uniform continuity of ϕ, there is a number Δ depending only on ε such that $|t_1-t_2| < \Delta$ implies $|\phi(t_1)-\phi(t_2)| < \varepsilon/(2\lambda)$. If we denote $\max_{t\in R} |\phi(t)|$ by M, our next step is to choose N large enough that for $n \ge N$, we obtain both $|\lambda_n-\lambda| < \varepsilon/(2M)$ and $\max_{t\in K} |\lambda_n(x_n-t)-\lambda(x-t)| < \Delta$. Then, for $t \in K$ and $n \ge N$,

$$\begin{aligned}
|\phi_{x_n,\lambda_n}(t)-\phi_{x,\lambda}(t)| &= |\lambda_n\phi[\lambda_n(x_n-t)] - \lambda\phi[\lambda(x-t)]| \\
&\le |\lambda_n-\lambda||\phi[\lambda_n(x_n-t)]| \\
&\qquad + \lambda|\phi[\lambda_n(x_n-t)] - \phi[\lambda(x-t)]| \\
&\le |\lambda_n-\lambda|M + \lambda|\phi[\lambda_n(x_n-t)] - \phi[\lambda(x-t)]| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $\lambda_n(x_n-t) \to \lambda(x-t)$ uniformly on K, and Δ was dependent only on ε, our choice of N is uniform for all $t \in K$. Thus, $\phi_{x_n,\lambda_n} \to \phi_{x,\lambda}$ uniformly on K which implies that $\phi_{x_n,\lambda_n} \to \phi_{x,\lambda}$ uniformly on R.

By noting that the only properties of ϕ used in the above argument were its uniform continuity and its boundedness, we see that a similar proof yields that $\phi^{(k)}_{x_n,\lambda_n} \to \phi^{(k)}_{x,\lambda}$ uniformly on R for all positive integers k. Hence $\phi_{x_n,\lambda_n} \to \phi_{x,\lambda}$ in $\mathcal{B}$. Q.E.D.

<u>Lemma 2.2</u>. If $\phi_{x_n,\lambda_n} \to \psi$ in $\mathcal{B}$ and there is a positive number λ_o such that $\lambda_n \geq \lambda_o$ for all n, then there is an element $(x,\lambda) \in R_2$ such that $(x_n,\lambda_n) \to (x,\lambda)$ in R_2 and further, $\psi = \phi_{x,\lambda}$.

Proof: Since $\phi_{x_n,\lambda_n} \to \psi$ in $\mathcal{B}$, we have that $\phi_{x_n,\lambda_n} \to \psi$ uniformly on R. Hence, $|\max_{t\in R} \phi_{x_n,\lambda_n}(t) - \max_{t\in R} \phi_{x_m,\lambda_m}(t)| \to 0$ as $m,n \to \infty$ which implies that $|\lambda_n\phi(0)-\lambda_m\phi(0)| \to 0$ as $m,n \to \infty$, by (2.1.6) and (2.2.2). Thus, $|\lambda_n-\lambda_m| \to 0$ as $m,n \to \infty$ since $\phi(0) > 0$. Therefore, $\{\lambda_n\}$ is a Cauchy sequence in R and there is a $\lambda \in R$ such that $\lambda_n \to \lambda$. Further, since each $\lambda_n \geq \lambda_o$, we have $\lambda \geq \lambda_o > 0$.

Again, using the fact that $\phi_{x_n,\lambda_n} \to \psi$ uniformly on R, we obtain that $\max_{t\in R} \phi_{x_n,\lambda_n}(t) \to \max_{t\in R} \psi(t)$. However, since $\max_{t\in R} \phi_{x_n,\lambda_n}(t) = \lambda_n\phi(0)$ for each n and $\lambda_n \to \lambda$, we also have that $\max_{t\in R} \phi_{x_n,\lambda_n}(t) \to \lambda\phi(0)$ which implies that $\max_{t\in R} \psi(t) = \lambda\phi(0)$. Thus, the fact that $\psi \in \mathcal{B}$ yields the existence of an $x \in R$ such that $\lambda\phi(0) = \max_{t\in R} \psi(t) = \psi(x)$. We will show that $x_n \to x$ and hence that $(x_n,\lambda_n) \to (x,\lambda)$ in R_2.

Since $\phi_{x_n,\lambda_n}(t) \to \psi(t)$ pointwise on R, then in particular, $\phi_{x_n,\lambda_n}(x) \to \psi(x) = \lambda\phi(0)$. Thus,

$\lambda_n\phi[\lambda_n(x_n-x)] \to \lambda\phi(0)$ as $n \to \infty$ which implies that

(*) $\phi[\lambda_n(x_n-x)] \to \phi(0)$ as $n \to \infty$

since $\lambda_n \to \lambda$ and each $\lambda_n \geq \lambda_o > 0$.

Now, suppose $x_n \not\to x$. Then, there is an $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k}-x| \geq \varepsilon$ for all k. Thus, for all k, $\lambda_{n_k}|x_{n_k}-x| \geq \lambda_{n_k}\varepsilon \geq \lambda_o\varepsilon$. Also, by (2.1.5) and (2.1.6), we obtain both $\phi(\lambda_o\varepsilon) < \phi(0)$ and $\phi[\lambda_{n_k}(x_{n_k}-x)] \leq \phi(\lambda_o\varepsilon)$ for all k. Therefore, $\limsup_{k\to\infty} \phi[\lambda_{n_k}(x_{n_k}-x)] \leq \phi(\lambda_o\varepsilon) < \phi(0)$, which is impossible by (*). Hence, $x_n \to x$ and $(x_n,\lambda_n) \to (x,\lambda)$ in R_2.

Since λ as well as all the λ_n are positive, we may apply Lemma 2.1 to obtain that $\phi_{x_n,\lambda_n} \to \phi_{x,\lambda}$ in $\mathcal{B}$. Thus, $\psi = \phi_{x,\lambda}$ and the proof is completed. Q.E.D.

Note that the proof of these lemmas relied heavily on the restriction of λ to values which were bounded away from zero. Since we will be primarily interested in working with the functions $\phi_{x,\lambda}$ when λ is a large positive number, we will impose a lower bound on the values of λ for all that follows. With this in mind, we will use the symbol R_2^+ to designate the space $\{(x,\lambda): x \in R$ and $\lambda \geq 1\}$.

By combining the previous lemmas, we obtain the following theorem:

<u>Theorem 2.3</u>. The map $\rho: R_2^+ \to \mathcal{B}$ defined by $\rho[(x,\lambda)] = \phi_{x,\lambda}$ is a homeomorphism from R_2^+ into $\mathcal{B}$.

Using the definition of the functions $\phi_{x,\lambda}$, we may express each of the regularizations $T*\phi_\lambda(x)$ as follows:

(2.3) $\quad T*\phi_\lambda(x) = \langle T_t, \phi_\lambda(x-t)\rangle = \langle T, \phi_{x,\lambda}\rangle.$

Thus, by Theorem 2.3, we may associate each $T \in \mathcal{D}'$ with a unique continuous map Γ_T from R_2^+ to R in this way:

(2.4) $\quad \Gamma_T[(x,\lambda)] = T \circ \rho[(x,\lambda)] = \langle T, \rho[(x,\lambda)]\rangle =$

$$= \langle T, \phi_{x,\lambda}\rangle = T*\phi_\lambda(x).$$

It is this continuity of the expression $T*\phi_\lambda(x)$ with respect to both x and λ which leads us to consider the net $\{T*\phi_\lambda\}$, rather than the sequence $\{T*\phi_n\}$, as the suitable representation for T.

CHAPTER III

DISTRIBUTIONS DEFINED BY PERRON INTEGRALS

In this chapter, we digress briefly to consider a certain collection of distributions which are not usually mentioned in the standard references. This family, which includes the subspace of locally Lebesgue integrable functions, will furnish us with a number of explicit examples of distributions other than the obvious specimens ordinarily cited. Moreover, we will utilize them to generalize a well known property of the regularizations of Lebesgue integrable functions.

We will base our approach on the generalized integral developed by Perron. Due to the complicated nature of this theory, we will not enter into a complete discussion of the Perron integral, except to mention that it is loosely founded on the notion of defining integration as the inverse operation of differentiation. The reader is referred to [5], Chapter VIII, for a full development of Perron integration.

Instead, we will state some of the main results concerning the Perron integral. For the most part, these theorems are in slightly weaker form than the versions given in the above source. We will use the notation "$P\int$" to denote Perron integration, while "$\int$" will pertain to Lebesgue integration.

<u>Theorem A</u>. If f_1 and f_2 are Perron integrable on the interval $[a, b]$ and k_1, k_2 are constants, then $k_1f_1+k_2f_2$ is Perron integrable on $[a, b]$ and $P\int_a^b [k_1f_1(x) + k_2f_2(x)]dx = k_1P\int_a^b f_1(x)dx + k_2P\int_a^b f_2(x)dx$.

<u>Theorem B</u>. If the function F is continuous on $[a, b]$ and F' is defined and finite on $]a, b[$, then F' is Perron integrable on $[a, b]$ and $P\int_a^x F'(t)dt = F(x)-F(a)$ for all $x \in [a, b]$.

<u>Theorem C</u>. If f is Lebesgue integrable on $[a, b]$, then f is also Perron integrable on $[a, b]$ and $P\int_a^b f(x)dx = \int_a^b f(x)dx$.

<u>Theorem D</u>. If f is Perron integrable on $[a, b]$ and $F(x) = P\int_a^x f(t)dt$, then F is continuous.

<u>Theorem E</u>. If f is Perron integrable on $[a, b]$ and $F(x) = P\int_a^x f(t)dt$, then $F'(x) = f(x)$ a.e. on $[a, b]$. (The notation "a.e." denotes "almost everywhere", i.e., except for a set of Lebesgue measure 0).

<u>Theorem F</u>. (Integration by Parts). If f is Perron integrable on $[a, b]$ and ψ is of bounded variation on $[a, b]$, then $f\psi$ is Perron integrable on $[a, b]$ and $P\int_a^b f(x)\psi(x)dx = F(b)\psi(b) - \int_a^b F(x)d\psi$, where $F(x) = P\int_a^x f(t)dt$ and $\int_a^b F(x)d\psi$ is a Stieltjes integral.

With these preliminaries over, we now proceed to use Perron integrals to define distributions. Let f be a function which is Perron integrable over every finite interval. By Theorem B, finite derivatives of continuous functions provide examples of such functions. For any test function $\psi \in \mathcal{D}$, Theorem F tells us that $f\psi$ is also Perron integrable over every finite interval. Further, since $f\psi$ has compact support, we may use the symbol $P\int_R f(x)\psi(x)dx$ without confusion. Using Theorem A, we see that the functional T^f defined by $\langle T^f,\psi\rangle = P\int_R f(x)\psi(x)dx$ is linear on $\mathcal{D}$. To prove that T^f is also continuous on $\mathcal{D}$, we make use of the function $G(x) = P\int_0^x f(t)dt$. Since Theorem D states that G is a continuous function, we may also consider G as a distribution. Therefore, we use Theorem F to obtain the following: $\langle T^f,\psi\rangle = P\int_R f(x)\psi(x)dx = -\int_R [G(x)+C]\psi'(x)dx$, where C is a constant. However, since $\psi \in \mathcal{D}$ implies that $\int_R \psi'(x)dx = 0$, we may write

$$(3.1) \qquad \langle T^f,\psi\rangle = -\int_R G(x)\psi'(x)dx = \langle DG,\psi\rangle.$$

Hence, the continuity of the functional DG on $\mathcal{D}$ insures the continuity of T^f. Further, (3.1) tells us that the distribution T^f is the distributional derivative of G, i.e.,

$$(3.2) \qquad T^f = DG.$$

Let us consider the case where g is a continuous function having a finite derivative g' everywhere on R. Then, the above arguments may be combined with Theorem B to obtain the following:

$$(3.3) \qquad T^{g'} = Dg.$$

In this way, we observe that for a continuous function g with a finite derivative g', the notions of distributional derivative and ordinary derivative are basically the same even though g' is not necessarily locally Lebesgue integrable.

We now address ourselves to the consideration of the net $\{T^f * \phi_\lambda\}$ of regularizations of T^f by the family $\{\phi_\lambda : \lambda \geq 1\}$ described in Chapter II. Recalling the definition of convolution product given in Chapter I, we see that

$$(3.4) \qquad T^f * \phi_\lambda(x) = \langle T_t^f, \phi_\lambda(x-t)\rangle = P\int_R f(t)\phi_\lambda(x-t)dt.$$

It is a well known classical result for the case where f is locally Lebesgue integrable that $f*\phi_\lambda(x) \to f(x)$ a.e. on R. These next two results will show that the same statement is true when f is locally Perron integrable.

<u>Lemma 3.1</u>. Let F be locally Lebesgue integrable. If F is differentiable at x_o, then $F*\phi_\lambda'(x_o) \to F'(x_o)$ as $\lambda \to \infty$.

Proof: If we set $\nu(t) = F(x_o) - F(x_o - t) - tF'(x_o)$, then $|\nu(t)/t| \to 0$ as $t \to 0$. Hence, if $\epsilon > 0$ is given, there is a $\Delta > 0$ such that $|\nu(t)| < \epsilon|t|$ whenever $|t| < \Delta$. Using (2.1.2), integration by parts, and the fact that $\int_{-1/\lambda}^{1/\lambda} \phi_\lambda'(t)dt = 0$, we obtain the following:

$$\begin{aligned}\int_{-1/\lambda}^{1/\lambda} \nu(t)\phi_\lambda'(t)dt &= F(x_o)\int_{-1/\lambda}^{1/\lambda} \phi_\lambda'(t)dt - \int_{-1/\lambda}^{1/\lambda} F(x_o-t)\phi_\lambda'(t)dt \\ &\qquad - F'(x_o)\int_{-1/\lambda}^{1/\lambda} t\phi_\lambda'(t)dt \\ &= -\int_{x_o-1/\lambda}^{x_o+1/\lambda} F(t)\phi_\lambda'(x_o-t)dt + F'(x_o)\int_{-1/\lambda}^{1/\lambda} \phi_\lambda(t)dt \\ &= F'(x_o) - F*\phi_\lambda'(x_o).\end{aligned}$$

Thus, for all λ, $|F*\phi_\lambda'(x_o) - F'(x_o)| \leq \int_{-1/\lambda}^{1/\lambda} |\nu(t)||\phi_\lambda'(t)|dt$.

In particular, for $\lambda > 1/\Delta$,

$$|F*\phi_\lambda'(x_0)-F'(x_0)| < \epsilon\int_{-1/\lambda}^{1/\lambda}|t||\phi'(t)|dt = -\epsilon\int_{-1/\lambda}^{1/\lambda}t\phi_\lambda'(t)dt \text{ by } (2.1.5)$$

$$= \epsilon\int_{-1/\lambda}^{1/\lambda}\phi_\lambda(t)dt = \epsilon.$$

Therefore, $F*\phi_\lambda'(x_0) \to F'(x_0)$ as $\lambda \to \infty$. Q.E.D.

<u>Theorem 3.2</u>. If f is locally Perron integrable on R, then $\lim_{\lambda\to\infty} T^f*\phi_\lambda(x) = f(x)$ a.e. on R. In particular, this result is true for all x such that $F'(x) = f(x)$, where $F(x) = P\int_0^x f(t)dt$.

Proof: By Lemma 3.1, $F*\phi_\lambda'(x) \to F'(x)$ whenever $F'(x)$ exists. Using (1.2) and (3.2), we obtain that $F*\phi_\lambda'(x) = DF*\phi_\lambda(x) = T^f*\phi_\lambda(x)$ for all λ. Therefore, $T^f*\phi_\lambda(x) \to F'(x)$ whenever $F'(x)$ exists, which implies that $T^f*\phi_\lambda(x) \to f(x)$ a.e. on R by Theorem E. Q.E.D.

We conclude this chapter by noting that an analogous result is also possible in the case of a measure μ, in that the net $\{\mu*\phi_\lambda(x)\}$ converges a.e. to a function which is closely related to μ. Although this result is not useful to us here, the reader may find it instructive to refer to the discussion of the Poisson integral given in [7], Chapter 11. The methods used there may be adapted to prove Theorem 3.2 if we use $\phi_\lambda(x-t)$ in place of the Poisson kernel and substitute integration by parts for the use of Fubini's Theorem in (Rudin's) Lemma 11.9.

CHAPTER IV

THE SETS $E^T_{\alpha\beta}$ AND $E^{T,N}_{\alpha\beta}$

We are now ready to set up the machinery for our analysis of individual distributions by means of their regularizations by the net $\{\phi_\lambda, \lambda \geq 1\}$. As was indicated in Chapter I, we will attempt to apply a Denjoy-type approach to each $T \in \mathcal{D}'$ by examining the sets $\{x: \alpha < T*\phi_\lambda(x) < \beta\}$ for large values of λ. However, in order to make the maximum use of the continuity of the expression $T*\phi_\lambda(x)$ with respect to both x and λ, we will find it more advantageous to consider the sets $\{(x,\lambda): \alpha < T*\phi_\lambda(x) < \beta, \lambda \geq N\}$ in R_2^+, as N becomes large.

With this in mind, we begin with the following definitions:

<u>Definition 4A</u>. If $T \in \mathcal{D}'$ and $\alpha < \beta$, then

1. $E^T_{\alpha\beta} = \{(x,\lambda): \alpha < T*\phi_\lambda(x) < \beta, \lambda \geq 1\}$
2. $E^{T,N}_{\alpha\beta} = \{(x,\lambda): \alpha < T*\phi_\lambda(x) < \beta, \lambda \geq N\}$.

Note that according to (2.4), $E^T_{\alpha\beta} = \Gamma_T^{-1}(]\alpha,\beta[)$, which implies that $E^T_{\alpha\beta}$ is an open set in R_2^+. Further, it is clear that $E^{T,N+1}_{\alpha\beta} \subseteq E^{T,N}_{\alpha\beta} \subseteq E^T_{\alpha\beta}$ for all $N \geq 1$.

The first series of results will deal with the essentials of the set $E^T_{\alpha\beta}$. The symbol m_2 will be used throughout to denote two-dimensional Lebesgue measure. This first lemma follows directly from the fact that each $E^T_{\alpha\beta}$ is an open set in R^+_2.

Lemma 4.1. For all real numbers α, β such that $\alpha < \beta$, we have either $E^T_{\alpha\beta} = \Phi$ or $m_2(E^T_{\alpha\beta}) > 0$.

A similar argument, using the properties of Γ_T restricted to $\{(x,\lambda): \lambda \geq N\}$, yields this corollary:

Corollary 4.2. If $N \geq 1$, then for all α, β such that $\alpha < \beta$, we have either $E^{T,N}_{\alpha\beta} = \Phi$ or $m_2(E^{T,N}_{\alpha\beta}) > 0$.

This next lemma, originally given by S. Lojasiewicz in [4], serves to show the relationship between a positive distribution T and the point functions $T*\phi_\lambda(x)$ which represent T. More specifically, it demonstrates that if a distribution is not positive, it fails because eventually the regularizations $T*\phi_\lambda(x)$ take on negative values. Although this result is used only sparingly in this chapter, we will find it to be crucial for many of the results of Chapter V.

Lemma 4.3. If there is a test function $\psi \in \mathcal{D}$ such that $\psi(x) \geq 0$ on R but $\langle T,\psi\rangle < 0$, then there is a number Λ such that corresponding to each $\lambda \geq \Lambda$, there is an x_λ in the support of ψ for which $T*\phi_\lambda(x_\lambda) < 0$.

Proof: Since $T*\phi_\lambda \to T$ in $\mathcal{D}'$ as $\lambda \to \infty$, we observe that $\langle T*\phi_\lambda,\psi\rangle = \int_R [T*\phi_\lambda(x)]\psi(x)dx \to \langle T,\psi\rangle$ as $\lambda \to \infty$. Thus, $\langle T,\psi\rangle < 0$ implies that there is a Λ such that $\int_R [T*\phi_\lambda(x)]\psi(x)dx < 0$ for all $\lambda \geq \Lambda$. The conclusion then follows from the fact that $\psi(x) \geq 0$ on R. Q.E.D.

We now produce two lemmas which will enable us to establish an interesting correspondence between the sets $E^T_{\alpha\beta}$ and the statements "$T \leq \alpha$", "$T \geq \beta$". Note that these statements make sense in that any constant γ may be considered as both the constant function $\gamma(t) \equiv \gamma$ and the constant distribution defined by $\langle\gamma,\psi\rangle = \gamma\int_R \psi(t)dt$ for all $\psi \in \mathcal{D}$.

<u>Lemma 4.4</u>. $E^T_{\alpha\beta} = \Phi$ iff either $\Gamma_T \geq \beta$ on R^+_2 or $\Gamma_T \leq \alpha$ on R^+_2.

Proof: The sufficiency is obvious, since $E^T_{\alpha\beta} = \Gamma_T^{-1}(]\alpha,\beta[)$.
Necessity: $R^+_2 = \Gamma_T^{-1}(]-\infty,\alpha]) \cup \Gamma_T^{-1}(]\alpha,\beta[) \cup \Gamma_T^{-1}([\beta,\infty[)$.
Therefore, if $\Gamma_T^{-1}(]\alpha,\beta[) = E^T_{\alpha\beta} = \Phi$, we have $R^+_2 = \Gamma_T^{-1}(]-\infty,\alpha]) \cup \Gamma_T^{-1}([\beta,\infty[)$. The continuity of Γ_T and the connectedness of R^+_2 then imply that either $\Gamma_T^{-1}(]-\infty,\alpha]) = \Phi$ or $\Gamma_T^{-1}([\beta,\infty[) = \Phi$ from which the conclusion follows. Q.E.D.

<u>Lemma 4.5</u>. $E^T_{\alpha\beta} = \Phi$ iff $T \leq \alpha$ or $T \geq \beta$.

Proof: By Lemma 4.4, $E^T_{\alpha\beta} = \Phi$ iff $T \leq \alpha$ on $\rho(R^+_2)$ or $T \geq \beta$ on $\rho(R^+_2)$, (see Theorem 2.3 and the relation (2.4)). Now, assume $T \leq \alpha$ on $\mathcal{D}$. Then, for each $(x,\lambda) \in R^+_2$, $\phi_{x,\lambda}(t) \geq 0$ on R which implies that $\langle T,\phi_{x,\lambda}\rangle \leq \alpha\int_R \phi_{x,\lambda}(t)dt$, or $T \leq \alpha$ on $\rho(R^+_2)$. Similarly, $T \geq \beta$ on $\mathcal{D}$ implies that $T \geq \beta$ on $\rho(R^+_2)$. Thus, either of the statements $T \leq \alpha$ or $T \geq \beta$ implies that $E^T_{\alpha\beta} = \Phi$. To prove the converse, we note that Lemma 4.3 assures us that $(\alpha - T) \geq 0$ on $\rho(R^+_2)$ implies that $(\alpha - T) \geq 0$ on $\mathcal{D}$ and that $(T-\beta) \geq 0$ on $\rho(R^+_2)$ implies that $(T-\beta) \geq 0$ on $\mathcal{D}$. Q.E.D.

We combine Lemma 4.1 with Lemma 4.5 to obtain our first theorem of this chapter.

Theorem 4.6. For all numbers α,β such that $\alpha < \beta$, exactly one of the following cases occurs:

1. $m_2(E^T_{\alpha\beta}) > 0$.
2. $T \le \alpha$.
3. $T \ge \beta$.

The remainder of this chapter will be devoted to proving that the properties of $E^T_{\alpha\beta}$ are essentially determined by the sets $E^{T,N}_{\alpha\beta}$, $N \ge 1$.

Lemma 4.7. $E^T_{\alpha\beta} = \Phi$ iff there is an N^* such that $E^{T,N^*}_{\alpha\beta} = \Phi$.

Proof: The necessity is true for all $N \ge 1$ since each $E^{T,N}_{\alpha\beta} \subseteq E^T_{\alpha\beta}$.

Sufficiency: Suppose there is an N^* such that $E^{T,N^*}_{\alpha\beta} = \Phi$. Then, since Γ_T restricted to $\{(x,\lambda): \lambda \ge N^*\}$ is still a continuous function, we may use a proof similar to that of Lemma 4.4 to obtain that $T \le \alpha$ on $\rho(\{(x,\lambda): \lambda \ge N^*\})$ or $T \ge \beta$ on $\rho(\{(x,\lambda): \lambda \ge N^*\})$. Again, Lemma 4.3 tells us that this is enough to insure that $T \le \alpha$ or $T \ge \beta$. The conclusion follows from Lemma 4.5. Q.E.D.

By applying Lemma 4.1, Corollary 4.2, and Lemma 4.7, we obtain the following corollary:

Corollary 4.8. $m_2(E^T_{\alpha\beta}) > 0$ iff $m_2(E^{T,N}_{\alpha\beta}) > 0$ for all $N \ge 1$.

Before presenting the main result of Chapter IV, we will need some notational conventions and an additional lemma concerning subsets of R_2^+.

<u>Definition 4B</u>. If G is a measurable subset of R_2^+, then $G^N = \{(x,\lambda) \in G: \lambda \geq N\}$.

<u>Definition 4C</u>. For all $N \geq 1$, we define $H_N = \{(x,\lambda): N \leq \lambda < N+1\}$.

<u>Lemma 4.9</u>. If G is a measurable subset of R_2^+, then either $m_2(G^N) = \infty$ for all N or $m_2(G^N) \to 0$ as $N \to \infty$.

Proof: It is clear from Definition 4C that for each N, we have $G^N = \bigcup_{K=N}^{\infty} (G \cap H_K)$ and $m_2(G^N) = \sum_{K=N}^{\infty} m_2(G \cap H_K)$. Thus, if there is an N^* such that $m_2(G^{N^*}) < \infty$, then $\sum_{K=N^*}^{\infty} m_2(G \cap H_K) < \infty$ which implies that $\lim_{N\to\infty} m_2(G^N) = \lim_{N\to\infty} [\sum_{K=N}^{\infty} m_2(G \cap H_K)] = 0$. Q.E.D.

Finally, we present the result which will form the foundation for the definition of the classes to be studied in Chapter V. This theorem follows directly from Theorem 4.6, Corollary 4.8, and Lemma 4.9.

<u>Theorem 4.10</u>. For all real numbers α,β such that $\alpha < \beta$, exactly one of the following cases occurs:

1. $E_{\alpha\beta}^T = \Phi$, i.e., $T \leq \alpha$ or $T \geq \beta$.
2. For all N, $m_2(E_{\alpha\beta}^{T,N}) > 0$, but $m_2(E_{\alpha\beta}^{T,N}) \to 0$ as $N \to \infty$.
3. For all N, $m_2(E_{\alpha\beta}^{T,N}) = \infty$.

CHAPTER V

THE θ-CLASSES

It is obvious that condition 1 of Theorem 4.10 cannot occur for all α and β. Further, there are numerous distributions for which this case never occurs. In particular, Theorem II of Chapter I tells us that any distribution which is not a measure can never satisfy this condition. For these reasons, we will rely on conditions 2 and 3 of Theorem 4.10 for our analysis of distributions. With this in mind, we make the following definitions:

Definition 5A. A distribution T is said to be in Class 0-S if for all α,β such that $\alpha < \beta$, exactly one of the following occurs:

1. $T \leq \alpha$ or $T \geq \beta$.
2. There is a measurable set $E \subset R$ and a number N' such that $m_1(E) > 0$ and $(E \times [N,\infty[) \subseteq E^{T,N}_{\alpha\beta}$ for all $N \geq N'$.

Definition 5B. A distribution T is in Class 0-W if for all α,β such that $\alpha < \beta$, exactly one of the following occurs:

1. $T \leq \alpha$ or $T \geq \beta$.
2. For each N, $m_2(E^{T,N}_{\alpha\beta}) = \infty$.

Definition 5C. A distribution T is in Class θ for $\theta > 0$, if for all α,β such that $\alpha < \beta$, exactly one of the following occurs:

1. $T \leq \alpha$ or $T \geq \beta$.

2. There exist numbers N' and K such that $m_2(E_{\alpha\beta}^{T,N}) \geq K(1/N)^{\theta}$ for all $N \geq N'$.

We may make some immediate observations about the above definitions.

Remark I. Suppose $T \in$ Class 0-S and condition 2 of Definition 5A is satisfied. Then, there is a set $E \subseteq R$ and a number $N' > 0$ such that $m_1(E) > 0$ and $E \times [N,\infty[\subseteq E_{\alpha\beta}^{T,N}$ for all $N \geq N'$. In particular, for $N \geq N'$, we apply Fubini's Theorem to obtain $m_2(E_{\alpha\beta}^{T,N}) \geq m_2(E \times [N,\infty[) = \int_N^{\infty} m_1(E)d\lambda = \infty$, since $m_1(E) > 0$. Further, since $E_{\alpha\beta}^{T,N'} \subseteq E_{\alpha\beta}^{T,N}$ for all $N \leq N'$, we have that $m_2(E_{\alpha\beta}^{T,N}) = \infty$ for all N. Thus, every T in Class 0-S is also in Class 0-W, i.e., Class 0-S $\subseteq$ Class 0-W.

Remark II. Clearly, Class 0-W $\subseteq$ Class θ for all $\theta > 0$.

Remark III. Suppose that $0 < \theta_1 < \theta_2$. If $T \in$ Class θ_1 and condition 2 of Definition 5C is satisfied, then there exist numbers N', K such that $m_2(E_{\alpha\beta}^{T,N}) \geq K(1/N)^{\theta_1}$ for all $N \geq N'$. However, since $(1/N)^{\theta_1} \geq (1/N)^{\theta_2}$, we have that $m_2(E_{\alpha\beta}^{T,N}) \geq K(1/N)^{\theta_2}$ also for $N \geq N'$. Thus, $T \in$ Class θ_2 which implies that Class $\theta_1 \subseteq$ Class θ_2 whenever $0 < \theta_1 < \theta_2$.

The remainder of this chapter will be devoted to the establishment of significant sufficient conditions for membership in the various classes. We begin by proving a result based on Theorem 3.2 of Chapter III.

Lemma 5.1. Let h be a locally Perron integrable function and $\alpha < \beta$. If $m_1(\{x: \alpha < h(x) < \beta\}) > 0$, then there is a set $E \subseteq R$ and a number N' such that $m_1(E) > 0$ and $(E \times [N,\infty[) \subseteq E_{\alpha\beta}^{h,N}$ for all $N \geq N'$. (Here, we use h to denote the distribution T^h as well as the point function $h(x)$ in order to alleviate the notational problem).

Proof: Since $\{x: \alpha < h(x) < \beta\} = \bigcup_{i=1}^{\infty} H_i$, where $H_i = \{x: \alpha + 1/i \leq h(x) \leq \beta-1/i\}$, there is a set H and a number $\eta > 0$ such that $0 < m_1(H) < \infty$ and $\alpha + \eta \leq h(x) \leq \beta-\eta$ for all $x \in H$. To simplify matters, let $h_\lambda(x) = h*\phi_\lambda(x)$. Then, since $h_\lambda(x) \to h(x)$ a.e. on H and $m_1(H) < \infty$, we apply Egoroff's Theorem to find a subset L of H such that $m_1(L) < [m_1(H)]/2$ and $h_\lambda(x) \to h(x)$ uniformly on $H\setminus L$, (the complement of L in H). The conclusion follows if we set $E = H\setminus L$ and choose N' large enough that $\max_{x\in E} |h_\lambda(x)-h(x)| < \eta/2$ for all $\lambda \geq N'$. Q.E.D.

<u>Definition 5D</u>. A locally Perron integrable function h will be called <u>admissible</u> if for all numbers α,β such that $\alpha < \beta$, we have $m_1(\{x: \alpha < h(x) < \beta\}) = 0$ iff either $h(x) \leq \alpha$ a.e. or $h(x) \geq \beta$ a.e.

If h is a finite derivative, then h satisfies the Darboux condition, i.e. $h(C)$ is a connected set whenever C is connected. This condition, together with Theorem B of Chapter III and Theorem I of Chapter I, tells us that finite derivatives furnish examples of admissible functions.

Definition 5D provides us with a family of distributions which belong to Class O-S.

<u>Theorem 5.2</u>. If h is an admissible function, then the distribution T^h is in Class 0-S.

Proof: If $h(x) \le \alpha$ a.e., then it is a well-known result that h is locally Lebesgue integrable, (c.f. [5], Chapter VIII, Theorem 62.1). Thus, for any test function $\psi \in \mathcal{D}$ such that $\psi(x) \ge 0$ on R, we have $\langle T^h, \psi\rangle = P\int_R h(x)\psi(x)dx = \int_R h(x)\psi(x)dx \le \alpha\int_R \psi(x)dx$, or $T^h \le \alpha$. Similarly, if $h(x) \ge \beta$ a.e., then $T^h \ge \beta$. Finally, if neither of these cases occurs, we have $m_1(\{x: \alpha < h(x) < \beta\}) > 0$ which implies condition 2 of Definition 5A by Lemma 5.1. Q.E.D.

In particular, Theorem 5.2 tells us that all continuous functions and finite derivatives are included in Class 0-S. To bring out our main result, we will depend on a series of lemmas which concern the local properties of the regularizations of T. For the rest of the chapter, we will use $f_\lambda(x)$ to denote $T*\phi_\lambda(x)$. Recall also, that we are assuming that $\lambda \ge 1$ in each case.

<u>Lemma 5.3</u>. Suppose $T = D^{(n)}g$, where g is a locally bounded function. Then, for each finite interval $[a, b]$, there is a constant K such that $|f_\lambda'(x)| \le K\lambda^{n+1}$ for all $x \in [a, b]$.

Proof: By (1.2), we have $f_\lambda'(x) = T*\phi_\lambda'(x) = D^{(n)}g*\phi_\lambda'(x) = g*\phi_\lambda^{(n+1)}(x)$ for all $x \in R$. In particular, if $|g(x)| \le M$ a.e. on $[a-1, b+1]$, we obtain the following for all $x \in [a, b]$.

$$|f_\lambda'(x)| \le \int_{x-1/\lambda}^{x+1/\lambda}|g(t)||\phi_\lambda^{(n+1)}(x-t)|dt \le M\int_{x-1/\lambda}^{x+1/\lambda}|\phi_\lambda^{(n+1)}(x-t)|dt$$

$$= M\lambda^{n+1}\int_{-1}^{1}|\phi^{(n+1)}(u)|du = K\lambda^{n+1}. \quad \text{Q.E.D.}$$

Lemma 5.4. Suppose $T = D^{(n)}g$, where g is a locally L^p function for $p > 1$. Then, for each finite interval $[a, b]$, there is a constant K such that $|f_\lambda'(x)| \leq K\lambda^{n+1+1/p}$ for all $x \in [a, b]$.

Proof: As in Lemma 5.3, $f_\lambda'(x) = g*\phi_\lambda^{(n+1)}(x)$ for all $x \in R$. Now, if q is the number for which $1/p+1/q = 1$, we apply Hölder's Inequality to derive the following for all $x \in [a, b]$.

$$|f_\lambda'(x)| \leq \int_{x-1/\lambda}^{x+1/\lambda} |g(t)||\phi_\lambda^{(n+1)}(x-t)|dt$$

$$\leq \{\int_{x-1/\lambda}^{x+1/\lambda} |g(t)|^p dt\}^{1/p} \{\int_{x-1/\lambda}^{x+1/\lambda} |\phi_\lambda^{(n+1)}(x-t)|^q dt\}^{1/q}$$

$$\leq \{\int_{a-1}^{b+1} |g(t)|^p dt\}^{1/p} \lambda^{n+2-1/q} \{\int_{-1}^{1} |\phi^{(n+1)}(u)|^q du\}^{1/q}$$

$$= K\lambda^{n+1+1/p}. \quad \text{Q.E.D.}$$

Lemma 5.5. Suppose $T = D^{(n)}\mu$, where μ is a measure. Then for each finite interval $[a, b]$, there is a constant K such that $|f_\lambda'(x)| \leq K\lambda^{n+2}$ for all $x \in [a, b]$.

Proof: As in the previous lemmas, $f_\lambda'(x) = \mu*\phi_\lambda^{(n+1)}(x)$ for all $x \in R$. Then, if $M = \max_{x\in R} |\phi^{(n+1)}(x)|$ and $|\mu|$ denotes the total variation of the measure μ, we have the following for all $x \in [a, b]$.

$$|f_\lambda'(x)| \leq \int_{x-1/\lambda}^{x+1/\lambda} |\phi_\lambda^{(n+1)}(x-t)|d|\mu(t)| \leq M\lambda^{n+2} \int_{x-1/\lambda}^{x+1/\lambda} d|\mu(t)|$$

$$\leq M|\mu|([a-1, b+1])\lambda^{n+2} = K\lambda^{n+2}. \quad \text{Q.E.D.}$$

Lemma 5.6. Suppose that T is not a constant distribution. If $T \not\leq \alpha$ and $T \not\geq \beta$, then there is a finite interval $[a, b]$ and numbers α', β', and N' such that:

1. $\alpha < \alpha' < \beta' < \beta$;

2. for each $\lambda \geq N'$, there is an interval $I_\lambda \subset [a, b]$ such that f_λ takes on the values α' and β' at the endpoints of I_λ while $\alpha' < f_\lambda(x) < \beta'$ on the interior of I_λ.

Proof: Since $T \not\leq \alpha$, there is a $\psi_1 \in \mathcal{D}$ such that $\psi_1(x) \geq 0$ for all $x \in R$, $\int_R \psi_1(x)dx = 1$, and $<T,\psi_1> > \alpha$. Also, $T \not\geq \beta$ implies that there is a $\psi_2 \in \mathcal{D}$ such that $\psi_2(x) \geq 0$ on R, $\int_R \psi_2(x)dx = 1$, and $<T,\psi_2> < \beta$. Further, since T is not constant, ψ_1 and ψ_2 may be chosen so that $<T,\psi_1> \neq <T,\psi_2>$. Now, choose α' and β' in such a way that

$$\max[\alpha, \min(<T,\psi_1>, <T,\psi_2>)] < \alpha' < \beta' < \min[\beta, \max(<T,\psi_1>, <T,\psi_2>)].$$

Since either $<T,\psi_1> < \alpha'$ or $<T,\psi_2> < \alpha'$, we apply Lemma 4.3 to find a Λ_1 such that for each $\lambda \geq \Lambda_1$ there is an $x_\lambda \in$ (support of ψ_1) $\cup$ (support of ψ_2) for which $f_\lambda(x_\lambda) < \alpha'$. Using similar reasoning for β', we obtain a Λ_2 such that for each $\lambda \geq \Lambda_2$, there is an $x_\lambda \in$ (support of ψ_1) $\cup$ (support of ψ_2) for which $f_\lambda(x_\lambda) > \beta'$. Choose $N' \geq \max(\Lambda_1, \Lambda_2)$ and a and b such that (support of ψ_1) $\cup$ (support of ψ_2) $\subseteq [a, b]$. Then, for each $\lambda \geq N'$, there exist two elements $x_\lambda^1, x_\lambda^2 \in [a,b]$ such that $f_\lambda(x_\lambda^1) < \alpha'$ while $f_\lambda(x_\lambda^2) > \beta'$. Finally, since $f_\lambda(x)$ is continuous in x, for all $\lambda \geq N'$ there is an interval $I_\lambda \subseteq [a, b]$ such that $f_\lambda(x)$ takes on the values α' and β' at the endpoints of I_λ while $\alpha' < f_\lambda(x) < \beta'$ on the interior of I_λ. Q.E.D.

Now, we use these last four lemmas to prove the main result of this work.

Theorem 5.7. If $T = D^{(n)}g$, where g is locally bounded, then $T \in$ Class n, (Class 0-W if $n = 0$); if $T = D^{(n)}g$, where g is a locally L^p function for $p \geq 1$, then $T \in$ Class $(n+1/p)$; if $T = D^{(n)}\mu$ for some measure μ, then $T \in$ Class $(n+1)$.

Proof: If T is a constant distribution, then $T \in$ Class 0-S by Theorem 5.2. Therefore, we will suppose that T is not constant. If $T \not\leq \alpha$ and $T \not\geq \beta$, we apply Lemma 5.6 to obtain a finite interval $[a, b]$ and numbers α', β', and N' such that: 1) $\alpha < \alpha' < \beta' < \beta$; and 2) for each $\lambda \geq N'$, there is an interval $I_\lambda \subset [a, b]$ such that $f_\lambda(x)$ takes on the values α' and β' at the endpoints of I_λ while $\alpha' < f_\lambda(x) < \beta'$ on the interior of I_λ. Therefore, for $N \geq N'$,

$$(5.1) \qquad m_2(E^{T,N}_{\alpha\beta}) \geq m_2(E^{T,N}_{\alpha'\beta'}) = \int_N^\infty m_1(\{x\colon \alpha' < f_\lambda(x) < \beta'\})d\lambda$$

$$\geq \int_N^\infty m_1(I_\lambda)d\lambda.$$

Further, for all $\lambda \geq N'$, we use the mean value theorem for derivatives to obtain an element $t_\lambda \in I_\lambda$ such that

$$(5.2) \qquad |f'_\lambda(t_\lambda)| = (\beta'-\alpha')/m_1(I_\lambda) \text{ or } m_1(I_\lambda) = (\beta'-\alpha')/|f'_\lambda(t_\lambda)|.$$

We complete the proof of this theorem by considering each of the hypotheses separately.

Case 1. If $T = D^{(n)}g$, where g is locally bounded, then since $I_\lambda \subseteq [a, b]$ for all $\lambda \geq N'$, we apply Lemma 5.3 to (5.2) above to obtain that $m_1(I_\lambda) \geq M'\lambda^{-(n+1)}$ for $\lambda \geq N'$, where

$M' = (\beta'-\alpha')/K$. For all $N \geq N'$, (5.1) implies that $m_2(E_{\alpha\beta}^{T,N}) \geq \int_N^\infty m_1(I_\lambda)d\lambda \geq M'\int_N^\infty d\lambda/\lambda^{n+1} = M(1/N)^n$, where $M = M'/n$. Therefore, if $T \not\leq \alpha$ and $T \not\geq \beta$, condition 2 of Definition 5C is satisfied for $\theta = n$, which implies that $T \in$ Class n.

Note: If $n = 0$, the application of Lemma 5.3 yields that $m_1(I_\lambda) \geq M'(1/\lambda)$ for $\lambda \geq N'$ which implies that $m_2(E_{\alpha\beta}^{T,N}) \geq M'\int_N^\infty d\lambda/\lambda = \infty$ for $N \geq N'$. Hence, when T is a locally bounded function, $T \in$ Class 0-W.

Case 2. If $T = D^{(n)}g$, where g is locally L^p for $p > 1$, then we use Lemma 5.4 and arguments similar to those of case 1 to obtain that $m_1(I_\lambda) \geq M'(1/\lambda^{n+1+1/p})$ for $\lambda \geq N'$. Hence, for all $N \geq N'$, $m_2(E_{\alpha\beta}^{T,N}) \geq M'\int_N^\infty d\lambda/\lambda^{n+1+1/p} = M(1/N)^{n+1/p}$, where $M = M'/(n+1/p)$. Again, this implies that $T \in$ Class $(n+1/p)$.

Case 3. If $T = D^{(n)}\mu$, Lemma 5.5 tells us that $m_1(I_\lambda) \geq M'(1/\lambda^{n+2})$ for $\lambda \geq N'$. Thus, for $N \geq N'$, $m_2(E_{\alpha\beta}^{T,N}) \geq M'\int_N^\infty d\lambda/\lambda^{n+2} = M(1/N)^{n+1}$, where $M = M'/(n+1)$. This is enough to insure that $T \in$ Class $(n+1)$. Finally, the case where $T = D^{(n)}g$, where g is locally L^1, may be handled as part of Case 3. Q.E.D.

As a result of Theorems 5.2 and 5.7, we can place a large segment of the space of distributions in the appropriate θ-Classes. In particular, continuous functions and finite derivatives belong in Class 0-S; locally bounded functions fit into Class 0-W; locally L^p functions, with $p \geq 1$, are settled into Class $(1/p)$; and measures are located in Class 1.

Further, if T is any distribution covered by the above cases and $T \in$ Class θ', we see that $D^{(n)}T$ is in Class $(\theta'+n)$. In Chapter VI, we will discuss some examples to show that these results are significant. We will also include some comments and conjectures in Chapter VII.

CHAPTER VI

EXAMPLES

In this chapter, we give some examples to illustrate that our previous results were not trivial. For instance, Theorem 5.7 would be completely useless if there were no distinctions between the various θ-classes. At present, we are able to give examples to show that the θ-classes are distinct for $\theta < 2$. It seems likely that similar examples exist for the higher classes also, but we are unable to exhibit them at this time. Some discussion of these cases is included at the end of the chapter.

We begin with an example to show that Class 0-S is a proper subset of Class 0-W.

Example 6.1. Consider the Heaviside function defined as follows:

$$H(x) = \begin{cases} 0 & \text{for} \quad x \le 0, \\ 1 & \text{for} \quad x > 0. \end{cases}$$

If we set $H_\lambda(x) = H*\phi_\lambda(x)$, then

$$H_\lambda(x) = \begin{cases} \int_{x-1/\lambda}^{x+1/\lambda} \phi_\lambda(x-t)dt = 1 & \text{for} \quad x \ge 1/\lambda; \\ \int_0^{x+1/\lambda} \phi_\lambda(x-t)dt = \int_{-1}^{\lambda x} \phi(u)du & \text{for} \quad -1/\lambda < x < 1/\lambda; \\ 0 \quad \text{for} \quad x \le -1/\lambda. \end{cases}$$

Our next step is to define $F(x) = \int_{-1}^{x} \phi(u)du$ for all $x \in [-1,1]$. Then, F is strictly increasing, $F(-1) = 0$, and $F(1) = 1$. Further, $H_\lambda(x) = F(\lambda x)$ for all $x \in]-1/\lambda, 1/\lambda[$. If we choose α and β such that $0 < \alpha < \beta < 1$, then $\{x: \alpha < H_\lambda(x) < \beta\} \subseteq]-1/\lambda, 1/\lambda[$ for all $\lambda \geq 1$. For $\lambda \geq 1$, we have $\alpha < H_\lambda(x) < \beta$ iff $\alpha < F(\lambda x) < \beta$ iff $(1/\lambda)F^{-1}(\alpha) < x < (1/\lambda)F^{-1}(\beta)$. Therefore, for $\lambda \geq 1$, $\{x: \alpha < H_\lambda(x) < \beta\} =]1/\lambda F^{-1}(\alpha), 1/\lambda F^{-1}(\beta)[$ which implies that $m_1(\{x: \alpha < H_\lambda(x) < \beta\}) = 1/\lambda[F^{-1}(\beta) - F^{-1}(\alpha)]$. Since $m_1(\{x: \alpha < H_\lambda(x) < \beta\}) \to 0$ as $\lambda \to \infty$, we see that H cannot satisfy condition 2 of Definition 5A when α and β satisfy $0 < \alpha < \beta < 1$. Further, H cannot satisfy condition 1 for these values of α and β because each H_λ takes on all values between 0 and 1. Hence, we have exhibited values of α and β for which H satisfies neither of the conditions of Definition 5A. We complete the example by noting that since H is locally bounded, $H \in$ Class 0-W by Theorem 5.7.

Up to this time, the only conditions we have placed on the function ϕ were those listed in (2.1). Unfortunately, in order to perform the computations required for these next examples, we are forced to add an additional restriction on ϕ.

(6.1) There is an element $a \in]0, 1[$ such that $\phi'''(x) = 0$ for all $x \in [-a, a]$, and ϕ''' has exactly one zero in $]a, 1[$.

Combining (6.1) with the properties of ϕ listed in (2.1), we see that the first three derivatives of ϕ may be pictured as in the illustrations on the following page.

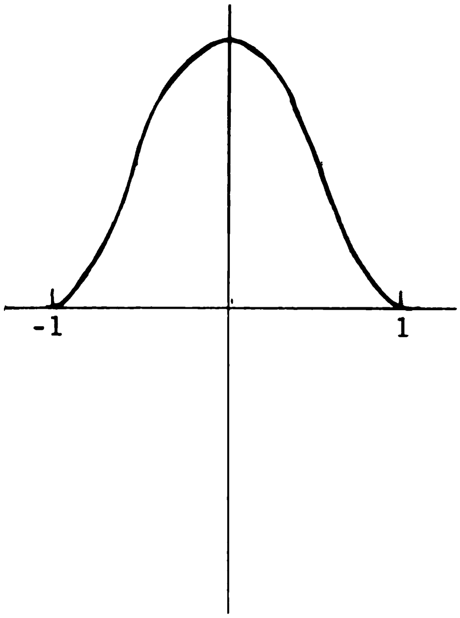

Figure 6.1 - ϕ

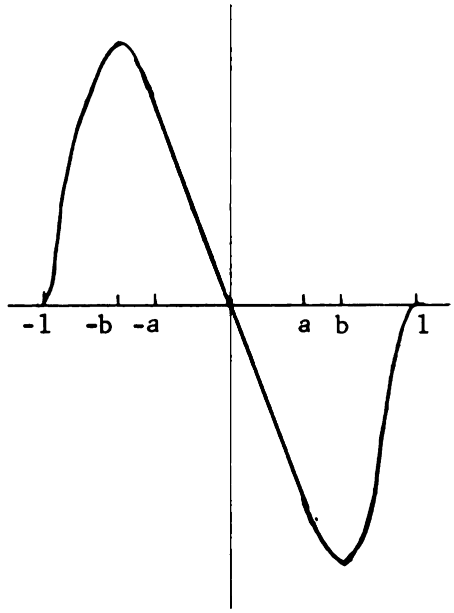

Figure 6.2 - ϕ'

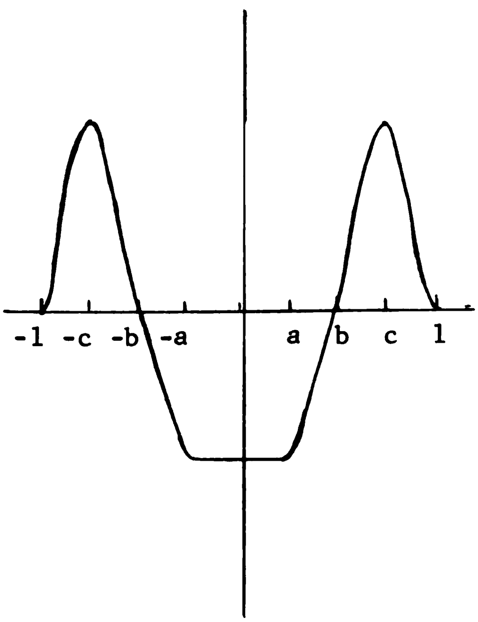

Figure 6.3 - ϕ''

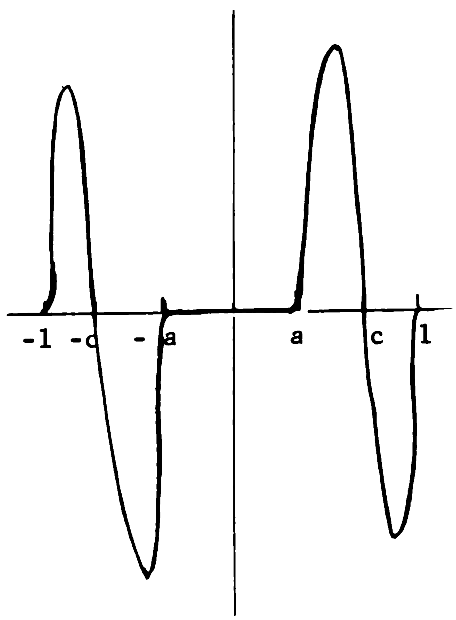

Figure 6.4 - ϕ'''

We will use the information shown in these figures to prove a series of lemmas which are necessary for the remaining examples.

Lemma 6.2. For $0 < \nu < 1$, the following are true:

1. $\int_0^1 t^{-\nu}\phi(t)dt > 0.$
2. $\int_0^1 t^{-\nu}\phi'(t)dt < 0.$
3. $\int_0^1 t^{-\nu}\phi''(t)dt < 0.$
4. $\int_0^1 t^{-\nu}\phi^{(n)}(t)dt > 0$ for $n \geq 3.$

Proof: 1 and 2 follow easily from the properties of ϕ and ϕ'.

$$\begin{aligned}\int_0^1 t^{-\nu}\phi''(t)dt &= \int_0^b t^{-\nu}\phi''(t)dt + \int_b^1 t^{-\nu}\phi''(t)dt \\ &< b^{-\nu}\int_0^b \phi''(t)dt + b^{-\nu}\int_b^1 \phi''(t)dt \\ &= 0 \quad \text{since} \quad \phi'(1) = \phi'(0) = 0.\end{aligned}$$

Therefore, $\int_0^1 t^{-\nu}\phi''(t)dt < 0.$

$$\begin{aligned}\int_0^1 t^{-\nu}\phi'''(t)dt &= \int_a^1 t^{-\nu}\phi'''(t)dt \\ &= \int_a^c t^{-\nu}\phi'''(t)dt + \int_c^1 t^{-\nu}\phi'''(t)dt \\ &> c^{-\nu}\int_a^c \phi'''(t)dt + c^{-\nu}\int_c^1 \phi'''(t)dt \\ &= c^{-\nu}[-\phi''(a)] \\ &> 0.\end{aligned}$$

Therefore, $\int_0^1 t^{-\nu}\phi'''(t)dt > 0.$

For $n > 3$, we use integration by parts $(n-3)$ times to obtain

$$\begin{aligned}\int_0^1 t^{-\nu}\phi^{(n)}(t)dt &= \int_a^1 t^{-\nu}\phi^{(n)}(t)dt \\ &= \nu(\nu + 1)\cdots(\nu + n - 4)\int_a^1 t^{-\nu-(n-3)}\phi'''(t)dt.\end{aligned}$$

Now, for any number $m > 0$, we have

$$\int_a^1 t^{-m}\phi'''(t)dt = \int_a^c t^{-m}\phi'''(t)dt + \int_c^1 t^{-m}\phi'''(t)dt$$

$$> c^{-m}\int_a^c\phi'''(t)dt + c^{-m}\int_c^1 \phi'''(t)dt$$

$$= c^{-m}[-\phi''(a)]$$

$$> 0.$$

Therefore, for $n > 3$, $\int_0^1 t^{-\nu}\phi^{(n)}(t)dt > 0$. Q.E.D.

We will omit the proof for this next lemma since it is almost identical to that of Lemma 6.2.

Lemma 6.3. The following inequalities hold:

1. $\int_0^1 \ln t\ \phi(t)dt < 0.$
2. $\int_0^1 \ln t\ \phi'(t)dt > 0.$
3. $\int_0^1 \ln t\ \phi''(t)dt > 0.$
4. $\int_0^1 \ln t\ \phi^{(n)}(t)dt < 0$ for $n \geq 3$.

Lemma 6.4. Suppose h is locally Lebesgue integrable and $\psi \in \mathcal{D}$. Then, if one of these functions is odd, while the other is even, the convolution product $h*\psi(x)$ is an odd function.

Proof: Choose n large enough that the support of ψ is contained in $[-n, n]$. Then, for all x, we have that $h*\psi(x) = \int_{x-n}^{x+n} h(t)\psi(x-t)dt$. Now, if we assume that h is odd and ψ is even, we obtain the following:

$$h*\psi(x) = \int_{-x-n}^{-x+n} h(-t)\psi(x + t)dt = -\int_{-x-n}^{-x+n} h(t)\psi(-x-t)dt$$

$$= -h*\psi(-x).$$

Thus, when h is odd and ψ is even, $h*\psi(x)$ is an odd function. The case where h is even and ψ is odd is proven similarly. Q.E.D.

Our second example of this chapter will serve to illustrate that $\bigcup_{\theta<\nu}$ Class $\theta \subsetneq$ Class ν, when ν satisfies $0 < \nu < 1$.

Example 6.5. We begin by introducing an auxiliary function g defined as follows:

$$g(x) = \begin{cases} -(-x)^{-\nu} & \text{for } x < 0; \\ x^{-\nu} & \text{for } x > 0. \end{cases}$$

Note that g is locally L^p for $1 \le p < 1/\nu$. We will prove a number of properties concerning g and then proceed to apply these to a second distribution related to g. It is this second distribution which will provide the desired example.

First of all, we let $G(x) = g*\phi(x) = \int_{x-1}^{x+1} g(t)\phi(x-t)dt$. In particular, for $x \in]-1,1[$, we have:

$$(6.2) \qquad G(x) = \int_{x-1}^{0} -(-t)^{-\nu}\phi(x-t)dt + \int_0^{x+1} t^{-\nu}\phi(x-t)dt$$

$$= -\int_x^1 (u-x)^{-\nu}\phi(u)du + \int_{-1}^{x} (x-u)^{-\nu}\phi(u)du.$$

Now, for $x \ge 1$, $G(x) > 0$ since $g(t) > 0$ on $]x-1, x+1[$.

For $x \in]0, 1[$, $G(x) = -\int_{x-1}^{0}(-t)^{-\nu}\phi(x-t)dt + \int_0^{x+1} t^{-\nu}\phi(x-t)dt$

$$= \int_0^{-x+1} t^{-\nu}[\phi(x-t)-\phi(x+t)]dt + \int_{-x+1}^{x+1} t^{-\nu}\phi(x-t)dt$$

$$= \int_x^1 (u-x)^{-\nu}[\phi(2x-u)-\phi(u)]du + \int_{-x+1}^{x+1} t^{-\nu}\phi(x-t)dt.$$

Since $x > 0$, we have $|2x - u| < u$ for all $u \in]x, 1[$ which implies that $\phi(2x - u) \geq \phi(u)$ for $u \in]x, 1[$. Therefore, for $x \in]0, 1[$, $G(x) > 0$ which assures us that $G(x) > 0$ for all $x > 0$. By Lemma 6.4, G is an odd function since g is odd and ϕ is even. Thus, using the previous information on G and its continuity, we obtain the following:

$$(6.3) \qquad G(x) \begin{cases} > 0 & \text{for} \quad x > 0, \\ = 0 & \text{for} \quad x = 0, \\ < 0 & \text{for} \quad x < 0. \end{cases}$$

Next, we consider the behavior of $G'(x)$ for x near 0. Note that ϕ' is an odd function and $g\phi'$ is even.

$$\begin{aligned} G'(0) = g*\phi'(0) &= \int_{-1}^{1} g(t)\phi'(-t)dt = -\int_{-1}^{1} g(t)\phi'(t)dt \\ &= -2\int_{0}^{1} g(t)\phi'(t)dt = -2\int_{0}^{1} t^{-\nu}\phi'(t)dt \\ &> 0 \text{ by condition 2 of Lemma 6.2.} \end{aligned}$$

Since $G'(x)$ is continuous, there is an $h > 0$ such that

(6.4) $G'(x) > 0$ on $]-h, h[$, i.e., G is strictly increasing on $]-h, h[$.

Now, using the fact that $g\phi''$ is an odd function, we obtain:

$$G''(0) = g*\phi''(0) = \int_{-1}^{1} g(t)\phi''(-t)dt = \int_{-1}^{1} g(t)\phi''(t)dt = 0.$$

Further, using condition 4 of Lemma 6.2 and an argument similar to that used for $G'(0)$, we have that $G'''(0) < 0$. By the continuity of G''', there is a $k > 0$ such that $G'''(x) < 0$ on $]-k, k[$. Combining this with the fact that

$G''(0) = 0$, we see that

(6.5) $G''(x) < 0$ on $]0, k[$, i.e., G' is strictly decreasing on $]0, k[$.

We will make one final observation about G. If we let

(6.6) $w = \min(h, k)$ and $v = \min\limits_{x \in]w,1[} G(x)$, then by (6.3), we have $v > 0$.

With all the preliminaries completed, we are now prepared to describe the desired distribution. Let f be the function defined as follows:

$$(6.7) \quad f(x) = \begin{cases} -1 & \text{for} \quad x \leq -1, \\ -(-x)^{-\nu} & \text{for} \quad -1 \leq x < 0, \\ x^{-\nu} & \text{for} \quad 0 < x \leq 1, \\ 1 & \text{for} \quad x \geq 1. \end{cases}$$

By Theorem 5.7, the fact that f is locally L^p for $1 \leq p < 1/\nu$ implies that $f \in$ Class $(1/p)$ for $\nu < 1/p \leq 1$. We will demonstrate that the distribution f is in Class ν, but $f \notin$ Class θ for any $\theta < \nu$.

Following our earlier convention, we will set $f_\lambda(x) = f*\phi_\lambda(x)$. Then, for $\lambda \geq 2$, we have $]x - 1/\lambda, x + 1/\lambda[\subseteq]-1, 1[$ whenever $x \in]-1/\lambda, 1/\lambda[$, and

$$f_\lambda(x) = \int_{x-1/\lambda}^{x+1/\lambda} f(t)\phi_\lambda(x-t)dt = \int_{x-1/\lambda}^{0} -(-t)^{-\nu}\phi_\lambda(x-t)dt$$
$$+ \int_0^{x+1/\lambda} t^{-\nu}\phi_\lambda(x-t)dt$$

$$= -\int_{\lambda x}^{1}[(u-\lambda x)/\lambda]^{-\nu}\phi(u)du + \int_{-1}^{\lambda x}[(\lambda x-u)/\lambda]^{-\nu}\phi(u)du$$

$$= \lambda^\nu[-\int_{\lambda x}^{1}(u-\lambda x)^{-\nu}\phi(u)du + \int_{-1}^{\lambda x}(\lambda x-u)^{-\nu}\phi(u)du].$$

(6.8) Therefore, by (6.2), we obtain that $f_\lambda(x) = \lambda^\nu G(\lambda x)$ whenever $\lambda \geq 2$ and $x \in]-1/\lambda, 1/\lambda[$.

Henceforth, we will assume that $\lambda \geq 2$. Then, for $x \geq 1/\lambda$, we have $f_\lambda(x) = \int_{x-1/\lambda}^{x+1/\lambda} f(t)\phi_\lambda(x-t)dt \geq \int_{x-1/\lambda}^{x+1/\lambda} \phi_\lambda(x-t)dt = 1$, since $f(t) \geq 1$ on $]x-1/\lambda, x+1/\lambda[$. Further, for $x \in]0, 1/\lambda[$, we use (6.3) and (6.8) to obtain that $f_\lambda(x) > 0$. Using this information, Lemma 6.4, and the continuity of f, we may write the following statement:

$$(6.9) \qquad f_\lambda(x) \begin{cases} \geq 1 & \text{for } x \geq 1/\lambda, \\ > 0 & \text{for } 0 < x < 1/\lambda, \\ = 0 & \text{for } x = 0, \\ < 0 & \text{for } -1/\lambda < x < 0, \\ \leq -1 & \text{for } x \leq -1/\lambda. \end{cases}$$

Let $0 < \alpha < \beta < 1$. It is obvious by (6.9) that f does not satisfy condition 1 of Definition 5C for these values of α and β. Also, (6.9) tells us that when $\alpha < f_\lambda(x) < \beta$, we must have $x \in]0, 1/\lambda[$. Now, choose N^* greater than $\max \{[\beta/G(k)]^{1/\nu}, (\beta/v)^{1/\nu}, 2\}$. Then, for $\lambda \geq N^*$, $\alpha < f_\lambda(x) < \beta$ iff $\alpha/\lambda^\nu < G(\lambda x) < \beta/\lambda^\nu$ and $x \in]0, 1/\lambda[$. Using (6.6), we see that when $\lambda \geq N^*$, and $G(\lambda x) < \beta/\lambda^\nu$, we have that $0 < \lambda x < h$. Therefore, $\alpha < f_\lambda(x) < \beta$ iff $(1/\lambda)G^{-1}(\alpha/\lambda^\nu) < x < (1/\lambda)G^{-1}(\beta/\lambda^\nu)$. Hence, for $\lambda \geq N^*$, $\{x: \alpha < f_\lambda(x) < \beta\} =](1/\lambda)G^{-1}(\alpha/\lambda^\nu), (1/\lambda)G^{-1}(\beta/\lambda^\nu)[$ which implies that $m_1(\{x: \alpha < f_\lambda(x) < \beta\}) = 1/\lambda[G^{-1}(\beta/\lambda^\nu)-G^{-1}(\alpha/\lambda^\nu)]$.

For $N \geq N^*$, $m_2(E^{f,N}_{\alpha\beta}) = \int_N^\infty m_1(\{x: \alpha < f_\lambda(x) < \beta\})d\lambda$

$$= \int_N^\infty [G^{-1}(\beta/\lambda^\nu) - G^{-1}(\alpha/\lambda^\nu)]d\lambda/\lambda.$$

(6.10) Further, when $\lambda \geq N^*$, we have $0 < G^{-1}(\alpha/\lambda^\nu) < G^{-1}(\beta/\lambda^\nu) < k$.

By the mean value theorem for derivatives, there is a $t_\lambda \in]G^{-1}(\alpha/\lambda^\nu), G^{-1}(\beta/\lambda^\nu)[$ such that $G'(t_\lambda) = [(\beta-\alpha)/\lambda^\nu]/[G^{-1}(\beta/\lambda^\nu) - G^{-1}(\alpha/\lambda^\nu)]$, or $G^{-1}(\beta/\lambda^\nu) - G^{-1}(\alpha/\lambda^\nu) = (\beta-\alpha)/[\lambda^\nu G'(t_\lambda)]$. Using (6.10) and (6.5), we obtain that $G'[G^{-1}(\beta/\lambda^\nu)] \leq G'(t_\lambda) \leq G'[G^{-1}(\alpha/\lambda^\nu)]$. Therefore, for all $\lambda \geq N^*$,

(6.11) $$(\beta-\alpha)/\{\lambda^\nu G'[G^{-1}(\alpha/\lambda^\nu)]\} \leq G^{-1}(\beta/\lambda^\nu) - G^{-1}(\alpha/\lambda^\nu) \leq (\beta-\alpha)/\{\lambda^\nu G'[G^{-1}(\beta/\lambda^\nu)]\}.$$

In particular, for $N \geq N^*$,

$$m_2(E^{f,N}_{\alpha\beta}) \leq (\beta-\alpha)\int_N^\infty 1/\{\lambda^\nu G'[G^{-1}(\beta/\lambda^\nu)]\}d\lambda/\lambda.$$

If we make the change of variables $u = G^{-1}(\beta/\lambda^\nu)$, then $G(u) = \beta/\lambda^\nu$ and $G'(u)du = (-\nu\beta/\lambda^{\nu+1})d\lambda$. Hence, for $N \geq N^*$,

$$m_2(E^{f,N}_{\alpha\beta}) \leq [(\beta-\alpha)/\nu\beta]\int_0^{G^{-1}(\beta/N^\nu)} 1/G'(u) \quad G'(u)du$$

$$= [(\beta-\alpha)/\nu\beta]G^{-1}(\beta/N^\nu).$$

To complete the proof that $f \notin$ Class θ for any $\theta < \nu$, we must show that for any $\theta < \nu$ and for any $M > 0$, we have $G^{-1}(\beta/N^\nu) < M(1/N)^\theta$ as N becomes large. Since $\theta < \nu$, we may write $\theta = \gamma\nu$ for some $\gamma \in]0, 1[$. Then, we consider the function n defined as follows:

$n(y) = G^{-1}(y) - (M/\beta^{\gamma})y^{\gamma}$ for $y \in [0, G(w)[$. By (6.3), $n(0) = 0$. Further, for $y \in]0, G(w)[$, we have $n'(y) = 1/G'[G^{-1}(y)] - (\gamma M)/(\beta^{\gamma}y^{1-\gamma})$ by the inverse function theorem. Since $G'(0) > 0$ and $G^{-1}(0) = 0$, we obtain that $\lim_{y\to 0} G'[G^{-1}(y)]/y^{1-\gamma} = \infty$. Therefore, there is a $j > 0$ such that $G'[G^{-1}(y)]/y^{1-\gamma} > \beta^{\gamma}/(\gamma M)$ for all $y \in]0, j[$, i.e., $n'(y) < 0$ on $]0, j[$. Combining this with the fact that $n(0) = 0$, we see that $n(y) < 0$ for all $y \in]0, j[$, or $G^{-1}(y) < (M/\beta^{\gamma})y^{\gamma}$ for $y \in]0, j[$.

Thus, for $N > (\beta/j)^{1/\nu}$, we have $\beta/N^{\nu} < j$ which implies that $G^{-1}(\beta/N^{\nu}) < (M/\beta^{\gamma})(\beta/N^{\nu})^{\gamma} = M(1/N)^{\gamma\nu} = M(1/N)^{\theta}$. In summation, we have demonstrated values of α and β such that for any $M > 0$ and any $\theta < \nu$, we have $m_2(E_{\alpha\beta}^{f,N}) < M(1/N)^{\theta}$ for large N. Together with our previous remarks, this shows that $f \notin$ Class θ for any $\theta < \nu$.

To complete the example, we must show that $f \in$ Class ν. It is clear that f fails to satisfy condition 1 of Definition 5C for any α and β. Hence, we must show that f satisfies condition 2 of Definition 5C for all α and β.

If $0 < \alpha < \beta < 1$, then by the left side of (6.11) and a series of steps similar to those used directly after (6.11), we obtain that for $N \geq N^*$, $m_2(E_{\alpha\beta}^{f,N}) \geq [(\beta-\alpha)/\alpha\nu]G^{-1}(\alpha/N^{\nu})$. Choose $M < 1/G'(0)$ and consider the function $q(y) = G^{-1}(y)-My$ for $y \in [0, G(w)[$. Using the same methods as needed for the function $n(y)$, we see that there is an $s > 0$ such that $q(y) > 0$ for $y \in]0, s[$, i.e., $G^{-1}(y) > My$ for $y \in]0, s[$.

Hence, for large values of N, we obtain the following:

$$m_2(E_{\alpha\beta}^{f,N}) \geq [(\beta-\alpha)/\alpha]G^{-1}(\alpha/N^{\nu}) > [(\beta-\alpha)/\alpha]M(\alpha/N^{\nu})=(\beta-\alpha)M(1/N)^{\nu}.$$

Thus, for $0 < \alpha < \beta < 1$, f satisfies condition 2 of Definition 5C.

When $1 \leq \alpha < \beta < \infty$, we have that $m_1(\{x: \alpha < f(x) < \beta\}) > 0$. By using local versions of Theorem 3.2 and Lemma 5.1, it can be shown that this is sufficient to insure that $m_2(E_{\alpha\beta}^{f,N}) = \infty$ for all N.

Any of the other possible cases for $0 \leq \alpha < \beta < \infty$ may be handled as subcases of the two mentioned above. Finally, since $f_\lambda(x)$ is an odd function for all λ, we see that symmetric results are obtained when $-\infty < \alpha < \beta \leq 0$. Therefore, when $-1 \leq \alpha < \beta \leq 1$, there are numbers N^* and M such that $m_2(E_{\alpha\beta}^{f,N}) \geq M(1/N)^{\nu}$ for $N \geq N^*$. For all other possible cases, we have $m_2(E_{\alpha\beta}^{f,N}) = \infty$ for all N. Thus, $f \in$ Class ν and the example is completed.

The next example illustrates that $\bigcup_{\theta<1}$ Class θ is a proper subset of Class 1. Since the actual details of the example are as intricate as those of the previous example, we will give a brief outline of the steps involved. Where possible, we will refer back to similar procedures used in Example 6.5.

<u>Example 6.6</u>. In this case, we begin by defining an auxiliary distribution S. Let $g(x) = \ln |x|$ for all $x \neq 0$. Then, since g is locally Lebesgue integrable, we may consider the

distribution $S = Dg$. (Note: this distribution is usually referred to as "Pv(1/x)" and is defined by $\langle pv(1/x), \psi\rangle = pv\int_R \psi(x)/x\, dx$, where "pv" denotes Cauchy principal value. For our purposes, it is more convenient to consider S in the above form. A more thorough discussion of this distribution is given in [8], pp. 84-85.).

The next step is to define $G(x) = S*\phi(x)$ and to show that G satisfies conditions similar to (6.3), (6.4), and (6.5). In these steps, we have to make use of Lemmas 6.3 and 6.4. With this accomplished, we define a function $f(x)$ as follows:

$$f(x) = \begin{cases} x - 1 & \text{for} \quad x \geq 1, \\ \ln x & \text{for} \quad 0 < x \leq 1, \\ \ln(-x) & \text{for} \quad -1 \leq x < 0, \\ -x - 1 & \text{for} \quad x \leq -1. \end{cases}$$

Finally, we prove that the distribution $T = Df$ provides the desired example. To do so, we set $f_\lambda(x) = T*\phi_\lambda(x)$ and show that f_λ satisfies conditions similar to those listed in (6.9). Further, it is easily shown that $f_\lambda(x) = \lambda G(\lambda x)$ when $x \in]-1/\lambda, 1/\lambda[$. The remainder of the steps are exactly the same as those of the preceding example. In the end, it can be shown that for $0 < \alpha < \beta < 1$, we have $T \not\leq \alpha$, $T \not\geq \beta$, and for any $\theta < 1$ and any constant M, $m_2(E^{T,N}_{\alpha\beta}) < M(1/N)^\theta$ for large values of N. This demonstrates that $T \notin \bigcup_{\theta<1} \text{Class } \theta$. Also, following the same procedure as used in Example 6.5, we can show that for $-1 \leq \alpha < \beta \leq 1$, there are numbers N^*, M such that $m_2(E_{\alpha\beta}^{T,N}) \geq M(1/N)$ for $N \geq N^*$. For all other cases for α and β, $m_2(E_{\alpha\beta}^{T,N}) = \infty$ for all N. Hence, $T \in$ Class 1 and the example is completed.

Using exactly the same procedures as above, we may also exhibit an example to show that $\bigcup_{\theta<\nu}$ Class $\theta \subsetneq$ Class ν when $1 < \nu < 2$. In this case, we will list only the pertinent distributions.

Example 6.7. Let $\epsilon = \nu - 1$ and define $f(x)$ as follows:

$$f(x) = \begin{cases} -x - (\epsilon + 1)/\epsilon & \text{for} \quad x \leq -1, \\ -[\epsilon(-x)^{\epsilon}]^{-1} & \text{for} \quad -1 \leq x < 0, \\ -[\epsilon x^{\epsilon}]^{-1} & \text{for} \quad 0 < x \leq 1, \\ x - (\epsilon + 1)/\epsilon & \text{for} \quad x \geq 1. \end{cases}$$

Since $0 < \epsilon < 1$, $f(x)$ is locally Lebesgue integrable. The distribution $T = Df$ provides the example for this case.

From the previous examples, it would seem that similar examples can be obtained for the higher classes using appropriate derivatives of adaptations of the functions $\ln |x|$ and $|x|^{-\nu}$. Unfortunately, this process fails for derivatives of order higher than 1. In these cases, it is impossible to adapt the above functions in such a way as to obtain results similar to (6.9). Since these conditions were crucial in the demonstration of the previous examples, it would appear that the approach used there will be ineffective in these cases. At present, the question of whether straightforward examples exist for the higher classes must remain open.

CHAPTER VII

CONCLUSIONS, COMMENTS AND CONJECTURES

As indicated in Chapter I, the purpose of this work was to study distributions by analyzing nets of infinitely differentiable functions which "represent" the various distributions. Following the example of Mikusinski and Temple, we saw that the natural representation to consider for a distribution T was the net $\{T*\phi_\lambda\}$, where $\{\phi_\lambda\}$ is the family defined by (2.1) and (2.2). We then showed that it is possible to define classes of distributions based on a Denjoy-type analysis of the net $\{T*\phi_\lambda\}$. The examples of Chapter VII serve to illustrate that these classes were actually distinct in at least certain important cases. The major result of this work was that a large number of distributions may be placed in the appropriate class solely on the basis of their primitives.

In essence, this paper represents an attempt to use a relatively new approach for the classification of distributions. Therefore, it is not unusual that the questions raised by this research far outnumber the answers found. With this in mind, we will discuss some of these questions and attempt to indicate their importance to the study. Where feasible, we will try to make known some possible approaches to solve these problems.

First of all, it is somewhat natural to question the dependence of this theory on the test function ϕ. For instance, if we used two different test functions satisfying (2.1), would we necessarily obtain the same classes in both cases? This question takes on added importance when we take into account the fact that we placed the additional condition (6.1) on ϕ in order to work the examples of the last chapter. If different ϕ's give us different classes, then it is possible that the distinctions between classes indicated by the examples of Chapter VI are still unsettled in the case where ϕ does not satisfy (6.1).

As far as we can tell, this problem cannot be answered at present. Although it is true that all the theory can be formulated for any test function ϕ satisfying (2.1), we can see nothing to guarantee that the individual classes will be the same in each case. However, it is certain that the results of Theorem 5.7 are valid independent of ϕ. The situation may well be analogous to that of the spaces $L^p(\mu)$, the L^p spaces defined by the measure μ. If μ is changed, it is possible that the L^p classes are changed. In any circumstance however, there are always some functions which are in a fixed L^p class regardless of the measure μ.

Thus, a partial answer to this question is that while a different theory may result from different choices of ϕ, we are sure that Theorem 5.7 holds for any ϕ satisfying (2.1). Further, there are at least some choices of ϕ for which the lower classes can be proven to be distinct. Hence, the approach does provide results which are consistent with our original aims.

A second question that arises refers back to the basic description of the methods used in this work. Essentially, the family $\{\phi_\lambda\}$, which was chosen for the regularizations, consists of extremely well behaved functions. In addition, the fact that all the ϕ_λ's were obtained by a change of variables from the single function ϕ was crucial to much of our work. A possible generalization of this approach would be to see how much of the theory can be retrieved if we begin with an arbitrary net $\{\phi_\lambda\}$ for which $T*\phi_\lambda \to T$ in $\mathcal{D}'$ as $\lambda \to \infty$ for each distribution T. The major problems encountered with this change are the following: 1. The continuity of the function Γ_T on R_2^+ is no longer obvious; 2. Lemmas 5.3, 5.4, and 5.5 become doubtful, and hence, Theorem 5.7 is in danger. This is not to say that such a theory is impossible. However, this slightly different approach will require new proofs for some of the crucial results demonstrated in our work.

Next, we turn to some more specific questions concerning the classes defined in our theory. For one thing, the trend of our results leans heavily on sufficient conditions for the various classes. It is unknown at present whether necessary conditions exist. Along the same lines, it is not apparent that the θ-classes are all-inclusive. On the basis of the research done up to now, it is entirely possible that there may exist a distribution which belongs to none of these classes. However, we are assured by Theorem 5.7 that every distribution which has a measure as one of its primitives may be placed in one of the θ-classes.

While we are on the subject, Theorem 5.7 suggests an obvious conjecture. The theorem tells us that if T is a measure or a function, then T satisfies the following statement:

(7.1) If $T \in \text{Class } \theta$, then $D^{(n)}T \in \text{Class } (\theta + n)$.

Whether (7.1) is true for an arbitrary distribution is still an open question.

Our final considerations pertain to the examples given in Chapter VI. Example 6.5 shows that for $0 < \nu < 1$, there is a continuous function h such that Dh is in Class ν, but $Dh \notin \bigcup_{\theta<\nu} \text{Class } \theta$. (The function h is the indefinite Lebesgue integral of the function f defined in (6.7).) Now, Theorem 5.7 tells us that all first derivatives of continuous functions lie in Class 1. Example 6.6 is based on the derivative of a locally L^1 function. Hence, it is natural to ask whether there is a continuous function $p(x)$ such that $Dp \in \text{Class } 1$ but $Dp \notin \bigcup_{\theta<1} \text{Class } \theta$. We conjecture that such an example might be fashioned from the function

$$p(x) = \begin{cases} 1/\ln |x| & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Unfortunately, for this function, the details of Example 6.6 are not readily accomplished. Thus, for the present, this remains as another unanswered question.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. J. Clarkson, "A property of derivatives", Bull. Amer. Math. Soc. 53 (1947) 124-125.

2. A. Denjoy, "Sur une propriété des fonctions dérivées", Enseignement Math. 18 (1916) 320-328.

3. J. Horváth, Topological Vector Spaces and Distributions, (Addison-Wesley, 1966).

4. S. Lojasiewicz, "Sur la valeur et la limite d'une distribution en un point", Studia Math. 16 (1957) 1-36.

5. E. J. McShane, Integration, (Princeton University Press, 1944).

6. J. Mikusinski, "Sur la méthode de généralisation de M. Laurent Schwartz et sur la convergence faible", Fund. Math. 35 (1948) 235-239.

7. W. Rudin, Real and Complex Analysis, (McGraw-Hill, Inc., 1966).

8. L. Schwartz, Mathematics for the Physical Sciences, (English translation: Addison-Wesley Publishing Company. Original French Edition: Hermann, 1966).

9. L. Schwartz, Théorie des Distributions, (Hermann, 1966).

10. G. Temple, "The theory of generalized functions", Proc. Roy. Soc. London. Ser. A. 228 (1955) 175-190.

11. F. Treves, Topological Vector Spaces, Distributions, and Kernels, (Academic Press, 1967).