COMPUTATION OF OPTIMAL CONTROLS FOR LINEAR SYSTEMS WITH CONTROL OUTAGE

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This is to certify that the

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ABSTRACT

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By

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In this study different aspects of optimal control problems are considered for a class of linear systems for the case in which the control temporarily fails to function but resumes normal operation after a certain time period (finite duration of control outage). Only the class of linear systems and controls with constrained amplitude are treated. The concepts of recoverability and cost constraint for the general linear optimal control problem are also discussed.

Because of the control outage, the control constraint set is time varying and piecewise continuous. The τ-reachable sets for linear systems with and without control outage are studied. It is proved that the reachable set in the event of outage is compact, convex, and varies continuously with time just as it does in the absence of outage.

Because of the similarity between the optimal controls for some singular systems and systems with control outage, the conditions for singularity in linear systems are studied, and the distinctions between the two cases are made explicit.

The τ-reachable sets are derived analytically for time-invariant systems using the switching time variation technique. This problem is treated as a series of subcases; e.g., second order, third order,..., scalar input, vector input, real eigenvalues, and complex eigenvalues. Variation of the τ-reachable set both with respect to the control outage starting time and duration of the control outage is studied. Also variation of the area inscribed by the boundary of the τ-reachable set for second order systems with respect to the control outage starting time and duration of the control outage is treated. The minimum regulation time for linear time-invariant systems with control outage and its variation with respect to control outage starting time, duration of control outage, and initial state of the system are investigated.

The convexity of the τ-reachable set for linear systems with control outage makes possible the use of known solution techniques based on the convexity of this set. Gilbert's method is applied to compute optimal controls for linear systems with a finite duration of control outage. The modification for calculation of the contact function for systems with control outage is shown. Several examples are solved to demonstrate the convergence of this algorithm. The convergence rate is slow for systems of order three and higher. Suggestions are given on methods for obtaining faster convergence.

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TO MY PARENTS

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CHAPTER 1

INTRODUCTION

There exists at present, a considerable amount of theoretical material, dealing with minimum-time control of linear systems, with constraint on control amplitude. Bellman, Glicksberg, Gross [2], Pontryagin [3] and, Neustadt [1] have shown that the minimal time control is in the familiar "bang-bang" form.

In some problems it is necessary to constrain the rate of control, due to inertial or other factors. Luh and Shafran [4] have studied this class of problems.

In all preceding studies, it is assumed that control can be exerted over the process throughout the time interval of interest. Another problem that can be considered, is that of loss of control for some time duration during the process. That is, what happens to a system if the controller has normal operation from an initial time, to some instant $\hat{t}$, fails to function during the interval $[\hat{t}, \hat{t} + \Delta T]$, and resumes normal operation after $\hat{t} + \Delta T$? This loss of control is called "control outage".

The idea of control outage forms the basis for a new class of optimal control problems, in which a "recoverability constraint" is another term to be considered in the design of

control systems. That is, a question exists as to whether a system can recover from a loss of control, and still hit the target before a specified time threshold is exceeded. Hauer and Hsu [5], were the first to develop the concept of recoverability constraint (control outage), but they only considered a specific two-dimensional example.

1.1 Review of the Literature

To be able to understand minimal-time problems with control outage, a knowledge of this class of problems without control outage is required. In order to apply the results of Pontryagin's Maximum Principle [3],some methods require a bounded state-variable process. Berkovitz [6], Gamkrelidze [3], and Dreyfus [7] have investigated this class of optimal control problem . Each author used a different method to obtain the necessary condition for the optimal open-loop control. Berkovitz used the calculus of variation approach, Gamkrelidze modified the maximum principle of Pontryagin, and Dreyfus applied the method of dynamic programming. In later work [8, 9], the equivalence between the variously-derived necessary conditions was shown.

Many computational techniques have also been developed to find the optimal solution to the general open-loop bounded state variable problem, [10, 11, 12, 13]. One technique involves first replacing the constraint by penalty functions. Then this new unconstrained problem is solved by using gradient methods. Another technique involves the direct application of

the necessary conditions. A certain set of initial conditions are guessed. The trajectory is then found by numerical integration, and the necessary conditions are tested. If they are not all satisfied, the initial guess is corrected, and the procedure is repeated.

There are also methods which are based on the convexity of the reachable set of the system states. Some of these methods require strict convexity of the reachable sets, and exhibit slow convergence [14].

Gilbert's method [22] does not require strict convexity of the reachable sets, nor is restricted to bounded state variable control problems, but exhibits slow convergence especially for higher-order systems. An extension of Gilbert's method by Barr [23], guarantees much faster convergence. Using Barr's technique, a wide variety of optimal control problems can be solved [24].

1.2 Organization of the Thesis

Structurally, there are five chapters in the main body of this dissertation. In Chapter 1, a general introduction, literature review, and the problem statement are discussed. Precise definitions, and mathematical backgrounds for the systems with and without control outage are presented in the second chapter. In Chapter 2, it is also proved that the reachable set for linear systems with control outage is compact, convex, and continuously varying with time. Chapter 3 is devoted to the analytical derivation of the reachable set of the linear time-invariant systems with control outage. Also time-optimal

control problems with control outage are studied in this chapter. In Chapter 4, Gilbert's method is used to find the optimal control for systems with a finite duration of control outage. Although a broad class of optimal control problems could be solved with this computational technique; in this study most emphasis is upon the time-optimal control problems. Finally, conclusions and suggestions for further study are given in Chapter 5.

1.3 Statement of the Problem

The class of linear time varying processes to be investigated are described by the system of differential equations

$$\underline{\dot{x}(t)} = A(t)\ \underline{x(t)} + B(t)\ \underline{u(t)} \quad \text{on} \quad t \in [t_o, T] \qquad (1.3.1)$$

where

$\underline{x(t)}$ is an n-dimensional state vector.

$\underline{u(t)}$ is an r-dimensional control vector.

$A(t)$ is an $n \times n$ matrix.

$B(t)$ is an $n \times r$ matrix.

$\underline{x(t_o)} = \underline{x_o}$ is an n-dimensional initial state vector.

$W(t) \subset R^n$ is a nonempty, compact, and continuously moving target set with respect to the real variable t on $t_o \le t \le T$. The matrices $A(t)$ and $B(t)$ are assumed to be piecewise continuous in t, on $t_o \le t \le T$.

The input vector $\underline{u(t)}$ is required to be admissible, so that it satisfies the following conditions:

a. $\underline{u(t)}$ is measurable

b. $\underline{u(t)} \in U(t) \subset R^r$, where

$$U(t) = \begin{cases} \mathcal{O} & \text{if} \quad t \in [\hat{t}, \hat{t} + \Delta T] \\ \mathcal{U} & \text{otherwise} . \end{cases} \quad (1.3.2)$$

$\mathcal{U}$ is a unit hypercube in R^r, and $\mathcal{O}$ is the zero vector in R^r.

$\hat{t}$ is an arbitrary element (time) in $[t_o, T]$, and ΔT is a finite duration of time such that

$$t_o \leq \hat{t} \leq \hat{t} + \Delta T \leq T . \quad (1.3.3)$$

The set $U(t)$ is a time varying (piecewise continuous), compact, convex restraint set, which indicates a control outage starting at $t = \hat{t}$, for a duration of ΔT, during the process. The unit hypercube $\mathcal{U}$ in R^r could be extended to an arbitrary convex, compact set, but unit hypercube is used for simplicity.

The general time-optimal control problem, which is studied in Chapter 3 of this thesis, can be described as follows:

Given a system defined by plant equation (1.3.1), with initial state $\underline{x(t_o)}$, and desired terminal state or target function $W(t)$. Find an admissible control $\underline{u(t)} \in U(t)$, where $U(t)$ is defined by equation (1.3.2), for $t \in [t_o, T]$, which makes $\underline{x(t)} \in W(t)$ in the smallest possible time, i.e. to^*. Such an admissible control, is called optimal control $\underline{u^*(t)}$. Here to^* corresponds to the minimum time which it takes the system to travel from $\underline{x(t_o)}$ to the final state at $\underline{x(to^*)} \in W(to^*)$, considering a control outage starting at $\hat{t}$, for a duration of

ΔT, such that

$$t_o \leq \hat{t} \leq \hat{t} + \Delta T \leq t_o^* .$$

The time-optimal control problem is considered a free terminal time problem. Fixed terminal-time, optimal control problems, which can be solved by the method described in Chapter 4, can be stated as follows:

Given a system, defined by plant equation (1.3.1), with initial state $\underline{x(t_o)}$ at t_o, and given a fixed final time $\tau \in [t_o, T]$. Find an admissible control $\underline{u^*(t)}$, which transfers the state of the system from $\underline{x(t_o)}$ to $W(\tau)$, such that:

$$JO_\tau(\underline{u}^*) = g(\underline{x_{u^*}(\tau)}) + \int_{t_o}^{\tau} [\underline{x'(\sigma)}Q(\sigma)\underline{x(\sigma)} + \underline{u^{*\prime}(\sigma)}R(\sigma)\underline{u^*(\sigma)}]d\sigma$$

$$\leq \min_{\underline{u(t)}\in U(t)} \{g(\underline{x_u(\tau)}) + \int_{t_o}^{\tau} [\underline{x'(\sigma)}Q(\sigma)\underline{x(\sigma)} + \underline{u'(\sigma)}R(\sigma)\underline{u(\sigma)}]d\sigma\} . \tag{1.3.4}$$

$JO_\tau(\underline{u})$ is called the cost functional for the systems with control outage, where the control constraint set is defined by equation (1.3.2). It should be noted for $\tau \in [t_o, T]$, $g(\underline{x_u(\tau)})$ is a given real continuous, convex function from R^n to R^1. $Q(t)$ and $R(t)$ are real $n \times n$ continuous symmetric matrices on $t_o \leq t \leq T$, and $Q(t)$ is assumed to be positive semidefinite and $R(t)$ is positive definite. Prime indicates the transpose. The assumptions made here are the same as used in [19], for optimal control problems without any control outage.

1.4 Optimal Control Problems with Recoverability Constraint

In fixed or free terminal-time optimal control problems, a cost functional $JO_t(\underline{u})$ is given of the form (1.3.4). In time-optimal control problems the cost functional is simply

$$JO_t(\underline{u}) = \int_0^t dt \ .$$

For given initial state $\underline{x(t_o)}$, and final state $\underline{x(t_{fo})} \in w(t_{fo})$ in R^n, where t_{fo} is the time when the target is reached, the cost functional $JO_{t_{fo}}(\underline{u}) \in R^1$ is a function of the control-outage starting time, $\hat{t}$, where $\hat{t} \in [t_o, T]$, and ΔT is fixed. Thus for fixed values of $\underline{x(t_o)}$, $\underline{x(t_{fo})}$, and ΔT; the following equation could be written

$$JO_{t_{fo}}(\underline{u}) = F(\hat{t}) \qquad (1.3.5)$$

where in fixed terminal-time problems t_{fo} is given, but $JO_{t_{fo}}(\underline{u})$ will vary as $\hat{t}$ changes. In free terminal-time problems both t_{fo} and $JO_{t_{fo}}(\underline{u})$ are functions of $\hat{t}$.

Given a cost constraint $JO_{cs} \in R^1$, there may be subintervals $I_1, I_2, \ldots$ in $[t_o, T]$, such that $F(\hat{t}) > JO_{cs}$, if $\hat{t} \in I_j$, $j = 1,2,\ldots$. The system is said to be <u>recoverable</u>, if the control-outage starting time, $\hat{t} \notin I_j$, $j = 1,2,\ldots$, (the system is not recoverable if $\hat{t} \in I_J$, $J = 1,2,\ldots$). In Figure (1.4.1) the system is not recoverable if $\hat{t} \in I_1 \cup I_2$, and is recoverable if $\hat{t} \in \{[t_o, T] - (I_1 \cup I_2)\}$.

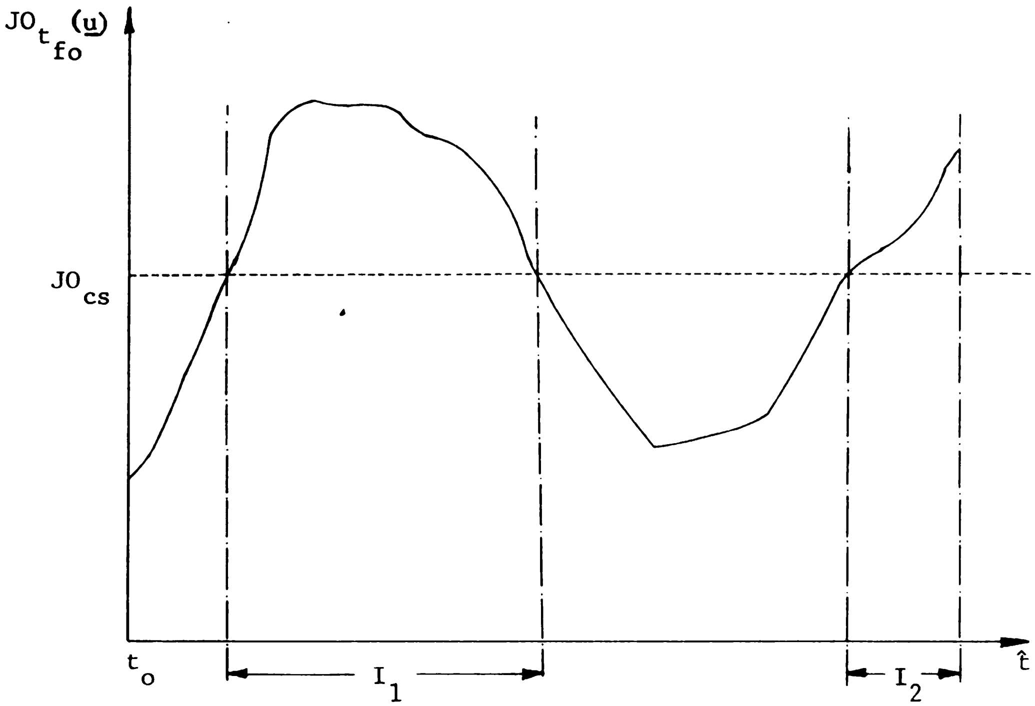


Figure (1.4.1): Variation of the Cost Functional with Respect to the control-outage starting time $\hat{t}$.

CHAPTER 2

DEFINITIONS AND GENERAL THEOREMS

In this chapter precise definitions of the τ-reachable, and τ-recoverable sets are given. The general properties of these sets for the cases with and without control outage are studied. The distinction between the term recoverability-constraint, which is defined in Chapter 1, and the set of recoverability is stated. Moreover, the basic theorems which will be useful for the remainder of the dissertation are presented. Those theorems which are proved in previous studies are simply stated, and the reader is referred to the original papers for proof. The compactness, convexity, and continuity with respect to time, of the τ-reachable set, for the linear time varying systems with control outage are proved in this chapter.

2.1 Definitions

All definitions in this section are given with implicit reference to the system defined by equation (1.3.1), and control constraint set given by equation (1.3.2).

DEFINITION 2.1.1. For the given class of the functions $\underline{u(t)} \in U(t)$, where $U(t)$ is defined by equation (1.3.2), and initial state $\underline{x(t_o)} = \underline{x_o} \in R^n$, and time $\tau \in [t_o, T]$, the set of all possible states that the system defined by equation (1.3.1) can reach in R^n, at time $\tau \in [t_o, T]$, by the use of the controls from the set $U(t)$, is called the $\underline{\tau\text{-reachable}}$ region, considering control outage. This set will be denoted by $RO(\tau, \hat{t}, \Delta T)$. The τ-reachable region in some of the literature is called the set of attainability at time τ.

DEFINITION 2.1.2. For the given system defined by equation (1.3.1) and given time $\tau \in [t_o, T]$ and final state $\underline{x(\tau)} = W(\tau)$; the τ-recoverable region with control outage, relative to $\underline{x(\tau)}$, is defined as the set of all initial conditions at time t_o that can be brought to the final state at time τ, using admissible controls defined by equation (1.3.2). This set will be denoted by $KO(\tau, \hat{t}, \Delta T)$. The relationship between $RO(\tau, \hat{t}, \Delta T)$, and $KO(\tau, \hat{t}, \Delta T)$ is shown later in this chapter.

Let Ω be the control constraint set with no control outage. That is, consider the admissible control $\underline{u(t)} \in \Omega$, where

$$\Omega = \{\mathcal{U} \quad \text{for all} \quad t \in [t_o, T]\} \tag{2.1.1}$$

where $\mathcal{U}$ is the unit hypercube in R^r.

Definitions 2.1.1, and 2.1.2, could be exactly repeated for the case with no control outage, using $\underline{u(t)} \in \Omega$, for admissible controls. In this study $R(\tau)$, and $K(\tau)$ will represent τ-reachable and τ-recoverable sets respectively, considering no control outage.

DEFINITION 2.1.3. A system defined by equation (1.3.1), is called completely controllable, if given any two states in R^n, there is a bounded control $\underline{u(t)}$, that will drive the system from one state to the other in finite time.

It is easy to see that if a system is completely controllable for the case with no control outage, it is also completely controllable for a finite duration of outage during the process.

2.2 Properties of the τ-reachable Regions Considering Control Outage

Given a system defined by equation (1.3.1) and a control set defined by equation (2.1.1), the reachable set considering no control outage with respect to the initial state $\underline{x(t_o)}$ is

$$R(t) = \{\underline{x}: \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \int_{t_o}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma ,$$

$$\underline{u(t)} \in \Omega \quad \text{and} \quad t \in [t_o, T]\}$$

where $\phi(t,t_o)$ is a fundamental solution matrix of the homogeneous differential equation given by equation (1.3.1).

It has been shown [19], that the reachable set with no control outage $R(t)$ is compact, convex, and continuously moving with t, where $t \in [t_o, T]$. Let $\hat{t} \in [t_o, T]$, be a given control outage starting time, and ΔT, a given duration of control outage, such that

$$t_o \leq \hat{t} \leq \hat{t} + \Delta T \leq T .$$

To determine the reachable set with control outage for the system defined by equation (1.3.1), and control set given by equation (1.3.2), consider the following three cases:

a)

$$t_o \leq t < \hat{t} .$$

Then

$$RO(t,\hat{t},\Delta T) = \{\underline{x}: \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \int_{t_o}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma,$$

$$\underline{u(t)} \in \Omega, \ t \in [t_o,\hat{t})\} . \tag{2.2.2}$$

Therefore

$$RO(t,\hat{t},\Delta T) = R(t) \quad \text{for} \quad t_o \leq t < \hat{t} .$$

b)

$$\hat{t} \leq t < \hat{t} + \Delta T .$$

Then

$$\underline{x(\hat{t})} = \phi(\hat{t},t_o)\underline{x(t_o)} + \int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma . \tag{2.2.3}$$

$$\underline{x(t)} = \phi(t,\hat{t})\underline{x(\hat{t})} .$$

Thus

$$RO(t,\hat{t},\Delta T) = \{\underline{x}: \underline{x(t)} = \phi(t,\hat{t})\underline{x(\hat{t})}, \text{ if } t \in [\hat{t},\hat{t} + \Delta T)\}$$

or

$$RO(t,\hat{t},\Delta T) = \phi(t,\hat{t})R(\hat{t}) . \tag{2.2.4}$$

Substituting equation (2.2.3) into equation (2.2.4), yields

$$RO(t,\hat{t},\Delta T) = \{\underline{x}: \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma$$

$$\underline{u(t)} \in \Omega, \ t \in [\hat{t},\hat{t} + \Delta T)\} . \tag{2.2.5}$$

c)

$$\hat{t} + \Delta T \leq t \leq T$$

$$\underline{x(t)} = \phi(t,\hat{t} + \Delta T)\underline{x(\hat{t} + \Delta T)} + \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma \quad (2.2.6)$$

where

$$\underline{x(\hat{t} + \Delta T)} = \phi(\hat{t} + \Delta T,\hat{t})\underline{x(\hat{t})} \ . \quad (2.2.7)$$

Substitution of equation (2.2.6) into (2.2.7) yields

$$\underline{x(t)} = \phi(t,\hat{t})\underline{x(\hat{t})} + \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma \ .$$

Thus

$$RO(t,\hat{t},\Delta T) = \{\underline{x}\colon \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma,$$
$$\underline{u(t)} \in \Omega,\ t \in [\hat{t} + \Delta T,T]\} \ . \quad (2.2.8)$$

Finally

$$RO(t,\hat{t},\Delta T) = \phi(t,\hat{t})R(\hat{t}) + \{\underline{y}\colon \underline{y(t)} = \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma,$$
$$\underline{u(t)} \in \Omega,\ \hat{t} + \Delta T \leq t \leq T\} \ . \quad (2.2.9)$$

Considering equations (2.2.2), (2.2.4), and (2.2.9), the following theorem can be stated.

THEOREM 2.2.1. For the time-varying linear system given by equation (1.3.1), the reachable set $R(t)$ with no control outage and the reachable set $RO(t,\hat{t},\Delta T)$, with a control outage starting at $\hat{t}$, and duration of ΔT, are related as follows:

$$RO(t,\hat{t},\Delta T) = \begin{cases} R(t) & \text{if} \quad t_o \le t < \hat{t} \\ \phi(t,\hat{t})R(\hat{t}) & \text{if} \quad \hat{t} < t < \hat{t} + \Delta T \\ \phi(t,\hat{t})R(\hat{t}) + \{\underline{y}: \underline{y(t)} = \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma, & \\ \qquad \underline{u(t)} \in \Omega,\ \hat{t} + \Delta T \le t \le T\}. & \end{cases} \quad (2.2.10)$$

For time-invariant linear systems

$$RO(t,\hat{t},\Delta T) = \begin{cases} R(t) & \text{if} \quad t_o \le t < \hat{t} \\ e^{A(t-\hat{t})}R(\hat{t}) & \text{if} \quad \hat{t} \le t < \hat{t} + \Delta T \\ e^{A(t-\hat{t})}R(\hat{t}) + \{\underline{y}: \underline{y(t)} = \int_{\hat{t}+\Delta T}^{t} e^{A(t-\sigma)}[B(\sigma)\underline{u(\sigma)}]d\sigma, & \\ \qquad \underline{u(t)} \in \Omega,\ \hat{t} + \Delta T \le t \le T\}. & \end{cases} \quad (2.2.11)$$

THEOREM 2.2.2. Consider the linear time-varying control process in R^n, given by equation (1.3.1). Suppose the control constraint set, $U(t)$, is given by equation (1.3.2) where $U(t)$ is a time-varying, piecewise continuous with respect to time, nonempty, and compact set in R^r. Given $\hat{t} \in [t_o,T]$, and ΔT, such that $t_o \le \hat{t} \le \hat{t} + \Delta T \le T$, then the reachable set with control outage $RO(t,\hat{t},\Delta T)$, is compact, convex, and varies continuously with respect to t, $\hat{t}$, and ΔT, for all $t \in [t_o,T]$.

Proof: Consider the following three cases:

a)

$$t_o \le t < \hat{t}\ .$$

Then by equation (2.2.10)

$$RO(t,\hat{t},\Delta T) = R(t).$$

Since, $R(t)$ is compact, convex, and continuous with respect to t [19], so is $RO(t,\hat{t},\Delta T)$, for $t \in [t_o,\hat{t})$.

b)

$$\hat{t} \le t < \hat{t} + \Delta T .$$

By equation (2.2.10)

$$RO(t,\hat{t},\Delta T) = \phi(t,\hat{t})R(\hat{t}).$$

Since $R(\hat{t})$ is compact and convex and $\phi(t,t_o)$ is a non-singular continuous transformation, it follows that $RO(t,\hat{t},\Delta T)$, is compact, convex, and continuous with respect to t, for $t \in [\hat{t},\hat{t} + \Delta T)$.

c)

$$\hat{t} + \Delta T \le t \le T .$$

Consider equation (2.2.8)

$$RO(t,\hat{t},\Delta T) = \{\underline{x}: \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma,$$
$$\underline{u(t)} \in \Omega\} . \qquad (2.2.12)$$

I - Compactness

To show, $RO(t,\hat{t},\Delta T)$ is compact in R^n, for $t \in [\hat{t} + \Delta T,T]$, and $t_o \le \hat{t} \le \hat{t} + \Delta T \le T$, we have to prove that every sequence of points in $RO(t,\hat{t},\Delta T)$ has a subsequence which converges to some limit point in $RO(t,\hat{t},\Delta T)$.

Let

$$\underline{x_r(t)} \in RO(t,\hat{t},\Delta T), \quad \text{for} \quad r = 1,2,...$$

where

$$\underline{x_r(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u_r(\sigma)}]d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u_r(\sigma)}]d\sigma, \quad \underline{u_r(t)} \in \Omega, \; r = 1,2,\ldots$$

The set of all controllers $\underline{u(t)} \in \Omega$, where Ω is defined by equation (2.1.1) is weakly compact [19]. Therefore there is a subsequence in Ω which converges to some $\underline{\hat{u}(t)} \in \Omega$. Let $\underline{\hat{x}(t)}$ correspond to $\underline{\hat{u}(t)}$, thus

$$\underline{\hat{x}(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{\hat{u}(\sigma)}]d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{\hat{u}(\sigma)}]d\sigma \quad .$$

Equation (2.2.12), shows that, $\underline{\hat{x}(t)} \in RO(t,\hat{t},\Delta T)$. Q.E.D.

II - Convexity

Consider $\underline{x_1}$, $\underline{x_2}$ in $RO(t,\hat{t},\Delta T)$. Then there are admissible controls $\underline{u_1(t)}$, and $\underline{u_2(t)}$ in Ω, such that

$$\underline{x_1(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u_1(\sigma)}]d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u_1(\sigma)}]d\sigma \quad .$$

$$\underline{x_2(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u_2(\sigma)}]d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u_2(\sigma)}]d\sigma \quad .$$

Now let

$$\underline{x_3(t)} = \lambda\, \underline{x_1(t)} + (1-\lambda)\underline{x_2(t)} \qquad 0 \le \lambda \le 1$$

$$= \phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)(\lambda\, \underline{u_1}(\sigma) + (1-\lambda)\underline{u_2(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)(\lambda\, \underline{u_1}(\sigma) + (1-\lambda)\underline{u_2(\sigma)})]d\sigma \,.$$

By virtue of the convexity of Ω, $\underline{u_3(t)} = \lambda\, \underline{u_1(t)} + (1-\lambda)\underline{u_2(t)}$ is in Ω for $t \in [t_o,T]$ and is admissible. Therefore $\underline{x_3(t)} \in RO(t,\hat{t},\Delta T)$ and $RO(t,\hat{t},\Delta T)$ is convex.

III - Continuity with respect to t

Let t_1, and t_2 in $[t_o, T]$, be such that $\hat{t} + \Delta T \le t_1 \le t_2 \le T$. From equation (2.2.12), it follows:

$$\underline{x(t_2)} - \underline{x(t_1)} = \phi(t_2,t_o)\underline{x(t_o)} + \phi(t_2,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{t_2} \phi(t_2,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \phi(t_1,t_o)\underline{x(t_o)}$$
$$-\phi(t_1,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \int_{\hat{t}+\Delta T}^{t_1} \phi(t_1,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma \,.$$

Adding, and subtacting the term, $\int_{\hat{t}+\Delta T}^{t_1} \phi(t_2,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma$ to the above equation, yields

$$|\underline{x(t_2)} - \underline{x(t_1)}| \le \left|\int_{t_1}^{t_2} \phi(t_2,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\right| + \left|\int_{\hat{t}+\Delta T}^{t_1} [\phi(t_2,\sigma) - \phi(t_1,\sigma)][B(\sigma)\underline{u(\sigma)}[d\sigma\right| + \left|[\phi(t_2,\hat{t}) - \phi(t_1,\hat{t})]\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\right| + \left|[\phi(t_2,t_o) - \phi(t_1,t_o)]\underline{x(t_o)}\right| \,.$$

By the continuity of $\phi(t,t_o)$, and boundedness of $\underline{u(t)}$, it can be shown

$$|\underline{x(t_2)} - \underline{x(t_1)}| < \underline{M}_1|t_2 - t_1| + \underline{M}_2|t_2 - t_1| + \underline{M}_3|t_2 - t_1| + \underline{M}_4|t_2 - t_1|$$

$$< \underline{M}|t_2 - t_1| = \underline{\varepsilon}, \text{ whenever } |t_2 - t_1| < \delta .$$

IV - Continuity with respect to $\hat{t}$

Suppose $t \in [\hat{t},\hat{t} + \Delta T)$, then, let $\underline{x_1} \in RO(t,\hat{t},\Delta T)$, and $\underline{x_2} \in RO(t,\hat{t} + \delta\hat{t},\Delta T)$ correspond to control $\underline{u(t)} \in \Omega$, (assume t, ΔT fixed), then

$$|\underline{x_2(t,\hat{t} + \delta\hat{t},\Delta T)} - \underline{x_1(t,\hat{t},\Delta T)}| = |\phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t} + \delta\hat{t}) \int_{t_o}^{\hat{t}+\delta\hat{t}} \phi(\hat{t} + \delta\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \phi(t,t_o)\underline{x(t_o)} - \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma| = |\int_{t_o}^{\hat{t}+\delta\hat{t}} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \int_{t_o}^{\hat{t}} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma|$$

$$\leq \underline{M}|\delta\hat{t}| = \underline{\varepsilon} \quad \text{whenever} \quad |\delta\hat{t}| = |\hat{t}_2 - \hat{t}_1| < \delta .$$

Suppose $t \in [\hat{t} + \Delta T,T]$, then

$$|\underline{x_2(t,\hat{t} + \delta\hat{t},\Delta T)} - \underline{x_1(t,\hat{t},\Delta T)}| = |\phi(t,t_o)\underline{x(t_o)} + \phi(t,\hat{t} + \delta\hat{t}) \int_{t_o}^{\hat{t}+\delta\hat{t}} \phi(\hat{t} + \delta\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma + \int_{\hat{t}+\delta\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \phi(t,t_o)\underline{x(t_o)} - \phi(t,\hat{t})\int_{t_o}^{\hat{t}} \phi(\hat{t},\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma| \leq |\int_{t_o}^{\hat{t}+\delta\hat{t}} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma - \int_{t_o}^{\hat{t}} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma| .$$

Thus

$$|\underline{x}_2 - \underline{x}_1| < 2\underline{M}|\delta\hat{t}| = \underline{\varepsilon} \quad \text{whenever} \quad |\delta\hat{t}| = |\hat{t}_2 - \hat{t}_1| < \delta \ .$$

V - Continuity with respect to ΔT

Equations (2.2.2) and (2.2.4) show that $RO(t,\hat{t},\Delta T)$, does not depend upon ΔT, if $t \in [t_o,\hat{t} + \Delta T)$. To show the continuity with respect to ΔT, when $t \in [\hat{t} + \Delta T, T]$, consider equation (2.2.12).

$$|\underline{x_2(t,\hat{t},\Delta T + \delta(\Delta T))} - \underline{x_1(t,\hat{t},\Delta T)}| = \left|\int_{\hat{t}+\Delta T+\delta(\Delta T)}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\right.$$

$$\left. - \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\right| = \left|\int_{\hat{t}+\Delta T+\delta(\Delta T)}^{\hat{t}+\Delta T} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\right| \ .$$

Let $|\phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]| \leq \underline{M}$.

Thus

$$|\underline{x_2(t,\hat{t},\Delta T + \delta(\Delta T))} - \underline{x_1(t,\hat{t},\Delta T)}| \leq \underline{M}\left|\int_{\hat{t}+\Delta T+\delta(\Delta T)}^{\hat{t}+\Delta T} \right| = \underline{M}|\delta(\Delta T)| = \underline{\varepsilon} \ .$$

Thus whenever $|\Delta T_2 - \Delta T_1| = \delta(\Delta T) < \delta$, then $|\underline{x}_2 - x_1| < \underline{\varepsilon}$. Q.E.D.

2.3 Boundary of the Reachable Set and Extremal Controls

Suppose $RO(t,\hat{t},\Delta T)$ is the reachable set, for a system with control outage, where $RO(t,\hat{t},\Delta T)$ is defined by equation (2.2.10). Let $\underline{x(t_o)} \in R^n$ be an initial point in the state space at time t_o, and assume, that the target can be reached in finite time using $\underline{u(t)} \in U(t)$. Since $RO(t,\hat{t},\Delta T)$ is moving continuously with t, for $t \in [t_o,T]$, there exists a first

time instant, say $\tau \in [\hat{t} + \Delta T, T]$ at which the trajectory of the system $\underline{x(\tau)}$ reaches the target set $W(\tau)$. It is assumed that there is a control outage during the process, that the system is recoverable, and that $\tau > \hat{t} + \Delta T$. It is clear that $\underline{x(\tau)} \in \partial RO(\tau, \hat{t}, \Delta T)$, the boundary of the τ-reachable set.

DEFINITION 2.3.1. An admissible control $\underline{\hat{u}(t)} \in U(t)$, where $U(t)$ is defined by equation (1.3.2) on $t \in [t_o, T]$, is said to be an <u>extremal</u> <u>control</u> if there exists a nontrivial solution $\eta(t) = [\eta_1(t), \ldots, \eta_n(t)]$ of the adjoint system

$$\dot{\eta}(t) = -\eta(t)A(t) \tag{2.3.1}$$

such that

$$\begin{aligned} \eta(t)B(t)\underline{\hat{u}(t)} &= \max_{\underline{u(t)} \in U(t)} \{\eta(t)B(t)\underline{u(t)}\} \quad \text{for almost all} \\ &\qquad\qquad t \in [t_o, \hat{t}) \cup (\hat{t} + \Delta T, \tau] \\ &= 0 \qquad\qquad \text{for almost all} \\ &\qquad\qquad t \in [\hat{t}, \hat{t} + \Delta T] \, . \end{aligned} \tag{2.3.2}$$

THEOREM 2.3.1. An admissible control $\underline{\hat{u}(t)} \in U(t)$, on $[t_o, \tau]$ is an extremal control if and only if, there exists a nontrivial solution of (2.3.1) such that, the trajectory corresponding to $\hat{u}$, has the property that, the state $\underline{\hat{x}(\tau)}$ belongs to the boundary of the τ-reachable set, and $\eta(\tau)$ is the outward normal to a support hyperplane of $RO(\tau, \hat{t}, \Delta T)$, at $\underline{\hat{x}(\tau)}$.

Proof: Let

$$\eta(t) = \eta(t_o)\phi(t_o, t) \tag{2.3.3}$$

be the solution for the adjoint differential equation, given by equation (2.3.1) with $\eta(t_o) = \eta_o = [\eta_{1o}, \eta_{2o}, \ldots, \eta_{no}]$. Let $\eta(\tau)$, be the outward normal to $RO(\tau,\hat{t},\Delta T)$ at $\underline{\hat{x}(\tau)}$. The set $RO(\tau,\hat{t},\Delta T)$ is convex (Theorem 2.2.2); then the scalar product

$$\eta(\tau)\cdot[\underline{\hat{x}(\tau)} - \underline{x(\tau)}] \geq 0 \quad \text{for all} \quad x(\tau) \in RO(\tau,\hat{t},\Delta T).$$

Considering equations (2.2.12), and (2.3.3), it will yield

$$[\int_{t_o}^{\hat{t}} \eta(\sigma)[B(\sigma)\underline{\hat{u}(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{\tau} \eta(\sigma)[B(\sigma)\underline{\hat{u}(\sigma)}]d\sigma] \geq$$
$$[\int_{t_o}^{\hat{t}} \eta(\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma + \int_{\hat{t}+\Delta T}^{\tau} \eta(\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma, \text{ for all } \underline{u(t)} \in \Omega] . \qquad (2.3.4)$$

The above equation implies

$$\eta(t)[B(t)\underline{\hat{u}(t)}] = \max_{\underline{u(t)}\in U(t)} \{\eta(t)[B(t)\underline{u(t)}]\} \quad \text{for} \quad t \in [t_o,\hat{t}) \cup (\hat{t} + \Delta T,\tau] .$$

Thus $\underline{\hat{u}(t)}$ is extremal.

Conversely let $\hat{u}(t)$ be an extremal which satisfies equation (2.3.2). Let $\underline{\hat{x}(\tau)}$ and $\eta(\tau)$, be the corresponding trajectory, and adjoint response which makes $\underline{\hat{u}(t)}$ an extremal. Then equation (2.3.4) holds, thus

$$\eta(\tau)\cdot[\underline{\hat{x}(\tau)} - \underline{x(\tau)}] \geq 0 \quad \text{for all} \quad \underline{x(\tau)} \in RO(\tau,\hat{t},\Delta T) .$$

By convexity of the $RO(\tau,\hat{t},\Delta T)$, $\underline{\hat{x}(\tau)} \in \partial RO(\tau,\hat{t},\Delta T)$. Q.E.D.

It can be shown, that if an extremal control exists, then there exists a bang-bang control which is also extremal.

To clarify the subject, let $\partial\mathcal{U}$, denote the boundary of a unit hypercube in R^r, and $\mathcal{O} = \partial\mathcal{O}$, the zero vector in R^r, then we can state the following definition.

DEFINITION 2.3.2. An admissible control $\underline{u(t)}$ is said to be a bang-bang control on $t \in [t_o,T]$, if $\underline{u(t)} \in \partial U(t)$, where

$$\partial U(t) = \begin{cases} \mathcal{O} & \text{if} \quad t \in [\hat{t},\hat{t} + \Delta T] \\ \partial\mathcal{U} & \text{otherwise} . \end{cases} \tag{2.3.5}$$

Consider the system defined by equation (1.3.1), and the constraint control set $U(t)$, given by equation (1.3.2). If there exists an extremal control $\underline{\hat{u}(t)} \in U(t)$ which satisfies equation (2.3.2), then it is clear there exists a bang-bang control which is also extremal. This result implies that if there is a unique extremal control $\underline{\hat{u}(t)}$ which satisfies the maximality condition (2.3.2), then it is a bang-bang control.

2.4 Normality and Singularity

The question of the uniqueness of the extremal control is answered by the concept of normality which is stated as follows:

DEFINITION 2.4.1. The system defined by equation (1.3.1) is normal if given two admissible controls $\underline{u_1(t)}$ and $\underline{u_2(t)}$ in $U(t)$ which drive the system from $\underline{x(t_o)}$ to the same final state $\underline{x(\tau)} \in \partial RO(\tau,\hat{t},\Delta T)$, then

$$\underline{u_1(t)} = \underline{u_2(t)} \quad \text{a.e.} \quad \text{on} \quad t_o \le t \le \tau .$$

This means that, if a system is normal, then the extremal control which satisfies equation (2.3.2) is almost everywhere unique.

It is known that [19], this unique extremal control must be a bang-bang control in the set defined by the equation (2.3.5). This bang-bang extremal control is called the optimal control for the case with a finite duration of control outage.

For normal systems, the τ-reachable set, $RO(\tau,\hat{t},\Delta T)$, (considering control outage) is strictly convex in R^n. The systems studied in Chapter 3, are assumed to be normal, and the τ-reachable sets are strictly convex. The technique used in Chapter 4 does not require strict convexity of the reachable sets; thus the system need not be normal.

There exists a similarity between the forms of the optimal controls for systems with control outage and for some singular systems (bang-bang, including $\underline{u(t)} \equiv 0$ for a duration of time). For this reason it is worth noting some behavior of the singular systems.

DEFINITION 2.4.2. Given a constraint control set Ω, where

$$\Omega = \{\underline{u}\colon |u_i(t)| \leq 1 \quad \text{for} \quad i = 1,\ldots,r, \text{ and } \quad t \in [t_o,T]\}. \qquad (2.4.1)$$

The system defined by equation (1.3.1) is called a singular system if the optimal control $\underline{u^*(t)} \in \Omega$, which takes the system from $\underline{x(t_o)}$ to $\underline{x(\tau)} \in \partial RO(\tau,\hat{t},\Delta T)$, and the corresponding adjoint equation have the following property:

There exists at least one interval $(t_1,t_2] \in [t_o,\tau]$, where

$$\eta(t)B(t)\underline{u^*(t)} = 0 \quad \text{for} \quad t \in (t_1,t_2] \,. \qquad (2.4.2)$$

Considering equations (2.3.2) and (2.4.2), the similarity between singular systems and the systems with control outage could be observed. Equation (2.4.2) is identically zero for $t \in (t_1, t_2]$ not because the control vector $\underline{u}^*(t)$ is zero in this interval, but because the nature of B(t) and the adjoint vector $\eta(t)$, which depends upon A(t), yields the equation (2.4.2). For systems with control outage, equation (2.3.2) is identically zero, for $t \in [\hat{t}, \hat{t} + \Delta T]$, because $\underline{u}(t) \equiv 0$ in this interval. Since normality is not required, the singular systems with control outage can be studied using the technique which is described in Chapter 4. For these systems, $\eta(t)B(t)\underline{\hat{u}}(t)$, can be zero for two different intervals, one for the singularity condition and one for the control outage period.

2.5 Controllability

In this section the controllability of the systems with control outage is investigated.

DEFINITION 2.5.1. The domain of null controllability C, consists of those initial states in R^n, that can be brought to the target in finite time using admissible controls.

THEOREM 2.5.1. Given the time-invariant linear system in R^n

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{2.5.1}$$

with control constraint set $\Omega = R^r$. The system is completely controllable if and only if the $n \times (n \times r)$ controllability matrix G

$$G = [B, AB, A^2B, \dots, A^{n-1}B] \tag{2.5.2}$$

has rank n [19].

THEOREM 2.5.2. Consider the time-invariant system in R^n

$$\underline{\dot{x}(t)} = A\ \underline{x(t)} + B\ \underline{u(t)} \tag{2.5.3}$$

where $\underline{u(t)} \in U(t)$, and

$$U(t) = \left\{ \underline{u}: \begin{array}{ll} u_i(t) = 0 & \text{for } t \in [\hat{t}, \hat{t} + \Delta T] \\ |u_i(t)| \leq 1 & \text{otherwise} \end{array} \quad i = 1, 2, \dots, r \right\} \tag{2.5.4}$$

Assume the G matrix for this system has rank n, and the system is stable, i.e. Re $\lambda_i < 0$, where λ_i are the eigenvalues of A. Then the domain of null controllability is $C = R^n$, or in other words the system is completely controllable.

Proof: For the case with no control outage the theorem is proved by [19]. For the case with control outage, let $\underline{x(\hat{t})} \in R^n$, be the state of the system at $t = \hat{t}$, where the control outage starts. Consider a finite duration of control outage ΔT. The state of the system at $t = \hat{t} + \Delta T$, i.e. $\underline{x(\hat{t} + \Delta T)}$ is in R^n. Thus, $\underline{x(\hat{t} + \Delta T)} \in R^n$, could be considered as a new initial state for the system with control constraint satisfying

$$U_2 = \{\underline{u}: |u_i(t)| \leq 1 \quad \text{for} \quad t \in [\hat{t} + \Delta T, T],\ i = 1, \dots, r\}. \tag{2.5.5}$$

The system is considered to be completely controllable, which implies thenew initial state $\underline{x(\hat{t} + \Delta T)} \in R^n$, can be driven to the target in a finite time, using an admissible control satisfying

equation (2.5.5). Thus $\underline{x(o)} \in R^n$ is controllable, using a control vector which satisfies equation (2.5.4). Q.E.D.

This proof illustrates that if a system is completely controllable for the case with no control outage, it is also completely controllable for finite durations of control outages occurring during the process.

THEOREM 2.5.3 [20]. Consider the time-invariant linear system in R^n

$$\underline{\dot{x}(t)} = A\,\underline{x(t)} + [b_1|b_2|\ldots|b_r]\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} . \tag{2.5.6}$$

The system is normal if the following holds

$$\text{Rank }(G_j) = \text{Rank }[b_j, Ab_j, A^2b_j, \ldots, A^{n-1}b_j] = n, \text{ for } j = 1,\ldots,r .$$

THEOREM 2.5.4. Given a time-invariant linear system in R^n

$$\underline{\dot{x}(t)} = A\,\underline{x(t)} + B\,\underline{u(t)} \tag{2.5.7}$$

with control constraint set

$$U(t) = \left\{ \underline{u}: \begin{array}{ll} u_i(t) = 0 & t \in [\hat{t}, \hat{t} + \Delta T] \\ |u_i(t)| \le 1 & \text{otherwise} \end{array} \quad i = 1,\ldots,r \right\} . \tag{2.5.8}$$

Suppose the rank of the controllability matrix G is less than n. Then there exists a unique linear subspace $C \subset R^n$, such that, only inside this region C, the system is completely controllable (i.e., no points outside C can be brought inside

C, or vice versa).

Proof: This theorem is proved for the case with no control outage by [19]. It will be shown here that $C \subset R^n$, the region of the controllability is the same for both cases, considering control outage, or no control outage. Let $\omega \in R^n$ be a target outside the region of the controllability C, where C is the region of the controllability considering no control outage. By the definition ω, cannot be reached using any admissible control, including $\underline{u(t)} = 0$. Thus, if a target outside C (the region of controllability), cannot be reached in finite time, with no control outage, it also cannot be reached if a control outage exists during the process. The same argument could be used, if the initial condition $\underline{x(t_o)}$ is outside C, (the controllability region), and the target moves inside the controllability region.

THEOREM 2.5.5. Consider the linear time-varying system in R^n

$$\underline{\dot{x}(t)} = A(t)\underline{x(t)} + B(t)\underline{u(t)} \tag{2.5.9}$$

with control constraint set defined by (2.5.8). The system is controllable if and only if the $n \times n$ matrix $W(t_o,T)$

$$W(t_o,T) = \int_{t_o}^{\hat{t}} \phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)\,d\sigma + \int_{\hat{t}+\Delta T}^{T} \phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)\,d\sigma \tag{2.5.10}$$

is nonsingular for all $t_o \le \hat{t} \le \hat{t} + \Delta T \le T$ in R^1. Here $\phi(t,t_o)$ is the fundamental solution of the homogeneous part of

equation (2.5.9), with $\phi(t_o,t_o) = I$.

Proof: For the case with no control outage, the corresponding W matrix is [19]

$$W(t_o,T) = \int_{t_o}^{T} \phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)d\sigma \qquad (2.5.11)$$

where admissible controls $\underline{u(t)} \in \Omega$ and Ω is defined by equation (2.1.1).

Suppose $W(t_o,T)$ is nonsingular. To prove the controllability of the system, let $\underline{x_o}$ and $\underline{x_f}$ be given initial and end points. Assume

$$\underline{u(t)} = B(t)'\phi(t_o,t)\underline{\xi} \quad \text{for} \quad t \in [t_o,\hat{t}) \cup (\hat{t} + \Delta T,T]$$

$$\underline{u(t)} = 0 \qquad \text{otherwise}$$

where

$$\underline{\xi} = W(t_o,t_f)^{-1}[\phi(t_o,t_f)\underline{x_f} - \underline{x_o}] \; .$$

Thus

$$\underline{x_f} = \phi(t_f,t_o)\underline{x_o} + \phi(t_f,t_o)W(t_o,t_f)\underline{\xi}$$

or

$$\underline{x_f} = \phi(t_f,t_o)\underline{x_o} + \phi(t_f,t_o)\{\int_{t_o}^{\hat{t}} \phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t_f} \phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)d\sigma\}\{B'(t)\phi(t_o,t)'\}^{-1}\underline{u(t)}$$

$$\underline{x_f} = \phi(t_f,t_o)\underline{x_o} + \phi(t_f,t_o)\{\int_{t_o}^{\hat{t}} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{t_f} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma\} \; .$$

Therefore the system is controllable.

Assume that the system is controllable. Suppose $W(t_o,t_f)$ is singular. Then there exists a nonzero constant vector $\underline{\eta}$ such that

$$\eta' W(t_o,t_f)\underline{\eta} = 0 .$$

Thus

$$\int_{t_o}^{\hat{t}} \eta'\phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)\underline{\eta}d\sigma + \int_{\hat{t}+\Delta T}^{t_f} \eta'\phi(t_o,\sigma)B(\sigma)B'(\sigma)\phi'(t_o,\sigma)\underline{\eta}d\sigma = 0. \tag{2.5.12}$$

The system is controllable, therefore the terminal point $\phi(t_f,t_o)\underline{\eta}$ can be reached from the origin such that

$$\phi(t_f,t_o)\underline{\eta} = \phi(t_f,t_o)\int_{t_o}^{\hat{t}} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma + \phi(t_f,t_o)\int_{\hat{t}+\Delta T}^{t_f} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma$$

or

$$\underline{\eta} = \int_{t_o}^{\hat{t}} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma + \int_{\hat{t}+\Delta T}^{t_f} \phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma .$$

But

$$\eta'\underline{\eta} = \int_{t_o}^{\hat{t}} \eta'\phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma + \int_{\hat{t}+\Delta T}^{t_f} \eta'\phi(t_o,\sigma)B(\sigma)\underline{u(\sigma)}d\sigma > 0. \tag{2.5.13}$$

Results (2.5.12) and (2.5.13) are a contradiction. Thus $W(t_o,T)$ is nonsingular. Q.E.D.

THEOREM 2.5.6. Given time-varying linear system in R^n

$$\underline{\dot{x}(t)} = A(t)\underline{x(t)} + B(t)\underline{u(t)} \tag{2.5.14}$$

where $\underline{u(t)} \in U(t)$, for $t_o \le t \le T$, and $U(t)$ is defined by equation (2.5.8). Let $\underline{v(t)} \in U(t)$ be such that $\underline{v(t)} = \underline{u(\tau - t)}$ on $0 \le t \le \tau - t_o$. Then the τ-reachable set for the system defined by equation (2.5.14) relative to initial state $\underline{x(t_o)}$ is the same as the $(\tau - t_o)$ recoverable set relative to $\underline{x(t_o)}$ for the same system with time reversed [18].

THEOREM 2.5.7 [18]. Given the time-invariant linear system in R^n

$$\underline{\dot{z}(\tau)} = A\ \underline{z(\tau)} + B\ \underline{u(\tau)}\ .$$

Assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real, distinct and positive eigenvalues of A. If the system is normal, then the canonical form of the system equation is

$$\underline{\dot{x}(t)} = P\ \underline{x(t)} + Q\ \underline{u(t)}$$

where

$$t = \lambda_1 \tau \quad \text{and} \quad Q = PK,$$

such that

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_2/\lambda_1 & \cdots & \\ \vdots & & & \\ 0 & \cdots & & \lambda_n/\lambda_1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & k_{12} & \cdots & k_{1r} \\ \vdots & & & \vdots \\ 1 & k_{n2} & \cdots & k_{nr} \end{bmatrix} .$$

THEOREM 2.5.8 [21]. Consider the time-varying linear system in R^n

$$\underline{\dot{x}(t)} = A(t)\underline{x(t)} + B(t)\underline{u(t)} \tag{2.5.16}$$

where $\underline{u(t)} \in U(t)$ is given by equation (1.3.2). Assume the equation for the τ-reachable set in R^{n+1} is given by $\phi(\underline{x},\tau)$. Then the τ-reachable set satisfies the following Hamilton-Jacobi partial differential equation

$$\frac{\partial}{\partial\tau}[\phi(\underline{x},\tau)] + \sum_{i=1}^{n} \frac{\partial\phi(\underline{x},\tau)}{\partial x_i}\,\dot{x}_i(t) = 0 \;. \tag{2.5.17}$$

CHAPTER 3

COMPUTATION OF THE τ-RECOVERABLE REGION AND OPTIMAL TIME FOR TIME-INVARIANT SYSTEMS

Sections 1, 2, and 3 of this chapter are concerned with the computation of the τ-reachable set of the linear time-invariant second-order systems, with real eigenvalues and scalar input, for the cases with and without control outage. By Theorem (2.5.6), the τ-recoverable set for the forward-time system is the same as the τ-reachable set for the reverse-time system. The computation of the τ-reachable set is more tractable, thus the τ-reachable set is studied in these sections.

Section 4 of this chapter considers the variation of the area inscribed by the boundary of the τ-reachable set, with respect to the control outage starting time $\hat{t}$, and with respect to the duration of the control outage ΔT.

Section 5 of this chapter studies the optimal time for the second order system considering a control outage. The relationships between the optimal time and the control outage starting time $\hat{t}$, and between the optimal time and duration of the control outage ΔT, are also studied in this section.

In Sections 6 and 7 of this chapter, the τ-reachable set for nth order systems with distinct real eigenvalues, with scalar and vector input respectively are computed. Finally Section 8 considers the computation of the τ-reachable set for nth-order

systems with complex eigenvalues.

3.1 Generation of the τ-reachable Set

According to Theorem (2.5.7), the normalized equation for the second order system, with positive, real, and distinct eigenvalues for the forward time is

$$\underline{x}^{o}(t) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2/\lambda_1 \end{bmatrix} \underline{x(t)} + \begin{bmatrix} 1 \\ \lambda_2/\lambda_1 \end{bmatrix} u(t) \qquad (3.1.1)$$

where λ_1 and λ_2 are eigenvalues, $u(t) \in \Omega$, which is defined by equation (2.1.1). It is easy to show that (3.1.1) is completely controllable and normal (Theorems 2.5.1 and 2.5.3).

The problem is:

Given $\underline{x(\tau)}$ as a final state, what is the set of all initial states for which there is a control in Ω, that drives the system to the given final state at given time τ? This set of initial states, which is called the τ-recoverable region related to $\underline{x(\tau)}$, is the same as the τ-reachable set if the system with time reversed is considered. To compute, the τ-recoverable set for (3.1.1) we consider the reverse time system given by

$$\underline{\dot{x}(t)} = P\,\underline{x(t)} + \underline{q}\,u(t) \qquad (3.1.2)$$

where

$$P = \begin{bmatrix} -1 & 0 \\ 0 & -\lambda_2/\lambda_1 \end{bmatrix}, \quad \underline{q} = \begin{bmatrix} -1 \\ -\lambda_2/\lambda_1 \end{bmatrix} \qquad (3.1.3)$$

If there is no control outage at most one switching is required to reach any boundary point of the τ-reachable set with no control outage $R(\tau)$ [19]. Let the time of switching be t_1. As t_1 ranges from 0 to τ, the locus of the end points of the corresponding trajectories generates the boundary $\partial R(\tau)$ of the τ-reachable set.

If the constraint set is $U(t)$, which is defined by equation (1.3.2), then given $\hat{t} \in [0,T]$, where the control outage starts, and ΔT, duration of the control outage, the boundary of the τ-reachable set with control outage $\partial RO(\tau,\hat{t},\Delta T)$ can be generated considering the following three cases for the switching time t_1:

a)

$$0 \le \hat{t} \le \hat{t} + \Delta T \le t_1 \le T \quad .$$

To find $\partial RO(\tau,\hat{t},\Delta T)$, let

$$\begin{aligned} u(t) &= k_o && \text{for} \quad t \in [0,\hat{t}) \cup [\hat{t} + \Delta T, t_1) \\ u(t) &\equiv 0 && \text{for} \quad t \in [\hat{t},\hat{t} + \Delta T) \\ u(t) &= -k_o && \text{otherwise} \end{aligned}$$

where $k_o = \pm 1$.

b)

$$0 \le \hat{t} \le t_1 \le \hat{t} + \Delta T \le T \ .$$

In this case let

$$\begin{aligned} u(t) &= k_o && \text{for} \quad t \in [0,\hat{t}) \\ u(t) &\equiv 0 && \text{for} \quad t \in [\hat{t},\hat{t} + \Delta T) \\ u(t) &= -k_o && \text{otherwise} \ . \end{aligned}$$

c)

$$0 \le t_1 \le \hat{t} \le \hat{t} + \Delta T \le T \; .$$

$u(t) = k_o$ for $t \in [0,t_1)$

$u(t) = -k_o$ for $t \in [t_1,\hat{t}) \cup [\hat{t} + \Delta T,T]$

$u(t) = 0$ for $t \in [\hat{t},\hat{t} + \Delta T)$.

3.2 Switching Curves

For the systems with no control outage the switching curve γ^+, corresponding to $u = +1$ can be generated by starting at $\underline{x(o)} = \underline{0}$, and letting the system run for τ seconds with $u = +1$. Similarly, given $\hat{t}$, and ΔT, the switching curve $\gamma 0^+$ for the systems with control outage can be generated by starting at the origin, and letting the system run for τ-seconds with $u(t) \equiv 0$, for $t \in [\hat{t},\hat{t} + \Delta T]$, and $u(t) = +1$ otherwise. The switching curve $\gamma 0^+$, for the systems with control outage consists of three segments $\gamma 0_1^+$, $\gamma 0_2^+$, and $\gamma 0_3^+$ (Figure 3.2.1), such that

$$\gamma 0^+ = \gamma 0_1^+ \cup \gamma 0_2^+ \cup \gamma 0_3^+ \tag{3.2.1}$$

where

$$\gamma 0_1^+ = \{\underline{x}: \underline{x(t)} = \int_0^t e^{P(t-\sigma)} \underline{q}\, u(\sigma)\, d\sigma = [e^{Pt} - I]P^{-1}\underline{q} \quad \text{for } 0 \le t \le \hat{t}\} \tag{3.2.2}$$

$$\gamma 0_2^+ = \{\underline{x}: \underline{x(t)} = e^{P(t-\hat{t})} x(\hat{t}) = [e^{Pt} - e^{P(t-\hat{t})}]P^{-1}\underline{q} \quad \text{for } \hat{t} \le t \le \hat{t} + \Delta T\} \tag{3.2.3}$$

$$\gamma 0_3^+ = \{\underline{x}: \underline{x(t)} = [e^{Pt} - e^{P(t-\hat{t})} + e^{P(t-\hat{t}-\Delta T)} - I]P^{-1}\underline{q} \quad \text{for } \hat{t} + \Delta T \le t \le T\} \; . \tag{3.2.4}$$

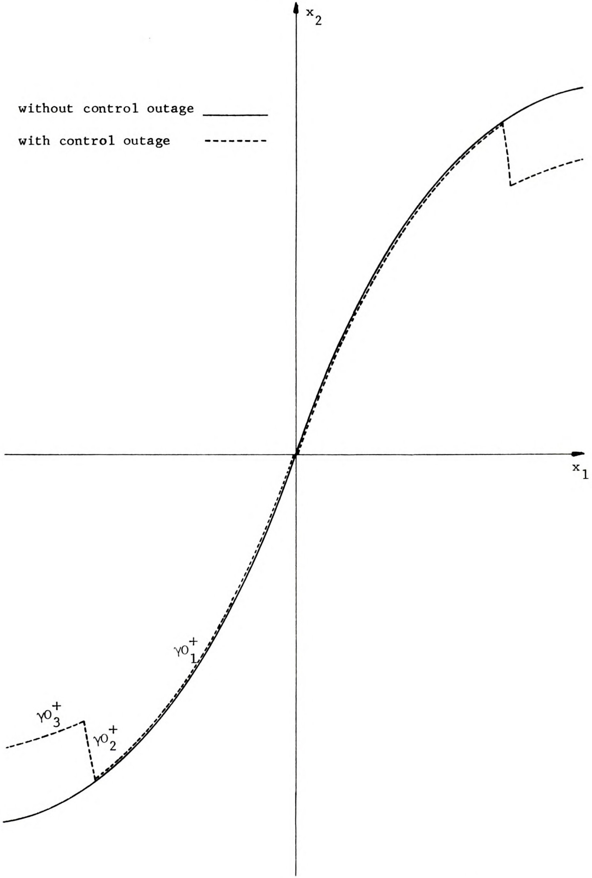


Figure (3.2.1): Switching curves for system defined by equation (3.1.2) for ΔT = .1 seconds, $\hat{t}$ = 1 second, and $r = \lambda_2/\lambda_1 = 2$.

The switching curve $\gamma 0^-$, corresponds to $u(t) = 0$, for $t \in [\hat{t}, \hat{t} + \Delta T]$ and $u(t) = -1$, otherwise. This $\gamma 0^-$ is symmetric to $\gamma 0^+$ with respect to the origin (Figure (3.2.1)) and can be described as a union of $\gamma 0_1^-$, $\gamma 0_2^-$, and $\gamma 0_3^-$.

Given $\hat{t}$, and ΔT, the equation of the switching curve for the system defined by equation (3.1.1) is:

$$\gamma 0_1^{\pm} = \left\{ \underline{x}:\ \underline{x(t)} = \pm \begin{bmatrix} e^{-t} - 1 \\ e^{-\lambda_2/\lambda_1} - 1 \end{bmatrix} \quad \text{for} \quad t \in [0, \hat{t}) \right\} \tag{3.2.5}$$

$$\gamma 0_2^{\pm} = \left\{ \underline{x}:\ \underline{x(t)} = \pm \begin{bmatrix} e^{-t} - e^{-t+\hat{t}} \\ e^{-\lambda_2/\lambda_1} - e^{-\lambda_2/\lambda_1 (t-\hat{t})} \end{bmatrix} \right\}$$

$$\text{for} \quad t \in [\hat{t}, \hat{t} + \Delta T) \tag{3.2.6}$$

$$\gamma 0_3^{\pm} = \left\{ \underline{x}:\ \underline{x(t)} = \pm \begin{bmatrix} e^{-t} - e^{-t+\hat{t}} + e^{-t+\hat{t}+\Delta T} - 1 \\ e^{-\lambda_2/\lambda_1 t} - e^{-\lambda_2/\lambda_1 (t-\hat{t})} + e^{-\lambda_2/\lambda_1 (t-\hat{t}-\Delta T)} - 1 \end{bmatrix} \right\} \tag{3.2.7}$$

Switching curves for the system given by equation (3.1.1) are shown in Figure (3.2.1) for the cases with no control outage and with control outage.

3.3 Calculation of the Boundary of the τ-Recoverable Set

The τ-reachable set for the reverse time system defined by equation (3.1.3) is calculated in this section. To compute $\partial RO(\tau, \hat{t}, \Delta T)$, one can begin at the origin in the reverse time

system and search for states that can be reached by admissible controls in time τ, along the minimum time trajectories. That is, search for the states that can be reached from the origin in precisely time τ, but cannot be reached in time less than τ. This set of states forms the boundary of the τ-reachable set for the reverse time system. Following the procedure given in Section (3.1), it follows:

If

$$0 \le \hat{t} \le \hat{t} + \Delta T \le t_1 \le \tau$$

where t_1 is the switching time, then

$$\underline{x(\hat{t})} = \pm [\int_0^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q} d\sigma] = \pm [e^{P\hat{t}} - I]P^{-1}\underline{q}$$

$$\underline{x(\hat{t} + \Delta T)} = e^{P(\hat{t}+\Delta T-\hat{t})} \underline{x(\hat{t})} = \pm e^{P\Delta T}[e^{P\hat{t}} - I]P^{-1}\underline{q}$$

$$\underline{x(t_1)} = e^{P(t_1-\hat{t}-\Delta T)} \underline{x(\hat{t} + \Delta T)} \pm \int_{\hat{t}+\Delta T}^{t_1} e^{P(t_1-\sigma)} \underline{q} d\sigma$$

$$= \pm [e^{Pt_1} - e^{P(t_1-\hat{t})} + e^{P(t_1-\hat{t}-\Delta T)} - I]P^{-1}\underline{q}$$

$$\underline{x(\tau)} = e^{P(\tau-t_1)} \underline{x(t_1)} \mp \int_{t_1}^{\tau} e^{P(\tau-\sigma)} \underline{q} d\sigma$$

where the upper sign corresponds to $u = +1$ initially and the lower sign to an initial choice of $u = -1$. Therefore the first segment of the boundary of the τ-reachable set with control outage, $\partial RO_1(\tau, \hat{t}, \Delta T)$, can be expressed as

$$\partial RO_1(\tau, \hat{t}, \Delta T) = \{\underline{x}: \underline{x(\tau)} = \pm [I + e^{P\tau} - 2e^{P(\tau-t_1)} + e^{P(\tau-\hat{t}-\Delta T)} - e^{P(\tau-\hat{t})}]P^{-1}\underline{q} \text{ for } 0 \le \hat{t} + \Delta T \le t_1\} . \qquad (3.3.1)$$

If

$$0 \le \hat{t} \le t_1 \le \hat{t} + \Delta T \le \tau$$

then

$$\underline{x(\hat{t})} = \pm [e^{P\hat{t}} - I]P^{-1}\underline{q}$$

$$\underline{x(\hat{t} + \Delta T)} = \pm e^{P\Delta T}[e^{P\hat{t}} - I]P^{-1}\underline{q}$$

$$\underline{x(\tau)} = e^{P(\tau-\hat{t}-\Delta T)} \underline{x(\hat{t} + \Delta T)} \mp \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)}\underline{q}d\sigma .$$

Thus the second part of the boundary of the τ-reachable set can be expressed as

$$\partial RO_2(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x(\tau)} = \pm [I + e^{P\tau} - e^{P(\tau-\hat{t}-\Delta T)} - e^{P(\tau-\hat{t})}]P^{-1}\underline{q}$$

$$\text{for} \quad \hat{t} \le t_1 \le \hat{t} + \Delta T\} . \qquad (3.3.2)$$

If

$$0 \le t_1 \le \hat{t} \le \hat{t} + \Delta T \le \tau,$$

then

$$\underline{x(t_1)} = \int_0^{t_1} e^{P(t_1-\sigma)}\underline{q}d\sigma = \pm [e^{Pt_1} - I]P^{-1}\underline{q}$$

$$\underline{x(\hat{t})} = e^{P(\hat{t}-t_1)}\underline{x(t_1)} \mp \int_{t_1}^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}d\sigma$$

$$= [e^{P\hat{t}} - 2e^{P(\hat{t}-t_1)} + I]P^{-1}\underline{q}$$

$$\underline{x(\hat{t} + \Delta T)} = \pm e^{P\Delta T}[e^{P\hat{t}} - 2e^{P(\hat{t}-t_1)} + I]P^{-1}\underline{q}$$

$$\underline{x(\tau)} = e^{P(\tau-\hat{t}-\Delta T)} \underline{x(\hat{t} + \Delta T)} \mp \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)}\underline{q}d\sigma$$

or

$$\partial RO_3(\tau,\hat{t},\Delta T) = \{\underline{x}\colon \underline{x(\tau)} = \pm [I + e^{P\tau} - 2e^{P(\tau-t_1)} - e^{P(\tau-\hat{t}-\Delta T)} + e^{P(\tau-\hat{t})}]P^{-1}\underline{q} \text{ for } 0 \le t_1 \le \hat{t}\}. \quad (3.3.3)$$

Thus the boundary of the τ-reachable set with control outage is

$$\partial RO(\tau,\hat{t},\Delta T) = \partial RO_1(\tau,\hat{t},\Delta T) \cup \partial RO_2(\tau,\hat{t},\Delta T) \cup \partial RO_3(\tau,\hat{t},\Delta T) .$$

For every value of $t_1 \in [0,\tau]$, there corresponds a state $\underline{x} \in \partial RO(\tau,\hat{t},\Delta T)$ in R^n. As t_1 ranges from 0 to $\hat{t}$, $\underline{x}$ will move to generate the boundary of the τ-reachable set $\partial RO(\tau,\hat{t},\Delta T)$. Suppose $\underline{x(\tau,\hat{t})}$ is a state corresponding to $t_1 = \hat{t}$. Then $\underline{x(\tau,\hat{t})}$ will be the same response for all $t_1 \in [\hat{t},\hat{t} + \Delta T]$. Thus the following lemma can be stated.

LEMMA 3.3.1. The transformation from R^1_+ (positive real's) to R^n (n-dimensional vector space for $\underline{x(\tau,t_1)}$) is not one to one for $t_1 \in [\hat{t},\hat{t} + \Delta T]$.

Proof: Equation (3.3.2) does not depend on t_1. Therefore for fixed values of τ, $\hat{t}$, and ΔT, one single state can be reached in the state space for all $t_1 \in [\hat{t},\hat{t} + \Delta T]$. This implies that the mapping from t_1 to R^n is many to one.

Considering equations (3.3.1), (3.3.2), and (3.3.3), one can deduce that the boundary of the τ-reachable set with control outage is symmetric with respect to the origin.

THEOREM 3.3.1. Let $\underline{x(\tau,t_1)}$ represent a point on the boundary, $\partial RO(\tau,\hat{t},\Delta T)$, of the τ-reachable set with control outage, generated by switching at t_1, where $0 \le t_1 \le \tau$. Then

$\underline{x}(\tau,t_1)$ moves continuously in R^n as t_1 ranges from 0 to τ.

Proof: Suppose $\underline{x'(\tau,t_1')}$ is an arbitrary state in $\partial RO_1(\tau,\hat{t},\Delta T)$, which is defined by equation (3.3.1). Let $\underline{x''(\tau,t_1'')}$ be an arbitrary point in $\partial RO_3(\tau,\hat{t},\Delta T)$, given by equation (3.3.3). To show the continuity, suppose $\epsilon > 0$ is given; then there exists a $\delta > 0$, such that

$$|\underline{x'(\tau,t_1')} - \underline{x''(\tau,t_1'')}| < \epsilon \quad \text{whenever} \quad |t_1' - t_1''| < \delta - \Delta T$$

for all $t_1' \in [\hat{t} + \Delta T,\tau]$, and $t_1'' \in [0,\hat{t}]$.

Considering equations (3.3.1) and (3.3.3), it follows

$$|[I + e^{P\tau} - 2e^{P(\tau-t_1')} + e^{P(\tau-\hat{t}-\Delta T)} - e^{P(\tau-\hat{t})} - I - e^{P\tau}$$

$$+ 2e^{P(\tau-t_1'')} + e^{P(\tau-\hat{t}-\Delta T)} - e^{P(\tau-\hat{t})}]P^{-1}\underline{q}|$$

$$\le 2|[e^{P(\tau-\hat{t}-\Delta T)} - e^{P(\tau-\hat{t})}]P^{-1}\underline{q}| + 2|[e^{P(\tau-t_1'')} - e^{P(\tau-t_1')}]P^{-1}\underline{q}|. \tag{3.3.4}$$

Introducing a vector norm e.g., $\sum_i |a_i|$, and applying the Taylor expansion to the right hand side of the inequality (3.3.4) for P and $\underline{q}$ given by equation (3.1.3) yields

$$|\underline{x'(\tau,t_1')} - \underline{x''(\tau,t''_1)}| < 2(1+r)(\Delta T + |t_1' - t_1''|) = \epsilon$$

where

$$r = \frac{\lambda_2}{\lambda_1} > 0 \ .$$

Thus, let

$$|t_1' - t_1''| = \frac{\epsilon}{2(1+r)} - \Delta T$$

and

$$\delta > \frac{\epsilon}{2(1+r)} > 0 \ . \qquad \text{Q.E.D.}$$

DEFINITION 3.3.1. The reachable set for infinite travel time $RO(\infty,\hat{t},\Delta T)$ is called the region of inescapability, which simply is

$$RO(\infty,\hat{t},\Delta T) = \underset{\tau\to\infty}{} RO(\tau,\hat{t},\Delta T) \ .$$

THEOREM 3.3.2. Given a system with positive, real, distinct eigenvalues, and finite $\hat{t}$ and ΔT. Suppose $R(\infty)$ denotes the region of inescapability for the case with no control outage and let $RO(\infty,\hat{t},\Delta T)$ be the region of inescapability for the case with control outage. Then

$$R(\infty) = RO(\infty,\hat{t},\Delta T) \ .$$

Proof: The equation for the boundary of the τ-reachable set with no control outage is:

$$\partial R(\tau) = \{\underline{x}\colon \underline{x} = \pm [I + e^{P\tau} - 2e^{P(\tau - t_1)}]P^{-1}\underline{q} \quad \text{for} \quad 0 \le t_1 \le \tau\} \ . \tag{3.3.5}$$

Let $r = \tau - t_1$, and considering $\lim_{\tau\to\infty} e^{P\tau} = 0$, it follows

$$\partial R(\infty) = \{\underline{x}\colon \underline{x} = \pm [I - 2e^{Pr}]P^{-1}\underline{q} \quad \text{for} \quad 0 \le r \le \infty\} \ . \tag{3.3.6}$$

The equation for the boundary of the region of the inescapability with control outage is

$$\lim_{\tau\to\infty} \partial RO(\tau,\hat{t},\Delta T) = \{\underline{x}\colon \underline{x} = \pm [I + e^{P\tau} - 2e^{Pr} \pm e^{P(\tau-\hat{t}-\Delta T)} \mp e^{P(\tau-\hat{t})}]P^{-1}\underline{q} \quad \text{for} \quad 0 \le r \le \tau\} \ . \tag{3.3.7}$$

For finite $\hat{t}$ and ΔT

$$\partial RO(\infty,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [I - 2e^{Pr}]P^{-1}\underline{q} \quad \text{for} \quad 0 \le r \le \infty\}. \qquad (3.3.8)$$

Comparing equations (3.3.6) and (3.3.8) proves the theorem.

If the normalized equation for the reverse time system is given by equation (3.1.2), then the equations for the boundary of the τ-reachable set with control outage are

$$\partial RO_1(\tau,\hat{t},\Delta T) = \left\{ \underline{x}: \underline{x} = \pm \begin{bmatrix} 1-2e^{-(\tau-t_1)}+e^{-\tau}+e^{-\tau+\hat{t}+\Delta T}-e^{-\tau+\hat{t}} \\ \\ 1-2e^{-r(\tau-t_1)}+e^{-r\tau}+e^{-r(\tau-\hat{t}-\Delta T)}-e^{-r(\tau-\hat{t})} \end{bmatrix} \right\}$$

$$\text{for} \quad \hat{t} \le \hat{t} + \Delta T \le t_1 \quad , \qquad (3.3.9)$$

$$\partial RO_2(\tau,\hat{t},\Delta T) = \left\{ \underline{x} : \underline{x} = \pm \begin{bmatrix} 1-2e^{-(\tau-t_1)}+e^{-\tau}-e^{-\tau+\hat{t}+\Delta T}+e^{-\tau+\hat{t}} \\ \\ 1-2e^{-r(\tau-t_1)}+e^{-r\tau}-e^{-r(\tau-\hat{t}-\Delta T)}+e^{-r(\tau-\hat{t})} \end{bmatrix} \right\}$$

$$\text{for} \quad 0 \le t_1 \le \hat{t} \quad , \qquad (3.3.10)$$

where $\frac{\lambda_2}{\lambda_1} = r > 1$. For $t_1 \in [\hat{t},\hat{t} + \Delta T]$, equation (3.3.9) with $t_1 = \hat{t} + \Delta T$ or equation (3.3.10) with $t_1 = \hat{t}$, can be used. Thus

$$\partial RO(\tau,\hat{t},\Delta T) = \partial RO_1(\tau,\hat{t},\Delta T) \cup \partial RO_2(\tau,\hat{t},\Delta T) .$$

Digital computer solutions for the $\partial RO(\tau,\hat{t},\Delta T)$ are shown in the following figures for different values of $\hat{t}$, ΔT, τ, and $r = \frac{\lambda_2}{\lambda_1}$.

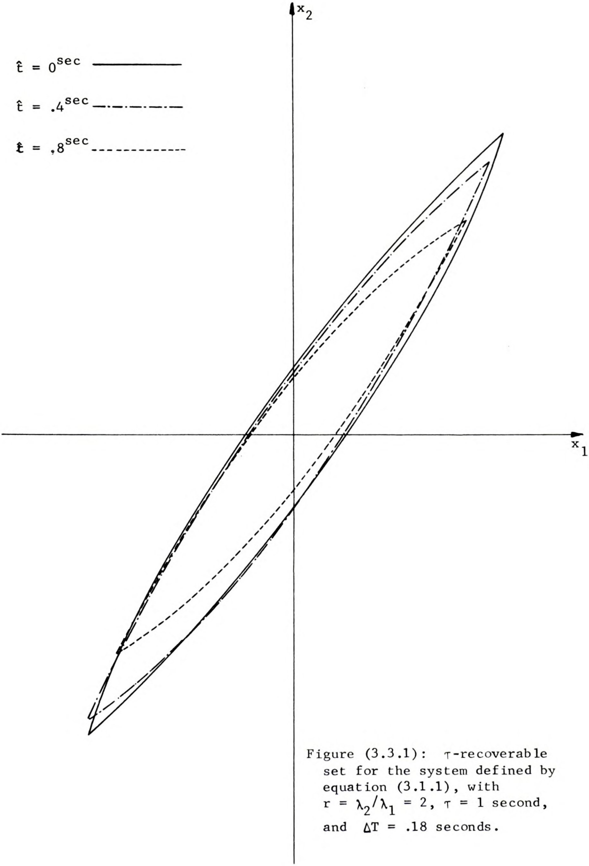


Figure (3.3.1): τ-recoverable set for the system defined by equation (3.1.1), with $r = \lambda_2/\lambda_1 = 2$, $\tau = 1$ second, and $\Delta T = .18$ seconds.

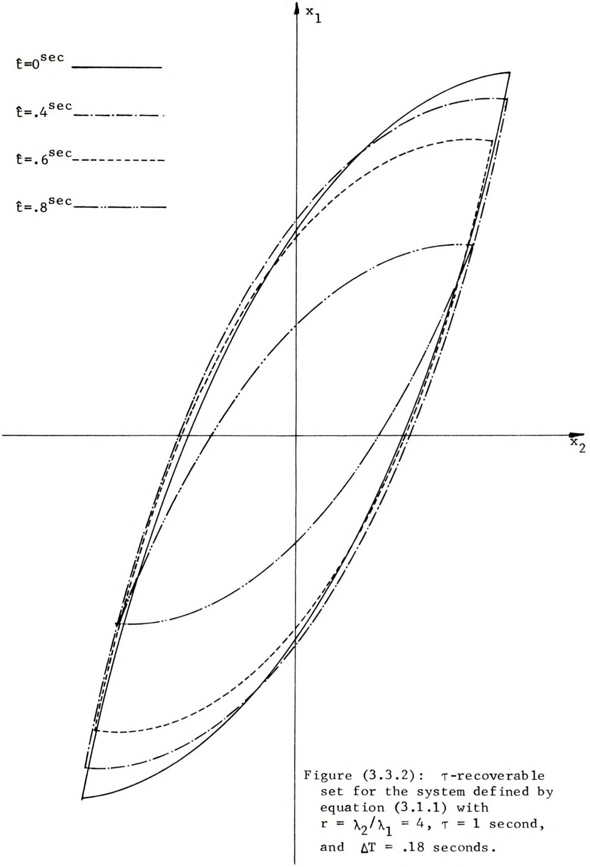


Figure (3.3.2): τ-recoverable set for the system defined by equation (3.1.1) with $r = \lambda_2/\lambda_1 = 4$, $\tau = 1$ second, and $\Delta T = .18$ seconds.

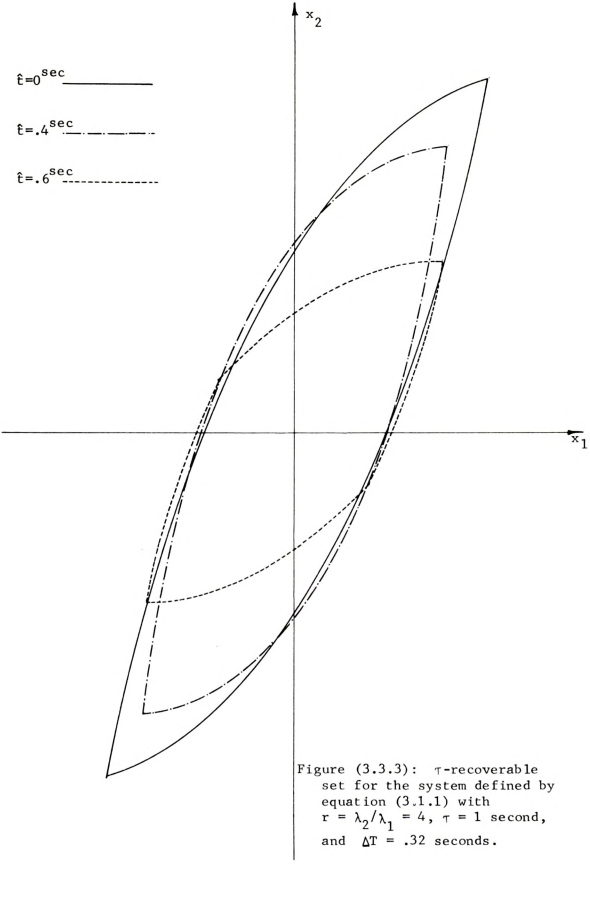


Figure (3.3.3): τ-recoverable set for the system defined by equation (3.1.1) with $r = \lambda_2/\lambda_1 = 4$, $\tau = 1$ second, and $\Delta T = .32$ seconds.

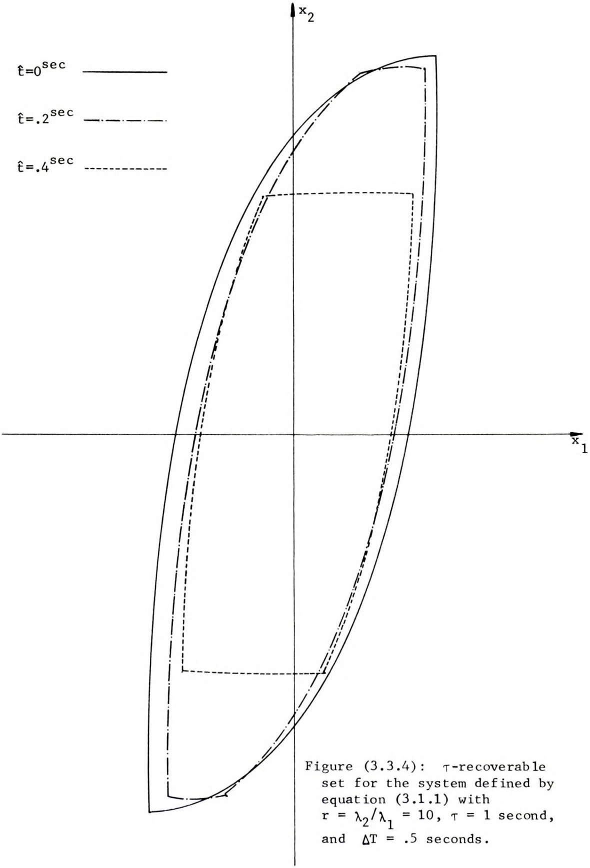


Figure (3.3.4): τ-recoverable set for the system defined by equation (3.1.1) with $r = \lambda_2/\lambda_1 = 10$, $\tau = 1$ second, and $\Delta T = .5$ seconds.

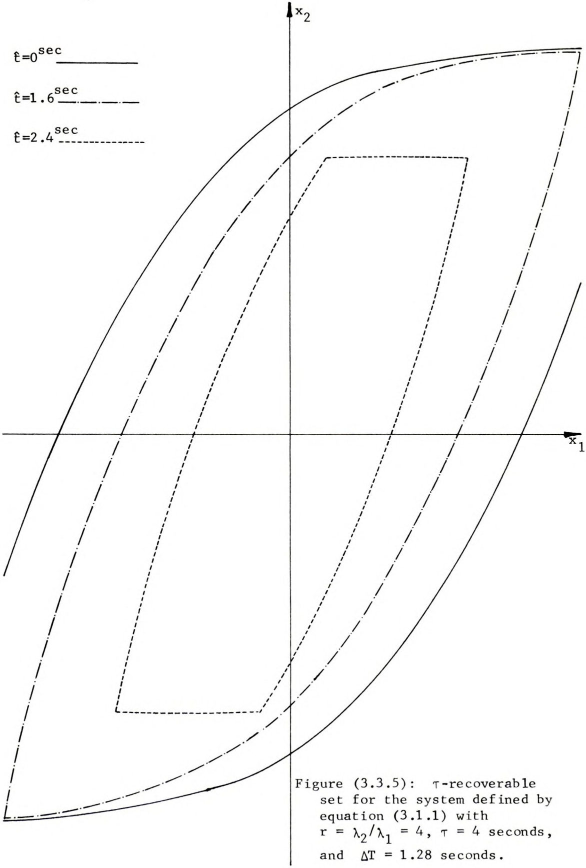


Figure (3.3.5): τ-recoverable set for the system defined by equation (3.1.1) with $r = \lambda_2/\lambda_1 = 4$, $\tau = 4$ seconds, and $\Delta T = 1.28$ seconds.

3.4 Variation of the Area of the τ-Reachable Set with Respect to ΔT and $\hat{t}$

For the system given by equation (3.1.2), the boundary of the τ-reachable set is given by equations (3.3.9) and (3.3.10):

$$\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \underline{x}^1(\tau,t_1) = \pm \begin{bmatrix} 1-2e^{-(\tau-t_1)}+e^{-\tau}-e^{-\tau+\hat{t}+\Delta T}+e^{-\tau+\hat{t}} \\ 1-2e^{-r(\tau-t_1)}+e^{-r\tau}-e^{-r(\tau-\hat{t}-\Delta T)}+e^{-r(\tau-\hat{t})} \end{bmatrix}$$

$$\text{for} \quad t_1 \in [0,\hat{t}] \qquad (3.4.1)$$

and

$$\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \underline{x}^2(\tau,t_1) = \pm \begin{bmatrix} 1-2e^{-(\tau-t_1)}+e^{-\tau}+e^{-\tau+\hat{t}+\Delta T}-e^{-\tau+\hat{t}} \\ 1-2e^{-r(\tau-t_1)}+e^{-r\tau}+e^{-r(\tau-\hat{t}-\Delta T)}-e^{-r(\tau-\hat{t})} \end{bmatrix}$$

$$\text{for} \quad t_1 \in [\hat{t}+\Delta T,\tau] \qquad (3.4.2)$$

where $r = \dfrac{\lambda_2}{\lambda_1} > 1.$

Considering the symmetry of the τ-reachable set with respect to the origin, the area of the τ-reachable set is

$$A = -2\left[\int_0^{\hat{t}} x_2^1 d(x_1^1) + \int_{\hat{t}+\Delta T}^{\tau} x_2^2 d(x_1^2)\right] . \qquad (3.4.3)$$

Substituting equations (3.4.1) and (3.4.2) into equation (3.4.3) yields

$$A = 4Q[e^{-(\tau-\hat{t})} - e^{-\tau}] + 4S[1 - e^{-(\tau-\hat{t}-\Delta T)}] - \frac{8}{1+r} P \qquad (3.4.4)$$

where

$$Q = [1 + e^{-r\tau} - e^{-r(\tau-\hat{t}-\Delta T)} + e^{-r(\tau-\hat{t})}]$$

$$S = [1 + e^{-r\tau} + e^{-r(\tau-\hat{t}-\Delta T)} - e^{-r(\tau-\hat{t})}]$$

$$P = [1 + e^{-(1+r)(\tau-\hat{t})} - e^{-(1+r)\tau} - e^{-(1+r)(\tau-\hat{t}-\Delta T)}] \,.$$

Suppose τ and $\hat{t}$ are fixed. Then

$$\frac{\partial A}{\partial(\Delta T)} = 4\{re^{-\tau}e^{-r(\tau-\hat{t}-\Delta T)}[e^{\tau} - e^{\hat{t}+\Delta T} - e^{\hat{t}} + 1]$$

$$- e^{-\tau+\hat{t}+\Delta T}[1 - e^{-r(\tau-\hat{t}-\Delta T)} + e^{-r\tau} - e^{-r(\tau-\hat{t})}]\} \qquad (3.4.5)$$

or

$$\frac{\partial A}{\partial(\Delta T)} = 4e^{-\tau}\cdot e^{-r(\tau-\hat{t}-\Delta T)}\cdot e^{\hat{t}+\Delta T}\{r[e^{\tau-\hat{t}-\Delta T} - 1 - e^{-\Delta T} + e^{-\hat{t}-\Delta T}]$$

$$- [e^{r(\tau-\hat{t}-\Delta T)} - 1 + e^{-r(\hat{t}+\Delta T)} - e^{-r\Delta T}]\} \,. \qquad (3.4.6)$$

Let

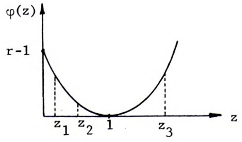
$$z_1 = e^{-\hat{t}-\Delta T} \le z_2 = e^{-\Delta T} \le 1 \le z_3 = e^{\tau-\hat{t}-\Delta T}$$

$$\varphi(z) = z^r - rz + r-1 \qquad (3.4.7)$$

then

$$\frac{\partial A}{\partial(\Delta T)} = 4z_3^{-(1+r)}[-\varphi(z_1) + \varphi(z_2) + \varphi(1) - \varphi(z_3)] \qquad (3.4.8)$$

Since $\dot{\varphi}(z) = r(z^{r-1}-1)$, it follows that for $r > 1$, $\varphi(z)$ decreases if $0 < z < 1$ and increases if $1 < z$, (see the following figure).



Hence

$$\varphi(z_1) > \varphi(z_2) \quad \text{and} \quad \varphi(1) < \varphi(z_3)$$

Thus

$\frac{\partial A}{\partial(\Delta T)} < 0$, for all values of $\hat{t}$, ΔT, and τ, satisfying

$$0 \le \hat{t} \le \hat{t} + \Delta T \le \tau \tag{3.4.9}$$

THEOREM 3.4.1. The area of the τ-reachable set with control outage is monotonically decreasing with respect to the duration of the outage (ΔT) if $\hat{t}$ and τ are assumed to be constants.

To observe the variation of the area of the τ-reachable region with respect to the control outage starting time $\hat{t}$, it is assumed that τ and ΔT are constants. Applying the following formulas:

$$F(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

and

$$F'(x) = \beta'(x) f(x,\beta(x)) - \alpha'(x) f(x,\alpha(x)) + \int_{\alpha(x)}^{\beta(x)} f_1(x,y)\,dy$$

to equation (3.4.3) yields

$$\frac{\partial A}{\partial \hat{t}} = 4e^{-\tau} \cdot e^{\hat{t}+\Delta T} \cdot e^{-r(\tau-\hat{t})}\{r(e^{-\Delta T} - e^{(r-1)\Delta T})(1-e^{-\hat{t}}$$
$$-e^{\tau-\hat{t}} + e^{\Delta T}) + (e^{-\Delta T} - 1)(e^{r(\tau-\hat{t})} - 1 - e^{r\Delta T} + e^{-r\hat{t}})\}. \tag{3.4.10}$$

For $\Delta T \approx 0$, it follows that $\frac{\partial A}{\partial \hat{t}} \approx 0$. This implies that the area of the τ-reachable set will not change significantly, no matter where the control outage occurs during the process if the duration of the control outage ΔT is small with respect to τ.

Depending upon $\hat{t}$, ΔT, τ, and r, the area of the τ-reachable set could be increasing to reach a maximum for some $0 \le \hat{t} \le \tau$. Finding such an extremum analytically is cumbersome. In fact Figure (3.4.1), shows that the area of the τ-reachable set is not always monotonically decreasing, with respect to $\hat{t}$. This contrasts with Theorem (3.4.1), which showed that the area of the τ-reachable set is always monotonically decreasing with respect to ΔT.

3.5 Study of the Minimum Time for Regulation of the System with Control Outage

Consider a system defined by equation (3.1.2), and an initial state $\underline{x(0)} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, at time $t_o = 0$. Assume t^* is the minimum time which it takes the system to travel from $\underline{x(0)}$ to the origin with no control outage. As is well known the switching curves with no control outage divide the state space into two regions, i.e., Q^+ and Q^-, where $u = +1$ is used when $\underline{x} \in Q^+$ and $u = -1$ is used when $\underline{x} \in Q^-$ (see Figure (3.5.1)). Suppose $\underline{x(0)} \in Q^-$, and let t_1 be the switching time. Thus

$$\underline{x(t_1)} = e^{Pt_1}\,\underline{x(0)} - [e^{Pt_1} - I]P^{-1}\underline{q}$$

and

$$\underline{0} = e^{P(t^*-t_1)}\,\underline{x(t_1)} + [e^{P(t^*-t_1)} - I]P^{-1}\underline{q}\ .$$

Thus

$$\underline{0} = e^{Pt^*}\,\underline{x(0)} + [2e^{P(t^*-t_1)} - e^{Pt^*} - I]P^{-1}\underline{q}\ . \qquad (3.5.1)$$

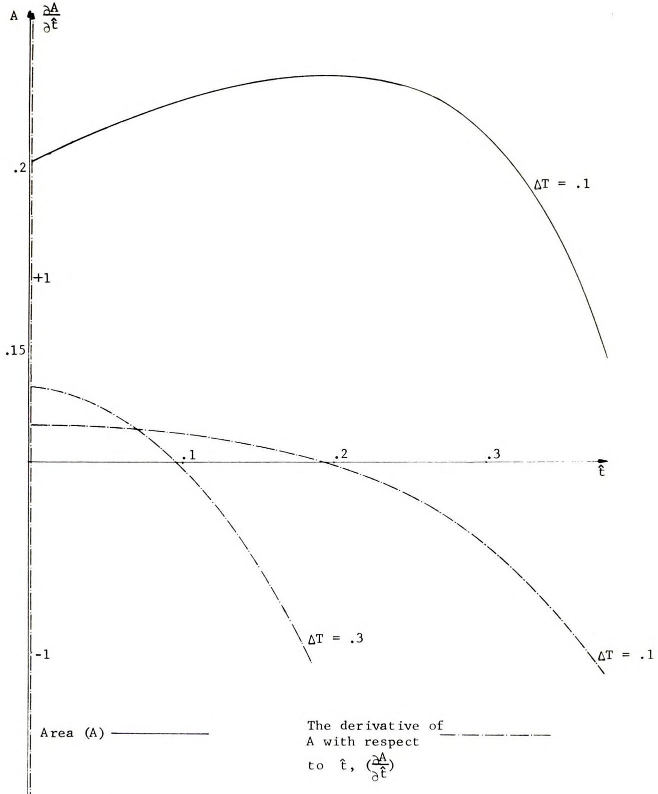


Figure (3.4.1): Variation of the area of the τ-recoverable set with respect to $\hat{t}$, when $r = \lambda_2/\lambda_1 = 4$, $\tau = .5$ seconds, for the system defined by equation (3.1.1).

For P and $\underline{q}$ given by equation (3.1.3), and $\frac{\lambda_2}{\lambda_1} = 2$, there will result

$$t^* = \text{Log}_e[x_1^2(0) + 2x_2(0) - 2x_1(0) - 1] - \text{Log}_e[x_1(0) - 1 + \sqrt{2x_1^2(0) - 4x_1(0) + 2x_2(0)}\ ] . \qquad (3.5.2)$$

If $\underline{x(0)} \in Q^+$, the same calculation yields

$$\underline{0} = e^{Pt^*}\underline{x(0)} + [e^{Pt^*} - 2e^{P(t^*-t_1)} + I]P^{-1}q . \qquad (3.5.3)$$

In the special case when $r = 2$

$$t^* = \text{Log}_e[x_1^2(0) + 2x_1(0) - 2x_2(0) - 1] - \text{Log}_e[-x_1(0) - 1 + \sqrt{2x_1^2(0) + 4x_1(0) - 2x_2(0)}\ ] . \qquad (3.5.4)$$

To calculate the minimum time when there exists a control outage, the following three cases are considered (three symmetrical cases occur for $\underline{x(0)} \in Q^+$).

Case a:

$$\underline{x(0)} \in \bar{Q}\ , \ 0 \le \hat{t} \le t_1, \text{ and } \underline{x(\hat{t} + \Delta T)} \in \bar{Q}$$

where t_1 is the switching time if there is no control outage. Thus

$$\underline{x(\hat{t})} = e^{P\hat{t}}\,\underline{x(0)} - [e^{P\hat{t}} - I]P^{-1}\underline{q} \qquad (3.5.5)$$

$$\underline{x(\hat{t} + \Delta T)} = e^{P\Delta T}\,\underline{x(\hat{t})} . \qquad (3.5.6)$$

Substitution of equation (3.5.5) into equation (3.5.6) gives

$$\underline{x(\hat{t} + \Delta T)} = e^{P(\hat{t}+\Delta T)} \underline{x(0)} - [e^{P(\hat{t}+\Delta T)} - e^{P\Delta T}]P^{-1}\underline{q} . \quad (3.5.7)$$

The minimum time for the system to move from the new initial state, i.e. $\underline{x(\hat{t} + \Delta T)}$, to the origin must satisfy equation (3.5.2). Let us denote this minimum time by t_f'. Thus t_f' satisfies the following equation

$$\underline{0} = e^{Pt_f'} \underline{x(\hat{t} + \Delta T)} + [2e^{P(t_f'-t_1')} - e^{Pt_f'} - I]P^{-1}\underline{q} \quad (3.5.8)$$

where t_1' is the new switching time. It is obvious that the total minimum time with control outage to^* is:

$$to^* = t_f' + \hat{t} + \Delta T . \quad (3.5.9)$$

Given $\underline{x(0)}$, $\hat{t}$, and ΔT, one can calculate $\underline{x(\hat{t} + \Delta T)}$ from equation (3.5.7). With a calculated value of $\underline{x(\hat{t} + \Delta T)}$ and equation (3.5.8), t_f' can be found. Finally equation (3.5.9) gives to^*, which is the minimum time that it takes the system to travel from $\underline{x(0)}$ to $\underline{x(to^*)} = \underline{0}$.

In special case when $r = \dfrac{\lambda_2}{\lambda_1} = 2$,

$$\underline{x(\hat{t} + \Delta T)} = \begin{bmatrix} e^{-(\hat{t}+\Delta T)}(x_1(0) - 1) + e^{-\Delta T} \\ e^{-2(\hat{t}+\Delta T)}(x_2(0) - 1) + e^{-2\Delta T} \end{bmatrix}$$

and

$$t_f' = \text{Log}_e[x_1^2(\hat{t} + \Delta T) + 2x_2(\hat{t} + \Delta T) - 2x_1(\hat{t} + \Delta T) - 1]$$
$$- \text{Log}_e[x_1(\hat{t} + \Delta T) - 1 + \sqrt{2x_1^2(\hat{t} + \Delta T) - 4x_1(\hat{t} + \Delta T) + 2x_2(\hat{t} + \Delta T)}\ ]. \quad (3.5.10)$$

The computer solution shows that, depending upon the initial state and ΔT, the variation of the t_o^* with respect to $\hat{t}$ is either monotonically decreasing, or reaches a maximum for some $\hat{t} \in [0,t_1]$. These results are depicted in Figure (3.5.2)

Case b:

Let $\underline{x(0)} \in \bar{Q}$, $0 \le \hat{t} \le t_1$, and $\underline{x(\hat{t} + \Delta T)} \in Q^+$. The state of the system at $\hat{t} + \Delta T$, $\underline{x(\hat{t} + \Delta T)}$, can be calculated from equation (3.5.7). Motion from $\underline{x(\hat{t} + \Delta T)}$ to the origin is described by equation (3.5.3). Let the time which it takes the system to travel from $\underline{x(\hat{t} + \Delta T)}$ to the origin be t_f'. Then

$$\underline{0} = e^{Pt_f'}\,\underline{x(\hat{t} + \Delta T)} + [e^{Pt_f'} - 2e^{P(t_f'-t_1')} + I]P^{-1}\underline{q} \;. \qquad (3.5.11)$$

Equation (3.5.9) gives the minimum time t_o^*. In the special case when $r = \frac{\lambda_2}{\lambda_1} = 2$, then

$$t_f' = \mathrm{Log}_e[x_1^2(\hat{t} + \Delta T) + 2x_1(\hat{t} + \Delta T) - 2x_2(\hat{t} + \Delta T) - 1]$$
$$- \log_e[-x_1(\hat{t} + \Delta T) - 1 + \sqrt{2x_1^2(\hat{t}+\Delta T) + 4x_1(\hat{t}+\Delta T) - 2x_2(\hat{t}+\Delta T)}\,]. \qquad (3.5.12)$$

The computer solution indicates that the variation of t_o^* with respect to $\hat{t}$ is monotonically increasing for $\hat{t} \in [0,t_1]$, and $\underline{x(\hat{t} + \Delta T)} \in Q^+$ (see Figure (3.5.2)).

Case c:

If $t_1 \le \hat{t} \le t^*$ and $\underline{x(\hat{t} + \Delta T)} \in Q^+$, it follows that

$$\underline{x(t_1)} = e^{Pt_1}\,\underline{x(0)} - [e^{Pt_1} - I]P^{-1}\underline{q} \;. \qquad (3.5.13)$$

Now $\underline{x(t_1)} \in \gamma^+$, where γ^+ satisfies the following equation

$$\gamma^+ = \{\underline{x}: x(\tau) = [e^{-P\tau} - I]P^{-1}\underline{q} \text{ for } 0 \le \tau \le T, \text{ where } \tau = t^* - t\}. \quad (3.5.14)$$

Therefore t_1 can be calculated by substituting equation (3.5.13) into equation (3.5.14). Thus

$$\underline{x(\hat{t})} = e^{P(\hat{t}-t_1)} \underline{x(t_1)} + [e^{P(\hat{t}-t_1)} - I]P^{-1}\underline{q} \quad (3.5.15)$$

and

$$\underline{x(\hat{t} + \Delta T)} = e^{P\Delta T} \underline{x(\hat{t})} = e^{P(\hat{t}-t_1+\Delta T)} \underline{x(t_1)} + [e^{P(\hat{t}-t_1+\Delta T)} - e^{P\Delta T}]P^{-1}\underline{q} . \quad (3.5.16)$$

If $\underline{x(\hat{t} + \Delta T)} \in Q^+$, then it satisfies equation (3.5.3), such that

$$\underline{0} = e^{Pt_f'} \underline{x(\hat{t} + \Delta T)} + [e^{Pt_f'} - 2e^{P(t_f'-t_1)} + I]P^{-1}\underline{q} \quad (3.5.17)$$

where t_f' is the minimum time which takes the system to travel from $\underline{x(\hat{t} + \Delta T)}$ to the origin. Here t_1' is the new switching time. Equation (3.5.9) gives the total minimum time t_o^* for the system to go from $\underline{x(0)}$ to the origin. In the special case, when $r = \frac{\lambda_2}{\lambda_1} = 2$

$$t_f' = \mathrm{Log}_e[x_1^2(\hat{t} + \Delta T) + 2x_1(\hat{t} + \Delta T) - 2x_2(\hat{t} + \Delta T) - 1]$$
$$- \mathrm{Log}_e[-x_1(\hat{t} + \Delta T) - 1 + \sqrt{2x_1^2(\hat{t}+\Delta T) + 4x_1(\hat{t}+\Delta T) - 2x_2(\hat{t}+\Delta T)}\,]. \quad (3.5.18)$$

The computer solution indicates that in this case t_o^* is monotonically decreasing for $\hat{t} \in [t_1,T]$ (see Figure (3.5.2)).

Consider now the analysis of these cases.

Suppose the initial state $\underline{x(0)}$ and duration of control outage ΔT are given. For different values of $\hat{t} \in [0,T]$, the minimum times for the system to travel from $\underline{x(0)}$ to the origin are different. Let us denote the maximum value of these different minimum times by $\text{Max}(to^*)$. This maximum value of to^* occurs when $\hat{t}$, the control outage starting time, is $\hat{t} = t_1 + \epsilon$. Here t_1 is the switching if there is no control outage and $\epsilon > 0$ is very small.

For given $\underline{x(0)}$, and ΔT, the maximum value of the minimum times, $\text{Max}(to^*)$, can be calculated. Equations (3.5.13) and (3.5.14) give t_1. Thus

$$\underline{x(t_1)} \simeq \underline{x(\hat{t})} \simeq e^{Pt_1}\,\underline{x(0)} - [e^{Pt_1} - I]P^{-1}\underline{q}$$

and

$$\underline{x(\hat{t} + \Delta T)} = e^{P(t_1+\Delta T)}\,\underline{x(0)} - [e^{P(t_1+\Delta T)} - e^{P\Delta T}]P^{-1}\underline{q}\ . \tag{3.5.19}$$

From equation (3.5.19) $\underline{x(\hat{t} + \Delta T)}$ is calculated. Equation (3.5.17) gives t_f', if $\underline{x(\hat{t} + \Delta T)}$ is considered to be a new initial point. Finally

$$\text{Max}(to^*) = t_f' + \hat{t} + \Delta T \tag{3.5.20}$$

where $\text{Max}(to^*)$ is the maximum value of minimum times, for different $\hat{t}$, as $\hat{t}$ changes from 0 to t^*. Thus

Given: $\underline{x(0)}$ the initial condition, ΔT the duration of control outage, and $t_{cs} > 0$, called the recoverability constraint. The system is said to be recoverable if $\text{Max}(to^*) \le t_{cs}$ for almost all $\hat{t} \in [0,T]$. Equation (3.5.20) gives a means of testing whether or not the system is recoverable.

3.6 Calculation of the Boundary of the τ-Reachable Set for Systems of Order n with Distinct Positive Real Eigenvalues and Scalar Input

Given any system with positive real and distinct eigenvalues, there exists a real similarity transformation that reduces the system equation to the following form (Theorem 2.5.7):

$$\dot{\underline{x}}(t) = P\,\underline{x}(t) + \underline{q}\,u(t) \tag{3.6.1}$$

where

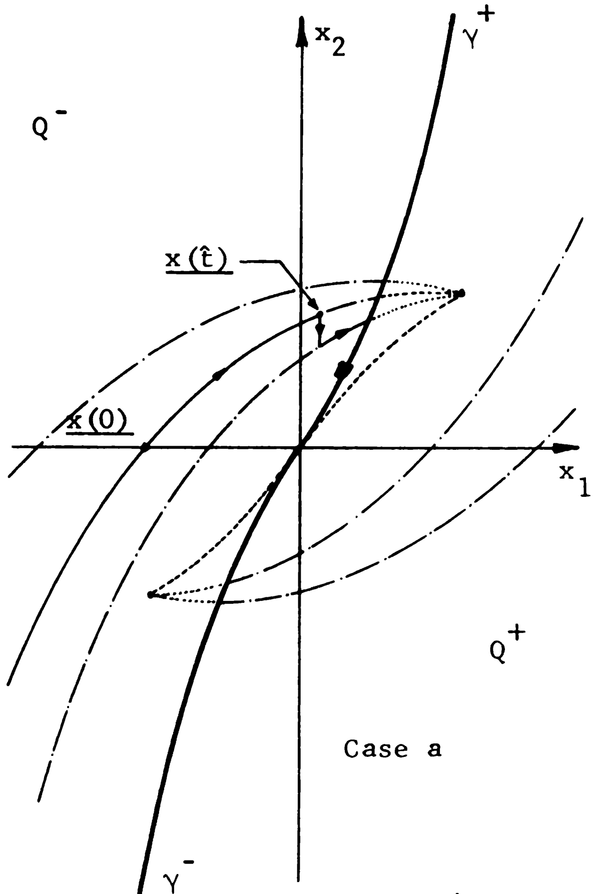
$$P = \begin{bmatrix} 1 & & & 0 \\ & \lambda_2/\lambda_1 & & \\ & 0 & \ddots & \\ & & & \lambda_n/\lambda_1 \end{bmatrix} \tag{3.6.2}$$

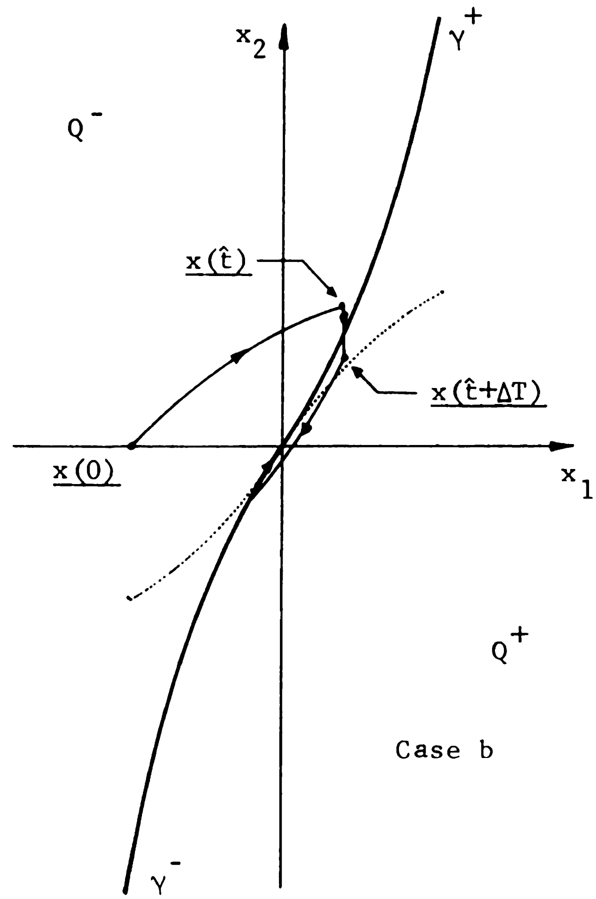
and

$$\underline{q} = \begin{bmatrix} 1 \\ \lambda_2/\lambda_1 \\ \vdots \\ \lambda_n/\lambda_1 \end{bmatrix} . \tag{3.6.3}$$

First third order systems are considered. For these systems to reach any point in the state space, at most two switching times, namely t_1 and t_2, are required, where

$$0 \le t_1 \le t_2 \le \tau . \tag{3.6.4}$$





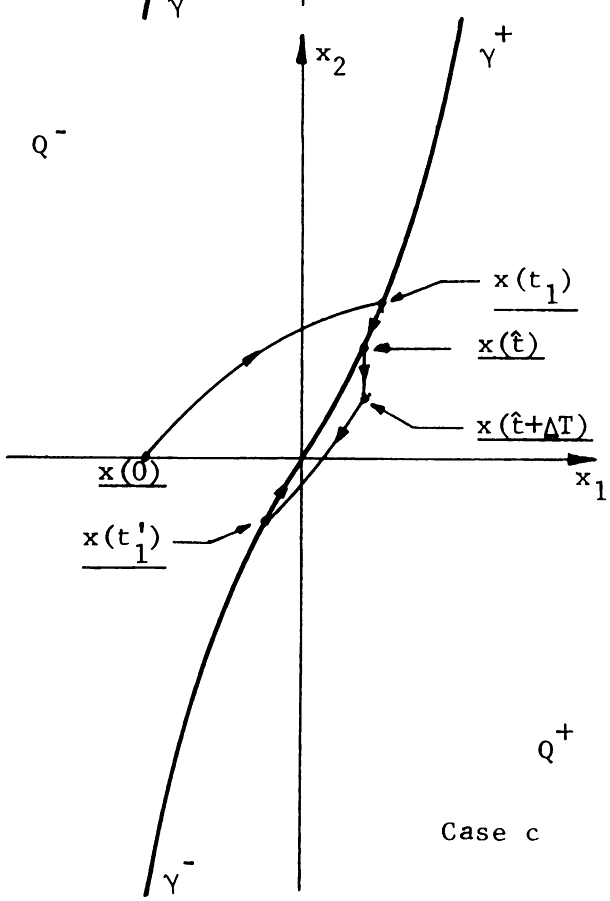


Figure (3.5.1): Switching curves and optimal-time trajectories for the system given by equation (3.1.2), and initial state $\underline{x(0)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

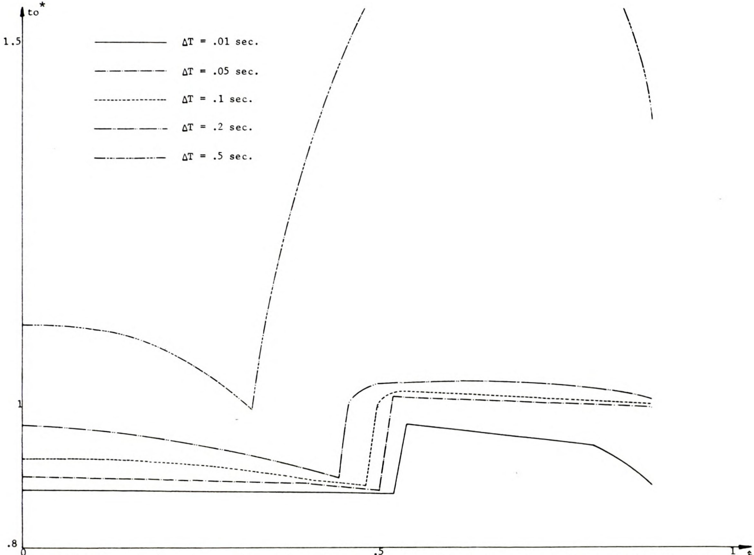


Figure (3.5.2): Variation of the minimum time with respect to $\hat{t}$ for the system defined by equation (3.1.2), with $\underline{x(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $r = 2$.

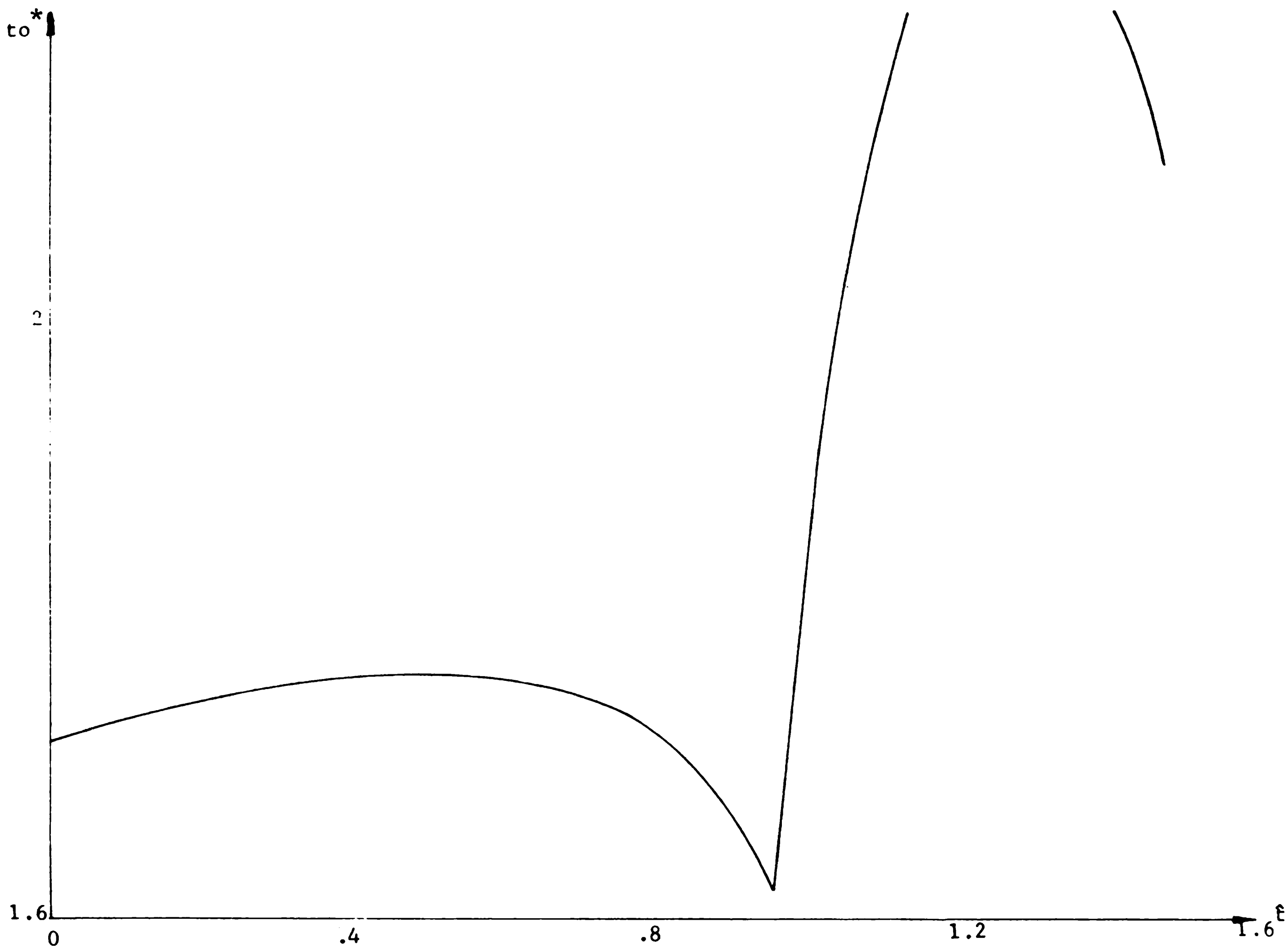


Figure (3.5.3): Variation of t_o^* with respect to $\hat{t}$ for the system defined by equation (3.1.2), when $\underline{x(0)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $r = 2$.

To calculate the boundary of the τ-reachable set with control outage $\partial RO(\tau,\hat{t},\Delta T)$, the sign of the control component should change at t_1 and t_2. For fixed values of $\hat{t}$ and ΔT, as t_1 and t_2 are ranging from 0 to τ, the six possible cases and control sequences are as follows (considering $u = +1$ as the initial value of the control component):

a)

$$0 \leq \hat{t} \leq \hat{t} + \Delta T \leq t_1 \leq t_2 \leq \tau$$

and $u(t)$ is $+1, 0, +1, -1, +1$.

b)

$$0 \leq \hat{t} \leq t_1 \leq \hat{t} + \Delta T \leq t_2 \leq \tau$$

$+1, 0, -1, +1$.

c)

$$0 \leq t_1 \leq \hat{t} \leq \hat{t} + \Delta T \leq t_2 \leq \tau$$

$+1, -1, 0, -1, +1$.

d)

$$0 \leq t_1 \leq \hat{t} \leq t_2 \leq \hat{t} + \Delta T \leq \tau$$

$+1, -1, 0, +1$.

e)

$$0 \leq t_1 \leq t_2 \leq \hat{t} \leq \hat{t} + \Delta T \leq \tau$$

$+1, -1, +1, 0, +1$.

f)

$$0 \leq \hat{t} \leq t_1 \leq t_2 \leq \hat{t} + \Delta T \leq \tau$$

$+1, 0, +1$.

For case a):

$$\underline{x(\hat{t})} = \int_0^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}d\sigma = [e^{P\hat{t}} - I]P^{-1}\underline{q},\ u(t) = +1 \quad \text{for} \quad t \in [0,\hat{t})$$

$$\underline{x(\hat{t} + \Delta T)} = e^{P\Delta T}\ \underline{x(\hat{t})} = [e^{P(\hat{t}+\Delta T)} - e^{P\Delta T}]P^{-1}\underline{q},\ u(t) \equiv 0$$

$$\text{for} \quad t \in [\hat{t},\hat{t} + \Delta T)$$

$$\underline{x(t_1)} = e^{P(t_1-\hat{t}-\Delta T)}[e^{P(\hat{t}+\Delta T)} - e^{P\Delta T}]P^{-1}\underline{q} + [e^{P(t_1-\hat{t}-\Delta T)} - I]P^{-1}\underline{q}$$

or

$$\underline{x(t_1)} = [e^{Pt_1} - e^{P(t_1-\hat{t})} + e^{P(t_1-\hat{t}-\Delta T)} - I]P^{-1}\underline{q},\ u(t) = +1$$

$$\text{for} \quad t \in [\hat{t} + \Delta T,t_1)$$

and considering $u(t) = -1$ for $t \in [t_1,t_2)$ yields

$$\underline{x(t_2)} = [e^{Pt_2} - e^{P(t_2-\hat{t})} + e^{P(t_2-\hat{t}-\Delta T)} - 2e^{P(t_2-t_1)} + I]P^{-1}\underline{q}$$

and

$$\underline{x(\tau)} = [e^{P\tau} - e^{P(\tau-\hat{t})} + e^{P(\tau-\hat{t}-\Delta T)} - 2e^{P(\tau-t_1)} + 2e^{P(\tau-t_2)} - I]P^{-1}\underline{q} .$$

Thus

$$\partial RO_1(\tau,\hat{t},\Delta T) = \{\underline{x}\colon \underline{x} = \pm [e^{P\tau} - e^{P(\tau-\hat{t})} + e^{P(\tau-\hat{t}-\Delta T)} - 2e^{P(\tau-t_1)} + 2e^{P(\tau-t_2)} - I]P^{-1}\underline{q}, \text{ for } 0 \le \hat{t} \le \hat{t} + \Delta T \le t_1 \le t_2 \le \tau\} . \quad (3.6.5)$$

For case b):

$$\underline{x(\tau)} = [e^{P\tau} - e^{P(\tau-\hat{t})} + 2e^{P(\tau-t_2)} - e^{P(\tau-\hat{t}-\Delta T)} - I]P^{-1}\underline{q}$$

$$\partial RO_2(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} - e^{P(\tau-\hat{t})} - e^{P(\tau-\hat{t}-\Delta T)} + 2e^{P(\tau-t_2)} - I]P^{-1}\underline{q} \text{ for } 0 \le \hat{t} \le t_1 \le \hat{t} + \Delta T \le t_2 \le \tau\} . \quad (3.6.6)$$

For case c):

$$\partial RO_3(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} + e^{P(\tau-\hat{t})} - e^{P(\tau-\hat{t}-\Delta T)} - 2e^{P(\tau-t_1)} + 2e^{P(\tau-t_2)} - I]P^{-1}\underline{q}, \ 0 \le t_1 \le \hat{t} \le \hat{t} + \Delta T \le t_2 \le \tau\} . \quad (3.6.7)$$

For case d):

$$\partial RO_4(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} + e^{P(\tau-\hat{t})} + e^{P(\tau-\hat{t}-\Delta T)} - 2e^{P(\tau-t_1)} - I]P^{-1}\underline{q} \text{ for } 0 \le t_1 \le \hat{t} \le t_2 \le \hat{t} + \Delta T \le \tau\} . \quad (3.6.8)$$

For case e):

$$\partial RO_5(\tau,\hat{t},\Delta T) = \partial RO_1(\tau,\hat{t},\Delta T) . \quad (3.6.9)$$

For case f

$$\partial RO_6(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} - e^{P(\tau-\hat{t})} + e^{P(\tau-\hat{t}-\Delta T)} - I]P^{-1}\underline{q} \text{ for } 0 \le \hat{t} \le t_1 \le t_2 \le \hat{t} + \Delta T \le \tau\} . \quad (3.6.10)$$

Equation (3.6.10) is a special case of equation (3.6.6) if $t_2 = \hat{t} + \Delta T$. Thus the boundary of the τ-reachable set for the third order system is

$$\partial RO(\tau,\hat{t},\Delta T) = \bigcup_{j=1}^{5} \partial RO_j(\tau,\hat{t},\Delta T) .$$

For the system of order n to reach any state in the state space;at most $(n-1)$ switching times, i.e., $t_1, t_2, \ldots, t_{n-1}$

are required, where

$$0 = t_0 \le t_1 \le \ldots \le t_j \le \ldots \le t_{n-1} \le t_n = \tau .$$

To calculate the boundary of the τ-reachable set with control outage $\partial RO(\tau,\hat{t},\Delta T)$, the sign of the control component should change at the switching times. Let $t_j \le \hat{t} \le \hat{t} + \Delta T \le t_{j+1}$, then

$$\partial RO_1^j(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} + (-1)^n I + (-1)^{j+1} e^{P(\tau-\hat{t})} + (-1)^j e^{P(\tau,\hat{t},\Delta T)} + 2 \sum_{i=1}^{n-1} (-1)^i e^{P(\tau-t_i)}] P^{-1} \underline{q} \text{ when}$$

$$t_j \le \hat{t} \le \hat{t} + \Delta T \le t_{j+1} \quad \text{for} \quad j = 0,1,\ldots,n-1\}. \tag{3.6.11}$$

If

$$\hat{t} \le t_j \le \hat{t} + \Delta T \le t_{j+1}$$

then

$$\partial RO_2^j(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm [e^{P\tau} + (-1)^n I + (-1)^j e^{P(\tau-\hat{t})} + (-1)^j e^{P(\tau-\hat{t}-\Delta T)} + 2((\sum_{i=1}^{n-1} (-1)^i e^{P(\tau-t_i)}) - (-1)^j e^{P(\tau-t_j)})] P^{-1} \underline{q}$$

$$\text{when} \quad \hat{t} \le t_j \le \hat{t} + \Delta T \le t_{j+1}, \text{ for } j = 1,\ldots,n-1\} . \tag{3.6.12}$$

Therefore the boundary of the τ-reachable set is

$$\partial RO(\tau,\hat{t},\Delta T) = (\bigcup_{j=0}^{n-1} \partial RO_1^j(\tau,\hat{t},\Delta T)) \cup (\bigcup_{j=1}^{n-1} \partial RO_2^j(\tau,\hat{t},\Delta T)) . \tag{3.6.13}$$

3.7 Calculation of the Boundary of the τ-Reachable Set for the System of Order n with Distinct Positive Real Eigenvalues and Vector Input

Given

$$\underline{\dot{x}(t)} = A\,\underline{x(t)} + B\,\underline{u(t)} \tag{3.7.1}$$

where $\underline{u(t)} \in U'(t)$, such that

$$U'(t) = \left\{ \underline{u}: \begin{array}{ll} |u_i(t)| \equiv 0 & \text{for } t \in [\hat{t},\hat{t} + \Delta T] \\ |u_i(t)| \le k_i & \text{otherwise} \end{array} \quad i = 1,2,\dots,r \right\}. \tag{3.7.2}$$

There exists a real similarity transformation that diagonalizes A, (Theorem 2.5.7). The resulting system equation is:

$$\underline{\dot{x}(t)} = P\,\underline{x(t)} + Q\,\underline{u(t)} \tag{3.7.3}$$

where

$$P = \begin{bmatrix} 1 & & & 0 \\ & \lambda_2/\lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_n/\lambda_1 \end{bmatrix} \tag{3.7.4}$$

and

$$Q = PK$$

such that:

$$K = \begin{bmatrix} 1 & k_{12} & \cdots & k_{1r} \\ 1 & k_{22} & \cdots & k_{2r} \\ \vdots & \vdots & & \vdots \\ 1 & k_{n2} & \cdots & k_{nr} \end{bmatrix} \quad . \tag{3.7.5}$$

In equation (3.7.3) $\underline{u(t)} \in U(t)$, where

$$U(t) = \left\{ \underline{u(t)} : \begin{array}{ll} |u_i(t)| \equiv 0 & \text{for} \quad t \in [\hat{t}, \hat{t} + \Delta T] \\ |u_i(t)| \leq 1 & \text{otherwise} \end{array} \quad i = 1,2,\ldots,r \right\}. \quad (3.7.6)$$

For the case of multiple inputs there still exist at most (n-1) switching times for each component of the control vector to reach any state in the state space. Assuming the same switching times for all components of the input vector, and following the same procedure applied in Section (3.6) the following can be obtained:

$$\partial RO_1^j(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x(\tau)} = \pm[e^{P\tau} + (-1)^n I + (-1)^{j+1} e^{P(\tau-\hat{t})}$$

$$+ (-1)^j e^{P(\tau-\hat{t}-\Delta T)} + 2\sum_{i=1}^{n-1} (-1)^i e^{P(\tau-t_i)}]P^{-1}Q$$

$$\text{for} \quad t_j \leq \hat{t} \leq \hat{t} + \Delta T \leq t_{j+1}, \text{ and } \quad j = 0,1,\ldots,n-1\} \quad (3.7.7)$$

and

$$\partial RO_2^j(\tau,\hat{t},\Delta T) = \{\underline{x}: \underline{x} = \pm[e^{P\tau} + (-1)^n I + (-1)^j e^{P(\tau-\hat{t})}$$

$$+ (-1)^j e^{P(\tau-\hat{t}-\Delta T)} + 2((\sum_{i=1}^{n-1} (-1)^i e^{P(\tau-t_i)}) - (-1)^j e^{P(\tau-t_j)})]P^{-1}Q$$

$$\text{for} \quad \hat{t} \leq t_j \leq \hat{t} + \Delta T \leq t_{j+1}, \; j = 1,\ldots,n-1\} . \quad (3.7.8)$$

The boundary of the τ-reachable set for this case is

$$\partial RO(\tau,\hat{t},\Delta T) = (\bigcup_{j=0}^{n-1} \partial RO_1^j(\tau,\hat{t},\Delta T)) \cup (\bigcup_{j=1}^{n-1} \partial RO_2^j(\tau,\hat{t},\Delta T)) . \quad (3.7.9)$$

For the case in which different components of $\underline{u(t)}$ have different switching times deriving an analytical expression for the boundary of the τ-reachable set following the procedure mentioned above is possible, but very cumbersome. Many different cases must be considered to include all possibilities.

3.8 Calculation of the Boundary of the τ-Reachable Set for Systems with Complex Eigenvalues and Scalar Input

Let τ^* be the time of the nth. switching of the control component. For all $\tau \le \tau^*$ equation (3.6.13) can be used to determine the boundary of the τ-reachable region. For values of τ, $\tau^* < \tau \le 2\tau^*$, the following theorem is used:

THEOREM 3.8.1: Given $\tau \in [\tau^*, 2\tau^*]$ the boundary of the τ-reachable set and τ^*-reachable set are related as follows:

a)

$$\text{if} \quad 0 \le \hat{t} \le \hat{t} + \Delta T \le \tau^* \le \tau \le 2\tau^*$$

then

$$\partial RO(\tau,\hat{t},\Delta T) = e^{P(\tau-\tau^*)} \partial RO(\tau^*,\hat{t},\Delta T) + \partial R(\tau-\tau^*) \quad . \tag{3.8.1}$$

b)

$$0 \le \tau^* \le \hat{t} \le \hat{t} + \Delta T \le \tau \le 2\tau^*$$

then

$$\partial RO(\tau,\hat{t},\Delta T) = e^{P(\tau-\tau^*)} \partial R(\tau^*) + \partial RO((\tau-\tau^*),\hat{t},\Delta T). \tag{3.8.2}$$

c)

$$\text{if} \quad 0 \le \hat{t} \le \tau^* \le \hat{t} + \Delta T \le \tau \le 2\tau^*$$

then

$$\partial RO(\tau,\hat{t},\Delta T) = e^{P(\tau-\hat{t})} \partial R(\hat{t}) + \partial R(\tau-\hat{t}-\Delta T) \tag{3.8.3}$$

where $R(\tau-\tau^*)$ indicates the reachable set from origin in $(\tau-\tau^*)$ seconds considering no control outage.

Proof: For case a) the variation of parameters formula results in

$$\underline{x(\tau)} = e^{P(\tau-\tau^*)}\underline{x(\tau^*)} + \int_{\tau^*}^{\tau} e^{P(\tau-\sigma)}\underline{q}\; u(\sigma)\,d\sigma \qquad (3.8.4)$$

where $\underline{x(\tau^*)}$ can be obtained from equations (3.6.11) or (3.6.12) depending upon the location of $\hat{t}$, and $\hat{t} + \Delta T$ in $[0,\tau^*]$. If τ^* is considered as the initial time and the origin as the initial state, then the boundary of the $(\tau-\tau^*)$ reachable set with no control outage is

$$\partial R(\tau-\tau^*) = \{\underline{x}\colon \underline{x} = \pm [\int_{\tau^*}^{\tau} e^{P(\tau-\sigma)}\underline{q}\; u(\sigma)\,d\sigma \quad . \qquad (3.8.5)$$

Equations (3.8.4) and (3.8.5) prove case a of this theorem.

For case b, since the control vector is present for $t \in [0,\hat{t}]$, where $\hat{t} > \tau^*$, then

$$\underline{x(\hat{t})} = \int_0^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}\; u(\sigma)\,d\sigma$$

$$\underline{x(\hat{t}+\Delta T)} = e^{P\Delta T}\;\underline{x(\hat{t})} = e^{P\Delta T}\int_0^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}\; u(\sigma)\,d\sigma$$

$$\begin{aligned}
\underline{x(\tau)} &= e^{P(\tau-\hat{t}-\Delta T)}\;\underline{x(\hat{t}+\Delta T)} + \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)}\underline{q}\; u(\sigma)\,d\sigma \\
&= e^{P(\tau-\hat{t})}\int_0^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}\; u(\sigma)\,d\sigma + \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)}\underline{q}\; u(\sigma)\,d\sigma \\
&= e^{P(\tau-\tau^*)}\int_0^{\tau^*} e^{P(\tau^*-\sigma)}\underline{q}\; u(\sigma)\,d\sigma + e^{P(\tau-\hat{t})}\int_0^{\hat{t}} e^{P(\hat{t}-\sigma)}\underline{q}\; u(\sigma)\,d\sigma \\
&\quad + \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)}\underline{q}u(\sigma)\,d\sigma \;. \qquad (3.8.6)
\end{aligned}$$

Since no control outage is assumed for $t \in [0,\tau^*]$, then

$$\partial R(\tau^*) = \{\underline{x}: \underline{x} = \int_0^{\tau^*} e^{P(\tau^*-\sigma)} \underline{q}\, u(\sigma)\, d\sigma,\ \underline{u} \in \Omega\} . \quad (3.8.7)$$

Assuming τ^* as initial time, the $(\tau-\tau^*)$ reachable set is calculated as follows

$$\underline{x(\hat{t}-\tau^*)} = \int_{\tau^*}^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma$$

$$\underline{x(\hat{t}+\Delta T-\tau^*)} = e^{P\Delta T} \int_{\tau^*}^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma$$

$$\underline{x(\tau-\tau^*)} = e^{P(\tau-\hat{t}-\Delta T)} e^{P\Delta T} \int_{\tau^*}^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma + \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)} \underline{q}\, u(\sigma)\, d\sigma . \quad (3.8.8)$$

Comparing equations (3.8.6), (3.8.7), and (3.8.8) proves part b .

For case c):

$$\underline{x(\hat{t})} = \int_0^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma$$

$$\underline{x(\hat{t}+\Delta T)} = e^{P\Delta T} \int_0^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma$$

$$\underline{x(\tau)} = e^{P(\tau-\hat{t})} \int_0^{\hat{t}} e^{P(\hat{t}-\sigma)} \underline{q}\, u(\sigma)\, d\sigma + \int_{\hat{t}+\Delta T}^{\tau} e^{P(\tau-\sigma)} \underline{q}\, u(\sigma)\, d\sigma .$$

Thus

$$\partial RO(\tau,\hat{t},\Delta T) = e^{P(\tau-\hat{t})} \partial R(\hat{t}) + \partial R(\tau-\hat{t}-\Delta T) . \qquad \text{Q.E.D.}$$

By repeated use of the above argument, Theorem (3.8.1) can be extended for $\tau \in [k\tau^*, (k+1)\tau^*]$, where $k > 1$ is a positive integer.

CHAPTER 4

COMPUTATION OF OPTIMAL CONTROLS FOR THE SYSTEMS WITH FINITE DURATION OF CONTROL OUTAGE

In Chapter 2 the convexity of the τ-reachable set for linear systems with a finite duration of control outage was proved. In Chapter 3, the analytical expressions for the τ-reachable set for linear time-invariant systems were calculated. The convexity of these sets, even when there is a control outage, leads us to the use of methods which are based upon this characteristic.

Gilbert [22] derived an iterative procedure which minimizes a quadratic form on a convex set. Barr [23] modified Gilbert's method to achieve much faster convergence. These methods require the convexity of the reachable set, which is also true for linear systems with control outage. In this chapter Gilbert's method is used to compute optimal controls for linear systems with a finite duration of control outage.

4.1 Basic Theory

In this section the general ideas of the iterative procedure are introduced [23].

BASIC PROBLEM (BP): Given $RO(\tau,\hat{t},\Delta T)$, the convex reachable set for linear systems with control outage in R^n, find a point $\underline{x}^* \in RO(\tau,\hat{t},\Delta T)$, such that

$$\|\underline{x}^*\|^2 = \min_{\underline{x} \in RO(\tau,\hat{t},\Delta T)} \|x\|^2 \quad . \tag{4.1.1}$$

To find $\underline{x}^*$, the following basic iterative procedure (BIP) is used [23]:

$$\underline{x_{k+1}} = \underline{x_k} + \alpha_k(\underline{s(-x_k)} - \underline{x_k}) \qquad k = 1,2,\ldots \tag{4.1.2}$$

where

$$\begin{cases} \alpha_k = \beta(\underline{x_k}) & \text{if} \quad 0 \le \beta(\underline{x_k}) \le 1 \\ \alpha_k = 1 & \text{if} \quad \beta(\underline{x_k}) > 1 \quad . \end{cases} \tag{4.1.3}$$

Here $\underline{s(-x_k)}$ is called a contact function evaluation for $-\underline{x_k}$, which is defined as follows. The mapping $\underline{s(y)}$ from R^n to the set $RO(\tau,\hat{t},\Delta T)$ is such that $\underline{y}\cdot\underline{s(y)} = \max_{\underline{x} \in RO(\tau,\hat{t},\Delta T)} \underline{y}\cdot\underline{x}$.

The expression for $\beta(\underline{x_k})$ is:

$$\begin{cases} \beta(\underline{x_k}) = \|\underline{x_k} - \underline{s(-x_k)}\|^{-2}\underline{x_k} \cdot (\underline{x_k} - \underline{s(-x_k)}) & \text{if} \quad \underline{x_k} - \underline{s(-x_k)} \neq 0 \\ \beta(\underline{x_k}) = 0 & \text{if} \quad \underline{x_k} - \underline{s(-x_k)} = 0 \end{cases} \tag{4.1.4}$$

where $\underline{x_k} \cdot (\underline{x_k} - \underline{s(-x_k)})$ is the scalar product of the two vectors. Another function $\gamma(\underline{x_k})$ is used to measure the "closeness" of $\underline{x_k}$ to $\underline{x}^*$ such that:

$$\begin{cases} \gamma(\underline{x_k}) = \|\underline{x_k}\|^{-2}\underline{x_k} \cdot \underline{s(-x_k)} & \text{if} \quad \|\underline{x_k}\| > 0 \text{ and } \underline{x_k}\cdot\underline{s(-x_k)} > 0 \\ \gamma(\underline{x_k}) = 0 & \text{if} \quad \|\underline{x_k}\| = 0 \text{ or } \|\underline{x_k}\| > 0, \text{ and} \\ & \qquad \underline{x_k}\cdot\underline{s(-x_k)} \le 0. \end{cases} \tag{4.1.5}$$

ABSTRACT PROBLEM (AP), [23]: Given $t \in [t_o,T]$ in R^1, and $RO(\tau,\hat{t},\Delta T)$, a compact, convex set in R^n, which moves continuously with t; find

$$t0^* = \underset{t\in[t_o,T],\ \underline{0}\in RO(t,\hat{t},\Delta T)}{\text{Min}}\ t \quad . \tag{4.1.6}$$

Note that $t0^*$ need not exist in general. It does exist and is unique if $\underline{x(0)}$ lies in the set of null controllability. Henceforth, in this chapter $\underline{x(0)}$ is assumed to have this property.

CONTACT FUNCTION: Consider the system defined by

$$\dot{\underline{x}}\underline{(t)} = A(t)\ \underline{x(t)} + B(t)\ \underline{u(t)} \tag{4.1.7}$$

where $\underline{u(t)} \in U(t)$. The time-varying control constraint set, is defined by equation (1.3.2). In Section (2.2), the reachable set for linear systems with control outage was derived as:

$$RO(t,\hat{t},\Delta T) = \begin{cases} R(t) & \text{if } t_o \le t \le \hat{t} \\ \phi(t,\hat{t})R(\hat{t}) & \text{if } \hat{t} < t < \hat{t} + \Delta T \\ \phi(t,\hat{t})R(\hat{t}) + \{\underline{z}\colon \underline{z(t)} = \int_{\hat{t}+\Delta T}^{t} \phi(t,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma & \\ & \text{if } \hat{t} + \Delta T \le t \le T \end{cases} \tag{4.1.8}$$

where

$$R(t) = \{\underline{x}\colon \underline{x(t)} = \phi(t,t_o)\underline{x(t_o)} + \phi(t,t_o)\int_{t_o}^{t} \phi(t_o,\sigma)[B(\sigma)\underline{u(\sigma)}]d\sigma\}. \tag{4.1.9}$$

It is shown in Section 2.2 that $RO(t,\hat{t},\Delta T)$ is compact, convex, and continuous with respect to t, $\hat{t}$, and ΔT; for almost all $t \in [t_o,T]$.

Given $\tau \in [t_o,T]$, $\hat{t} \in [t_o,T]$, ΔT, the duration of the control outage, and a vector $\underline{y} \in R^n$, a contact function corresponding to the vector $\underline{y}$ can be found by considering the following three cases [24]:

a)

$$t_o \leq \tau \leq \hat{t} .$$

For given $\underline{y} \in R^n$, a contact function is:

$$\underline{s(\underline{y},\tau)} = \phi(\tau,t_o)\underline{x(t_o)} + \phi(\tau,t_o)\int_{t_o}^{\tau} \phi(t_o,\sigma)[B(\sigma)\underline{u(\sigma,\underline{y})}]d\sigma \tag{4.1.10}$$

where

$$\underline{u(t,\underline{y})} = \text{sgn}\{B'(t)\cdot\underline{\eta'(t,\underline{y})}\} \tag{4.1.11}$$

and $\eta(t,\underline{y})$ is the solution of the adjoint differential equation

$$\dot{\eta}(t) = -\eta(t)A(t), \text{ with } \eta(\tau) = \underline{y}' . \tag{4.1.12}$$

b)

$$\hat{t} < \tau < \hat{t} + \Delta T$$

$$\underline{s(\underline{y},\tau)} = \phi(\tau,t_o)x(t_o) + \int_{t_o}^{\hat{t}} \phi(\tau,\sigma)[B(\sigma)\underline{u(\sigma,\underline{y})}]d\sigma . \tag{4.1.13}$$

c)

$$\hat{t} + \Delta T \leq \tau \leq T$$

$$\underline{s(\underline{y},\tau)} = \phi(\tau,t_o)\underline{x(t_o)} + \int_{t_o}^{\hat{t}} \phi(\tau,\sigma)[B(\sigma)\underline{u(\sigma,\underline{y})}]d\sigma$$

$$+ \int_{\hat{t}+\Delta T}^{\tau} \phi(\tau,\sigma)[B(\sigma)\underline{u(\sigma,\underline{y})}]d\sigma . \tag{4.1.14}$$

For time-invariant linear systems, a contact function for a given $\underline{y} \in R^n$ is:

$$\underline{s(y,\tau)} = \begin{cases} e^{A\tau}\{\underline{x(0)} + \int_0^{\tau} e^{-A\sigma}B\ \underline{u(\sigma,y)}d\sigma\} & \text{for} \quad 0 \le \tau \le \hat{t} \\ e^{A\tau}\{\underline{x(0)} + \int_0^{\hat{t}} e^{-A\sigma}B\ \underline{u(\sigma,y)}d\sigma\} & \text{for} \quad \hat{t} < \tau < \hat{t} + \Delta T \\ e^{A\tau}\{\underline{x(0)} + \int_0^{\hat{t}} e^{-A\sigma}B\ \underline{u(\sigma,y)}d\sigma + \int_{\hat{t}+\Delta T}^{\tau} e^{-A\sigma}B\ \underline{u(\sigma,y)}d\sigma\} & \\ & \text{for} \quad \hat{t} + \Delta T \le \tau \le T. \end{cases} \tag{4.1.15}$$

4.2 Computer Evaluation of Contact Function and of $t0^*$

In equation (4.1.15), the expression for the control $\underline{u(t,y)}$ is $\underline{u(t,y)} = \text{sgn}\{B'(t)\cdot\underline{\eta'(t,y)}\}$. Moreover, for time-invariant linear systems, the solution of the adjoint differential equation with final value $\underline{\eta'(\tau)} = \underline{y} = -\underline{x_k}$, is

$$\underline{\eta'(t,y)} = e^{-A'(t-\tau)}\underline{y} \quad \text{for} \quad t \in [0,\tau] \ . \tag{4.2.1}$$

In hybrid computation, the solutions of the state and adjoint equations can be implemented on an analog computer. If only a digital computer is used, the solution of the adjoint differential equation for time-invariant linear system is:

$$\begin{aligned} \underline{\eta'(t,y)} = {} & [Z'_{11} + Z'_{12}(\tau-t) + \ldots + Z'_{1r_1}\frac{(\tau-t)^{r_1-1}}{(r_1-1)!}]\underline{y}\, e^{\lambda_1(\tau-t)} + \\ & [Z'_{21} + Z'_{22}(\tau-t) + \ldots + Z'_{2r_2}\frac{(\tau-t)^{r_2-1}}{(r_2-1)!}]\underline{y}\, e^{\lambda_2(\tau-t)} + \\ & \vdots \\ & [Z'_{m1} + Z'_{m2}(\tau-t) + \ldots + Z'_{mr_m}\frac{(\tau-t)^{r_m-1}}{(r_m-1)!}]\underline{y}\, e^{\lambda_m(\tau-t)} \end{aligned} \tag{4.2.2}$$

where Z_{ij} are constituent matrices of the matrix A, and

$\lambda_1, \lambda_2, \ldots, \lambda_m$ are the distinct eigenvalues of A. The scalar coefficients of the characteristic equation for the square matrix A can be determined by the Frame method [25, page 212].

The constituent matrices can be calculated from the following equation:

$$\begin{bmatrix} Z_{11} \\ Z_{12} \\ \vdots \\ Z_{21} \\ \vdots \\ Z_{mr_m} \end{bmatrix} = (V^T)^{-1} \begin{bmatrix} U \\ A \\ A^2 \\ \vdots \\ A^{n-1} \end{bmatrix} \qquad (4.2.3)$$

where

$$(V^T)^{-1} = (G^{-1})VW . \qquad (4.2.4)$$

Entries of both matrices VW and G can be calculated by using the synthetic division algorithm for the characteristic polynomial [26]. Reduction of [G, VW] matrix to the form $[U, (G^{-1})VW]$ by elementary row operations produces $(V^T)^{-1}$.

After calculation of Z_{ij}, $\lambda_1, \ldots, \lambda_m$, equation (4.2.2) gives the forward-time solution of the adjoint differential equation with final value $\underline{\eta'(\tau)} = y$. The control which generates $\underline{s(y)}$ is

$$\underline{u(t,y)} = \mathrm{sgn}[B' \cdot \underline{\eta'(t,y)}] . \qquad (4.2.5)$$

Substitution of equation (4.2.5) into equation (4.1.15) yields a contact function evaluation. The ability to make this evaluation is fundamental to the computing procedure for solving (BP) and (AP).

In Chapter 3, the minimum time to reach the origin for some linear time-invariant systems with control outage was calculated. Derivation of an analytical expression for $t0^*$ for an arbitrary linear system is impossible. The iterative procedure described in this chapter is capable of solving any time-optimal regulator problem with a convex reachable set. The normality condition is not necessary; thus $RO(t,\hat{t},\Delta T)$ is not required to be strictly convex.

ITERATIVE PROCEDURE FOR (AP). Suppose a linear system defined by equation (4.1.7) and an initial condition $\underline{x(0)}$ is given. To find the optimal time $t0^*$, first consider a given value of time, i.e., $\tau \in [t_o,T]$ where τ is chosen to be less than $t0^*$. For example τ can be chosen to be δt, where δt is a small preassigned time increment. The basic iterative procedure (BIP) is used to find an approximation to the point $x^* \in RO(\tau,\hat{t},\Delta T)$.

Given $\underline{x_k}$, the use of equation (4.1.2) in (BIP) to compute $\underline{x_{k+1}}$ requires the value of $\underline{s(-\underline{x_k})}$, of a contact function corresponding to $-\underline{x_k}$. This value is calculated from equation (4.1.15). For initialization of $\underline{x_k}$, let $-\underline{x_1} = \underline{y} = -\underline{x(0)}$, where $\underline{x(0)}$ is the given initial state. Using equations (4.1.2) and (4.1.5) the values of $\underline{x_{k+1}}$ and $\gamma(\underline{x_k})$ are determined. If $\|x_{k+1}\| \leq \varepsilon$ (ε a preassigned positive value), the program is terminated and $t0^*$ is viewed as equal to τ.

If $\|\underline{x}_{k+1}\| > \varepsilon$, then $\gamma(\underline{x}_k)$ is checked. If $\gamma(\underline{x}_k) < \theta$, $(0 < \theta < 1$ is a preassigned number) with new values of $\underline{x}_k = \underline{x}_{k+1}$, and $\underline{y} = -\underline{x}_{k+1}$ another iteration of (BP) is repeated. For values of $\gamma(\underline{x}_k) \geq \theta$, the time τ, is increased by an increment of δt. The support function h [23] with $\underline{x}_k = \underline{x}_{k+1}$ is constructed:

$$h(-\underline{x}_k,\tau) = -\underline{x}_k \cdot s(-\underline{x}_k,\tau) \ .$$

For fixed value of $\underline{x}_k$, the value of the support function $h(-\underline{x}_k,\tau)$, is calculated for increasing τ in increments of δt. This is continued until for some value of τ, the support function $h(-\underline{x}_k,\tau) \geq 0$. For this value of τ, the basic iterative procedure (BIP) is used again to determine an approximation to $\underline{x}^* \in RO(\tau,\hat{t},\Delta T)$, considering $\underline{x}_k = \underline{s(-x_k,\tau)}$, and $\underline{y} = \underline{-s(-x_k,\tau)}$.

The summary of the iterative procedure is shown in Figure (4.2.1). Several second-order, time-invariant systems without control outage were solved successfully with the above method. Tables (4.2.1), (4.2.2), (4.2.3), and (4.2.4) show the highlight of the runs for the specified problems.

The system defined by $\dddot{x} = u(t)$ was tried with initial condition [2,0,0], on the Control Data 6500 (Table (4.2.4)). The number of iterations for (BP) is 214, and for finding the zero crossing for h, the support function, is 17. The total time is 200.6 seconds and the total computing cost is $10.38.

To improve the convergence rate, more than one contact point at each iteration can be used. Such an improved iterative procedure (IIP) is discussed by Barr [23].

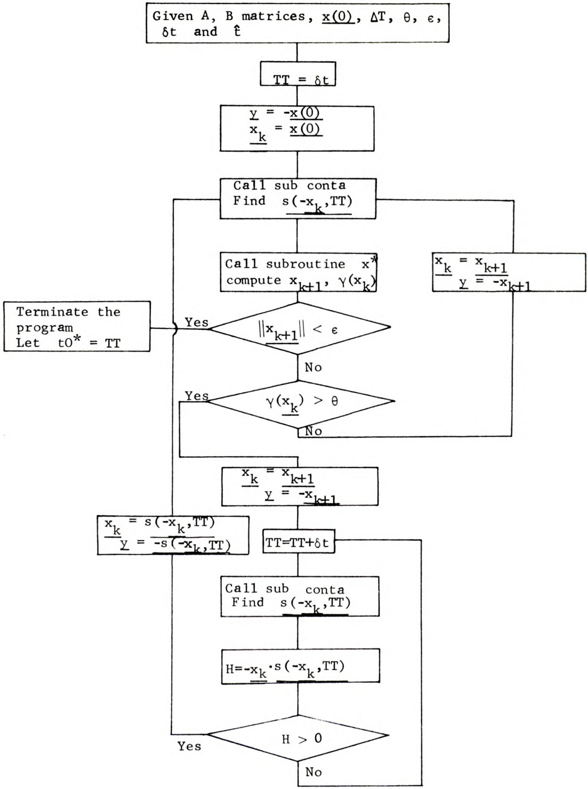


Figure (4.2.1): Flow Chart for Finding $t0^*$, using BIP and Support Function.

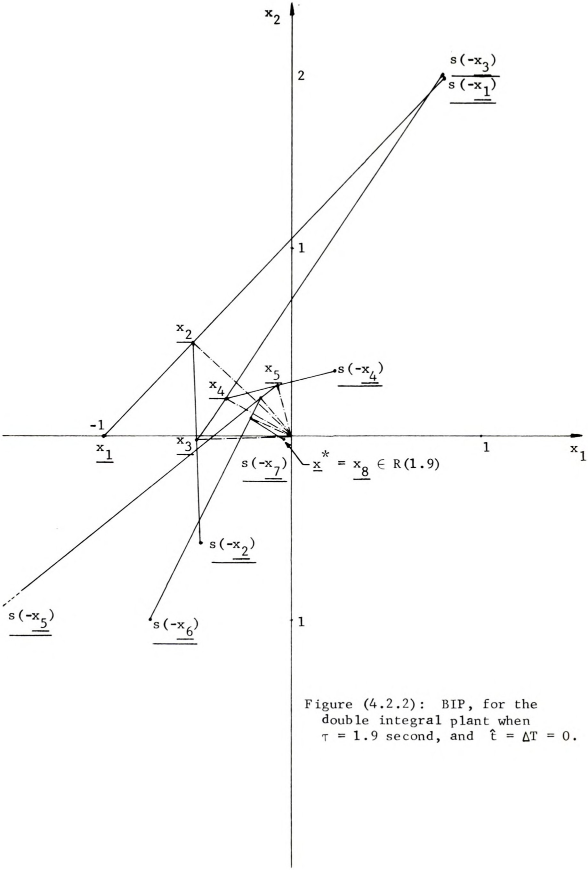


Figure (4.2.2): BIP, for the double integral plant when $\tau = 1.9$ second, and $\hat{t} = \Delta T = 0$.

BASIC ITERATIVE PROCEDURE (BIP), TT = .1 sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, .1)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
1	0 , 1	-.0951, .6374	.637	1	-.095,.637

I	$\underline{x}^*$	TT	$H=-\underline{x}^*.s$
1	-.95, .637	.15	-.32
4	-.95, .637	.3	-.0866
5	-.95, .637	.35	-.0088

BASIC ITERATIVE PROCEDURE (BIP), TT = .35 sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, .35)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
2	-.95, .637	-.296, -.0088	.054	.857	-.267,.083
4	-.23, .14	-.296, -.008	.915	.238	-.248,.104

I	$\underline{x}^*$	TT	H
15	-.037, .028	.85	-.00063

BASIC ITERATIVE PROCEDURE (BIP), TT = 85 = t^* sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, .85)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
27	-.037, .028	-.226, -.283	.284	.012	-.040,.024
28	-.040, .024	-.0038, .0199	.289	1	-.0038,.019

Table (4.2.1): Highlights of the runs, using (BIP) for the system given by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \ \underline{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \underline{y} = -\underline{x}(0), \text{ and } \underline{x}_1 = \underline{x}(0)$$

with $\theta = .5$, $\epsilon = \|\underline{x}^*\|^2 = .001$, TT = TT + I * DELT, δt = DELT = .05

BASIC ITERATIVE PROCEDURE (BIP), TT = 1.8 sec.					
k	x_k	$s(-x_k, 1.8)$	$\gamma(x_k)$	$\alpha(x_k)$	x_{k+1}
1	4 , 0	.204, -2.339	.051	.763	1.101,-.786
2	1.101, -.786	.783, -1.307	.726	1	.783,-1.307

I	x^*	TT	H
1	.783, -1.307	1.85	-2.166
10	.783, -1.307	2.3	-.765
18	.783, -1.307	2.7	-1.498
21	.783, -1.307	2.85	.070

BASIC ITERATIVE PROCEDURE (BIP), TT = 2.85 sec. = t^*					
k	x_k	$s(-x_k, 2.85)$	$\gamma(x_k)$	$\alpha(x_k)$	x_{k+1}
3	.783, -1.307	-1.491, -.840	0	.443	-.225, -1.1
4	-.225, -1.100	-.0246, .082	0	.9355	-.037,.0064
5	-.037, .0064	.734, -.537	0	.0364	.009,-.013

Table (4.2.2): Highlights of the runs to find t^*, using (BIP), for the system given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{x}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\theta = .5, \quad \epsilon = \|\underline{x}^*\|^2 = .001, \quad \delta t = .05 \text{ sec.}$$

BASIC ITERATIVE PROCEDURE (BIP), TT = 1 sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, .1)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
1	-1 , 0	-.994, .099	.994	.498	-.997,.0498

I	$\underline{x}^*$	TT	H
1	-.997,-.0498	.15	-.991
20	-.997,-.0498	1.1	-1.596
28	-.997,-.0498	1.5	-.436
32	-.997,-.0498	1.7	-.034

BASIC ITERATIVE PROCEDURE (BIP), TT = 1.7 sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, 1.7)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
2	-.997, -.0498	-.0174, .339	.034	.921	-.093,.317
4	-.196, .229	-.971, -.679	.385	.0393	-.227,.193
5	-.227, .193	-.595, -.339	.780	.046	-.244,.168

I	$\underline{x}^*$	TT	H
37	-.097, .0	1.95	.0878

BASIC ITERATIVE PROCEDURE (BIP), TT = 1.95 sec.					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, 1.95)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
16	-.097, .0	.901, 1.949	0	.020	-.077,.0395
20	-.039, .019	.559, .779	0	.009	-.034,.0268
21	-.034, .026	-.049, .0	.894	.208	-.037,.0212

I	$\underline{x}^*$	TT	H
38	-.037, .021	1.999	.0068

BASIC ITERATIVE PROCEDURE (BIP), TT = 1.999 sec = t^*					
k	$\underline{x}_k$	$\underline{s}(-\underline{x}_k, 1.999)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
22	-.037,.0212	.639, .799	0	.008	-.0317,.027
23	-.0317,.0276	0 , 0	0	1	0 , 0

Table (4.2.3): (BIP) applied to the system given by

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\theta = .5$, $\epsilon = .001$, $\delta t = .05$ sec.

BASIC ITERATIVE PROCEDURE (BIP), TT = .85 sec.					
k	$\underline{x}_k$	$s(-\underline{x}_k, .85)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
1	1, 0, 0	-.0028,-.36,-.8	0	.01191	.099,-.004,
4	.088,-.026,.0	.068,-.007,.34	.7673	.0169	.088,-.026,0

I	$\underline{x}^*$	TT	H
1	.088,-.026,.00	.9	-.0025

BASIC ITERATIVE PROCEDURE (BIP), TT = .9 sec.					
k	$\underline{x}_k$	$s(-\underline{x}_k, .9)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
5	.088,-.026,.00	-.022,-.4,-.9	.302	.006	.087,-.028,0
8	.085,-.03,.001	.055,-.009,.38	.685	.018	.08,-.03,.00

I	$\underline{x}^*$	TT	H
7	.08,-.03,.014	1.2	.0108

BASIC ITERATIVE PROCEDURE (BIP), TT = 1.3					
k	$\underline{x}_k$	$s(-\underline{x}_k, 1.3)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
42	.04,-.03,.006	-.035,-.17,-.26	.508	.01309	.04,-.03,.0

I	$\underline{x}^*$	TT	H
11	.034,-.02,.00	1.4	-.0004

BASIC ITERATIVE PROCEDURE (BIP), TT = t^* = 1.4					
k	$\underline{x}_k$	$s(-\underline{x}_k, 1.4)$	$\gamma(\underline{x}_k)$	$\alpha(\underline{x}_k)$	$\underline{x}_{k+1}$
64	.034,-.02,.006	-.07,-.2,-.3	.238	.01	.03,-.02,0
97	.03,-.015,.003	-.05,-.098,0	.134	.068	.02,-.02,0

Table (4.2.4): BIP applied to the system given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \theta = .5,$$

$\epsilon = .001$, $\delta t = .05$

4.3 Numerical Results for BIP Applied to a Minimum-Time Example with a Control Outage

The system to be considered is the "double-integral" plant. The system equation is:

$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (4.3.1)$$

where $u(t) \in U(t)$, which $U(t)$ is defined by equation (1.3.2). The initial, and final values of the system are

$$\underline{x(0)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad , \quad \underline{x(tO^*)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (4.3.2)$$

The solution of this problem is known. Hauer and Hsu [5] have shown that

$$tO^*(\hat{t},\Delta T) = \Delta T + 2[1 - \hat{t}\Delta T]^{\frac{1}{2}} \quad \text{if } 0 \le \hat{t} \le 1,\ Q \le 0 \qquad (4.3.3)$$

$$tO^*(\hat{t},\Delta T) = 2\hat{t} + \Delta T + 2[\hat{t}^2 + \hat{t}\Delta T - 1]^{\frac{1}{2}} \quad \text{if } 0 \le \hat{t} \le 1,\ Q > 0 \qquad (4.3.4)$$

$$tO^*(\hat{t},\Delta T) = 2 + \Delta T + 2[\Delta T(2-\hat{t})]^{\frac{1}{2}} \quad \text{if } 1 < \hat{t} \le 2,\ Q > 0 \qquad (4.3.5)$$

where

$$Q = Q(x(\hat{t}),\Delta T) = -1 \qquad \text{if } x_2(\hat{t}) \le 0 \qquad (4.3.6)$$

$$Q = \frac{x_2^2(\hat{t})}{2} + x_1(\hat{t}) + x_2(\hat{t})\cdot\Delta T \qquad \text{if } x_2(\hat{t}) > 0 \qquad (4.3.7)$$

and

$$\underline{x(\hat{t})} = \begin{bmatrix} \frac{\hat{t}^2}{2} - 1 \\ \hat{t} \end{bmatrix} \qquad \text{if } \hat{t} \in [0,1) \qquad (4.3.8)$$

$$\underline{x(\hat{t})} = \begin{bmatrix} -\frac{(2-\hat{t})^2}{2} \\ \\ 2-\hat{t} \end{bmatrix} \qquad \text{if} \quad \hat{t} \in [1,2] \ . \qquad (4.3.9)$$

It is easy to show that in equations (4.3.3) and (4.3.5), $t0^*$ is monotonically decreasing with respect to $\hat{t}$, but in equation (4.3.4) $t0^*$ is monotonically increasing with respect to $\hat{t}$. Figure (4.3.1) represents the variation of $t0^*$ (minimum-time with control outage) with respect to the control outage starting time $\hat{t}$, such that $0 \le \hat{t} \le t^*$. Here t^* is the minimum time for this system, with the same initial, and final state, if there is no control outage during the process.

In Figure (4.3.1), the dotted line indicates the theoretical value of $t0^*$ with respect to changes in $\hat{t}$, which follows from equations (4.3.3), (4.3.4), and (4.3.5). The solid line represents the computed value of $t0^*$ using Gilbert's method as described in Sections (4.1) and (4.2). The control outage duration ΔT is assumed to be .2 seconds. The minimum-time $t0^*$ is computed for steps of .1 seconds in $\hat{t}$. Figure (4.3.1) also indicates that the computed value is very close to theoretical one. The speed of convergence for BIP is slow and could be improved considerably by using additional contact points as suggested by Barr [24].

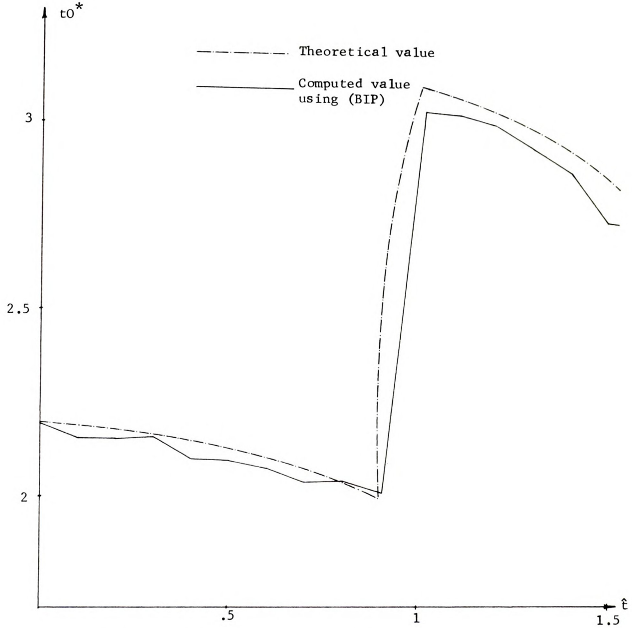


Figure (4.3.1): Theoretical and Computed value of Minimum time $t0^*$, for different values of $\hat{t}$, when $\Delta T = .2$ sec. The system is a double integral plant with $\underline{x(0)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

4.4 Application of Gilbert's Technique to Optimal Control Problems with Control Outage

The iterative procedures for time-optimal control which are described in Sections 4.1, 4.2, and 4.3 can be applied to many other optimal control problems in which there exists a finite duration of control outage during the process. Consider the system defined by

$$\underline{\dot{x}(t)} = A(t)\ \underline{x(t)} + B(t)\ \underline{u(t)} \quad \text{on} \quad t \in [t_o, T] \tag{4.4.1}$$

where $\underline{u(t)} \in U(t)$. Here $U(t)$ is a time-varying compact control set which is defined by equation (1.3.2). The matrix functions $A(t)$ and $B(t)$ are continuous with respect to time. The continuity of these matrices insure that equation (4.4.1) has a unique solution $\underline{x(t,\underline{u})}$ for each admissible $\underline{u(t)}$ and initial condition $\underline{x(t_o)} = \underline{x_o}$, where $t_o \le t \le T$.

The family of closed sets $W(t) \subset E^n$ which are defined for every $t \in [t_o,T]$ are called the target sets. It is assumed that $W(t)$ is convex and continuous in t. In many applications $W(t)$ either consists of a single point $w(t)$ for each t or is a k-dimensional linear subspace with $1 \le k \le n-1$.

A cost functional of the following form is considered:

$$JO_\tau(\underline{u}) = g_1(\underline{x_{\underline{u}}(\tau)}) + \int_{t_o}^{\tau} [\underline{a'(\sigma)}\ \underline{x(\sigma)} + g_2(\underline{u(\sigma)},\sigma)]d\sigma \tag{4.4.2}$$

where $g_1(\underline{x_{\underline{u}}(\tau)})$ is a given continuous, convex function from R^n to R^1 for almost all $\tau \in [t_o,T]$, $a'(\sigma)$ is a continuous row vector on $[t_o,T]$ and $g_2(\underline{u},\sigma)$ is a continuous function from $R^m \times [t_o,T]$ to R^1.

An admissible control $\underline{\hat{u}(t)}$ defined for $t \in [t_o, \tau]$, where $\tau \in (t_o, T]$, is said to transfer the system state from $\underline{x(t_o)} = \underline{x_o}$ to $W(\tau)$ in time τ if $\underline{x(\tau, \hat{u})} \in W(\tau)$. An optimal control problem will now be described for the two following cases:

Case a):

Let $\tau > t_o$ be a fixed point in $(t_o, T]$. The problem is to find an admissible control $\underline{u^*(t)}$, (it is assumed that there exists one), which transfers the system state from $\underline{x(t_o)}$ to $W(\tau)$ and

$$JO_\tau(\underline{u}^*) \leq \min_{u(t) \in U(t)} (JO_\tau(\underline{u})).$$

The above problem is called the fixed terminal time optimal control problem.

Case b):

In the free terminal time optimal control problem, the objective is to find an admissible control $\underline{u^*(t)}$ and an optimal time $tO^* \in [t_o, T]$ such that $\underline{u^*(t)}$ transfers the system state from $\underline{x(t_o)}$ to $W(tO^*)$ in time tO^* and

$$JO_{tO^*}(\underline{u}^*) \leq \min_{u(t) \in U(t)} (JO_t(\underline{u})),\ t \in [t_o, T].$$

Barr and Gilbert [23] show that by extending the abstract problem AP an entire class of optimal control problems including minimum fuel, minimum effort, and minimum error rendezvous can be solved by sequentially applying BIP. The minimum time problem with moving target set can also be treated. The essential features required are convexity the ability to compute a contact function. Theorem (2.2.2) shows that the reachable set for the linear system with control outage is compact, convex, and continuous in t, $\hat{t}$, and ΔT. A contact function can be evaluated using equation (2.2.10). Thus optimal computations may be performed for these broad classes of problems when there is control outage.

CHAPTER V

SUMMARY AND CONCLUSIONS

The problem of computing optimal controls for a class of linear systems with amplitude bounded inputs and finite duration of control outage has been considered in this work. It is shown that in the event of control outage the control constraint set is time-varying and piecewise continuous with respect to time. The structure of this thesis is based upon Theorem (2.2.2) which guarantees the compactness, convexity, and continuity of the reachable set for linear systems in the case of outage.

It is not possible to calculate a general expression for the boundary of the τ-reachable set for linear systems with control outage. Analytical expressions for the boundary of these sets are obtained for linear time-invariant systems with positive distinct real eigenvalues or complex eigenvalues. These expressions are restricted to the case where the optimal control for the cases with and without outage have the same number of switchings (the change in control from $\pm$ 1 to 0 or from 0 to $\pm$ 1 is not considered a switching).

The minimum regulation times for linear time-invariant systems with a finite duration of control outage are investigated. Specifically, an expression for the minimum regulation time for a second-order, time-invariant system with negative distinct real eigenvalues is derived. The variations of this minimum time with respect to initial state, control outage starting time, and duration of outage are shown. The recoverability of the system is investigated. Equation (3.5.20) determines for which value of control outage starting time the system is recoverable.

There are many computational methods for computing optimal controls for linear optimal control problems based on convexity of the reachable set of system states. Theorem (2.2.2) makes possible the use of Gilbert's technique which utilizes the convexity of reachable set to compute optimal controls for linear systems with control outage. A modified contact function (for the case of outage) is given by equation (4.1.15). Several examples for the case with no control outage are solved successfully with the above techniques.

The minimum regulation time for a double integral plant is computed using Gilbert's method and considering different values of control outage starting time. For higher-order systems and faster convergence rate, Barr's technique [23] can be used.

There are a number of extensions which can be considered for linear systems with a finite duration of control outage.

I. Linear time-optimal control problems with control outage, bounded amplitude control, and rate saturation.

II. Extension of Chapter 4 to other linear optimal control problems, i.e., minimum-error regulation, minimum-fuel, minimum effort, etc.

III. Application of other computational techniques based on the convexity of the reachable set, and comparison of the

convergence rates, computing times, etc.

IV. Multiple control outages.

V. Consideration of linear systems with a disturbance in input vector, where this disturbance can be a known input or an unknown realization of a random process.

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