GENERALIZATION OF TAUB'S RELATIVISTIC RANKINE-HUGONIOT EQUATIONS

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ABSTRACT

GENERALIZATION OF TAUB'S RELATIVISTIC RANKINE-HUGONIOT EQUATIONS

by Ahmed S. El-Ariny

The present work refers to the relativistic hydrodynamics in the presence of the gravitational field. The velocity of the propagation of signals is assumed to be a variable in accordance with the proposition by Einstein (1907), Fok (1955) and others. The present approach is a generalization of Taub's work in vacuo. The fluid is considered to be a collection of particles in random motion and under the influence of a gravitational field. Only ideal fluid is taken into account. In this case transport physical aspects like viscosity, heat conductivity, etc., are to be disregarded. The flow governing equations (continuity, momentum and energy) are based on the Kinetic theory of gases by means of Boltzmann equation. The space-time based on a variable velocity of propagation of signals is necessarily Riemannian one. Due to the difficulties that arise in establishing solutions of flow problems in the Riemannian space-time, an approximation is suggested by introducing a piece-wise constant velocity of signals. This enables us to obtain, in the Euclidean space-time, solutions which are valid only at a point. The entire formalism of Taub is transferred to the Euclidean space-time where the velocity of the propagation of signals is less than that in vacuo. It is shown that actually the Taub procedure is transferable to the present case with small modification involving constant parameters. To demonstrate the discrepancy between the present approach and the possible one in the Riemannian space-time, in Chapter III are some equations which show clearly the simplification which must be applied to reduce the problem to the Euclidean space-time. To illustrate the theory a numerical example is calculated.

GENERALIZATION OF TAUB'S RELATIVISTIC RANKINE-HUGONIOT EQUATIONS

By

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To my wife

Malak,

with deep appreciation

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NOMENCLATURE

a	velocity of sound
$a_{\sigma\rho}$	covariant form of the metric tensor of the Riemannian space-time-{x}
$\bar{a}_{\sigma\rho}$	covariant form of the metric tensor of the Riemannian space-time-{$\bar{x}$}
$A_{\sigma\rho}$	covariant form of the metric tensor of the Riemannian space-time-{X}
$(b^{\sigma}_{\rho})_{o}$	constant coefficients evaluated at point "0", (p. 16)
c	velocity of light signal in vacuo.
$D_{\pm}$	operators (p. 39)
$f(x^{j},t,\zeta^{j})$	the distribution function in the Riemannian space-time-{x}
$f(y^{j},t,\xi^{j})$	the distribution function in the Riemannian space-time-{y}
$f(\phi)$	an arbitrary function of ϕ (p. 41)
$F^{\alpha}_{(x)}$	four-force vector in the space-time-{x}
$F^{\alpha}_{(\bar{x})}$	four-force vector in the space-time-{$\bar{x}$}
$F^{\alpha}_{(X)}$	four-force vector in the space-time-{X}
$F^{\alpha}_{(y)}$	four-force vector in the space-time-{y}
$F^{\alpha}_{(\bar{y})}$	four-force vector in the space-time-{$\bar{y}$}
$F^{\alpha}_{(Y)}$	four-force vector in the space-time-{Y}
$\bar{F}^{\alpha}$	components of the forces and the work done by them obtained by differentiation with respect to the time t

$\mathcal{F}^{*\alpha}$	modified four-force vector
$\mathcal{F}^{j}$	components of the external force per unit mass (p. 26)
$g_{\sigma\rho}$	covariant form of the metric tensor of the Euclidean space-time-{y}
$\overline{g}_{\sigma\rho}$	covariant form of the metric tensor of the Euclidean space-time-{$\overline{y}$}
$G_{\sigma\rho}$	covariant form of the metric tensor of the Euclidean space-time-{Y}
I	velocity of propagation of signals in the presence of the gravitational field
I_o	velocity of propagation of signals I evaluated at a particular point "0"
K_o	constant of integration (p. 78)
i,j,k,l	indices running over the range 1,2,3
$\mathcal{L}$	Langrangian
lb_f	pound force
lb_m	pound mass
m_o	rest mass of a particle in vacuo
m	mass of a particle in space-time-{y}
$\overline{m}$	mass of a particle in space-time-{$\overline{y}$}
m_w	molecular weight of the gas
M	mass of a particle in the space-time-{x}
M	mass flux across the shock (p. 43)
$\overline{M}$	mass of a particle in the space-time-{$\overline{x}$}
n	an integer = 1,2,3,...(p. 74)
n	number of particles per unit volume ($n=\int f d_3\xi$) as measured with respect to fixed coordinates - {y} (or {x})

n^{o}	number density of the particles as measured by an observer moving with the velocity $\bar{u}^{j}$ with respect to the fixed coordinates $\{y\}$ (or $\{x\}$)
p	hydrostatic pressure (p. 34)
p^{α}	four-momentum vector in the space-time-$\{y\}$
$\bar{p}^{\alpha}$	four-momentum vector in the space-time-$\{\bar{y}\}$
P^{α}	four-momentum vector in the space-time-$\{x\}$
$\bar{P}^{\alpha}$	four-momentum vector in the space-time-$\{\bar{x}\}$
q	magnitude of the particle spatial velocity whose components = q^{j}
q^{j}	the particle velocity components measured with respect to the x-coordinates
$\bar{q}$	magnitude of the particle spatial velocity whose components = $\bar{q}^{j}$
$\bar{q}^{j}$	the particle velocity components measured with respect to the $\bar{x}$-coordinates
Q	heat input into the system from the outside
°R	degree Rankine
$\mathcal{R}$	universal gas constant
r,s	local relativistic analogs of the Riemann functions
$\mathscr{s}$	$d\mathscr{s}$ = element of arc length in the four-dimensional space-time
S	entropy as measured by an observer at rest with respect to the fluid
$T^{\alpha\beta}$	contravariant form of the energy-momentum tensor
$T^{*\alpha\beta}$	contravariant form of the energy-momentum tensor including the effect of the external forces

u	dimensionless velocity in one dimensional flow in the Euclidean space-time referred to I_o $(u = \bar{u}^1 I_o^{-1})$
u^α	dimensionless four-velocity vector of a flow, $(u^\alpha = n^{o-1} U^\alpha)$
$\bar{u}^j$	mean value of the spatial components of the particle velocity v^j
$U^\alpha{}_{(x)}$	mass-current vector in the space-time-{x}
$U^\alpha{}_{(y)}$	mass-current vector in the space-time-{y}
v	magnitude of the particle spatial velocity whose components = v^j
v^j	the particle velocity components measured with respect to the y-coordinates
$\bar{v}$	magnitude of the particle spatial velocity whose components = $\bar{v}^j$
$\bar{v}^j$	the particle velocity components measured with respect to the $\bar{y}$-coordinates
$V^\alpha{}_{(X)}$	the four-velocity vector in the Riemannian space-time-{X}
$V^\alpha{}_{(Y)}$	the four-velocity vector in the Riemannian space-time-{Y}
w	dimensionless velocity in one dimensional flow referred to I in the space-time-{x} $(w = \bar{w}^1 I^{-1})$
$\bar{w}^j$	mean value of the spatial components of the physical velocity q^j of the gas particles
w^α	dimensionless four-velocity vector of a flow in the space-time-{x} $(w^\alpha = n^{o-1} U^\alpha(x))$
x^α	coordinates of the Riemannian space-time-{x}, $x^4 = t$
$\bar{x}^\alpha$	coordinates of the Riemannian space-time-{$\bar{x}$}, $\bar{x}^4 = ct$
X^α	coordinates of the Riemannian space-time-{X}, $X^j = c^{-1} x^j$, $X^4 = x^4$

y^{α}	coordinates of the Euclidean space-time-$\{y\}$, $y^4 = t$
$\bar{y}^{\alpha}$	coordinates of the Euclidean space-time-$\{\bar{y}\}$, $\bar{y}^4 = ct$
Y^{α}	coordinates of the Euclidean space-time-$\{Y\}$, $Y^j = c^{-1}y^j$, $Y^4 = y^4$
$Y^{(n)}$	distance from the celestial body (p. 79)
Y'^{α}	coordinates of the Euclidean space in vacuo (corresponding to Taub's work)
α	dimensionless sound velocity referred to I_o
$\alpha,\beta,\lambda,$ μ,ν,ρ,σ	integers running over the range 1,2,3,4
β	a function of the shock parameters (p. 44)
γ	specific heat ratio
$\Gamma_{\pm}$	the characteristics of the governing equations of the one-dimensional flow in $(y,I_o t)$-plane
$\delta_{\pm}$	operators (p. 71)
ε	Internal energy of the fluid per unit mass
$\varepsilon^*(\bar{\varepsilon}^*)$	real energy of the particle in x- or y-coordinates ($\bar{x}$- or $\bar{y}$-coordinates)
$\varepsilon_t^*(\bar{\varepsilon}_t^*)$	total energy of the particle in x- or y-coordinates ($\bar{x}$- or $\bar{y}$-coordinates)
ζ^{α}	four-velocity vector in the space-time-$\{x\}$
$\bar{\zeta}^{\alpha}$	four-velocity vector in the space-time-$\{\bar{x}\}$
η	density ratio ($\eta = \rho_+^o \rho_-^{o-1}$)
Θ	temperature
μ	$\mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1})$, (p. 37)
	$d\mu = (1 + \xi^2 c^{-2})^{-\frac{1}{2}} f d_3\xi$,
	$[d\mu = (1 + \zeta^2 c^{-2})^{-\frac{1}{2}} f d_3\zeta]$ (pp. 28 and 58)

ξ	pressure ratio ($\xi = p_+ \, p_-^{-1}$)
ξ^α	four-velocity vector in the space-time-$\{y\}$
$\bar{\xi}^\alpha$	four-velocity vector in the space-time-$\{\bar{y}\}$
$\Pi^{\alpha\beta}$	a second order tensor whose covariant derivative represents the external forces
ρ	density of the gas as measured with respect to a fixed coordinate system, $\rho = nm_o$
ρ^o	density of the gas as measured by an observer moving with the mean velocity with respect to a fixed coordinate system $\rho^o = n^o m_o$
ρ^{o*}	normalized density
τ	$d\tau$ = arc length in the four-dimension space-time
Φ	$\Phi = \Phi\,(\chi^j, t, \zeta^j)$ a transport quantity in the $\{x\}$-space time
Φ	$\Phi = \Phi\,(y^j, t, \xi^j)$ a transport quantity in the $\{y\}$-space time
ϕ	$d\phi = \alpha\rho^{o-1} d\rho^o$
χ	gravitational potential
ψ^σ	summational invariant functions

INTRODUCTION

Relativistic theory of fluid dynamics with the reference velocity of propagation of signals in vacuo has been developed by Taub [18] and Synge [15]. The former based his work on the Kinetic theory by means of the relativistic Boltzmann transport equation, while the latter based his formulations on Maxwell's Boltzmann statistics. Additional aspects of the microscopic approach with the reference velocity of propagation of signals in vacuo have been investigated by various authors: Goto [8], Vlasov [20] and others.

Attention is called to the meaning of the word "vacuo" used in the above literature as it was demonstrated by Pauli [14] that in perfectly empty space no gravitational field exists. According to Einstein's article in 1907 [6], and the discussion presented by Fok [7], the effect of the gravitational field upon the form of the metric of the space-time may be taken into account. Thus, the constant reference velocity of propagation of signals in vacuo is to be replaced by a variable one in the presence of the gravitational field.

It must be emphasized very strongly that the symbols c and I do not necessarily denote and refer to the velocity

of light. Actually, they may refer to the velocity of any other signal. It may be optical, electromagnetic waves, etc. Already Pauli [14] mentioned about other possible signals. It is well known that the velocity of electromagnetic waves depends very strongly upon environmental conditions.

In the present work, we introduce a generalization to Taub's work under the influence of the gravitational field. Thus, the present approach refers to a space-time with a reference velocity of signals which is considered to be a function of the spatial coordinates. Following the analysis of Synge and Schild [17], local rectangular coordinates are used in the Euclidean space-time to provide simplified solutions that are valid only at a point. Therefore, by introducing a piece-wise constant reference velocity of signals, an approximate solution in the Riemannian space-time is obtained from solutions involving constant parameters in the Euclidean space-time.

In Chapter I, we demonstrate the basic principle upon which the configuration of the space-time in question is based. The relativistic quantities, being used in the following chapters are developed in accordance with the chosen forms of the metric of the space-time. In addition, several transformations of coordinates are established. The modification of Taub's work in accordance with the present theory is presented in Chapter II. The fundamental governing equations of a flow in the Riemannian space-time

are derived in Chapter III by making use of the Boltzmann equation. It is shown that the Riemannian space-time, due to its curvature, results in the appearance of additional terms in the differential equations governing a flow. These additional terms present the main difference between the formalism in the Riemannian space-time and its correspondence in the Euclidean one. However, it is shown also that the entire formulations obtained in Chapter III are reducible to their correspondence in Chapter II upon replacing the variable reference velocity of signals by a constant one evaluated at a particular point, i.e., by a piece-wise constant reference velocity.

A numerical solution of an initial value problem in one dimensional flow and the calculations of some shock parameters are presented in Chapter IV as an illustrative example demonstrating the influence of the gravitational field.

CHAPTER I

FUNDAMENTALS OF THE APPLIED RELATIVISTIC MODEL

1.1. Second Einstein Model of the Special Relativity

In 1905 Einstein proposed his first model of the special relativity and in 1907 his second one in which he took into account the effect of the gravitational field on the velocity of the propagation of signals. Einstein [6][1] derived the formula for the metric of the space-time using the expression

$$(ds)^2 = -(dx^2 + dy^2 + dz^2) + [c(1 + \chi c^{-2})]^2 \, dt^2, \quad (1.1.1)$$

where,

χ = gravitational potential; x, y, z being the space coordinates,

t = the time coordinate, and c is the velocity of signals in vacuo.

1.2. Fok's Model

Fok [7] proposes the following two metrics of the space-time in the presence of the gravitational field

---

[1]Numbers in square brackets refer to the bibliography of standard works.

$$(ds)^2 = -(dx^2 + dy^2 + dz^2) + (c^2 - 2\chi)dt^2, \quad (1.2.1)$$

or

$$(ds)^2 = -(1 + 2\chi c^{-2})(dx^2 + dy^2 + dz^2) + (c^2 - 2\chi)dt^2. \quad (1.2.2)$$

The gravitational potential, χ, appearing in the above metrics, as considered by Fok, should be small. The first metric, (1.2.1), has a special interest in the present work.

1.3. The Metric Tensor of the Four-dimensional Space-Time

Following the models of the special relativity discussed above, let us introduce the function $I = I(\bar{x}^j)$, $j = 1,2,3$, defined as

$$I^2 = c^2 - 2\chi. \quad (1.3.1)$$

The function I represents the maximum velocity of the propagation of signals in the presence of the external fields of action.

The metric of the four-dimensional $\{\bar{x}\}$-space-time is chosen in the form

$$-(ds)^2 = \bar{a}_{jk} d\bar{x}^j d\bar{x}^k - c^{-2} I^2 (d\bar{x}^4)^2$$

$$= \bar{a}_{\sigma\rho} d\bar{x}^\sigma d\bar{x}^\rho, \quad (1.3.2)$$

where

$$\bar{a}_{11} = \bar{a}_{22} = \bar{a}_{33} = 1,\ \bar{a}_{44} = -c^{-2}I^2,\ \bar{a}_{\sigma\rho} = 0$$

$$\text{for } \sigma \neq \rho . \tag{1.3.3}$$

$\bar{x}^j$ being the spatial coordinates, and

$\bar{x}^4 = ct.$

Throughout the present work, we use Latin suffixes for the range 1, 2, 3, and Greek suffixes for the range 1, 2, 3, 4, unless otherwise stated. The contravariant components of the metric tensor, $\bar{a}^{\sigma\rho}$, in $\bar{a}^{\sigma\rho}\, d\bar{x}_\sigma d\bar{x}_\rho$ are

$$\bar{a}^{11} = \bar{a}^{22} = \bar{a}^{33} = 1,\ \bar{a}^{44} = -\, c^2 I^{-2},\ \bar{a}^{\sigma\rho} = 0$$

$$\text{for } \sigma \neq \rho . \tag{1.3.4}$$

From (1.3.2) we obtain

$$\frac{dt}{ds} = [I(1 - \bar{q}^2 I^{-2})^{\frac{1}{2}}]^{-1},\ \bar{q}^2 = \bar{a}_{jk}\bar{q}^j\bar{q}^k,\ \bar{q}^j = \frac{d\bar{x}^j}{dt}. \tag{1.3.5}$$

1.4. Fundamental Mathematical Operations

Assume a metric in $\{x\}$-space-time:

$$(ds)^2 = a_{\sigma\rho} dx^\sigma dx^\rho , \tag{1.4.1}$$

with a metric tensor, $a_{\sigma\rho}$, whose components are functions of the coordinates, x^σ. The absolute derivative, denoted by the symbol $\frac{\delta}{\delta s}$; (s being a parameter), of a first order contravariant tensor, T^σ, is defined by

$$\frac{\delta T^{\sigma}}{\delta s} = \frac{dT\sigma}{ds} + \{^{\sigma}_{\mu\nu}\}T^{\mu}\frac{dx^{\nu}}{ds}\ , \qquad (1.4.2)$$

where, $\frac{dT\sigma}{ds}$, denotes the ordinary derivative of T^{σ} with respect to s, whereas, $\{^{\sigma}_{\mu\nu}\}$, stands for the Christoffel symbol of the second kind which is defined by

$$\{^{\sigma}_{\mu\nu}\} = \tfrac{1}{2}a^{\sigma\rho}\left(\frac{\partial a_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial a_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial a_{\mu\nu}}{\partial x^{\rho}}\right) = a^{\sigma\rho}[\mu\nu,\rho], \qquad (1.4.3)$$

where $[\mu\nu,\rho]$ = the Christoffel symbol of the first kind. The absolute derivative of the corresponding covariant tensor, T_{σ}, is

$$\frac{\delta T_{\sigma}}{\delta s} = \frac{dT_{\sigma}}{ds} - \{^{\mu}_{\sigma\rho}\}T_{\mu}\frac{dx^{\nu}}{ds}\ . \qquad (1.4.4)$$

The absolute derivatives of the second order contravariant and covariant tensors, $T^{\sigma\rho}$ and $T_{\sigma\rho}$, are given respectively by

$$\frac{\delta T^{\sigma\rho}}{\delta s} = \frac{dT^{\sigma\rho}}{ds} + \{^{\sigma}_{\mu\nu}\}\ T^{\mu\rho}\frac{dx^{\nu}}{ds} + \{^{\rho}_{\mu\nu}\}\ T^{\mu\sigma}\frac{dx^{\nu}}{ds}\ ; \qquad (1.4.5)$$

$$\frac{\delta T_{\sigma\rho}}{\delta s} = \frac{dT_{\sigma\rho}}{ds} - \{^{\mu}_{\sigma\nu}\}\ T_{\mu\rho}\frac{dx^{\nu}}{ds} - \{^{\mu}_{\rho\nu}\}\ T_{\mu\sigma}\frac{dx^{\nu}}{ds}\ . \qquad (1.4.6)$$

The covariant derivatives of the tensors T^{σ}, T_{σ}, $T^{\sigma\rho}$ and $T_{\sigma\rho}$, with respect to the coordinate x^{ν} are defined respectively by, (notice the proper notation):

$$T^{\sigma}|_{\nu} = T^{\sigma}_{,\nu} + \{^{\sigma}_{\mu\nu}\}\, T^{\mu}, \quad \frac{\partial T^{\sigma}}{\partial x^{\nu}} \equiv T,^{\sigma}_{x\nu} \equiv T^{\sigma}_{,\nu} \;; \tag{1.4.7}$$

$$T_{\sigma}|_{\nu} = T_{\sigma,\nu} - \{^{\mu}_{\sigma\nu}\}\, T_{\mu} \;; \tag{1.4.8}$$

$$T^{\sigma\rho}|_{\nu} = T^{\sigma\rho}{}_{,\nu} + \{^{\sigma}_{\mu\nu}\}\, T^{\mu\rho} + \{^{\rho}_{\mu\nu}\}\, T^{\mu\sigma} \;; \tag{1.4.9}$$

$$T_{\sigma\rho}|_{\nu} = T_{\sigma\rho,\nu} - \{^{\mu}_{\sigma\nu}\}\, T_{\mu\rho} - \{^{\mu}_{\rho\nu}\}\, T_{\mu\sigma} \,. \tag{1.4.10}$$

When the metric of the space-time $\{x\}$ assumes the form (1.3.2), i.e.,

$$-(d\mathscr{s})^2 = \bar{a}_{jk} d\bar{x}^j d\bar{x}^k - c^{-2} I^2 (d\bar{x}^4)^2$$

$$= \bar{a}_{\sigma\rho} d\bar{x}^{\sigma} d\bar{x}^{\rho}, I = I(\bar{x}^j) \quad , \tag{1.4.11}$$

the computation of the non-vanishing components of the Christoffel symbols gives

$$\{^{j}_{44}\} = \tfrac{1}{2}\bar{a}^{kj}\,(c^{-2}I^2)_{,k}, \{^{4}_{4j}\} = \tfrac{1}{2}(c^2 I^{-2})(c^{-2}I^2)_{,j}$$

$$= \tfrac{1}{2}[\ln(c^{-2}I^2)^2]_{,j} \;. \tag{1.4.12}$$

Hence, carrying out the computations of the components of the absolute derivative (1.4.2) using (1.4.12) we obtain

$$\frac{\delta T^j}{\delta \mathscr{s}} = \frac{dT^j}{d\mathscr{s}} + \frac{1}{2}\,\bar{a}^{kj}(c^{-2}I^2)_{,k} \;; \tag{1.4.13}$$

$$\frac{\delta T^4}{\delta s} = \frac{dT^4}{ds} + \frac{1}{2}\,[\ln(c^{-2}I^2)]_{,j}\; T^j\,\frac{d\bar{x}^4}{ds}$$

$$+ \frac{1}{2}\,[\ln(c^{-2}I^2)]_{,j}\; T^4\,\frac{d\bar{x}^j}{ds}\;. \qquad (1.4.14)$$

Similarly,

$$\frac{\delta T_j}{\delta s} = \frac{dT_j}{ds} - \frac{1}{2}\,[\ln(c^{-2}I^2)]_{,j}\; T_4\,\frac{d\bar{x}^4}{ds}\;; \qquad (1.4.15)$$

$$\frac{\delta T_4}{\delta s} = \frac{dT_4}{ds} - \frac{1}{2}\,\bar{a}^{kj}(c^{-2}I^2)_{,k}T_j\,\frac{d\bar{x}^4}{ds}\;. \qquad (1.4.16)$$

The covariant derivatives in (1.4.7) to (1.4.9), with the contraction $\nu = \sigma$, become

$$T^\sigma|_\sigma = T^\sigma{}_{,\sigma} + \frac{1}{2}\,[\ln(c^{-2}I^2)]_{,j}T^j\;; \qquad (1.4.17)$$

$$T^{k\sigma}|_\sigma = T^{k\sigma}{}_{,\sigma} + \frac{1}{2}\,[\ln(c^{-2}I^2)]_{,j}T^{kj}$$

$$+ \frac{1}{2}\,\bar{a}^{kj}(c^{-2}I^2)_{,j}T^{44}\;; \qquad (1.4.18)$$

$$T^{4\sigma}|_\sigma = T^{4\sigma}{}_{,\sigma} + \frac{3}{2}\,[\ln(c^{-2}I^2)]_{,j}T^{4j}\;. \qquad (1.4.19)$$

1.5. Relativistic Mechanics of a Particle in the Four-Dimensional Space-Time

The metric, (1.3.2), of the four-dimensional space-time $\{\bar{x}\}$ is used primarily in this section.

(i) Velocity

The contravariant four-velocity vector is defined by

$$\bar{\zeta}^{\lambda} = \frac{d\bar{x}^{\lambda}}{d\mathcal{s}} \quad . \tag{1.5.1}$$

Using (1.3.5) and (1.5.1) the contravariant and covariant components of the four-felocity vector are

$$\bar{\zeta}^{j} = \bar{q}^{j}[I(1 - \bar{q}^{2}I^{-2})^{\frac{1}{2}}]^{-1} \quad ,$$

$$\bar{\zeta}^{4} = \bar{q}^{4}[I(1 - \bar{q}^{2}I^{-2})^{\frac{1}{2}}]^{-1} \quad ; \tag{1.5.2}$$

$$\bar{\zeta}_{\sigma} = \bar{a}_{\sigma\rho}\bar{\zeta}^{\rho} \quad , \tag{1.5.3}$$

where, $\bar{q}^{4} = \frac{d\bar{x}^{4}}{dt} = c$.

The magnitude of the four-velocity vector is given by

$$\bar{a}_{\sigma\rho}\bar{\zeta}^{\sigma}\bar{\zeta}^{\rho} = -1 \quad . \tag{1.5.4}$$

Since the absolute derivative of the metric tensor vanishes, the absolute derivative of (1.5.4) with respect to $\mathcal{s}$ gives

$$\bar{\zeta}^{\sigma}\frac{\delta\bar{\zeta}_{\sigma}}{\delta\mathcal{s}} = 0 \quad . \tag{1.5.5}$$

(ii) Lagrangian and Momentum

Let us introduce the Lagrangian function in the form

$$\mathcal{L} = \frac{1}{2}\, m_{o}\, c^{2}\bar{a}_{\sigma\rho}\bar{\zeta}^{\sigma}\bar{\zeta}^{\rho} \quad . \tag{1.5.6}$$

The four-momentum vector is defined by

$$\overline{P}_\sigma = \frac{\partial \mathcal{L}}{\partial \overline{\zeta}^\sigma} = m_o c^2 \overline{a}_{\sigma\rho} \overline{\zeta}^\rho \quad . \tag{1.5.7}$$

Substituting (1.3.3) and (1.5.2) into (1.5.7) we obtain

$$\overline{P}_j = cm_o[c^{-1}I(1 - \overline{q}^2 I^{-2})^{\frac{1}{2}}]^{-1}\overline{q}_j \quad ;$$

$$\overline{P}_4 = -m_o[c^{-1}I(1 - \overline{q}^2 I^{-2})^{\frac{1}{2}}]^{-1} I^2 \quad . \tag{1.5.8}$$

Let us define the inertial relativistic mass by

$$\overline{M} = m_o[c^{-1}I(1 - \overline{q}^2 I^{-2})^{\frac{1}{2}}]^{-1} \quad . \tag{1.5.9}$$

Hence, the components of the four-momentum vector take forms

$$\overline{P}_j = c\overline{M}\overline{q}_j \quad , \quad \overline{P}_4 = -\overline{M}I^2 \quad ; \tag{1.5.10}$$

$$\overline{P}^j = \overline{a}^{j\sigma}\overline{P}_\sigma = m_o c^2 \overline{\zeta}^j = c\overline{M}\overline{q}^j \quad , \quad \overline{P}^4 = \overline{a}^{4\sigma}\,\overline{P}_\sigma = \overline{a}^{44}\overline{P}_4 = \overline{M}c^2 \quad . \tag{1.5.11}$$

(iii) <u>Force</u>

The contravariant four-force vector is defined by

$$F(\overset{\sigma}{\overline{x}}) = \frac{\delta \overline{P}^\sigma}{\delta \mathcal{J}} \quad . \tag{1.5.12}$$

Using (1.4.13), with T^j replaced by $\overline{P}^j$, the first three contravariant components of the four-force vector are

$$F^j_{(\bar{x})} = \frac{d\bar{P}^j}{d\mathcal{s}} + \frac{1}{2}\,\bar{a}^{jk}(c^{-2}I^2)_{,k}\bar{P}^4\frac{d\bar{x}^4}{d\mathcal{s}}$$

$$= [\frac{d\bar{P}^j}{dt} + \frac{1}{2}\,\bar{a}^{jk}(c^{-2}I^2)_{,k}\bar{P}^4 c]\frac{dt}{d\mathcal{s}} \quad . \qquad (1.5.13)$$

Substituting (1.3.5) and (1.5.11) for $\frac{dt}{d\mathcal{s}}$ and $\bar{P}^\sigma$ in (1.5.13) we obtain

$$F^j_{(\bar{x})} = [c^{-1}I(1 - \bar{q}^2I^{-2})^{\frac{1}{2}}]^{-1}[\frac{d}{dt}(\overline{Mq}^j) + \frac{1}{2}\,\bar{a}^{jk}\bar{M}(I^2)_{,k}]$$

$$= [c^{-1}I(1 - \bar{q}^2I^{-2})^{\frac{1}{2}}]^{-1}\bar{F}^j_{(\bar{x})} \quad , \qquad (1.5.14)$$

where we define the physical spatial force by

$$\bar{F}^j_{(x)} = \frac{d}{dt}(\overline{Mq}^j) + \frac{1}{2}\,\bar{a}^{jk}\bar{M}(I^2)_{,k} = \frac{\delta}{\delta t}(\overline{Mq}^j) \quad . \qquad (1.5.15)$$

The first three covariant components of the four-force vector are

$$F_j(\bar{x}) = \bar{a}_{j\sigma}\overset{\sigma}{F}(\bar{x}) = [c^{-1}I(1 - \bar{q}^2I^{-2})^{\frac{1}{2}}]^{-1}\bar{F}_j(\bar{x}) \quad , \qquad (1.5.16)$$

where

$$\bar{F}_j(\bar{x}) = \bar{a}_{j\sigma}\overset{\sigma}{\bar{F}}(\bar{x}) = \frac{\delta}{\delta t}(\overline{Mq}_j) \quad . \qquad (1.5.17)$$

Similarly, using (1.4.16), with $\bar{P}_4$ replacing T_4, the covariant fourth component of the four-force vector is

$$F_4(\bar{x}) = \frac{d\bar{P}_4}{d\mathcal{s}} - \frac{1}{2}\,\bar{a}^{jk}(c^{-2}I^2)_{,k}\bar{P}_j\frac{d\bar{x}^4}{d\mathcal{s}} - \frac{1}{2}\,c^2I^{-2}(c^{-2}I^2)_{,j}\bar{P}_4\frac{d\bar{x}^j}{d\mathcal{s}} \quad . \tag{1.5.18}$$

Substituting (1.3.3), (1.5.1), (1.5.7) into (1.5.18) we get

$$F_4(\bar{x}) = \frac{d\bar{P}_4}{d\mathcal{s}} - \frac{1}{2}(c^{-2}I^2)_{,k}(m_o c^2\bar{\zeta}^k\bar{\zeta}^4)$$

$$+ \frac{1}{2}(c^{-2}I^2)_{,k}(m_o c^2\bar{\zeta}^k\bar{\zeta}^4) = \frac{d\bar{P}_4}{d\mathcal{s}} \quad . \tag{1.5.19}$$

Hence,

$$F_4(\bar{x}) = \frac{d\bar{P}_4}{d\mathcal{s}} = \frac{d\bar{P}_4}{dt}\frac{dt}{d\mathcal{s}} = -[\frac{d}{dt}(\bar{M}I^2)]\frac{dt}{d\mathcal{s}}$$

$$= [I(1 - \bar{q}^2I^{-2})^{\frac{1}{2}}]^{-1}\bar{F}_4(\bar{x}) \quad , \tag{1.5.20}$$

where we define

$$\bar{F}_4(\bar{x}) = -\frac{d}{dt}(\bar{M}I^2) \quad . \tag{1.5.21}$$

The contravariant fourth component of the four-force vector is given by

$$\overset{4}{F}(\bar{x}) = \bar{a}^{4\sigma}F_\sigma(\bar{x}) = \bar{a}^{44}F_4(\bar{x}) = -c^2I^{-2}F_4(\bar{x})$$

$$= [I(1 - \bar{q}^2I^{-2})^{\frac{1}{2}}]^{-1}c^2I^{-2}\frac{d}{dt}(\bar{M}I^2) \quad , \tag{1.5.22}$$

or, we may write it in the form

$$\overset{4}{F}(\bar{x}) = [I(1 - \bar{q}^2 I^{-2})^{\frac{1}{2}}]^{-1}\bar{F}^4(x), \bar{F}(\overset{4}{\bar{x}}) = c^2 I^{-2}\frac{d}{dt}(\bar{M}I^2) \quad . \qquad (1.5.23)$$

(iv) <u>Energy</u>

From (1.5.5), (1.5.7) and (1.5.12) we obtain

$$\bar{\zeta}^{\sigma} F_{\sigma}(\bar{x}) = 0 \quad . \qquad (1.5.24)$$

Substituting (1.5.2), (1.5.16) and (1.5.20) into (1.5.24) and simplifying we get

$$\bar{q}^j \bar{F}_j(\bar{x}) = \frac{d}{dt}(\bar{M}I^2) \quad . \qquad (1.5.25)$$

Eq. (1.5.25) suggests that we may define the real energy of a particle by

$$\bar{\varepsilon}^* = \bar{M}I^2 \quad . \qquad (1.5.26)$$

From (1.5.10) and (1.5.26) we have

$$\bar{P}_4 = -\bar{\varepsilon}^* \quad . \qquad (1.5.27)$$

i.e., the fourth covariant component of the four-momentum vector is equal to the negative value of the real energy of the particle.

Similarly, from (1.5.11), (1.5.26) and (1.5.27), we may define the total relative energy $\bar{\varepsilon}_t^*$ by

$$\bar{\varepsilon}_t^* = \bar{P}^4 = -c^2 I^{-2}\bar{P}_4 = c^2 I^{-2}\bar{\varepsilon}^* = \bar{M}c^2 \quad . \qquad (1.5.28)$$

which represents the maximum energy the particle may possess.

1.6. Local Orthogonal Coordinates

Consider a Riemannian space-time $\{\bar{x}\} \equiv \{\bar{x}^j, \bar{x}^4\}$, $\bar{x}^4 = ct$, with the metric

$$-(ds)^2 = \bar{a}_{\sigma\rho} d\bar{x}^\sigma d\bar{x}^\rho \quad . \qquad (1.6.1)$$

The terms that contain $d\bar{x}^1$ are, in general,

$$\bar{a}_{11}(d\bar{x}^1)^2 + 2\bar{a}_{12}d\bar{x}^1 d\bar{x}^2 + 2\bar{a}_{13}d\bar{x}^1 d\bar{x}^2 + 2\bar{a}_{14}d\bar{x}^1 d\bar{x}^4 \;, \qquad (1.6.2)$$

where we assume that $\bar{a}_{\sigma\rho}$ is symmetric. Similar expressions hold for $d\bar{x}^2$, $d\bar{x}^3$ and $d\bar{x}^4$. The right hand side of (1.6.1) can be reduced to a sum of four squares, each having as a coefficient, the corresponding diagonal component of the metric tensor $\bar{a}_{\sigma\rho}$. In that respect, we follow [17] and write (1.6.1) in the form

$$-(ds)^2 = \bar{a}_{11}(\psi^1)^2 + \bar{a}_{22}(\psi^2)^2$$

$$+ \bar{a}_{33}(\psi^3)^2 + \bar{a}_{44}(\psi^4)^2 \quad , \qquad (1.6.3)$$

where, for example, ψ^1 takes the form

$$\psi^1 = d\bar{x}^1 + \bar{a}_{12}\bar{a}_{11}^{-1}d\bar{x}^2 + \bar{a}_{13}\bar{a}_{11}^{-1}d\bar{x}^3$$

$$+ \bar{a}_{14}\bar{a}_{11}^{-1}d\bar{x}^4 \quad , \qquad (1.6.4)$$

and where we assume $\bar{a}_{11} \neq 0$.

Hence, we may write

$$\psi^{\sigma} = b^{\sigma}{}_{\rho} d\bar{x}^{\rho} \quad , \quad \bar{a}_{(\sigma\sigma)} \neq 0 \quad .^{1} \qquad (1.6.5)$$

If we consider a given point, 0, with coordinates a^{ρ} we may write

$$\bar{y}^{\sigma} = (b^{\sigma}{}_{\rho})_{o}(\bar{x}^{\rho} - a^{\rho}) \quad , \qquad (1.6.6)$$

where the subscript, o, denotes evaluation at a particular point 0. Hence, from (1.6.3), (1.6.4), (1.6.5) and (1.6.6) we obtain at "0":

$$-(ds)^2 = \bar{g}_{11}(d\bar{y}^1)^2 + \bar{g}_{22}(d\bar{y}^2)^2 + \bar{g}_{33}(d\bar{y}^3)^2 + \bar{g}_{44}(d\bar{y}^4)^2 \quad , \qquad (1.6.7)$$

where $\bar{g}_{11}$, $\bar{g}_{11}$, $\bar{g}_{33}$ and $\bar{g}_{44}$ are evaluated at the point "0". In the present work, the components of the metric tensor, $\bar{a}_{\sigma\rho}$, of (1.6.1) are

$$\bar{a}_{11} = \bar{a}_{22} = \bar{a}_{33} = 1 \;, \quad \bar{a}_{44} = -c^{-2}I^2 \quad , \quad \bar{a}_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \;. \qquad (1.6.8)$$

Using (1.6.4), (1.6.5), (1.6.6), (1.6.7) and (1.6.8) we obtain:

$$\psi^{j} = d\bar{y}^{j} = d\bar{x}^{j} \quad , \quad \psi^4 = d\bar{y}^4 = d\bar{x}^4 = cdt \;; \qquad (1.6.9)$$

$$\bar{g}_{11} = \bar{g}_{22} = \bar{g}_{33} = 1 \;, \quad \bar{g}_{44} = -c^{-2}I_{o}^{2}, \; \bar{g}_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \quad . \qquad (1.6.10)$$

---

[1] Indices between brackets do not follow the summation rule.

Hence, (1.6.7) becomes

$$-(d\mathscr{s})^2 = \bar{g}_{\sigma\rho}d\bar{y}^{\sigma}d\bar{y}^{\rho} = \bar{g}_{jk}d\bar{y}^{j}d\bar{y}^{k} - I_o{}^2dt^2 \ . \qquad (1.6.11)$$

In the local $\{\bar{y}\}$-space-time coordinates with the metric (1.6.11), the four-velocity vector is defined by

$$\bar{\xi}^{\lambda} = \frac{d\bar{y}^{\lambda}}{d\mathscr{s}} = \frac{d\bar{y}^{\lambda}}{dt}\frac{dt}{d\mathscr{s}} \ , \qquad (1.6.12)$$

$$\bar{\xi}^{j} = \bar{v}^{j}[I_o(1 - \bar{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1} \ ,$$

$$\bar{\xi}^{4} = [c^{-1}I_o(1 - \bar{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1} \ , \qquad (1.6.13)$$

with

$$\frac{dt}{d\mathscr{s}} = [I_o(1 - \bar{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1}, \bar{v}^{j} = \frac{d\bar{y}^{j}}{dt}, \bar{v}^2 = \bar{g}_{jk}\bar{v}^{j}\bar{v}^{k}$$

(1.6.14)

If, in Section 1.5, I is replaced by I_o, $\bar{q}^{j}$ by $\bar{v}^{j}$, keeping in mind that all derivatives of I_c with respect to the space and time coordinates vanish, we obtain the corresponding quantities in the local $\{\bar{y}\}$-space-time coordinates. The components of the four-momentum vector in $\{\bar{y}\}$-space-time take the form [see (1.5.10) and 1.5.11)]:

$$\bar{P}_j = c\bar{m}\bar{v}_j \ , \quad \bar{P}_4 = -\bar{m}I_o{}^2 \ ; \quad \bar{P}^{j} = c\bar{m}\bar{v}^{j} \ , \quad \bar{P}^4 = \bar{m}c^2 \ , \qquad (1.6.15)$$

where the relativistic mass, $\overline{m}$, in the $\{\overline{y}\}$-space-time is defined by

$$\overline{m} = m_o[c^{-1}I_o(1 - \overline{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1} \quad . \qquad (1.6.16)$$

The components of the four-force vector in the $\{\overline{y}\}$-space-time become

$$F_j(\overline{y}) = [c^{-1}I_o(1 - \overline{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1}\overline{F}_j(\overline{y}), \ \overline{F}_j(\overline{y}) = \frac{d}{dt}(\overline{mv_j}), \qquad (1.6.17)$$

$$F_4(\overline{y}) = [I_o(1 - \overline{v}^2 I_o{}^{-2})^{\frac{1}{2}}]^{-1}\overline{F}_4(\overline{y}) \ , \quad \overline{F}_4(\overline{y}) = -\frac{d}{dt}(mI_o{}^2) \quad . \qquad (1.6.18)$$

The energy equation in the $\overline{x}$ -space-time, (1.5.25), has its correspondence in the $\overline{y}$ -space-time in the form

$$\overline{v}^j \overline{F}_j(\overline{y}) = \frac{d}{dt}(\overline{m}I_o{}^2) \quad . \qquad (1.6.19)$$

1.7. Second Form of the Four-dimensional Space-time

Here we define the "world point" as a point at a certain time with three coordinates x^1, x^2, x^3, having the dimensions of length, and a fourth coordinate x^4, having the dimensions of time t, i.e., $x^4 = t$. The metric of the space-time (1.3.2) is replaced by

$$-(d\tau)^2 = a_{jk}dx^j dx^k - c^{-2}I^2(dx^4)^2 = a_{\sigma\rho}dx^\sigma dx^\rho \ ; \qquad (1.7.1)$$

where $d\tau = c^{-1}ds$ and has the dimensions of time.

Four-velocity

$$\zeta^\lambda = \frac{dx^\lambda}{d\tau} \; ; \; \zeta^j = q^j[c^{-1}I(1 - q^2I^{-2})^{\frac{1}{2}}]^{-1}, q^j = \frac{dx^j}{dt} , \quad (1.7.2)$$

$$\zeta^4 = [c^{-1}I(1 - q^2I^{-2})^{\frac{1}{2}}]^{-1} \quad . \qquad (1.7.3)$$

Four-momentum

$$P_j = m_o c^2 \zeta_j = c^2 M q_j \; , \quad P_4 = -MI^2 ; \quad P^j = m_o c^2 \zeta^j = c^2 M q^j ,$$

$$P^4 = Mc^2 = \varepsilon^*_t \; , \quad M = m_o[c^{-1}I(1 - q^2I^{-2})^{\frac{1}{2}}]^{-1} \; . \qquad (1.7.4)$$

Four-force

$$F_\sigma(x) = \frac{\delta P_\sigma}{\delta\tau} = m_o c^2 \frac{\delta\zeta_\sigma}{\delta\tau} \; , \quad F_j(x) = [c^{-1}I(1 - q^2I^{-2})^{\frac{1}{2}}]^{-1}\overline{F}_j(x) ,$$

$$\overline{F}_j(x) = c^2\frac{\delta}{\delta t}(Mq_j) = c^2[\frac{d}{dt}(Mq_j) + \frac{1}{2}M(I^2)_{,j}] \quad ; \qquad (1.7.5)$$

$$F_4(x) = [c^{-1}I(1 - q^2I^{-2})^{\frac{1}{2}}]^{-1}\overline{F}_4(x) ,$$

$$\overline{F}_4(x) = -\frac{d}{dt}(MI^2) \quad . \qquad (1.7.6)$$

Energy

$$q^j\overline{F}_j(x) = \frac{d}{dt}(MI^2) = \frac{d\varepsilon^*}{dt} \; , \; q_j\overline{F}^j(x) = c^{-2}I^2\overline{F}^4(x) ,$$

$$\overline{F}^4(x) = c^2I^{-2}\frac{d}{dt}(MI^2) \qquad (1.7.7)$$

The following relation (to be used below) is obtained from (1.4.13) and (1.7.5):

$$\frac{\delta\zeta^j}{\delta t} = c^{-2}m_o{}^{-1}\bar{F}^j(x) = \frac{d\zeta^j}{dt} + \frac{1}{2}\,m_o{}^{-1}Ma^{kj}(c^{-2}I^2)_{,k} \; . \quad (1.7.8)$$

Similarly, the metric (1.6.11), in the {y}-space-time, takes the form

$$-(d\tau)^2 = g_{\sigma\rho}dy^\sigma dy^\rho = g_{jk}dy^j dy^k - c^{-2}I_o{}^2(dy^4)^2, dy^4 = dt, \quad (1.7.9)$$

where;

$$g_{11} = g_{22} = g_{33} = c^{-2},\; g_{44} = -c^{-2}I_o{}^2,\; g_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \; . \quad (1.7.10)$$

The four-velocity

$$\xi^\lambda = \frac{dy^\lambda}{d\tau}\;;\; \xi^j = v^j[c^{-1}I_o(1 - v^2I_o{}^{-2})^{\frac{1}{2}}]^{-1},\; v^j = \frac{dy^j}{dt}\;, \quad (1.7.11)$$

$$\xi^4 = [c^{-1}I_o(1 - v^2I_o{}^{-2})^{\frac{1}{2}}]^{-1} \; . \quad (1.7.12)$$

The four-momentum

$$p_j = m_o c^2\xi_j = c^2mv_j\;,\quad p_4 = -mI_o{}^2 = -\varepsilon^*(y)\;; \quad (1.7.13)$$

$$p_4 = mc^2 = \varepsilon_t^*(y)\;,\quad m = m_o[c^{-1}I_o(1 - v^2I_o{}^{-2})^{\frac{1}{2}}]^{-1} \; . \quad (1.7.14)$$

The four-force

$$F_\sigma(y) = \frac{dp_\sigma}{d\tau} = m_o c^2 \frac{d\xi_\sigma}{d\tau}, \; F_j(y) = [c^{-1} I_o (1 - v^2 I_o^{-2})^{\frac{1}{2}}]^{-1} \bar{F}_j(y),$$

(1.7.15)

$$\bar{F}_j(y) = c^2 \frac{d}{dt}(mv_j) \quad , \quad F_4(y) = [c^{-1} I_o (1 - v^2 I_o^{-2})^{\frac{1}{2}}]^{-1} \bar{F}_4(y) \; ,$$

(1.7.16)

$$\bar{F}_4(y) = - \frac{d}{dt}(mI_o{}^2) \quad . \qquad (1.7.17)$$

The energy

$$v^j \bar{F}_j(y) = \frac{d}{dt}(mI_o{}^2) = \frac{d\varepsilon^*(y)}{dt} \; , \; v_j \bar{F}^j(y) = c^{-2} I_o{}^2 \bar{F}^4(y),$$

$$\bar{F}^4(y) = c^2 I_o{}^{-2} \frac{d}{dt}(MI_o{}^2) \quad . \qquad (1.7.18)$$

1.8. Transformation of Coordinates

A simple form of the metric, (1.7.1), can be obtained by introducing the transformation of coordinates

$$X^j = c^{-1} x^j \quad , \quad X^4 = x^4 = t \quad . \qquad (1.8.1)$$

Hence, the metric (1.7.1) becomes

$$d\tau^2 = -A_{jk} dX^j dX^k + c^{-2} I^2 (dX^4)^2 = -A_{\sigma\rho} dX^\sigma dX^\rho \; , \qquad (1.8.2)$$

where,

$$A_{jk} = 1 \text{ for } j{=}k \quad , \quad A_{44} = -c^{-2} I^2, \; A_{\sigma\rho} = 0, \text{ for } \sigma \neq \rho \; .$$

(1.8.3)

The components of the velocity vector in the {X}-space-time are

$$V^j(X) = \frac{dX^j}{d\tau} = \zeta^j c^{-1}, V^4(X) = \frac{dX^4}{d\tau} = cI^{-1}(1 + V^2(X))^{\frac{1}{2}}$$

$$= cI^{-1}(1 + \zeta^2 c^{-2})^{\frac{1}{2}};\ V_j = c\zeta_j \quad . \qquad (1.8.4)$$

where,

$$V^2(X) = A_{jk}V^j(X)V^k(X) \quad , \quad A_{\sigma\rho}V^\sigma(X)V^\rho(X) = -1 \quad . \qquad (1.8.5)$$

The spatial force

$$F^j(X) = \frac{\delta}{\delta\tau}\,[m_o c^2 V^j(X)] = \frac{\delta}{\delta\tau}\,(m_o c^2 \zeta^j c^{-1}) = c^{-1}F^j(x); \quad (1.8.6)$$

$$\overline{F}^j(X) = c^{-1}\overline{F}^j(x) \quad , \quad F_j(X) = cF_j(x) \quad . \qquad (1.8.7)$$

Similarly, in the local orthogonal coordinates we apply the transformation

$$Y^j = c^{-1}y^j \quad , \quad Y^4 = y^4 = t \quad . \qquad (1.8.8)$$

The metric (1.7.9) has its correspondence in the form

$$-(d\tau)^2 = G_{\sigma\rho}dY^\sigma dY^\rho = G_{jk}dY^j dY^k - c^{-2}I_o{}^2(dY^4)^2 \quad , \quad (1.8.9)$$

where

$$G_{jk} = 1 \text{ for } j{=}k \quad , \quad G_{44} = -c^{-2}I_o{}^2, G_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \ .$$

$$(1.8.10)$$

The components of the velocity vector are

$$V^j(Y) = \frac{dY^j}{d\tau} = \xi^j c^{-1} \quad , \quad V^4(Y) = \frac{dY^4}{d\tau} = cI_o^{-1}[1 + V^2(Y)]^{\frac{1}{2}}$$

$$= cI_o^{-1}(1 + \xi^2 c^{-2})^{\frac{1}{2}}, \; V_j(Y) = c\xi_j \quad ; \tag{1.8.11}$$

$$V^2(Y) = G_{jk}V^j(Y)V^k(Y) \quad , \quad G_{\sigma\rho}V^\sigma(Y)V^\rho(Y) = -1 \; . \tag{1.8.12}$$

The spatial force

$$F^j(Y) = c^{-1}F^j(y) \quad , \quad \bar{F}^j(Y) = c^{-1}\bar{F}^j(y), F_j(Y) = cF_j(y) \quad . \tag{1.8.13}$$

CHAPTER II

RELATIVISTIC FLUID DYNAMICS

2.1. Fundamental Aspects

In classical relativistic theories of fluid dynamics, the fluid is characterized by its internal energy per unit mass, ε, measured by an observer at rest with respect to the element of the fluid as a function of the pressure, p, and the rest density, ρ^{o}.

In the present modified relativistic theory of fluid dynamics, additional aspects are taken into account due to the presence of a gravitational field. We assume a certain domain filled out by a fluid considered as a collection of particles with rest mass m_o. The system in question possesses certain amount of Kenetic energy, potential energy and is subject to the work of the external force fields.

2.2. The Hydrodynamical Equations

The fundamentals of the relativistic fluid in a flat space with a reference velocity of propagation of signals in vacuo were derived in [18].[1] In this work we derive the generalized formalism corresponding to that in [18]

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[1] Numbers in square brackets refer to the bibliography of standard works.

but using a piece-wise constant velocity of the propagation of signals. The concept of the variable velocity of propagation of signals leads to the necessity of dealing with Riemannian spaces. As discussed in Chapter I, due to insurmountable difficulties in dealing with Riemannian spaces, we, from the very beginning, introduce an approximation in the form of Euclidean space. Therefore, we introduce the local orthogonal coordinates, {y}, (1.7.9). We begin with the hydrodynamical equations described in terms of a rectangular system of coordinates, y^σ, fixed in the space-time {y}. The particle random velocity components v^j are measured with respect to {y}. The spatial components of the relativistic velocity vector are given in the {y}-space-time by (1.7.11), i.e.,

$$\xi^j = v^j[c^{-1}I_o(1 - v^2 I_o^{-2})^{\frac{1}{2}}]^{-1} , \qquad (2.2.1)$$

from which we obtain

$$v^j = c^{-1}I_o\xi^j(1 + c^{-2}\xi^2)^{-\frac{1}{2}}, c^{-2}\xi^2 = g_{jk}\xi^j\xi^k ,$$

$$v^2c^{-2} = g_{jk}v^jv^k , \qquad (2.2.2)$$

where, $g_{jk} = c^{-2}$ for $j=k$ and $g_{jk} = 0$ for $j \neq k$.

(i) Boltzmann Equation

Let us introduce the distribution function $f(y^j,t,\xi^j)$ in the orthogonal phase-space with coordinates y^j and velocities ξ^j.

As shown in [18] the Boltzmann equation for f is

$$Df \equiv \frac{\partial f}{\partial t} + v^j \frac{\partial f}{\partial y^j} + \mathcal{F}^j \frac{\partial f}{\partial \xi^j} = \Delta_e f \quad , \qquad (2.2.3)$$

or, substituting (2.2.2) into (2.2.3) we get

$$Df \equiv \frac{\partial f}{\partial t} + c^{-1} I_o \xi^j (1 + \xi^2 c^{-2})^{-\frac{1}{2}} \frac{\partial f}{\partial y^j} + \mathcal{F}^j \frac{\partial f}{\partial \xi^j} = \Delta_e f \ , \qquad (2.2.4)$$

where, $\mathcal{F}^j = m_o^{-1} c^{-2} \bar{F}^j(y)$ = the external force per unit mass; $\bar{F}^j(y)$ is given by (1.7.16), whereas $\Delta_e f$ = the time rate of change in f due to encounters between the particles.

We define the mean value of a function G by

$$n\langle G\rangle = \int G f d_3\xi ; \quad n = \int f d_3\xi , \ \langle G\rangle \equiv \text{mean value of } G ,$$

$$d_3\xi = d\xi^1 \xi^2 \xi^3 \quad . \qquad (2.2.5)$$

Multiplying (2.2.4) by any transport quantity $\Phi(y^j, t, \xi^j)$, and integrating over the entire volume of the (ξ^1, ξ^2, ξ^3)-space we get

$$\int \Phi D f d_3\xi \equiv \int \Phi \left[\frac{\partial f}{\partial t} + c^{-1} I_o \xi^j (1 + c^{-2}\xi^2)^{-\frac{1}{2}} \frac{\partial f}{\partial y^j} + \mathcal{F}^j \frac{\partial f}{\partial \xi^j}\right] d_3\xi$$

$$= \int \Phi \Delta_e f d_3\xi \ . \qquad (2.2.6)$$

Integrating (2.2.6) by parts, with the usual assumptions that products of the form $(f\Phi)$ tend to zero as ξ^j tends to $\pm\infty$, and that $\mathcal{F}^j$ is independent of ξ^j, after some algebraic rearrangements, we obtain

$$\int \Phi D f d_3\xi \equiv \frac{\partial}{\partial t}(n\langle\Phi\rangle) + c^{-1} I_o \frac{\partial}{\partial y^j}[n\langle\Phi\xi^j(1 + c^{-2}\xi^2)^{-\frac{1}{2}}]$$

$$-n[\langle\frac{\partial\Phi}{\partial t}\rangle + \langle c^{-1} I_o \xi^j (1 + c^{-2}\xi^2)^{-\frac{1}{2}} \frac{\partial\Phi}{\partial y^j}\rangle + \mathcal{F}^j \langle\frac{\partial\Phi}{\partial\xi^j}\rangle]$$

$$= \int \Phi \Delta_e f d_3\xi \quad . \tag{2.2.7}$$

(ii) The summational invariants

Let us associate with (2.2.6) the form

$$\int \Phi D f d_3\xi \equiv n\Delta\overline{\Phi} = \int \Phi \Delta_e f d_\varepsilon \xi \quad , \quad \overline{\Phi} \equiv \text{mean value of } \Phi. \tag{2.2.8}$$

There is a certain class of functions, Ψ, characterized by some conservation properties during encounters in the sense that the sum of these properties for all the particles involved in an encounter undergoes no change by the encounter. Hence, the variation $\Delta\overline{\Psi} = 0$, (see [2] and [9]). Such functions are called symmational invariants.

For a gas we may have five summational invariants corresponding to the physical conservation laws with $\Psi^\sigma, \sigma = 0,1,2,3,4$, inserted for Φ in (2.2.8):

$$\Psi^o = m_o \quad , \quad \Psi^j = m_o \xi^j \quad , \quad \Psi^4 = E \quad , \tag{2.2.9}$$

where E = total energy of a particle, denoted below by Ψ^4 in (2.2.11).

Relation (2.2.8) for such functions takes the form:

$$\int \Psi^{\sigma} D f d_3 \xi \equiv n \Delta \overline{\Psi}^{\sigma} = \sigma \ , \quad (\sigma = 0,1,2,3,4) \quad . \qquad (2.2.10)$$

The condition $\Delta\overline{\Psi}^{\sigma} = 0$ expresses the conservation of mass during the encounter, $\Delta\overline{\Psi}^{j} = 0$ expresses the principle of conservation of momentum, while $\Delta\overline{\Psi}^{4} = 0$ expresses that of the conservation of energy.

In analogy to the classical relativistic theory, [7], Ψ^4 is assumed to be given by [see (1.7.14)]:

$$\Psi^4 = \overset{*}{\varepsilon}_t(y) = p^4 = mc^2 = c^2 m_o [c^{-1} I_o (1 - v^2 I_o^{-2})^{\frac{1}{2}}]^{-1} \quad . \qquad (2.2.11)$$

Inserting (2.2.2) into (2.2.11), we have

$$\Psi^4 = c^2 m_o c I_o^{-1} (1 + \xi^2 c^{-2})^{\frac{1}{2}} \quad . \qquad (2.2.12)$$

(iii) Law of Conservation of Mass

In this section we operate interchangebly in both $\{y^j\}$ and $\{Y^j\}$ coordinates which differ only by the factor c.

Let us, first, introduce the mass current vector defined by

$$U^{\alpha} = \int V^{\alpha} d\mu \quad , \quad d\mu = (1 + \xi^2 c^{-2})^{-\frac{1}{2}} f d_3 \xi \quad , \qquad (2.2.13)$$

where V^{α} is given by (1.8.11).

Substituting $\Phi = \Psi^{o} = m_o$ = constant into (2.2.7), using the transformation (1.8.8), simplifying and rearranging we obtain in the {Y}-space-time:

$$m_o U^{\alpha}|_{\alpha} = 0 \quad . \tag{2.2.14}$$

Eq. (2.2.14) expresses the law of conservation of mass.

Let us introduce the notation of the average velocity

$$\bar{u}^j = n^{-1}\int v^j f d_3\xi = n^{-1}\int c^{-1} I_o \xi^j (1 + \xi^2 c^{-2})^{-\frac{1}{2}} f d_3\xi \quad . \tag{2.2.15}$$

Defining $\bar{u}^2 = \sum_{j=1}^{3} \bar{u}^j \bar{u}^j$, and making use of (2.2.5), (2.2.13) and (2.2.15), then simplifying and rearranging, we obtain

$$(1 - \bar{u}^2 I_o^{-2})^{\frac{1}{2}} = (-n^{-2} U^{\alpha} U_{\alpha})^{\frac{1}{2}} \quad . \tag{2.2.16}$$

Let the number density as measured by an observer moving with velocity $\bar{u}^j$ with respect to the fixed coordinates (Y^j), taking into account the relativistic aspects, be defined by

$$n^{o2} = n^2(1 - \bar{u}^2 I_o^{-2}) = -U^{\alpha} U_{\alpha}, \text{ and } \rho^o = n^o m_o . \tag{2.2.17}$$

Furthermore, we define a dimensionless velocity, u^{α}, by

$$u^{\alpha} = n^{o-1} U^{\alpha} \quad . \tag{2.2.18}$$

Hence, from (2.2.17) and (2.2.18) we have

$$u^{\alpha} u_{\alpha} = -1 \quad . \tag{2.2.19}$$

The law of conservation of mass, (2.2.14), expressed in terms of u^α and ρ^o then takes the form

$$(\rho^o u^\alpha)|_\alpha = 0 \quad . \tag{2.2.20}$$

(iv) Laws of Conservation of Energy and Momentum

If we use $(1 - \bar{u}^2 I_o^{-2})^{-\frac{1}{2}}$ as a fundamental factor instead of $(1 - v^2 I_o^{-2})^{-\frac{1}{2}}$, a modified force per unit rest mass, corresponding to (1.7.15) may be introduced in the form

$$\mathcal{F}^{*j}(Y) = m_o^{-1} c I_o^{-1}(1 - \bar{u}^2 I_o^{-2})^{-\frac{1}{2}} \bar{F}^j(Y) \quad . \tag{2.2.21}$$

From (1.7.18), we have

$$v_j \bar{F}^j(Y) = c^{-2} I_o^{\,2} \bar{F}^4(Y) \quad . \tag{2.2.22}$$

Substituting, $v_j = c^{-1} I_o \xi_j (1 + \xi^2 c^{-2})^{-\frac{1}{2}}$ into (2.2.22) and taking average we get

$$\bar{F}^j(Y) \int c^{-1} I_o \xi_j (1 + \xi^2 c^{-2})^{-\frac{1}{2}} f d_3\xi = c^{-2} I_o^{\,2} \int \bar{F}^4(Y) f d_3\xi, \tag{2.2.23}$$

or, by making use of (2.2.15), we obtain

$$n \bar{F}^j(Y) \bar{u}_j = c^{-2} I_o^{\,2} n \langle \bar{F}^4(Y) \rangle \quad . \tag{2.2.24}$$

Multiplying both sides of (2.2.24) by $n^{-1} m_o^{-1} (c I_o^{-1}) \cdot$ $\cdot\, (1 - u^{-2} I_o^{-2})^{-\frac{1}{2}}$, using (2.2.21), we get

$$\mathcal{F}^{*j} \bar{u}_j = c^{-2} I_o^{\,2} \mathcal{F}^{*4} \quad , \tag{2.2.25}$$

where we define

$$\mathcal{F}^{*4} = m_o^{-1} c I_o^{-1} (1 - \bar{u}^2 I_o^{-2})^{-\frac{1}{2}} \langle \bar{F}^4(Y) \rangle \quad . \qquad (2.2.26)$$

Remodelling (2.2.23), we get

$$\int \bar{F}^j(Y) V_j d\mu \equiv \bar{F}^j(Y) \int V_j d\mu = c^{-1} I_o \int \bar{F}^4(Y) f d_3\xi$$

$$= c^{-1} I_o n \langle \bar{F}^4(Y) \rangle \quad . \qquad (2.2.27)$$

From (1.8.11) and (2.2.5) we obtain:

$$c^{-1} I_o n \langle \bar{F}^4(Y) \rangle = c^{-2} I_o^{\ 2} n \langle \bar{F}^4 \rangle n^{-1} \int c I_o^{-1} (1 + \xi^2 c^{-2})^{\frac{1}{2}} \ .$$

$$. \ (1 + \xi^2 c^{-2})^{-\frac{1}{2}} f d_3\xi = -\langle \bar{F}^4 \rangle \int V_4 d\mu \quad . \qquad (2.2.28)$$

Inserting (2.2.28) into (2.2.27), making use of (2.2.13) and rearranging we have

$$\bar{F}^j(Y) U_j + \langle \bar{F}^4 \rangle U_4 = 0 \quad . \qquad (2.2.29)$$

Using (2.2.21), (2.2.26) and (2.2.18) into (2.2.29), after some algebraic rearrangements we obtain

$$\mathcal{F}^{*\alpha} u_\alpha = 0 \quad . \qquad (2.2.30)$$

Eq. (2.2.27) may also be rewritten in the following form

$$\int \bar{F}^j V_j d\mu = c^{-1} I_o n (1 - \bar{u}^2 I_o^{-2})^{\frac{1}{2}} (1 - \bar{u}^2 I_o^{-2})^{-\frac{1}{2}} \langle \bar{F}^4 \rangle \quad ,$$

$$(2.2.31)$$

or, using (2.2.17) and (2.2.26);

$$\int \bar{F}^j V_j d\mu = c^{-2} I_o{}^2 n^o m_o \mathcal{F}^{*4}. \tag{2.2.32}$$

Substituting $\Phi(\xi^k) = \Psi^\sigma$, $\sigma = 1, 2, 3, 4$, [see (2.2.9) and (2.2.13], into (2.2.7), introducing the transformation of coordinates (1.8.8), using (2.2.21), (2.2.26) and (2.2.32), simplifying and rearranging we obtain the equations of conservation of mementum and energy

$$T^{\alpha\beta}|_\beta = \rho^o \mathcal{F}^{*\alpha} , \tag{2.2.33}$$

Where we define the energy momentum tensor by

$$T^{\alpha\beta} = m_o c^2 \int V^\alpha V^\beta d\mu \quad . \tag{2.2.34}$$

The right hand side of (2.2.33) represents the external forces and the work done by them on the fluid.

In order to bring the forces, $\rho^o \mathcal{F}^{*\alpha}$, to the same form as in the left hand side of (2.2.33), let us assume the existence of a second order tensor $\Pi^{\alpha\beta}$ such that

$$\rho^o \mathcal{F}^{*\alpha} = \Pi^{\alpha\beta}|_\beta \quad . \tag{2.2.35}$$

The form of $\Pi^{\alpha\beta}$ is chosen below.

Hence, an energy-momentum tensor, $T^{*\alpha\beta}$, can be introduced in the form

$$T^{*\alpha\beta} \equiv T^{\alpha\beta} - \Pi^{\alpha\beta} . \tag{2.2.36}$$

As a consequence of (2.2.35) and (2.2.36), the equations of conservation of momentum and energy, (2.2.33), Take simple form

$$T^{*\alpha\beta}|_{\beta} = 0 \quad . \tag{2.2.37}$$

2.3. Specific Internal Energy

According to the manipulations and the discussion presented in the work [18], we define the internal energy of the fluid, ε, per unit mass in the $\{Y\}$-coordinates by:

$$m_o{}^2(\rho^o)^{-2}T_{\alpha\beta}U^\alpha U^\beta = T_{\alpha\beta}u^\alpha u^\beta = \rho^o(c^2 + \varepsilon) \quad , \tag{2.3.1}$$

where we used (2.2.17) and (2.2.18) to obtain the second invariant quantity in (2.3.1). The tensor $T_{\alpha\beta}$ is the covariant form of the energy-momentum tensor, $T^{\alpha\beta}$, defined by (2.2.34).

2.4. The Fundamental Inequality

The internal energy, ε, per unit mass of the fluid defined by (2.3.1) undergoes certain restrictions when it is considered as a function of the pressure and the rest density. The restriction imposed on ε appears in the form of an inequality derived by [18] in $\{Y\}$-space-time coordinates:

$$\varepsilon \geqslant \tfrac{3}{2}p\rho^{o-1} + c^2\{[1 + \tfrac{9}{4}(c^{-2}p\rho^{o-1})^2]^{\frac{1}{2}} - 1\} \quad . \tag{2.4.1}$$

The inequality (2.4.1) holds also in the $\{y\}$-space-time coordinates.

As stated in [18], the significance of the inequality (2.4.1) for a flow, is that it imposes a restriction on the types of functions, $\varepsilon(p,\rho^o)$, furnished by the relativistic

kinetic theory of gases. This contradicts the macroscopic viewpoint which allows ε to be any function of p and ρ^o.

2.5. Case of an Ideal Gas

As in [18], we assume in {Y}-space-time coordinates:

$$T^{\alpha\beta} = \rho^o c^2[1 + c^{-2}(\varepsilon + p\rho^{o-1})]u^\alpha u^\beta + pG^{\alpha\beta} \quad . \tag{2.5.1}$$

where p is the hydrostatic pressure.

Let us choose the tensor, $\Pi^{\alpha\beta}$, given by (2.2.35), in the form

$$\Pi^{\alpha\beta} = \chi G^{\alpha\beta} \quad . \tag{2.5.2}$$

As proposed in Chapter I, [see (1.3.1)], the function I depends on the gravitational potential. Since $I = I_o$ = constant in the {Y}-space-time, it follows that:

$$\chi = \chi_o = \text{constant} \quad , \tag{2.5.3}$$

where χ_o is evaluated at point 0.

Inserting (2.5.1) and (2.5.2) into (2.2.36), using (2.5.3), we have:

$$T^{*\alpha\beta} = T^{\alpha\beta} - \Pi^{\alpha\beta} = \rho^o c^2[1 + c^{-2}(\varepsilon + p\rho^{o-1})]u^\alpha u^\beta$$

$$+ (p - \chi_o)G^{\alpha\beta} \quad . \tag{2.5.4}$$

The equations governing the motion of the fluid are [see (2.2.20) and (2.2.37)]:

$$(\rho^{o}u^{\alpha})|_{\alpha} = 0 \quad ; \tag{2.5.5}$$

$$T^{*\alpha\beta}|_{\beta} = 0 \quad . \tag{2.5.6}$$

Inserting (2.5.4) into (2.5.6), taking into account (2.5.3) and (2.5.5), simplifying and rearranging we obtain:

$$\rho^{o}c^{2}u^{\beta}[\mu u^{\alpha}]|_{\beta} + p,_{\beta}G^{\alpha\beta} = 0 \quad , \tag{2.5.7}$$

where we define

$$\mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1}) \quad . \tag{2.5.8}$$

Multiplying (2.5.7) by $(-u_{\alpha})$, using (2.2.19), (2.5.5) and simplifying we obtain

$$\rho^{o}[\varepsilon,_{\beta}u^{\beta} + p(\rho^{o-1}),_{\beta}u^{\beta}] = 0 \quad , \tag{2.5.9}$$

or, we write (2.5.9) in the form:

$$d\varepsilon + pd(\rho^{o-1}) = 0 \ , \ d\varepsilon = \varepsilon,_{\beta}u^{\beta}, d(\rho^{o-1}) = (\rho^{o-1}),_{\beta}u^{\beta} \quad . \tag{2.5.10}$$

Eq. (2.5.10) expresses the first law of thermodynamics with $dQ = 0$, (dQ being the elementary heat input into the system from the outside). Accordingly, we may introduce the notion of the absolute temperature Θ, and the specific entropy S, as measured by an observer at rest with respect to the fluid, such that:

$$\Theta dS = dQ = 0 \;, \; dS = S,_{\beta}u^{\beta}, dQ = Q,_{\beta}u^{\beta} \;. \qquad (2.5.11)$$

Combining (2.5.10) and (2.5.11) we have

$$d\varepsilon + pd(\rho^{o-1}) = \Theta dS \;. \qquad (2.5.12)$$

2.6. One-Dimensional Motion

All quantities in one-dimensional motion are assumed to be functions of the local orthogonal coordinates, Y^1 and $Y^4 = t$, which are introduced in Section 1.6.

Eq. (2.2.15) can be remodelled by making use of (2.2.13), (2.2.17) and (2.2.18) as follows

$$\bar{u}^j = n^{-1}\int c^{-1}I_o\xi^j(1 + c^{-2}\xi^2)^{-\frac{1}{2}}fd_3\xi = n^{-1}I_o\int V^j d\mu = n^{-1}I_oU^j$$

$$= (n^{-1}n_o)n_o^{-1}I_oU^j = I_o(1 - u^{-2}I_o^{-2})^{\frac{1}{2}}u^j \;, \qquad (2.6.1)$$

or, rearranging we get

$$u^j = \bar{u}^j I_o^{-1}(1 - \bar{u}^2 I_o^{-2})^{-\frac{1}{2}} \;, \; \bar{u}^2 = G_{jk}\bar{u}^j\bar{u}^k \;. \qquad (2.6.2)$$

Let us denote the dimensionless velocity $\bar{u}^1 I_o^{-1}$ by the symbol u, then Eq. (2.6.2.), for j = 1, becomes

$$u^1 = u(1 - u^2)^{-\frac{1}{2}} \;. \qquad (2.6.3)$$

Substituting, $u^1 = u(1 - u^2)^{-\frac{1}{2}}$, $u^2 \equiv u^3 \equiv 0$, into (2.2.19) and rearranging we get

$$u^4 = cI_o^{-1}(1 - u^2)^{-\frac{1}{2}} \;. \qquad (2.6.4)$$

Substituting (2.6.3) and (2.6.4) into (2.5.5) and (2.5.6), using (2.5.4), and rearranging we obtain in the (y,t)-plane:

$$[I_o^{-1}\rho^o(1 - u^2)^{-\frac{1}{2}}]_{,4} + [\rho^o u(1 - u^2)^{-\frac{1}{2}}]_{,1} = 0 \ ; \qquad (2.6.5)$$

$$[I_o^{-1}c^2\rho^o\mu u(1 - u^2)^{-1}]_{,4} + [\rho^o c^2\mu u^2(1 - u^2)^{-1} + (p - \chi_o)]_{,1} = 0 \quad . \qquad (2.6.6)$$

Carrying out the differentiations in (2.6.5) and (2.6.6), simplifying and rearranging we obtain in the (y,t)-plane:

$$(1 - u^2)(I_o^{-1}\rho^{o-1}\rho^o{}_{,t} + u\rho^{o-1}\rho^o{}_{,y}) + I_o^{-1}uu_{,t} + u_{,y} = 0 \ ; \qquad (2.6.7)$$

$$u(1 - u^2)(I_o^{-1}\mu^{-1}\mu_{,t} + u\mu^{-1}\mu_{,y}) + I_o^{-1}u_{,t} + uu_{,y} + (1 - u^2)^2\rho^{o-1}c^{-2}\mu^{-1}p_{,y} = 0 \quad , \qquad (2.6.8)$$

where

$$\mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1}) \quad . \qquad (2.6.9)$$

Differentiating (2.6.9) we get

$$c^2 d\mu = d\varepsilon + pd(\rho^{o-1}) + \rho^{o-1}dp \quad . \qquad (2.6.10)$$

Substituting (2.5.10) into (2.6.10) we get

$$c^2 d\mu = \rho^{o-1} dp \quad . \qquad (2.6.11)$$

Let us introduce the auxilliary quantity:

$$\alpha^2 = \rho^o \mu^{-1} \frac{d\mu}{d\rho^o} \ , \text{ i.e., } \alpha^2 \rho^{o-1} d\rho^o = \mu^{-1} d\mu \quad . \qquad (2.6.12)$$

Multiplying (2.6.11) by $c^{-2}\mu^{-1}$, using (2.6.12), we get

$$\alpha^2 \rho^{o-1} d\rho^o = c^{-2} \mu^{-1} \rho^{o-1} dp \quad . \qquad (2.6.13)$$

Similarly, we introduce the auxilliary function, ϕ, defined by

$$d\phi = \alpha \rho^{o-1} d\rho^o \quad . \qquad (2.6.14)$$

Hence, we may write (2.6.12) and (2.6.13) in the forms:

$$\mu^{-1} d\mu = \alpha d\phi \quad ; \qquad (2.6.15)$$

$$c^{-2} \rho^{o-1} \mu^{-1} dp = \alpha d\phi \quad . \qquad (2.6.16)$$

Substituting (2.6.15) and (2.6.16) into (2.6.7) and (2.6.8) and simplifying we obtain:

$$(1 - u^2)(I_o^{-1} \phi_{,t} + u\phi_{,y}) + \alpha(I_o^{-1} uu_{,t} + u_{,y}) = 0 \ , \qquad (2.6.17)$$

$$\alpha(1 - u^2)(I_o^{-1} u\phi_{,t} + \phi_{,y}) + I_o^{-1} u_{,t} + uu_{,y} = 0 \quad . \qquad (2.6.18)$$

Hence, Eqs. (2.6.17) and (2.6.18) constitute a system of partial differential equations for the unknown variables ϕ and u.

Addition and subtraction of (2.6.17) and (2.6.18) yields, respectively,

$$(1 - u^2)D_+\phi + D_+u = 0 \quad ; \qquad (2.6.19)$$

$$(1 - u^2)D_-\phi - D_-u = 0 \quad , \qquad (2.6.20)$$

where

$$D_+ = (1 + \alpha u)I_o^{-1}\frac{\partial}{\partial t} + (\alpha + u)\frac{\partial}{\partial y} \quad , \qquad (2.6.21)$$

$$D_- = (1 - \alpha u)I_o^{-1}\frac{\partial}{\partial t} - (\alpha - u)\frac{\partial}{\partial y} \quad . \qquad (2.6.22)$$

Let us introduce the identity

$$(1 - u^2)^{-1}D_\pm u = D_\pm \ln[(1 + u)(1 - u)^{-1}]^{\frac{1}{2}} \quad . \qquad (2.6.23)$$

Hence, substituting (2.6.23) into (2.6.19) and (2.6.20) we obtain respectively

$$D_+ r = 0 \quad , \qquad (2.6.24)$$

$$D_- s = 0 \quad . \qquad (2.6.25)$$

where,

$$r = \phi + \ln[(1 + u)(1 - u)^{-1}]^{\frac{1}{2}}, \; s = \phi - \ln[(1 + u)(1 - u)^{-1}]^{\frac{1}{2}} \qquad (2.6.26)$$

As stated in [18], the functions r and s are the local relativistic analogs of the Riemann functions which occur in the classical theory of propagation of one-dimensional waves of finite amplitude.

The characteristic curves of (2.6.24) and (2.6.25) along which r and s are constants, are respectively, given by

$$\left(\frac{dy}{dt}\right)_{I} = (\alpha + u)(1 + \alpha u)^{-1} I_o \quad , \tag{2.6.27}$$

$$\left(\frac{dy}{dt}\right)_{II} = -(\alpha - u)(1 - \alpha u)^{-1} I_o \ . \tag{2.6.28}$$

Let us define

$$\alpha = a I_o^{-1} \quad . \tag{2.6.29}$$

It is shown below that the quantity, a, represents the velocity of sound, whereas α is its dimensionless form referred to I_o.

2.7. Progressive Waves

According to the definition of [18], a disturbance is said to propagate as a progressive wave if either r or s is constant.

If we assume that:

$$s = \phi_o = \text{constant}; \tag{2.7.1}$$

we then obtain from (2.6.26)

$$u = \tanh(\phi - \phi_o) \quad . \tag{2.7.2}$$

Inserting (2.7.2) into (2.6.24), carrying out the differentiation and simplifying we get:

$$I_o^{-1}\phi_{,t} + \Gamma(\phi)\phi_{,y} = 0 \quad , \tag{2.7.3}$$

where

$$\Gamma(\phi) = (\alpha + u)(1 + \alpha u)^{-1} \quad , \tag{2.7.4}$$

whereas α is a function of ϕ determined by (2.6.14).

The general solution of (2.7.3) is of the form:

$$f(\phi) = y - \Gamma(\phi)\; I_o t \quad , \tag{2.7.5}$$

where $f(\phi)$ is an arbitrary function.

It follows from (2.7.5) that ϕ is constant along the straight lines of slope $\Gamma(\phi)$ in the $(y, I_o t)$-plane. Hence α represents the dimensionless velocity of propagation of a sound wave referred to I_o; the form of α is given below. According to the classical theory, the internal energy, ε, for a perfect gas can be written in the form (see [4]).

$$\varepsilon = (\gamma - 1)^{-1} p\rho^{o-1} \quad . \tag{2.7.6}$$

Differentiating (2.7.6), using (2.5.10), and simplifying we obtain:

$$\frac{dp}{d\rho^{o}} = \gamma p \rho^{o-1} \quad . \tag{2.7.7}$$

Substituting (2.7.7) into (2.6.13), making use of (2.6.9), we get after some rearrangements:

$$\alpha^2 = c^{-2}\gamma p\rho^{o-1}[1 + c^{-2}\gamma(\gamma - 1)^{-1}p\rho^{o-1}]^{-1} \quad . \tag{2.7.8}$$

If we consider a medium of high temperature, for which $c^{-2}p\rho^{o-1}$ is large compared to one, Eq. (2.7.8) then approximately becomes

$$\alpha \longrightarrow (\gamma - 1)^{\frac{1}{2}} \quad . \tag{2.7.9}$$

Hence, for $\gamma > 2$, sound waves propagate with velocity greater than the maximum velocity of propagation, i.e., I_o. This contradiction implies that the equation for ε, (2.7.6) for $\gamma > 2$ is not a possible one.

A physically possible flow, for which (2.7.5) is a solution, exists only if the curves ϕ = const. do not intersect in the $(y, I_o t)$-plane ([4] and [12]). If this condition is not satisfied, one-dimensional motion will suffer a discontinuity in the form of shock waves according to the classical theory.

2.8. Rankine-Hugoniot Equations

The relativistic Rankine-Hugoniot equations were derived by [18] in the flat space-time with the reference velocity of propagation of signals "c" in vacuo. Similar equations, having identical forms as those of [18], are

obtained in the {Y}-space-time coordinates with a reference velocity "I_o". We assume that both I_o and χ_o remain constant at their corresponding values at a point "0". Only the flow variables ρ^o, μ^α, p and ε are subject to jump discontinuities across the shock. We choose our coordinate system in such a way that the discontinuity is at rest and is perpendicular to the Y^1-axis of the {Y}-space-time. We put down the relativistic Rankine-Hugoniot equations without derivation as obtained by [18] in one-dimensional flow:

$$\text{(mass):} \quad \rho^o{}_+ u_+ (1 - u_+{}^2)^{-\frac{1}{2}} = \rho^o{}_- u_- (1 - u_-{}^2)^{-\frac{1}{2}} = M \quad ; \tag{2.8.1}$$

(from momentum):

$$M = c^{-1}[(p_+ - p_-)(\mu_- \rho^o{}_-{}^{1} - \mu_+ \rho^o{}_+{}^{-1})^{-1}]^{\frac{1}{2}} \quad ; \tag{2.8.2}$$

(energy):

$$M^2 c^2 (\mu_+{}^2 - \mu_-{}^2) = M^2 (p_+ - p_-)(\mu_+ \rho^{o-1}_+ + \mu_- \rho^{o-1}_-) \ . \tag{2.8.3}$$

In the above formulations, we assume that the fluid moves from right to left across the fixed shock. Quantities on the right side of the shock are denoted by the subscript (-) whereas those on the left side are denoted by the subscript (+).

2.9. The Shock Velocity

Following [18], we introduce the quantities:

$$\xi = p_+ p_-^{-1} \;, \quad \eta = \rho_+^o \rho_-^{o-1} \;, \quad \beta = \gamma_+(\gamma_+ - 1)^{-1} c^{-2} p_- \rho_-^{o-1} \;. \tag{2.9.1}$$

Rewriting (2.6.9) in terms of quantities (2.9.1) making use of (2.7.6) we have:

$$\mu_+ = 1 + \beta\xi\eta^{-1} \;, \quad \mu_- = 1 + \gamma_-\gamma_+^{-1}(\gamma_+ - 1)(\gamma_- - 1)^{-1}\beta \;. \tag{2.9.2}$$

As stated in [18], γ_+ and hence β may be functions of $p_+\rho_+^{o-1}$. However, they are assumed to be slowly varying functions and for the purposes of the discussion below, it is sufficient to consider γ_+ to be a constant.

Hence, the second of (2.9.2) becomes (with $\gamma_+ = \gamma_-$):

$$\mu_- = 1 + \beta \;. \tag{2.9.3}$$

From the inequality (2.4.1) and the fact that $\varepsilon > 0$, it follows that:

$$5/3 \geqslant \gamma_+ > 1 \;. \tag{2.9.4}$$

Substituting (2.9.1) and the first of (2.9.2) into (2.8.3) we obtain after some algebraic rearrangements:

$$\xi(\xi + \gamma_+ - 1)\beta^2\eta^{-2} + [(\gamma_+ + 1)\xi + (\gamma_+ - 1)]\beta\eta^{-1}$$

$$- \{[\beta(\gamma_+ - 1)(\xi - 1) + \mu_-\gamma_+]\mu_- - \gamma_+\} = 0 \quad . \tag{2.9.5}$$

Eq. (2.9.5) is a quadratic form for the quantity $\beta\eta^{-1}$. Consequently, if we solve for the positive value of $\beta\eta^{-1}$ we have:

$$\beta\eta^{-1} = \{R - [(\gamma_+ + 1)\xi + (\gamma_+ - 1)]\}\{2\xi[\xi + (\gamma_+ - 1)]\}^{-1} \quad , \tag{2.9.6}$$

where

$$R = \{(\gamma_+ - 1)^2(\xi - 1)^2 + 4\xi(\xi + \gamma_+ - 1) \cdot$$

$$\cdot [\gamma_+\mu_-^2 + \beta\mu_-(\gamma_+ - 1)(\xi - 1)]\}^{\frac{1}{2}} \quad . \tag{2.9.7}$$

After some manipulations, the author of [18] obtained the following inequality:

$$\mu_- - \mu_+\eta^{-1} \gtrless \mu_-(\xi + \gamma_+ - 1)^{-1}[(\xi - 1)(2 - \gamma_+)$$

$$+ (\gamma_- - \gamma_+)(\gamma_- - 1)^{-1}] \quad . \tag{2.9.8}$$

Substituting (2.8.1) into (2.8.2), using (2.9.1) and rearranging we get:

$$u_-(1 - u_-^2)^{-\frac{1}{2}} = [(\gamma_+ - 1)\beta(\xi - 1)]^{\frac{1}{2}}[\gamma_+(\mu_- - \mu_+\eta^{-1})]^{-\frac{1}{2}} \tag{2.9.9}$$

As mentioned in [18], u_- is less than one whenever the right hand side of the inequality (2.9.8) is positive. According to the convention presented in Section 2.8, the gas moves from right to left across a fixed shock. The velocity of the gas on the right side of the shock is denoted by $\bar{u}^1_-$, whereas that on the left side is denoted by $\bar{u}^1_+$. The shock is considered to be stationary with respect to a suitably chosen coordinates $\{Y\}$. Let us assume now that the fluid on the right side of the shock is at rest and the shock moves across the medium. In order to find the shock velocity, let us superimpose the velocity of the magnitude $\bar{u}^1_-$ upon the entire system in the direction opposite to the moving fluid. The gas will be at rest in the moving new system $\{Y^*\}$, and the shock will move with the velocity $\bar{u}^1_-$ from the left to the right.

The transformation of coordinates $\{Y^*\}\to\{Y\}$ is of the Lorentzian type:

$$Y^{*1} = (Y^1 - c^{-1}\bar{u}^1_- t)[1 - (\bar{u}^1_-)^2 I_o^{-2}]^{-\frac{1}{2}} \;,\; Y^{*2} = Y^2,\; Y^{*3} = Y^3 \;,$$

$$t^* = (t - cI_o^{-2}\bar{u}^1_- Y^1)[1 - (\bar{u}^1_-)^2 I_o^{-2}]^{-\frac{1}{2}} \;,\; \bar{u}^1_- = u_- I_o \;. \tag{2.9.10}$$

which leaves $(d\tau)^2$ invariant in the four-dimensional space-time, i.e.,

$$-(d\tau)^2 = (dY^1)^2 + (dY^2)^2 + (dY^3)^2 - c^{-2}I_o{}^2(dt)^2$$

$$= (dY^{*1})^2 + (dY^{*2})^2 + (dY^{*3})^2 - c^{-2}I_o{}^2(dt^*)^2 \quad . \tag{2.9.11}$$

In our main problem of the association between Riemannian and Euclidean spaces we solve the problem of shock in rectangular coordinates related piece-wise to the curvilinear coordinates. Hence, the velocity $\bar{u}^1_-$ is considered to be momentarily constant. This implies that the above transformations (2.9.10) is valid momentarily in a piece-wise sense. In conclusion, the velocity of the shock relative to the gas into which it is traveling is less than the signal velocity I_o. The remaining reasonings of the discussion that follows in the work [18] are valid in the present approach.

2.10. Concluding Remarks

A passage from the present work in the {Y}-space-time, with reference velocity I_o, to that of [18] in the {Y'}-space-time, with reference velocity c, can be made through the transformation of coordinates,

$$Y'^j = Y^j \quad , \quad Y'^4 = c^{-1}I_oY^4 \quad . \tag{2.10.1}$$

As a consequence of the coordinate transformation (2.10.1), the relation between quantities in the above reference frames are presented in Table 1. It follows from this table that the flow variables are independent of the above coordinate transformation. This is due to the fact that the fundamental factors $(1 - u'^2)^{\frac{1}{2}}$ and $(1 - u^2)^{\frac{1}{2}}$ are equal and that the distribution function $f(Y,t,\xi^j)$ is an invariant under (2.10.1) (see [8]). The magnitudes ($\bar{u}'^1_-$ and $\bar{u}^1_-$) of the velocities of the shock waves, in the above frames of references, relative to the gas into which they are traveling are governed by the relation

$$\bar{u}'^1_- = cI_o^{-1}\bar{u}^1_- \quad . \tag{2.10.2}$$

which shows that $\bar{u}'^1_- > \bar{u}^1_-$.

However, their dimensionless magnitudes, u'_- and u_- referred to c and I_o, are equal. The same argument holds for the velocities of sound a' and a. Thus, in conclusion, only $\bar{u}^1_-$ and a are affected by introducing I_o in place of c.

TABLE 1.--Comparison between flow quantities in the work [18] and those of the present work (considering isentropic flow).

Item	Quantities in the Work [18] ({Y'}-Space-Time)	Quantities in the Present Work ({Y}-Space-Time)	Relation Between Quantities in {Y'}- and {Y}-Space-Times
Velocity Vector	$V'^j = c^{-1}\xi'^j \ , \ V'^4 = (1 + c^{-2}\xi'^2)^{\frac{1}{2}}$	$V^j = c^{-1}\xi^j \ ; \ V^4 = cI_o^{-1}(1 + c^{-2}\xi^2)^{\frac{1}{2}}$	$\xi'^j = \xi^j \ , \ V'^j = V^j \ V'^4 = c^{-1}I_oV^4$
Mass-Current Vector	$U'^\alpha = \int V'^\alpha d\mu' \ ,$	$U^\alpha = \int V^\alpha d\mu \ ,$	$U'^j = U^j \ , \ U'^4 = c^{-1}I_oU^4$
	$d\mu' = f(Y'^j,t',\xi'^j)(1 + c^{-2}\xi'^2)^{-\frac{1}{2}}d_3\xi'$	$d\mu = f(Y^j,t,\xi^j)(1 + c^{-2}\xi^2)^{-\frac{1}{2}}d_3\xi$	
Average Velocity	$\bar{u}'^j = n'^{-1}\int v'^j f d_3\xi = n'^{-1}\int \xi'^j d\mu' \ ,$	$\bar{u}^j = n^{-1}\int v^j f d_3\xi = n^{-1}\int c^{-1}I_o\xi^j d\mu \ ,$	$\bar{u}'^j = cI_o^{-1}\bar{u}^j$
	$\bar{u}' = G_{jk}\bar{u}'^j\bar{u}^k$	$\bar{u} = G_{jk}\bar{u}^j\bar{u}^k$	$\bar{u}'^2 = c^2I_o^{-2}\bar{u}^2$
Dimensionless (one-dimensional) average velocity	$u' = \bar{u}'^1c^{-1}$	$u = \bar{u}^1I_o^{-1}$	$u' = u$
Number Density	$n' = \int f(Y'^j,t',\xi'^j)d_3\xi'$	$n = \int f(Y^j,t,\xi^j)d_3\xi$	$n' = n$
Number Density at rest	$n'^o = n'(1 - \bar{u}'^2c^{-2})^{\frac{1}{2}} = (-U'^\alpha U'_\alpha)^{\frac{1}{2}}$	$n^o = n(1 - \bar{u}^2I_o^{-2})^{\frac{1}{2}} = (-U^\alpha U_\alpha)^{\frac{1}{2}}$	$n^{o\prime} = n^o$
Dimensionless Velocity Vector	$u'^\alpha = n^{o\prime -1}U'^\alpha$	$u^\alpha = n^{o-1}U^\alpha$	$u'^j = u^j \ , \ u'^4 = c^{-1}I_ou^4$

Density at rest	$\rho'^{o} = n'^{o} m_o$	$\rho^{o} = n^{o} m_o$	$\rho^{o\prime} = \rho^{o}$
Pressure	$p' p_1'^{-1} = (\rho'^{o} \rho_1'^{o-1})^{\alpha}$	$p p_1^{-1} = (\rho^{o} \rho_1^{o-1})^{\alpha}$	$p' = p$
Internal Energy	$\varepsilon' = (\gamma - 1)^{-1} p' \rho^{o,-1}$	$\varepsilon = (\gamma - 1)^{-1} p \rho^{o-1}$	$\varepsilon' = \varepsilon$
μ	$\mu' = 1 + c^{-2}(\varepsilon' + p\rho^{o,-1})$	$\mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1})$	$\mu' = \mu$
Dimensionless Velocity of sound	$\alpha'^2 = c^{-2} \gamma p' \rho'^{o-1} [1 + c^{-2} \gamma (\gamma - 1)^{-1} p' \rho^{o,-1}]^{-1}$	$\alpha^2 = c^2 \gamma p \rho^{o-1} [1 + c^{-2} \gamma (\gamma - 1)^{-1} p \rho^{o-1}]^{-1}$	$\alpha' = \alpha$
Velocity of sound	$a' = \alpha' c$	$a = \alpha I_o$	$a' = c I_o^{-1} a$
Shock Velocity Relative to the gas	$\dfrac{u_-'}{(1 - u_-'^2)^{1/2}} = \left[\dfrac{(\gamma_+' - 1)\beta'(\xi' - 1)}{\gamma_+'(u_-' - \mu_+' n'^{-1})}\right]^{1/2}$	$\dfrac{u_-}{(1 - u_-)^{1/2}} = \left[\dfrac{(\gamma_+ - 1)\beta(\xi - 1)}{\gamma_+(u_- - \mu_+ n^{-1})}\right]^{1/2}$	$u_-' = u_-$, $(\bar{u}_-'^{1} = c I_o^{-1} \bar{u}_-^{1})$

CHAPTER III

DERIVATION OF THE HYDRODYNAMICAL EQUATIONS IN THE RIEMANNIAN SPACE-TIME

3.1. Introduction

In this chapter, we derive the hydrodynamical equations in the Riemannian space-time, $\{x\}$, with the reference velocity of signals $I = I(x^j)$. We demonstrate below that these equations reduce to their corresponding equations in Chapter II, when we set $I = I_o$ = constant and $\chi = \chi_o$ = constant.

The hydrodynamical equations are described in terms of a curvilinear system of coordinates, x^σ, fixed in the space-time $\{x\}$, (1.7.1). The particle random velocity components q^j are measured with respect to $\{x\}$.

3.2. Boltzmann Equation

Let us introduce the distribution function $f(x^j,t,\zeta^j)$ in the Riemannian phase-space, with coordinates, x^j, and velocities, ζ^j. As stated in [8] the distribution function $f(x^j,t,\zeta^j)$ is an invariant. The variation in the number of particles during the interval of time dt is

$$[f(x^j + dx^j,t + dt,\zeta^j + d\zeta^j) - f(x^j,t,\zeta^j)]d_3xd_3\zeta$$

$$= \Delta_e f d_3xd_3\zeta dt \quad , \qquad (3.2.1)$$

where, $d_3x = dx^1dx^2dx^3$, $d_3\zeta = d\zeta^1d\zeta^2d\zeta^3$; whereas $\Delta_e f$ = the time rate of change in f due to encounters between the particles.

Expanding the first term on the left hand side of (3.2.1) in Taylor series around (x^j, t, ζ^j), retaining only the first order differential terms, dividing all through by $d_3xd_3\zeta dt$ and rearranging we get

$$\frac{\partial f}{\partial t} + \frac{dx^j}{dt}\frac{\partial f}{\partial x^j} + \frac{d\zeta^j}{dt}\frac{\partial f}{\partial \zeta^j} = \Delta_e f \quad . \qquad (3.2.2)$$

The validity of the operations of the ordinary differentiation carried out in (3.2.2) follows from the fact that the ordinary derivative of a scalar (an invariant) is identical with its absolute derivative (see [17]).

Solving (1.7.2) for q^j in terms of ζ^j we get

$$q^j = c^{-1}I\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}} , \quad c^{-2}\zeta^2 = a_{jk}\zeta^j\zeta^k \quad . \qquad (3.2.3)$$

From (1.7.8) we obtain

$$\frac{d\zeta^j}{dt} = c^{-2}m_o{}^{-1}[\overline{F}^j(x) - \tfrac{1}{2}Ma^{kj}(I^2)_{,k}] \quad . \qquad (3.2.4)$$

Substituting for $\frac{dx^j}{dt} = q^j$ using (3.2.3) and (3.2.4) into (3.2.2) we obtain

$$Df \equiv \frac{\partial f}{\partial t} + c^{-1}I\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\frac{\partial f}{\partial x^j} + c^{-2}m_o^{-1}[\bar{F}^j(x)$$

$$- \frac{1}{2} Ma^{jk}(I^2)_{,k}]\frac{\partial f}{\partial \zeta^j} = \Delta_e f \quad . \qquad (3.2.5)$$

Similar to (2.2.5), we define the mean value of a function G by:

$$n\langle G\rangle = \int Gfd_3\zeta \ , \quad n = \int fd_3\zeta \ , \quad \langle G\rangle \equiv \text{mean value of } G. \qquad (3.2.6)$$

Multiplying (3.2.5) by any transport quantity $\Phi(x^j,t,\zeta^j)$ and integrating over the entire volume of the $(\zeta^1, \zeta^2, \zeta^3)$-space we get

$$\int \Phi Dfd = \int \Phi\{\frac{\partial f}{\partial t} + c^{-1}I\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\frac{\partial f}{\partial x^j} + c^{-2}m_o^{-1}[\bar{F}^j(x)$$

$$- \frac{1}{2} a^{jk}M(I^2)_{,k}]\frac{\partial f}{\partial \zeta^j}\}d_3\zeta = \int \Phi\Delta_e fd_3\zeta \ . \qquad (3.2.7)$$

Integrating by parts making use of (3.2.6) we obtain

$$\int \Phi\frac{\partial f}{\partial t}d_3\zeta = \frac{\partial}{\partial t}\int \Phi fd_3\zeta - \int \frac{\partial \Phi}{\partial t}fd_3\zeta = \frac{\partial}{\partial t}(n\langle\Phi\rangle) - n\langle\frac{\partial \Phi}{\partial t}\rangle; \quad (3.2.8)$$

$$\int \Phi\zeta^j c^{-1} I(1 + c^{-2}\zeta^2)^{-\frac{1}{2}} \frac{\partial f}{\partial x^j} d_3\zeta = \frac{\partial}{\partial x^j}[\int c^{-1} I \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta]$$

$$- \int \{\frac{\partial}{\partial x^j}[c^{-1} I \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}}]\} f d_3\zeta$$

$$= (c^{-1} I) \frac{\partial}{\partial x^j}[\int \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta]$$

$$+ [\frac{\partial}{\partial x^j}(c^{-1} I)] \int \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta$$

$$- \int \frac{\partial}{\partial x^j}[c^{-1} I \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}}] f d_3\zeta$$

$$= (c^{-1} I)[n\langle \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} \rangle]_{,j}$$

$$+ (c^{-1} I)_{,j}[n\langle \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} \rangle]$$

$$- n\langle [(c^{-1} I) \Phi \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}}]_{,j} \rangle \quad . \tag{3.29}$$

For $j = 1$, we have

$$\iiint \{c^{-2} m_o^{-1} \Phi [\overline{F}^1(x) - \tfrac{1}{2} a^{i1} M(I^2)_{,i}] \frac{\partial f}{\partial \zeta^1}\} d\zeta^1 d\zeta^2 d\zeta^3$$

$$= \iint \{c^{-2} f \Phi m_o [\overline{F}^1(x) - \tfrac{1}{2} a^{i1} M(I^2)_{,i}]\}\Big|_{\zeta^1=-\infty}^{\zeta^1=+\infty} d\zeta^2 d\zeta^3$$

$$- \iiint f \frac{\partial}{\partial \zeta^1}\{c^{-2} m_o^{-1} \Phi [\overline{F}^1(x) - \tfrac{1}{2} a^{i1} M(I^2)_{,j}]\} d_3\zeta \quad . \tag{3.2.10}$$

The same is valid for $j = 2,3$.

As mentioned before, we assume that products of the form $(f\Phi)$ tend to zero as ζ^j tends to $\pm\infty$. Adding Eq. (3.2.10) for $j = 1$ and its correspondence for $j = 2,3$, we obtain:

$$\int c^{-2}m_o^{-1}\Phi[\bar{F}^j(x) - \tfrac{1}{2}a^{jk}M(I^2)_{,k}]\frac{\partial f}{\partial \zeta^j}d_3\zeta$$

$$= - \int f\frac{\partial}{\partial \zeta^j}\{c^{-2}\Phi[m_o^{-1}\bar{F}^j(x) - \tfrac{1}{2}m_o^{-1}Ma^{jk}(I^2)_{,k}]\}d_3\zeta$$

$$= - \int\{c^{-2}m_o^{-1}[\bar{F}^j(x) - \tfrac{1}{2}Ma^{jk}(I^2)_{,k}]\frac{\partial \Phi}{\partial \zeta^j}$$

$$+ \Phi\frac{\partial}{\partial \zeta^j}[c^{-2}m_o^{-1}\bar{F}^j(x) - \tfrac{1}{2}c^{-2}m_o^{-1}Ma^{jk}(I^2)_{,k}]\}fd_3\zeta$$

$$= -n\langle c^{-2}m_o^{-1}\bar{F}^j(x)\frac{\partial \Phi}{\partial \zeta^j}\rangle + n\langle\tfrac{1}{2}m_o^{-1}Ma^{jk}(I^2)_{,k}\frac{\partial \Phi}{\partial \zeta^j}\rangle$$

$$- n\langle\Phi\frac{\partial}{\partial \zeta^j}[c^{-2}m_o^{-1}\bar{F}^j(x)]\rangle + n\langle\Phi\frac{\partial}{\partial \zeta^j}[\tfrac{1}{2}c^{-2}m_o^{-1}Ma^{jk}(I^2)_{,k}]\rangle \ .$$

(3.2.11)

The mass, M, can be expressed in terms of ζ^2 and I as follows:

$$M = m_o cI^{-1}(1 - q2I^{-2})^{-\frac{1}{2}} = m_o cI^{-1}(1 + c^{-2}\zeta^2)^{\frac{1}{2}} \ . \quad (3.2.12)$$

Hence, with $I = I(\overset{j}{\dot{x}})$, we have

$$\frac{\partial}{\partial \zeta^j}[\tfrac{1}{2}m_o^{-1}Ma^{jk}(c^{-2}I^2)_{,k}] = \tfrac{1}{2}a^{jk}(c^{-2}I^2)_{,k}\frac{\partial}{\partial \zeta^j}[cI^{-1} \cdot$$

$$\cdot (1 + c^{-2}\zeta^2)^{\frac{1}{2}}] = (c^{-1}I)_{,j}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}} \ . \qquad (3.2.13)$$

Substituting (3.2.8), (3.2.9), (3.2.11) and (3.2.13) into (3.2.7) and rearranging we obtain

$$\int \Phi D f d_3\zeta = \frac{\partial}{\partial t}(n\langle\Phi\rangle) + (c^{-1}I)[n\langle\Phi\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j}$$

$$+ (c^{-1}I)_{,j}[n\langle\Phi\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] - \{n\langle\frac{\partial\Phi}{\partial t}\rangle$$

$$+ n\langle[(c^{-1}I)\Phi\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}]_{,j}\rangle$$

$$+ n\langle c^{-2}m_o^{-1}\bar{F}^j(x)\frac{\partial\Phi}{\partial\zeta^j}\rangle - n\langle(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\frac{\partial\Phi}{\partial\zeta^j}\rangle\, a^{jk}(c^{-1}I)_{,k}$$

$$+ n\langle c^{-2}m_o^{-1}\Phi\frac{\partial\bar{F}^j}{\partial\zeta^j}\rangle - n\langle\Phi\zeta^j(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\rangle(c^{-1}I)_{,j}\}$$

$$= \int \Phi\Delta_e f d_3\zeta \quad . \qquad (3.2.14)$$

3.3. The Summational Invariants

The summational invariants ψ^σ (y), (2.2.9) and (2.2.12), in the flat space-time, {y}, have their correspondence in the Riemannian space-time, {x}, in the form

$$\psi^o(x) = m_o \quad , \quad \psi^j(x) = m_o\zeta^j \quad ,$$

$$\psi^4(x) = c^2 m_o(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}} \quad . \qquad (3.3.1)$$

3.4. Law of Conservation of Mass

Substituting $\Phi(\zeta^j) = \psi^o = m_o$ = constant into (3.2.14) we obtain

$$(n\langle m_o\rangle)_{,t} + (c^{-1}I)[n\langle m_o\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j}$$

$$+ (c^{-1}I)_{,j}[n\langle m_o\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]$$

$$- \{n\langle m_{o,t}\rangle + n\langle[(c^{-1}I)m_o\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}]_{,j}\rangle$$

$$+ n\langle c^{-2}m_o^{-1}\overline{F}^j(x)m_{o,\zeta^j}\rangle$$

$$- n\langle(1 + c^{-2}\zeta^2)^{\frac{1}{2}}a^{jk}m_{o,\zeta^j}(c^{-1}I)_{,k}$$

$$+ n\langle c^{-2}m_o^{-1}m_o\overline{F}^j_{,\zeta^j}\rangle$$

$$- n\langle m_o\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle(c^{-1}I)_{,j}\} = 0 \quad . \qquad (3.4.1)$$

Simplifying and rearranging we obtain

$$m_o\{n_{,4} + c^{-1}I[n\langle\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j}$$

$$+ (c^{-1}I)_{,j}n\langle\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle\} = c^{-2}n\langle\overline{F}_{,\zeta^j}\rangle \quad .$$

$$(3.4.2)$$

Introducing the transformation of coordinates (1.8.1), with the usual assumption that the force is independent of the velocity ζ^j and with $I = I(X^j)$, Eq. (3.4.2) takes the form

$$m_o[c^{-1}I(\int V^4 d\mu)_{,4} + c^{-1}I(\int V^j d\mu)_{,j} + (c^{-1}I)_{,j}\int V^j d\mu] = 0,$$

$$(3.4.3)$$

where,

$$d\mu = (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta \quad . \tag{3.4.4}$$

Let us define

$$U^\alpha(X) = \int V^\alpha(X) d\mu \quad , \tag{3.4.5}$$

Eq. (3.4.3) can be rewritten in the form, (after multiplying all through by cI^{-1}):

$$m_o\{U^4{}_{,4} + U^j{}_{,j} + \tfrac{1}{2}[\ln(c^{-2}I^2)]_{,j}U^j\} = 0 \quad . \tag{3.4.6}$$

Comparison of (1.4.17) and the left hand side of (3.4.6) suggests that U^α can be considered as a contravariant four-vector, (the mass current vector), in the Riemannian {X}-space-time, so that we may write

$$m_o U^\alpha|_\alpha = 0 \quad . \tag{3.4.7}$$

For operations below, we need to introduce the notion of the average velocity defined by

$$\overline{w}^j = n^{-1}\int q^j f d_3\zeta = n^{-1}\int c^{-1} I \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta \;, \tag{3.4.8}$$

where we used (3.2.3) in (3.4.8).

Defining $\overline{w}^2 = A_{jk}\overline{w}^j\overline{w}^k$, using (3.2.6), (3.4.5) and (3.4.8) we have

$$(1 - \bar{w}^2 I^{-2}) = 1 - A_{jk}\bar{w}^j\bar{w}^k I^{-2} = (n^{-1}\int f d_3\zeta)(n^{-1}\int f d_3\zeta)$$

$$- A_{jk} I^{-2}[n^{-1}\int c^{-1} I \zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta] \cdot$$

$$\cdot [n^{-1}\int c^{-1} I \zeta^k (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta]$$

$$= n^{-2} c^{-2} I^2 [\int c I^{-1}(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta] \cdot$$

$$\cdot\ [\int c I^{-1}(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta]$$

$$- A_{jk} n^{-2}(\int c^{-1}\zeta^j d\mu)(\int c^{-1}\zeta^k d\mu)$$

$$= -n^{-2}[(\int V^4 d\mu)(\int -c^{-2} I^2 V^4 d\mu)$$

$$+ A_{jk}(\int V^j d\mu)(\int V^k d\mu)] = -n^{-2}[(\int V^4 d\mu)(\int V_4 d\mu)$$

$$+ (\int V^j d\mu)(\int V_j d\mu)] = -n^{-2} U^\alpha U_\alpha \quad . \qquad (3.4.9)$$

or,

$$(1 - \bar{w}^2 I^{-2})^{\frac{1}{2}} = (-n^{-2} U^\alpha U_\alpha)^{\frac{1}{2}} \ . \qquad (3.4.10)$$

Let the number density as measured by an observer moving with velocity $\bar{w}^j$ with respect to the fixed coordinates (X^j), taking into account the relativistic aspects, be defined by

$$n^{o2}(X) = n^2(X)(1 - \bar{w}^2 I^{-2}) = -U^\alpha(X) U_\alpha(X) \ . \qquad (3.4.11)$$

The corresponding density, ρ^{o}, is

$$\rho^{o} = n^{o}m_{o} \quad . \tag{3.4.12}$$

Let us, further, define a dimensionless velocity

$$w^{\alpha} = n^{o-1}U^{\alpha} \quad . \tag{3.4.13}$$

Using (3.4.11) and (3.4.13) we get

$$w_{\alpha}w^{\alpha} = -1 \quad . \tag{3.4.14}$$

Similarly, inserting (3.4.12), (3.4.13) into (3.4.7) we get

$$(\rho^{o}w^{\alpha})|_{\alpha} = 0 \quad . \tag{3.4.15}$$

Hence, Eqs. (3.4.7) and (3.4.15) are alternative expressions for the law of conservation of mass.

3.5. Laws of Conservation of Energy and Momentum

As discussed in Chapter II, we introduce the corresponding modified force-vector, [see (2.2.21) and (2.3.26)]:

$$\mathcal{F}^{*\alpha} = m_{o}^{-1}cI^{-1}(1 + \overline{w}^{2}I^{-2})^{-\frac{1}{2}}\langle\overline{F}^{\alpha}(X)\rangle \quad , \quad \langle\overline{F}^{j}\rangle = \overline{F}^{j} \quad . \tag{3.5.1}$$

Similar manipulations to those presented in Chapter II lead to the expressions:

$$\mathcal{F}^{*}(X)w_{\alpha} = 0 \quad ; \tag{3.5.2}$$

$$\int\overline{F}^{j}(X)V_{j}d\mu = c^{-2}I^{2}n^{o}m_{o}\mathcal{F}^{*4}(X) \quad . \tag{3.5.3}$$

Substituting $\Phi(\zeta^k) = \Psi^k = m_o\zeta^k$ into (3.2.14), we obtain

$$(n\langle m_o\zeta^k\rangle)_{,t} + (c^{-1}I)[n\langle m_o\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j}$$

$$+ (c^{-1}I)_{,j}[n\langle m_o\zeta^j\zeta^k(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] - \{n\langle(m_o\zeta^k)_{,t}\rangle$$

$$+ n\langle[c^{-1}Im_o\zeta^j\zeta^k(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}]_{,j}\rangle + n\langle c^{-2}m_o\bar{F}^j(x)(m_o\zeta^k)_{,\zeta^j}\rangle$$

$$- n\langle(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(m_o\zeta^k)_{,\zeta^j}\rangle a^{ij}(c^{-1}I)_{,j} + n\langle c^{-2}m_o{}^{-1}m_o\zeta^k\bar{F}^j{}_{,\zeta^j}\rangle$$

$$- n\langle m_o\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}(c^{-1}I)_{,j}\rangle\} = 0 \quad . \tag{3.5.4}$$

Simplifying and rearranging, with the assumption that $\bar{F}^j$ is independent of ζ^j, we get

$$[m_o\int\zeta^k(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta]_{,4}$$

$$+ c^{-1}I[m_o\int\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta]_{,j}$$

$$+ (c^{-1}I)_{,j}[m_o\int\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta]$$

$$- (c^{-1}I)_{,j}[m_o\int\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta]$$

$$+ (c^{-1}I)_{,j}[m_o\int\zeta^k\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta]$$

$$+ (c^{-1}I)_{,j}a^{jk}\int m_o(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}fd_3\zeta$$

$$= c^{-2}\rho\ m_o{}^{-1}\bar{F}^k(x) \quad , \qquad \rho = nm_o \quad . \tag{3.5.5}$$

Applying the transformation of coordinates, (1.8.1), with $I = I(X^j)$, Eq. (3.5.5) reduces to

$$c[(c^{-1}I)(m_o\int V^4 V^k d\mu)_{,4} + (c^{-1}I)(m_o\int V^j V^k d\mu)_{,X^j}]$$

$$+ c^{-1}(c^{-2}I^2)(c^{-1}I)_{,X^j}A^{jk}m_o\int V^4V^4 d\mu + c(c^{-1}I)_{,X^j}m_o\int V^i V^{-k} d\mu$$

$$= c^{-1}\rho m_o^{-1}\bar{F}^k(X) \quad . \qquad (3.5.6)$$

Let us define

$$T^{k\alpha} = m_o c^2 \int V^k V^\alpha d\mu \quad , \qquad (3.5.7)$$

Eq. (3.5.6), after we multiply all through by $(c^2 I^{-1})$, becomes

$$T^{4k}{}_{,4} + T^{jk}{}_{,j} + \tfrac{1}{2}(c^{-2}I^2)_{,j}A^{jk}T^{44}$$

$$+ \tfrac{1}{2}[\ln(c^{-2}I^2)]_{,j}T^{jk} = \rho^\circ \mathcal{F}^{*k}(X),$$

$$cI^{-1}\rho m_o^{-1}\bar{F}^k(X) = \rho^\circ \mathcal{F}^{*k}(X) \quad , \qquad (3.5.8)$$

where we used (3.4.11) and (3.5.1) to obtain the right side of (3.5.8).

Similarly, if we substitute $\Phi(\zeta^j) = \Psi^4 = m_o c^2(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}} = Mc^2$, (3.3.1), into (3.2.14), simplifying, keeping in mind that $\bar{F}^j$ is independent of ζ^j, and re-arranging we get

$$
\begin{aligned}
&[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\rangle]_{,4} \\
&+ (c^{-1}I)[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j\ (1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j} \\
&+ (c^{-1}I)_{,j}[m_c c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] \\
&- m_o m_o{}^{-1} n\langle\overline{F}^j \zeta_j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle(cI^{-1}) \\
&+ (c^{-1}I)_{,k} a^{jk} m_o c^2 n\langle(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(cI^{-1})\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle \\
&+ m_o c^2 n\langle cI^{-1}(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle\ (c^{-1}I)_{,j} = 0 \quad .
\end{aligned}
\tag{3.5.9}
$$

or, rearranging (3.5.9), with $I = I(x^j)$, we get

$$
\begin{aligned}
&(c^{-1}I)[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,4} \\
&\quad + (c^{-1}I)[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle]_{,j} \\
&\quad + (c^{-1}I)_{,j}[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] \\
&\quad - cI^{-1} n\langle\overline{F}^j(x)\zeta_j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle \\
&\quad + (c^{-1}I)_{,j}[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] \\
&\quad + (c^{-1}I)_{,j}[m_o c^2 n\langle(cI^{-1})(1 + c^{-2}\zeta^2)^{\frac{1}{2}}\zeta^j(1 + c^{-2}\zeta^2)^{-\frac{1}{2}}\rangle] = 0 \quad .
\end{aligned}
\tag{3.5.10}
$$

Applying the transformation $\{x\}\to\{X\}$, (1.8.1), dividing all through by $(c^{-1}I)$ and simplifying we get

$$(m_o c^2 \int V^4 V^4 d\mu)_{,4} + (m_o c^2 \int V^4 V^j d\mu)_{,j}$$

$$+ \tfrac{3}{2}[\ln(c^{-2}I^2)]_{,j}(m_o c^2 \int V^4 V^j d\mu) = \rho^o \mathcal{F}^{*4},$$

$$c^2 I^{-2} \int \bar{F}^j (X) V_j d\mu = \rho^o \mathcal{F}^{*4}(X) \quad , \qquad (3.5.11)$$

where, we used (3.5.3) to obtain the right hand side of (3.5.11).

Let us define

$$T^{4\sigma} = m_o c^2 \int V^4 V^\sigma d\mu \ , \qquad (3.5.12)$$

Eq. (3.5.11) then becomes

$$T^{44}{}_{,4} + T^{4j}{}_{,j} + \tfrac{3}{2}[\ln(c^{-2}I^2)]_{,j} T^{4j} = \rho^o \mathcal{F}^{*4}(X) \ . \qquad (3.5.13)$$

Comparison of (1.4.18) and (1.4.19) with the left hand sides of (3.5.8) and (3.5.13), suggests that $T^{\alpha\beta}$ can be regarded as a second order contravariant tensor in the Riemannian {X}-space-time.

Therefore, (3.5.8) and (3.5.13) are combined into the tensorial form

$$T^{\alpha\beta}|_\beta = \rho^o \mathcal{F}^{*\alpha} \quad , \qquad (3.5.14)$$

where the energy-momentum tensor is defined by

$$T^{\alpha\beta} = m_o c^2 \int V^\alpha V^\beta d\mu \quad . \tag{3.5.15}$$

The right hand side of (3.5.14) represents the external forces and the work done by them on the fluid.

In order to bring the forces, $\rho^o \mathcal{F}^{*\alpha}$, to the same form as in the left hand side of (3.5.14), let us assume the existence of a second order tensor $\Pi^{\alpha\beta}$ such that

$$\rho^o \mathcal{F}^{*\alpha} = \Pi^{\alpha\beta}|_\beta \quad . \tag{3.5.16}$$

The form of $\Pi^{\alpha\beta}$ is chosen below.

Hence, an energy-momentum tensor, $T^{*\alpha\beta}$, can be introduced in the form

$$T^{*\alpha\beta} \equiv T^{\alpha\beta} - \Pi^{\alpha\beta} \quad . \tag{3.5.17}$$

Hence, from (3.5.14), (3.5.16), (3.5.17) we get

$$T^{*\alpha\beta}|_\beta = 0 \quad . \tag{3.5.18}$$

3.6. Specific Internal Energy

Similar discussion as that presented in Section 2.3 holds here. The internal energy of the fluid, ε, per unit mass, similarly, is defined by:

$$T_{\alpha\beta} w^\alpha w^\beta = \rho^o (c^2 + \varepsilon) \quad . \tag{3.6.1}$$

3.7. Case of An Ideal Gas

Following [18], we assume:

$$T^{\alpha\beta} = \rho^{o}c^{2}[1 + c^{-2}(\varepsilon + p\rho^{o-1})]w^{\alpha}w^{\alpha} + pA^{\alpha\beta} \quad . \tag{3.7.1}$$

where p = the hydrostatic pressure.

Let us choose the tensor, $\Pi^{\alpha\beta}$, given by (3.5.16), in the form

$$\Pi^{\alpha\beta} = \chi A^{\alpha\beta} \quad . \tag{3.7.2}$$

Hence, inserting (3.7.1), (3.7.2) into (3.5.17) we get

$$T^{*\alpha\beta} = \rho^{o}c^{2}[1 + c^{-2}(\varepsilon + p\rho^{o-1})]\, w^{\alpha}w^{\beta} + (p - \chi)A^{\alpha\beta} \quad . \tag{3.7.3}$$

The equations governing the motion of the fluid are [see (3.4.15) and (3.5.18)]:

$$(\rho^{o}w^{\alpha})|_{\alpha} = 0 \quad , \tag{3.7.4}$$

$$T^{*\alpha\beta}|_{\beta} = 0 \quad . \tag{3.7.5}$$

Substituting (3.7.3) into (3.7.5), taking into account (3.5.16), (3.7.2) and (3.7.4), simplifying and rearranging we get:

$$\rho^{o}c^{2}w^{\beta}(\mu w^{\alpha})|_{\beta} + p_{,\beta}A^{\alpha\beta} = \rho^{o}\mathcal{F}^{*\alpha} , \quad \mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1}) \quad . \tag{3.7.6}$$

Multiplying (3.7.6) by $(-w_\alpha)$, using (3.4.14), (3.5.2) and (3.7.4), and simplifying we get:

$$\rho^{\circ}[\varepsilon_{,\beta}w^{\beta} + p(\rho^{\circ -1})_{,\beta}w^{\beta}] = 0 \quad , \qquad (3.7.7)$$

or, we write (3.7.7) in the form:

$$d\varepsilon + pd(\rho^{\circ -1}) = 0 \ , \quad d\varepsilon = \varepsilon_{,\beta}w^{\beta} , \ d(\rho^{\circ -1}) = (\rho^{\circ -1})_{,\beta}w^{\beta} \ . \qquad (3.7.8)$$

Eq. (3.7.8) expresses the first law of thermodynamics with $dQ = 0$, where dQ = the elementary heat input into the system from the outside.

If we introduce the notion of the absolute temperature Θ and the specific entropy S, as measured by an observer at rest with respect to the fluid, such that

$$\Theta dS = dQ = 0 \ , \ dS = S_{,\beta}w^{\beta}, \ dQ = Q_{,\beta}w^{\beta} \ , \qquad (3.7.9)$$

Eq. (3.7.8) combined with (3.7.9) then becomes

$$d\varepsilon + pd(\rho^{\circ -1}) = \Theta dS \quad . \qquad (3.7.10)$$

3.8. One-Dimensional Motion

All quantities in one-dimensional motion are considered to be functions of the coordinates X^1 and $X^4 = t$, also we assume $w^2 = w^3 = 0$. Eq. (3.4.8) can be remodelled by making use of (3.4.5), (3.4.11), (3.4.13) as follows:

$$\bar{w}^j = n^{-1}\int c^{-1} I\zeta^j (1 + c^{-2}\zeta^2)^{-\frac{1}{2}} f d_3\zeta = n^{-1}\int IV^j d\mu$$

$$= n^{-1} IU^j = (n^{-1}n_o)n_o{}^{-1}IU^j = (1 - \bar{w}^2 I^{-2})^{\frac{1}{2}} w^j I \ .$$

$$(3.8.1)$$

or, rearranging we get

$$w^j = \bar{w}^j I^{-1}(1 - \bar{w}^2 I^{-2})^{-\frac{1}{2}}, \ \bar{w}^2 = A_{jk}\bar{w}^j\bar{w}^k \ . \qquad (3.8.2)$$

Let us denote the dimensionless velocity $\bar{w}^1 I^{-1}$ by the symbol w, then Eq. (3.8.2) for j = 1, becomes

$$w^1 = w(1 - w^2)^{-\frac{1}{2}} \qquad (3.8.3)$$

Substituting $w^1 = w(1 - w^2)^{-\frac{1}{2}}, w^2 \equiv w^3 \equiv 0$ into (3.4.14), and rearranging we get

$$w^4 = cI^{-1}(1 - w^2)^{-\frac{1}{2}} \ . \qquad (3.8.4)$$

Writing (3.7.4) in full, making use of (1.4.17) we get

$$(\rho^o w^1)_{,1} + (\rho^o w^4)_{,4} + \tfrac{1}{2}[\ln(c^{-2}I^2)]_{,1}\,\rho^o w^1 = 0 \ . \qquad (3.8.5)$$

Substituting (3.8.3) and (3.8.4) into (3.8.5), simplifying and rearranging we get in the (x,t)-plane:

$$(1 - w^2)(I^{-1}\rho^{o-1}\rho^o{}_{,t} + w\,\rho^{o-1}\rho^o{}_{,x}) + I^{-1}w\,w_{,t}$$

$$+ w_{,x} = -\tfrac{1}{2}w(1 - w^2)(\ln I^2)_{,x} \ . \qquad (3.8.6)$$

Similarly, writing (3.7.5) in full making use of (1.4.17), (3.5.16), (3.7.2), (3.7.3), (3.8.3), (3.8.4), simplifying and rearranging we obtain in the (x,t)-plane:

$$w(1 - w^2)(I^{-1}\mu^{-1}\mu_{,t} + w\mu^{-1}\mu_{,x}) + I^{-1}w_{,t} + ww_{,x}$$
$$+ (1 - w^2)^2\rho^{o-1}c^{-2}\mu^{-1}p_{,x} = -\frac{1}{2}(1 - w^2)(\ln I^2)_{,x}$$
$$+ (1 - w^2)^2\rho^{o-1}c^{-2}\mu^{-1}\chi_{,x} \quad . \qquad (3.8.7)$$

where

$$\mu = 1 + c^{-2}(\varepsilon + p\rho^{o-1}) \quad . \qquad (3.8.8)$$

Differentiating (3.8.8) we get

$$c^2 d\mu = d\varepsilon + pd(\rho^{o-1}) + \rho^{o-1}dp \quad . \qquad (3.8.9)$$

Inserting (3.7.9) into (3.8.9) we get

$$c^2 d\mu = \rho^{o-1}dp \quad . \qquad (3.8.10)$$

Let us introduce the auxiliary function, μ, defined by

$$\mu^{-1}d\mu = \alpha^2\rho^{o-1}d\rho^o \quad . \qquad (3.8.11)$$

From (3.8.10) and (3.8.11) we obtain

$$c^{-2}\mu^{-1}\rho^{o-1}dp = \alpha^2\rho^{o-1}d\rho^o \quad . \qquad (3.8.12)$$

Substituting (3.8.12) into (3.8.7) and rearranging we get

$$\alpha^2(1 - w^2)(wI^{-1}\rho^{\sigma-1}\rho^{\sigma}_{,t} + \rho^{\sigma-1}\rho^{\sigma}_{,x}) + I^{-1}w_{,t} + ww_{,x}$$

$$= (1 - w^2)[\rho^{\sigma-1}c^{-2}\mu^{-1}(1 - w^2)\chi_{,x} - \tfrac{1}{2}(\ln I^2)_{,x}] \; . \tag{3.8.13}$$

Furthermore, let us define the auxiliary function ϕ by

$$d\phi = \alpha\rho^{\sigma-1}d\rho^{\sigma} \quad , \tag{3.8.14}$$

Eqs. (3.8.6) and (3.8.13) in terms of ϕ become respectively

$$(1 - w^2)(I^{-1}\phi_{,t} + w\phi_{,x}) + \alpha(I^{-1}ww_{,t} + w_{,x})$$

$$= -\tfrac{1}{2}\alpha w(1 - w^2)[\ln(I^2)]_{,x} \quad , \tag{3.8.15}$$

$$(1 - w^2)(wI^{-1}\phi_{,t} + \phi_{,x}) + I^{-1}w_{,t} + ww_{,x}$$

$$= (1 - w^2)[c^{-2}\rho^{\sigma-1}\mu^{-1}(1 - w^2)\chi_{,x} - \tfrac{1}{2}(\ln I^2)_{,x}] \; . \tag{3.8.16}$$

Adding and subtracting (3.8.16) we obtain respectively

$$(1 - w^2)\delta_{+}\phi + \delta_{+}w^2 = -(1 - w^2)[\tfrac{1}{2}(1 + \alpha w)(\ln I^2)_{,x}$$

$$- c^{-2}\rho^{\sigma-1}\mu^{-1}(1 - w^2)\chi_{,x}] \; ; \tag{3.8.17}$$

$$(1 - w^2)\delta_-\phi - \delta_- w = (1 - w^2)[\tfrac{1}{2}(1 - \alpha w)(\ln I^2)_{,x}$$

$$- c^{-2}\rho^{0-1}\mu^{-1}(1 - w^2)\chi_{,x}] \quad . \qquad (3.8.18)$$

where,

$$\delta_+ = (1 + \alpha w)I^{-1}\frac{\partial}{\partial t} + (\alpha + w)\frac{\partial}{\partial x} \quad ,$$

$$\delta_- = (1 - \alpha w)I^{-1}\frac{\partial}{\partial t} - (\alpha - w)\frac{\partial}{\partial x} \quad . \qquad (3.8.19)$$

Let us introduce the identity

$$(1 - w^2)^{-1}\delta_\pm w = \delta_\pm \ln[(1 + w)(1 - w)^{-1}]^{\frac{1}{2}} \quad . \qquad (3.8.20)$$

Substituting (3.8.20) into (3.8.17) and (3.8.18) we obtain respectively

$$\delta_+ r = \rho^{0-1}c^{-2}\mu^{-1}(1 - w^2)\chi_{,x} - \tfrac{1}{2}(1 + \alpha w)(\ln I^2)_{,x}; \qquad (3.8.21)$$

$$\delta_- s = \tfrac{1}{2}(1 - \alpha w)(\ln I^2)_{,x} - \rho^{0-1}c^{-2}\mu^{-1}(1 - w^2)\chi_{,x}. \qquad (3.8.22)$$

where

$$r = \phi + \ln[(1 + w)^{\frac{1}{2}}(1 - w)^{-\frac{1}{2}}] \quad ; \qquad (3.8.23)$$

$$s = \phi - \ln[(1 + w)^{\frac{1}{2}}(1 - w)^{-\frac{1}{2}}] \quad . \qquad (3.8.24)$$

3.9. Concluding Remarks

The one-dimensional hydrodynamical equations (3.8.15) and (3.8.16) or their modified forms (3.8.21) and (3.8.22) in the Riemannian space-time can be reduced to (2.6.17), (2.6.18), (2.6.24), (2.6.25) in the flat space-time if we set $I = I_0$ = constant, and $\chi = \chi_0$ = constant.

All derivatives of I and χ then vanish and the right hand sides of the above equations (3.8.15), (3.8.16), (3.8.21) and (3.8.22) are equal to zero.

CHAPTER IV

APPLICATION

4.1. Gravitational Potential

Let us assume that there exists a celestial body with a magnitude of the gravitational potential at its surface:

$$\chi^{(1)} = 8.5(10^4)\chi_s \quad . \tag{4.1.1}$$

where

χ_s = gravitational potential of the sun at its surface,

$$\chi_s = 7.34(10^4) \ \ \mathrm{mi}^2 \ \mathrm{sec}^{-2} \quad . \tag{4.1.2}$$

The velocity of propagation of light signal in vacuo is

$$c = 1.86272(10^5) \ \mathrm{mi} \ \mathrm{sec}^{-1} \quad . \tag{4.1.3}$$

Using (4.1.1) and (4.1.3) to calculate the dimensionless quantity

$$2c^{-2}\chi^{(1)} = 0.36 \quad . \tag{4.1.4}$$

Substituting (4.1.4) into (1.3.1) and calculating, $c^{-1}I_o^{(1)}$, we obtain:

$$c^{-1}I_o^{(1)} = 0.8 \quad , \qquad (4.1.5)$$

where $I_o^{(1)}$ denotes the signal velocity at the surface of the body.

The gravitational potential at points $R^{(n)} = n\,R^{(1)}$, measured outward from the surface of the body is given by

$$\chi^{(n)} = n^{-1}\chi^{(1)} \quad , \qquad (4.1.6)$$

where, $R^{(1)}$ = radius of the body, $n = 1,2,3,\ldots.$

Consequently, the velocity of propagation of signals at points $R^{(n)}$ are calculated using the formula [obtained from (1.3.1) and (4.1.6)]:

$$I_o^{(n)} = (1 - 0.36n^{-1})^{\frac{1}{2}}c. \qquad (4.1.7)$$

Table 2 shows the calculations of the quantities $c^{-2}\chi^{(n)}$ and $c^{-1}I_o^{(n)}$ at the points $R^{(n)}$.

The quantities $c^{-2}\chi$ and $c^{-1}I_o$, considered as functions of the distance Y from the surface of the body, are shown in Figure 1.

TABLE 2.--The calculations of the quantities $c^{-2}\chi^{(n)}$ and $c^{-1}I_o^{(n)}$ at the points $R^{(n)}$.

$R^{(n)}$	$R^{(1)}$	$2\,R^{(1)}$	$3\,R^{(1)}$	$4\,R^{(1)}$	$5\,R^{(1)}$	$6\,R^{(1)}$	$7\,R^{(1)}$	$8\,R^{(1)}$	$9\,R^{(1)}$	$10\,R^{(1)}$
$c^{-2}\chi^{(n)}$	0.180	0.090	0.060	0.045	0.036	0.030	0.0255	0.0225	0.020	0.018
$c^{-1}I_o^{(n)}$	0.800	0.905	0.938	0.954	0.963	0.070	0.974	0.978	0.980	0.983

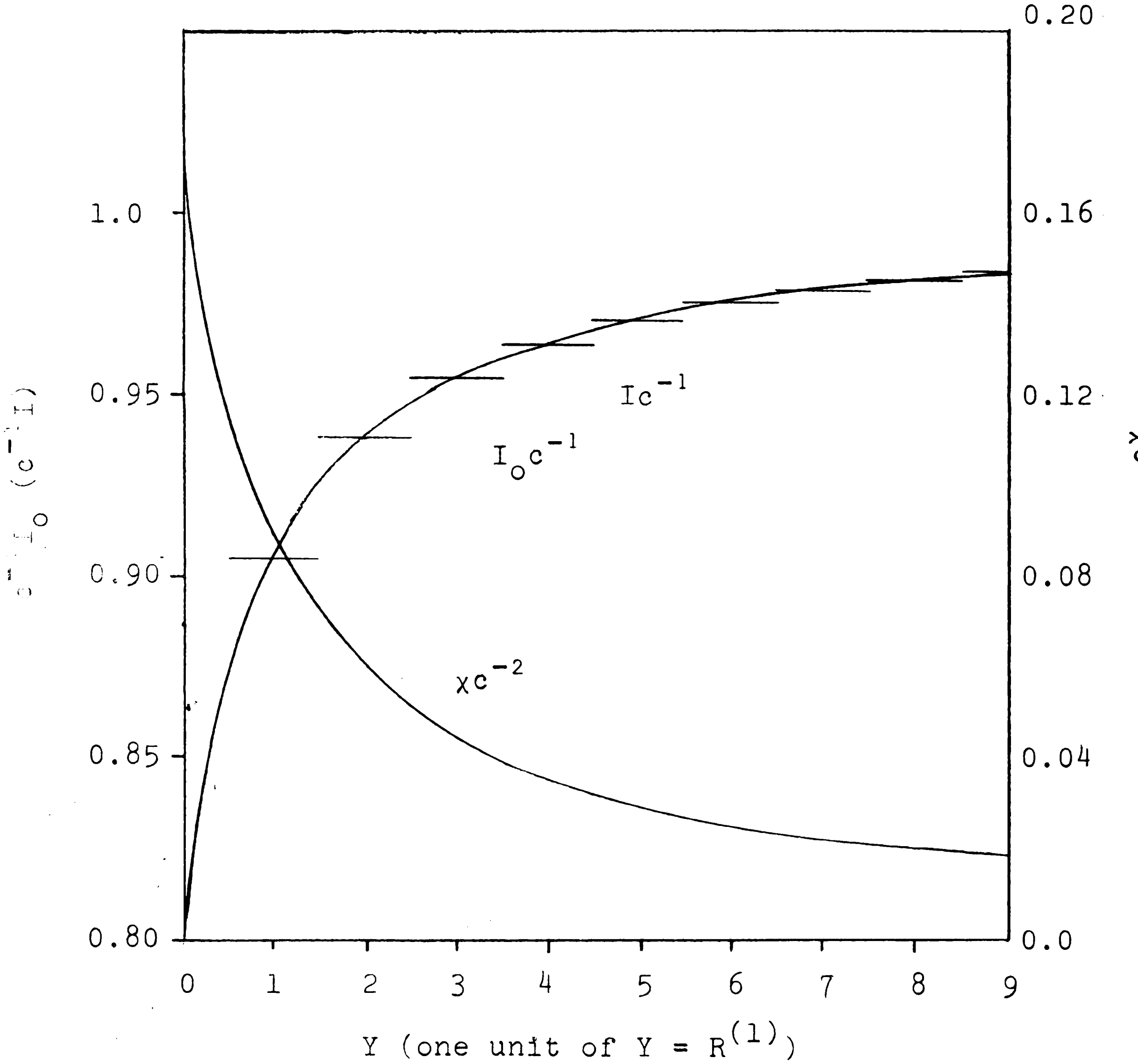

Figure 1.--The dimensionless gravitational potential $c^{-2}\chi$ and the dimensionless velocity $c^{-1}I_o(c^{-1}I)$ as functions of the Y-coordinate.

4.2. Gas Model

We assume a hypothetical medium consisting only of electrons at a very high temperature. The governing equation of state is assumed to follow that of the perfect gas law, i.e.,

$$p\rho^{o-1} = \mathcal{R} m_w^{-1}\theta \quad , \qquad (4.2.1)$$

where

$\mathcal{R}$ = universal constant = 1545.33 ft lb_f $mole^{-1}\,{}^oR^{-1}$;

m_w = molecular weight of the gas, m_w = $(1836)^{-1}$ lb_m for electrons;

θ = temperature of the gas in degrees Rankine;

p = pressure of the gas ($lb_f ft^{-2}$);

ρ^o = density of the gas ($lb_m ft^{-3}$).

The flow of the electron gas is assumed to be governed by (2.6.17) and (2.6.18) in the {Y}-space-time with the following initial conditions at t = 0:

$$u(Y,0) = u_o[1 + Y(1 + Y)^{-1}] \quad , \qquad (4.2.2)$$

$$\phi(Y,0) = \frac{2}{3}\phi_o[\frac{3}{2} - Y(1 + Y)^{-1}] \quad , \qquad (4.2.3)$$

where u_o and ϕ_o are constants whose values are given below. The origin of the {Y}-coordinate is located at the surface of the celestial body described in Section 4.1.

At t = 0 and Y = 0, we assume

$$u(0,0) \equiv u_o = 0.2 \quad ; \qquad (4.2.4)$$

$$\rho^{o}(0,0) \equiv \rho^{o}_{o} = 10^{-15}\ lb_m\ ft^{-3} \quad ; \tag{4.2.5}$$

$$\Theta(0,0) \equiv \Theta_o = 1.225(10^9)^{o}R \quad . \tag{4.2 6}$$

Using (4.2.1), (4.2.5) and (4.2.6) we calculate the following quantity at $t = 0$ and $Y = 0$:

$$c^{-2} p_o \rho^{o-1}_{o} = 0.115 \quad . \tag{4.2.7}$$

From (2.4.1), (2.7.6) and (4.2.7) we find that

$$\gamma_o < 1.614 \quad . \tag{4.2.8}$$

As mentioned in Section 2.9 γ is considered to be a constant. We take γ to be equal to 1.614 throughout the calculations below.

Integrating (2.7.7), making use of (4.2.5) and (4.2.7), we obtain the isentropic relation:

$$p = K_o \rho^{o\gamma} \quad , \quad c^{-2} K_o = 1.8676(10^8) \quad . \tag{4.2.9}$$

Normalizing ρ^{o} with respect to its value at $t = 0$ and $Y = 0$, i.e., ρ^{o}_{o}, we write:

$$\rho^{o*} = \rho^{o} \rho^{o-1}_{o} \quad . \tag{4.2.10}$$

From (2.7.8), (4.2.9) and (4.2.10) we obtain α as a function of ρ^{o*} as shown in Figure 2. Similarly, by integrating (2.6.14), making use of (2.7.8), (4.2.9) and

(4.2.10) with the requirement that $\phi = 0$ when $\rho^{o*} = 0$ (see [4]), we determine ϕ as a function of ρ^{o*} as shown in Figure 3. The quantity ϕ_o, (4.2.3), is determined from Figure 3 corresponding to the value $\rho^{o*}_o = 1$, ($\phi_o = 0.354466$).

From Figures 2 and 3, the quantity α, to be used in the calculations below, is determined as a function of ϕ as shown in Figure 4.

Figures 5 and 6 represent the numerical solutions of Eqs. (2.6.17) and (2.6.18) with the initial conditions (4.2.2) and (4.2.3) for ϕ and u for different constant parameters $I_o^{(n)}$ at a particular instant $t = t_o$ for the range of $Y = [0,0.35]$.

Using Figure 1 we determine the positions, $Y = Y^{(n)} = R^{(n)}$ at which the values of the parameter $I_o = I_o^{(n)}$ are chosen (see Table 1). In Figures 5 and 6, vertical lines are drawn at each point $Y^{(n)}$. Points of intersections of these vertical lines with the corresponding curves drawn for the corresponding parameter $I_o = I_o^{(n)}$ are determined. Due to the small range of Y for which diagrams 5 and 6 are drawn, only one point of intersection (corresponding to the vertical line Y = 0) is shown on each of these diagrams. However, the numerical values of ϕ and u at the points of intersections can be obtained, alternatively, using the diagonal numbers of Tables 3 and 4. Curves drawn through these points representing approximate graphical solutions for ϕ and u in the case of a

variable reference velocity I_o are marked in Figures 5 and 6 as dashed lines for the range of $Y = [0,0.35]$. The same curves representing the approximate solutions for ϕ and u are drawn in Figures 7 and 8 for the range of $Y = [0,7]$.

TABLE 3.--Solution of eqs. (2.6.17) and (2.6.18) for ϕ at the particular instant $t = t_o = 0.4$ sec. for Different values of the parameter $I^{(n)} c^{-1}$ and the corresponding approximate solution for ϕ.

Y \ $c^{-1}I_o^{(n)}$	0.800	0.905	0.938	0.954	0.963	0.970	0.974	0.978	0.980	0.983	0.99982
0	0.34056	0.33921	0.33882	0.33863	0.33852	0.33844	0.33839	0.33836	0.33833	0.33830	0.33810
1	0.23384	0.23360	0.23353	0.23350	0.23348	0.23347	0.23346	0.23345	0.23344	0.23344	0.23340
2	0.19596	0.19587	0.19584	0.19582	0.19581	0.19581	0.19580	0.19580	0.19580	0.19580	0.19578
3	0.17673	0.17667	0.17666	0.17665	0.17665	0.17664	0.17446	0.17664	0.17664	0.17664	0.17663
4	0.16511	0.16507	0.16506	0.16506	0.16506	0.16505	0.16505	0.16505	0.16505	0.16505	0.16504
5	0.15733	0.15731	0.15730	0.15730	0.15730	0.15729	0.15729	0.15729	0.15729	0.15729	0.15729
6	0.15176	0.15175	0.15174	0.15174	0.15174	0.15174	0.15174	0.15174	0.15174	0.15174	0.15173
7	0.14758	0.14757	0.14756	0.14756	0.14756	0.14756	0.14756	0.14756	0.14756	0.14756	0.14756
8	0.14432	0.14431	0.14431	0.14431	0.14431	0.14431	0.14431	0.14431	0.14431	01.4431	01.4430
9	0.14172	0.14171	0.14170	0.14170	0.14170	0.14170	0.14170	0.14170	0.14170	0.14170	0.14170
999	0.13978	0.13980	0.13981	0.13982	0.13982	0.13982	0.13982	0.13982	0.13982	0.13982	0.13983

TABLE 4.--Solution of eqs. (2.6.17) and (2.6.18) for u at the particular instant $t = t_o = 0.4$ sec. for different values of the parameter $I_o^{(n)} c^{-1}$ and the corresponding approximate solution for u (diagonal numbers of this table).

Y \ $c^{-1}I_o^{(n)}$	0.800	0.905	0.938	0.954	0.963	0.970	0.974	0.978	0.980	0.983	0.99982
0	0.23374	0.23786	0.23911	0.23972	0.24008	0.24032	0.24049	0.24061	0.24071	0.24079	0.24147
1	0.31019	0.31148	0.31187	0.31207	0.31218	0.31225	0.31231	0.31235	0.31238	0.31240	0.31262
2	0.33815	0.33877	0.33896	0.33905	0.33911	0.33914	0.33917	0.33919	0.33920	0.33921	0.33932
3	0.35280	0.35316	0.35327	0.35332	0.35336	0.35338	0.35339	0.35340	0.35341	0.34342	0.35348
4	0.36183	0.36206	0.36213	0.36217	0.36219	0.36220	0.36221	0.36222	0.36223	0.36223	0.36227
5	0.36795	0.36812	0.36817	0.36819	0.36821	0.36822	0.36872	0.36823	0.36823	0.36824	0.36827
6	0.37238	0.37250	0.37254	0.37756	0.37257	0.37258	0.37258	0.37259	0.37259	0.37259	0.37261
7	0.37573	0.37583	0.37586	0.37587	0.37588	0.37589	0.37589	0.37589	0.37590	0.37590	0.37591
8	0.37836	0.37844	0.37846	0.37847	0.37848	0.37848	0.37849	0.37849	0.37849	0.37849	0.37850
9	0.38047	0.3805	0.38056	0.38056	0.38057	0.38057	0.38058	0.38058	0.38058	0.38058	0.38059
999	0.38200	0.38203	0.38204	0.38205	0.38205	0.38205	0.38205	0.38205	0.38206	0.38206	0.38206

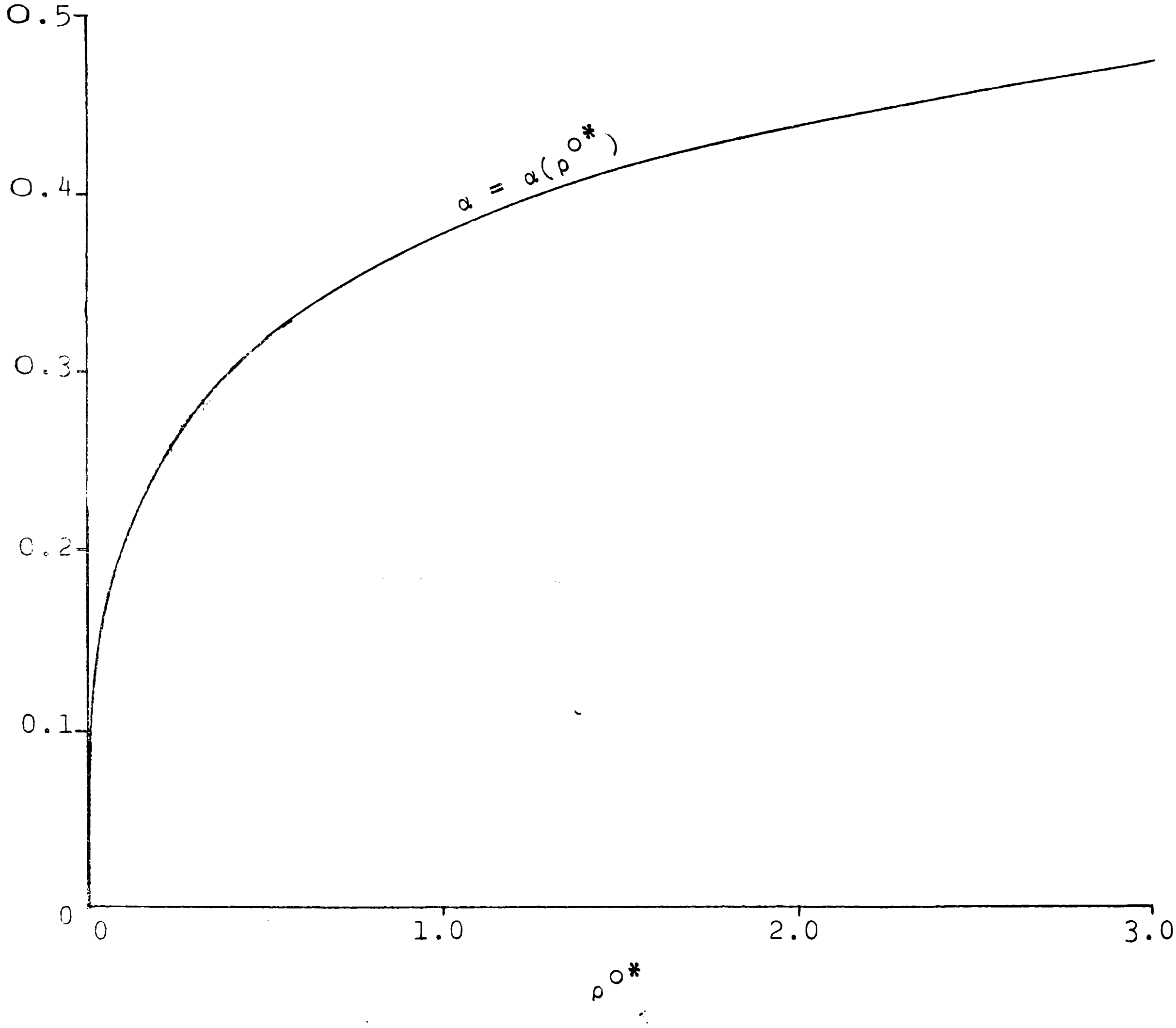

Figure 2.--The dimensionless sound velocity α as a function of the normalized density ρ^{o*}.

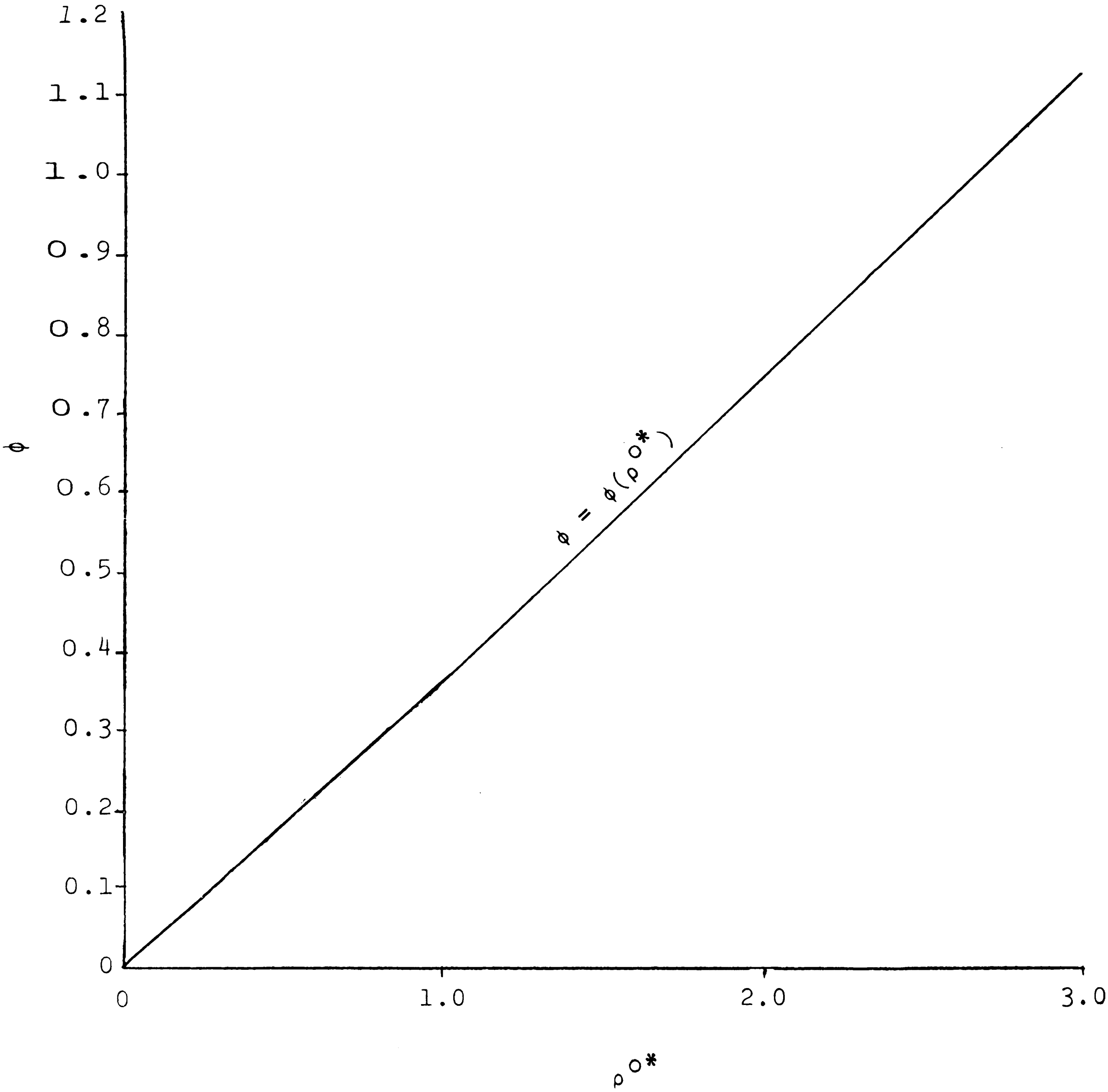

Figure 3.--The quantity ϕ as a function of the normalized density ρ_0^*.

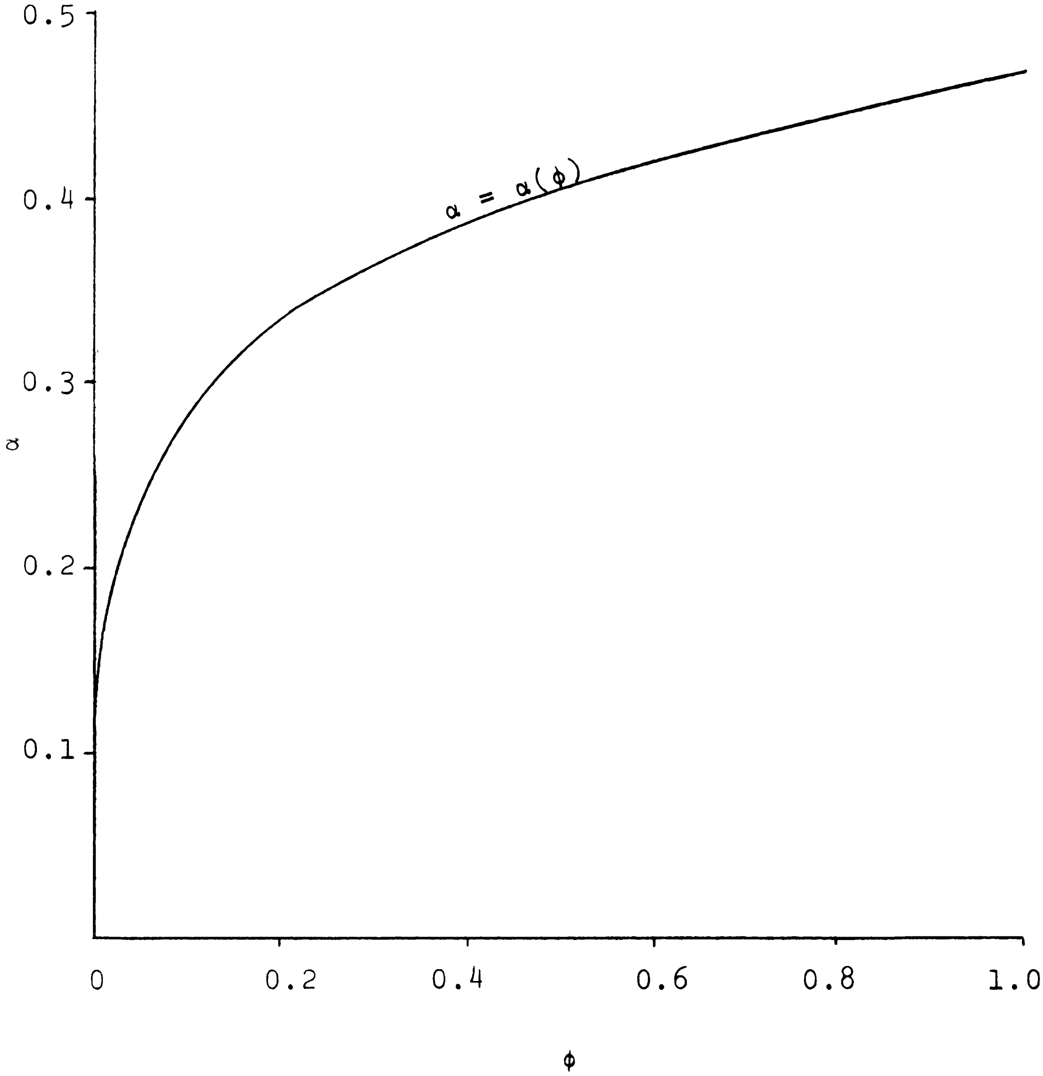

Figure 4.--The dimensionless sound velocity α as a function of the quantity ϕ.

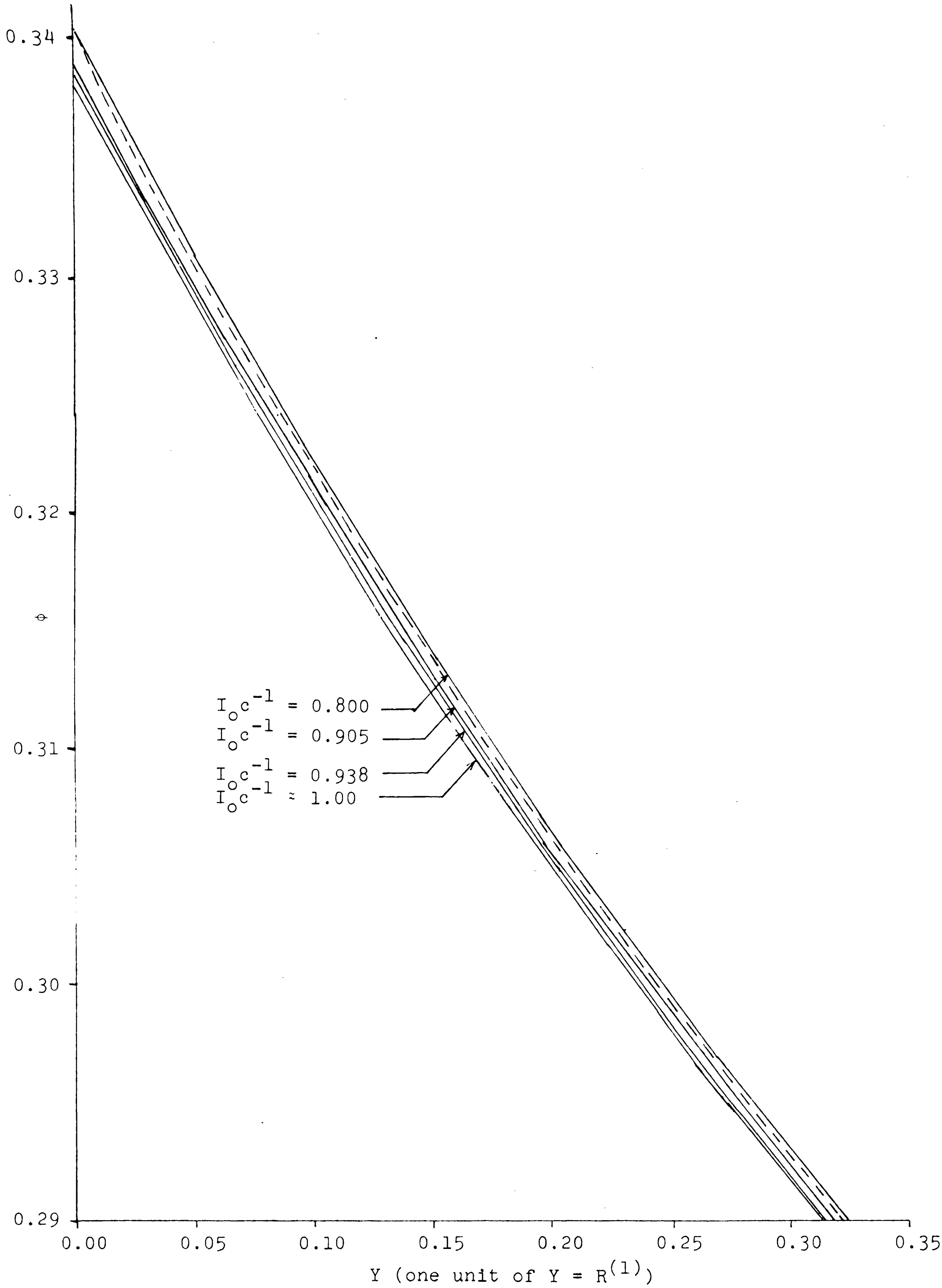

Figure 5.--Solution of eqs. (2.6.17) and (2.6.18) for ϕ at the particular instant $t = t_o = 0.4$ for different values of the parameter $I_o c^{-1}$ and the corresponding approximate solution for ϕ (shown in dashed line) in the range of Y [0,0.325].

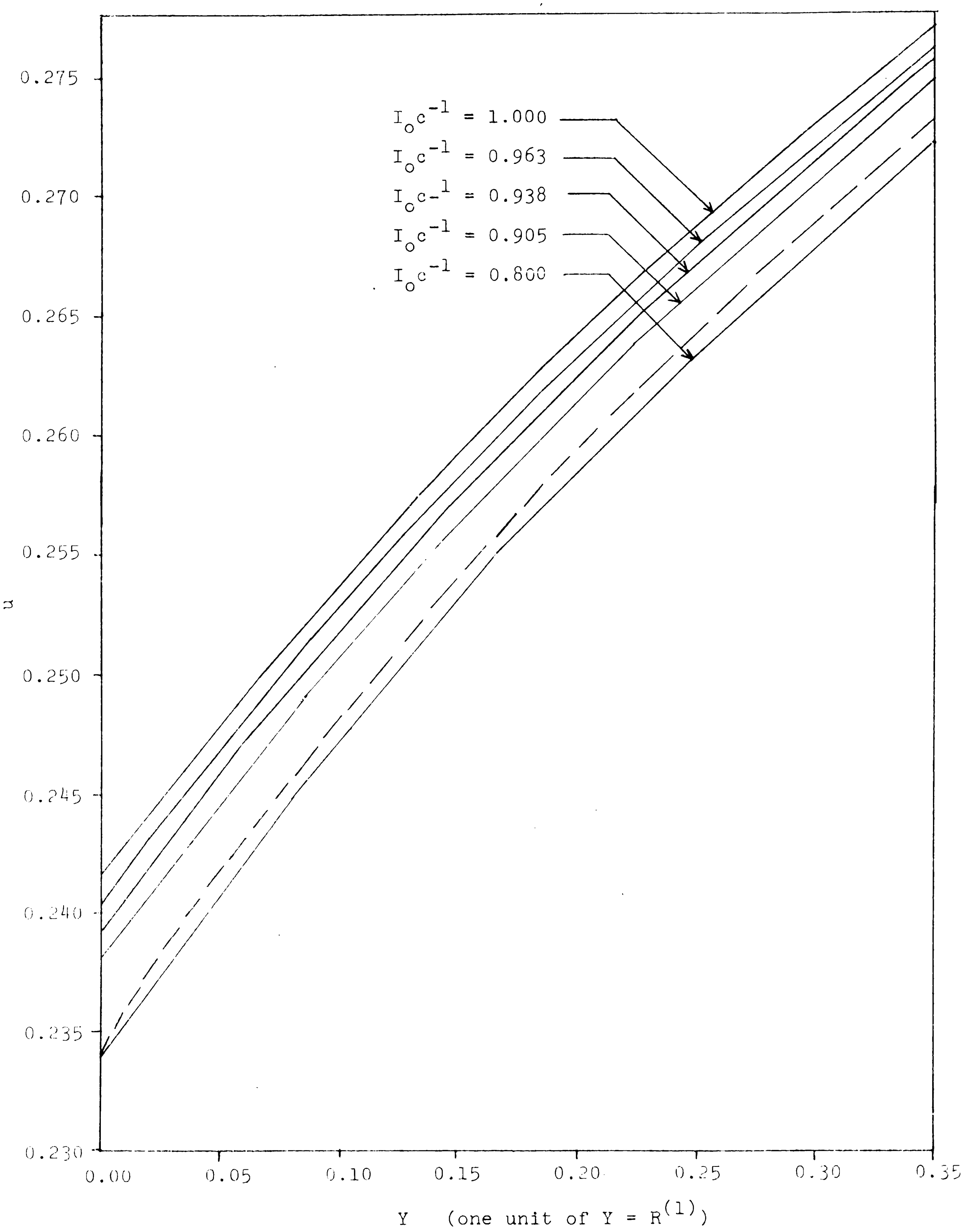

Figure 6.--Solution of eqs. (2.6.17) and (2.6.18) for u at the particular instant $t = t_o = 0.4$ for different values of the parameter I_oc^{-1} and the corresponding approximate solution for u (shown in dashed line) in the range of Y = [0 , 0.35].

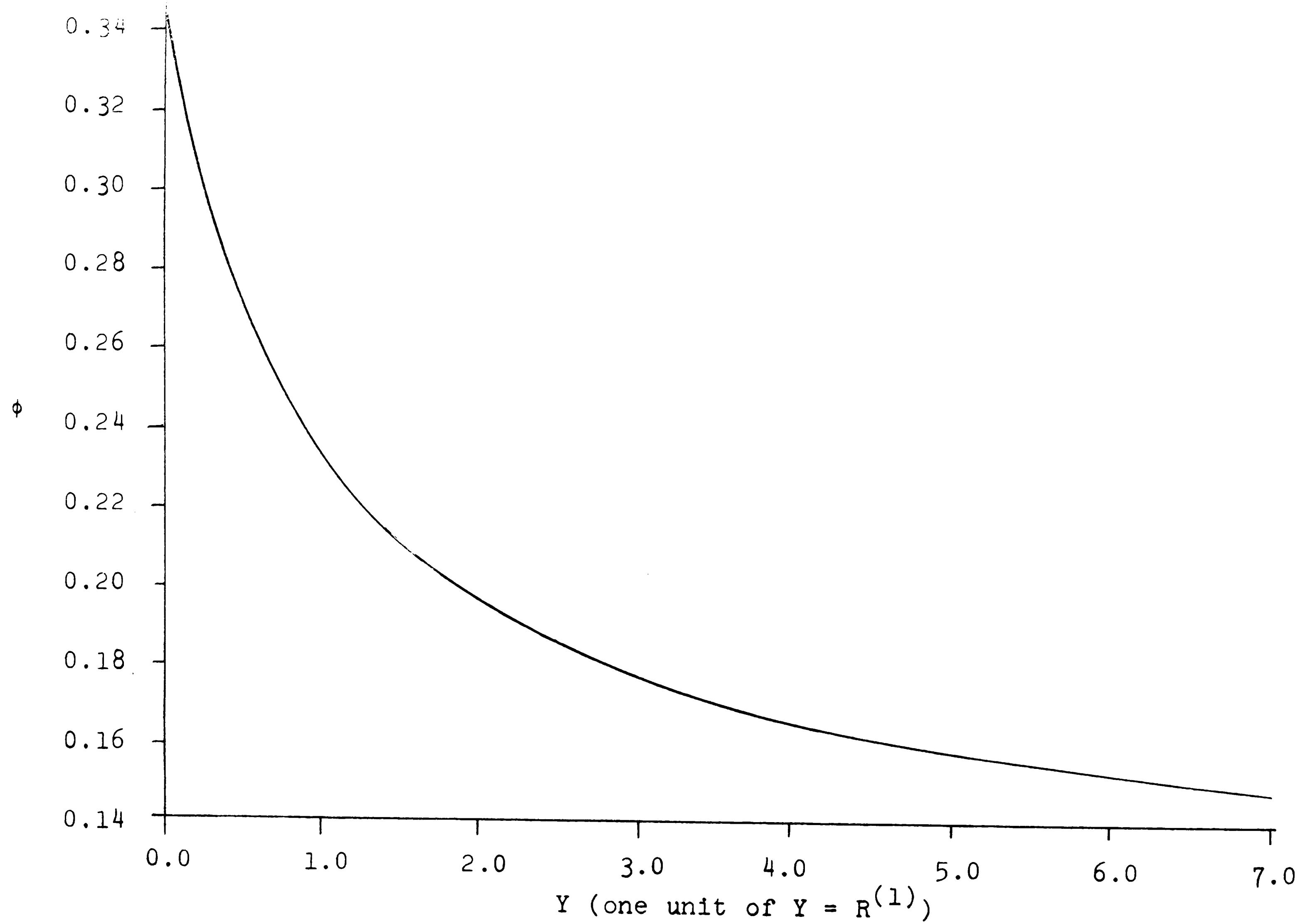

Figure 7.--Approximate solution of eqs. (2.6.17) and (2.6.18) for $\phi = \phi(Y, 0.4)$ using the diagonal numerical values in Table 3. The variation of the curve ϕ as a function of Y at $t = 0.4$ due to the variations of the parameter $I_o c^{-1}$ is so small that it cannot be shown clearly on this diagram.

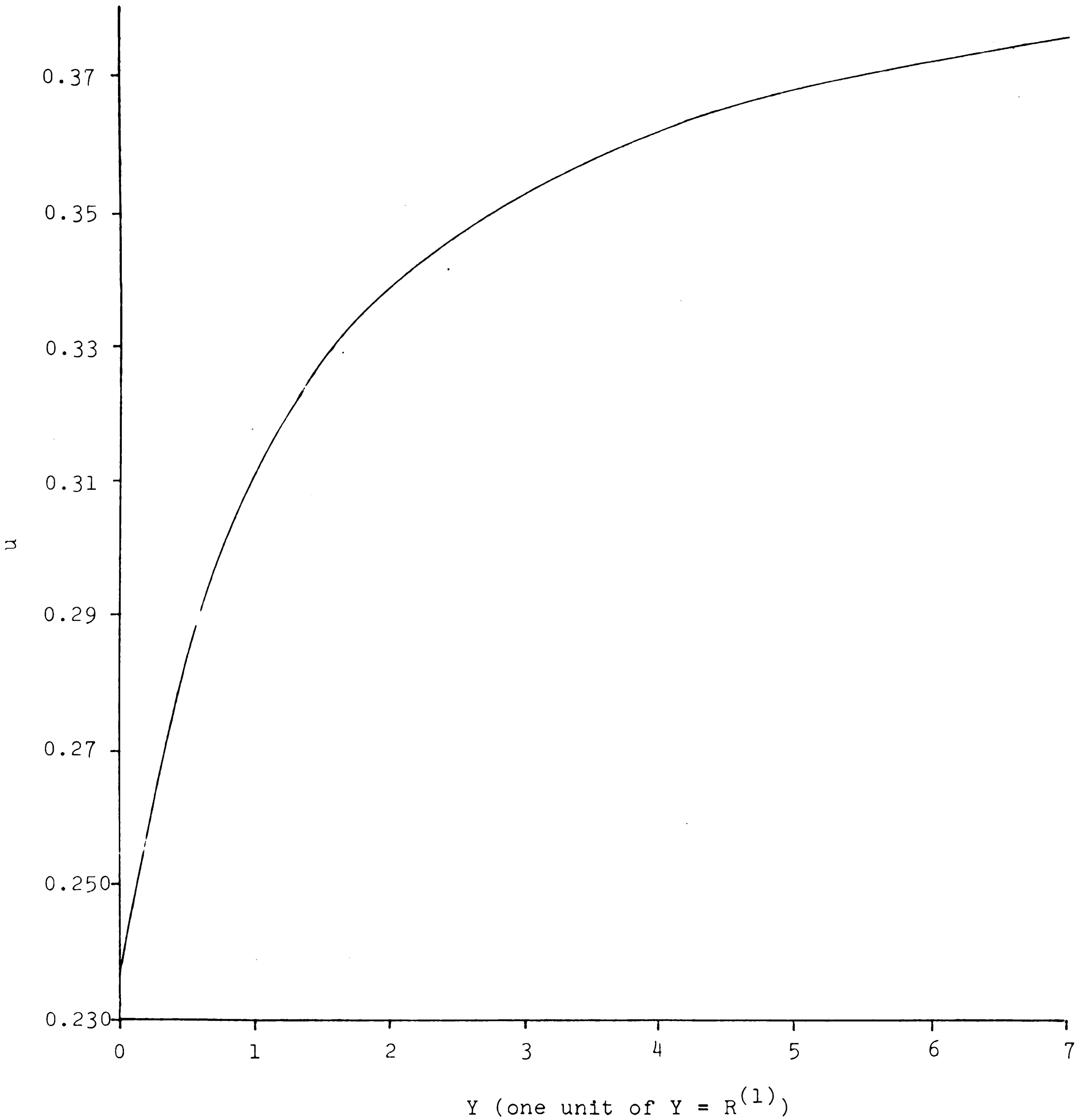

Figure 8.--Approximate solution of eqs. (2.6.17) and (2.6.18) for $u = u(Y,0.4)$ using the diagonal numerical values in Table 4. The variation of the curve u as a function of Y at t = 0.4 due to the variations of the parameter I_0c^{-1} is so small that it cannot be shown clearly on this diagram.

4.3. Shock Model

Let us assume that the shock is moving away from the celestial body described in Section 4.1. As discussed in Sections 2.8 and 2.9 a coordinate system $\{Y\}$ is introduced such that the shock becomes stationary and perpendicular to the Y^1-axis. As in many practical problems, we specify the shock parameters ρ_-^o and p_- on the right side of the shock, and we choose either the pressure p_+ on the left side or the pressure ratio ξ as an additional parameter describing the strength of the shock, (see [9]). The remaining shock parameters (ρ_+^o or η, u_- and u_+) are calculated from (2.8.1), (2.9.6), (2.9.9), taking into account (2.9.1), (2.9.2) and (2.9.3).

For a chosen constant value of the quantity $p_-\rho_-^{o-1}$, (i.e., the temperature on the right side of the shock is kept constant), Figures 9, 10 and 11 show the relations $(\xi, \bar{u}_-^1)$, $(\eta, \bar{u}_-^1)$ and $(\bar{u}_+^1, \bar{u}_-^1)$ respectively, for different constant values of the velocity parameter I_o. The linear relation between $\bar{u}_\pm^1$ and $u_\pm$ for different values of I_o is also shown in Figure 12.

In conclusion, Figures 9 and 10 indicate that the shock parameters ξ and η increase as the gravitational potential χ increases or I_o decreases keeping $\bar{u}_-^1$ constant; or, for fixed values of the shock parameters ξ and η, the velocity $\bar{u}_-^1$ increases as χ decreases or I_o increases. Similarly, Figure 11 indicates that the velocity

$\bar{u}^1_+$ increases as χ decreases or I_o increases keeping $\bar{u}^1_-$ = constant.

A critical shock strength line "$\xi_{cr.}$" is drawn in Figure 11. This line shows that the value of $\bar{u}^1_-$ on the left hand side increases as χ decreases or I_o increases, whereas $\bar{u}^1_-$ on the right hand side of that line increases as χ increases or I_o decreases, keeping $\bar{u}^1_+$ = constant in both cases. The latter case has not been investigated in the present work.

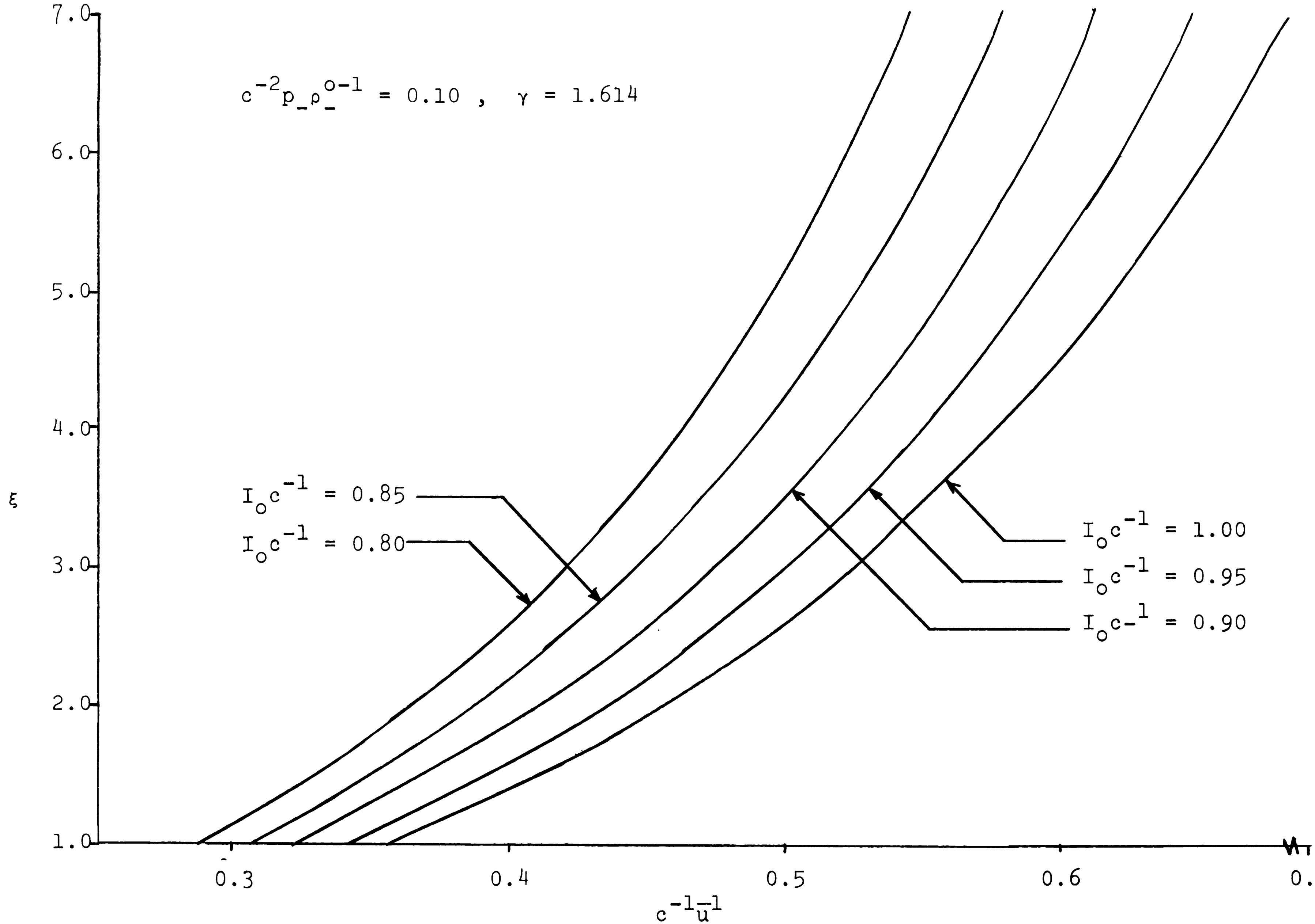

Figure 9.--The pressure ratio $\xi = p_{+}p_{-}^{-1}$ vs. the dimensionless shock velocity $c^{-1}\bar{u}_{-}^{1}$ for different values of the parameter $c^{-1}I_o$ when $c^{-2}p_{-}\rho_{-}^{o-1} = 0.10$ and $\gamma = 1.614$.

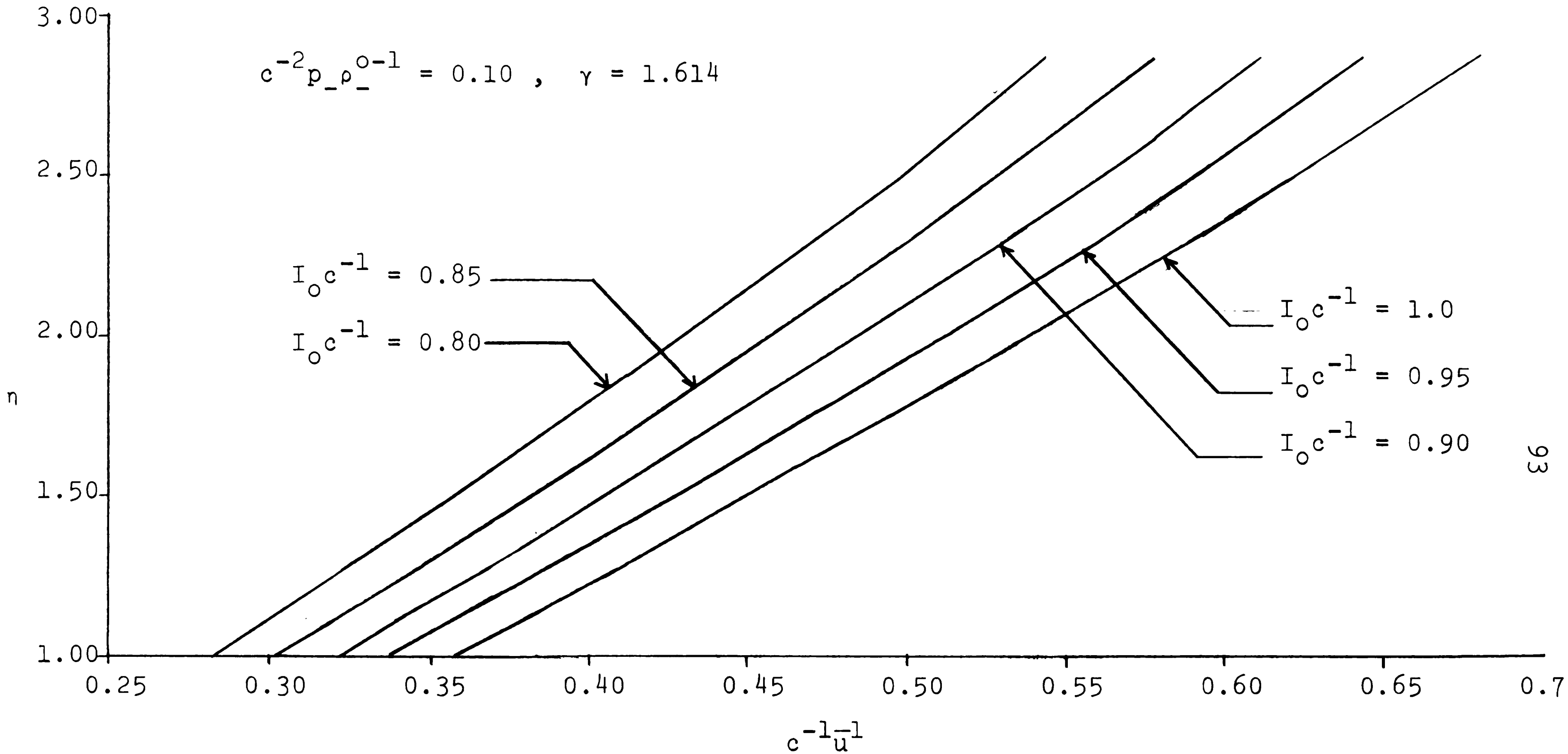

Figure 10. The density ratio $\eta = \rho_{+}^{o}\rho_{-}^{o-1}$ vs. $c^{-1}\bar{u}_{-}^{1}$ for different values of the parameter $c^{-1}I_o$ when $c^{-2}p_{-}\rho_{-}^{o-1} = 0.10$ and $\gamma = 1.614$.

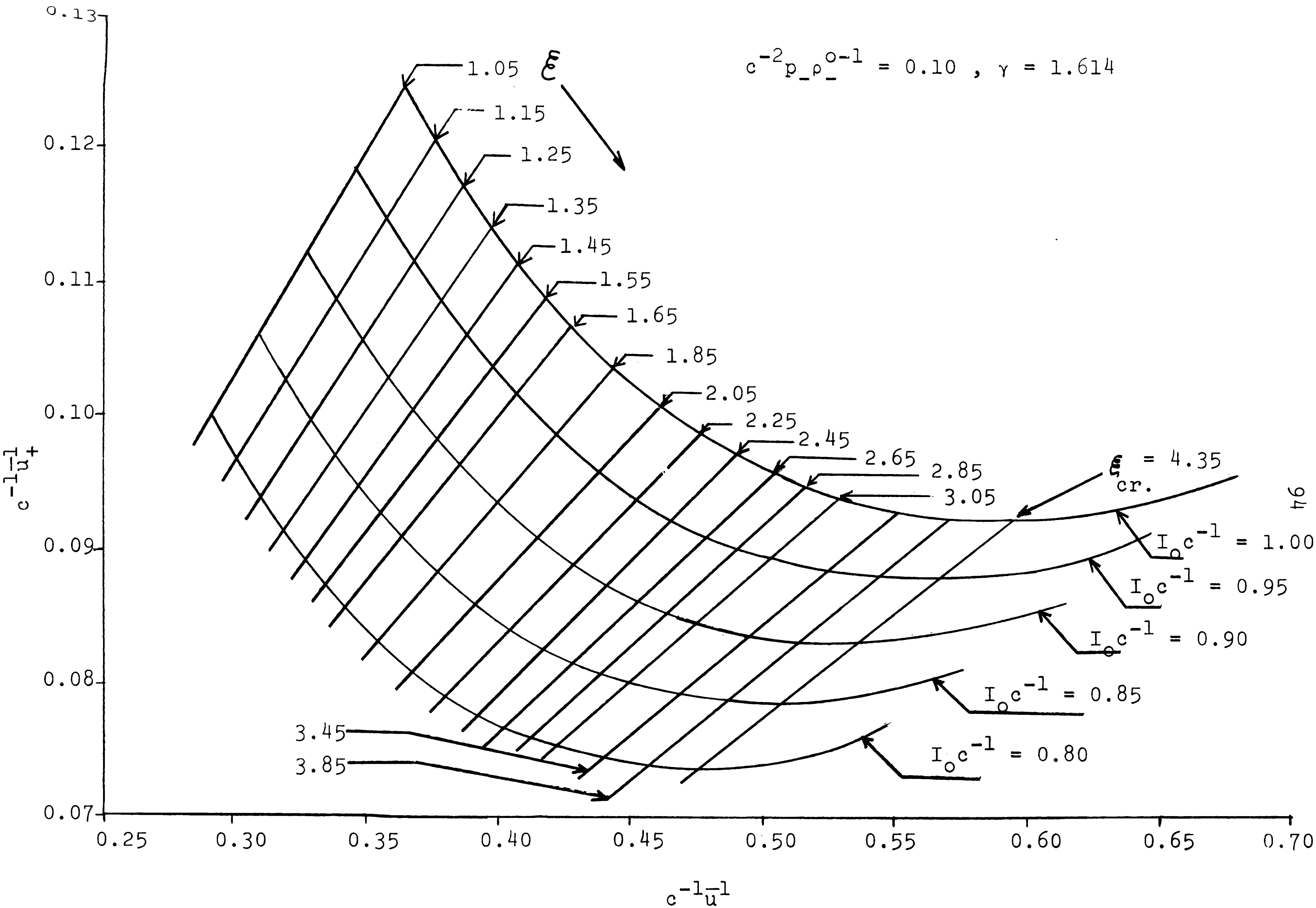

Figure 11.--The dimensionless velocity $c^{-1}\bar{u}_+^{-1}$ on the right side of the shock vs. the dimensionless velocity $c^{-1}\bar{u}_-^{-1}$ on the left side of the shock for different values of the parameter $c^{-1}I_o$ when $c^{-2}p_-\rho_-^{o-1} = 0.10$ and $\gamma = 1.614$. Constant pressure ratio lines are also shown.

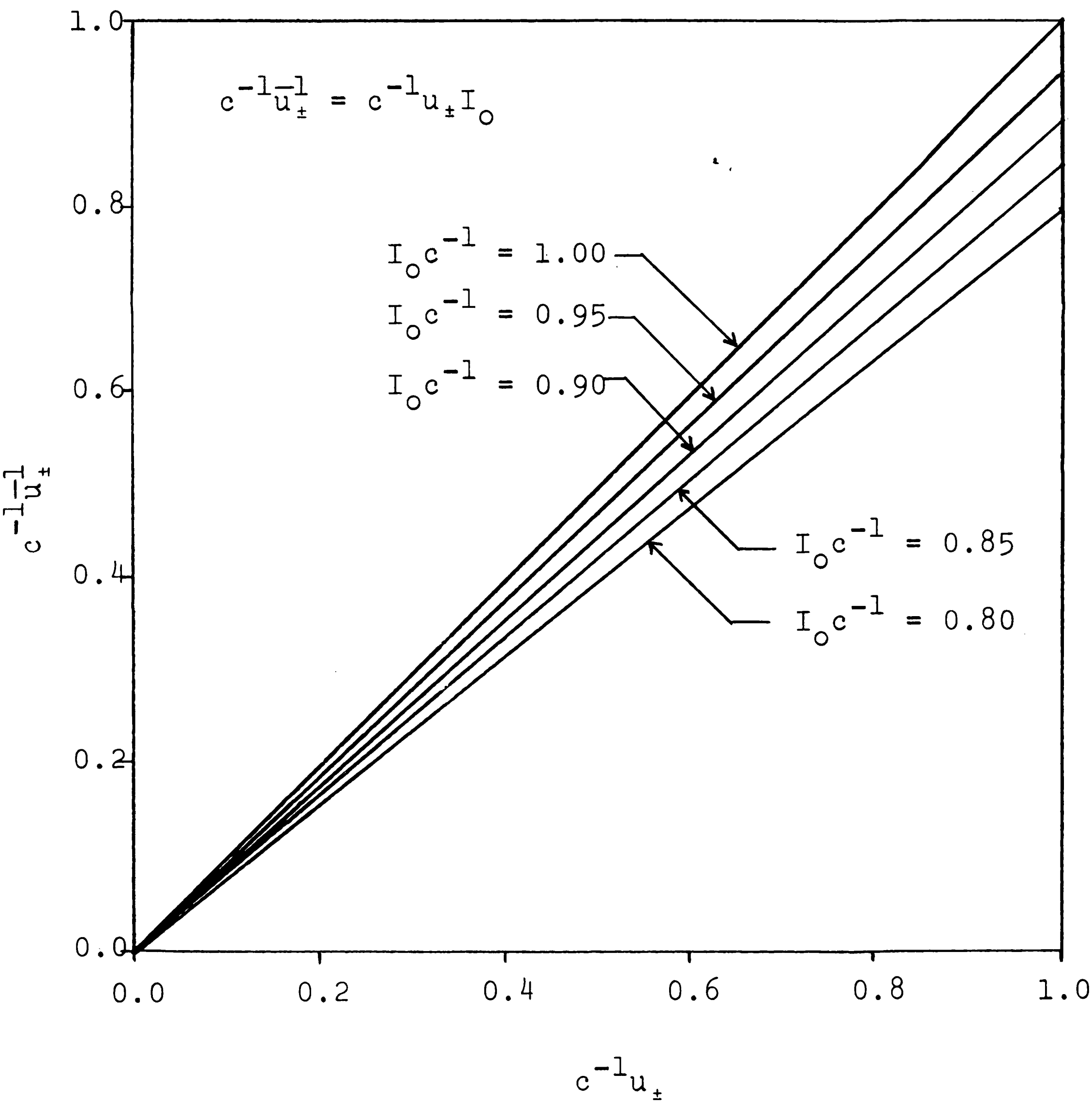

Figure 12.--$c^{-1}\bar{u}_{\pm}^{1}$ vs. $c^{-1}u_{\pm}$ for different values of the parameter $c^{-1}I_o$.

REFERENCES

1. Aller, L. H. The Atmosphere of the Sun and Stars. The Ronald Press Company, second ed., 1963.

2. Chapman, S. and Cowling, T. G. The Mathematical Theory of Non-uniform Gases. Cambridge at the University Press, ed., 1961.

3. Collatz, L. The Numerical Treatment of Differential Equations. Springer-Verlag, Berlin, third ed., 1960.

4. Courant, R. and Friedrichs, K. O. Supersonic Flows and Shock Waves. Interscience Publishers, Inc., 1948.

5. Eckart, C. Thermodynamics of Irreversible Processes. III, Relativistic Theory of the Simple Fluid. Phys. Rev., Vol. 58, No. 15, 1940, pp. 919-924.

6. Einstein, A. Ueber das Relativitaetsprinzip und die aus demselben gezogenen Folgerungen. Jahrb. d. Radioakt und Elektronik, 4, 1907, pp. 411-457.

7. Fok, V. A. The Theory of Space-Time and Gravitation. Pergamon Press, 1959, (Russian ed., 1955).

8. Goto, K. Relativistic Magnetohydrodynamics. Progress of Theoretical Phys., Vol. 20, No. 1, 1958, pp. 1-14.

9. Hirschfelder, J. O., Curtis, C. F. and Bird, R. B. Molecular Theory of Gases and Liquids. Wiley, Inc., 1954.

10. Krzywoblocki, M. Z. v. On the General Form of the Special Theory of Relativity. I, II, III, IV. Acta Physica Austriaca, Vol. 13, No. 4, 1960, pp. 387-394; Vol. 14, No. 1, 1961, pp. 22-28; Vol. 14, No. 1, 1961, pp. 39-49; Vol. 14, No. 2, 1961, pp. 239-241.

11. Krzywoblocki, M. Z. v. Special Relativity--A Particular Energy Formulation in Newtonian Mechanics? I, II. Acta Physica Austriaca, Vol. 15, No. 3, 1962, pp. 201-212, 251-261.

12. Mises, V. R. Mathematical Theory of Compressible Fluid Flow. Academic Press, Inc., 1958.

13. Moller, C. The Theory of Relativity. Oxford at the Clarendon Press, 1952.

14. Pauli, W. Theory of Relativity. Pergamon Press, 1958.

15. Synge, J. L. The Relativistic Gas. Interscience Publishers, Inc., 1957.

16. Synge, J. L. Relativity: The Special Theory. North-Holland Publishing Company, Amsterdam, 1955.

17. Synge, J. L. and Schild, A. Tensor Calculus. University of Toronto Press, 1949.

18. Taub, A. H. Relativistic Rankine-Hugoniot Equations. Phys. Rev., Vol. 74, No. 3, 1948, pp. 328-334.

19. Tolman, R. C. Relativity, Thermodynamics and Cosmology. Oxford, The Clarendon Press, 1934.

20. Vlasov, A. A. Many-Particle Theory and its Application to Plasma. Gordon and Breach Science Publishers, Inc., 1961.