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dissertation entitled

DEFORMATION AND LOAD-CARRYING CAPACITY OF HOLLOW ELASTIC CYLINDERS SUBMERGED IN A FLUID

presented by

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has been accepted towards fulfillment of the requirements for

Ph.D. degree in Mathematics

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Major professor

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DEFORMATION AND LOAD-CARRYING CAPACITY OF HOLLOW ELASTIC CYLINDERS SUBMERGED IN A FLUID

By

Mohamed Belal Mohamed Elgindi

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
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ABSTRACT

DEFORMATION AND LOAD-CARRYING CAPACITY OF HOLLOW ELASTIC CYLINDERS SUBMERGED IN A FLUID

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This thesis investigates the equilibrium states of a thin-walled, elastic, cylindrical shell fully or partially submerged in a fluid. Previous studies on the deformation of the shell have assumed that the pressure due to the fluid is uniform. The present study takes into consideration the non-uniformity of the pressure. The consideration of the pressure gradient brings an additional parameter to the problem and is essential to the load-carrying capacity of the shell.

Several formulations of the problem for the shell displacement under non-uniform pressure are given, which are followed to prove existence of solution. The problem is solved analytically using perturbation methods and the analysis shows that there exist critical values of the pressure near which drastic changes in the deflection patterns occur which in turn affect the load-carrying capacity of the shell. Mathematically these correspond to bifurcation and perturbed bifurcation problems. Some numerical methods for treating these "pitchfork" bifurcation and perturbed bifurcation problems are introduced. Numerical solutions based on Newton's iteration and shooting methods are obtained for both the fully and partially submerged cases. The results show that given a pressure gradient, there is an upper limit for the load-carrying capacity of the shell and this maximum occurs with the shell being partially submerged. The load-carrying capacity decreases with the depth when the shell is fully submerged and thus the corresponding equilibrium solution is unstable.

DEDICATION

TO THE SOUL OF MY FATHER

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The work presented in this thesis would have been much more difficult had it not been for the constant love and encouragement of the author's wife, Hala, to whom much credit is due.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd.)

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

This thesis is concerned with the problem of a thin-walled, elastic, cylindrical shell fully or partially submerged in a fluid. The problem obviously serves as a model for many problems with engineering importance. Previous studies on the deformation of the shell under the assumption of a uniform pressure have been done by I. Tadjbakhsh [15], J.E. Flaherty [6] and C.Y. Wang [17], among others. These studies show that there exists a critical uniform pressure beyond which the shell will buckle. Beyond this "buckling" pressure the deformation of the shell differs drastically from the circular and the calculation of the post-buckling shape requires in general the solution of a mathematical nonlinear bifurcation problem. The post-buckling shapes have been determined numerically for the case of uniform pressure in [15] and [6].

The present study is concerned with a non-uniform fluid pressure by taking account of the effect of gravity. The consideration of a pressure gradient brings an additional parameter to the problem and is essential to the load-carrying capacity of the shell. The analysis shows, however, that critical pressures still exist near which drastic changes in the deflection patterns occur which in turn affect the load-carrying capacity of the shell. Mathematically one now has to contend with the so-called perturbed bifurcation problems.

Due to the two-dimensional nature of the problem it suffices to consider a typical cross-section of the shell, i.e. to consider a thin ring.

The ring will be modeled as an elastica.

In Chapter 1 the problem for the shell displacement under non-uniform pressure is formulated along two lines. The first leads to a nonlinear boundary value problem with two physical parameters λ and τ representing respectively the depth and the non-uniformity of the pressure. The second formulation is in the form of a variational problem based upon Hamilton's principle of least energy. In Chapter 2, the two variational formulations of Chapter 1 are followed to prove the existence of solution. The proof is based upon a work on variational methods for nonlinear elliptic eigenvalue problems by F.E. Browder [1]. In Chapter 3, regular and singular perturbation methods are used to obtain analytic solutions of the problem for small values of the non-uniformity τ. A singular or modified perturbation method is necessary to treat the case when λ is near one or a set of critical values and the regular perturbation method breaks down. In Chapter 4, a numerical method for determining both the bifurcation and perturbed bifurcation curves is introduced. This method is based upon the Liapunov-Schmidt reduction technique and Newton's method. The use of Newton's method in bifurcation problems was considered by D.W. Decker and H.B. Keller [5]. In their studies both the solution and the bifurcation parameter are further parameterized by a new parameter (an approximation of the arc length). In [5] Decker and Keller treated the bifurcation parameter as an unknown to be determined along with the solution. Although their method is general it suffers the drawback that in practical problems the bifurcation parameter is often a physical quantity and solutions are required for some given values of this parameter. In the present study, convergence of Newton's iterations to

the solution paths parameterized with the bifurcation parameter near a bifurcation point is shown for some general bifurcation problems under appropriate conditions. It is also shown that Newton's method can be used to determine the perturbed bifurcation points and the perturbed bifurcation paths. In Chapter 5 the boundary value problem formulated in Chapter 1 is discretized by the shooting method. This leads to a finite dimensional bifurcation problem. The finite dimensional problem is then solved by the numerical method introduced in Chapter 4. Finally, numerical results and interpretations are given and discussed. In Chapter 6 the conclusions of Chapters 1 - 5 are summarized and discussed.

CHAPTER 1

MATHEMATICAL FORMULATION

1.0 Introduction

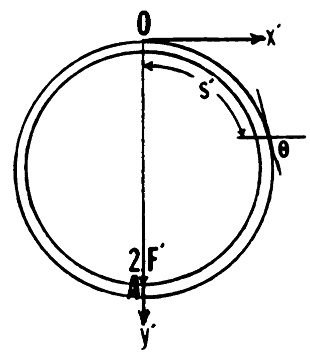
The physical assumptions upon which the mathematical formulation is based are given in Section 1.1. The problem for the cylinder displacement is formulated along two different lines. The first formulation given in Section 1.2 is the classical one. It leads to a nonlinear boundary value problem with two physical parameters λ and τ representing respectively the depth and the non-uniformity of the pressure. This formulation will be followed in Chapter 2 to prove the existence of solutions with arbitrary norm. In Section 1.3 the formulation presented in Section 1.2 is modified for the case of partially submerged cylinder. A second and equivalent formulation based upon Hamilton's principle is given in Section 1.4. It is in the form of a minimization problem which is followed in Chapter 2 to prove the existence of solutions for each given value of λ and τ.

1.1 Physical assumptions and notations

Consider a hollow elastic cylinder submerged in a fluid with its axis lying horizontally. Because of the two-dimensional nature of the problem it suffices to consider a typical cross section, i.e. a circular elastic ring. The thickness of the ring will be assumed small compared

with other length parameters of the problem. Under these assumptions the ring will be treated as an elastica with the local bending moment being proportional to the local curvature.

Let (x',y') be Cartesian coordinates with the origin at O, s' be the arc length measured from O and θ be the local angle (see Figure 1.1.1).



A typical cross section.

Figure 1.1.1

The following notations will be used:

p_o ≡ The pressure difference between the inside and the outside of the ring at the point O.

H' ≡ The horizontal component of the internal force at the point O.

$2F'$ ≡ An external vertical force applied at the point A.

L ≡ Half the perimeter of the ring.

EI ≡ The flexural rigidity of the ring.

ρ ≡ The density of the fluid outside the ring.

g ≡ The gravitational acceleration.

m ≡ The local bending moment per longitudinal length.

1.2 Formulation for the fully submerged case

Consider an element of length ds'. The hydrostatic pressure p is given by

$$p = p_o + \rho g\, y' \tag{1.2.1}$$

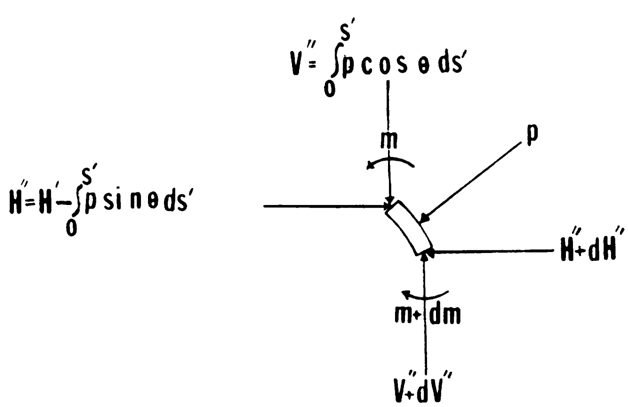
A moment balance on the element ds' gives (see Figure 1.2.1)

$$m + dm = m + \left(-H' + \int_0^{s'} p \sin\theta\, ds'\right) ds' \sin\theta + \left(\int_0^{s'} p\cos\theta\, ds'\right) ds' \cos\theta$$

or

$$\frac{dm}{ds'} = \left(-H' + \int_0^{s'} p \sin\theta\, ds'\right)\sin\theta + \left(\int_0^{s'} (p\cos\theta\, ds'\right)\cos\theta, \tag{1.2.2}$$

where p is given by (1.2.1).



Internal forces and local bending moments acting on an element of length ds'

Figure 1.2.1

The Cartesian coordinates are related to the arc length by

$$\frac{dx'}{ds'} = \cos\theta, \quad \frac{dy'}{ds'} = \sin\theta \tag{1.2.3}$$

The physical assumption of an elastica means m is proportional to $\frac{d\theta}{ds'}$, the proportionality constant being EI

$$m = EI\,\frac{d\theta}{ds'} \tag{1.2.4}$$

Using (1.2.1), (1.2.3) and (1.2.4), equation (1.2.2) then becomes

$$EI\,\frac{d^2\theta}{ds^2} = (-H' + p_0 y' + \frac{1}{2}\rho g y'^2)\sin\theta + (p_0 x' + \rho g \int_0^{s'} y' \cos\theta\, ds')\cos\theta \tag{1.2.5}$$

The following non-dimensional quantities are now used

$$s = \frac{s'}{L},\ x = \frac{x'}{L},\ y = \frac{y'}{L},\ \lambda = \frac{p_0 L^3}{EI},\ \tau = \frac{\rho g L^4}{EI},\ H = \frac{H' L^2}{EI},\ F = \frac{F' L^2}{EI}.$$

Equation (1.2.5) becomes

$$\theta'' = (-H + \lambda y + \frac{1}{2}\tau y^2)\sin\theta + (\lambda x + \tau u)\cos\theta \tag{1.2.6}$$

where u is defined by

$$u(s) = \int_0^s y(t)\cos\theta(t)\,dt.$$

The equations of equilibrium for the ring are now:

$$\begin{aligned}
&(a)\quad \theta_{ss} = (-H + \lambda y + \frac{1}{2}\tau y^2)\sin\theta + (\lambda x + \tau u)\cos\theta \\
&(b)\quad x_s = \cos\theta \\
&(c)\quad y_s = \sin\theta \\
&(d)\quad u_s = y\cos\theta
\end{aligned} \tag{1.2.7}$$

with the boundary conditions

$$\begin{aligned} &\theta(0) = x(0) = y(0) = u(0) = 0 \\ &x(1) = 0 \quad \text{and} \quad \theta(1) = \pi \end{aligned} \tag{1.2.8}$$

and H in (1.2.7) is to be determined.

Observe that the vertical force F' is regarded as a parameter to be determined by the shape of the ring as follows. A balance of the vertical forces gives

$$F' + \int_0^L p \cos\theta \, ds' = 0$$

and therefore

$$\begin{aligned} F = \frac{EI\,F'}{L^2} &= \frac{EI}{L^2} \int_0^L (p_o + \rho g y')\cos\theta \, ds \\ &= -\tau \int_0^1 y \cos\theta \, ds \\ &= -\tau\, u(1). \end{aligned}$$

When $\tau = 0$ (i.e. when the ring is under uniform pressure λ) the boundary value problem (1.2.7), (1.2.8) has the following (basic) solution:

$$\begin{aligned} \theta_o &= \pi s \\ x_o &= \frac{1}{\pi} \sin \pi s \\ y_o &= \frac{1}{\pi} (1-\cos \pi s) \\ u_o &= \frac{1}{\pi} \sin \pi s - \frac{1}{4\pi^2} \sin 2\pi s - \frac{1}{2\pi} s \\ H_o &= \frac{\lambda}{\pi}, \end{aligned} \tag{1.2.9}$$

for each value of λ. This means that the ring remaining circular is one solution for each value of λ when $\tau = 0$. For $\tau \neq 0$, however, (1.2.9) will no longer be a solution and the nature of the problem will be completely different.

For Chapters 3, 4 and 5 it will be convenient to write the boundary value problem (1.2.7) and (1.2.8) in the form

$$G(X,\lambda,\tau) = LX - f(X,\lambda) - \tau N(X) = 0 \tag{1.2.10}$$

$$B[X] = B_0 X(0) + B_1 X(1) = 0 \tag{1.2.11}$$

where $X = (x_1,x_2,x_3,x_4,x_5,x_6)^T$, $x_1 = \theta - \pi s$, $x_2 = \theta_s - \pi$,

$x_3 = x - \frac{1}{\pi} \sin \pi s$, $x_4 = y - \frac{1}{\pi}(1 - \cos \pi s)$,

$x_5 = u - (\frac{1}{\pi^2} \sin \pi s - \frac{1}{4\pi^2} \sin 2\pi s - \frac{1}{2\pi} s)$, $x_6 = H - \frac{\lambda}{\pi}$,

$LX = \frac{d}{ds} X$,

$$f(X,\lambda) = \begin{bmatrix} x_2 \\ [-x_6+\lambda(x_4-\frac{1}{\pi}\cos \pi s)]\sin(x_1+\pi s) + \lambda(x_3+\frac{1}{\pi}\sin \pi s)\cos(x_1+\pi s) \\ \cos(x_1+\pi s) - \cos \pi s \\ \sin(x_1+\pi s) - \sin \pi s \\ (x_4+\frac{1}{\pi}(1-\cos \pi s))\cos(x_1+\pi s) - \frac{1}{\pi}(1-\cos \pi s)\cos \pi s \\ 0 \end{bmatrix}$$

$$N(X) = \begin{bmatrix} 0 \\ \frac{1}{2}[x_4 + \frac{1}{\pi}(1-\cos \pi s)]^2 \sin(x_1+\pi s) + [x_5 + \frac{\sin \pi s}{\pi^2} - \frac{\sin 2\pi s}{4\pi^2} - \frac{s}{2\pi}] \\ \cos(x_1+\pi s) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and B_0, B_1 are the constant matrices

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

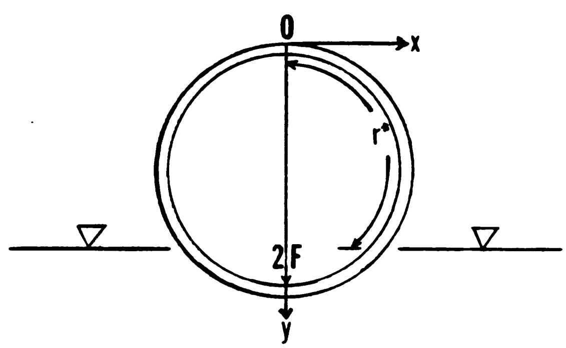
Note again that when $\tau = 0$ the boundary value problem (1.2.10), (1.2.11) has the trivial (basic) solution $X_0(\lambda) = 0$ for each λ.

Remark: Observe that in deriving the formulation (1.2.7) the top point of the ring O was kept fixed (see Figure 1.1.1). Alternatively one may arrive at a similar formulation by keeping the bottom point of the ring fixed. Obviously the resulting equations will not be the same as those in (1.2.7). However, they differ by a constant shift in the uniform part of the pressure λ (due to the two different locations of the top point of the ring in the two cases). The variational formulation to be derived in Section 1.4 is equivalent to the one obtained by balancing the moments while the bottom point of the ring is held fixed.

The load carrying capacity γ ($\equiv$ F) then does not enter the energy expression in the variational formulation and therefore may be treated as an unknown to be determined by the shape of the ring. The variational formulation is followed in Chapter 2 to prove the existence of a weak solution for each given value of λ and τ. The variational formulation with the top point of the ring being fixed will be given in Section 2.3, and used to prove the existence of equilibrium states for each given value of γ and τ with $|\frac{\gamma}{\tau}| < \frac{1}{2\pi}$.

1.3 Formulation for the partially submerged case

Let $0 \leqslant r^* \leqslant 1$ denotes the arc length of the non-wetted part of the ring (see Figure 1.3.1).



A partially submerged ring

Figure 1.3.1

In this case the wetted and non-wetted parts of the ring will have different equilibrium equations. Note also that the pressure difference at 0 is zero, i.e. $\lambda = 0$. Let $y^* = y(r^*)$, $x^* = x(r^*)$ and

$u = \int_{r^*}^{s} y \cos\theta \, dt$. Using the same notations as in Sections 1.1 and 1.2 the equations of equilibrium for the non-wetted part may be written as $(0 \le s \le r^*)$

$$\begin{aligned} &(a) \quad \theta_{ss} = -H \sin\theta \\ &(b) \quad x_s = \cos\theta \\ &(c) \quad y_s = \sin\theta \end{aligned} \tag{1.3.1}$$

and for the wetted part $(r^* \le s \le 1)$

$$\begin{aligned} &(a) \quad \theta_{ss} = (-H + \tfrac{1}{2}\tau(y-y^*)^2)\sin\theta + \tau(u\cos\theta - y^*(x-x^*)\cos\theta) \\ &(b) \quad x_s = \cos\theta \\ &(c) \quad y_s = \sin\theta \end{aligned} \tag{1.3.2}$$

with the conditions that x, y, u, θ and θ_s are continuous at r^* and satisfy

$$\begin{aligned} &x(0) = y(0) = \theta(0) = u(r^*) = 0 \\ &x(1) = 0, \quad \theta(1) = \pi \end{aligned} \tag{1.3.3}$$

Observe that when $\tau = 0$ the boundary value problem (1.3.1), (1.3.2) and (1.3.3) has the (basic) solution

$$\theta_0 = \pi s$$

$$x_0 = \frac{1}{\pi}\sin\pi s$$

$$y_0 = \frac{1}{\pi}(1 - \cos\pi s)$$

$$u_0 = \begin{cases} 0 & , \ 0 \le s \le r^* \\ \frac{1}{\pi^2}(\sin\pi s - \sin\pi r^*) - \frac{1}{4\pi^2}(\sin 2\pi s - \sin 2\pi r^*) - \frac{1}{2\pi}(s - r^*), & r^* \le s \le 1 \end{cases}$$

$$H_0 = 0.$$

1.4 Variational formulation

Hamilton's principle of least energy is used here to give a variational formulation for the problem previously formulated in Section 1.2. The variational formulation will be followed in Section 2.2 to prove the existence of a weak solution for each value of λ and τ and the "load carrying capacity" γ to be determined.

Let (x,y) be Cartesian coordinate with origin taken at the point 0, s be the (normalized) arc length measured from 0 and θ be the local angle (see Figure 1.1.1)

Let λ, τ and γ ($\equiv$ F) be as in Section 1.2. Note that in the present case the bottom point of the ring will be kept fixed while the top point is free to move vertically. In this case the coordinates x, y of a point on the ring are related to the local angle θ by the relations:

$$x(s) = \int_{-1}^{s} \cos\theta(t)dt$$

$$y(s) = \frac{2}{\pi} + \int_{-1}^{s} \sin\theta(t)dt.$$

Let $\vec{n}$ denote the normal vector $(y_s,-x_s)^T$ and $\vec{z}$ denote the position vector $(x,y)^T$. Then the strain energy W and the potential energy P are given by

$$W = \frac{1}{2}(\theta_s - \pi)^2$$

$$P = -\frac{1}{2}(\lambda + \frac{2}{3}\tau y)\,\vec{n}\cdot\vec{z}.$$

Let $y^*(s) = \int_{-1}^{s} \sin\theta(t)dt$, so that $y(s) = y^*(s) + \frac{2}{\pi}$.

Then P takes the form

$$P = -\frac{1}{2}(\lambda + \frac{2}{3}\tau y^* + \frac{4\tau}{3\pi})[\int_{-1}^{s}\sin(\theta(s) - \theta(\xi))d\xi] + \frac{1}{\pi}(\lambda + \frac{2}{3}\tau y^* + \frac{4\tau}{3\pi})\cos\theta$$

Hamilton's principle leads to the following variational formulation.

Minimize $(V(\theta) \equiv \int_{-1}^{1} (W-P)ds)$

$$V(\theta) = -\pi^2 + \frac{1}{2}\int_{-1}^{1} [\theta_s^2 + (\lambda + \frac{2}{3}\tau y^* + \frac{4\tau}{3\pi})\int_{-1}^{s}\sin(\theta(s)-\theta(\xi))d\xi]ds$$

$$-\frac{1}{\pi}\int_{-1}^{1}(\lambda + \frac{2}{3}\tau y^* + \frac{4\tau}{3\pi})\cos\theta\, ds \tag{1.4.1}$$

over all θ on $[-1,1]$ satisfying

$$\theta(-1) = -\pi, \quad \theta(1) = \pi, \quad \int_{-1}^{1}\cos\theta(s)ds = \int_{-1}^{1}\sin\theta(s)ds = 0 \tag{1.4.2}$$

Let η denote an "admissible direction" i.e.

$$\eta(1) = \eta(-1) = 0 \quad \text{and} \quad \int_{-1}^{1}\eta\sin\theta\, ds = \int_{-1}^{1}\eta\cos\theta\, ds = 0.$$

Then the variation $\delta V(\theta)$ in the direction of any such η vanishes and this leads to Euler's equation. The variation of $V(\theta)$ in the direction of η is defined by $V(\theta) = \frac{d}{d\varepsilon}V(\theta + \varepsilon\eta)\Big|_{\varepsilon=0}$.

It is easy to verify the following relations:

(1) $$\delta\int_{-1}^{1}\theta_s^2\, ds = -2\varepsilon\int_{-1}^{1}\eta\,\theta_{ss}\, ds$$

(2) $$\delta\int_{-1}^{1}\int_{-1}^{s}\sin(\theta(s)-\theta(\xi))d\xi ds = 2\varepsilon\int_{-1}^{1}\eta[x\cos\theta + y^*\sin\theta]ds$$

$$(3)\quad \delta \int_{-1}^{1} y^*(s) \int_{-1}^{s} \sin(\theta(s)-\theta(\xi))d\xi ds = \frac{3}{2}\,\varepsilon \int_{-1}^{1} \eta \sin\theta\, y^{*2}\, ds$$

$$+ 3\varepsilon \int_{-1}^{1} \eta \cos\theta\, u\, ds - 3u(1)\varepsilon \int_{-1}^{1} \eta \cos\theta\, ds$$

where $u(s)$ is defined by $u(s) \equiv \int_{-1}^{s} y^*(t) \cos\theta(t)\, dt$

$$(4)\quad \delta \int_{-1}^{1} \cos\theta\, ds = -\,\varepsilon \int_{-1}^{1} \eta \sin\theta\, ds,$$

$$(5)\quad \delta \int_{-1}^{1} y^* \cos\theta\, ds = -\,\varepsilon \int_{-1}^{1} \eta[y^* \sin\theta + x\cos\theta]ds$$

It follows from the relations (1)-(5) that the Euler equation is:

$$-\theta_{ss} + \lambda(x\cos\theta + y^*\sin\theta) + \frac{1}{2}\,\tau\, y^{*2} \sin\theta + \tau\, u \cos\theta$$

$$-\tau u(1)\cos\theta + \frac{1}{\pi}\,(\lambda + \frac{4}{3\pi}\,\tau)\sin\theta + \frac{2\tau}{\pi}\,[y^*\sin\theta + x\cos\theta]$$

$$= \mu_1 \sin\theta + \mu_2 \cos\theta. \tag{1.4.3}$$

where μ_1 and μ_2 are arbitrary constants.

Remark: It can be shown that the Euler's equation (1.4.3) is identical with the equilibrium equation resulting from balancing the moments while keeping the bottom point of the ring fixed. It follows that any solution of the variational problem (1.4.1), (1.4.2) will be an equilibrium solution.

CHAPTER 2

EXISTENCE OF SOLUTION

2.0 Introduction

This chapter is concerned with the proof of existence of the solution for the problem formulated in Chapter 1. The proofs are based upon a theorem, due to F. Browder, which is used in the proof of existence in the uniform pressure case [15]. In Section 2.1 some mathematical preliminaries due to Browder [1] are given and two corollaries of Sobolev's and Rellich's Imbedding Theorems [7] are stated. In Section 2.2 the Browder's theory outlined in Section 2.1 is used to prove the existence of a weak solution to the variational problem (1.4.1), (1.4.2) for each given value of λ and τ. In Section 2.3 a variational formulation, with the top of the ring being fixed is stated and used to prove the existence of a weak solution for each given value of γ and τ with $|\frac{\gamma}{\tau}| < \frac{1}{2\pi}$. In Section 2.4 it is shown that for each γ and $k > 0$ there exists a weak solution θ, H, λ, τ of the boundary value problem (1.2.7) satisfying $\|\theta_S\|_{L_2} = \pi^2 + k$, and that for each value of γ, λ and $k > 0$ there exists a weak solution θ, H, τ of the boundary value problem (1.2.7) satisfying $\|\theta_S\|_{L_2} = \pi^2 + k$.

2.1 Abstract variational problems

Let H be a real Banach space and $\{w_j\}_{j=1}^{\infty}$ be a sequence in H.

Then $w_j \longrightarrow w$ will denote the convergence of the sequence $\{w_j\}_{j=1}^{\infty}$ to w in the H-norm and $w_j \rightharpoonup w$ will denote the convergence of the sequence $\{f^*(w_j)\}_{j=1}^{\infty}$ to $f^*(w)$ for every $f^* \in H^*$, the dual space of H. A subset C of H is said to be weakly closed if whenever a sequence $\{w_j\}_{j=1}^{\infty}$ in C converges weakly to w, then w is also in C. Note that every closed convex subset of a reflexive Banach space H is weakly closed.

The following definitions and theorem are due to Browder [1].

Definition 2.1.1: A function $\Phi : H \times H \longrightarrow R^1$ is said to be semi-convex if it satisfies the following three conditions:

(a) For each w in H and each c in R^1 the subset $S_{c,w}$ defined by

$$S_{c,w} = \{v \in H : \Phi(v,w) \leq c\}$$

is convex, i.e. for each w in H the function

$$\Phi(\cdot,w) : H \longrightarrow R^1$$

is convex function.

(b) For each bounded subset B of H and each sequence $\{w_j\}_{j=1}^{\infty}$ in H with $w_j \rightharpoonup w$, $\Phi(v,w_j) \longrightarrow \Phi(v,w)$ uniformly for v in B,

(c) For each w in H, $\Phi(\cdot,w)$ is continuous as a function from H to R^1.

Definition 2.1.2: A function $g : H \longrightarrow R^1$ is said to be differentiable at w_0 in H if there exists an element $g'(w_0)$ in H^* such that for all h in H

$$g(w_0+h) - g(w_0) - g'(w_0)(h) = o(\|h\|) \quad \text{as} \quad \|h\| \longrightarrow 0.$$

A function $\Phi : H \times H \longrightarrow R^1$ is differentiable at (w_1,w_2) if there exists a pair f_1^*, f_2^* in H^* such that for all h_1, h_2 in H $\Phi(w_1+h_1,\ w_2+h_2) - \Phi(w_1,w_2) - f_1^*(h_1) - f_2^*(h_2) = o(\|h_1\| + \|h_2\|)$ as $\|h_1\| \longrightarrow 0$ and $\|h_2\| \longrightarrow 0$, and we set $f_1^* = \Phi_1'(w_1,w_2)$, $f_2^* = \Phi_2'(w_1,w_2)$.

Remark: If $\Phi : H \times H \longrightarrow R^1$ is differentiable at (w_0,w_0) then $E(w) = \Phi(w,w)$ is differentiable at w_0 and

$$E'(w_0) = \Phi_1'(w_0,w_0) + \Phi_2'(w_0,w_0).$$

Theorem 2.1.1: Let H be a real reflexive Banach space, $\Phi : H \times H \longrightarrow R^1$ be semi-convex and $E(V) = \Phi(V,V)$. Let C be a weakly closed bounded subset of H. Then E is bounded below on C and assumes it minimum on C.

Let H_1 denote the Hilbert space

$$H_1([0,1],R^1) = \{w : w, w_s \in L_2([0,1],R^1)\}$$

with the inner product

$$\langle v,w\rangle = \int_0^1 (vw + v_s w_s)ds,$$

where derivatives are being understood in the weak sense.

Let $\overset{o}{H}_1$ denote the subspace of H_1

$$\overset{o}{H}_1 = \{w \in H_1 : w(0) = w(1) = 0\}.$$

The following two theorems are corollaries of Sobolev's and Rellich's Imbedding Theorems, respectively. The statements and proofs of Sobolev's and Rellich's Imbedding Theorems can be found in any

standard book on elliptic partial differential equations (for example see [7]).

Theorem 2.1.2: Each $w \in H_1$ is equal almost everywhere to a continuous function on $[0,1]$, and the imbedding of H_1 in the space of continuous functions on $[0,1]$, with the maximum norm, is continuous.

Theorem 2.1.3: The imbedding of H_1 in L_2 is compact.

2.2 Existence of solution for fixed λ and τ

In this section it is shown that the variational problem (1.4.1), (1.4.2) has a weak solution for each given value of λ and τ.

Let us start by recalling the variational problem formulated in Section 1.4. It is required to minimize the functional

$$V(\theta) = -\pi^2 + \frac{1}{2}\int_{-1}^{1} [\theta_s^2 + (\lambda + \frac{2}{3}\tau y)(\frac{-2}{\pi}\cos\theta + \int_{-1}^{s}\sin(\theta(s) - \theta(\xi))d\xi)]ds \tag{2.2.1}$$

overall θ on $[-1,1]$ which satisfy the conditions

$$\theta(-1) = -\pi, \quad \theta(1) = \pi, \quad \int_{-1}^{1} \sin\theta(s)ds = \int_{-1}^{1} \cos\theta(s)ds = 0. \tag{2.2.2}$$

Upon using the transformation $u(s) = \theta(s) - \pi s$, (2.2.1) and (2.2.2) reduce, respectively, to:

$$V(u) = \frac{1}{2}\int_{-1}^{1} [u_s^2 + (\lambda + \frac{2}{3}\tau y(u))h(u)]ds \tag{2.2.3}$$

and

$$u(-1) = 0, \quad u(1) = 0, \quad \int_{-1}^{1}\sin(u + \pi s)ds = \int_{-1}^{1}\cos(u + \pi s)ds = 0 \tag{2.2.4}$$

where

$$\text{(a)} \quad y(u) = \frac{2}{\pi} + \int_{-1}^{s} \sin(u(t) + \pi t)dt \qquad (2.2.5)$$

$$\text{(b)} \quad h(u) = \frac{-2}{\pi}\cos(u + \pi s) + \int_{-1}^{s} \sin(u(s) - u(\xi) + \pi s - \pi\xi)d\xi$$

Browder's theory outlined in Section 2.1 will be used to prove the existence of a solution of the above variational problem for each given value of λ and τ. The proof simply consists of verifying the monotonicity and smoothness conditions of Theorem 2.1.1. In order to define the problem more precisely, let $\overset{o}{H}_1$ be the Hilbert space defined in Section 2.1 and S be the set

$$S = \{v \in \overset{o}{H}_1 : \int_{-1}^{1} \sin(v + \pi s)ds = \int_{-1}^{1} \cos(v + \pi s)ds \doteq 0\} \qquad (2.2.6)$$

and $\Phi : H \times H \longrightarrow R^1$ be the functional

$$\Phi(u,v) \equiv \frac{1}{2}\int_{-1}^{1} [u_s^2 + (\lambda + \frac{2}{3}\tau y(v))h(v)]ds \qquad (2.2.7)$$

<u>**Lemma 2.2.1**</u>: The set S defined by (2.2.6) is weakly closed in $\overset{o}{H}_1$.

<u>**Proof**</u>: Let $\{v_j\}_{j=1}^{\infty}$ be a sequence in S which converges weakly to v in $\overset{o}{H}_1$. Since every weakly convergent sequence in $\overset{o}{H}_1$ is bounded, it follows from Theorem 2.1.3 that $\{v_j\}_{j=1}^{\infty}$ has a subsequence which converges to v in the L_2-norm, which may still be called $\{v_j\}_{j=1}^{\infty}$. But

$$\left| \int_{-1}^{1} \sin(v(t) + \pi t)dt - \int_{-1}^{1} \sin(v_j(t) + \pi t)dt \right| \leq c \, \|v_j - v\|_{L_2}$$

for some constant c. It follows that $\int_{-1}^{1} \sin(v(t) + \pi t)dt = 0$. Similarly $\int_{-1}^{1} \cos(v(t) + \pi t)dt = 0$. It follows that S is weakly closed.

Lemma 2.2.2: The function Φ defined by (2.2.7) is semi-convex and differentiable on $\overset{o}{H}_1 \times \overset{o}{H}_1$.

Proof: The differentiability of Φ is clear from its definition. The proof of the semi-convexity of Φ consists of checking the three conditions of Definition 2.1.1.

(a) For each fixed v, it is required to prove that the function $\Phi(\cdot,v)$ is convex. For $t \in [0,1]$ and u_0, u_1 in $\overset{o}{H}_1$, set $u_t = tu_1 + (1-t)u_0$, and define

$$h(t) = \Phi(u_t,v) - t\Phi(u_1,v) - (1-t)\Phi(u_0,v).$$

It is required to prove that $h(t) \leq 0$ on $[0,1]$. Since $h(0) = h(1) = 0$, it is enough to prove that $h'(t)$ is nondecreasing. But

$$h'(t) = \Phi_u(u_t,v)(u_1 - u_0) - \Phi(u_1,v) + \Phi(u_0,v)$$

and for any $0 \leq t < \xi \leq 1$

$$h'(\xi) - h'(t) = \frac{1}{\xi-t} [\Phi_u(u_\xi,v) - \Phi_u(u_t,v)](u_\xi - u_t)$$

$$= \frac{1}{\xi-t} \int_0^1 [(u_\xi - u_t)_s]^2 ds \geq 0.$$

(b) Suppose that $\{v_j\}_{j=1}^{\infty}$ is a sequence in $\overset{o}{H}_1$ which converges weakly to v. Then, as in Lemma 2.2.1 the sequence $\{v_j\}_{j=1}^{\infty}$ can be

replaced by a subsequence which converges to v in the L_2-norm. Now

$$2|\Phi(u,v) - \Phi(u,v_j)| \leq |\lambda| \left| \int_{-1}^{1} y(v_j) - y(v)ds \right|$$

$$+ \frac{2|\tau|}{3} \left| \int_{-1}^{1} y(v_j)(h(v_j) - h(v))ds \right|$$

$$+ \frac{2|\tau|}{3} \left| \int_{-1}^{1} h(v)(y(v_j) - y(v))ds \right|$$

$$\leq c\|v_j - v\|_{L_2} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.$$

The last inequality follows from the boundedness of the functions $y(u)$, $h(u)$ and the Mean Value Theorem.

(b) Let $\{u_j\}_{j=1}^{\infty}$ be any sequence which converges to some u in $\overset{o}{H}_1$. Then

$$|\Phi(u,v) - \Phi(u_j,v)| = \frac{1}{2} \left| \int_{-1}^{1} (u_s^2 - u_{js}^2)ds \right|$$

$$\leq c\|u_s - u_{js}\|_{L_2} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.$$

Lemma 2.2.3: The functional $V(u) \equiv \Phi(u,u)$ is differentiable on $\overset{o}{H}_1$ and satisfies $V(u) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$.

Proof: The differentiability of V follows from that of Φ, and since $\|u\| \longrightarrow \infty$ in $\overset{o}{H}_1$ if and only if $\|u_s\| \longrightarrow \infty$ the lemma follows.

Now by Lemma 2.2.3 there exists a large enough number K such that $V(u) > V(0)$ for any u not in the closed ball D of radius K. Since the closed ball D is weakly closed, it follows from Lemma 2.2.1 that the set $C \equiv D \cap S$ is weakly closed bounded. It follows from Theorem

2.1.1 that V is bounded below and assumes its minimum on C. This proves the following

Theorem 2.2.1: For each value of λ and τ the variational problem (2.2.3), (2.2.4) has a solution in $\overset{o}{H}_1$.

2.3 Existence of solution for fixed γ and τ

In this section the problem of existence of solution for a given value of γ, τ is considered. With the top point of the ring being fixed, the expression for the total energy becomes

$$V(\theta) = -\pi^2 + \frac{1}{2}\int_0^2 [\theta_s^2 + (\lambda + \frac{2}{3}\tau y - 4\gamma\delta(s-1)\int_0^s \sin(\theta(s) - \theta(\xi))d\xi]ds. \tag{2.3.1}$$

where $\delta(s-1)$ is the Dirac function.

It is required to minimize $V(\theta)$ sugject to the constrains

$$\theta(0) = 0, \quad \theta(2) = 2\pi \quad \text{and} \quad \int_0^2 \cos\theta\, ds = \int_0^2 \sin\theta\, ds = 0 \tag{2.3.2}$$

With the transformation $u = \theta - \pi s$, $V(\theta)$ becomes

$$V(u) = \frac{1}{2}\int_0^2 [u_s^2 + (\lambda + \frac{2}{3}\tau y(u) - 4\gamma\delta(s-1))h(u)]ds \tag{2.3.3}$$

where

$$y(u) \equiv \int_0^s \sin(u + \pi s)ds,$$

$$h(u) \equiv \int_0^s \sin(u(s) - u(\xi) + \pi s - \pi\xi)d\xi.$$

The above variational problem can now be reformulated in the following form. Given τ, γ find λ_0 in R^1 and $u_0 \in \overset{o}{H}_1$ such that

the variation $\delta V(u_0)$ vanishes identically in the direction of any $\eta \in H_1^0$ with $\int_0^2 \eta \sin(u_0 + \pi s)ds = \int_0^2 \eta \cos(u_0 + \pi s)ds = 0$, and u_0 satisfies the conditions

$$\begin{aligned} &\int_0^2 \sin(u_0 + \pi s)ds = \int_0^2 \cos(u_0 + \pi s)ds = 0 \\ &\tau \int_0^2 y(u_0) \cos(u_0 + \pi s)ds = -2\gamma. \end{aligned} \tag{2.3.4}$$

Define $\Phi : H_1^0 \times H_1^0 \longrightarrow R'$, $E : H_1^0 \longrightarrow R'$, $g : H_1^0 \longrightarrow R'$ and the subset $S \subseteq H_1^0$ by

(a) $\Phi(u,v) \equiv \frac{1}{2} \int_0^2 u_s^2 \, ds + \frac{1}{2} \int_0^2 [\frac{2}{3} \tau y(v) - 4\gamma\delta(s-1)]h(v)ds$

(b) $E(u) \equiv \Phi(u,u)$ (2.3.5)

(c) $g(u) \equiv \int_0^2 y(u) \cos(u + \pi s)ds$

(d) $S \equiv \{u \in H_1^0 : \int_0^2 \sin(u + \pi s)ds = \int_0^2 \cos(u + \pi s)ds = 0\}$

__Lemma 2.3.2__: The function Φ defined by (a) of (2.3.5) is differentiable and semi-convex on H_1^0.

__Lemma 2.3.2__: The function E defined by (b) of (2.3.5) is differentiable and $E(u) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$.

The proofs of the above two lemmas are similar to the proofs of Lemmas 2.2.2 and 2.2.3 and are therefore omitted. Also it follows from Lemma 2.2.1 that the set S is weakly closed.

Lemma 2.3.3: For $\tau \neq 0$ the subset

$$C \equiv \{u \in \overset{\circ}{H}_1 : g(u) = -\frac{\gamma}{\tau}\}$$

is weakly closed, where g is as in (c) of (2.3.5).

Proof: It is enough to show that the function g is weakly continuous. For this let $\{u_n\}_{n=1}^{\infty}$ be any sequence in $\overset{\circ}{H}_1$ with $u_n \longrightarrow u$ for some $u \in \overset{\circ}{H}_1$. As before, $\{u_n\}_{n=1}^{\infty}$ has a subsequence which converges to u in the L_2-norm. It follows from the Mean Value Theorem and the following inequality that $g(u_n) \rightarrow g(u)$.

$$|g(u_n)-g(u)| \leq \int_0^2 |y(u_n)| \; |\cos(u_n + \pi s) - \cos(u + \pi s)|ds$$

$$+ \int_0^2 |\cos(u_n + \pi s)| \; |y(u_n)-y(u)|ds.$$

Lemma 2.3.4: For any γ, $\tau \neq 0$, the functional E assumes its minimum value on the set $S \cup C$.

Proof: Since Φ is semi-convex, $S \cap C$ is weakly closed and $E(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$, it follows from Theorem 2.1.1 that E is bounded below on $S \cap C$ and assumes its minimum value on $S \cap C$.

Theorem 2.3.1: For any γ, $\tau \neq 0$ for which $S \cap C \neq \phi$ there exist λ, μ_1, μ_2 in R^1 such that if u_0 is the minimizer of E on $S \cap C$ then

$$E'(u_0) + \lambda \; g'(u_0) = \mu_1 \int_0^2 \sin(u_0 + \pi s)ds$$

$$+ \mu_2 \int_0^2 \cos(u_0 + \pi s)ds$$

Proof: Follows from Ljusternik's Theorem (since all the constraints under consideration are smooth).

Remark: It is easy to see that the function g, for the problem under consideration, satisfies

$$|g(u)| < \frac{1}{\pi} . \qquad (2.3.6)$$

Inequality (2.3.6) restricts the range of γ and τ in Theorem 2.3.1. Observe that for $\gamma, \tau \neq 0$ with $|\frac{\gamma}{\tau}| < \frac{1}{2\pi}$ it is clear, geometrically, that the set $S \cap C$ will not be empty. Finally, note that Theorem 2.3.1 implies the existence of $u_0 \in \overset{0}{H}_1$ and $\lambda_0 \in R^1$ such that u_0 satisfies the conditions (2.3.4), and the variation $\delta V(u_0)$ vanishes identically in the direction of every admissible function η.

2.4 Existence of solution for fixed γ

In this section the problem of existence of solution for a prescribed value of γ is considered. Let us first derive the equilibrium equations which will be used here. The variables H, x and u can be eliminated from equation (a) of (1.2.7) to give:

$$\theta_{sss} - c\theta_s + \frac{\theta_s^3}{2} = \gamma + \tau \int_0^2 \sin\theta \, dt, \qquad (2.4.1)$$

where c is an arbitrary constant of integration. Upon using the transformation $w = \theta_s - \pi$, equation (2.4.1) reduces to

$$w_{ss} + \nu w = \delta - f(w) - \tau \int_0^2 \sin(\pi\tau + \int_0^t w(\xi)d\xi)dt \qquad (2.4.2)$$

where $\nu = \frac{3}{2}\pi^2 - c$, $\delta = \lambda + c\pi - \frac{\pi^3}{2}$ and $f(w) = \frac{1}{2}w^3 + \frac{3}{2}\pi w^2$.

Formulate the problem as follows. Given γ in R^1 and $k > 0$, find $\tau, \delta, \nu \in R^1$ and $w \in H_1$ with w being a weak solution of (2.4.2) and satisfies the following conditions

$$\begin{aligned} & w_S(0) = 0, \quad w_S(1) = \gamma, \quad \|w\|_{L_2} = k, \\ & \int_0^1 w\,ds = 0, \quad \int_0^1 \cos(\pi s + \int_0^s w(\xi)d\xi)ds = 0. \end{aligned} \tag{2.4.3}$$

Let H denote the Hilbert space

$$H = \{w \in H_1 : \int_0^1 w\,ds = 0\}.$$

Observe that H is the kernel of a bounded linear functional on H_1.

Define $\Phi : H \times H \to R^1$, $E : H \to R$, $g_1 : H \to R$ and $g_2 : H \to R$ by

$$\begin{aligned} &(a) \quad \Phi(v,w) \equiv \frac{1}{2}\int_0^1 v_S^2\,ds - \int_0^1 [\frac{1}{8}w^4 + \frac{1}{2}\pi\, w^3]ds - \gamma\, v(1) \\ &(b) \quad E(w) \equiv \Phi(w,w) \\ &(c) \quad g_1(w) \equiv \int_0^1 w^2\,ds \\ &(d) \quad g_2(w) \equiv \int_0^1 \cos(\pi s + \int_0^s w(\xi)d\xi)ds. \end{aligned} \tag{2.4.4}$$

<u>**Lemma 2.4.1**</u>: The function Φ defined by (a) of (2.4.4) is differentiable and semi-convex on $H \times H$. The functions g_1, g_2 defined respectively by (c), (d) of (2.4.4) are differentiable and weakly continuous on H.

Proof: The differentiability of Φ, g_1 and g_2 is clear. That Φ is convex in v for fixed w follows from Lemma 2.2.2. The second condition for the semi-convexity of Φ follows from the fact that the imbedding of H in L_2 is compact. The third condition is clear. Finally g_1 and g_2 are clearly weakly continuous.

Lemma 2.4.2: The functional E defined by (b) of (2.4.4) is differentiable and satisfies

$$E(w) \to +\infty \quad \text{as} \quad \|w\| \to \infty$$

on the set $C \equiv \{w \in H : g_1(w) = k\}$, for any $k > 0$.

Proof: It is enough to consider smooth $w \in H$. Observe that for any $w \in C$

$$\int_0^1 w^4 \, ds \leq \max(w^2) \quad g_1(w) \leq k \max(w^2).$$

By the definition of H, there exists $\xi \in [0,1]$ such that $w(\xi) = 0$ and so

$$w^2(s) = 2 \int_\xi^s w \, w_s \, ds \leq 2 \, k^{1/2} \left(\int_0^1 w_s^2 \, ds \right)^{1/2}.$$

It follows that

$$\int_0^1 w^4 \, ds \leq 2 \, k^{3/2} \left(\int_0^1 w_s^2 \, ds \right)^{1/2}, \qquad \text{and}$$

$$\int_0^1 w^3 \, ds \leq 2^{1/2} \, k^{5/4} \left(\int_0^1 w_s^2 \, ds \right)^{1/4}.$$

From the definition of E and the above inequalities it follows that

$$E(w) \geq \frac{1}{2} \left(\int_0^1 w_s^2 \, ds \right) - \frac{1}{4} k^{3/2} \left(\int_0^1 w_s^2 \, ds \right)^{1/2}$$

$$- 2^{1/2} k^{5/4} \left(\int_0^1 w_s^2 \, ds \right)^{1/4} - 2|\gamma| \, k^{1/2} \left(\int_0^1 w_s^2 \, ds \right)^{1/2}.$$

As $\|w\| \to \infty$ with $g_1(w) = k$, $\int_0^1 w_s^2 \, ds \to \infty$ must hold. Therefore, the last inequalities give the result.

Let $S \equiv \{w \in H : g_2(w) = 0\}$.

It follows from Lemma 2.4.1 that the set S and the set C defined in Lemma 2.4.2 are both weakly closed. Then $S \cap C$ is weakly closed. Lemma 2.4.2 allows us to restrict to a bounded set. It follows from Theorem 2.1.1 that E assumes its minimum on $S \cap C$, say at w_0, and hence by Ljusternik's Theorem there exists ν_0 and τ_0 in R^1 such that

$$E'(w_0) + \nu_0 \, g_1'(w_0) + \tau_0 \, g_2'(w_0) = 0.$$

From the definition of H, it follows that there exists δ_0 in R^1 such that

$$E'(w_0) + \nu_0 \, g_1'(w_0) + \tau_0 \, g_2'(w_0) = \delta_0 \int_0^1 w_0 \, ds \tag{2.4.5}$$

Therefore w_0 is a weak solution of (2.4.2) satisfying conditions (2.4.3).

The conclusion of the above calculations is summarized in the following

Theorem 2.4.1: For any γ in R^1 and $k > 0$ for which the intersection $S \cap C$ is non-empty, there exists ν_0, δ_0, τ_0 in R^1 and $w_0 \in H_1$ such that $\|w_0\| = k$ and w_0 is a weak solution of (2.4.2) satisfying conditions (2.4.3) with $\nu = \nu_0$, $\delta = \delta_0$ and $\tau = \tau_0$.

Finally, observe that the proof for the following result is similar to the proof of Theorem 2.4.1 and is therefore omitted.

Theorem 2.4.2: For any γ and λ and R^1 and $k > 0$ there exists c_0, τ_0 in R^1 and $\theta \in H_1$ satisfying equation (2.4.1) with $c = c_0$, $\tau = \tau_0$ and the conditions $\theta(0) = 0$, $\theta(1) = \pi$, $\theta_{ss}(0) = 0$, $\theta_{ss}(1) = \gamma$, $\int_0^1 \cos\theta \, ds = 0$ and $\int_0^1 \theta_s^2 \, ds = k + \pi^2$.

CHAPTER 3

PERTURBATION METHODS

3.0 Introduction

This chapter is concerned with approximating the solution curves of the boundary value problem (1.2.10), (1.2.11) in the neighborhood of the trivial solution curve $(0,\lambda,0)$. A point $(0,\lambda_0,0)$ of the trivial solution curve through which there passes a non-trivial smooth solution curve will be called a bifurcation point. If $(0,\lambda_0,0)$ is a bifurcation point it follows from the Implicit Function Theorem that $G_X(0,\lambda_0,0)$ must be singular.

In Section 3.1 some basic facts about bifurcation from the trivial solution are presented and sufficient conditions for a singular point of the trivial solution curve to be a bifurcation point are given. All the bifurcation points of the above boundary value problem are found and asymptotic expansions of the bifurcating solution curves near the bifurcation points are obtained using the Liapunov-Schmidt theory [16]. Similar results were obtained in [15].

In Section 3.2, for $\tau \neq 0$ but small, the regular perturbation method is used to obtain asymptotic expansions for the solution curves of (1.2.10), (1.2.11) in the neighborhood of a point $(0,\lambda,0)$ for which $G_X(0,\lambda,0)$ has bounded inverse. These expansions are not valid near the bifurcation points. Also a perturbation solution for the case of partially submerged ring is given.

The behavior of the bifurcation points of a general bifurcation problem under certain perturbation $\tau \neq 0$ is examined in Section 3.3. It is shown, under some assumptions, that there is locally a family of "limit points" through each of which there passes a curve of nontrivial solutions.

Finally in Section 3.4, for a small $\tau \neq 0$, the singular perturbation method [13] is used to obtain valid asymptotic expansions of the solution curves of (1.2.10), (1.2.11) in the neighborhood of each bifurcation point.

3.1 Bifurcation from the trivial solution

Let H be a real Hilbert space, $G : H \times R^1 \to H$ be a smooth mapping and consider the equation

$$G(X,\lambda) = 0. \tag{3.1.1}$$

Assume that equation (3.1.1) has the trivial solution $X = 0$ for all $\lambda \in R^1$. If the Frechet derivative $G_X(0,\lambda_0)$ has a bounded inverse for some $\lambda_0 \in R^1$, the Implicit Function Theorem can be applied to guarantee the local existence and uniqueness of a smooth solution curve $X(\lambda)$ of (3.1.1) passing through $(0,\lambda_0)$. By uniqueness it must, then, be the trivial solution curve $X(\lambda) = 0$. Bifurcation theory, however, is concerned with the study of the solution set of (3.1.1) in a neighborhood of $(0,\lambda_0)$ where $G_X(0,\lambda_0)$ is not invertible. In the neighborhood of such a point there may exist non-trivial solution curves passing through $(0,\lambda_0)$.

A point $(0,\lambda_0)$ is called a bifurcation point if there is a smooth solution curve different from the trivial one, defined in a neighborhood

of $(0,\lambda_o)$ and passing through it. In this case such a non-trivial curve is called a bifurcating branch. From the above discussion a necessary condition for bifurcation to occur at $(0,\lambda_o)$ is that $G_X(0,\lambda_o)$ is not invertible.

Let $N(L)$ denote the null space, $R(L)$ denotes the range and L^* denotes the adjoint of a bounded linear operator L on **H**. For some $\lambda_o \in R$, the following assumptions are made about $G_X^o = G_X(0,\lambda_o)$

(a) $N(G_X^o)$ is one-dimensional spanned by Φ,

(b) $N(G_X^{o*})$ is one-dimensional spanned by Ψ, (3.1.2)

(c) $R(G_X^o) = N(G_X^{o*})^\perp$ and $R(G_X^{o*}) = N(G_X^o)^\perp$.

Thus G_X^o is assumed to be a Fredholm operator with zero index.

Conditions (3.1.2), however, are not sufficient to ensure that $(0,\lambda_o)$ is a bifurcation point. In addition to (3.1.2) the following assumption is made

$$a \equiv \langle \Psi, G_{X\lambda}(0,\lambda_o)\Phi \rangle \neq 0. \tag{3.1.3}$$

The number a in (3.1.3) is usually called the bifurcation coefficient. Under the assumptions (3.1.2) and (3.1.3) it will be shown, using the classical Liapunov-Schmidt method [16], that there exists a unique smooth non-trivial solution curve passing through $(0,\lambda_o)$ (and therefore $(0,\lambda_o)$ is a bifurcation point).

Using the assumptions of (3.1.2) the space H can be decomposed into the direct sums

$$\text{(a)} \quad H = \langle \Phi \rangle \oplus \langle \Phi \rangle^{\perp} \tag{3.1.4}$$

$$\text{(b)} \quad H = \langle \Psi \rangle \oplus R(G_X^o)$$

where $\langle X \rangle$ denotes the linear span of $X \in H$. The decomposition (a) allows us to write each $X \in H$ in the form $X = \varepsilon\Phi + w$ for unique $\varepsilon \in R^1$ and $w \in \langle \Phi \rangle^{\perp}$. Using the decomposition (b), equation (3.1.1) can be decomposed into the two equations

$$\text{(a)} \quad G(\varepsilon\Phi + w, \lambda_0 + \mu) - \langle \Psi, G(\varepsilon\Phi + w, \lambda_0 + \mu) \rangle \frac{\Psi}{\|\Psi\|^2} = 0 \tag{3.1.5}$$

$$\text{(b)} \quad \langle \Psi, G(\varepsilon\Phi + w, \lambda_0 + \mu) \rangle = 0$$

where $\lambda = \lambda_0 + \mu$.

Let $F : \langle \Phi \rangle^{\perp} \times R^1 \times R^1 \to H$ be defined by

$$F(w,\varepsilon,\mu) = G(\varepsilon\Phi + w, \lambda_0 + \mu) - \langle \Psi, G(\varepsilon\Phi + w, \lambda_0 + \mu) \rangle \frac{\Psi}{\|\Psi\|^2} .$$

Then $F(0,0,0) = 0$ and $F_w(0,0,0) = G_X^o$, which maps $\langle \Phi \rangle^{\perp}$ one to one onto $R(G_X^o)$. By the Open Mapping Theorem it follows that $F_w(0,0,0)$ has a bounded inverse and hence the Implicit Function Theorem can be applied to F. It follows that there exists a unique smooth curve $w = w(\varepsilon,\mu)$ defined on some neighborhood N of (0,0) such that

$$\text{(a)} \quad w(0,0) = 0 \tag{3.1.6}$$

$$\text{(b)} \quad F(w(\varepsilon,\mu),\varepsilon,\mu) = 0$$

for each (ε,μ) in N. This gives a unique solution to equation (a) of (3.1.5). Furthermore, the smoothness of the solution $w(\varepsilon,\mu)$ allow us to

expand it about (0,0) and compute its coefficients by successive differentiation of equation (b) of (3.1.6) with respect to ε and μ. This leads to

$$w(\varepsilon,\mu) = \varepsilon\mu w_1 + \varepsilon^2 w_2 + \varepsilon\, O[(|\varepsilon| + |\mu|)^2],$$

$$w_1 = A'\Phi - \overset{o}{G}_X{}^{-1}(\overset{o}{G}_{X\lambda}\Phi - a\frac{\Psi}{\|\Psi\|^2}), \tag{3.1.7}$$

$$w_2 = A''\Phi - \overset{o}{G}_X{}^{-1}(\overset{o}{G}_{XX}\Phi - \langle\Psi,\overset{o}{G}_{XX}\Phi\Phi\rangle\frac{\Psi}{\|\Psi\|^2}),$$

where $\overset{o}{G}_{X\lambda} = G_{X\lambda}(0,\lambda_0)$, $\overset{o}{G}_{XX} = G_{XX}(0,\lambda_0)$ and A', A'' are constants determined by the conditions $\langle\Phi,w_1\rangle = 0$, $\langle\Phi,w_2\rangle = 0$ respectively.

Having obtained the unique solution (3.1.7) of equation (a) of (3.1.5) one must substitute it into equation (b) of (3.1.5) to obtain the "bifurcation" equation

$$\varepsilon[E_0\varepsilon + a\mu + E_1\varepsilon\mu + E_2\mu^2 + E_3\varepsilon^2 + T(\varepsilon,\mu)] = 0 \tag{3.1.8}$$

(a) $E_0 = \frac{1}{2}\langle\Psi,\overset{o}{G}_{XX}\Phi\Phi\rangle$

(b) $E_1 = \frac{1}{2}\langle\Psi,\overset{o}{G}_{XX}\Phi w_1\rangle + \frac{1}{2}\langle\Psi,\overset{o}{G}_{XX}w_1\Phi\rangle + \frac{1}{2}\langle\Psi,\overset{o}{G}_{XX\lambda}\Phi\Phi\rangle + \langle\Psi,\overset{o}{G}_{X\lambda}w_2\rangle$

(c) $E_2 = \langle\Psi,\overset{o}{G}_{X\lambda}w_1\rangle$ (3.1.9)

(d) $E_3 = \frac{2}{3}\langle\Psi,\overset{o}{G}_{XX}\Phi w_2\rangle + \frac{1}{3}\langle\Psi,\overset{o}{G}_{XX}w_2\Phi\rangle + \frac{1}{6}\langle\Psi,\overset{o}{G}_{XXX}\Phi\Phi\Phi\rangle$

(e) $T(\varepsilon,\mu) = O[(|\varepsilon| + |\mu|)^3]$

where $\overset{o}{G}_{XX\lambda} = G_{XX\lambda}(0,\lambda_0)$ and $\overset{o}{G}_{XXX} = G_{XXX}(0,\lambda_0)$.

Solutions of the bifurcation equation (3.1.8) are, then, in one-to-one correspondance with the solutions of (3.1.1) in a heighborhood of $(0,\lambda_0)$. Dividing equation (3.1.8) through by the factor ε (which corresponds to the trivial solution) leads to the equation

$$g(\varepsilon,\mu) \equiv E_0\varepsilon + a\mu + E_1\varepsilon\mu + E_2\mu^2 + E_3\varepsilon^2 + T(\varepsilon,\mu) = 0 \tag{3.1.10}$$

whose solutions correspond to the non-trivial solutions of (3.1.1). Since $g(0,0) = 0$ and $g_\mu(0,0) = a$, condition (3.1.3) allows the application of the Implicit Function Theorem to $g(\varepsilon,\mu)$. It guarantees the existence and uniqueness of a smooth solution curve $\mu = \mu(\varepsilon)$ of (3.1.10) defined in some neighborhood N_1 of $\varepsilon = 0$ such that

$$\begin{aligned} &(1)\quad \mu(0) = 0 \\ &(2)\quad g(\varepsilon,\mu(\varepsilon)) = 0 \end{aligned} \tag{3.1.11}$$

for each ε in N_1. Expanding $\mu(\varepsilon)$ about $\varepsilon = 0$ and computing its coefficients leads to

$$\mu(\varepsilon) = -\frac{E_0}{a}\varepsilon - \frac{E_3 a + E_1 E_0 + E_2 E_0}{a^2}\varepsilon^2 + O(\varepsilon^3). \tag{3.1.12}$$

The above calculations proves the following

<u>Theorem 3.1.1</u>: A point $(0,\lambda_0)$ on the trivial solution curve of (3.1.1) satisfying conditions (3.1.2) and (3.1.3) is a bifurcation point. Furthermore, there exists exactly one bifurcation branch passing through $(0,\lambda_0)$ which is given (locally) by

$$X(\varepsilon) = \varepsilon\,\Phi + w(\varepsilon,\mu(\varepsilon))$$

$$\lambda(\varepsilon) = \lambda_0 + \mu(\varepsilon)$$

where $w(\varepsilon,\mu(\varepsilon))$ and $\mu(\varepsilon)$ are given by (3.1.7) and (3.1.12) respectively.

In what follows, the Liapunov-Schmidt theory outlined above, is applied to the ring problem formulated in Chapter 1 for the case when the ring is under a uniform pressure, i.e. $\tau = 0$.

Let $G(X,\lambda)$ denotes the mapping defined by (1.2.10) when $\tau = 0$, with domain

$$D = \{X \in C_6^1[0,1] \;:\; B[X] = 0\}$$

where B denotes the boundary operator defined by (1.2.11). The equation $G(X,\lambda) = 0$ has the trivial solution $X(\lambda) = 0$ for all $\lambda \in R^1$ and

$$G_X(0,\lambda) = L - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\lambda}{\pi} & 0 & \lambda\cos\pi s & \lambda\sin\pi s & 0 & -\sin\pi s \\ -\sin\pi s & 0 & 0 & 0 & 0 & 0 \\ \cos\pi s & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\pi}(1-\cos\pi s)\sin\pi s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is non-invertible if and only if $\lambda = \lambda_n = \pi^3(n^2-1)$, $n = 2,3,\ldots,$ in which case its null space has dimension one and is generated by $\Phi^{(n)}$

$$\Phi^{(n)} = \begin{bmatrix} \sin n\pi s \\ n\,\pi\cos n\pi s \\ -\left[\frac{\sin\pi(n-1)s}{2\pi(n-1)} - \frac{\sin\pi(n+1)s}{2\pi(n+1)}\right] \\ -\left[\frac{\cos\pi(n-1)s}{2\pi(n-1)} + \frac{\cos\pi(n+1)s}{2\pi(n+1)}\right] + \frac{n}{\pi(n^2-1)} \\ -\frac{1}{\pi}\left[\frac{\sin\pi(n-1)s}{2\pi(n-1)} - \frac{\sin\pi(n+1)s}{2\pi(n+1)}\right] - \frac{\sin\pi s}{\pi}\left[\frac{\cos\pi(n-1)s}{2\pi(n-1)} + \frac{\cos\pi(n+1)s}{2\pi(n+1)}\right] + \frac{n\sin\pi s}{\pi^3(n^2-1)} \\ n\pi^2 \end{bmatrix}$$

The adjoint G_X^{o*} of $G_X^o = G_X(0,\lambda_n)$ is given by

$$G_X^{o*}Z = - LZ - f_x{}^T(0,\lambda_n)Z = 0$$

with domain

$$D^* = \{Z \in C_6^1[0,1] : B^*[Z] = 0\}$$

where $B^*[Z] = B_0^*Z(0) + B_1^*Z(1)$, B_0^* and B_1^* are any 6×6 constant matrices with $\text{rank}[B_0^* : B_1^*] = 6$ and $B_0^*B_0^T = B_1^*B_1^T$.

It is well known [12] that the range of G_X^o is closed, $R(G_X^o) = N(G_X^{o*})^\perp$ and $R(G_X^{o*}) = N(G_X^o)^\perp$. The null space of G_X^{o*} is one dimensional generated by

$$\Psi(n) = \begin{bmatrix} -n\pi \cos n\pi s \\ \sin n\pi s \\ \lambda_n \left[\dfrac{\cos \pi(n-1)s}{2\pi(n-1)} + \dfrac{\cos \pi(n+1)s}{2\pi(n+1)}\right] \\ -\lambda_n \left[\dfrac{\sin \pi(n-1)s}{2\pi(n-1)} - \dfrac{\sin \pi(n+1)s}{2\pi(n+1)}\right] \\ 0 \\ \dfrac{\sin \pi(n-1)s}{2\pi(n-1)} - \dfrac{\sin \pi(n+1)s}{2\pi(n+1)} \end{bmatrix}$$

and the bifurcation coefficient $\langle\Psi(n), G_{X\lambda}^o\Phi(n)\rangle$ is given by $a(n) = \dfrac{n^2}{2\pi(n^2-1)}$.

It follows from the above calculations that the conditions (3.1.2) and (3.1.3) hold and, hence, by Theorem 3.1.1 each $(0,\lambda_n)$ is a bifurcation point.

For the problem under consideration $E_0 = 0$ and $E_3 = - 3/16\ \pi^2 n^2$, and therefore an approximation for the unique bifurcating

branch near $(0,\lambda_n)$ is given by

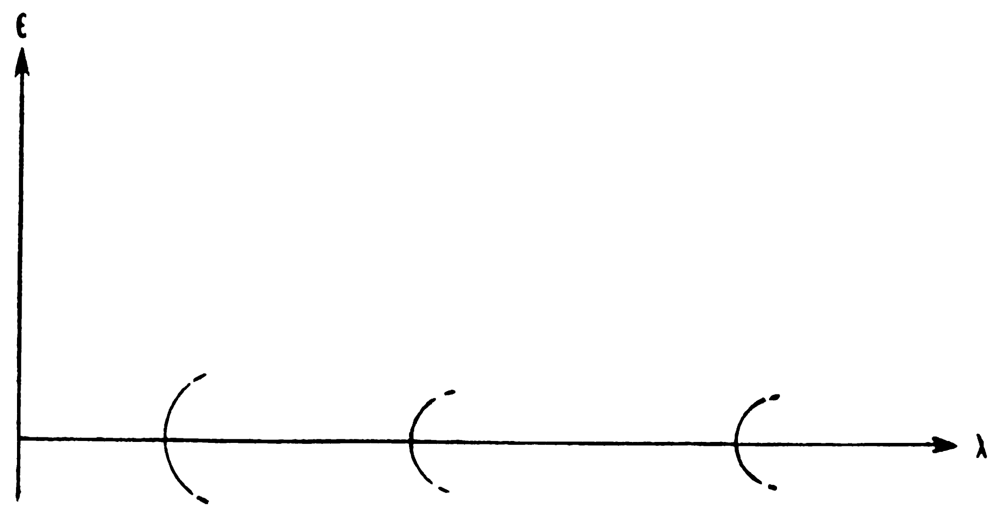
$$X = \varepsilon\, \phi^{(n)} + O(\varepsilon^2)$$

$$\lambda = \lambda_n + \frac{3\pi^3}{8}(n^2-1)\varepsilon^2 + O(\varepsilon^3).$$

Since the coefficient of ε^2, in the second expression above, is not zero, the implicit function theorem yields to

$$\varepsilon = \pm \left[\left(\frac{8}{3\pi^3(n^2-1)}\right)^{1/2} (\lambda-\lambda_n)^{1/2} + O((\lambda-\lambda_n))\right]$$

which shows that for each $\lambda > \lambda_n$ with $\lambda - \lambda_n$ small, there are two non-trivial solutions for the equation $G(X,\lambda) = 0$ (see Figure 3.1.1). The type of bifurcation occuring in this problem is known as "pitchfork bifurcation". This type of bifurcation frequently occurs in elasticity problems (for example the deformation of elastic rods and plates under an external applied edge thrust).



Pitchfork bifurcation at λ_n, $n = 2,3,\ldots$

<u>Figure 3.1.1</u>

3.2 Regular perturbation $(\tau \neq 0)$

Let G denote the mapping defined by (1.2.10) with domain

$$D = \{X \in C_6^1[0,1] : B[X] = 0\}$$

where B[X] is defined as in (1.2.11).

In this section the regular perturbation method [13] is used to obtain valid asymptotic expansions for the solution of

$$G(X,\lambda,\tau) = 0 \tag{3.2.1}$$

for small $\tau \neq 0$ in the neighborhood of a solution $(0,\lambda_o,0)$ where $G_X(0,\lambda_o,0)$ is invertible. For this one seeks asymptotic expansions as $\tau \to 0$ of the solutions $X(\lambda,\tau)$ of (3.2.1) near $(0,\lambda_o,0)$ in the form

$$X(\lambda,\tau) = \sum_{j=1}^{\infty} X_j(\lambda)\,\tau^j . \tag{3.2.2}$$

The coefficients $X_j(\lambda)$ are determined by inserting (3.2.2) into (3.2.1) and equating each power of τ to zero. This leads to a system of equations

$$\begin{aligned} &\text{(a)} \quad G_X(0,\lambda_o,0)X_1(\lambda) = -\,G_\tau(0,\lambda_o,0) \\ &\text{(b)} \quad G_X(0,\lambda_o,0)X_2(\lambda) = -\frac{1}{2}\Big[G_{XX}(0,\lambda_o,0)X_1(\lambda)X_1(\lambda) + 2G_{X\tau}(0,\lambda_o,0) + G_{\tau\tau}(0,\lambda_o,0)\Big] \end{aligned} \tag{3.2.3}$$

and so on, which can be solved successively since $G_X^{-1}(0,\lambda_o,0)$ exists.

Observe that expansions (3.2.2) are valid only in a neighborhood of each solution $(0,\lambda_o,0)$ where $G_X^{-1}(0,\lambda_o,0)$ exists. This neighborhood shrinks to zero as a bifurcation point $(0,\lambda_n,0)$ is approached.

An asymptotic expansions which is valid in a neighborhood of such a point will be obtained in Section 3.3 using the singular perturbation.

Returning to the case when $G_X^{-1}(0,\lambda_0,0)$ exists let us write $X_1 = (x_{11},x_{21},x_{31},x_{41},x_{51},x_{61})$. From the definition of G it follows that

$$G_T(0,\lambda_0,0) = (0, \frac{-\sin \pi s}{2\pi}, \frac{2 \cos \pi s}{2\pi},0,0,0,0)^T.$$

Solving equation (a) of (3.2.3) for $X_1(\lambda)$ one obtains

<u>Case 1</u>: $\lambda \neq 0$

$$x_{11} = \frac{-\lambda}{\pi^2}(x_{31} \cos \pi s + x_{41} \sin \pi s) + \frac{x_{61}}{\pi^2} \sin \pi s - \frac{3}{2\pi^4} \sin \pi s$$
$$+ \frac{s}{2\pi^3}(1 + \cos \pi s)$$

$$x_{21} = \frac{-1}{2\pi\ \mu \sin \mu} \cos \mu s - \frac{1}{\lambda} \cos \pi s + \frac{1}{2\pi\ \mu^2}$$

$$x_{31} = \frac{1}{2\pi\ \mu^2 \sin \mu}\left[\frac{\sin(\mu-\pi)s}{2(\mu-\pi)} - \frac{\sin(\mu+\pi)s}{2(\mu+\pi)}\right] + \frac{1}{2\pi\lambda}\left[s - \frac{\sin 2\pi s}{2\pi}\right]$$
$$+ \frac{1}{2\pi^2\ \mu^2}\left[s \cos \pi s - \frac{\sin \pi s}{\pi}\right]$$

$$x_{41} = \frac{1}{2\pi\ \mu^2 \sin \mu}\left[\frac{\cos(\mu-\pi)s}{2(\mu-\pi)} + \frac{\cos(\mu+\pi)s}{2(\mu+\pi)}\right] + \frac{1}{4\pi^2\lambda} \cos 2\pi\ s$$
$$+ \frac{1}{2\pi^2\ \mu^2}\left[s \sin \pi s + \frac{\cos \pi s}{\pi}\right] - \frac{1}{4\pi^2\lambda} - \frac{1}{2\pi^3\ \mu^2} - \frac{1}{2\mu\lambda \sin \mu}.$$

$$x_{51} = \frac{1}{\pi}(x_{41} \sin \pi s + x_{31}).$$

$$x_{61} = \frac{1}{2\mu^2} - \frac{1}{2\mu \sin \mu} - \frac{\pi}{\lambda} + \frac{1}{2\pi^2}$$

where $\mu = \sqrt{\pi^2 + \frac{\lambda}{\pi}}$.

<u>**Case 2:**</u> $\lambda = 0$

$$x_{11} = \frac{-3}{4\pi^4} \sin \pi s + \frac{1}{2\pi^3} s \cos \pi s + \frac{s}{2\pi^3}$$

$$x_{21} = \frac{-1}{4\pi^3} \cos \pi s - \frac{1}{2\pi^2} s \sin \pi s + \frac{1}{2\pi^3}$$

$$x_{31} = \frac{3}{8\pi^4} s - \frac{1}{4\pi^5} \sin 2\pi s + \frac{1}{8\pi^4} s \cos 2\pi s + \frac{1}{4\pi^4} s \cos \pi s - \frac{1}{2\pi^5} \sin \pi s$$

$$x_{41} = \frac{1}{8\pi^3} s^2 + \frac{1}{4\pi^5} \cos 2\pi s + \frac{1}{8\pi^4} s \sin 2\pi s + \frac{1}{2\pi^4} s \sin \pi s + \frac{1}{2\pi^5} \cos \pi s - \frac{3}{4\pi^5}$$

$$x_{51} = \frac{1}{\pi} (x_{41} \sin \pi s + x_{31}).$$

$$x_{61} = \frac{3}{4\pi^2} .$$

The above expansions are used to give approximations to $x_{21}(0)$ for different values of λ with $\tau = .001$ (see Table 3.2.1). These approximations become invalid near the bifurcation point $(0,\lambda_2,0)$ (see Figure 3.2.1).

The regular perturbation may also be used to obtain approximation to the solution for the case when the ring is partially submerged. Let

$0 \leq r^* \leq 1$ denotes the arc length of the non-wetted part of the ring, and $X(r^*,\tau)$ denotes the solution vector $(\theta,\theta',x,y,u,H)^T$ of the boundary value problem (1.3.1), (1.3.2) and (1.3.3). Write $X(r^*,\tau) = X_0(r^*) + \tau X_1(r^*) + O(\tau^2)$, where $X_0(r^*)$ denotes the basic solution (1.3.4). The first component x_{11} of $X_1(r^*)$ is given by

$$x_{11}(s) = \begin{cases} \dfrac{x_{61}}{\pi^2} \sin \pi s + C_1 s & 0 \leq s \leq r^* \\[2ex] (\dfrac{x_{61}}{\pi^2} - \dfrac{5}{4\pi^4}) \sin \pi s + \dfrac{1}{2\pi^3} s \cos \pi s - \dfrac{r^*}{2\pi^3} \cos \pi s & r^* \leq s \leq 1 \\[2ex] - \dfrac{1}{4\pi^4} \sin(s-2r^*) + C_2 s + C_3 & \end{cases}$$

where

$$x_{61} = \frac{1}{2\pi^2} (1-r^*) + \frac{1}{4\pi^2} (1-r^*) \cos 2\pi r^* + \frac{3}{8\pi^4} \sin 2\pi r^*,$$

$$C_1 = \frac{1}{4\pi^4} \sin 2\pi r^* - \frac{1}{\pi^4} \sin \pi r^* - (\frac{r^*-1}{2\pi^3})[1 - 2 \cos \pi r^*],$$

$$C_2 = \frac{\sin 2\pi r^*}{4\pi^4} - \frac{\sin \pi r^*}{\pi^4} - \frac{r^*}{2\pi^3} [1 - 2 \cos \pi r^*] + \frac{1}{2\pi^3} ,$$

$$C_3 = \frac{\sin \pi r^*}{\pi^4} - \frac{r^* \cos \pi r^*}{\pi^3} .$$

Due to the complexity of the other components of $X_1(r^*)$ they will not be presented here.

The above expansions are used to approximate $x_{21}(0)$ at different values of r^* with $\tau = .001$ (see Table 3.2.2 and Figure 3.2.1).

Table 3.2.1

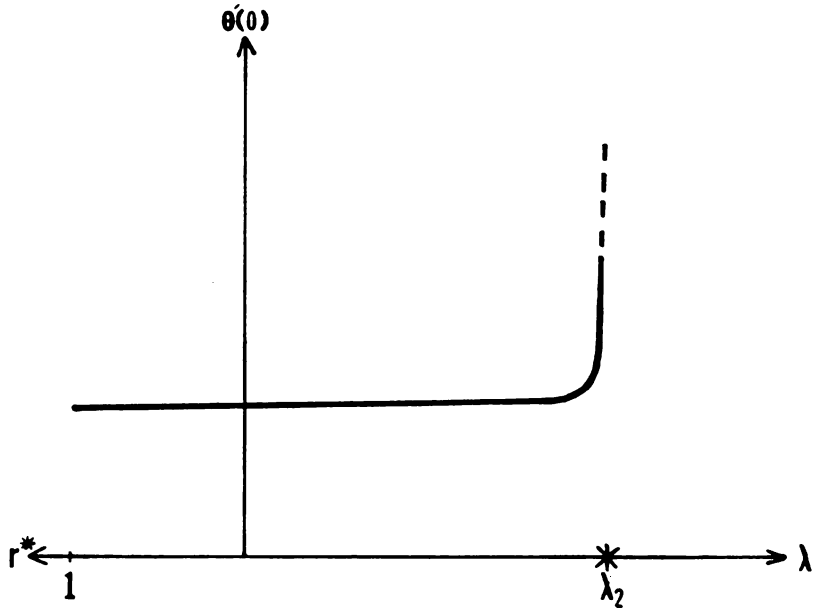
Fully submerged, $\tau = .001$

λ	$x_{21}(0) = \theta'(0)$
0	3.141601
20	3.141603
40	3.141608
60	3.141620
80	3.141665
90	3.141919
92	3.142570
93	3.191903

Table 3.2.2

Partially submerged, $\tau = .001$

r^*	$x_{21}(0) = \theta'(0)$
1.0	3.141593
0.8	3.141593
0.6	3.141593
0.4	3.141596
0.2	3.141600
0.0	3.141601



Solution curve obtained by regular perturbation when $\tau = .001$

Figure 3.2.1

3.3 Perturbed bifurcation

Let $G : H \times R^1 \times R^1 \to H$ be a smooth mapping, where H is a real Hilbert space, and consider the equation

$$G(X,\lambda,\tau) = 0 \qquad (3.3.1)$$

Assume that when $\tau = 0$ (3.3.1) has the trivial solution $(0,\lambda,0)$ for all $\lambda \in R^1$, and that the perturbation is in such a way that $G_\tau(0,\lambda,0) \neq 0$. The last condition ensures that (3.3.1) no longer has the trivial solution when $\tau \neq 0$. Assume further that there is λ_0 such that the Frechet derivative $G_X^o = G_X(0,\lambda_0,0)$ satisfies the conditions

(a) $N(G_X^o)$ is one-dimensional spanned by $\Phi \neq 0$

(b) $N(G_X^{o*})$ is one-dimensional spanned by $\Psi \neq 0$

(c) $R(G_X^o) = N(G_X^{o*})^\perp$ and $R(G_X^{o*}) = N(G_X^o)^\perp$ (3.3.2)

(d) $a \equiv \langle \Psi, G_{X\lambda}^o \Phi\rangle \neq 0$

where $G_{X\lambda}^o = G_{X\lambda}(0,\lambda_0,0)$.

By the Liapunov-Schmidt theory outlined in Section 3.1 it follows that $(0,\lambda_0,0)$ is a bifurcation point for the unperturbed problem

$$G(X,\lambda,0) = 0 \qquad (3.3.3)$$

In this section the behavior of the bifurcation point $(0,\lambda_0,0)$ under the perturbation $\tau \neq 0$ is examined. It is shown under the additional assumption

$$b \equiv \langle\Psi,G_\tau\rangle \neq 0 \qquad (3.3.4)$$

where $G_\tau^o = G_\tau(0,\lambda_0,0)$, that there is locally a family of "limit points"

through each of which there is a family of non-trivial solutions of (3.3.1). The Implicit Function Theorem together with the perturbation theory of linear operators [9] will be used to obtain these results.

Let Y denotes the space $H \times H \times R^1 \times R^1$ and $F : Y \times R^1 \to Y$ be defined by

$$F(y,\varepsilon) = \begin{bmatrix} G(\varepsilon\Phi + y_1,\ \lambda_0 + r_1, r_2) \\ G_X(\varepsilon\Phi + y_1,\ \lambda_0 + r_1, r_2)(\Phi + y_2) \\ \langle \Phi,\ y_1 \rangle \\ \langle \Phi,\ y_2 \rangle \end{bmatrix} \qquad (3.3.5)$$

where $y = (y_1, y_2, r_1, r_2)$, and consider the equation

$$F(y,\varepsilon) = 0. \qquad (3.3.5)$$

Equation (3.3.5) has the trivial solution (0,0) and the Frechet derivative of F with respect to y at (0,0) is given by

$$F_y(0,0) = \begin{bmatrix} \overset{o}{G}_X & 0 & 0 & \overset{o}{G}_\tau \\ \overset{o}{G}_{XX}\Phi & \overset{o}{G}_X & \overset{o}{G}_{X\lambda}\Phi & \overset{o}{G}_{X\tau}\Phi \\ \langle \Phi, \cdot \rangle & 0 & 0 & 0 \\ 0 & \langle \Phi, \cdot \rangle & 0 & 0 \end{bmatrix},$$

where $\overset{o}{G}_{XX}$, $\overset{o}{G}_{X\tau}$ are $G_{XX}(0,\lambda_0,0)$ and $G_{X\tau}(0,\lambda_0,0)$ respectively.

<u>Lemma 3.3.1</u>: The bounded linear operator $F_y(0,0)$ has a bounded inverse.

Proof: This will follow from the Open Mapping Theorem once it is shown that $F_y(0,0)$ is one-to-one onto. To show that $F_y(0,0)$ is one-to-one let $y = (y_1,y_2,r_1,r_2)$ be such that $F_y(0,0)y = 0$. Then

(a) $\overset{o}{G}_X y_1 + r_2\overset{o}{G}_\tau = 0$

(b) $\overset{o}{G}_{XX}\phi y_1 + \overset{o}{G}_X y_2 + r_1\overset{o}{G}_{X\lambda}\phi + r_2\overset{o}{G}_{X\tau}\phi = 0$

(c) $\langle\phi,y_1\rangle = 0$

(d) $\langle\phi,y_2\rangle = 0$.

From equation (a) it follows that $r_2 b = 0$ and therefore condition (3.3.4) implies that $r_2 = 0$. Hence $y_1 = A\phi$ for some constant A. By equation (c) A must be zero and hence $y_1 = 0$. Equation (b), therefore, becomes $\overset{o}{G}_X y_2 + r_1\overset{o}{G}_{X\lambda}\phi = 0$, which gives that $r_1 a = 0$, and so $r_1 = 0$ and $y_2 = 0$ also. Thus $y = 0$ and $F_y(0,0)$ is one-to-one. Since $F_y(0,0)$ is clearly onto, the lemma is proved.

Theorem 3.3.1: There exists a unique smooth curve $y = y(\varepsilon)$ defined on $|\varepsilon| \leq \varepsilon_0$ for some small $\varepsilon_0 > 0$ with $y(0) = 0$ and $F(y(\varepsilon),\varepsilon) = 0$ for each ε.

Proof: The proof follows immediately from Lemma 3.3.1 and the Implicit Function Theorem.

For each $|\varepsilon| \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is the number provided by Theorem 3.3.1, let $X(\varepsilon) = \varepsilon\phi + y_1(\varepsilon)$, $\phi(\varepsilon) = \phi + y_2(\varepsilon)$, $\lambda(\varepsilon) = \lambda_0 + r_1(\varepsilon)$ and $\tau(\varepsilon) = r_2(\varepsilon)$. Then $G(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon)) = 0$ and $G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))\phi(\varepsilon) = 0$. It follows from the perturbation theory of linear operators [9] that the following conditons will hold for small enough ε_0

(1) $R(G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon)))$ is closed.

(2) $\dim(N(G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))) = 1$

(3) $\dim(N(G_X^*(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))) = 1$

It follows for each $|\varepsilon| \leq \varepsilon_0$, where ε_0 is small enough, that the operator $G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ is a Fredholm operator with index zero. Furthermore, if $N(G_X^*(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon)))$ is spanned by $\Psi(\varepsilon)$ then $a(\varepsilon) = \langle\Psi(\varepsilon),G_{X\lambda}(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon)) \neq 0$ and hence, by a similar argument to that of Section 3.1, it follows that there exists a unique smooth solution curve $\Gamma(\varepsilon)$ of (3.3.1) branching from $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$. It holds further that $d(\varepsilon) = \langle\Psi(\varepsilon), G_\lambda(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))\rangle \neq 0$ for $\varepsilon \neq 0$. This shows that the "perturbed bifurcation point" $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ is a limit point. The conclusion of the above calculations is summarized in the following

<u>Theorem 3.3.2</u>: There exists a unique smooth curve $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ defined on $|\varepsilon| \leq \varepsilon_0$, for some $\varepsilon_0 > 0$, of solutions of (3.3.1) passing through the bifurcation point $(0,\lambda_0,0)$. For $\varepsilon \neq 0$ $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ is a limit point through which there passes a unique smooth curve of non-trivial solutions of (3.3.1).

<u>Remark</u>: The use of the Implicit Function Theorem in the treatment of the perturbed bifurcation problems was suggested by Keener and Keller [10], where they treated the problem using different technics.

Finally, the above theory is applied to the ring problem formulated in Chapter 1 and approximations to the limit points $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ are obtained for this particular problem. An appxoximation to the solution curves through the limit points will be obtained using the

singular perturbation method in Section 3.5.

Let $G(X,\lambda,\tau)$ denote the mapping defined by (1.2.10) with domain

$$D = \{X \in C_6^1[0,1] : B[X] = 0\}$$

where the operator $B[X]$ is defined as in (1.2.11). Let λ_0 take the value λ_n of any of the buckling loads obtained in Section 3.1. In this case, a and b are given by $\frac{n^2}{2\pi(n^2-1)}$ and $\frac{(-1)^n n}{2\pi^2(n^2-1)}$ respectively. Therefore the conditions for Theorems 3.3.1 and 3.3.2 are satisfied. The smoothness of the solution $y(\varepsilon)$ of Theorem 3.3.1 as a function of ε allows us to write $y(\varepsilon) = \varepsilon\, y_\varepsilon(0) + \frac{1}{2}\varepsilon^2 y_{\varepsilon\varepsilon}(0) + \cdots$. The coefficients $y_\varepsilon(0)$, $y_{\varepsilon\varepsilon}(0)$, $\cdots$ can be obtained by differentiating the equation $F(y(\varepsilon),\varepsilon) = 0$ with respect to ε and setting $\varepsilon = 0$. This leads to:

$$y(\varepsilon) = \begin{bmatrix} X(\varepsilon) \\ \Phi(\varepsilon) \\ \lambda(\varepsilon) \\ \tau(\varepsilon) \end{bmatrix} = \begin{bmatrix} \varepsilon\Phi + w_2\varepsilon^2 + O(\varepsilon^3) \\ \Phi + 2w_2\varepsilon + O(\varepsilon^2) \\ \lambda_n + \beta\varepsilon^2 + O(\varepsilon^3) \\ \gamma\varepsilon^3 + O(\varepsilon^4) \end{bmatrix}$$

Where $w_2 = A'\Phi - \frac{1}{2}\overset{o}{G}{}_X^{-1}[\overset{o}{G}_{XX}\Phi\Phi]$, $\beta = \frac{-3E_3}{a}$ and $\gamma = \frac{2E_3}{b}$, for some constant A' to be determined by the condition $\langle\Phi,w_2\rangle = 0$.

3.4 Singular perturbation of bifurcation

In this section the singular perturbation method [13] is used to obtain a valid asymptotic expansion for the solution curves in a neighborhood of the bifurcation points λ_n found in Section 3.1. Recall that the regular perturbation method of Section 3.2 fails to describe the solution curves near each λ_n.

Let $G(X,\lambda,\tau)$ denote the mapping defined by (1.2.10) with with domain $D = \{X \in C_6^1[0,1] : B[X] = 0\}$ where B is the operator defined by (1.2.11), and let λ_0 denote λ_n for some n. Due to the "corner" at the bifurcation point (see Figure 3.1.1) the perturbation is singular and one has to stretch the neighborhood of the bifurcation point through a transformation of the form

$$\lambda = \lambda_0 + \xi\ \varepsilon^\alpha + \sum_{i=2}^{\infty} \xi_i(\varepsilon^\alpha)^i \tag{3.4.1}$$

where $\alpha > 0$ is a constant to be determined and ε is a small parameter defined by

$$\tau = \varepsilon^\beta \tag{3.4.2}$$

where β is a constant to be determined by the nonlinearity of G near λ_0. For fixed $\tau \neq 0$ the solution X of $G(X,\lambda,\tau) = 0$ is assumed to be a smooth function of ε

$$X = \sum_{j=1}^{\infty} X_j\ \varepsilon^j \tag{3.4.3}$$

where the X_j's are functions to be determined by substituting (3.4.1), (3.4.2) and (3.4.3) into $G(X,\lambda,\tau) = 0$, equating the coefficients of each power of ε to zero and solving the resulting linear boundary value problems successively for the X_j's. Also it is required that the derivatives of λ and τ with respect to ε will remain bounded as the bifurcation point λ_0 is approached. This forces both α and β to be positive integers.

Equating to zero the powers of ε leads to equations of the form

$$\overset{o}{G}_X\ X_j = K_j, \qquad j = 1,2,2,\ldots \tag{3.4.4}$$

where K_j depends only on X_i, $i \leq j-1$ for each j. The solvability conditions of (3.4.4) are

$$\langle \Psi, K_j \rangle = 0. \qquad (3.4.5)$$

For $j = 1$, equation (3.4.4) is

$$G_X^o X_1 = -\tau_\varepsilon(0)\, G_\tau^o$$

and since $\langle \Psi, G_\tau^o \rangle \neq 0$, the solvability condition (3.4.5) implies that $\tau_\varepsilon(0) = 0$ (hence $\beta \geq 2$) and $X_1 = A\Phi$, for some constant A.

For $j = 2$, equation (3.4.4) is

$$G_X^o X_2 = -\frac{1}{2} A^2 G_{XX}^o \Phi\Phi - A G_{X\lambda}^o \Phi \lambda_\varepsilon(0) - \frac{1}{2} G_\tau^o \tau_{\varepsilon\varepsilon}(0)$$

whose solvability condition is

$$A\langle \Psi, G_{X\lambda}^o \Phi \rangle \lambda_\varepsilon(0) + \frac{1}{2} \langle \Psi, G_\tau^o \rangle \tau_{\varepsilon\varepsilon}(0) = 0$$

and since A is required to remain bounded as λ approaches λ_o, $\langle \Psi, G_{X\lambda}^o \Phi \rangle \neq 0$ and $\langle \Psi, G_\tau^o \rangle \neq 0$ it follows that $\lambda_\varepsilon(0)$ and $\tau_{\varepsilon\varepsilon}(0)$ must both be zero. Hence $\alpha \geq 2$, $\beta \geq 3$ and X_2 is given by

$$X_2 = A'\Phi - \frac{A^2}{2} G_X^{o^{-1}} (G_{XX}^o \Phi\Phi).$$

where A' is a constant.

For $j = 3$ equation (3.4.4) is

$$G_X^o X_3 = -\frac{1}{6} \{A^3 G_{XXX}^o \Phi\Phi\Phi + 4A G_{XX}^o \Phi X_2 + 2A G_{XX}^o X_2 \Phi + 3 G_{X\lambda}^o \Phi \lambda_{\varepsilon\varepsilon}(0)$$
$$+ G_\tau^o \tau_{\varepsilon\varepsilon\varepsilon}(0)\},$$

and the solvability condition (3.4.5) gives

$$E_3A^3 + a\xi A + b = 0 \qquad (3.4.6)$$

where E_3, a and b are given by

$$E_3 = -\frac{3}{16}\pi^2 n^2 ,$$

$$a = \frac{n^2}{2\pi(n^2-1)} ,$$

$$b = \frac{(-1)^n n}{2\pi^2(n^2-1)} .$$

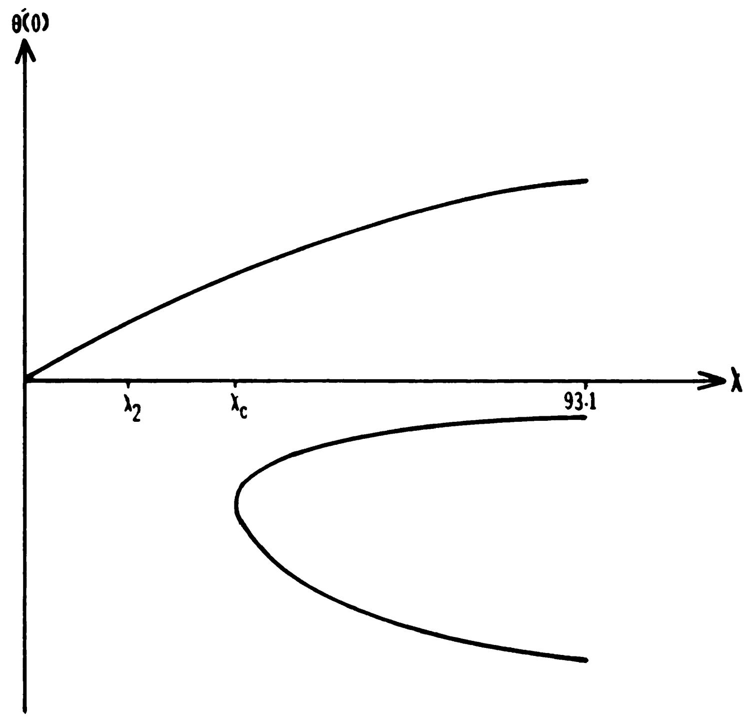
Equations (3.4.6) determines A as a function of ξ. It has a unique real root if $\xi < \frac{-3}{2a}\sqrt[3]{2b^2E_3}$, two real roots if $\xi = \frac{-3}{2a}\sqrt[3]{2b^2E_3}$ and three real roots if $\xi > \frac{-3}{2a}\sqrt[3]{2b^2E_3}$. It follows that at $\lambda_C \equiv \lambda_0 - \frac{3}{2a}\sqrt[3]{2b^2E_3}\,\varepsilon^2 + O(\varepsilon^3)$ there are exactly two solutions of the equation $G(X,\lambda,\tau) = 0$. This gives an approximation $\lambda_0 - \frac{3}{2a}\sqrt[3]{2b^2E_3}\,\varepsilon^2$ of the limit points which agrees with the results in Section 3.3.

The above expansions are used to approximate the values of $\theta'(0)$ near λ_2 for $\tau = .001$ (see Table 3.4.1). These approximate values will be compared with the numerical values in Chapter 5. The solution curves near λ_2 are shown in Figure 3.4.1.

Table 3.4.1

Perturbation solution for λ near λ_2 when $\tau = .001$

λ	$[\theta'(0)]_1$	$[\theta'(0)]_2$	$[\theta'(0)]_3$
93.0	3.189530	--	--
93.0188	3.245700	--	--
$\lambda_2 = 3\pi^3$	3.245810	--	--
93.036	3.304070	--	--
93.03697	3.307560	3.059780	3.058420
93.043	3.324720	3.096970	3.003091
93.046	3.333060	3.102901	2.988810
93.1	3.450670	3.129250	2.844860



Solution curves in neighborhood of λ_2 obtained by singular perturbation when $\tau = .001$

Figure 3.4.1

Remark: The expansions obtained above are valid only in a neighborhood of the bifurcation point λ_0. Away from λ_0 the expansion of Section 3.2 obtained by regular perturbation are valid. These two expansions can be "matched" using the method of matched asymptotic expansions to give a uniform representation of the solution curves. Due to the complexity of this matching process it will not be presented here.

Observe that only first order approximation is used in Table 3.4.1. This is enough to describe the qualitative behavior of the solution curves near λ_2. However, second order approximation is needed for the numerical methods of Chapter 4, where the perturbation solution will be used as initial guess for Newton's iteration. The second order approximation for $\lambda_0 = \lambda_2$ is:

$$x_{12} = \frac{A^2}{16} \sin 4\pi s ,$$

$$x_{22} = \frac{\pi A^2}{4} \cos 4\pi s ,$$

$$x_{32} = \frac{A^2}{16\pi} \sin 4\pi s \cos \pi s - \frac{A^2}{4\pi} \sin \pi s ,$$

$$x_{42} = \frac{A^2}{16\pi} \sin 4\pi s \sin \pi s - \frac{A^2}{4\pi} (1-\cos \pi s) ,$$

$$x_{52} = \frac{1}{\pi} (x_{42} \sin \pi s + x_{32}) + x_{41}x_{31} + \frac{3A^2}{8\pi} [s - \frac{\sin 4\pi s}{4\pi}]$$
$$- \frac{A^2}{24\pi} [s - \frac{\sin 6\pi}{6\pi}] - \frac{A^2}{12\pi} [\frac{\sin 2\pi s}{4\pi} - \frac{\sin 4\pi s}{8\pi}] ,$$

$$x_{61} = \frac{\xi}{\pi} - \frac{3\pi^2 A^2}{4} .$$

CHAPTER 4

NUMERICAL DETERMINATION OF BIFURCATION AND PERTURBED BIFURCATION SOLUTIONS

4.0 Introduction

This chapter is concerned with the study of constructive methods for determining bifurcation and perturbed bifurcation solutions in the neighborhood of a simple bifurcation point λ_0.

In Section 4.1 some general constructive methods for constructing the bifurcation solution curves in the neighborhood of λ_0 are discussed. In these methods the solution X as well as the bifurcation parameter λ are parameterized using a different parameter ε. However, in practical problems λ is often a physical parameter and the solution X may be required for some given values of this parameter. On the other hand, there can not be a general method in which λ is used as a parameter since for some bifurcation problems non-trivial solutions may exist only at λ_0.

In Section 4.2 the basic theorems for the convergence of Newton's and chord methods are stated.

In Sections 4.3 and 4.4 some numerical schemes, based upon Newton's and chord methods, in which λ is used as a parameter, are introduced and shown to converge under appropriate conditions.

Sections 4.5 and 4.6 deal with the perturbed bifurcation problem. It is shown in Section 4.5 that Newton's and chord methods can be used to

compute the perturbed bifurcation points as well as the solution curves through them. In Section 4.6 Newton's and chord methods are used to determine all the perturbed solution curves in the neighborhood of λ_0.

4.1 Numerical methods for bifurcation problems

Let $G : H \times R' \to H$ be a smooth mapping, where H is a real Hilbert space and consider the equation

$$G(X,\lambda) = 0. \tag{4.1.1}$$

Using the same notations as in Section 3.1 it is assumed that (4.1.1) has the trivial solution $X = 0$ for all λ in R' and that for some λ_0 in R the following conditions hold:

$$\begin{aligned}
&\text{(a)} \quad N(\overset{o}{G}_X) \text{ is one-dimensional spanned by } \Phi, \ \langle\Phi,\Phi\rangle = 1,\\
&\text{(b)} \quad N(\overset{o}{G}{}_X^*) \text{ is one-dimensional spanned by } \Psi, \ \Psi \neq 0,\\
&\text{(c)} \quad R(\overset{o}{G}_X) = N(\overset{o}{G}{}_X^*)^{\perp} \quad \text{and} \quad R(\overset{o}{G}{}_X^*) = N(\overset{o}{G}_X)^{\perp},\\
&\text{(d)} \quad a \equiv \langle\Psi, \overset{o}{G}_{X\lambda}\Phi\rangle \neq 0,\\
&\text{(e)} \quad \langle\Psi,\Phi\rangle = 1.
\end{aligned} \tag{4.1.2}$$

A bounded linear operator L on a Hilbert space H satisfying conditions (a) - (c) of (4.1.2) is said to be a Fredholm operator with zero index. Note that condition (e) of (4.1.2) implies that the algebraic multiplicity of the zero eigenvalue of $\overset{o}{G}_X$ is one.

Under these conditions the Liapunov-Schmidt theory outlined in Section 3.1 ensures the existence and uniqueness of the nontrivial smooth solution curve $(X(\varepsilon),\lambda(\varepsilon))$ passing through $(0,\lambda_0)$. In this

section, some general constructive methods for determining the nontrivial solution curve are discussed. In all these methods the solution X as well as the "bifurcation" parameter λ are parameterized using a different parameter ε.

The Liapunov-Schmidt method presented in Section 3.1 can be used to construct the nontrivial solution curves near the bifurcation point $(0,\lambda_0)$. However, this method is seldom used for actual numerical calculations due to its disadvantage of involving two iterations processes.

In the last decade other iterative methods have been developed which have shown to be of great value for the development of the numerical methods for bifurcation problems, see Keller and Langford [11], Crandall and Rabinowitz [4], Chen and Demoulin [2], and Decker and Keller [5].

In [5], Decker and Keller introduced a method for constructing the bifurcation branches near $(0,\lambda_0)$. They replaced the single equation

$$G(\varepsilon\Phi + w, \lambda_0 + \mu) = 0$$

by an "inflated" system

$$F(y,\varepsilon) = \begin{bmatrix} G(\varepsilon\Phi + w, \lambda_0 + \mu) \\ \langle\Phi, w\rangle \end{bmatrix} = 0$$

where $y = (w,\mu) \in H \times R'$, introduced an initial "guess" $y^0 = (0,\varepsilon)$ and proved the convergence of Newton's and chord iterates for a given small value of $\varepsilon \neq 0$. Their proof is based upon Newton-Kantorovich Theorem [8].

Remark: Although the above methods have the advantage of being general, the use of ε as a parameter can be a disadvantage in practical problems where λ represents a physical quantity and the solution X is required for some given values of λ. For such problems it is more convenient to use λ as a given parameter and compute the solution $X(\lambda)$. On the other hand this may not be possible in general, for example, when the nontrivial solutions occur only at λ_0 in which case a different parameter must be used. In Sections 4.3 and 4.4 an approach due to Decker and Keller [5] is followed to prove the convergence of Newton's and chord iterates, for certain type of bifurcation problem, with λ being fixed near the bifurcation point λ_0. The perturbation methods of Chapter 3 will be used to obtain an initial "guess" for these iterates.

4.2 Newton and chord methods

The basic convergence result for Newton's method is the following theorem due to Kantorovich [8].

Theorem 4.2.1: (Newton-Kantorovich Theorem)

Let $F : B_1 \to B_2$ be a continuously differentiable mapping, where B_1 and B_2 are Banach spaces such that $F_U(U^0)$ has a bounded inverse for some $U^0 \in B_1$ which satisfies

(a) $\|F_U(U^0)^{-1} F(U^0)\| \leq \eta$

(b) $\|F_U(U^0)^{-1}[F_U(U) - F_U(V)]\| \leq L\|U-V\|$, for all $U,V \in B_R(U^0)$,

an open ball of radius R about U^0. Then if $h = L\eta \leq \frac{1}{2}$ and

$\frac{\eta(1-\sqrt{1-2h})}{h} \le R$, the equation $F(U) = 0$ has a solution $U^* \in B_R(U^o)$ to which Newton's iterates

$$U^{n+1} = U^n - F_U(U^n)^{-1} F(U^n)$$

converges with a rate of convergence

$$\|U^* - U^n\| \le (2h)^{2^n-1} \frac{\eta}{2^{n-1}} .$$

Furthermore, if $R < \frac{\eta(1+\sqrt{1-2h})}{h}$, U^* is unique in $B_R(U^o)$.

The form of the Newton-Kantorovich Theorem suitable for bifurcation problems is the following theorem due to G. Moore [14].

Theorem 4.2.2: Let $F(U,\delta)$ be a mapping from $B_1 \times R^1$ to B_2, where B_1 and B_2 are Banach spaces. Assume that F is continuously differentiable with respect to U and continuous with respect to δ. Let $U^o(\delta)$ be a continuous mapping from R^+ to B_1 such that for some $\delta_1 > 0$ $F_U(U^o(\delta),\delta)$ has a bounded inverse for all $0 < \delta < \delta_1$ which satisfies

(a) $\|F_U(U^o(\delta),\delta)^{-1} F(U^o(\delta),\delta)\| \le \eta(\delta)$

(b) $\|F_U(U^o(\delta),\delta)^{-1} [F_U(U,\delta) - F_U(V,\delta)]\| \le L(\delta)\|U-V\|$

for all $U,V \in B_{2\eta(\delta)}(U^o(\delta))$.

For some $\eta(\delta)$ and $L(\delta)$ satisfying $h(\delta) = \eta(\delta)L(\delta) = 0(\delta)$ as $\delta \to 0$, then there exists $\delta_2 > 0$, $\delta_2 \le \delta_1$ and a continuous mapping $U^*(\delta)$ from $(0,\delta_2)$ to B_1 such that $U^*(\delta)$ is the unique solution of $F(U,\delta) = 0$ in $B_{2\eta(\delta)}(U^o(\delta))$ and the Newton's iterates

$$U^{n+1} = U^n - F_U(U^n,\delta)^{-1} F(U^n,\delta)$$

converge to $U^*(\delta)$, for each $\delta \in (0,\delta_2)$.

The following theorem, due to G. Moore [14], is the basic result for the convergence of chord method.

<u>Theorem 4.2.3</u>: Let $F(U,\delta)$ be a mapping from $B_1 \times R^1$ to B_2 where B_1 and B_2 are Banach spaces. Assume that F is continuously differentiable with respect to U and continuous with respect to δ. Let $U^o(\delta)$ be a continuous map from R^+ to B_1 such that for some $\delta_1 > 0$ $F_U(U^o(\delta),\delta)$ has a bounded inverse for all $\delta \in (0,\delta_1)$ which satisfies

(a) $\|F_U(U^o(\delta),\delta)^{-1} F(U^o(\delta),\delta)\| \leq \eta(\delta)$

(b) $\|F_U(U^o(\delta),\delta)^{-1} [F_U(U^o(\delta),\delta)(U-V) - (F(U,\delta) - F(V,\delta))]\| \leq$

$$q(\delta)\|U-V\|, \quad \text{for } U,V \in B_{2\eta(\delta)}(U^o(\delta)),$$

for some $\eta(\delta)$ and $q(\delta)$ satisfying $q(\delta) = 0(\delta)$ as $\delta \to 0$, then there exists $\delta_2 > 0$, $\delta_2 \leq \delta_1$ and a continuous mapping $U^*(\delta)$ from $(0,\delta_2)$ to B_1 such that $U^*(\delta)$ is a unique solution of the equation $F(U,\delta) = 0$ in the ball $B_{2\eta(\delta)}(U^o(\delta))$ and the chord iterates

$$U^{n+1} = U^n - F_U(U^o(\delta),\delta)^{-1} F(U^n,\delta)$$

converge to $U^*(\delta)$ for each $\delta \in (0,\delta_2)$.

4.3 Newton's and chord methods with λ as parameter

In addition to the assumptions of (4.1.2) it is assumed here that

$$E_0 \equiv \frac{1}{2} \langle \Psi, G^0_{XX}\Phi\Phi \rangle \neq 0.$$

For a given λ near λ_0 an initial guess X^0 is chosen, by the perturbation method of Section 3.1, to be

$$X^0(\tilde{\varepsilon}) = \tilde{\varepsilon}\Phi$$

where $\tilde{\varepsilon} = \frac{-a}{E_0}(\lambda - \lambda_0)$.

The following notations will be used throughout this section

$$G(\tilde{\varepsilon}) \equiv G(X^0(\tilde{\varepsilon}),\lambda)$$

$$G_X(\tilde{\varepsilon}) \equiv G_X(X^0(\tilde{\varepsilon}),\lambda).$$

In this section, it is shown that for fixed λ close enough to λ_0 Newton's and chord iterates, with X^0 as an initial guess, converge to a unique nontrivial solution of the equation

$$G(X,\lambda) = 0. \qquad (4.3.1)$$

It is shown that $G_X^{-1}(\tilde{\varepsilon})$ exists for all $\tilde{\varepsilon}$ in some small punctured neighborhood of zero and that $\|G_X^{-1}(\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{-1})$ as $\tilde{\varepsilon} \to 0$. This will be the conclusion of the following lemmas.

Using simple modifications the following three lemmas follow from Theorem 5.1 [5] and the remarks following it. They are stated here in forms suitable for the present situation.

Lemma 4.3.1: There exists smooth functions $\Phi(\tilde{\varepsilon})$ and $\beta(\tilde{\varepsilon})$ defined on $|\tilde{\varepsilon}| \leq \delta$, for some $\delta > 0$, such that

$$G_X(\tilde{\varepsilon})\Phi(\tilde{\varepsilon}) = \beta(\tilde{\varepsilon})\Phi(\tilde{\varepsilon}),\ \Phi(0) = \Phi \text{ and } \beta(0) = 0.$$

Lemma 4.3.2: There exists smooth functions $\Psi(\tilde{\varepsilon})$ and $\hat{\beta}(\tilde{\varepsilon})$ defined on $|\tilde{\varepsilon}| \leq \delta$, for some $\delta > 0$ such that $G_X(\tilde{\varepsilon})\Psi(\tilde{\varepsilon}) = \hat{\beta}(\tilde{\varepsilon})\Psi(\tilde{\varepsilon})$, $\Psi(0) = \Psi$ and $\hat{\beta}(0) = 0$.

Let δ be the smaller of the two δ's provided by Lemmas 4.3.1 and 4.3.2. Define for each $\tilde{\varepsilon}$ in $|\tilde{\varepsilon}| \leq \delta$ the subspaces

$$N(\tilde{\varepsilon}) = \langle\Phi(\tilde{\varepsilon})\rangle$$

$$H(\tilde{\varepsilon}) = \{X \in H : \langle\Psi(\tilde{\varepsilon}),X\rangle = 0\}.$$

This gives the following decompositions of H

$$H = N(\tilde{\varepsilon}) \oplus H(\tilde{\varepsilon}).$$

Finally, let $P(\tilde{\varepsilon})$ be the projection onto $N(\tilde{\varepsilon})$ and $Q(\tilde{\varepsilon}) = I - P(\tilde{\varepsilon})$, where I is the identity on H.

Lemma 4.3.3: There exists $\delta > 0$ such that for each $\tilde{\varepsilon}$ in $|\tilde{\varepsilon}| \leq \delta$, $G_X(\tilde{\varepsilon})$ maps $H(\tilde{\varepsilon})$ one to one onto itself.

Remark: Lemma 4.3.3 implies that $G_X(\tilde{\varepsilon})$ has a bounded inverse on $H(\tilde{\varepsilon})$ and hence the behavior of $G_X^{-1}(\tilde{\varepsilon})$ on H will be determined by its behavior on the subspace $N(\tilde{\varepsilon})$. This is studied in the following

Lemma 4.3.4: The restriction of $G_X(\tilde{\varepsilon})$ on the subspace $N(\tilde{\varepsilon})$ has one and only one eigenvalue, namely $\beta(\tilde{\varepsilon})$. Furthermore, $\beta(\tilde{\varepsilon})$ satisfies

$$\beta(\tilde{\varepsilon}) = E_0\tilde{\varepsilon} + O(|\tilde{\varepsilon}|^2)$$

as $\tilde{\varepsilon} \to 0$.

Proof: The first statement follows from Lemma 4.3.1. To prove the second statement, differentiate the equation

$$G_X(\tilde{\varepsilon})\Phi(\tilde{\varepsilon}) = \beta(\tilde{\varepsilon})\Phi(\tilde{\varepsilon})$$

with respect to $\tilde{\varepsilon}$ and set $\tilde{\varepsilon} = 0$. This gives the equation

$$G^o_{XX}\Phi\Phi - \frac{E_0}{a} G^o_{X\lambda}\Phi + G^o_X \dot{\Phi}(0) = \dot{\beta}(0)\Phi$$

whose solvability condition implies

$$E_0 = \dot{\beta}(0).$$

The following theorem is an easy consequence of Lemmas 4.3.3 and 4.3.4.

Theorem 4.3.1: There exists $\delta > 0$ such that for each $\tilde{\varepsilon}$ in $0 < |\tilde{\varepsilon}| \leq \delta$, $G_X^{-1}(\tilde{\varepsilon})$ exists and

$$\|G_X^{-1}(\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{-1}) \quad \text{as} \quad \tilde{\varepsilon} \to 0.$$

The convergence of Newton's and chord iterates follows from the following

Lemma 4.3.5: For each $0 < |\tilde{\varepsilon}| \leq \delta$, where $\delta > 0$ is as in Lemma 4.3.1, the following holds

$$\|G_X^{-1}(\tilde{\varepsilon})G(\tilde{\varepsilon})\| = O(|\tilde{\varepsilon}|^2).$$

Proof: First observe that

$$G(\tilde{\varepsilon}) = \frac{1}{2}\,[G^o_{XX}\Phi\Phi - \frac{2E_o}{a}\,G^o_{X\lambda}\Phi]\tilde{\varepsilon}^2 + O(\tilde{\varepsilon}^3).$$

Since $\langle\Psi, G^o_{XX}\Phi\Phi - \frac{2E_o}{a}\,G^o_{X\lambda}\Phi\rangle = 0$, it follows that

$$\langle\Psi(\tilde{\varepsilon}), G(\tilde{\varepsilon})\rangle = \langle\Psi + O(\tilde{\varepsilon}), G(\tilde{\varepsilon})\rangle$$

$$= O(\tilde{\varepsilon}^3).$$

But, then

$$\|G_X^{-1}(\tilde{\varepsilon})G(\tilde{\varepsilon})\| \le \|G_X^{-1}(\tilde{\varepsilon})P(\tilde{\varepsilon})G(\tilde{\varepsilon})\| + \|G_X^{-1}(\tilde{\varepsilon})Q(\tilde{\varepsilon})G(\tilde{\varepsilon})\|$$

$$= O(|\varepsilon|^2) + O(|\tilde{\varepsilon}|^2)$$

$$= O(|\tilde{\varepsilon}|^2).$$

Theorem 4.3.2: There exists $\delta > 0$ such that for each $\bar{\lambda}$ with $0 < |\tilde{\varepsilon}| \le \delta$, where $\tilde{\varepsilon} = \frac{-a(\bar{\lambda}-\lambda_o)}{E_o}$, Newton's and chord iterates with initial guess $X^o = \tilde{\varepsilon}\Phi$ converge to unique nontrivial solution X^* of the equation

$$G(X,\bar{\lambda}) = 0.$$

Proof: Theorem 4.3.1 and Lemma 4.3.5 imply that $\eta(\tilde{\varepsilon})$, $L(\tilde{\varepsilon})$ and $q(\tilde{\varepsilon})$ of Theorems 4.2.2 and 4.2.3 satisfy

$$\eta(\tilde{\varepsilon}) = O(|\tilde{\varepsilon}|^2),\; L(\tilde{\varepsilon}) = O(|\tilde{\varepsilon}|^{-1}) \text{ and } q(\tilde{\varepsilon}) = O(|\tilde{\varepsilon}|).$$

The proof now follows by applying Theorems 4.2.2 and 4.2.3.

Remark: The case when $E_0 \neq 0$ is known as the non-degenerate case. The degenerate case is when $E_0 = 0$ and $E_3 \neq 0$ (where E_3 is as defined in Section 3.1). If $\frac{E_3}{a} < 0$ then for each $\bar{\lambda} > \lambda_0$ there are two nontrivial solutions of

$$G(X,\bar{\lambda}) = 0.$$

For this case, using the analysis of Section 3.1 one may choose

$$X^0(\tilde{\varepsilon}) = \tilde{\varepsilon}\phi + \tilde{\varepsilon}^2 w_2$$

as the initial guesses for the solutions at $\lambda = \bar{\lambda}$, where

$$\tilde{\varepsilon} = \pm \sqrt{\frac{-a(\bar{\lambda}-\lambda_0)}{E_3}} .$$

Using a similar argument to that used for the non-degenerate case it can be shown that $\eta(\tilde{\varepsilon})$, $L(\tilde{\varepsilon})$ of Theorem 4.2.2 satisfy

$$\eta(\tilde{\varepsilon}) = O(|\tilde{\varepsilon}|^2),\ L(\tilde{\varepsilon}) = O(|\tilde{\varepsilon}|^{-2}),$$

which show that the convergence of Newton's and chord iterates is not guaranteed. In Section 4.4 the equation $G(X,\lambda) = 0$ is replaced by the "inflated" system $F(U,\lambda) = 0$ [5] which will improve the convergence rate for the non-degenerate case and guarantee it for the degenerate case.

4.4 Newton's and chord methods applied to an "inflated" system with λ as parameter

In this section it is assumed, in addition to the conditions of (4.1.2), that either of the following conditions hold

(i) $E_0 \neq 0$ (non-degenerate case)

(ii) $E_0 = 0$ but $E_3 \neq 0$ and $\frac{a}{E_3} < 0$ (degenerate case).

Let H_1 denote the Hilbert space $H \times R^1$ with inner product

$$\langle \binom{w_1}{\varepsilon_1}, \binom{w_2}{\varepsilon_2} \rangle = \langle w_1, w_2 \rangle + \varepsilon_1 \varepsilon_2,$$

and define $F : H_1 \times R^1 \to H_1$ by

$$F(U,\lambda) = \begin{bmatrix} G(\varepsilon\Phi + w, \lambda \\ \langle \Phi, w \rangle \end{bmatrix} \qquad (4.4.1)$$

for $U = \binom{w}{\varepsilon}$ in H_1 and λ in R^1. Finally, define the initial guesses for cases (i) and (ii) by

$$\text{(i)} \quad U^0(\lambda) = \binom{0}{\tilde{\varepsilon}}, \quad \text{where } \tilde{\varepsilon} = \frac{-a(\lambda-\lambda_0)}{E_0}, \quad \lambda \in R^1$$

$$(4.4.2)$$

$$\text{(ii)} \quad U^0(\lambda) = \begin{pmatrix} \tilde{\varepsilon}^2 & w_2 \\ & \tilde{\varepsilon} \end{pmatrix}, \quad \text{where } \tilde{\varepsilon} = \pm \sqrt{\frac{-a(\lambda-\lambda_0)}{E_3}}, \quad \lambda \geq \lambda_0.$$

For a given λ near λ_0 it is shown that Newton's and chord iterates with the initial guesses (4.4.2) converge to a unique nontrivial solution U^* of the equation

$$F(U,\lambda) = 0 \tag{4.4.3}$$

Notations: Write $F(\tilde{\varepsilon})$, $F_U(\tilde{\varepsilon})$ and $\overset{o}{F_U}$ for $F(U^o,\lambda)$, $F_U(U^o,\lambda)$ and $F_U(0,\lambda_o)$ respectively.

Observe that

$$\overset{o}{F_U} = \begin{bmatrix} \overset{o}{G_X} & 0 \\ \langle \Phi, \cdot \rangle & 0 \end{bmatrix}. \tag{4.4.4}$$

Using simple modifications the following theorem and the three lemmas following it follow from Lemma 5.6 [5] and the remarks following it.

Theorem 4.4.1: $\overset{o}{F_U}$ is a Fredholm operator with zero index, $N(\overset{o}{F_U})$ is spanned by $\Phi_1 = \binom{0}{1}$ and $N(\overset{o}{F_U}{}^*)$ is spanned by $\Psi_1 = \binom{\Psi}{0}$. The algebraic multiplicity of the zero eigenvalue of $\overset{o}{F_U}$ is two.

Lemma 4.4.1: There exist smooth functions $\Phi_1(\tilde{\varepsilon})$, $\Phi_2(\tilde{\varepsilon})$ and $B(\tilde{\varepsilon}) = \begin{bmatrix} b_{11}(\tilde{\varepsilon}) & b_{12}(\tilde{\varepsilon}) \\ b_{21}(\tilde{\varepsilon}) & b_{22}(\tilde{\varepsilon}) \end{bmatrix}$ defined on $|\tilde{\varepsilon}| \leq \delta$, for some $\delta > 0$,

such that

$$\begin{bmatrix} F_U(\tilde{\varepsilon}) & 0 \\ 0 & F_U(\tilde{\varepsilon}) \end{bmatrix} \begin{bmatrix} \Phi_1(\tilde{\varepsilon}) \\ \Phi_2(\tilde{\varepsilon}) \end{bmatrix} = B(\tilde{\varepsilon}) \begin{bmatrix} \Phi_1(\tilde{\varepsilon}) \\ \Phi_2(\tilde{\varepsilon}) \end{bmatrix}$$

$$\Phi_1(0) = \Phi_1, \quad \Phi_2(0) = \Phi_2 \quad \text{and} \quad B(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Lemma 4.4.2: There exist smooth functions $\Psi_1(\tilde{\varepsilon})$, $\Psi_2(\tilde{\varepsilon})$ and $\hat{B}(\tilde{\varepsilon}) = \begin{bmatrix} \hat{b}_{11}(\tilde{\varepsilon}) & \hat{b}_{12}(\tilde{\varepsilon}) \\ \hat{b}_{21}(\tilde{\varepsilon}) & \hat{b}_{22}(\tilde{\varepsilon}) \end{bmatrix}$ defined on $|\tilde{\varepsilon}| \leq \delta$, for some $\delta > 0$ such that

$$\begin{bmatrix} F_U^{o*}(\tilde{\varepsilon}) & 0 \\ 0 & F_U^{o*}(\tilde{\varepsilon}) \end{bmatrix} \begin{bmatrix} \Psi_1(\tilde{\varepsilon}) \\ \Psi_2(\tilde{\varepsilon}) \end{bmatrix} = \hat{B}(\tilde{\varepsilon}) \begin{bmatrix} \Psi_1(\tilde{\varepsilon}) \\ \Psi_2(\tilde{\varepsilon}) \end{bmatrix},$$

$$\Psi_1(0) = \Psi_1, \quad \Psi_2(0) = \Psi_2 \quad \text{and} \quad \hat{B}(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Definition 4.4.1: Let δ denote the smaller of the two δ's provided by Lemmas 4.4.1 and 4.4.2. For each $\tilde{\varepsilon}$ in $|\tilde{\varepsilon}| \leq \delta$ define

$$N(\tilde{\varepsilon}) = \langle \Phi_1(\tilde{\varepsilon}), \Phi_2(\tilde{\varepsilon}) \rangle$$

$$H_1(\tilde{\varepsilon}) = \{U \in H_1 : \langle \Psi_i(\tilde{\varepsilon}), U \rangle = 0, \ i = 1,2\} .$$

For each $\tilde{\varepsilon}$ in $|\tilde{\varepsilon}| \leq \delta$ it follows that H_1 can be decomposed into the sum

$$H_1 = N(\tilde{\varepsilon}) \oplus H_1(\tilde{\varepsilon})$$

provided that δ is small enough. For each such $\tilde{\varepsilon}$ let $P(\tilde{\varepsilon})$ denote the projection onto $N(\tilde{\varepsilon})$ and $Q(\tilde{\varepsilon}) = I - P(\tilde{\varepsilon})$, where I is the identity on H_1.

In order to examine the convergence of Newton's and chord iterates one has to examine the invertibility of $F_U(\tilde{\varepsilon})$ and determine the rate

at which $\|F_U^{-1}(\tilde{\varepsilon})\|$ tends to infinity as $\tilde{\varepsilon}$ tends to zero. The following lemma shows that by making δ small enough the restriction of $F_U(\tilde{\varepsilon})$ to the subspace $H_1(\tilde{\varepsilon})$ has a bounded inverse and hence reduces the study of the behavior of $F_U(\tilde{\varepsilon})$ to the study of its restriction to the subspace $N(\tilde{\varepsilon})$.

Lemma 4.4.3: There exists $\delta > 0$ such that for each $\tilde{\varepsilon}$ in $|\tilde{\varepsilon}| \leq \delta$, $F_U(\tilde{\varepsilon})$ maps $H_1(\tilde{\varepsilon})$ one to one onto itself.

The behavior of the restriction $\overline{F_U(\tilde{\varepsilon})}$ of $F_U(\tilde{\varepsilon})$ to $N(\tilde{\varepsilon})$ is studied in the following

Lemma 4.4.4: $\overline{F_U(\tilde{\varepsilon})}$ has exactly two eigenvalues which are the same as those of $B(\tilde{\varepsilon})$.

Proof: Let $a_1\Phi_1(\tilde{\varepsilon}) + a_2\Phi_2(\tilde{\varepsilon})$ denote an eigenvector corresponding to an eigenvalue α of $\overline{F_U(\tilde{\varepsilon})}$. By Lemma 4.4.1 it follows that

$$B(\tilde{\varepsilon})^T \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

and hence α is an eigenvalue of $B(\tilde{\varepsilon})$.

The eigenvalues of $B(\tilde{\varepsilon})$ are studied in the following

Lemma 4.4.5: The two eigenvalues of $B(\tilde{\varepsilon})$ are of the form

(i) $\alpha = C_1 \tilde{\varepsilon}^{1/2} + 0(\tilde{\varepsilon})$

for the non-degenerate case, and

(ii) $\alpha = C_2 \tilde{\varepsilon} + 0(\tilde{\varepsilon}^2)$

for the degenerate case, where C_1 and C_2 are non-zero constants.

Proof: The eigenvalues of $B(\tilde{\varepsilon})$ are given by

$$\alpha = \frac{\beta \pm \sqrt{\beta^2-4\gamma}}{2}$$

where $\beta = b_{11}(\tilde{\varepsilon}) + b_{22}(\tilde{\varepsilon})$ and $\gamma = b_{11}(\tilde{\varepsilon})b_{22}(\tilde{\varepsilon}) - b_{12}(\tilde{\varepsilon})b_{21}(\tilde{\varepsilon})$.

From Lemma 4.4.1 $F_U(\tilde{\varepsilon})$ satisfies

(a) $F_U(\tilde{\varepsilon})\Phi_1(\tilde{\varepsilon}) = b_{11}(\tilde{\varepsilon})\Phi_1(\tilde{\varepsilon}) + b_{12}(\tilde{\varepsilon})\Phi_2(\tilde{\varepsilon})$.

Differentiating equation (a) with respect to $\tilde{\varepsilon}$ and setting $\tilde{\varepsilon} = 0$ gives

(b) $$\begin{bmatrix} G^o_{XX}\Phi\Phi - \frac{E_o}{a} G^o_{X\lambda}\Phi \\ 0 \end{bmatrix} + F^o_U \dot{\Phi}_1(0) = \dot{b}_{11}(0)\Phi_1 + \dot{b}_{12}(0)\Phi_2$$

for the non-degenerate case, and

(c) $$\begin{bmatrix} G^o_{XX}\Phi\Phi \\ 0 \end{bmatrix} + F^o_U \dot{\Phi}_1(0) = \dot{b}_{11}(0)\Phi_1 + \dot{b}_{12}(0)\Phi_2$$

for the degenerate case. The solvability condition of equation (b) is

$$\dot{b}_{12}(0) = E_o$$

which proves (i). While the solvability condition of equation (c) is

$$\dot{b}_{12}(0) = \dot{b}_{11}(0) = 0$$

and, hence

$$\dot{\Phi}_1(0) = \binom{2w_2}{\varepsilon}$$

for some constant ε. Differentiating equation (a) once more with respect to $\tilde{\varepsilon}$ and setting $\tilde{\varepsilon} = 0$ leads to (for the degenerate case)

$$\begin{bmatrix} 4G^o_{XX}\Phi w_2 + 2G^o_{XX}w_2\Phi + G^o_{XXX}\Phi\Phi\Phi - \frac{2E_3}{a} G^o_{X\lambda}\Phi \\ 0 \end{bmatrix} +$$

$$F^o_U \ddot{\Phi}_1(0) = \ddot{b}_{11}(0)\Phi_1 + \ddot{b}_{12}(0)\Phi_2$$

whose solvability condition is

$$\ddot{b}_{12}(0) = 4E_3$$

which proves (ii).

The following theorem is an easy consequence of the Lemmas 4.4.3, 4.4.4 and 4.4.5.

Theorem 4.4.2: There exists $\delta > 0$ such that for each $\tilde{\varepsilon}$ in $0 < |\tilde{\varepsilon}| \leq \delta$, $F_U^{-1}(\tilde{\varepsilon})$ exists and

$$\|F_U^{-1}(\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{-\gamma})$$

where γ is $\frac{1}{2}$ for the non-degenerate case and is 1 for the degenerate case.

One more lemma is needed to prove the convergence of Newton's and chord iterates. In this lemma $\|F_U^{-1}(\tilde{\varepsilon})F(\tilde{\varepsilon})\|$ is estimated.

Lemma 4.4.6:

$$\|F_U^{-1}(\tilde{\varepsilon})F(\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{\gamma})$$

where γ is $\frac{5}{2}$ for the non-degenerate case and 3 for the degenerate case.

Proof: For the non-degenerate case

$$F(\tilde{\varepsilon}) = \begin{bmatrix} \tilde{\varepsilon}^2[\frac{1}{2} G^o_{XX}\Phi\Phi - \frac{E_o}{a} G^o_{X\lambda}\Phi] \\ 0 \end{bmatrix} + O(\tilde{\varepsilon}^3)$$

while for the degenerate case

$$F(\tilde{\varepsilon}) = \begin{bmatrix} \frac{\tilde{\varepsilon}^3}{6} [4G^o_{XX}\Phi w_2 + 2G^o_{XX}w_2\Phi + G^o_{XXX}\Phi\Phi\Phi - \frac{6E_3}{a} G^o_{X\lambda}\Phi] \\ 0 \end{bmatrix} + O(\tilde{\varepsilon}^4)$$

It follows that

$$\langle \Psi_i(\tilde{\varepsilon}), F(\tilde{\varepsilon}) \rangle = O(\tilde{\varepsilon}^\gamma), \quad i = 1,2$$

where γ is as defined in the statement of the lemma. Since $F_U(\tilde{\varepsilon})$ has a bounded inverse on $H_1(\tilde{\varepsilon})$ the lemma follows.

It follows from Theorem 4.4.2 and Lemma 4.4.6 above that the functions $h(\tilde{\varepsilon})$ and $q(\tilde{\varepsilon})$ of Theorems 4.2.2 and 4.4.3 satisfy

$$h(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^2) \quad \text{and} \quad q(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^2)$$

for both the non-degenerate and the degenerate cases. This gives the proof for the following

Theorem 4.4.3: There exists $\delta > 0$ such that for each $\tilde{\varepsilon}$ in $0 < |\tilde{\varepsilon}| \leq \delta$ Newton's and chord iterates with initial guess $U^o(\tilde{\varepsilon})$ converge to a unique nontrivial solution U^* of the equation

$$F(U, \bar{\lambda}) = 0$$

where $\bar{\lambda}$ is defined as in (4.4.2).

4.5 Computing the perturbed bifurcation points and the solution curves through them

Let $G : H \times R^1 \times R^1 \to H$ be a smooth mapping, where H is a real Hilbert space and consider the "perturbed" problem

$$G(X,\lambda,\tau) = 0. \qquad (4.5.1)$$

Assume that

$$\begin{aligned} &(a) \quad G(0,\lambda,0) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad (4.5.2) \\ &(b) \quad G_\tau(0,\lambda,0) \neq 0 \end{aligned}$$

for each λ in R^1. Condition (a) of (4.5.2) implies that the trivial solution solves the "unperturbed" problem

$$G(X,\lambda,0) = 0 \qquad (4.5.3)$$

for each $\lambda \in R^1$, while (b) implies that (4.5.1) does not have the trivial solution for any λ in R^1. Thus the solution set of (4.5.1) is completely different from that of (4.5.3). Assume further that $G_X^0 \equiv G_X(0,\lambda_0,0)$ satisfies the conditions in (4.1.2) for some λ_0 in R^1 and that $G_\tau^0 \equiv G_\tau(0,\lambda_0,0)$ satisfies

$$b \equiv \langle \Psi, G_\tau^0 \rangle \neq 0. \qquad (4.5.4)$$

Under these assumptions Theorem 3.3.2 implies that there exists a smooth solution curve $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ of (4.5.1) defined in the neighborhood of $(0,\lambda_0,0)$ and passing through it, that each $G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ is a Fredholm operator with zero index, that for $\varepsilon \neq 0$

$$(a)\quad a(\varepsilon) \equiv \langle \psi(\varepsilon), G_{X\lambda}(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))\phi(\varepsilon)\rangle \neq 0$$

$$(4.5.5)$$

$$(b)\quad d(\varepsilon) \equiv \langle \psi(\varepsilon), G_{\lambda}(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))\rangle \neq 0$$

where $N(G_X(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))) = \langle \phi(\varepsilon)\rangle$ and $N(G_X^*(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))) = \langle \psi(\varepsilon)\rangle$, and that through each $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ there passes a unique smooth solution curve of (4.5.1). The solutions $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ are sometimes called perturbed bifurcation points.

In this section it is shown that Newton's and chord methods can be used to compute the singular solutions $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$ as well as the solution curves through them.

Let Y denote the space $H \times H \times R^1 \times R^1$ and $F : Y \times R^1 \to Y$ be defined by

$$F(y,\varepsilon) = \begin{bmatrix} G(\varepsilon\phi + y_1,\ \lambda_0 + r_1,\ r_2) \\ G_X(\varepsilon\phi + y_1,\ \lambda_0 + r_1,\ r_2)(\phi + y_2) \\ \langle \phi, y_1\rangle \\ \langle \phi, y_2\rangle \end{bmatrix}$$

where $y = (y_1,y_2,r_1,r_2)^T$ and consider the equation

$$F(y,\varepsilon) = 0. \qquad (4.5.6)$$

It was shown in Lemma 3.3.1 that $F_y(0,0)$ has a bounded inverse. Hence for small enough ε, $F_y(0,\varepsilon)$ will have a bounded inverse. Taking $y^0 = 0$ as an initial guess the convergence of Newton's and chord iterates, follows from Theorems 4.4.2 and 4.4.3.

Assume now that the unique solution $y^* = (y_1^*, y_2^*, r_1^*, r_2^*)$ of (4.5.6) corresponding to some $\varepsilon \neq 0$ has been determined and consider the equation

$$G(X, \lambda, \tau) = 0 \tag{4.5.7}$$

with $\tau = r_2^*$ being fixed and (X, λ) near (X^*, λ^*), where $X^* = \varepsilon\Phi + y_1^*$ and $\lambda^* = \lambda_0 + r_1^*$. Let $\Phi^* = \Phi + y_2^*$ and define

$$K(U, \varepsilon) = \begin{bmatrix} G(X^* + \varepsilon\Phi^* + w,\ \lambda^* + \mu,\ \tau) \\ \langle \Phi^*, w \rangle \end{bmatrix}$$

where $U = (w, \mu)^T \in H \times R^1$. Since

$$K_U(0,0) = \begin{bmatrix} G_X(X^*, \lambda^*, \tau) & G_\lambda(X^*, \lambda^*, \tau^*) \\ \langle \Phi^*, \cdot \rangle & 0 \end{bmatrix},$$

it follows from (b) of (4.5.5) that it is invertible, and hence that $K_U(0, \varepsilon)$ is also invertible for small enough ε. But

$$K(0, \varepsilon) = 0(\varepsilon^2)$$

and so the conditions of Theorems 4.2.2 and 4.2.3 are satisfied with $U^0 = 0$ as an initial guess. It follows that for each small $\varepsilon \neq 0$ Newton's and chord iterates with initial guess $U^0 = 0$ converge to a unique solution of the equation

$$K(U, \varepsilon) = 0.$$

Remark: In some cases (4.5.1) may have solution curves which do not pass through the singular solutions $(X(\varepsilon),\lambda(\varepsilon),\tau(\varepsilon))$. Such solution curves cannot be determined by the above method. In the next section it is shown, under additional assumptions, that Newton's and chord methods can be used to determined all the solution curves of (4.5.1) in some neighborhood of $(0,\lambda_0,0)$.

4.6 Newton's and chord methods for the perturbed problem

This section is concerned with the numerical computation of all the solution curves of equation (4.5.1) in a neighborhood of the bifucation point $(0,\lambda_0,0)$. In addition to the assumptions of Section 4.5 it is assumed that either of the following conditions hold

(i) $E_0 \neq 0$ (non-degenerate case)

(ii) $E_0 = 0$ but $E_3 \neq 0$ and $\frac{a}{E_3} < 0$ (degenerate case).

The initial guess for Newton's and chord methods is obtained using the singular perturbation method [13], which leads to the following:

Case 1: (Non-degenerate Case)

For a given $\tau \neq 0$ the solution curves of (4.5.1) near $(0,\lambda_0,0)$ can be written as

$$X(\tilde{\varepsilon}) = A\tilde{\varepsilon}\phi + O(\tilde{\varepsilon}^2)$$
$$\lambda(\tilde{\varepsilon}) = \lambda_0 + \xi\tilde{\varepsilon} + O(\tilde{\varepsilon}^2) \qquad (4.6.1)$$

where $\tilde{\varepsilon}^2 = |\tau|$ and A is determined by the equation

$$E_0A^2 + a\xi A + b = 0. \tag{4.6.2}$$

<u>Case 2</u>: (Degenerate Case)

In this case the solution curves of (4.5.1) near $(0,\lambda_0 0)$ can be written as:

$$\begin{aligned} X &= A\tilde{\varepsilon}\Phi + A^2\tilde{\varepsilon}^2 w_2 + O(\tilde{\varepsilon}^3) \\ \lambda &= \lambda_0 + \xi\tilde{\varepsilon}^2 + O(\tilde{\varepsilon}^3) \end{aligned} \tag{4.6.3}$$

where $\tilde{\varepsilon}^3 = |\tau|$ and A is determined by

$$E_3A^3 + a\xi A + b = 0. \tag{4.6.4}$$

<u>Definition 4.6.1</u>: Let H_1 denote the Hilbert space $H \times R^1$ and define $F^{(i)} : H_1 \times R^1 \times R^1 \to H_1$, $i = 1,2$, by

$$F^1(U,\xi,\tilde{\varepsilon}) = \begin{bmatrix} G(A\tilde{\varepsilon}\Phi + w,\ \lambda_0 + \xi\tilde{\varepsilon} + \mu,\ \tilde{\varepsilon}^2) \\ \\ \langle\Phi,w\rangle \end{bmatrix} \tag{4.6.5}$$

where A is determined by (4.6.2),

$$F^2(U,\xi,\tilde{\varepsilon}) = \begin{bmatrix} G(A\tilde{\varepsilon}\Phi + A^2\tilde{\varepsilon}^2 w_2 + w,\ \lambda_0 + \xi\tilde{\varepsilon}^2 + \mu,\ \tilde{\varepsilon}^3) \\ \\ \langle\Phi,w\rangle \end{bmatrix} \tag{4.6.6}$$

where A is determined by (4.6.4). In (4.6.5) and (4.6.6) $U = \binom{w}{\mu} \in H_1$. Finally, let $U^0 = 0$ denote the initial guess.

Consider the equation

$$F^{i}(U,\xi,\tilde{\varepsilon}) = 0. \tag{4.6.7}$$

For any $\xi \in R^1$

$$F_U^{(i)}(0,\xi,0) = \begin{bmatrix} G_X^o & 0 \\ & \\ \langle \Phi, \cdot \rangle & 0 \end{bmatrix}, \qquad i = 1,2.$$

Therefore $F_U^{(i)}(0,\xi,0)$, $i = 1,2$, has the same properties as those of F_U^o of Theorem 4.4.1. Furthermore for any constant $C > 0$ there is a number $\delta(C) > 0$ such that $F_U^{(i)}(0,\xi,\tilde{\varepsilon})$ satisfies the conclusions of Lemmas 4.4.1, 4.4.2, 4.4.3 and 4.4.4 for $|\xi| \leq C$, $|\tilde{\varepsilon}| \leq \delta(C)$ and $i = 1,2$. The equivalent lemma to Lemma 4.4.5 is

<u>Lemma 4.6.1</u>: The eigenvalues α of $B(\tilde{\varepsilon})$ satisfy

$$\alpha = C_1\, \tilde{\varepsilon}^{1/2} + 0(\tilde{\varepsilon})$$

for some constant $C_1 \neq 0$.

<u>Proof</u>: Differentiating

$$F_U^{(i)}(0,\xi,\tilde{\varepsilon})\Phi_1(\tilde{\varepsilon}) = b_{11}(\tilde{\varepsilon})\Phi_1(\tilde{\varepsilon}) + b_{12}(\tilde{\varepsilon})\Phi_2(\tilde{\varepsilon})$$

with respect to $\tilde{\varepsilon}$ and setting $\tilde{\varepsilon} = 0$ gives

$$\begin{pmatrix} AG_{X\lambda}^o\Phi \\ 0 \end{pmatrix} + F_U^{(i)}(0,\xi,0)\dot{\Phi}_1(0) = \dot{b}_{11}(0)\Phi_1 + \dot{b}_{12}(0)\Phi_2$$

whose solvability condition gives

$$\dot{b}_{12}(0) = Aa \neq 0.$$

Lemma 4.6.1 implies the following

Theorem 4.6.1: For each $C > 0$ there exists $\delta(C) > 0$ such that $F_U^{(i)^{-1}}(0,\xi,\tilde{\varepsilon})$ exists and

$$\|F_U^{(i)^{-1}}(0,\xi,\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{-1/2})$$

for each $|\xi| \leq C$, $\tilde{\varepsilon}$ in $0 < |\tilde{\varepsilon}| \leq \delta(C)$, and $i = 1,2$.

The equivalent lemma to Lemma 4.4.6 is

Lemma 4.6.2:

(a) $\|F_U^{(1)^{-1}}(0,\xi,\tilde{\varepsilon})F^{(1)}(0,\xi,\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{5/2})$

(b) $\|F_U^{(2)^{-1}}(0,\xi,\tilde{\varepsilon})F^{(2)}(0,\xi,\tilde{\varepsilon})\| = O(\tilde{\varepsilon}^{7/2})$.

Proof:

$$F^{(1)}(0,\xi,\tilde{\varepsilon}) = \begin{bmatrix} (A^2\overset{0}{G}_{XX}\Phi\Phi + 2A\xi\overset{0}{G}_{X\lambda}\Phi + 2\overset{0}{G}_{\tau})\tilde{\varepsilon}^2 \\ \\ 0 \end{bmatrix} + O(\tilde{\varepsilon}^3).$$

Since A satisfies (4.6.2) it follows that

$$\langle\Psi(\tilde{\varepsilon}),F^{(1)}(0,\xi,\tilde{\varepsilon})\rangle = O(\tilde{\varepsilon}^{5/2}),$$

which gives (a).

Similarly

$$\langle\Psi(\tilde{\varepsilon}),F^{(2)}(0,\xi,\tilde{\varepsilon})\rangle = O(\tilde{\varepsilon}^{7/2})$$

and this implies (b).

It follows from the above caluculation that the functions h and q of Theorems 4.2.2 and 4.2.3 satisfy

$$h(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^2), \; q(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^2)$$

for the non-degenerate case, and

$$h(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^3), \; q(\tilde{\varepsilon}) = O(\tilde{\varepsilon}^3)$$

for the degenerate case.

This gives the proof of the following

<u>Theorem 4.6.2</u>: For each constant $C > 0$ there exists $\delta(C) > 0$ such that for each ξ and $\tilde{\varepsilon}$ with $|\xi| \leq C$ and $0 < |\tilde{\varepsilon}| \leq \delta(C)$ Newton's and chord iterates with initial guess $U^0 = 0$ converge to a unique non-trivial solution U^* of

$$F^{(i)}(U,\xi,\tilde{\varepsilon}) = 0, \qquad i = 1,2.$$

CHAPTER 5

NUMERICAL METHODS

5.0 Introduction

In Section 3.1 the shooting algorithm for a non-linear boundary value problem (BVP) of the form $X'(s) = g(s,X,\lambda,\tau)$, $B_0X(0) + B_1X(1) = 0$, where λ and τ are parameters, is described. The shooting method reduces the BVP to a problem of finding zeroes of a non-linear equation with two parameters.

In Section 3.2 the application of the shooting method to bifurcation and perturbed bifurcation problems [18] is described. This leads to a finite dimensional bifurcation and perturbed bifurcation problems. It is also shown in Section 3.2 that the numerical schemes presented in Chapter 4 can be applied to the finite dimensional problem corresponding to the BVP (1.2.10), (1.2.11).

The numerical solution of the BVP (1.2.10), (1.2.11) are presented and discussed in Section 5.3. It is found that the numerical schemes of Chapter 4 give accurate results in the neighborhood of the bifurcation points.

Finally, the perturbation solution obtained in Chapter 3 is compared with the numerical solution in Section 5.4.

5.1 Basic definitions and results

Let $g : [0,1] \times \Omega \times R^1 \times R^1 \to R^n$ denotes a smooth mapping, where

$0 \in \Omega \subset R^n$ is an open domain, let B_0, B_1 denote any $n \times n$ constant matrices with $\text{rank}[B_0 : B_1] = n$ and consider the BVP

$$X' = g(s,X,\lambda,\tau)$$
$$B[X] = B_0X(0) + B_1X(1) = 0. \qquad (5.1.1)$$

The shooting method is the most common used method for solving such a BVP. It reduces (5.1.1) to a finite dimensional problem as follows. For each $A \in R^n$ the smoothness of g ensures the existence and uniqueness of solution $X(s,A,\lambda,\tau)$ of the initial value problem (IVP)

$$X' = g(s,X,\lambda,\tau)$$
$$X(0) = A \qquad (5.1.2)$$

which depends smoothly on A. Define a mapping $K : R^n \times R \times R \to R^n$ by

$$K(A,\lambda,\tau) \equiv B_0A + B_1X(1,A,\lambda,\tau).$$

The smoothness of g implies that K is smooth mapping. The following theorem shows that the BVP (5.1.1) is equivalent to the problem of finding the roots of the (finite dimensional) equation

$$K(A,\lambda,\tau) = 0 \qquad (5.1.3)$$

and gives the relation between the solutions of (5.1.1) and (5.1.3).

<u>**Theorem 5.1.1**</u>: The BVP (5.1.1) has as many solutions as there are distinct roots of (5.1.3).

Proof: If A is a root of (5.1.3), then there is a unique solution X of the IVP (5.1.2) satisfying $X(0) = A$. Since $K(A,\lambda,\tau) = 0$, it follows that $B[X] = 0$ and hence X is a solution of the BVP (5.1.1). Conversely, if X is a solution of the BVP (5.1.1) then $A = X(0)$ is a root of (5.1.3). This sets up a one to one correspondence between the set of solutions of BVP (5.1.1) and the roots of (5.1.3).

Let X_0 denotes a solution of the BVP (5.1.1), A_0 be the corresponding solution of (5.1.3) for some (λ,τ) and M be the linear operator

$$M(X) \equiv X' - g_X(s,X_0,\lambda,\tau)X \tag{5.1.4}$$

with domain

$$D \equiv \{X \in C_n^1[0,1] : B[X] = 0\}.$$

It is well known [12] that the adjoint M^* of M is the linear operator

$$M^*(Z) \equiv - Z' - g_X^T(s,X_0,\lambda,\tau)Z \tag{5.1.5}$$

with domain

$$D^* = \{Z \in C_n^1[0,1] : B^*[Z] = B_0^*Z(0) + B_1^*Z(1) = 0\}$$

where B_0^* and B_1^* are any $n \times n$ constant matrices with $\text{rank}[B_0^* : B_1^*] = n$ and

$$B_0 B_0^{*T} = B_1 B_1^{*T}. \tag{5.1.6}$$

For $X,Y \in C_n'[0,1]$, let $\langle X,Y\rangle$ denotes the L_2-inner product. The proof of the following theorem can be found in many standard ordinary differential equations texts (for example see [3]).

Theorem 5.1.2: The ranges of the operators M and M^* are closed and satisfy

$$R(M) = N(M^*)^{\perp}$$

$$R(M^*) = N(M)^{\perp} .$$

The following theorem gives the relationship between $N(M)$ and $N(K_A(A_0,\lambda,\tau))$ (see [12] for the proof).

Theorem 5.1.3: The mapping $E : N(M) \to N(K_A(A_0,\lambda,\tau))$ defined by $E(\phi) \equiv \phi(0)$ is one to one onto and

$$\dim(N(M)) = \dim(N(K_A(A_0,\lambda,\tau)).$$

5.2 Shooting method for bifurcation and perturbed bifurcation problems

The theory described in this section is due to H. Weber [18] (see [18] for the proofs of the lemmas and theorems).

Consider the BVP (5.1.1) under the additional assumptions

(a) $g(s,0,\lambda,0) = 0$ (b) $g_\tau(s,0,\lambda,0) \neq 0$

for all λ in R^1. So that $X = 0$ is a solution of (5.1.1) for each λ when $\tau = 0$.

It follows from the Implicit Function Theorem that nontrivial solutions of (5.1.1) may branch off from the trivial solution when $\tau = 0$ only at eigenvalues of the linearized problem. Assume for some $\lambda_0 \in R$

that the following conditions hold

(a) $N(M_0)$ is one dimensional generated by Φ

(b) $a \equiv \langle \Psi, g_{X\lambda}(\cdot,0,\lambda_0,0)\Phi \rangle \neq 0$ (5.2.1)

(c) $b \equiv \langle \Psi, g_T(\cdot,0,\lambda_0,0) \rangle \neq 0$

where M_0 denotes the linear operator defined by (5.1.4) with $X_0 = 0$ and Ψ generates $N(M_0^*)$.

Under these assumptions it follows from Theorem 3.1.1 that $(0,\lambda_0,0)$ is a bifurcation point. Now, Theorem 5.1.1 reduces the study of the solution set of (5.1.1) in a neighborhood of $(0,\lambda_0,0)$ to the study of the zeros of the non-linear equation (5.1.3) in the neighborhood of $(0,\lambda_0,0)$. It follows from Theorems 5.1.1 and 5.1.3 that $K(0,\lambda,0) = 0$, $K_T(0,\lambda,0) \neq 0$ for all λ in R^1, that $N(K_A(0,\lambda_0 0))$ is one-dimensional and that it is generated by $\Phi(0)$. In what follows it is shown that the mapping K satisfies the conditions of (5.2.1).

<u>Definition 5.2.1</u>: Let $P \in R^n$ be such that $\|P\| \neq 0$ and

$$P^T K_A(0,\lambda_0,0) = 0.$$

<u>Definition 5.2.2</u>: Let Y and Z denote the fundamental matrices defined (uniquely) by

(a) $Y' = g_X(s,0,\lambda_0 0)Y$ (b) $Z' = -g_X^T(s,0,\lambda_0,0)Z$

$Y(0) = I$ $Z(1) = I.$

<u>Lemma 5.2.1</u>:

(1) $Z(0)^T = Z(s)^T Y(s) = Y(1)$

(2) $\Phi(s) = Y(s)\Phi(0)$

(3) $N(M^*) = \langle Z(s)\ B_1^T P\rangle$

(4) $\frac{\partial}{\partial\lambda} Y(1,0,\lambda_0,0) = Y(1) \int_0^1 Y^{-1}(t) g_{X\lambda}(t,0,\lambda_0,0) Y(t)dt.$

<u>**Theorem 5.2.1**</u>:

(1) $\langle P, K_{A\lambda}(0,\lambda_0,0)\Phi(0)\rangle = \langle \hat{\Psi}, g_{X\lambda}(s,0,\lambda_0,0)\Phi\rangle$

(2) $\langle P, K_T(0,\lambda_0,0)\rangle = \langle \hat{\Psi}, g_T(s,0,\lambda_0,0)\rangle$

where $\hat{\Psi}(s) = Z(s)\ B_1^T P$.

It follows from Theorem 5.2.1 that the conditions (5.2.1) are satisfied for the finite dimensional problem (5.1.3).

The numerical schemes of Chapter 4 can be used to solve (5.1.3) in the neighborhood of $(0,\lambda_0,0)$ provided that the algebraic multiplicity of zero eigenvalue of $K_A(0,\lambda_0 0)$ is one. That is, provided that

$$\langle P, \Phi(0)\rangle \neq 0. \tag{5.2.2}$$

It is shown below that condition (5.2.2) holds for the BVP (1.2.10), (1.2.11) and therefore allows the application of the numerical schemes of Chapter 4 to this problem.

To see that (5.2.2) holds for BVP (1.2.10), (1.2.11) let $P = (P_1,P_2,P_3,P_4,P_5,P_6)^T$ be such that $\|P\| \neq 0$ and

$$(B_0 + y(1)B_1)^T P = 0. \tag{5.2.3}$$

It can be shown easily that

$$y(1) = \begin{bmatrix} 1-3\beta\pi^2 & 1-3\alpha\pi^2 & 0 & 6\beta\pi^3 & 0 & -2\beta \\ -3\pi^2+9\alpha\pi^4 & 1-3\beta\pi^2 & 0 & 6\pi^3-18\alpha\pi^5 & 0 & -2+6\alpha\pi^2 \\ -2+6\alpha\pi^2 & -2\beta & 1 & -12\alpha\pi^3 & 0 & 4\alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{2}{\pi}+6\alpha\pi & -\frac{2}{\pi}\beta & 0 & -12\alpha\pi^2 & 1 & \frac{4}{\pi}\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{(3\pi^2)^n}{(2n+3)!}$, $\beta = \sum_{n=0}^{\infty} (-1)^n \frac{(3\pi^2)^n}{(2n+2)!}$,

and, therefore, (5.2.3) is equivalent to

$$\begin{aligned} &\text{(a)} \quad P_1 + (1 - 3\beta\pi^2)P_2 + (-2 + 6\alpha\pi^2)P_4 = 0 \\ &\text{(b)} \quad (1 - 3\alpha\pi^2)P_2 - 2\beta P_4 = 0 \\ &\text{(c)} \quad P_3 + P_4 = 0 \\ &\text{(d)} \quad 6\beta\pi^3 P_2 - 12\alpha\pi^3 P_4 + P_5 = 0 \\ &\text{(e)} \quad P_6 = 0 \\ &\text{(f)} \quad -2\beta P_2 + 4\alpha P_4 = 0. \end{aligned} \qquad (5.3.2)$$

It is easy to show that $\alpha > 0$, $\beta > 0$. Hence if $P_2 = 0$ equation (f) of (5.3.2) would imply that $P_4 = 0$, hence $P_1 = 0$ by (a), $P_3 = 0$ by (c), $P_5 = 0$ by (d) and $P_6 = 0$ by (e). This contradicts the fact

that $\|P\| \neq 0$. This shows that $P_2 \neq 0$ and $P_6 = 0$. Since

$$\dot{\Phi}(n)(0) = (0, n\pi, 0, 0, 0, n\pi^2)$$

it follows that $\langle P, \dot{\Phi}(n)(0) \rangle \neq 0$ and hence the zero eigenvalue of the linearized finite dimensional problem corresponding to the BVP (1.2.10), (1.2.11) has algebraic multiplicity one.

5.3 The numerical solution

In this section the equilibrium states of an elastic shell under a uniform and non-uniform pressure fully or partially submerged in a fluid are determined. The load-carrying capacity of the shell F is computed for various values of the parameter τ (see Figure 5.3.3). It is found that the maximum of F occur at a point where the shell is partially submerged in the fluid.

The numerical solution is obtained as follows. The BVP (1.2.10), (1.2.11) is discritized using the shooting method described in the previous two sections. This results in a non-linear finite dimensional problem with the two parameters λ and τ. The numerical schemes of Sections 4.4 and 4.5 are used to solve the resulting bifurcation and perturbed bifurcation problems in the neighborhood of the bifurcation points. Away from the bifurcation points, the finite dimensional equation is solved by Newton's method.

It is found that the numerical results compare well with the perturbation approximations of Chapter 3. This shows that the numerical schemes of Chapter 4 are able to produce accurate results in

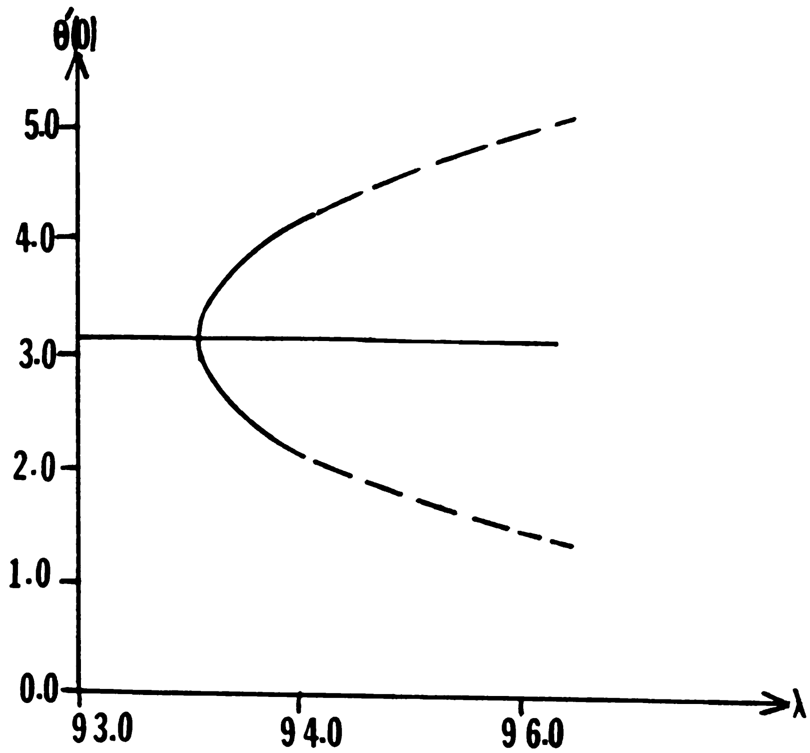
the neighborhood of a bifurcation point.

Table 5.3.1 gives the numerical values of $\theta'(0)$ corresponding to some values of λ near the first bifurcation point $\lambda_2 = 3\pi^3$, for the case when $\tau = 0$, i.e. when the shell is under a uniform pressure. These results agree with the numerical results previously obtained in [15] and [6]. Figure 5.3.1 shows the solution curves branching from the bifurcation point λ_2.

<u>Table 5.3.1</u>

Numerical solution for λ near $3\pi^3 \approx 93.01883$ when $\tau = 0$

λ	$\theta_1'(0)$	$\theta_2'(0)$
93.00	3.14159	--
93.018	3.14159	--
93.01883	3.14159	--
93.02	3.15070	3.13703
93.03	3.16757	3.12258
93.04	3.26680	3.10718
93.05	3.30614	2.98577
93.06	3.34218	2.95345
93.07	3.36170	2.92241
93.08	3.38592	2.90160
93.09	3.40547	2.88449
93.10	3.42700	2.85913



Bifurcation curves branching from $\lambda_2 = 3\pi^3$

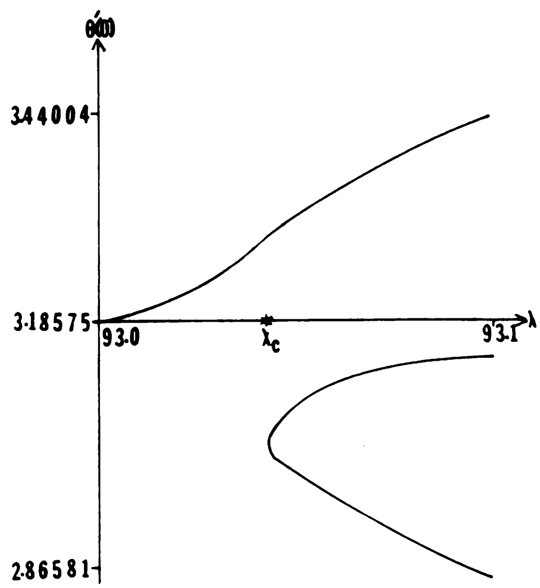
Figure 5.3.1

Table 5.3.2 and Figure 5.3.2 give numerical values of $\theta'(0)$ corresponding to some values of λ near λ_2 for $\tau = .001$. It appears from Figure 5.3.2 that there is still a critical pressure λ_c beyond which a drastic change occurs in the shape of the shell which in turn affect the load carrying capacity of the shell.

Table 5.3.2

Numerical solution for λ near $3\pi^3 \approx 93.01883$ when $\tau = .001$

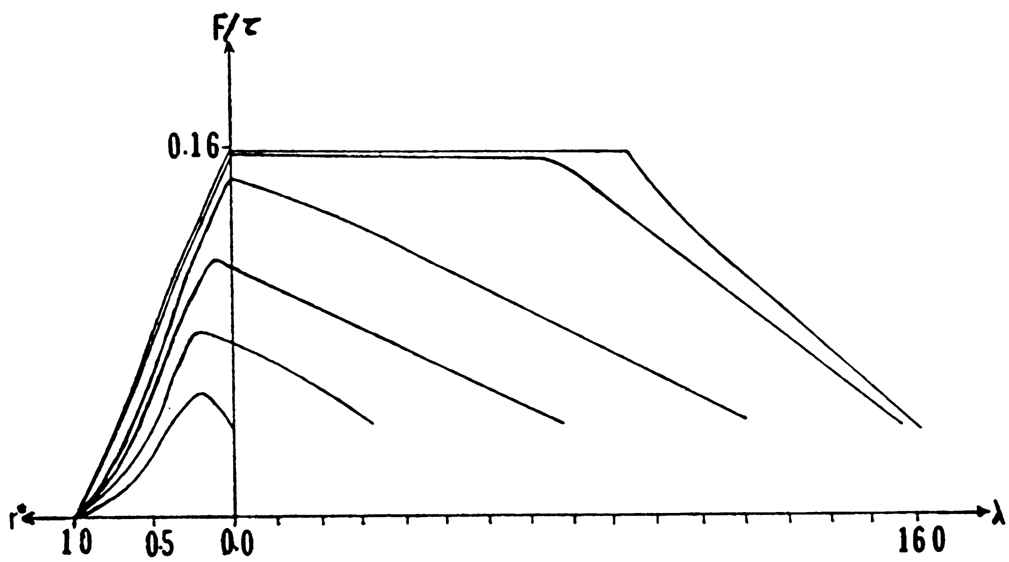
λ	$\theta_1'(0)$	$\theta_2'(0)$	$\theta_3'(0)$
93.00	3.18575	--	--
93.01883	3.21531	--	--
93.036	3.27039	--	--
93.03697	3.27352	--	--
93.04272	3.28746	3.02756	3.02756
93.043	3.29975	3.04625	3.03192
93.046	3.30758	3.06385	3.00181
93.06	3.34433	3.11090	2.97097
93.08	3.39265	3.12337	2.91125
93.1	3.44004	3.13745	2.86581



Solution curves of the perturbed bifurcation problem near the bifurcation point $\lambda_2 = 3\pi^3$ when $\tau = .001$

Figure 5.3.2

Figure 5.3.3 gives the relation between the load-carrying capcity of the shell F and the pressure λ for various values of τ. It also describes F as a function of r^* in the case when the shell is partially submerged in the fluid, where r^* is the arc length of the non-wetted part of the shell. The last point of each curve in Figure 5.3.3 represents the equilibrium state when there is a point contact between the sides of the shell (see Figures 5.3.5 - 5.3.7). For large values of τ (τ > 385) side contact occurs at a partially submerged state (see Figure 5.3.7). To study the behavior of the shell beyond this contact point a different formulation is needed.

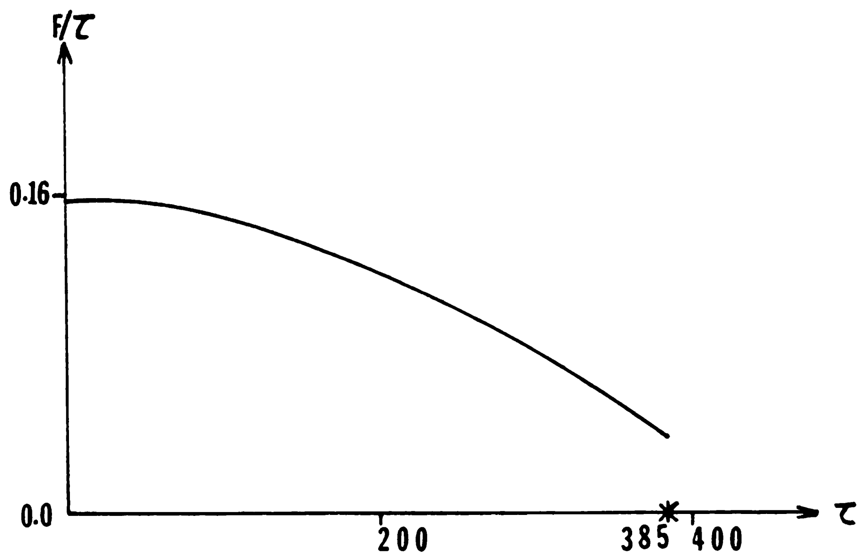


The load-carrying capacity of the shell for the values of
τ = 0,10,100,200,300,385

<u>Figure 5.3.3</u>

Remark: Note that for each values of τ the maximum of F occurs at some $r_c^*(\tau)$ (see Figure 5.3.3) corresponding to an equilibrium state where the shell is partially submerged. Note also that the load-carrying capacity of the shell is zero in the case when the shell is under a uniform pressure, i.e. $\tau = 0$.

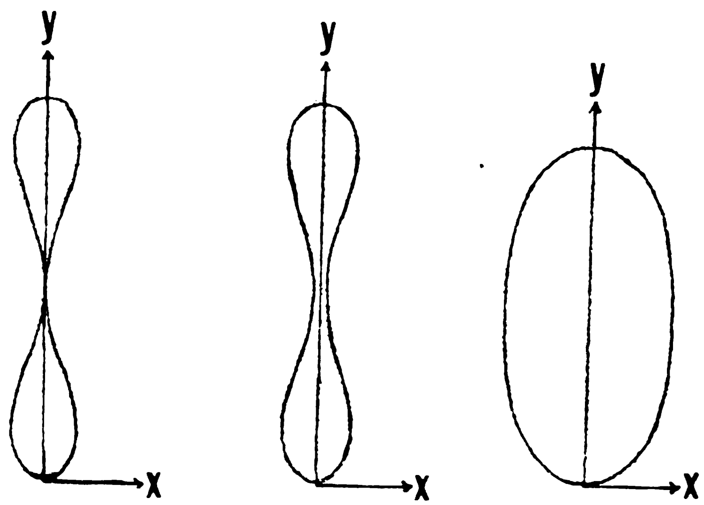
Figure 5.3.4 describes F as a function of τ while λ is kept zero, i.e. when the pressure difference between the inside and the outside of the shell is zero.



The load-carrying capacity of the shell as a function of τ when $\lambda = 0$

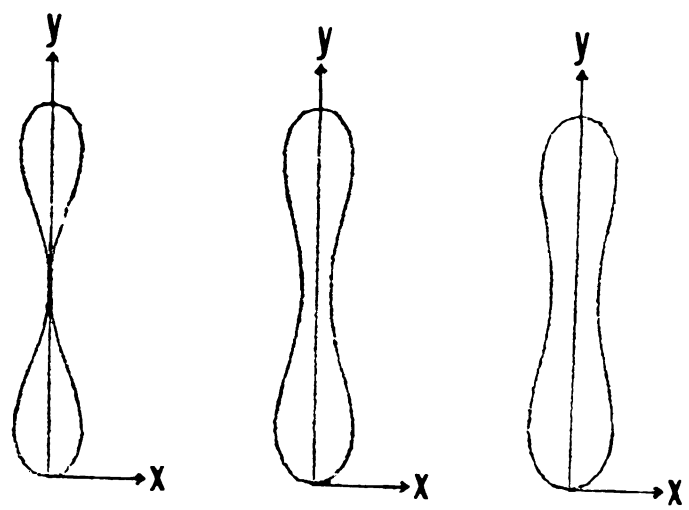
Figure 5.3.4

Finally, Figures 5.3.5 - 5.3.8 give the shape of the shell for various values of λ and τ.



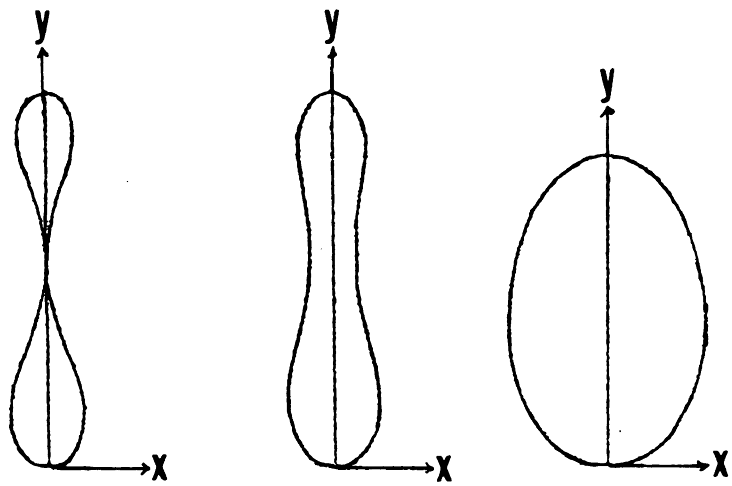
The deformation of the shell for $\tau = 10$ and $\lambda = 158,150,92$

Figure 5.3.5



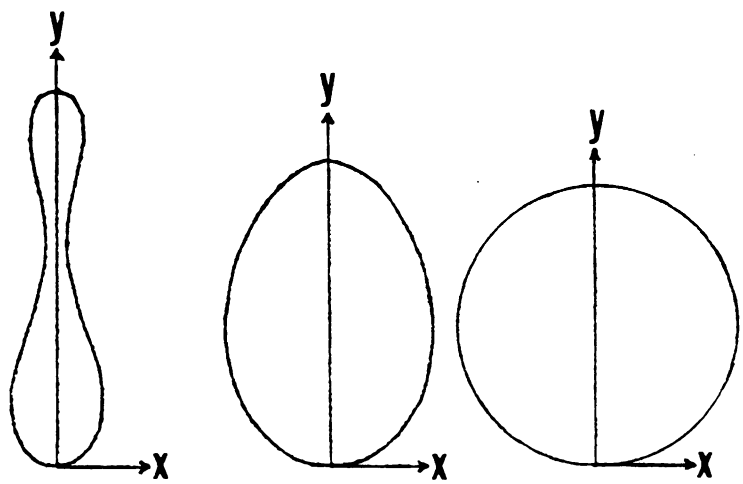
The deformation of the shell for $\tau = 100$, $\lambda = 119,100,90$

Figure 5.3.6



The deformation of a shell with no pressure difference and large values of τ = 385,300,100

Figure 5.3.7



The deformation of a partially submerged shell for τ = 700 and r^* = .3,.6,1

Figure 5.3.8

5.4 Error in the perturbation solution

In this section the perturbation solution obtained in Chapter 3 is compared with the numerical solution. The percentage of the error in the perturbation solution is computed in Tables 5.4.1 - 5.4.4 for $\tau = .001$ and $\tau = 1$. Away from the first bifurcation point λ_2 the regular perturbation expansions of Section 3.2 are used to approximate $\theta'(0)$ while the perturbation expansions of Section 3.4 are used to approximate $\theta'(0)$ for λ near λ_2. In Tables 5.4.1 - 5.4.4, $\theta'_p(0)$ and $\theta'_N(0)$ represent perturbation and numerical solutions respectively.

Table 5.4.1

Error in the perturbation solution for the partially submerged case when $\tau = .001$

r^*	$\theta'_p(0)$	$\theta'_N(0)$	Error percent
1.0	3.141593	3.141593	0.000000
.8	3.141593	3.141593	0.000000
.6	3.141593	3.141593	0.000000
.4	3.141596	3.141596	0.000000
.2	3.141600	3.141600	0.000000
0	3.141601	3.141601	0.000000

Table 5.4.2

Error in the perturbation solution for the fully submerged case when τ = .001

λ	$\theta'_P(0)$	$\theta'_N(0)$	Error percent
0	3.141601	3.141601	0.000000
20	3.141603	3.141606	0.000095
40	3.141608	3.141617	0.000286
60	3.141620	3.141630	0.000318
80	3.141665	3.141678	0.000414
90	3.141919	3.141911	0.000255
92	3.142574	3.142506	0.000022
93	3.189530	3.185750	0.118650
93.0188	3.245700	3.215310	0.945470
93.0360	3.304070	3.270390	1.029847
93.0430	3.324720	3.299750	0.756720
93.0460	3.333060	3.307580	0.770350
93.1000	3.450670	3.440040	0.309000

Table 5.4.3

Error in the perturbation solution for the partially submerged case when $\tau = 1$

r^*	$\theta'_P(0)$	$\theta'_N(0)$	Error percent
1.0	3.141593	3.141593	0.000000
.8	3.141627	3.141627	0.000000
.6	3.142336	3.142337	0.000032
.4	3.145058	3.145059	0.000032
.2	3.148582	3.148589	0.000222
0.0	3.149656	3.149680	0.000762

Table 5.4.4

Error in the perturbation solution for the fully submerged case when $\tau = 1$

λ	$\theta'_P(0)$	$\theta'_N(0)$	Error percent
0	3.149656	3.149680	0.000762
20	3.152293	3.152360	0.002125
40	3.157093	3.157220	0.004023
60	3.168193	3.168350	0.004955
80	3.214093	3.215390	0.040337
90	3.468255	3.481110	0.369270
92	3.834543	3.904634	1.795000
93	4.176900	4.270300	2.187200
93.0188	4.183705	4.277370	2.189780
93.0360	4.189932	4.283830	2.191917
93.0430	4.192466	4.286450	2.192584
93.0460	4.193552	4.287580	2.193032
93.1000	4.213093	4.307820	2.198955

CHAPTER 6

CONCLUSIONS

In this chapter results in the previous chapters are summarized and discussed.

In Chapter 1 the physical problem treated in this thesis was formulated along two lines. The first formulation was obtained by balancing the moments, keeping the top point of the ring fixed. This led to a non-linear boundary value problem for one half of the ring involving two physical parameters λ and τ representing, respectively, the depth of the ring and the non-uniformity of the pressure. A second formulation was obtained using Hamilton's principle of least energy in which the bottom point of the ring was kept fixed. This led to a variational problem for the entire ring. Since the bottom point of the ring was kept fixed the load carrying capacity 2γ did not enter the energy expression and therefore was treated as a parameter to be determined along with the shape of the ring. A similar variational formulation in which 2γ was treated as a given parameter was also given and this used to prove the existence of solution for a given value of 2γ in Chapter 2.

In Chapter 2 several existence results were obtained. The variational formulation obtained in Chapter 1 was used to prove the existence of an equilibrium state of the ring for each value of λ and τ, while 2γ was treated as a parameter to be determined (i.e. the shape determines the load carrying capacity 2γ). Three existence results in which 2γ was treated as a given parameter were given. In this case, the load carrying capacity 2γ determins the parameters λ and τ and, hence, the shape of the ring. The proofs of the existence results of Chapter 2 were based upon a paper by F.E. Browder [1] and are extensions to the results obtained by I. Tadjbakhsh [15] for the uniform pressure case.

In Chapter 3 all the singular or bifurcation points λ_n of the basic solution curve (the circular solution) were determined for the case when $\tau = 0$ (the uniform pressure case). The Liapunov-Schmidt theory was used to prove that a pitchfork bifurcation occurs at each λ_n. Valid asymptotic expansions for the bifurcating solution curves near each λ_n were obtained. The behavior of the basic solution, the bifurcation points and the bifurcating solution curves under the perturbation $\tau \neq 0$ were examined. It is found that each λ_n goes into a "perturbed" bifurcation point and that at each such perturbed bifurcation point a limit point bifurcation occurs. The proof of this result was based upon the Implicit Function Theorem and, hence, can be used to construct the perturbed bifurcation points curve as well as the solution curves through each of them. The regular and singular perturbation methods were used to obtain valid asymptotic expansions for the solution curves when $\tau \neq 0$ but small.

In Chapter 4 the convergence of Newton's iterates near the bifurcation point was considered. The use of Newton's method to determine the bifurcating branches near a simple bifurcation point was originally considered by Decker and Keller [5]. They replaced the equation $G(\lambda,u) = 0$ by an "inflated" system $F(X,\varepsilon) = 0$, where

$X = \binom{u}{\lambda}$, parametrized X by a different parameter ε (an approximation to the arc length), introduced an initial guess $X^0(\varepsilon)$ and proved the convergence of Newton's iterates to a solution $X(\varepsilon)$ of the inflated system $F(X,\varepsilon) = 0$ for each small enough ε. In practical problems, however, λ is often a physical parameter and it is required to compute u as a function of λ, though it may not always be possible in general to compute u as function λ (since bifurcating solutions may occur only at the bifurcation point). It is shown in Chapter 4 that for the pitchfork bifurcation problem Newton's method can be used to compute u for a given λ near the bifurcation point while the asymptotic expansions for the bifurcating curves of Chapter 3 were used to produce the initial guess for Newton's iterates. It was also shown in Chapter 4 that Newton's method can be used to determine the perturbed bifurcation points and the solution curves for the perturbed bifurcation problem provided that τ is small enough, with the initial guess for Newton's iterates being obtained from the singular perturbation expansions of Chapter 3.

In Chapter 5 the boundary value problem formulated in Chapter 1 was discretized by the shooting method. This led to a finite-dimensional, yet non-linear, problem. Away from the bifurcation points Newton's method was used and near the bifurcation points the numerical schemes of Chapter 4 were used to solve the finite-dimensional problem. It was found that the load carrying capacity 2γ attains its maximum value at a partially submerged case, that for each $0 \leq \tau \leq k$ $(k \approx 385)$ the deformation of the ring will increase as λ increases from 0 until at $\lambda = \lambda_c(\tau)$, when opposite sides of the ring touch at one pair of points (Figures 5.3.5, 5.3.6), and that for $\tau > k$ opposite sides of the

ring touch at a partially submerged case. Finally, the perturbation and the numerical solutions were compared for τ = .001 and τ = 1. It is found that the percentage of error in the perturbation solution is less than 3%.

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