REDUCTION OF BACKSCATTERING FROM THE INFINITE CONE AND THE SPHERE BY IMPEDANCE LOADING

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ABSTRACT

REDUCTION OF BACKSCATTERING FROM THE INFINITE CONE AND THE SPHERE BY IMPEDANCE LOADING

by Carl Kenneth Dudley

The theoretical study on the reduction of the backscattering of the infinite cone and the sphere by the impedance loading method is presented. An infinite cone is assumed to be illuminated by a plane wave at the nose-on incidence. The backscattering of the cone is minimized by loading the cone surface with (1) a circumferential slot backed by an impedance, (2) a radial slot backed by an impedance and (3) a patch impedance. The sphere is given a similar treatment with the impedance area being in the shape of the wedge and the patch. For both the infinite cone and the sphere, it is shown that the optimum impedance for each case can be found to make the backscattering vanish.

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INTRODUCTION

Interest is being shown lately in the reduction of Radar back-scattering of metallic objects. The sphere and the cylinder have already been analyzed.

In this work the infinite cone and some further aspects of the sphere are investigated and in each case an expression is found for various impedances which will eliminate the backscattering from the structure. Both the circumferential and longitudinal slot are used separately and other forms of impedances are also dealt with.

A loaded cone such as shown in Figure 1 is illuminated by a plane wave normal to the axis of the cone. The surface of the cone is considered as an ideal conductor except for an area in which an impedance is placed. The presence of the impedance causes the radar echo to be zero in the direction from which the incident wave is coming.

In the method used here the problem is divided into parts. In the first part the cone is illuminated with no impedance in its surface and both the surface current I_S and the scattered field E_S are noted. Then in the second part the cone is considered without illumination but with the area in which the impedance is to be placed having a specified arbitrary field E_o. Again the surface current I_R and the radiation field E_R are noted as functions of E_o. Next the field E_o is specified in such a way as to make $E_R = - E_S$ in the direction from which the incident wave is coming. Finally the two situations are combined using the super position principle to give the desired condition of zero back scattering. The value of the impedance is the ratio of the field E_o in the area of the impedance to the total surface current there.

The impedance thus found may have to be active. It has the expression

$$Z = \frac{E_o}{I_R + I_S} \tag{1}$$

An advantage of this method is that it incorporates existing solutions as parts of the overall problem.

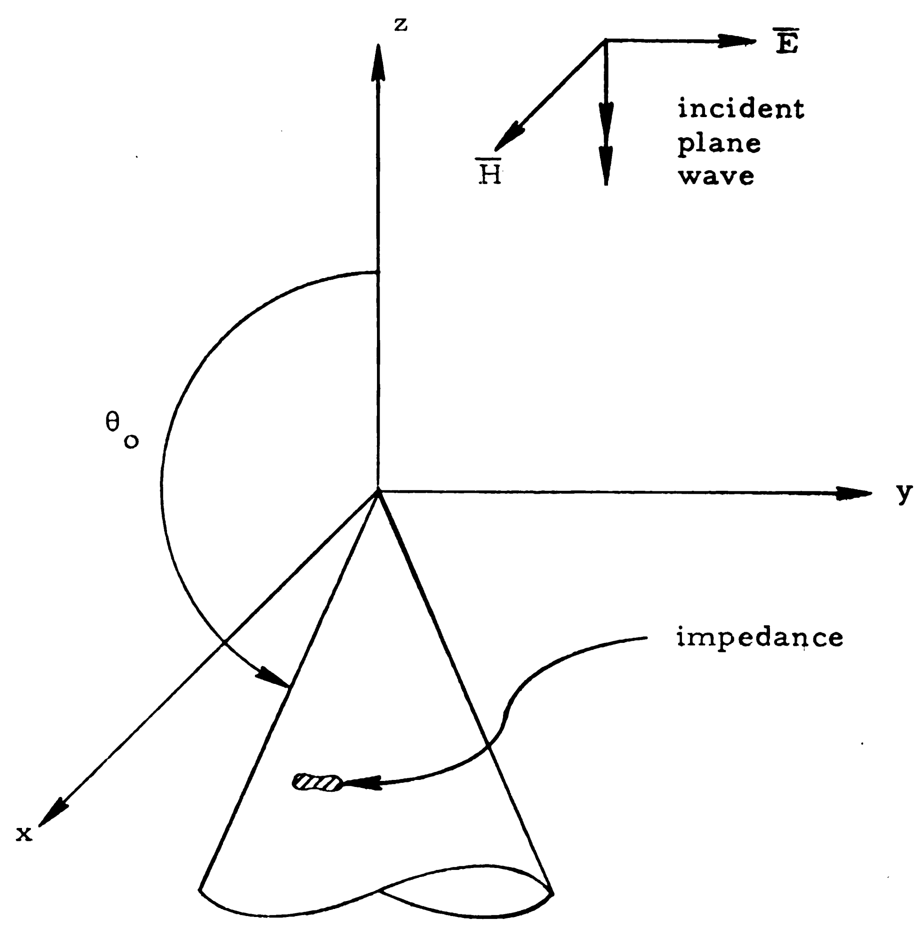

Figure 1. A loaded cone illuminated by a plane wave.

CHAPTER I

SCATTERING FROM A PERFECTLY CONDUCTING UNALTERED CONE SURFACE

1.1. Spherical Mode Functions

Consider first the illumination of a cone surface without the impedance. The incident wave is a plane wave traveling in the negative z direction with the electric field in the y - z plane. The cone surface is at an angle θ_o from the z axis. The geometry of the problem is shown in Figure 2. A surface current I_S is induced and a scattered field E_S is produced. The cone is a coordinate surface of the spherical coordinate system so the spherical mode functions are used (see equations (35) in Appendix I).

1.2. Boundary Conditions

In the equations (35) of Appendix I the constants $A_{m\nu}$, $B_{m\nu}$, $C_{m\nu'}$, and $D_{m\nu'}$ are all zero except for m = 1. This is seen by the geometry of the problem.

Also the only radial functions used are the spherical Bessel functions j_ν and $j_{\nu'}$ since the region includes the origin where n_ν, h_ν^1, and h_ν^2 are singular. The boundary conditions on the surface of the cone which is considered an ideal conductor are

$$E_r \Big|_{\theta = \theta_o} = E_\phi \Big|_{\theta = \theta_o} = 0 \qquad (2)$$

Equation (2) applies to the total of both the incident and reflected parts of the electric field intensity.

Examination of equations (35) of Appendix I, specifically the equations for E_r and E_ϕ

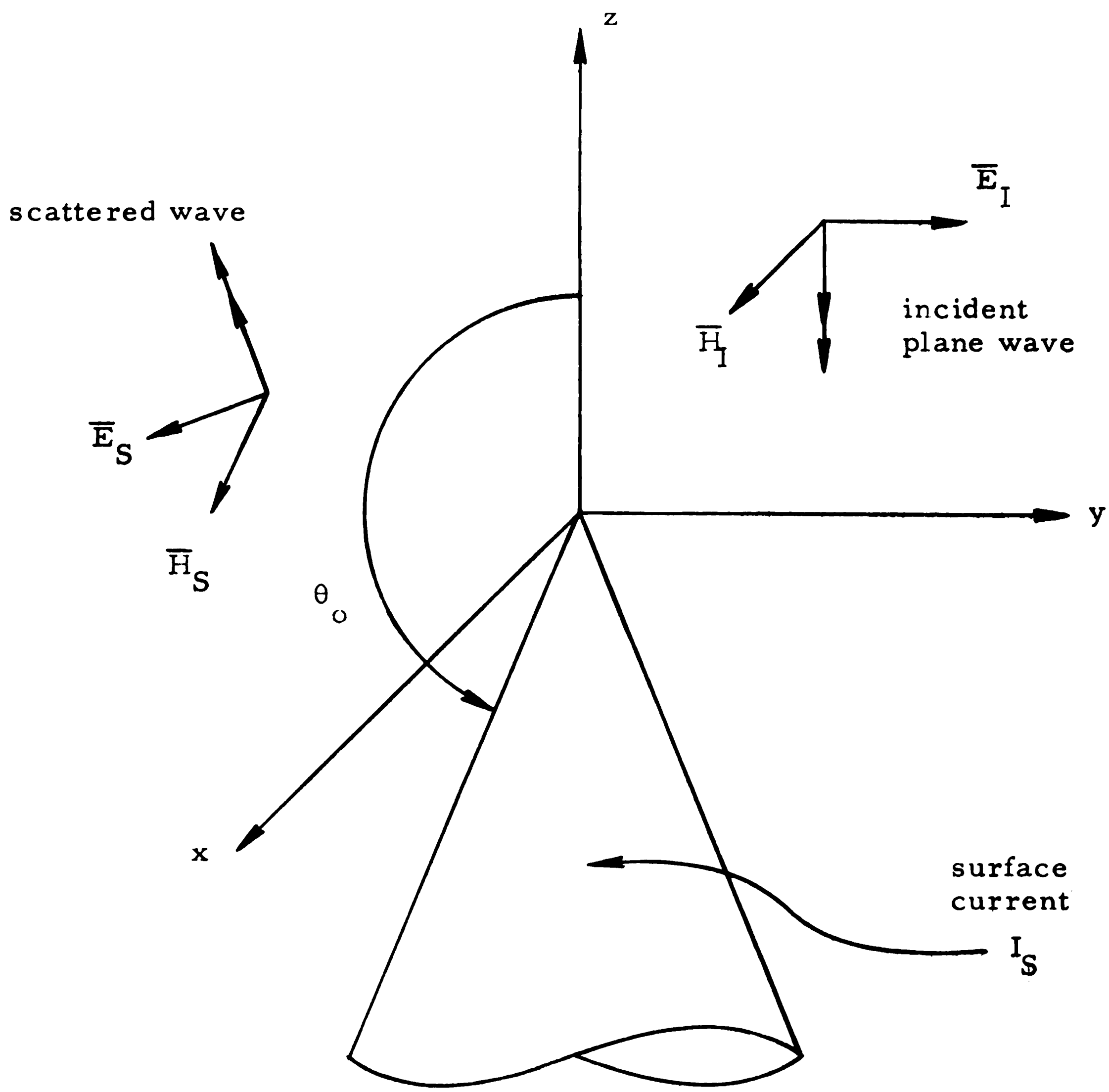

Figure 2. A conducting unaltered cone illuminated by a plane wave.

$$E_r = \sum_m \sum_\nu \frac{\nu(\nu+1)}{r} z_\nu(kr)\, P_\nu^m(\cos\theta) \left[A_{m\nu}\cos m\phi + B_{m\nu}\sin m\phi\right]$$

$$E_\phi = \sum_m \sum_\nu \frac{-m}{r\sin\theta} \frac{d}{dr}\left[r z_\nu\right] P_\nu^m \left[A_{m\nu}\sin m\phi - B_{m\nu}\cos m\phi\right]$$

$$+ j\omega\mu \sum_m \sum_{\nu'} z_{\nu'} \frac{d}{d\theta} P_{\nu'}^m \left[C_{m\nu'}\cos m\phi + D_{m\nu'}\sin m\phi\right]$$

shows that equation (2) is satisfied if

$$P_\nu^1(\cos\theta_o) = \frac{d}{d\theta} P_{\nu'}^1(\cos\theta_o) = 0 \tag{3}$$

Equations (3) determine the eigenvalues ν_i and ν'_i which are a countable set and may be ordered $\nu_o, \nu_1, \nu_2, \ldots$ and $\nu'_o, \nu'_1, \nu'_2, \ldots$ respectively.

With these changes equations (35) become

$$E_r = \sum_{i=0}^{\infty} \frac{\nu_i(\nu_i+1)}{r} j_{\nu_i} P_{\nu_i}^1 \left[A_i\cos\phi + B_i\sin\phi\right]$$

$$E_\theta = \sum_{i=0}^{\infty} \frac{1}{r}\frac{d}{dr}\left[r j_{\nu_i}\right] \frac{d}{d\theta} P_{\nu_i}^1 \left[A_i\cos\phi + B_i\sin\phi\right]$$

$$+ j\omega\mu \sum_{i=0}^{\infty} \frac{1}{\sin\theta} j_{\nu'_i} P_{\nu'_i}^1 \left[C_i\sin\phi - D_i\cos\phi\right].$$

$$E_\phi = \sum_{i=0}^{\infty} \frac{-1}{r\sin\theta}\frac{d}{dr}\left[r j_{\nu_i}\right] P_{\nu_i}^1 \left[A_i\sin\phi - B_i\cos\phi\right]$$

$$+ j\omega\mu \sum_{i=0}^{\infty} j_{\nu'_i} \frac{d}{d\theta} P_{\nu'_i}^1 \left[C_i\cos\phi + D_i\sin\phi\right]$$

$$H_r = \sum_{i=0}^{\infty} \frac{\nu'_i(\nu'_i + 1)}{r} j_{\nu'_i} P_{\nu'_i}^{1} \left[C_i \cos\phi + D_i \sin\phi\right]$$

$$H_\theta = \sum_{i=0}^{\infty} \frac{1}{r}\frac{d}{dr}\left[r\, j_{\nu'_i}\right] \frac{d}{d\theta} P_{\nu'_i}^{1} \left[C_i \cos\phi + D_i \sin\phi\right]$$

$$- j\omega\epsilon \sum_{i=0}^{\infty} \frac{\cdot 1}{\sin\theta} j_{\nu_i} P_{\nu_i}^{1} \left[A_i \sin\phi - B_i \cos\phi\right]$$

$$H_\phi = \sum_{i=0}^{\infty} \frac{-1}{r \sin\theta}\frac{d}{dr}\left[r\, j_{\nu'_i}\right] P_{\nu'_i}^{1} \left[C_i \sin\phi - D_i \cos\phi\right]$$

$$- j\omega\epsilon \sum_{i=0}^{\infty} j_{\nu_i} \frac{d}{d\theta} P_{\nu_i}^{1} \left[A_i \cos\phi + B_i \sin\phi\right] \tag{4}$$

1.3. Determination of the Expansion Coefficients

The expansion coefficients B_i and C_i must be found using the condition that as r becomes very large the total field approaches the incident field. This condition is actually only true for sharp cones. For example if the cone angle θ_o approaches 90^o the cone becomes an infinite sheet the field of which is a standing wave everywhere. This problem is eliminated later however and need not be a matter of concern here.

The fact that the incident wave is oriented with the electric field intensity in the y - z plane and that the remainder of the geometry is independent of ϕ indicates that in the x - z plane the radial component of electric field intensity is zero, and in the y - z plane the radial component of magnetic field intensity is zero. Examination of equations (4) reveals that this can be true only if $A_i = D_i = 0$.

The scalar functions Π and Π^* for the total field from equations (25) and (34) in Appendix I are

$$\Pi = \sum_{i=0}^{\infty} B_i \, j_{\nu_i}(kr) \, P_{\nu_i}^{1}(\cos\theta) \sin\phi$$

$$\Pi^* = \sum_{i=0}^{\infty} C_i \, j_{\nu'_i}(kr) \, P_{\nu'_i}^{1}(\cos\theta) \cos\phi \qquad (5)$$

The incident electric field intensity is expressed by the equation

$$\overline{E}_I = e^{jkz} \, \hat{y} \qquad (6)$$

In spherical coordinates the radial components of electric and magnetic field intensities from equation (6) are

$$E_{Ir} = e^{jkr\cos\theta} \sin\theta \sin\phi$$

$$H_{Ir} = Y e^{jkr\cos\theta} \sin\theta \cos\phi \qquad (7)$$

where $Y = \sqrt{\frac{\epsilon}{\mu}}$

In order to find the expansion coefficients B_i and C_i the orthogonality property of $P_{\nu_i}^{1}(\cos\theta)$ and $\frac{d}{d\theta} P_{\nu'_i}^{1}(\cos\theta)$ in the interval $(0, \theta_o)$ are used. (See Appendix II, equation (37)).

Using equations (5) the radial components of the total field are found to be

$$E_r = \sum_{i=0}^{\infty} B_i \frac{\nu_i(\nu_i+1)}{r} j_{\nu_i} P_{\nu_i}^1 \sin\phi$$

$$H_r = \sum_{i=0}^{\infty} C_i \frac{\nu_i'(\nu_i'+1)}{r} j_{\nu'_i} P_{\nu'_i}^1 \cos\phi \qquad (8)$$

For very large r the radial components of the total field as expressed by equation (8) approach the radial components of the incident field as expressed by Equation (7).

Equating the right-hand sides of equations (7) and (8) and multiplication by $P_{\nu_i}^1 \sin\theta$ or $P_{\nu'_i}^1 \sin\theta$ and integrating over $\theta = 0$ to $\theta = \theta_o$ for large r yields

$$\int_o^{\theta_o} e^{ikr\cos\theta} \sin^2\theta \, P_{\nu_i}^1 (\cos\theta)\, d\theta$$

$$\doteq \frac{1}{r} \nu_i(\nu_i+1) B_i \, j_{\nu_i}(kr) \int_o^{\theta_o} \left[P_{\nu_i}^1(\cos\theta)\right]^2 \sin\theta \, d\theta$$

$$(9)$$

$$\sqrt{\frac{\epsilon}{\mu}} \int_o^{\theta_o} e^{ikr\cos\theta} \sin^2\theta \, P_{\nu'_i}^1 (\cos\theta)\, d\theta$$

$$\doteq \frac{\nu'_i(\nu'_i+1)}{r} C_i \, j_{\nu'_i}(kr) \int_o^{\theta_o} \left[P_{\nu'_i}^1(\cos\theta)\right]^2 \sin\theta \, d\theta$$

The dots above the equal signs in equations (9) indicate that the relations are only approximate. On the left-hand side is the expression for the incident field for large r, while on the right-hand side is the expression for the total field for large r. This approximation may be improved upon by using the asymptotic forms for j_ν and recognizing which part is the incident field and which part is the reflected field. Equating only the incident parts will eliminate the problem referred to previously which occurs for cone angles θ_o not greater than 90^o.

For large r the asymptotic form for j_ν is

$$j_\nu (kr) \to \frac{1}{kr} \cos \left(kr - \frac{\nu + 1}{2} \pi \right) \qquad kr >> \nu \tag{10}$$

In another form relation (10) is

$$j_\nu (kr) \to \frac{1}{2kr} \left[e^{j(kr - \frac{\nu+1}{2} \pi)} + e^{-j(kr - \frac{\nu+1}{2} \pi)} \right] \qquad kr >> \nu \tag{11}$$

The integrals on the left-hand sides of equations (9) can be evaluated by the method of stationary phase and the results are given here as

$$\int_o^{\theta_o} e^{jkr \cos \theta} \sin^2 \theta \, P_{\nu_i}^1 (\cos \theta) \, d\theta = + \frac{\nu_i(\nu_i + 1)}{(jkr)^2} e^{jkr} \qquad r \to \infty$$

$$\sqrt{\frac{\epsilon}{\mu}} \int_o^{\theta_o} e^{jkr \cos \theta} \sin^2 \theta \, P_{\nu'_i}^1 (\cos \theta) \, d\theta = + \sqrt{\frac{\epsilon}{\mu}} \frac{\nu'_i(\nu'_i+1)}{(jkr)^2} e^{jkr} \qquad r \to \infty \tag{12}$$

When equations (12) are combined with equations (9) and replacing j_ν with its asymptotic form the result is

$$\frac{e^{jkr}}{(jkr)^2} \doteq \frac{1}{r} B_i \frac{1}{2kr}\left[e^{j\left(kr-\frac{\nu_i+1}{2}\pi\right)} + e^{-j\left(kr-\frac{\nu_i+1}{2}\pi\right)}\right]$$

$$\cdot \int_o^{\theta_o} \left[P_{\nu_i}^{1}(\cos\theta)\right]^2 \sin\theta \, d\theta \quad ,$$

$$\sqrt{\frac{\epsilon}{\mu}} \frac{e^{jkr}}{(jkr)^2} \doteq \frac{1}{r} C_i \frac{1}{2kr}\left[e^{j\left(kr-\frac{\nu'_i+1}{2}\pi\right)} + e^{-j\left(kr-\frac{\nu'_i+1}{2}\pi\right)}\right] \tag{13}$$

$$\cdot \int_o^{\theta_o} \left[P_{\nu'_i}^{1}(\cos\theta)\right]^2 \sin\theta \, d\theta$$

Reducing equations (13) gives

$$\frac{-e^{jkr}}{k} \doteq \frac{B_i}{2}\left[e^{jkr} e^{-j\frac{\nu_i+1}{2}\pi} + e^{-jkr} e^{+j\frac{\nu_i+1}{2}\pi}\right]$$

$$\cdot \int_o^{\theta_o} \left[P_{\nu_i}^{1}(\cos\theta)\right]^2 \sin\theta \, d\theta \quad ,$$

$$-\sqrt{\frac{\epsilon}{\mu}} \frac{e^{jkr}}{k} \doteq \frac{C_i}{2}\left[e^{jkr} e^{-j\frac{\nu'_i+1}{2}\pi} + e^{-jkr} e^{+j\frac{\nu'_i+1}{2}\pi}\right]$$

$$\cdot \int_o^{\theta_o} \left[P_{\nu'_i}^{1}(\cos\theta)\right]^2 \sin\theta \, d\theta$$

$$r \to \infty \tag{14}$$

Here in equations (14) the incident parts are identified by observing the terms with r dependence. The left-hand side is a function of e^{jkr} and is strictly due to the incident wave. The bracket on the right contains two terms e^{jkr}, which compares with the left-hand side and therefore is due to the incident part, and e^{-jkr} which is due to the reflected part.

Equating only the incident parts of equations (14) gives an accurate statement in the limit as $r \to \infty$. Thus the expansion coefficients B_i and C_i are found.

$$B_i = \frac{-2e^{j(\nu_i+1)\frac{\pi}{2}}}{k\int_0^{\theta_o}\left[P^1_{\nu_i}(\cos\theta)\right]^2 \sin\theta\, d\theta}$$

$$C_i = \frac{-2\sqrt{\frac{\epsilon}{\mu}}\; e^{j(\nu'_i+1)\frac{\pi}{2}}}{k\int_0^{\theta_o}\left[P^1_{\nu'_i}(\cos\theta)\right]^2 \sin\theta\, d\theta} \tag{15}$$

1.4. Total Field Components

It can be shown that

$$\int_0^{\theta_o}\left[P^m_{\nu_i}(\cos\theta)\right]^2 \sin\theta\, d\theta = -\frac{\sin\theta_o}{2\nu_i+1}\left[\frac{dP^m_{\nu_i}(\cos\theta)}{d\theta}\right]_{\theta_o}\left[\frac{dP^m_{\nu}(\cos\theta_o)}{d\nu}\right]_{\nu_i}$$

$$\int_0^{\theta_o}\left[P^m_{\nu'_i}(\cos\theta)\right]^2 \sin\theta\, d\theta = \frac{\sin\theta_o}{2\nu'_i+1} P^m_{\nu'_i}(\cos\theta_o)\left[\frac{\partial^2 P^m_{\nu}(\cos\theta)}{\partial\theta\,\partial\nu}\right]_{\substack{\theta=\theta_o\\ \nu=\nu'_i}} \tag{16}$$

In equation (15) m = 1. In order to facilitate the notation, new definitions will be given for the following constants. These definitions will be used throughout the remainder of this work.

$$
\begin{aligned}
Y &= \sqrt{\frac{\epsilon}{\mu}} \\
U_{mi} &= \frac{2\nu_i+1}{\left[\dfrac{d\,P_\nu^m(\cos\theta_o)}{d\nu}\right]_{\nu_i}} \\
U'_{mi} &= \frac{2\nu'_i+1}{\left[\dfrac{\partial^2 P_\nu^m(\cos\theta)}{\partial\theta\,\partial\nu}\right]_{\substack{\theta=\theta_o \\ \nu=\nu'_i}}} \\
V_i &= \nu_i(\nu_i+1) \\
V'_i &= \nu'_i(\nu'_i+1) \\
W_{mi} &= \frac{U_{mi}}{\left[\dfrac{d}{d\theta} P_{\nu_i}^m(\cos\theta)\right]_{\theta_o} \sin\theta_o} \\
W'_{mi} &= \frac{U'_{mi}}{P_{\nu'_i}^m(\cos\theta_o)\sin\theta_o}
\end{aligned}
\qquad (17)
$$

With substitutions from equations (16) and (17) equations (15) become

$$B_i = 2e^{j(\nu_i+1)\frac{\pi}{2}} W_{1i}/k$$

$$C_i = -2Ye^{j(\nu'_i+1)\frac{\pi}{2}} W'_{1i}/k$$

The total field components can now be found using these coefficients in equations (4). The total field components for the illuminated infinite cone are

$$E_r = \frac{-2\sin\phi}{kr}\sum_{i=0}^{\infty} V_i W_{1i}\, e^{j(\nu_i+1)\frac{\pi}{2}}\, j_{\nu_i}(kr)\, P^1_{\nu_i}(\cos\theta)$$

$$E_\theta = \frac{-2\sin\phi}{kr\sin\theta}\sum_{i=0}^{\infty} \sin\theta\, W_{1i}\, e^{j(\nu_i+1)\frac{\pi}{2}} \frac{d}{dr}\left[r j_{\nu_i}(kr)\right]$$

$$\cdot \frac{d}{d\theta} P^1_{\nu_i}(\cos\theta) + kr\, W'_{1i}\, e^{j\nu'_i\frac{\pi}{2}}\, j_{\nu'_i}(kr) P^1_{\nu'_i}(\cos\theta)$$

$$E_\phi = \frac{-2\cos\phi}{kr\sin\theta}\sum_{i=0}^{\infty} W_{1i}\, e^{j(\nu_i+1)\frac{\pi}{2}} \frac{d}{dr}\left[r j_{\nu_i}(kr)\right] P^1_{\nu_i}(\cos\theta)$$

$$- kr\sin\theta\, W'_{1i}\, e^{j\nu'_i\frac{\pi}{2}}\, j_{\nu'_i}(kr) \frac{d}{d\theta} P^1_{\nu'_i}(\cos\theta)$$

$$H_r = \frac{2Y\cos\phi}{kr} \sum_{i=0}^{\infty} V'_i W'_{1i}\, e^{j(\nu'_i+1)\frac{\pi}{2}}\, j_{\nu'_i}(kr)\, P^1_{\nu'_i}(\cos\theta)$$

$$H_\theta = \frac{2Y\cos\phi}{kr\sin\theta} \sum_{i=0}^{\infty} \sin\theta\, W'_{1i}\, e^{j(\nu'_i+1)\frac{\pi}{2}}\, \frac{d}{dr}\left[r j_{\nu'_i}(kr)\right]$$

$$\cdot \frac{d}{d\theta} P^1_{\nu'_i}(\cos\theta) + kr\, W_{1i}\, e^{j\nu_i\frac{\pi}{2}}\, j_{\nu_i}(kr)\, P^1_{\nu_i}(\cos\theta)$$

$$H_\phi = -\frac{2Y\sin\phi}{kr\sin\theta} \sum_{i=0}^{\infty} W'_{1i}\, e^{j(\nu'_i+1)\frac{\pi}{2}}\, \frac{d}{dr}\left[r j_{\nu'_i}(kr)\right] P^1_{\nu'_i}(\cos\theta)$$

$$+ kr\sin\theta\, W'_{1i}\, e^{j\nu_i\frac{\pi}{2}}\, j_{\nu_i}(kr)\, \frac{d}{d\theta} P^1_{\nu_i}(\cos\theta)$$

(18)

For the distant field components with $r \to \infty$ the asymptotic form may be substituted for $j_\nu(kr)$ and $h_\nu^{(2)}(kr)$ as follows.

$$j_\nu(kr) \to \frac{1}{kr}\cos\left(kr - \frac{(\nu+1)\pi}{2}\right) \qquad r \to \infty \qquad (10)$$

$$h_\nu^{(2)}(kr) \to e^{j(\nu+1)\frac{\pi}{2}}\, \frac{e^{-jkr}}{kr}$$

CHAPTER II

RADIATION FROM AN APERTURE ON A CONDUCTING CONE

2.1. Procedure

In this chapter the field equations due to a specified electric field intensity over an arbitrary portion of the surface of a conducting cone are sought.

To begin with a point generator is assumed which occupies no area and lies on the surface of the cone. After the fields due to the point generator are found they are used as differential elements of the fields due to the active aperture area and integration yields the fields due to the entire aperture. The point generator $\overline{E}_o$ is located at (r', θ_o, ϕ') and consists of components tangential to the cone surface.

$$E_{o\theta} = 0$$

2.2. Boundary Conditions

The cone is considered as an ideal conductor and thus the homogeneous boundary conditions still apply, and require that except for (r', θ_o, ϕ')

$$E_r = E_\phi = 0 \qquad \text{at } \theta = \theta_o$$

To represent the fields due to this point generator both TM and TE modes are used. The scalar potentials from equations (25) and (34) in Appendix I are or

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$$\Pi = \sum_{m} \sum_{\nu} z_{\nu}(kr)\, P_{\nu}^{m}(\cos\theta) \left[A_{m\nu} \cos m\phi + B_{m\nu} \sin m\phi\right]$$

$$\Pi^{*} = \sum_{m} \sum_{\nu'} z_{\nu'}(kr)\, P_{\nu'}^{m}(\cos\theta) \left[C_{m\nu'} \cos m\phi + D_{m\nu'} \sin m\phi\right] \tag{19}$$

As in the case of the illuminated cone the boundary conditions on the surface of the cone again require that

$$P_{\nu_i}^{m}(\cos\theta_o) = 0$$

$$\frac{d\, P_{\nu'_i}^{m}(\cos\theta_o)}{d\theta} = 0 \tag{20}$$

Thus if m = 1 ν_i and ν'_i are defined the same as in Chapter I. In Appendix II it is shown that ν_i, $i = 0, 1, 2, \ldots$ are a different set for each m. For this reason there should be two subscripts on ν as ν_{mi} when the boundary conditions (20) are used. The m subscript will be dropped however for the sake of simpler notation. When these ν_i are to be found it must be known to which m they belong.

In the case considered in this chapter there is no incident field and therefore the radiation condition is satisfied. That is, for large r the outgoing wave approaches a spherical wave with no radial component and the r dependence becomes

$$\left. \frac{e^{-jkr}}{kr} \right|_{r \to \infty}$$

2.3. Lorentz Reciprocity Theorem

The exterior of the cone is divided into two regions as shown in Figure 3

$$\begin{aligned} &\text{Region I :} && r \leq r_o \\ &\text{Region II :} && r > r_o \end{aligned} \tag{21}$$

In statement (21) above r_o is assumed less than r'. Since the origin is in region I the radial function $z_\nu(kr)$ is replaced with $j_\nu(kr)$ and in region II the radiation condition requires the use of $h_\nu^{(2)}(kr)$.

The expansion coefficients will be different in each region.

Region I is bounded by surfaces S_0 and S_1 and region II is bounded by S_0, S_2, and S_∞, the infinite sphere.

Now the Lorentz reciprocity theorem is applied to region II which is

$$\oint_S (\overline{E}_1 \times \overline{H}_2) \cdot d\overline{S} = \oint_S (\overline{E}_2 \times \overline{H}_1) \cdot d\overline{S} \tag{22}$$

In equation (22) the integral is over the entire bounding surface, and $d\overline{S}$ is an element of that surface in the outward normal direction.

$$S = S_0 + S_2 + S_\infty \tag{23}$$

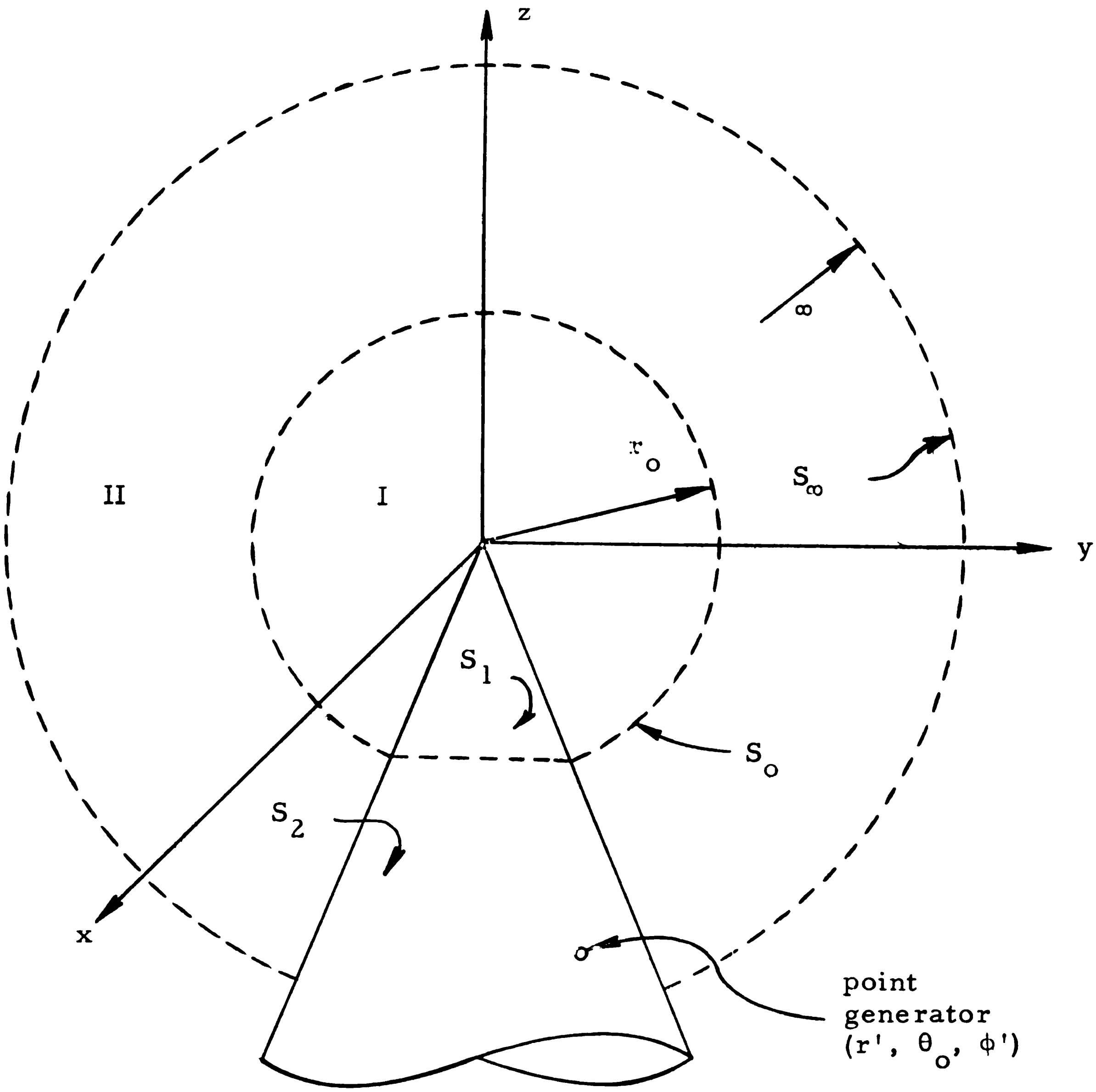

Figure 3. A point generator on the cone surface.

Equation (22) with (23) becomes six integrals

$$\int_{S_0} (\overline{E}_1 \times \overline{H}_2) \cdot (-\hat{r})\, dS_0 + \int_{S_2} (\overline{E}_1 \times \overline{H}_2) \cdot (\hat{\theta})\, dS_2$$

$$+ \int_{S_\infty} (\overline{E}_1 \times \overline{H}_2) \cdot (\hat{r})\, dS_\infty = \int_{S_0} (\overline{E}_2 \times \overline{H}_1) \cdot (-\hat{r})\, dS_0$$

$$+ \int_{S_2} (\overline{E}_2 \times \overline{H}_1) \cdot (\hat{\theta})\, dS_2 + \int_{S_\infty} (\overline{E}_2 \times \overline{H}_1) \cdot (\hat{r})\, dS_\infty \tag{24}$$

Performing the triple products in equation (24) yields

$$\int_{S_0} (E_{1\phi} H_{2\theta} - E_{1\theta} H_{2\phi})\, dS_0 + \int_{S_2} (E_{1\phi} H_{2r} - E_{1r} H_{2\phi})\, dS_2$$

$$+ \int_{S_\infty} (E_{1\theta} H_{2\phi} - E_{1\phi} H_{2\theta})\, dS_\infty = \int_{S_0} (E_{2\phi} H_{1\theta} - E_{2\theta} H_{1\phi})\, dS_0$$

$$+ \int_{S_2} (E_{2\phi} H_{1r} - E_{2r} H_{1\phi})\, dS_2 + \int_{S_\infty} (E_{2\theta} H_{1\phi} - E_{2\phi} H_{1\theta})\, dS_\infty \tag{25}$$

At this point the fields $\overline{E}_1$ and $\overline{H}_1$ are considered to be the actual fields due to the point generator while $\overline{E}_2$ and $\overline{H}_2$ are the fields of one mode with unit coefficient. Referring to equations (19) $\overline{E}_2$ and $\overline{H}_2$ will be due to each of the scalar potentials listed below in turn.

$$z_{\nu_i}(kr)\, P_{\nu_i}^{m}(\cos\theta)\cos m\phi \quad ; \ i = 0, 1, 2, \ldots ; \ m = 0, 1, 2, \ldots$$

$$z_{\nu_i}(kr)\, P_{\nu_i}^{m}(\cos\theta)\sin m\phi \quad ; \ i = 0, 1, 2, \ldots ; \ m = 1, 2, 3, \ldots$$

$$z_{\nu'_i}(kr)\, P_{\nu'_i}^{m}(\cos\theta)\cos m\phi \quad ; \ i = 0, 1, 2, \ldots ; \ m = 0, 1, 2, \ldots$$

$$z_{\nu'_i}(kr)\, P_{\nu'_i}^{m}(\cos\theta)\sin m\phi \quad ; \ i = 0, 1, 2, \ldots ; \ m = 1, 2, 3, \ldots$$

The integration of equation (25) will allow the orthogonal properties of the Legendre and sinusoidal functions to eliminate all but one coefficient. Thus an expression for each coefficient can be found.

In equation (25) it is noticed that two integrals cancel each other. For very large r it is possible to use the relations

$$\left.\begin{aligned} H_{2\phi} &= \sqrt{\frac{\epsilon}{\mu}}\, E_{2\theta} & \qquad H_{2\theta} &= -\sqrt{\frac{\epsilon}{\mu}}\, E_{2\phi} \\ H_{1\phi} &= \sqrt{\frac{\epsilon}{\mu}}\, E_{1\theta} & \qquad H_{1\theta} &= -\sqrt{\frac{\epsilon}{\mu}}\, E_{1\phi} \end{aligned}\right\} \quad r \to \infty \tag{26}$$

The integrals over the S_∞ surface become equal when equations (26) are used and they are dropped, leaving

$$\int_{S_0} (E_{1\theta}H_{2\phi} - E_{1\phi}H_{2\theta})\, dS_0 - \int_{S_0} (E_{2\theta}H_{1\phi} - E_{2\phi}H_{1\theta})\, dS_0 = K \tag{27}$$

where K is defined as

$$K = \int_{S_2} (E_{1\phi}H_{2r} - E_{1r}H_{2\phi})\, dS_2 - \int_{S_2} (E_{2\phi}H_{1r} - E_{2r}H_{1\phi})\, dS_2 \tag{28}$$

Because of the manner in which ν_i and ν'_i were chosen $E_{2r} = E_{2\phi} = 0$ on surface S_2 making the second integral in equation (28) zero. For the same reason E_{1r} and $E_{1\phi}$ are zero over all of S_2 except at the point generator where they are E_{0r} and $E_{0\phi}$ which will be expressed as a Dirac delta function. The eigenvalues ν_i and ν'_i were chosen to make the tangential components of electric field intensity zero on the cone surface but a discontinuity exists in the scalar potentials at the point generator making it possible for the integrals to be non-zero. The value of K may be found by integrating only over the source point since the integrand is zero elsewhere on S_2. Thus equation (28) becomes

$$K = - \int_{\text{source}} (E_{1r}H_{2\phi} - E_{1\phi}H_{2r})\, r \sin\theta d\phi dr \tag{29}$$

2.4. Evaluation of the Unknown Coefficients, Region I

Equation (27) will be used to evaluate the expansion coefficients. It is convenient to break this equation into parts, defining K_1, K_2, K_3, and K_4 as follows.

$$
\begin{aligned}
K_1 &= \int_{S_0} E_{1\theta} H_{2\phi} dS_0 \\
K_2 &= -\int_{S_0} E_{1\phi} H_{2\theta} dS_0 \\
K_3 &= -\int_{S_0} E_{2\theta} H_{1\phi} dS_0 \\
K_4 &= \int_{S_0} E_{2\phi} H_{1\theta} dS_0
\end{aligned}
\tag{30}
$$

Thus equation (27) becomes

$$K = K_1 + K_2 + K_3 + K_4 \tag{31}$$

The components of $\overline{E}_1$ and $\overline{H}_1$ are expressed in the most general terms and are taken from equations (35) in Appendix I with the radial function $j_\nu(kr)$ replacing $z_\nu(kr)$.

2.5. The TM Modes, Even Terms

In order to find each coefficient $\overline{E}_2$ and $\overline{H}_2$ are considered to be due to one mode having unit coefficient. For example to find a particular $A_{m'\ell}$ the scalar potentials Π and Π^* are chosen as

$$\Pi = h_{\nu_\ell}^{(2)}(kr)\, P_{\nu_\ell}^{m'}(\cos\theta)\cos m'\phi$$

$$\Pi^* = 0 \tag{32}$$

Using equations (32) in equations (33) in Appendix I gives the components of $\overline{E}_2$ and $\overline{H}_2$ as follows.

$$E_{2r} = \frac{V_\ell}{r}\, h_{\nu_\ell}^{(2)}(kr)\, P_{\nu_\ell}^{m'}(\cos\theta)\cos m'\phi$$

$$E_{2\theta} = \frac{1}{r}\frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}(kr)\right]\frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}(\cos\theta)\right]\cos m'\phi$$

$$E_{2\phi} = -m'\frac{1}{r}\frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}(kr)\right]\frac{1}{\sin\theta}\, P_{\nu_\ell}^{m'}(\cos\theta)\sin m'\phi$$

$$H_{2r} = 0$$

$$H_{2\theta} = -j\omega\epsilon\, m'\, h_{\nu_\ell}^{(2)}(kr)\frac{1}{\sin\theta}\, P_{\nu_\ell}^{m'}(\cos\theta)\sin m'\phi$$

$$H_{2\phi} = -j\omega\epsilon\, h_{\nu_\ell}^{(2)}(kr)\frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}(\cos\theta)\right]\cos m'\phi \tag{33}$$

In the manner just described equations (30) become

$$K_1 = \int_0^{2\pi}\int_0^{\theta_o}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\left\{-j\omega\epsilon\, r_o\, h_{\nu_\ell}^{(2)}(kr_o)\frac{d}{dr}\left[rj_{\nu_i}\right]_{r_o}\right.$$

$$\cdot\frac{d}{d\theta}\left[P_{\nu_i}^{m}\right]\frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}\right]\sin\theta\left[A_{mi}\cos m\phi + B_{mi}\sin m\phi\right]\cos m'\phi$$

$$+\omega^2\mu\epsilon\, m r_o^2\, h_{\nu_\ell}^{(2)}(kr_o)\, j_{\nu'_i}(kr_o)\, P_{\nu'_i}^{m}\frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}\right]$$

$$\left.\cdot\left[C_{mi}\sin m\phi - D_{mi}\cos m\phi\right]\cos m'\phi\right\}d\theta\, d\phi$$

$$K_2 = \int_0^{2\pi}\int_0^{\theta_o}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\left\{-j\omega\epsilon\, m m'\, r_o\frac{d}{dr}\left[rj_{\nu_i}\right]_{r_o} h_{\nu_\ell}^{(2)}(kr_o)\right.$$

$$\cdot\frac{1}{\sin\theta}P_{\nu_i}^{m}P_{\nu_\ell}^{m'}\left[A_{mi}\sin m\phi - B_{mi}\cos m\phi\right]\sin m'\phi$$

$$-\omega^2\mu\epsilon\, m' r_o^2\, j_{\nu'_i}(kr_o)\, h_{\nu_\ell}^{(2)}(kr_o)\frac{d}{d\theta}\left[P_{\nu'_i}^{m}\right]P_{\nu_\ell}^{m'}$$

$$\left.\cdot\left[C_{mi}\cos m\phi + D_{mi}\sin m\phi\right]\sin m'\phi\right\}d\theta\, d\phi$$

$$K_3 = \int_0^{2\pi}\int_0^{\theta_o} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left\{ m \frac{d}{dr}\left[r h_{\nu_\ell}^{(2)}\right]_{r_o} \frac{d}{dr}\left[r j_{\nu'_i}\right]_{r_o} \right.$$

$$\cdot \frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}\right] P_{\nu'_i}^{m} \left[C_{mi} \sin m\phi - D_{mi} \cos m\phi\right] \cos m'\phi$$

$$+ j\omega\epsilon \; r_o \frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}\right]_{r_o} j_{\nu_i}(kr_o) \sin\theta \frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}\right] \frac{d}{d\theta}\left[P_{\nu_i}^{m}\right]$$

$$\left. \cdot \left[A_{mi} \cos m\phi + B_{mi} \sin m\phi\right] \cos m'\phi \right\} d\theta \, d\phi$$

$$K_4 = \int_0^{2\pi}\int_0^{\theta_o} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left\{ -m' \frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}\right]_{r_o} \frac{d}{dr}\left[r j_{\nu'_i}\right]_{r_o} \right.$$

$$\cdot P_{\nu_\ell}^{m'} \frac{d}{d\theta}\left[P_{\nu'_i}^{m}\right] \left[C_{mi} \cos m\phi + D_{mi} \sin m\phi\right] \sin m'\phi$$

$$+ j\omega\epsilon \; m' m\, r_o \frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}\right]_{r_o} j_{\nu_i}(kr_o) \frac{1}{\sin\theta} P_{\nu_\ell}^{m'} \; P_{\nu_i}^{m}$$

$$\left. \cdot \left[A_{mi} \sin m\phi - B_{mi} \cos m\phi\right] \sin m'\phi \right\} d\theta \, d\phi$$

(34)

2.6. Application of Orthogonality to the Sum

In equations (34) the orthogonal properties of the sine and cosine functions eliminate all the coefficients except $A_{m'i}$ and $D_{m'i}$; $i = 0, 1, 2, \ldots$ and the orthogonal property of the Legendre functions (see Appendix II) eliminate all the $D_{m'i}$ ($m' \neq 0$) and all the $A_{m'i}$ except $A_{m'\ell}$. If $m' = 0$ then $D_{m'\ell}$ is eliminated by the fact that in equations (34) each term containing D_{mi} has a factor either m or m'.

With these changes equations (34) become

$$K_1 = -j\omega\epsilon\, A_{m'\ell}\, r_o\, h_{\nu_\ell}^{(2)}(kr_o) \frac{d}{dr}\left[r\, j_{\nu_\ell} \right]_{r_o}$$

$$\cdot \int_0^{\theta_o} \left\{ \frac{d}{d\theta}\left[P_{\nu_\ell}^{m'}(\cos\theta) \right] \right\}^2 \sin\theta\, d\theta \int_0^{2\pi} \cos^2 m'\phi\, d\phi$$

$$K_2 = -j\omega\epsilon\, A_{m'\ell}\, m'^2\, r_o\, h_{\nu_\ell}^{(2)}(kr_o) \frac{d}{dr}\left[r\, j_{\nu_\ell} \right]_{r_o}$$

$$\cdot \int_0^{\theta_o} \frac{1}{\sin\theta}\left[P_{\nu_\ell}^{m'}(\cos\theta) \right]^2 d\theta \int_0^{2\pi} \sin^2 m'\phi\, d\phi$$

$$K_3 = j\omega\epsilon\, A_{m'\ell}\, r_o\, \frac{d}{dr}\left[r h_{\nu_\ell}^{(2)}\right]_{r_o} j_{\nu_\ell}(kr_o)$$

$$\cdot \int_0^{\theta_o} \left[\frac{d}{d\theta}\left(P_{\nu_\ell}^{m'}(\cos\theta)\right)\right]^2 \sin\theta\, d\theta \int_0^{2\pi} \cos^2 m'\phi\, d\phi$$

$$K_4 = j\omega\epsilon\, A_{m'\ell}\, m'^2\, r_o\, \frac{d}{dr}\left[r h_{\nu_\ell}^{(2)}\right]_{r_o} j_{\nu_\ell}(kr_o)$$

$$\cdot \int_0^{\theta_o} \frac{1}{\sin\theta}\left[P_{\nu_\ell}^{m'}(\cos\theta)\right]^2 d\theta \int_\theta^{2\pi} \sin^2 m'\phi\, d\phi$$

(35)

In equations (35) the quantities

$$\int_0^{2\pi} \sin^2 m'\phi\, d\phi \qquad \text{and} \qquad \int_0^{2\pi} \cos^2 m'\phi\, d\phi \tag{36}$$

may be replaced by their equivalent values $\pi(1 - \delta_{om'})$ and $\pi(1 + \delta_{om'})$ respectively where $\delta_{om'}$ is the Krondecker delta

$$\delta_{om'} = \begin{cases} 1 & \text{if } m' = 0 \\ 0 & \text{if } m' \neq 0 \end{cases}$$

Upon examination of the equations (35) it is seen that there is a factor m'^2 where ever the quantity

$$\int_0^{2\pi} \sin^2 m'\phi \, d\phi \qquad \text{is found.}$$

This makes it possible to use $\pi(1 + \delta_{om})$ for both the integrals (36) since for each value of m'

$$m'^2 \pi(1 - \delta_{om}) = m'^2 \pi(1 + \delta_{om})$$

$$m' = 0, 1, 2, \ldots$$

Thus equations (35) may be totaled and simplified to give

$$K = j\omega\,\epsilon\, A_{m'\ell}\, \pi(1 + \delta_{om'}) \left\{ r_o\, j_{\nu_\ell}(kr_o) \frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)}\right]_{r_o} - r_o\, h_{\nu_\ell}^{(2)}(kr_o) \frac{d}{dr}\left[r\, j_{\nu_\ell}\right]_{r_o} \right\} \int_0^{\theta_o} \left\{ \left[\frac{d}{d\theta}\left(P_{\nu_\ell}^{m'}(\cos\theta)\right)\right]^2 + \left[\frac{m'\, P_{\nu_\ell}^{m'}(\cos\theta)}{\sin\theta}\right]^2 \right\} \sin\theta \, d\theta \tag{37}$$

The bracket which contains the r dependence of equation (37) may be expressed as

$$\left\{ r_o\, j_{\nu_\ell}(kr_o) \frac{d}{dr}\left[r\, h_{\nu_\ell}^{(2)} \right]_{r_o} - r_o\, h_{\nu_\ell}^{(2)}(kr_o) \frac{d}{dr}\left[r\, j_{\nu_\ell} \right]_{r_o} \right\}$$

$$= r_o^{\,2} \left[j_{\nu_\ell}(kr) \frac{d}{dr}\left(h_{\nu_\ell}^{(2)} \right) - h_{\nu_\ell}^{(2)}(kr) \frac{d}{dr}\left(j_{\nu_\ell} \right) \right]_{r = r_o}$$

$$= r_o^{\,2}\, W\left(j_{\nu_\ell},\, h_{\nu_\ell}^{(2)} \right) \Bigg|_{r_o} = -\frac{j}{k} \tag{38}$$

where $W\left(j_{\nu_\ell},\, h_{\nu_\ell}^{(2)}\right)$ is the Wronskian of the functions j_{ν_ℓ} and $h_{\nu_\ell}^{(2)}$.

The integral of equation (37) can be reduced as follows (see equation (46) in Appendix II).

$$\int_0^{\theta_o} \left\{ \left[\frac{d}{d\theta} \left[P_{\nu_\ell}^{m'}(\cos\theta) \right] \right]^2 + \left[\frac{m'\, P_{\nu_\ell}^{m'}(\cos\theta)}{\sin\theta} \right]^2 \right\} \sin\theta\, d\theta$$

$$= V_\ell \int_0^{\theta_o} \left[P_{\nu_\ell}^{m'}(\cos\theta) \right]^2 \sin\theta\, d\,\theta$$

$$= \frac{V_\ell}{W_{m'_\ell}} \tag{39}$$

With the substitutions indicated in equations (38) and (39) equation (37) becomes

$$K = A_{m'\ell} \, Y \, \pi(1 + \delta_{om'}) \frac{V_\ell}{W_{m'\ell}} \tag{40}$$

It is convenient to abbreviate equation (40) by defining constants $N_{\nu_i m}$ as

$$N_{\nu_i m} = \pi(1 + \delta_{om}) \frac{V_i}{W_{m'\ell}} \tag{41}$$

With the use of definition (41) the coefficients A_{mi} are found to be

$$A_{m\ell} = \frac{KY}{N_{\nu_i m}}$$

This coefficient is for region I defined by relation (21) as $r \leq r_o$ and represents the even term of the TM mode part of the field set up by the point generator. This can be seen from the fact that Π is an even function of ϕ and it produces no r component of magnetic field intensity (see equations (32) and (33)).

To evaluate K the same set of fields defined by equations (33) should be used, and since in these equations $H_{2r} = 0$ the K becomes

$$K = \int_{\text{source}} - H_{2\phi} \, E_{1r} \, \sin\theta \, d\phi \, dr$$

At this point it becomes clear that the ϕ component of the E field of the point generator does not maintain a TM mode field.

The source point is located at (r', θ_o, ϕ') and if the Dirac delta function for E_{1r} is defined as

$$E_{1r} = \frac{E_{or}\ \delta(r - r')\ \delta(\phi - \phi')}{r \sin\theta_o}$$

and $H_{2\phi}$ is taken from equations (33), then K becomes

$$K = \int_{\text{source}} j\omega\epsilon E_{or} \frac{\delta(r-r')\ \delta(\phi-\phi')}{r \sin\theta_o} h_{\nu_i}^{(2)}(kr) \frac{d}{d\theta}\left[P_{\nu_i}^{m}(\cos\theta)\right]_{\theta_o}$$

$$\cdot\ \cos m\phi\ r \sin\theta\ d\phi\, dr \tag{42}$$

When the integration in equation (42) is carried out the result is

$$K = j\omega\epsilon\ E_{or}\ h_{\nu_i}^{(2)}(kr') \frac{d}{d\theta}\left[P_{\nu_i}^{m}(\cos\theta)\right]_{\theta_o} \cos m\phi' \tag{43}$$

The final expression for A_{mi} for region I is found by using equations (43) and (41).

$$A_{mi}^{I} = \frac{jk\ E_{or}\ U_{mi}\ h_{\nu_i}^{(2)}(kr')\ \cos m\phi'}{\pi(1 + \delta_{om})\ V_i} \tag{44}$$

The definitions for U_{mi}, V_i and W_{mi} were given in definitions (17), Section 1.4.

2.7. Remaining Coefficients

The procedure for finding the remaining coefficients is precisely the same as it was for A_{mi} and it is not necessary to show all the steps involved for each of the other three coefficients. For comparison purposes the four resulting expressions for K are shown as follows

$$K_{A_{mi}} = j\omega\epsilon\, A_{mi}\, \pi\, (1+\delta_{om}) r_o^2 W\left(j_{\nu_i}, h_{\nu_i}^{(2)}\right)_{r_o}$$

$$\cdot \int_0^{\theta_o} \left\{ \left[\frac{d\, P_{\nu_i}^m}{d\theta}\right]^2 + \left[\frac{m\, P_{\nu_i}^m}{\sin\theta}\right]^2 \right\} \sin\theta\, d\theta$$

$$K_{B_{mi}} = j\omega\epsilon\, B_{mi} \pi\, r_o^2\, W\left(j_{\nu_i}, h_{\nu_i}^{(2)}\right)_{r_o}$$

$$\cdot \int_0^{\theta_o} \left\{ \left[\frac{d\, P_{\nu_i}^m}{d\theta}\right]^2 + \left[\frac{m\, P_{\nu_i}^m}{\sin\theta}\right]^2 \right\} \sin\theta\, d\theta$$

$$K_{C_{mi}} = -j\omega\mu\, C_{mi} \pi\, (1+\delta_{om})\, r_o^2\, W\left(j_{\nu'_i}, h_{\nu'_i}^{(2)}\right)_{r_o}$$

$$\cdot \int_0^{\theta_o} \left\{ \left[\frac{d\, P_{\nu'_i}^m}{d\theta}\right]^2 + \left[\frac{m\, P_{\nu'_i}^m}{\sin\theta}\right]^2 \right\} \sin\theta\, d\theta$$

$$K_{D_{mi}} = -j\omega\mu\, D_{mi} \pi\, r_o^2\, W\left(j_{\nu'_i}, h_{\nu'_i}^{(2)}\right)_{r_o}$$

$$\cdot \int_0^{\theta_o} \left\{ \left[\frac{d\, P_{\nu'_i}^m}{d\theta}\right]^2 + \left[\frac{m\, P_{\nu'_i}^m}{\sin\theta}\right]^2 \right\} \sin\theta\, d\theta \qquad (45)$$

If may be observed that in equations (45) the factor $(1 + \delta_{om})$ is not in the equations for $K_{B_{mi}}$ and $K_{D_{mi}}$. The reason for this is that B_{mi} and D_{mi} need not be defined for $m = 0$ since in the expressions for the field components, equations (35) Appendix I, each of these coefficients are multiplied by either m or $\sin m\phi$ and both of these are zero when $m = 0$. However for the sake of uniformity in the equations the factor $(1 + \delta_{om})$ will be included. This will not change the values of the coefficients for $m = 0$ but will give the convenient relations

$$\left.\begin{aligned} \frac{K_{A_{mi}}}{A_{mi}} &= \frac{K_{B_{mi}}}{B_{mi}} \\ \\ \frac{K_{C_{mi}}}{C_{mi}} &= \frac{K_{D_{mi}}}{D_{mi}} \end{aligned}\right\} \quad m = 0, 1, 2, \ldots \tag{46}$$

It is also seen in equations (45) that the first two equations as a pair are similar in form to the last two. If in the last two the μ factors were replaced with $(-\epsilon)$ and ν'_i with ν_i, then the forms would be the same. This change in ν_i would not affect the values of

$$W\,(j_{\nu'_i}\,,\ h_{\nu'_i}^{(2)}\,)$$

but it does change the values of the integrals. The values of the integrals of equations (45) can be shown to be

$$\int_0^{\theta_o} \left\{ \left[\frac{d P_{\nu_i}^m}{d\theta}\right]^2 + \left[\frac{m P_{\nu_i}^m}{\sin\theta}\right]^2 \right\} \sin\theta \, d\theta$$

$$= \nu_i (\nu_i + 1) \int_0^{\theta_o} \left[P_{\nu_i}^m (\cos\theta)\right]^2 \sin\theta \, d\theta$$

$$= \frac{V_i}{W_{mi}} \tag{47}$$

$$\int_0^{\theta_o} \left\{ \left[\frac{d P_{\nu'_i}^m}{d\theta}\right]^2 + \left[\frac{m P_{\nu'_i}}{\sin\theta}\right]^2 \right\} \sin\theta \, d\theta$$

$$= \nu'_i (\nu'_i + 1) \int_0^{\theta_o} \left[P_{\nu'_i}^m (\cos\theta)\right]^2 \sin\theta \, d\theta$$

$$= \frac{V'_i}{W'_{mi}} \tag{48}$$

Because of the similarity between the first two equations in (45) it is possible to make them into a single combined expression. If the pair A_{mi} and B_{mi} are designated as

$$\left.\begin{matrix} A_{mi}^{I} \\ B_{mi}^{I} \end{matrix}\right\} = A_{mi\,{}^e_o}^{I}$$

where the "e" and the "o" indicate that the coefficient is associated with the cos $m\phi$ and sin $m\phi$ which are even and odd functions respectively, then the expression for the scalar potential Π from equation (25) in Appendix I becomes

$$\Pi^{I} = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} A^{I}_{mi\substack{e\\o}} \, j_{\nu_i}(kr) \, P^{m}_{\nu_i}(\cos\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \qquad (49)$$

where

$$A^{I}_{mi\substack{e\\o}} = \frac{jk\, E_{or}\, U_{mi}\, h^{(2)}_{\nu_i}(kr') \begin{Bmatrix} \cos m\phi' \\ \sin m\phi' \end{Bmatrix}}{(1+\delta_{om})\, V_i \sin\theta_o} \qquad (50)$$

In equation (50) the bracket

$$\begin{Bmatrix} \cos m\phi' \\ \sin m\phi' \end{Bmatrix}$$

is due to the fact that the integral over the source point corresponding to equation (42) becomes

$$K_{A_{mi\substack{e\\o}}} = \int_{\text{source}} j\omega\epsilon\, E_{or}\, \delta(r - r')\, \delta(\phi - \phi')\, h^{(2)}_{\nu_i}(kr)$$

$$\cdot \frac{d}{d\theta} \left[P^{m}_{\nu_i}(\cos\theta) \right]_{\theta = \theta_o} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \, d\phi\, dr$$

(51)

2.8. The TE Modes

If C_{mi} and D_{mi} are treated in a similar manner as A_{mi} and B_{mi} were they will be combined as one expression

$$\left.\begin{matrix} C_{mi}^{I} \\ \\ D_{mi}^{I} \end{matrix}\right\} = C_{mi\,e_o}^{I}$$

and the resulting expression for $K_{C_{mi\,e_o}}$ is found by combining the last two equations of (45) to give

$$K_{C_{mi\,e_o}} = C_{mi\,e_o} \; Y \pi (1 + \delta_{om}) \; \frac{V'_i}{W'_{mi}} \tag{52}$$

The constants $K_{C_{mi\,e_o}}$ are also found by integrating over the source point as indicated by equation (29)

$$K = \int_{\substack{\text{source}\\\text{point}}} (E_{1\phi} H_{2r} - E_{1r} H_{2\phi}) \; r \sin\theta_o \; d\phi \; d r \tag{29}$$

It was found while working with $K_{A_{mi\,e_o}}$ that equation (29) was reduced by the fact that if $\Pi^* = 0$ then $H_{2r} = 0$. Here however, both components H_{2r} and $H_{2\phi}$ are present and integration of equation (29) gives two parts.

$$K_{C_{mi\,o}^{\,e}} = \frac{E_{o\phi}\; V'_i\; h^{(2)}_{\nu'_i}(kr')\; P^{m}_{\nu'_i}(\cos\theta_o)}{r'} \begin{Bmatrix} \cos m\phi' \\ \sin m\phi' \end{Bmatrix}$$

$$+ \frac{E_{or}\; m \frac{d}{dr}\left[r\, h^{(2)}_{\nu'_i}(kr)\right]_{r'} P^{m}_{\nu'_i}(\cos\theta_o)}{r' \sin\theta_o} \begin{Bmatrix} \sin m\phi' \\ -\cos m\phi' \end{Bmatrix} \tag{53}$$

This implies that both ϕ and r components of the E field of the point generator contribute to maintain a TE mode of wave.

Equating the right hand sides of equations (52) and (53) and solving for $C^{I}_{mi\,o}{}^{e}$ gives

$$C^{I}_{mi\,o}{}^{e} = \frac{E_{or}\; m\; Y\; U'_{mi} \frac{d}{dr}\left[r\, h^{(2)}_{\nu'_i}(kr)\right]_{r'}}{r' \sin^2\theta_o\; V'_i\; \pi(1+\delta_{om})} \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

$$+ \frac{E_{o\phi}\; Y\; U'_{mi}\; h^{(2)}_{\nu'_i}(kr')}{r' \sin\theta_o\; \pi(1+\delta_{om})} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \tag{54}$$

2. 9. Determination of Expansion Coefficients, Region II

The expressions for the expansion coefficients for region I are now found, and next the expressions for the coefficients for region II are to be sought. Before they are, however, it will be convenient to let the value of r_o become equal to r' making the point source on the boundary of the two regions. Since r_o does not appear in the expression for the coefficients they will be unaffected.

In order to determine the coefficients for region II it is only necessary to apply the condition of continuity of the scalar potentials at the boundary between the two regions. This condition is stated as

$$\left.\begin{aligned} \Pi^{I} &= \Pi^{II} \\ \Pi^{*I} &= \Pi^{*II} \end{aligned}\right\} \quad r = r' \tag{55}$$

The only way in which condition (55) can be fulfilled for all values of θ and ϕ is to make the coefficients satisfy the conditions

$$\begin{aligned} A^{I}_{mi\,{e \atop o}}\, j_{\nu_i}(kr') &= A^{II}_{mi\,{e \atop o}}\, h^{(2)}_{\nu_i}(kr') \\ C^{I}_{mi\,{e \atop o}}\, j_{\nu'_i}(kr') &= C^{II}_{mi\,{e \atop o}}\, h^{(2)}_{\nu'_i}(kr') \end{aligned} \tag{56}$$

Solving equations (56) for $A^{II}_{mi\,{e \atop o}}$ and $C^{II}_{mi\,{e \atop o}}$ and substituting for $A^{I}_{mi\,{e \atop o}}$ and $C^{I}_{mi\,{e \atop o}}$ from equations (50) and (54) gives the expressions

$$A^{II}_{mi\,{e \atop o}} = \frac{jk\,E_{or}\,U_{mi}\,j_{\nu_i}(kr')}{\pi\,(1+\delta_{om})\,V_i\,\sin\theta_o} \begin{Bmatrix} \cos m\phi' \\ \sin m\phi' \end{Bmatrix} \tag{57}$$

$$\begin{aligned} C^{II}_{mi\,{e \atop o}} = {} & \frac{E_{or}\,mYU'_{mi}\,j_{\nu'_i}(kr')\,\frac{d}{dr}\left[r\,h^{(2)}_{\nu'_i}(kr)\right]_{r'}}{\pi\,(1+\delta_{om})\,V'_i\,\sin^2\theta_o\,r'\,h^{(2)}_{\nu'_i}(kr')} \begin{Bmatrix} \sin m\phi' \\ -\cos m\phi' \end{Bmatrix} \\ & + \frac{E_{o\phi}\,Y\,U'_{mi}\,j_{\nu'_i}(kr')}{\pi\,(1+\delta_{om})\,r'\,\sin\theta_o} \begin{Bmatrix} \cos m\phi' \\ \sin m\phi' \end{Bmatrix} \end{aligned} \tag{58}$$

2.10. Scalar Potentials Due to the Point Source

Substitution of equations (50), (54), (57), and (58) into equation (25) and (34) in Appendix I for Π and Π^* gives

$$\Pi = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{jk\, E_{or}\, U_{mi}\, j_{\nu_i}(kr_<)\, h_{\nu_i}^{(2)}(kr_>)\, P_{\nu_i}^{m}(\cos\theta) \cos m(\phi - \phi')}{\pi\,(1+\delta_{om})\, V_i \sin\theta_o} \tag{59}$$

$$\Pi^* = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} - \frac{E_{or}\, mYU'_{mi} \frac{d}{dr}\left[r\, h_{\nu'_i}^{(2)}(kr)\right]_{r'} j_{\nu'_i}(kr_<)\, h_{\nu'_i}^{(2)}(kr_>)}{\pi\,(1+\delta_{om})\, V'_i \sin^2\theta_o\, r'\, h_{\nu'_i}^{(2)}(kr')}$$

$$\cdot\; P_{\nu'_i}^{m}(\cos\theta) \sin m(\phi - \phi')$$

$$+ \frac{E_{o\phi}\, YU'_{mi}\, j_{\nu'_i}(kr_<)\, h_{\nu'_i}^{(2)}(kr_>)\, P_{\nu'_i}^{m}(\cos\theta) \cos m(\phi - \phi')}{\pi\,(1+\delta_{om})\, r' \sin\theta_o} \tag{60}$$

In equations (59) and (60) the notations $r_<$ and $r_>$ have been introduced which are defined as follows. The quantity $r_<$ is the lesser of the two quantities r and r' while $r_>$ is the greater of the two.

2.11. The Non-Infinitesimal Active Source

With the finding of the completed expressions for Π and Π^* the field components due to a point generator at (r', θ_o, ϕ') may be found from equations (33) in Appendix I. If however, instead of a point source the fields are due to a specified electric field intensity over an arbitrary area on the surface of the cone, then the Π^* and Π of equations (59) and (60) are used as differential elements of the total scalar potential and integration yields the total fields. If the field over the active area is expressed as

$$\overline{E}_o(r'\phi') = E_{or}(r'\phi')\,\hat{r} + E_{o\phi}(r', \phi')\,\hat{\phi} \tag{61}$$

then the final expression for the total scalar potentials Π^* and Π are as follows.

$$\Pi^* = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{mY\,U'_{mi}\,P_{\nu'_i}^{m}(\cos\theta)}{\pi(1+\delta_{om})\,V'_i \sin\theta_o}$$

$$\cdot \iint_{\substack{\text{Active}\\ \text{Area}}} \frac{E_{or}(r', \phi')\sin m(\phi'-\phi)\frac{d}{dr'}\left[r' h_{\nu'_i}^{(2)}(kr')\right] h_{\nu'_i}^{(2)}(kr_>)\,j_{\nu'_i}(kr_<)\,dr'd\phi'}{h_{\nu'_i}^{(2)}(kr')}$$

$$+ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{Y\,U'_{mi}\,P_{\nu'_i}^{m}(\cos\theta)}{\pi(1+\delta_{om})}$$

$$\cdot \iint_{\substack{\text{Active}\\ \text{Area}}} E_{o\phi}(r', \phi')\cos m(\phi'-\phi)\,j_{\nu'_i}(kr_<)\,h_{\nu'_i}^{(2)}(kr_>)\,dr'd\phi'$$

$$\Pi = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{jk\, U_{mi}\, P_{\nu_i}^{m}(\cos\theta)}{\pi(1+\delta_{om})\, V_i}$$

$$\cdot \iint_{\text{Source}} E_{or}(r', \phi') \cos m(\phi' - \phi)\, j_{\nu_i}(kr_<)\, h_{\nu_i}^{(2)}(kr_>)\, r' dr' d\phi' \qquad (62)$$

2.12. Field Components due to a Specified Electric Field Intensity in an Aperture on a Cone.

$$E_r(r, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{jk\, U_{mi} P_{\nu_i}^{m}(\cos\theta)}{\pi(1+\delta_{om}) r} \iint_{\text{Source}} E_{or}(r', \phi') \cos m(\phi' - \phi)$$

$$\cdot\; r' j_{\nu_i}(kr_<)\, h_{\nu_i}^{(2)}(kr_>)\, dr'\, d\phi'$$

$$E_\theta(r, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{jk}{\pi(1+\delta_{om}) r} \left\{ \frac{U_{mi} \frac{d}{d\theta}\left[P_{\nu_i}^{m}(\cos\theta)\right]}{V_i} \right.$$

$$\cdot \iint_{\text{Source}} E_{or}(r', \phi') \cos m(\phi' - \phi)\, r' \frac{d}{dr}\left[r j_{\nu_i}(kr_<) h_{\nu_i}^{(2)}(kr_>)\right] dr' d\phi'$$

$$+ \frac{m\, U'_{mi} r P_{\nu'_i}^{m}(\cos\theta)}{\sin\theta \sin\theta_o\, V'_i} \iint_{\text{Source}} \frac{j_{\nu'_i}(kr_<)\, h_{\nu'_i}^{(2)}(kr_>)}{h_{\nu'_i}^{(2)}(kr')}$$

$$\cdot \left[E_{or}(r', \phi')\, m \cos m(\phi' - \phi) \frac{d}{dr'}\left(r' h_{\nu'_i}^{(2)}(kr')\right)\right.$$

$$\left.\left. - V'_i E_{o\phi}(r', \phi') \sin m(\phi' - \phi) \sin\theta_o\, h_{\nu'_i}^{(2)}(kr')\right] dr'\, d\phi' \right\}$$

$$E_\phi(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{jk}{\pi(1+\delta_{om}) r \sin\theta \sin\theta_o} \left\{ \frac{mU_{mi} \sin\theta_o P_{\nu_i}^m(\cos\theta)}{V_i} \right.$$

$$\cdot \iint_{\text{Source}} E_{or}(r',\phi') \sin m(\phi'-\phi) r' \frac{d}{dr}\left[r j_{\nu_i}(kr_<) h_{\nu_i}^{(2)}(kr_>)\right] dr'\, d\phi'$$

$$+ rU'_{mi} \sin\theta \frac{d}{d\theta}\left[P_{\nu'_i}^m(\cos\theta)\right] \iint_{\text{Source}} j_{\nu'_i}(kr_<)\, h_{\nu'_i}^{(2)}(kr_>)$$

$$\cdot \left[\frac{E_{or}(r',\phi') m \sin m(\phi'-\phi) \frac{d}{dr'}\left(r' h_{\nu'_i}^{(2)}(kr')\right)}{V'_i h_{\nu'_i}^{(2)}(kr')} \right.$$

$$\left. \left. + E_{o\phi}(r',\phi') \sin\theta_o \cos m(\phi'-\phi) \right] dr'\, d\phi' \right\} \tag{63}$$

$$H_r(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{YU'_{mi} P_{\nu'_i}^m(\cos\theta)}{\pi(1+\delta_{om}) r}$$

$$\cdot \iint_{\text{Source}} j_{\nu'_i}(kr_<) h_{\nu'_i}^{(2)}(kr_>) \left[\frac{mE_{or}(r',\phi) \sin m(\phi'-\phi) \frac{d}{dr'}\left(r' h_{\nu'_i}^{(2)}(kr')\right)}{\sin\theta_o h_{\nu'_i}^{(2)}(kr')} \right.$$

$$\left. + E_{o\phi}(r',\phi') V'_i \cos m(\phi'-\phi) \right] dr'\, d\phi'$$

$$H_\theta(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{Y}{\pi(1+\delta_{om})r} \iint_{\text{Source}} dr'd\phi' \Bigg\{ U'_{mi} \frac{d}{d\theta}\left[P_{\nu'_i}^{m}(\cos\theta)\right]$$

$$\cdot \frac{d}{dr}\left[rj_{\nu'_i}(kr_<)h_{\nu'_i}^{(2)}(kr_>)\right]\left[\frac{mE_{or}(r',\phi')\sin m(\phi'-\phi)\frac{d}{dr'}\left[r'h_{\nu'_i}^{(2)}(kr')\right]}{\sin\theta_o \quad V'_i\, h_{\nu'_i}^{(2)}(kr')}\right.$$

$$\left. + E_{o\phi}(r',\phi')\cos m(\phi'-\phi)\right]$$

$$- \frac{mk^2\, U_{mi}\, rP_{\nu_i}^{m}(\cos\theta)}{\sin\theta \quad V_i} \quad E_{or}(r',\phi')\sin m(\phi'-\phi)r'j_{\nu_i}(kr_<)h_{\nu_i}^{(2)}(kr_>)\Bigg\}$$

$$H_\phi(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} \frac{Y}{\pi(1+\delta_{om})r} \iint_{\text{Source}} dr'd\phi' \Bigg\{ \frac{mU'_{mi}P_{\nu'_i}^{m}(\cos\theta)}{\sin\theta}$$

$$\cdot \frac{d}{dr}\left[rj_{\nu'_i}(kr_<)h_{\nu'_i}^{(2)}(kr_>)\right]\left[E_{o\phi}(r',\phi')\sin m(\phi'-\phi)\right.$$

$$\left. - \frac{mE_{or}(r',\phi')\cos m(\phi'-\phi)\frac{d}{dr'}\left[r'\,h_{\nu'_i}^{(2)}(kr')\right]}{\sin\theta_o\; V_i\, h_{\nu'_i}^{(2)}(kr')}\right]$$

$$+ k^2\, U_{mi}\, rr' \frac{d}{d\theta}\left[P_{\nu_i}^{m}(\cos\theta)\right] E_{or}(r',\phi')\cos m(\phi'-\phi)\, j_{\nu_i}(kr_<)h_{\nu_i}^{(2)}(kr_>)\Bigg\} \tag{64}$$

CHAPTER III

SCATTERING FROM A CONE WITH A CIRCUMFERENTIAL SLOT IMPEDANCE

3.1. Combination of Previous Results

In the Introduction it was shown that the problem of finding an impedance which, when placed on the surface of a cone, would make the nose-on echo vanish, can be divided into parts. In the first chapter the illuminated unaltered cone was analyzed while in the second chapter the cone was analyzed with an active aperture of arbitrary shape.

In this chapter the results will be taken from the first two chapters and they will be combined to give the desired condition using a circumferential slot active area. The equation for the impedance was shown in Equation (1) to be

$$Z = \frac{E_o}{I_R + I_S} \tag{65}$$

The quantities involved in equation (65) are defined as follows:

$\overline{E}_o$ is the aperture field necessary to make

$$\overline{E}_R(\infty, 0, \phi) = -\overline{E}_S(\infty, 0, \phi) \tag{66}$$

$\overline{E}_R$ is the electric field intensity due to E_o on the aperture.

$\overline{E}_S$ is the scattered field due to illumination.

I_S is the surface current across the position of the aperture due to illumination.

I_R is the surface current across the aperture due only to the aperture source field $\overline{E}_o$.

The direction of $\overline{E}_o$ is the same as that of $\overline{I}_S$.

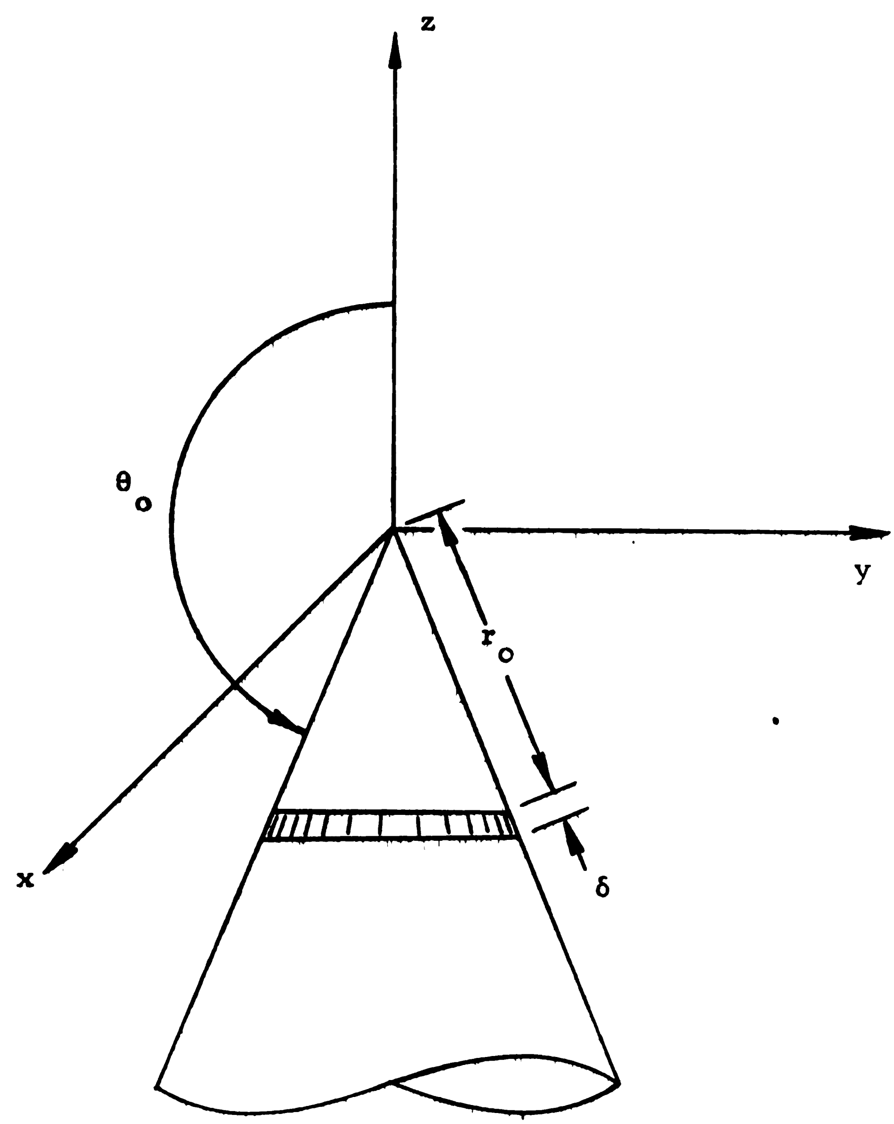

Figure 4. A cone with a circumferential slot.

The surface currents are both found from the tangential components of magnetic field intensity at the location of the aperture. For the circumferential slot the only pertinent current is the radial component at (r', θ_o, ϕ') and this is due to the magnetic field component $-H_\phi(r', \theta_o, \phi')$.

3.2. The Circumferential Slot as an Active Area

Equations (63) and (64) in section 2.12 give the fields due to a specified electric field intensity in an arbitrary active area on the surface of the cone.

Here the area is specified as a slot around the axis of the cone having width δ and located at a distance r_o along a radial line from the apex of the cone.

The equations (63) and (64) for E_θ and H_ϕ require integration over the slot area before they can be used. In this case the slot area is described as

$$\begin{aligned} \phi' &= 0 \text{ to } 2\pi \\ r' &= r_o - \frac{\delta}{2} \quad \text{to } r_o + \frac{\delta}{2} \end{aligned} \tag{67}$$

The field in the gap will be proportional to the surface current due to the illuminating wave. Examination of the equations (18) for H_ϕ shows that the current across the gap is proportional to $\sin\phi$ so it is reasonable to call the voltage across the gap

$$v = v_o \sin\phi'$$

The field $\overline{E}_o$ in the gap accordingly is

$$\overline{E}_o = \frac{v\,r}{\delta} = \frac{v_o \sin\phi' \, r}{\delta} \tag{68}$$

At this point it is convenient to introduce certain new functions which will assist the manipulation of quantities which would otherwise be somewhat cumbersome. These functions are as follows:

$$\begin{aligned} \rho &= kr \\ \xi_\nu(\rho) &= \rho\, h_\nu^{(2)}(\rho) \\ \psi_\nu(\rho) &= \rho\, j_\nu(\rho) \end{aligned} \tag{69}$$

3. 3. Surface Current Due to the Active Slot

The surface current in the area of the slot is radial and is found to be equal to $-H_\phi(r_o, \theta_o, \phi)$ from equation (64).

Substitution of equation (67) into equation (64) with only the ϕ component considered gives H_ϕ. With substitutions from definitions (69) this is found for region I to be

$$H_{R\phi} = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{Y v_o}{\delta\pi(1+\delta_{om})\rho} \int_0^{2\pi} \int_{\rho_0 - \frac{k\delta}{2}}^{\rho_0 + \frac{k\delta}{2}} d\rho' \, d\phi'$$

$$\cdot \left[\frac{U_{mi} \frac{d}{d\theta} P_{\nu_i}^m (\cos\theta) \sin\phi' \cos m(\phi' - \phi) \psi_{\nu_i}(\rho)\, \xi_{\nu_i}(\rho')}{V_i} \right.$$

$$\left. - \frac{m^2 U'_{mi} P_{\nu'_i}^m (\cos\theta) \sin\phi' \cos m(\phi' - \phi) \psi'_{\nu'_i}(\rho)\, \xi'_{\nu'_i}(\rho')}{V'_i \sin\theta \sin\theta_o} \right] \tag{70}$$

Because of the orthogonality of the sine and cosine functions the integral in equation (70) is zero for $m \neq 1$ and for $m = 1$ integration of the ϕ' dependent parts gives

$$\int_0^{2\pi} \sin\phi' \cos m(\varphi' - \phi)\, d\varphi' = \begin{cases} 0 & \text{for } m \neq 1 \\ \pi \sin\phi & \text{for } m = 1 \end{cases} \tag{71}$$

The integration of the ρ' dependent parts of equation (70) will be done with the aid of the fact that for vanishingly small δ and a continuous function $f(\rho')$ that

$$\int_{\rho_o - \frac{k\delta}{2}}^{\rho_0 + \frac{k\delta}{2}} f(\rho')\, dr' = f(\rho_o)\, k\delta \tag{72}$$

With the aid of relations (71) and (72) equation (70) becomes

$$H_{R\phi}(r,\theta,\varphi) = \frac{kYv_o \sin\phi}{\rho} \sum_{i=0}^{\infty} \left[\frac{U_{1i} \frac{d}{d\theta} P_{\nu_i}^{1}(\cos\theta)\, \xi_{\nu_i}(\rho_o)\psi_{\nu_i}(\rho)}{V_i} - \frac{U'_{1i} P_{\nu'_i}^{1}(\cos\theta)\, \psi'_{\nu'_i}(\rho)\, \xi'_{\nu'_i}(\rho_o)}{V'_i \sin\theta_o \sin\theta} \right] \tag{73}$$

The current across the gap is equal in magnitude and opposite in sign to $H_{R\varphi}(r_o, \theta_o, \phi)$. If the point (r_o, θ_o, ϕ) is substituted into equation (73) it takes the form

$$I_R = \frac{v_o}{\delta} \sin\phi\, Y_R \tag{74}$$

In equation (74) the quantity Y_R is the admittance of the gap and from equation (73) has the expression

$$Y_R = \sum_{i=0}^{\infty} \frac{k\delta Y}{\rho_0} \left[\frac{U'_{1i} P^1_{\nu'_i}(\cos\theta_0)\, \psi'_{\nu'_i}(\rho_0)\, \xi'_{\nu'_i}(\rho_0)}{V'_i \sin^2\theta_0} - \frac{U_{1i} \frac{d}{d\theta}\left[P^1_{\nu_i}(\cos\theta)\right]_{\theta_0} \psi_{\nu_i}(\rho_0)\, \xi_{\nu_i}(\rho_0)}{V_i} \right] \tag{75}$$

3.4. The Distant Field on the Cone Axis

In order to find the distant field due to the slot source E_o it is necessary to use the θ component of $\overline{E}$ from equation (63). This equation must be integrated over the slot area as was done in finding I_R and then the substitution must be made

$$\begin{aligned} r &\to \infty \\ \theta &\to 0 \\ \phi &\to \frac{\pi}{2} \end{aligned} \tag{76}$$

The result of the integration of equation (63) is

$$E_\theta(r, \theta, \phi) = jkv_o \sin\phi \sum_{i=0}^{\infty} \left[\frac{U_{1i} \frac{d}{d\theta} P^1_{\nu_i}(\cos\theta)\, \psi_{\nu_i}(\rho_0)\, \xi_{\nu_i}(\rho)}{\rho\, V_i} + \frac{U'_{1i} P^1_{\nu'_i}(\cos\theta)\, \psi_{\nu'_i}(\rho_0)\, \xi_{\nu'_i}(\rho)\, \xi'_{\nu'_i}(\rho_0)}{\rho_0 V'_i \sin\theta \sin\theta_o\, \xi_{\nu'_i}(\rho_0)} \right] \tag{77}$$

Substitution of the point $(\infty, 0, \phi)$ into equation (77) yields the distant radiated axial field E_R of equation (66) as

$$E_R(\infty, 0, \phi) = \frac{kv_o \sin\phi}{2} \left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty}$$

$$\cdot \sum_{i=0}^{\infty} \left[(j)^{\nu_i+1} U_{1i} \psi_{\nu_i}(\rho_0) - \frac{(j)^{\nu_i'} U'_{1i} \psi_{\nu'_i}(\rho_0) \xi'_{\nu'_i}(\rho_0)}{\sin\theta_o \, \xi_{\nu'_i}(\rho_0)} \right] \tag{78}$$

In equation (78) the fact was used that

$$\left[\frac{P_\nu^1(\cos\theta)}{\sin\theta}\right]_{\theta=0} = \left[\frac{d\,P_\nu^1(\cos\theta)}{d\theta}\right]_{\theta=0} = \frac{\nu(\nu+1)}{2} = \frac{V}{2}$$

3.5. Cancellation of the Distant Fields

In order to find the surface current and the radiation field due to the illuminated cone it is necessary to substitute the points (r_0, θ_o, ϕ) and $(\infty, 0, \phi)$ into the equations for H_ϕ and E_θ respectively from equations (18) in section 1.4. With this done the result for $I_S(r_o, \theta_o, \phi)$ and $E_\theta(\infty, 0, \phi)$ are

$$I_S = \frac{2Y\sin\phi}{\rho_o \sin^2\theta_o} \sum_{i=0}^{\infty} \left[U'_{1i}\, e^{j(\nu'_i+1)\frac{\pi}{2}} \psi'_{\nu'_i}(\rho_o) \right.$$

$$\left. + \, U_{1i}\, e^{j\nu_i\frac{\pi}{2}} \sin\theta_o \, \psi_{\nu_i}(\rho_o) \right] \tag{79}$$

$$E_\theta(\infty, 0, \phi) = \sin\phi \sum_{i=0}^{\infty} \left\{ W_{1i} V_i \, e^{j(\nu_i + 1)\frac{\pi}{2}} \cdot \left[\frac{1}{\rho} \sin\left(\rho - \frac{\nu_i + 1}{2}\pi\right)\right]_{\rho \to \infty} - W'_{1i} V'_i \, e^{j\nu'_i \frac{\pi}{2}} \left[\frac{1}{\rho} \cos\left(\rho - \frac{\nu'_i + 1}{2}\pi\right)\right]_{\rho \to \infty} \right\} \tag{80}$$

Where W_{mi} and W'_{mi} are defined in definitions (17).

The θ component of electric field intensity of equation (80) is for the total of both the incident and reflected parts. In this section only the reflected part is desired. In equation (80) the sinusoidal asymptotic forms were used for the radial functions. If these are replaced by their exponential forms then the reflected part of the expression is easily recognized as that part having the factor $e^{-j\rho}$. Thus the scattered distant axial field E_S is

$$E_S = \left.\frac{\sin\phi \, e^{-j\rho}}{2\rho}\right|_{\rho \to \infty} \sum_{i=0}^{\infty} \left[W_{1i} V_i \, e^{j(\nu_i - \frac{1}{2})\pi} - W'_{1i} V'_i \, e^{j(\nu'_i + \frac{1}{2})\pi} \right] \tag{81}$$

Equating the right-hand side of equation (78) to the negative of the right hand side of equation (81) according to equation (66) which is

$$E_R = - E_S \tag{66}$$

and solving for v_o gives the slot voltage required to give a cancellation of the two fields when they both are present. The expression for this slot voltage is

$$v_o = \frac{M}{N}\,\delta \tag{82}$$

where the constants M and N have been introduced for convenience. They are taken from equations (78) and (81) as follows.

$$M = \sum_{i=0}^{\infty} \left[W_{1i} V_i \, e^{j\nu_i \pi} + W'_{1i} V'_i \, e^{j\nu'_i \pi} \right] \tag{83}$$

$$N = k\delta \sum_{i=0}^{\infty} \left[U_{1i} \, e^{j\nu_i \frac{\pi}{2}} \; \psi_{\nu_i}(\rho_o) + \frac{U'_{1i} \, e^{j(\nu'_i + 1)\frac{\pi}{2}} \; \psi_{\nu'_i}(\rho_o)\, \xi'_{\nu'_i}(\rho_o)}{\sin\theta_o \;\; \xi_{\nu'_i}(\rho_o)} \right] \tag{84}$$

3.6. Circumferential Slot Impedance

Equation (79) is the expression for the surface current I_S due to illumination of the cone by an incident wave of unit magnitude. For convenience the constant Y_S is introduced as follows.

$$Y_S = \frac{2Y}{\rho_o \sin^2 \theta_o} \sum_{i=0}^{\infty} \left[U_{1i} \, e^{j\nu_i \frac{\pi}{2}} \sin \theta_o \, \psi_{\nu_i} (\rho_o) \right.$$

$$\left. + \; U'_{1i} \, e^{j(\nu'_i + 1) \frac{\pi}{2}} \, \psi'_{\nu'_i} (\rho_o) \right] \qquad (85)$$

With the use of definition (85) the expression (79) for I_S becomes

$$I_S = \sin \phi \, Y_S \qquad (86)$$

Equation (65) gives the expression for the slot impedance Z which is defined as the ratio of the electric field intensity in the slot divided by the surface current density across the slot and has for its unit the ohm. If substitution is made into equation (65) for E_o from equation (68), I_R from equation (74), and I_S from equation (86) then the impedance takes the form

$$Z = \frac{v_o \sin \phi}{\delta \left[\frac{v_o}{\delta} \sin \phi \, Y_R + \sin \phi \, Y_S \right]} \qquad (87)$$

After substitution for v_o as indicated by equation (82), and cancellation of the sin ϕ, the final expression for the circumferential slot impedance which will eliminate the nose-on echo from the cone is as follows.

$$Z = \frac{M}{MY_R + NY_S} \tag{88}$$

The constants of equation (88) are given below.

$$M = \sum_{i=0}^{\infty} \left[W_{1i} V_i \, e^{j\nu_i \pi} + W'_{1i} V'_i \, e^{j\nu'_i \pi} \right] \tag{83}$$

$$N = k\delta \sum_{i=0}^{\infty} \left[U_{1i} \, e^{j\nu_i \frac{\pi}{2}} \psi_{\nu_i}(\rho_o) + \frac{U'_{1i} \, e^{j(\nu'_i + 1)\frac{\pi}{2}} \psi_{\nu'_i}(\rho_o) \xi'_{\nu'_i}(\rho_o)}{\sin \theta_o \, \xi_{\nu'_i}(\rho_o)} \right] \tag{84}$$

$$Y_R = \frac{k\delta Y}{\rho_o} \sum_{i=0}^{\infty} \left[\frac{U'_{1i} \, P_{\nu'_i}^{1} (\cos \theta_o) \, \psi'_{\nu'_i}(\rho_o) \, \xi'_{\nu'_i}(\rho_o)}{V'_i \sin^2 \theta_o} \right.$$

$$\left. - \frac{U_{1i} \left[\frac{d}{d\theta} P_{\nu_i}^{1} (\cos \theta) \right]_{\theta_o} \psi_{\nu_i}(\rho_o) \, \xi_{\nu_i}(\rho_o)}{V_i} \right] \tag{75}$$

$$Y_S = \frac{2Y}{\rho_o \sin^2 \theta_o} \sum_{i=0}^{\infty} \left[U_{1i} \, e^{j\nu_i \frac{\pi}{2}} \sin \theta_o \, \psi_{\nu_i}(\rho_o) \right.$$

$$\left. + U'_{1i} \, e^{j(\nu'_i + 1) \frac{\pi}{2}} \psi'_{\nu'_i}(\rho_o) \right] \tag{85}$$

The quantity δ of equations (75) and (84) is the width of the slot and is considered very small compared to the wave length and the quantity r_o which is the r coordinate of the slot impedance. The other quantities used in the above equations are defined in equations (17) and (89).

If the quantities in above equations can be found and if the impedance found by using these quantities according to equation (88) is realizable it will eliminate the backscattering from the infinite cone along the axis of the cone.

CHAPTER IV

SCATTERING FROM A CONE WITH A RADIAL SLOT IMPEDANCE

4.1. The Radial Slot Source

The radial slot impedance will not be as easy to work with as the circumferential slot impedance. The reason is that since the radial slot does not lie entirely in the plane transverse to the incident wave the impedance in the radial slot will be a function of its position.

The slot will be defined as existing between $\phi_o - \frac{\Delta\phi}{2}$ and $\phi_o + \frac{\Delta\phi}{2}$ so that its width at r is $r \sin \theta_o \Delta\phi$ and it is centered at ϕ_o. It is bordered by radial lines on the cone. The geometry is shown in Figure 5.

From equation (81) in section 3.5 the reflected field from an unaltered illuminated cone was found to be

$$E_S(\infty, 0, \phi) = \frac{\sin\phi}{2} \left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty}$$

$$\cdot \sum \left[W_{1i} V_i \, e^{j(\nu_i - \frac{1}{2})\pi} - W'_{1i} V'_i \, e^{j(\nu'_i + \frac{1}{2})\pi} \right] \tag{81}$$

The radiation from a cone with a radial slot source is found by integrating the θ component of $\overline{E}$ of equation (63) over the slot area using $\overline{E}_o = E_{o\phi}(\rho') \hat{\phi}$ and substituting as the point of observation the point $(\infty, 0, \phi)$.

This field is

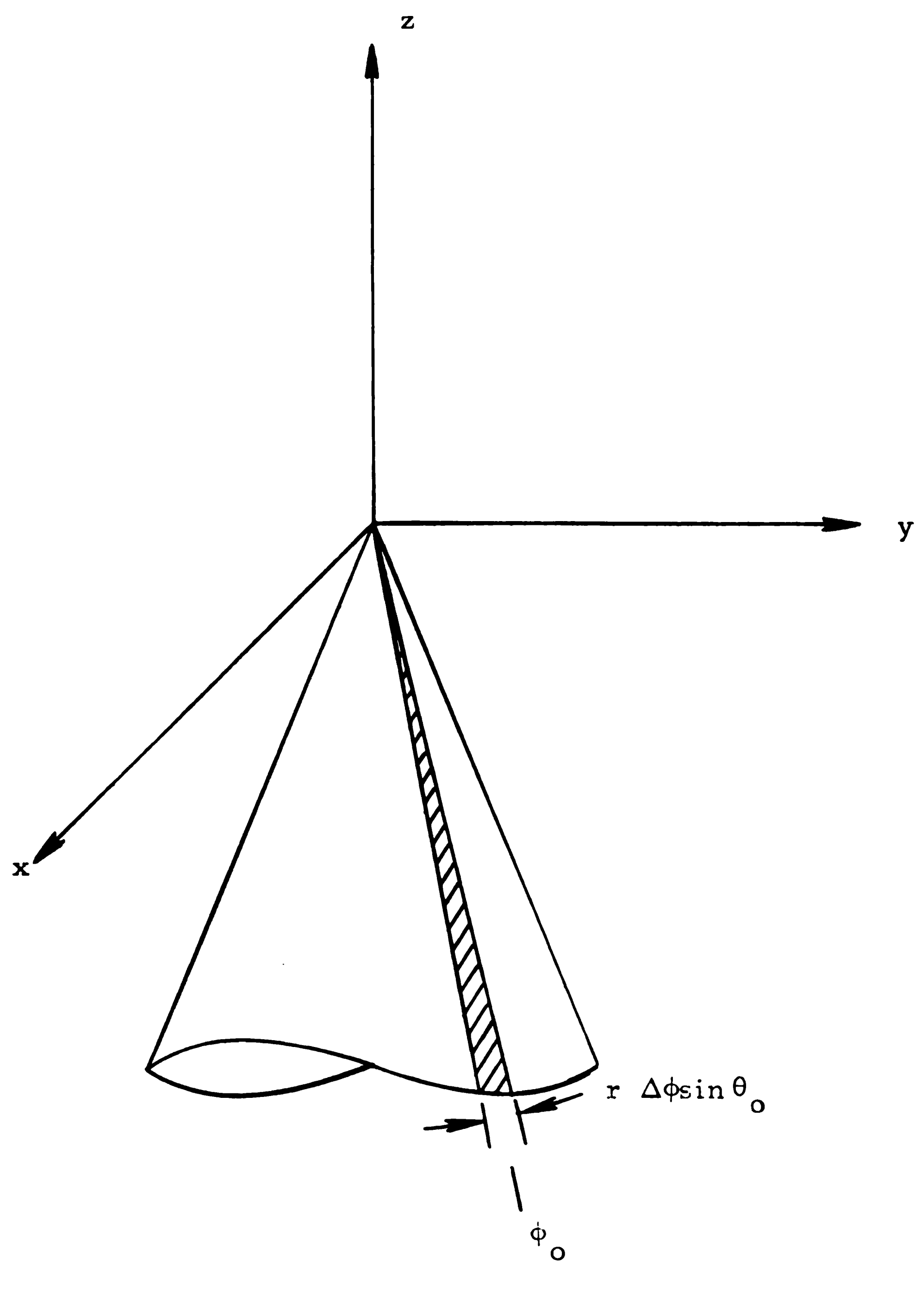

Figure 5. A cone with a longitudinal slot.

$$E_R(\infty, 0, \phi) = -\left.\frac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty} \frac{\Delta\phi}{2\pi} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} U'_{mi} V'_i \, e^{j\nu'_i \frac{\pi}{2}}$$

$$\cdot \frac{\sin m(\phi - \phi_o)}{(1 + \delta_{om})} \int_0^{\infty} E_{o\phi}(\rho)\, j_{\nu'_i}(\rho)\, d\rho \tag{89}$$

In equation (89) the r dependent part is the same as that of equation (81), namely

$$\left.\frac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty}$$

Thus when the condition $E_S(\infty, 0, \phi) = - E_R(\infty, 0, \phi)$ is imposed the r dependence will drop out. This must also be true with respect to the ϕ dependence. The boundary condition for the θ dependent parts at $\theta = 0$ is discussed in Appendix II and is such that the following condition is true.

$$\left[\frac{P_{\nu'_i}^{m}(\cos\theta)}{\sin\theta}\right]_{\theta=0} = \begin{cases} \dfrac{\nu'_i(\nu'_i + 1)}{2} & \text{if } m = 1 \\[2ex] 0 & \text{if } m \neq 1 \end{cases}$$

This condition is necessary in order that the field components are single valued and finite on the axis of the cone. For this reason it is necessary only to carry one term in the sum over m in equation (89) with $\theta = 0$, that term being the one for which $m = 1$.

Even with this condition the ϕ dependent parts still will not match for an arbitrary ϕ_o. The ϕ dependent part of $E_S(\infty, 0, \phi)$ is $\sin\phi$ while the ϕ dependence of $E_R(\infty, 0, \phi)$ is $\sin(\phi - \phi_o)$ which equals $\sin\phi$ only if $\phi_o = 0$. It is therefore necessary in using the radial slot to place it at $\phi_o = 0$ when the incident field is polarized in the $\hat{y}$ direction. Now with all this in mind if the condition $E_S(\infty, 0, \phi) = - E_R(\infty, 0, \phi)$ is imposed one obtains the following result in terms of $E_{o\phi}(\rho)$.

$$\sum_{i=0}^{\infty} e^{j\nu'_i \frac{\pi}{2}} U'_{1i} V'_i \int_0^{\infty} E_{o\phi}(\rho)\, j_{\nu'_i}(\rho)\, d\rho$$

$$= \frac{\pi}{\Delta\phi} \sum_{i=0}^{\infty} \left[e^{j(\nu_i - \frac{1}{2})\pi} W_{1i} V_i + e^{j(\nu'_i - \frac{1}{2})\pi} W'_{1i} V'_i \right] \tag{90}$$

It is necessary to obtain the $E_{o\phi}(\rho)$ which is sufficient to satisfy equation (90). This $E_{o\phi}$ may not be unique but it must be such that the integral can be evaluated. It is sufficient here to say that $E_{o\phi}(\rho)$ is any function such that F_i exists if F_i is defined as follows.

$$F_i = \int_0^{\infty} E_{o\phi}(\rho)\; j_{\nu'_i}(\rho)\, d\rho \tag{91}$$

4.2. The Surface Currents

The surface current across the slot will be found from the tangential magnetic field intensity.

$$I_S = H_r\ (r, \theta_o, 0)$$

For the illuminated unaltered cone equation (18) of section 1.4 gives

$$I_S = \sum_{i=0}^{\infty} \frac{2Y\, U'_{1i}\, e^{j(\nu'_i + 1)\frac{\pi}{2}}\, j_{\nu'_i}(\rho)}{\rho \sin\theta_o} \tag{92}$$

After the proper slot source $E_{o\phi}(r')$ is chosen the surface current in the slot will be found using equation (64) of section 2.12 as follows.

$$I_R = \frac{\Delta\phi Y}{\pi\rho} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{V'_i U'_{mi}\, P^m_{\nu'_i}(\cos\theta_o)}{(1 + \delta_{om})}$$

$$\cdot \left[h^{(2)}_{\nu'_i}(\rho) \int_0^{kr} E_{o\phi}(\rho') j_{\nu'_i}(\rho') d\rho' + j_{\nu'_i}(\rho) \int_{kr}^{\infty} E_{o\phi}(\rho') h^{(2)}_{\nu'_i}(\rho') d\rho' \right] \tag{93}$$

4.3. The Radial Slot Impedance

Equation (65) of section 3.1 shows how the expressions (90), (92), and (93) are to be used to find the impedance which will eliminate the radar echo from the cone along the cone axis.

$$Z = \frac{E_o}{I_R + I_S} \tag{65}$$

The choice of E_o determines the type of impedance which will result. It would be convenient if Z could be a positive constant or perhaps an imaginary constant. Since this requires that the total current across the slot be proportional to the slot voltage the form of $E_{o\phi}$ must be the same as that for $I_R + I_S$. While I_S is independent of $E_{o\phi}$ it is seen that I_R depends on $E_{o\phi}$. Pursuing this kind of solution leads to an equation for the unknown Z as a power series.

$$0 = \sum_{i=0}^{\infty} A_i Z^i$$

in which the A_i are an increasingly complicated mixture of integrals and infinite sums. If $E_{o\phi}$ is assumed to be constant then it may be taken out of the integral and summation signs of equation (90) to give:

$$E_{o\phi} = \frac{M}{N} \tag{94}$$

where

$$M = \sum_{i=0}^{\infty} \left[W_{1i} V_i \, e^{j(\nu_i - \frac{1}{2})\pi} + W'_{1i} V'_i \, e^{j(\nu'_i - \frac{1}{2})\pi} \right] \tag{95}$$

$$N = \frac{\Delta\phi}{\pi} \sum_{i=0}^{\infty} U'_{1i} V'_i \, e^{j \nu'_i \frac{\pi}{2}} \int_0^{\infty} j_{\nu'_i}(\rho) \, d\rho \tag{96}$$

If Y_R and Y_S are defined here in a similar manner as was done for the circumferential slot impedance then equation (65) takes the form as follows.

$$Z = \frac{M}{NY_S + MY_R} \tag{97}$$

The constants M and N are defined in equations (95) and (96) while the quantities Y_S and Y_R are obtained from equations (92) and (93) to give

$$Y_S = \frac{2Y}{\rho \sin\theta_o} \sum_{i=0}^{\infty} U'_{1i}\, e^{j(\nu'_i + 1)\frac{\pi}{2}}\, j_{\nu'_i}(\rho) \tag{98}$$

$$Y_R = \frac{\Delta\phi Y}{\rho\pi} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{U'_{mi} V'_i P_{\nu'_i}^{m}(\cos\theta_o)}{(1 + \delta_{om})}$$

$$\cdot \left[h_{\nu'_i}^{(2)}(\rho) \int_0^{\rho} j_{\nu'_i}(\rho')d\rho' + j_{\nu'_i}(\rho) \int_{\rho}^{\infty} h_{\nu'_i}^{(2)}(\rho')\, d\rho' \right] \tag{99}$$

4.4. The Finite Radial Slot Impedance

It is possible to modify equations (96) and (99) to make them apply to the finite radial slot as shown in Figure 6.

If the slot extends from r_1 to r_2 then $E_{o\phi} = 0$ for $r < r_1$ and $r > r_2$ making the limits on the integrals $\rho_1 = kr_1$ and $\rho_2 = kr_2$ instead of 0 and infinity. With this change the impedance is as follows.

$$Z = \frac{M}{N'Y_S + MY'_R}$$

where N' and Y'_R are as follows.

$$N' = \frac{\Delta\phi}{\pi} \sum_{i=0}^{\infty} U'_{1i} V'_i \, e^{j\nu'_i \frac{\pi}{2}} \int_{\rho_1}^{\rho_2} j_{\nu'_i}(\rho') \, d\rho' \tag{100}$$

$$Y'_R = \frac{\Delta\phi Y}{\rho\pi} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{U'_{mi} V'_i \, P^m_{\nu'_i}(\cos\theta_o)}{(1+\delta_{om})}$$

$$\cdot \left[h^{(2)}_{\nu'_i}(\rho) \int_{\rho_1}^{\rho} j_{\nu'_i}(\rho') d\rho' + j_{\nu'_i}(\rho) \int_{\rho}^{\rho_2} h^{(2)}_{\nu'_i}(\rho') \, d\rho' \right] \tag{101}$$

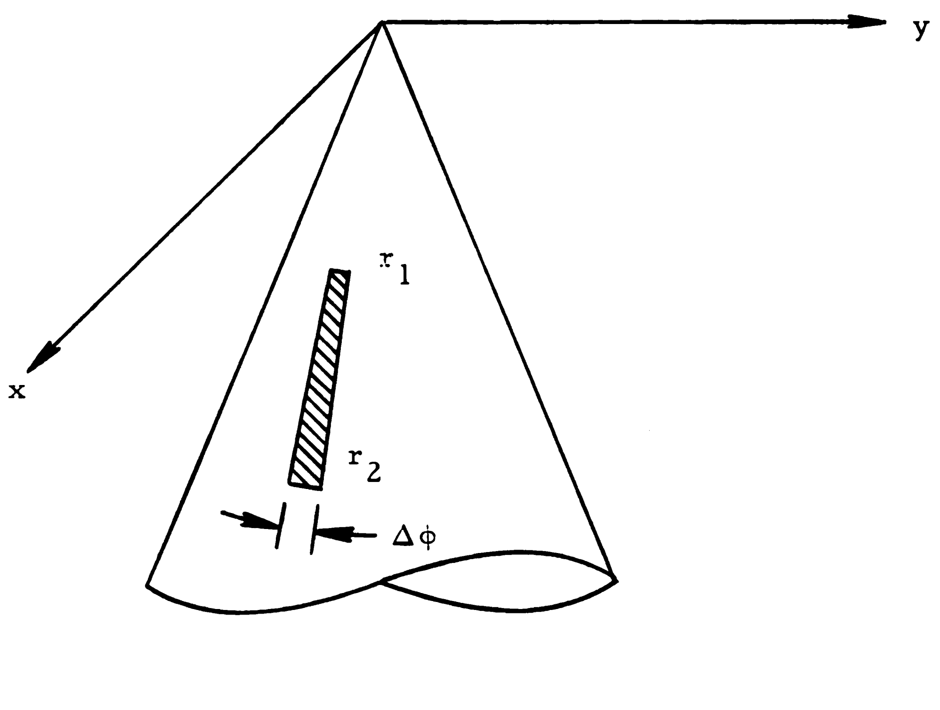

Figure 6. A cone with a finite slot.

CHAPTER V

PATCH IMPEDANCE ON THE CONE

5.1. The Isotropic Impedance

The patch impedance discussed in this section is like the finite radial slot discussed at the end of the preceding section except that its width is not made infinitesimal. An attempt will be made to let its position and size be completely arbitrary. Only the shape will be prescribed as shown in Figure 7.

It is bounded on the sides by radial lines at $\phi = \phi_1$ and ϕ_2 and on the ends by circle segments at $r = r_1$ and r_2 or in terms of $\rho = kr$ the patch lies between ρ_1 and ρ_2 respectively.

The electric field intensity in the patch area will be designated

$$\overline{E}_o = E_{or}\,\hat{r} + E_{o\phi}\,\hat{\phi}$$

$$= E_{or}\left[\hat{r} + K\,\hat{\phi}\right]$$

where $$K = \frac{E_{o\phi}}{E_{or}}$$

The components of $\overline{E}_o$ are assumed to be constant and the constant K will be used to make the impedance have its current in the same direction as its electric field so that an isotropic impedance can be used. Because of the need for considerable manipulation of cumbersome expressions a set of new admittance symbols will be used.

If $\overline{H}_R$ is the tangential magnetic field intensity at the surface of the cone and is due to the patch source $\overline{E}_o$ then each component of

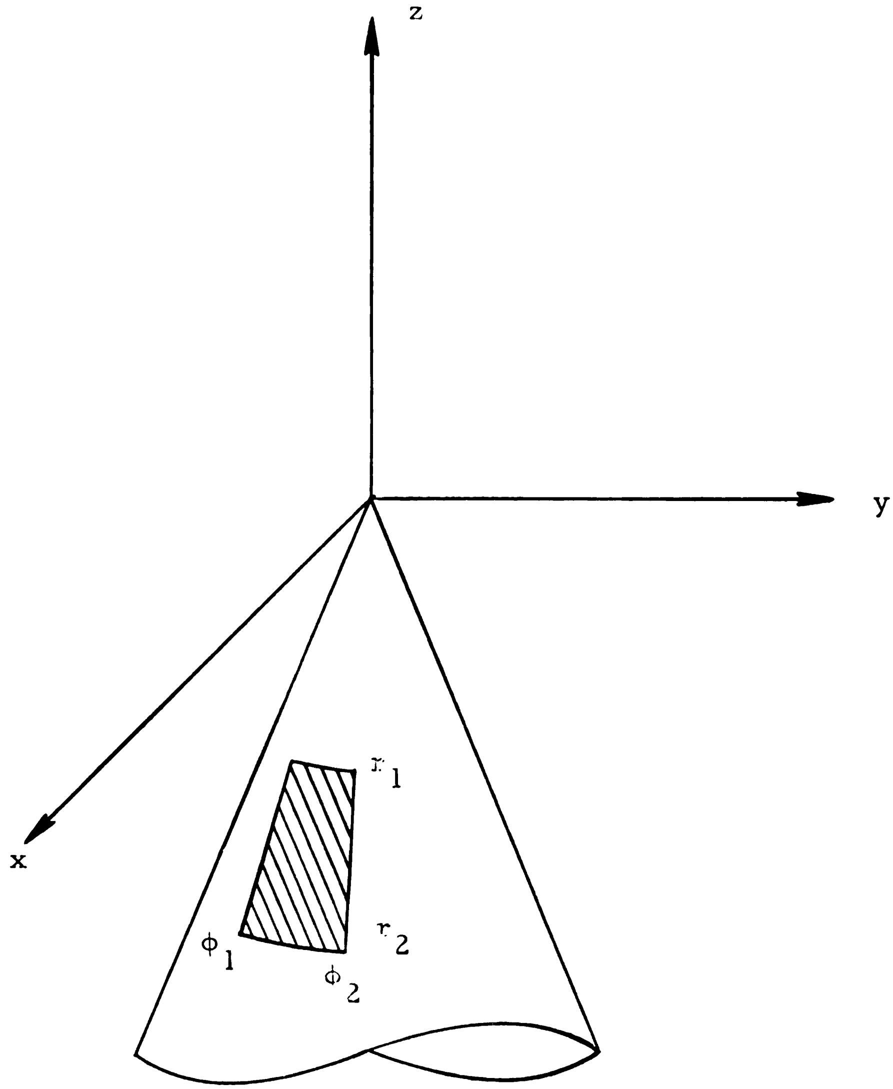

Figure 7. A cone with a patch impedance.

$\overline{H}_R$ can be subdivided into two parts, one due solely to E_{or} and the other due to $E_{o\phi}$. The last division will be indicated by the middle subscript thus

$$\overline{H}_R = (H_{Rrr} + H_{R\phi r})\,\hat{r} + (H_{Rr\phi} + H_{R\phi\phi})\,\hat{\phi}$$

In dealing with the currents and electric field intensities on the surface there are four admittances which are important. They are

$$\begin{aligned} Y_{rr} &= \frac{H_{Rrr}}{E_{or}} \\ Y_{r\phi} &= \frac{H_{Rr\phi}}{E_{or}} \\ Y_{\phi r} &= \frac{H_{R\phi r}}{E_{o\phi}} \\ Y_{\phi\phi} &= \frac{H_{R\phi\phi}}{E_{o\phi}} \end{aligned} \tag{102}$$

In terms of these admittances the current $\overline{I}_R$ has the expression

$$\begin{aligned} \overline{I}_R &= -(E_{or}\,Y_{r\phi} + E_{o\phi}\,Y_{\phi\phi})\,\hat{r} \\ &\quad + (E_{or}\,Y_{rr} + E_{o\phi}\,Y_{\phi r})\,\hat{\phi} \\ &= E_{or}\left[-(Y_{r\phi} + KY_{\phi\phi})\,\hat{r} + (Y_{rr} + KY_{\phi r})\,\hat{\phi}\right] \end{aligned}$$

The total current, $\overline{I}_T$, in the impedance is the sum of the currents $\overline{I}_R$ due to $\overline{E}_o$ and $\overline{I}_S$ due to the radar illumination of the unaltered cone.

$$\overline{I}_T = \overline{I}_S + \overline{I}_R$$

$$= E_{or}\left[\left(\frac{I_{Sr}}{E_{or}} - Y_{r\phi} - KY_{\phi\phi}\right)\hat{r} + \left(\frac{I_{S\phi}}{E_{or}} + Y_{rr} + KY_{\phi r}\right)\right]\hat{\phi} \qquad (103)$$

This total current will be made to have the same direction as the total electric field intensity in the area of the impedance which is as it should be for a simple isotropic impedance. This is done by causing the following condition to hold.

$$\frac{I_{T\phi}}{I_{Tr}} = \frac{E_{o\phi}}{E_{or}} = K \qquad (104)$$

Substitution into condition (104) from equation (103) and after rearrangement gives.

$$K^2\, Y_{\phi\phi} + K\left(Y_{r\phi} + Y_{\phi r} - \frac{I_{Sr}}{E_{or}}\right) + \frac{I_{S\phi}}{E_{or}} + Y_{rr} = 0 \qquad (105)$$

Next an expression for E_{or} will be obtained from the c ndition that the field $\overline{E}_o$ cancels the backscattered field due to the incident radar wave. This condition is stated below.

$$E_{S\theta}(\infty, 0, \phi) = -E_{R\theta}(\infty, 0, \phi) \tag{106}$$

$$= -E_{Rr\theta}(\infty, 0, \phi) - E_{R\phi\theta}(\infty, 0, \phi) \tag{107}$$

The two terms on the right hand side of equation (107) are due to E_{or} and $E_{o\phi}$ separately.

The expressions for E_R are found from section 2.12, equation (63), the two parts of which will be abbreviated as follows

$$\begin{aligned} E_{Rr\theta}(\infty, 0\ \phi) &= -E_{or}\, C_3 \\ E_{R\phi\theta}(\infty, 0, \phi) &= E_{o\phi}\, C_1 = E_{or}\, KC_1 \end{aligned} \tag{108}$$

Where C_1 and C_3 are defined later in definitions (111). Substitution of equations (108) into condition (107) and solving for E_{or} gives

$$E_{or} = \frac{E_S}{C_3 - KC_1} \tag{109}$$

Equation (109) must now be combined with equation (105) and the two solved simultaneously for both K and E_{or}.

The result is a quadratic equation in K which when solved may be expressed as follows.

$$K = \frac{(C_1C_2 + C_3C_4 - Y_{\phi r} - Y_{r\phi})}{2(C_1C_4 + Y_{\phi\phi})}$$

$$\pm \frac{\sqrt{(C_1C_2 + C_3C_4 - Y_{\phi r} - Y_{r\phi})^2 - 4(C_1C_4 + Y_{\phi\phi})(C_2C_3 + Y_{rr})}}{2(C_1C_4 + Y_{\phi\phi})} \tag{110}$$

where:

$$C_1 = E_{R\phi\theta}(\infty, 0, \phi) / E_{o\phi}$$

$$C_2 = I_{S\phi} / E_S(\infty, 0, \phi) = H_{Sr} / E_S(\infty, 0, \phi)$$

$$C_3 = -E_{Rr\theta}(\infty, 0, \phi) / E_{or}$$

$$C_4 = I_{Sr} / E_S(\infty, 0, \phi) = -H_{S\phi} / E_S(\infty, 0, \phi) \tag{111}$$

Having obtained condition (104) the impedance is isotropic and may now be expressed as

$$Z = \frac{E_{or}}{I_{Tr}} \tag{112}$$

Using equations (103), (109) and (111) it is possible to express the impedance of equation (112) as

$$Z = \frac{1}{C_3 C_4 - Y_{r\phi} - K(C_1 C_4 + Y_{\phi\phi})} \tag{113}$$

Of the two possible values for K the best one is that which makes Z more easy to realize.

The expressions for the symbols (102) and (111) may be found in a similar way as the necessary quantities were found for Z in Chapters III and IV. The work is straight forward and the results are listed as follows.

From section 3.5 equation (81),

$$E_S(\infty, 0, \phi) = \frac{\sin\phi\, e^{-j\rho}}{2\rho}\Bigg|_{\rho\to\infty} \sum_{i=0}^{\infty} \left[W_{1i} V_i\, e^{j(\nu_i - \frac{1}{2})\pi} \right.$$

$$\left. + W'_{1i} V'_i\, e^{j(\nu'_i - \frac{1}{2})\pi} \right] \tag{114}$$

From section 2.12 equation (63),

$$C_1 = E_{R\phi\theta}(\infty, 0, \phi)/E_{o\phi} = \frac{e^{-j\rho}}{\rho}\Bigg|_{\rho\to\infty} \left[\sin\phi \right.$$

$$\left. \cdot (\sin\phi_2 - \sin\phi_1) + \cos\phi\,(\cos\phi_2 - \cos\phi_1) \right] S_1$$

$$C_3 = E_{Rr\theta}(\infty, 0, \phi)\,/E_{or} = \frac{e^{-j\rho}}{\rho}\Bigg|_{\rho\to\infty} \left[\sin\phi \right.$$

$$\left. \cdot (\cos\phi_1 - \cos\phi_2) + \cos\phi\,(\sin\phi_2 - \sin\phi_1) \right] S_3 \tag{115}$$

where

$$S_1 = \sum_{i=0}^{\infty} e^{j\nu'_i \frac{\pi}{2}}\, U'_{1i}\, V'_i \int_{\rho_1}^{\rho_2} \frac{\psi_{\nu'_i}(\rho')\, d\rho'}{\rho'}$$

$$S_3 = \sum_{i=0}^{\infty} \left[\frac{e^{j(\nu_i+1)\frac{\pi}{2}} U_{1i}}{2\pi} \int_{\rho_1}^{\rho_2} \psi_{\nu_i}(\rho')\, d\rho' \right.$$

$$\left. - \frac{e^{j\nu'_i \frac{\pi}{2}} U_{1i}}{2\pi \sin\theta_o} \int_{\rho_1}^{\rho_2} \frac{\psi_{\nu'_i}(\rho')\, \xi'_{\nu'_i}(\rho')\, d\rho'}{\xi_{\nu'_i}(\rho')} \right] \tag{117}$$

Equations (115), (116) and (117) complete the definitions of C_1 and C_3. In these equations were used the definitions (69) and the relationships developed in Appendix II shown below.

$$\left[\frac{P^1_{\nu_i}(\cos\theta)}{\sin\theta} \right]_{\theta=0} = \left[\frac{d\, P^1_{\nu_i}(\cos\theta)}{d\theta} \right]_{\theta=0} = \frac{V_i}{2} \tag{118}$$

From section 1.4 equation (18),

$$H_{Sr}(r, \theta_o, \phi) = \frac{2Y\cos\phi}{\rho^2 \sin\theta_o} \sum_{i=0}^{\infty} U'_{1i} V'_i\, e^{j(\nu'_i+1)\frac{\pi}{2}} \psi_{\nu'_i}(\rho) \tag{119}$$

$$H_{S\phi}(r, \theta_o, \phi) = - \frac{2Y\sin\phi}{\rho \sin^2\theta_o} \sum_{i=0}^{\infty} \left[U'_{1i}\, e^{j(\nu'_i+1)\frac{\pi}{2}} \psi'_{\nu'_i}(\rho) \right.$$

$$\left. + \sin\theta_o\, U_{1i}\, e^{j\nu_i \frac{\pi}{2}} \psi_{\nu_i}(\rho) \right] \tag{120}$$

From section 2.12 equation (64),

$$Y_{rr} = \sum_{m=0}^{\infty} \Big[\cos m\phi \,(\cos m\phi_1 - \cos m\phi_2)$$

$$+ \sin m\phi \,(\sin m\phi_1 - \sin m\phi_2) \Big] T_{1m}$$

$$Y_{\phi r} = \sum_{m=0}^{\infty} \Big[\cos m\phi \,(\sin m\phi_2 - \sin m\phi_1)$$

$$+ \sin m\phi \,(\cos m\phi_1 - \cos m\phi_2) \Big] T_{2m}$$

$$Y_{\phi\phi} = \sum_{m=0}^{\infty} \Big[\cos m\phi \,(\cos m\phi_1 - \cos m\phi_2)$$

$$+ \sin m\phi \,(\sin m\phi_1 - \sin m\phi_2) \Big] T_{3m}$$

$$Y_{r\phi} = \sum_{m=0}^{\infty} \Big[\cos m\phi \,(\sin m\phi_2 - \sin m\phi_1)$$

$$+ \sin m\phi \,(\cos m\phi_1 - \cos m\phi_2) \Big] T_{4m}$$

(121)

Where the factors T_{im}, $i = 1, 2, 3, 4$; are defined:

$$T_{1m} = \frac{m\,Y}{\pi(1+\delta_{om})\,\rho^2 \sin\theta_o} \sum_{i=0}^{\infty} U'_{mi}\, P_{\nu'_i}^{m}(\cos\theta_o)$$

$$\cdot \left[\xi_{\nu'_i}(\rho) \int_{\rho_1}^{\rho} \frac{\psi_{\nu'_i}(\rho')\,\xi'_{\nu'_i}(\rho')\,d\rho'}{\xi_{\nu'_i}(\rho')} + \psi_{\nu'_i}(\rho) \int_{\rho}^{\rho_2} \xi'_{\nu'_i}(\rho')\,d\rho' \right]$$

$$T_{2m} = \frac{Y}{\pi(1+\delta_{om})\,\rho^2} \sum_{i=0}^{\infty} U'_{mi}\, V'_i\, P_{\nu'_i}^{m}(\cos\theta_o)$$

$$\cdot \left[\xi_{\nu'_i}(\rho) \int_{\rho_1}^{\rho} \frac{\psi_{\nu'_i}(\rho')\,d\rho'}{\rho'} + \psi_{\nu'_i}(\rho) \int_{\rho}^{\rho_2} \frac{\xi_{\nu'_i}(\rho')\,d\rho'}{\rho'} \right]$$

$$T_{3m} = \frac{Y\,m}{\pi(1+\delta_{om})\,\rho \sin\theta_o} \sum_{i=0}^{\infty} U'_{mi}\, P_{\nu'_i}^{m}(\cos\theta_o)$$

$$\cdot \left[\xi'_{\nu'_i}(\rho) \int_{\rho_1}^{\rho} \frac{\psi_{\nu'_i}(\rho')\,d\rho'}{\rho'} + \psi'_{\nu'_i}(\rho) \int_{\rho}^{\rho_2} \frac{\xi_{\nu'_i}(\rho')\,d\rho'}{\rho'} \right]$$

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$$\mathcal{T}_{4m} = \frac{-Y}{\pi(1+\delta_{om})\rho} \sum_{i=0}^{\infty} \left\{ \frac{m^2 U'_{mi} P_{\nu'_i}^m (\cos\theta_o)}{V'_i \sin^2\theta_o} \right.$$

$$\cdot \left[\xi'_{\nu'_i}(\rho) \int_{\rho_1}^{\rho} \frac{\psi_{\nu'_i}(\rho')\, \xi'_{\nu'_i}(\rho')\, d\rho'}{\xi_{\nu'_i}(\rho')} + \psi'_{\nu'_i}(\rho) \int_{\rho}^{\rho_2} \xi'_{\nu'_i}(\rho')\, d\rho' \right]$$

$$- \frac{U_{mi}}{V_i} \left[\frac{d}{d\theta} P_{\nu_i}^m (\cos\theta) \right]_{\theta_o} \left[\xi_{\nu_i}(\rho) \int_{\rho_1}^{\rho} \psi_{\nu_i}(\rho')\, d\rho' \right.$$

$$\left. \left. + \psi_{\nu_i}(\rho) \int_{\rho}^{\rho_2} \xi_{\nu_i}(\rho')\, d\rho' \right] \right\} \qquad (122)$$

5.2. Polarity Rotation by Impedance Loading

A careful examination of equations (114), and (115) shows that the r dependent parts as $r \to \infty$ are equal as they should be to satisfy the condition

$$\bar{E}_S = -\bar{E}_R \quad \text{at} \quad (\infty, 0, \phi)$$

The ϕ dependence however, does not match unless some condition is enforced so that $E_{Rr\theta}$ and $E_{R\phi\theta}$ are proportional to sin ϕ.

If all that is necessary is to eliminate the component of backscattered field which is polarized in the same direction as the incident radar field then one needs only to use that part of E_R at $(\infty, 0, \phi)$ which has sin ϕ as a factor.

For this purpose C_1 and C_3 are redefined as C_1' and C_3' as indicated below.

$$C_1' = \left.\frac{e^{-j\rho}}{\rho}\right|_{\rho \to \infty} (\sin\phi_2 - \sin\phi_1)\sin\phi \ S_1$$

$$C_3' = \left.\frac{e^{-j\rho}}{\rho}\right|_{\rho \to \infty} (\cos\phi_1 - \cos\phi_2)\sin\phi \ S_3 \tag{123}$$

Equations (110) and (113) should be used with the indicated substitutions, and the result is that the backscattered field will contain only the component whose polarity is rotated 90° from that of the incident field. Since the incident electric field was polarized in the $\hat{y}$ direction the field reflected back will be entirely polarized in the $\hat{x}$ direction.

This uncancelled backscattered field will be

$$\overline{E} = \frac{\hat{x}\ E_S(\infty, 0, \phi)}{C_3' - KC_1'} \left.\frac{e^{-j\rho}}{\rho}\right|_{\rho \to \infty} \left[(\sin\phi_2 - \sin\phi_1) + K(\cos\phi_2 - \cos\phi_1)\right] \tag{124}$$

5.3. Complete Cancellation using Symmetric Patch Impedances

If it is necessary to eliminate entirely all components of the backscattered field then the position and size of the impedance cannot be kept completely arbitrary. There may be many sets of conditions on size and position which will make $E_{R\theta}$ proportional to $\sin\phi$ but all are not easily obtained.

For example, if the impedance is symmetric across the $\phi = 0$ line then

$$C_1 = 2 \left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty} \sin \phi_2 \sin \phi \; S_1$$

$$C_3 = 2 \left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty} \sin \phi_2 \cos \phi \; S_3 \tag{125}$$

Now if ρ_1 and ρ_2 can be found such that $S_3 = 0$ then the impedance so placed will accomplish the desired result. Since these values for ρ_1 and ρ_2 are difficult to find, a more desirable method for eliminating the total backscattering is needed. Such a method might be obtained by the use of more than one patch area. For example consider the case of two patch impedances placed symmetrically on either side of the $\phi = \pi/2$ plane, bordered by the lines $\rho = \rho_1$ and ρ_2, the first lying between $\phi = \phi_1$ and ϕ_2 and the second between $\phi = (\pi - \phi_2)$ and $(\pi - \phi_1)$

Equations (119) and (120) reveal that the radial currents in the area of patch 1 and 2 are the same while the ϕ component of current in patch 1 is opposite in sign to that of patch 2. It is reasonable therefore in order that the impedance be the same for the two patches to make $E_{o\phi}$ be opposite in sign in the two areas while E_{or} is made to have the same sign. This is a slight departure from the specifications given for $E_{o\phi}$ prior to this time since, it will be remembered, that $E_{o\phi}$ and E_{or} were formerly considered constant over the impedance area. Here $E_{o\phi}$ has two constant values. Thus the electric field intensities $\overline{E}_{o1}$ and $\overline{E}_{o2}$ in the two areas are

$$\overline{E}_{o1} = E_{or}\,\hat{r} + E_{o\phi}\,\hat{\phi}$$

$$= E_{or}\,(\hat{r} + K\hat{\phi}) \tag{126}$$

$$\overline{E}_{o2} = E_{or}\,\hat{r} - E_{o\phi}\,\hat{\phi}$$

$$= E_{or}\,(\hat{r} - K\hat{\phi}) \tag{127}$$

Using equations (115) for each patch area and combining the results gives

$$\begin{aligned} E_{R\theta}(\infty, 0, \phi) = \frac{e^{-j\rho}}{\rho}\Bigg|_{\rho\to\infty} E_{or}\Bigg\{S_3\Bigg[\sin\phi\Big(\cos\phi_1 - \cos\phi_2 \\ + \cos(\pi - \phi_2) - \cos(\pi - \phi_1)\Big) + \cos\phi\Big(\sin\phi_2 - \sin\phi_1 \\ + \sin(\pi - \phi_1) - \sin(\pi - \phi_2)\Big)\Bigg] + S_1 K\Bigg[\sin\phi\Big(\sin\phi_2 \\ - \sin\phi_1 - \sin(\pi - \phi_1) + \sin(\pi - \phi_2)\Big) + \cos\phi\Big(\cos\phi_2 - \cos\phi_1 \\ - \cos(\pi - \phi_1) + \cos(\pi - \phi_2)\Big)\Bigg]\Bigg\} \end{aligned} \tag{128}$$

Equation (128) may be reduced by applying the following identities

$$\cos(\pi - \phi) = -\cos\phi$$
$$\sin(\pi - \phi) = \sin\phi \tag{129}$$

This makes $E_{R\theta}$ independent of $\cos\phi$.

$$E_{R\theta} = 2\left.\frac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty} E_{or}\sin\phi\left[S_3(\cos\phi_1 - \cos\phi_2)\right.$$
$$\left. + KS_1(\sin\phi_2 - \sin\phi_1)\right] \tag{130}$$

From equation (130) it is noticed that C_1 and C_3 for the present case are $2C'_1$ and $2C'_3$ which are

$$C_1 = 2\left.\frac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty} \sin\phi\, S_1(\sin\phi_2 - \sin\phi_1)$$
$$\tag{131}$$
$$C_3 = 2\left.\frac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty} \sin\phi\, S_3(\cos\phi_1 - \cos\phi_2)$$

The four admittances corresponding to equations (121) must be changed as follows.

$$Y_{rr} = 2\sum_{m=0}^{\infty} \cos m\phi\,(\cos m\phi_1 - \cos m\phi_2)\; T_{1m}$$

$$Y_{\phi r} = 2\sum_{m=0}^{\infty} \cos m\phi\,(\sin m\phi_1 - \sin m\phi_2)\; T_{2m}$$

$$Y_{\phi\phi} = 2 \sum_{m=0}^{\infty} \sin m\phi \,(\sin m\phi_1 - \sin m\phi_2)\; T_{3m}$$

$$Y_{r\phi} = 2 \sum_{m=0}^{\infty} \sin m\phi \,(\cos m\phi_2 - \cos m\phi_1)\; T_{4m} \qquad (132)$$

5.4. Generalization and Conclusions

Of all the impedances discussed thus far only one has the advantage of being constant over the impedance area, that being the circumferential slot discussed in Chapter III. All others were found to be functions of the variables r or ϕ or both. Another advantage possessed by the circumferential slot impedance is that due to its axial symmetry as seen in Figure 4 page 45 the back-scattered field will vanish for any polarization of the incident wave. A rotation of the cone doesn't change the problem and therefore leaves the solution unchanged.

The patch impedance has the advantage that it leaves the cone structurally more sound. If the impedance is produced by a cavity it wouldn't interfere with the strength of the structure as much as the circumferential slot especially if the patch area is small and it could be more easily produced and adjusted.

By using small patches it may be possible to obtain all the advantages mentioned for both the circumferential slot and the patch. If the patch is made narrow enough in the ϕ direction so that the ϕ

dependence of the impedance is nearly constant in the area of the patch then it may be replaced with its average value. Four such patches equally spaced in a circumferential band could be used to cause the backscattering to vanish regardless of the polarity of the incident wave. That this is true may be seen from the following logic. With four identical constant patch impedances placed at $\frac{n\pi}{4}$; $n = 1, 3, 5,$ and 7; there is symmetry across both the x axis and the y axis. If the incident wave is expressed as the sum of two components

$$\overline{E}_I = E_{Ix}\,\hat{x} + E_{Iy}\,\hat{y}$$

and the value of the impedance is found using only the y component, E_{Iy}, as if it existed alone, each of the four impedances would be half the magnitude of that found for the two impedances discussed in the preceding section. Examination of equations (119), (120) and (132) reveals that the two patches centered at $\phi = \frac{5\pi}{4}$ and $\phi = \frac{7\pi}{4}$ have exactly the same effect as the two placed at $\phi = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$. It is as though each pair cancelled half of the y component of the backscattered field. Now to see that the x component is automatically cancelled by the presence of these four impedances one may rotate the cone by 90° about its axis and it is found that nothing is changed. The transformation from ϕ to $\phi + \frac{\pi}{2}$ leaves the impedance expression and locations unchanged and consequently the solution is good for both components. If the impedance could not be considered as being independent of ϕ over its area then the transformation would not in general leave the impedance expression unchanged and even though symmetry existed across both the x axis and the y axis complete cancellation is not guaranteed. It is noted here that either the finite or the infinite longitudinal slot impedance discussed in Chapter IV could be used in such an arrangement.

If the patch area is also small in the r direction the impedance may be constant over the entire area in which case it may be possible to alter the shape of the patch from that shown in Figure 7 on page 65 to a more desirable shape such as a square or a circle provided the area is kept the same. This would facilitate both construction and adjustment of the cavity impedance. A simple plunger can be used to adjust the depth of the cavity and thus the impedance. This type of arrangement usually requires that the impedances be reactive and if they are centered on the circle $\rho = \rho_o$ with ρ_o chosen so as to insure the reactive nature of the impedances then ρ_o must be found by setting the real part of Z equal to zero. The solution of the resulting equation in ρ_o presents an extremely difficult task and one may find that the experimental method for determining ρ_o is more desirable. In such a method a change in frequency would effectively change the value of ρ_o.

Experimentation in this problem of backscattering reduction from the infinite cone is complicated by at least two difficulties, the first being the need for the spherical mode functions with non integer expansion parameters. This difficulty is discussed in more detail in Appendix II. The second is that of approximating the infinite cone with some kind of finite structure. Both of these difficulties are eliminated if the structure being examined is the sphere.

CHAPTER VI

THE FIELDS OF THE ILLUMINATED UNLOADED SPHERE

In this and the next three chapters the sphere will be analyzed in a similar manner as was the cone. Since the sphere is a coordinate surface of the spherical coordinate system the spherical mode functions will again be used. Here, however, because both polar axis are included in the region of consideration the subscripts ν_i and ν'_i are integers and the letter n will be used in their place.

$$\nu_i = \nu'_i = n \tag{129}$$

In many respects the analysis of the sphere is similar to that of the cone in the preceding chapters. The scalar potentials Π and Π^* giving T.M. and T.E. modes respectively have the form

$$\Pi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_n(kr)\, P_n^m(\cos\theta) \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi \right]$$

$$\Pi^* = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_n(kr)\, P_n^m(\cos\theta) \left[C_{mn} \cos m\phi + D_{mn} \sin m\phi \right]$$

The field components consequently have the form shown in Appendix I equation (35) with the substitution indicated by equation

(129) above.

The expression for the impedance located on the surface of the sphere will be of the same form as that of the cone

$$Z = \frac{E_o}{I_R + I_S}$$

with E_o, the aperture source, adjusted so that its radiated field E_R is equal to $-E_S$ the scattered field on the positive z axis.

In this chapter the unloaded sphere of radius r_o is examined while being illuminated by a plane wave incident from the positive z axis as shown in Figure 8.

The current I_S on the surface of the sphere and the scattered electric field $E_S(\infty, 0, \phi)$ are found for use in the expression for the impedance.

6. 1 - The Incident Field

The incident electric and magnetic fields are expressed as

$$\begin{aligned} \overline{E}_I &= E_{Iy}\,\hat{y} = \hat{y}\, e^{jkr\cos\theta} \\ \overline{H}_I &= H_{Ix}\,\hat{x} = \hat{x}\, Y e^{jkr\cos\theta} \end{aligned} \tag{130}$$

The subscript I indicates the incident nature of the field.

Stratton[6] has shown that the factor $e^{jkr\cos\theta}$ can be expanded in terms of the spherical mode functions.

$$e^{jkr\cos\theta} = \sum_{n=0}^{\infty} (j)^n (2n+1)\, j_n(kr)\, P_n(\cos\theta) \tag{131}$$

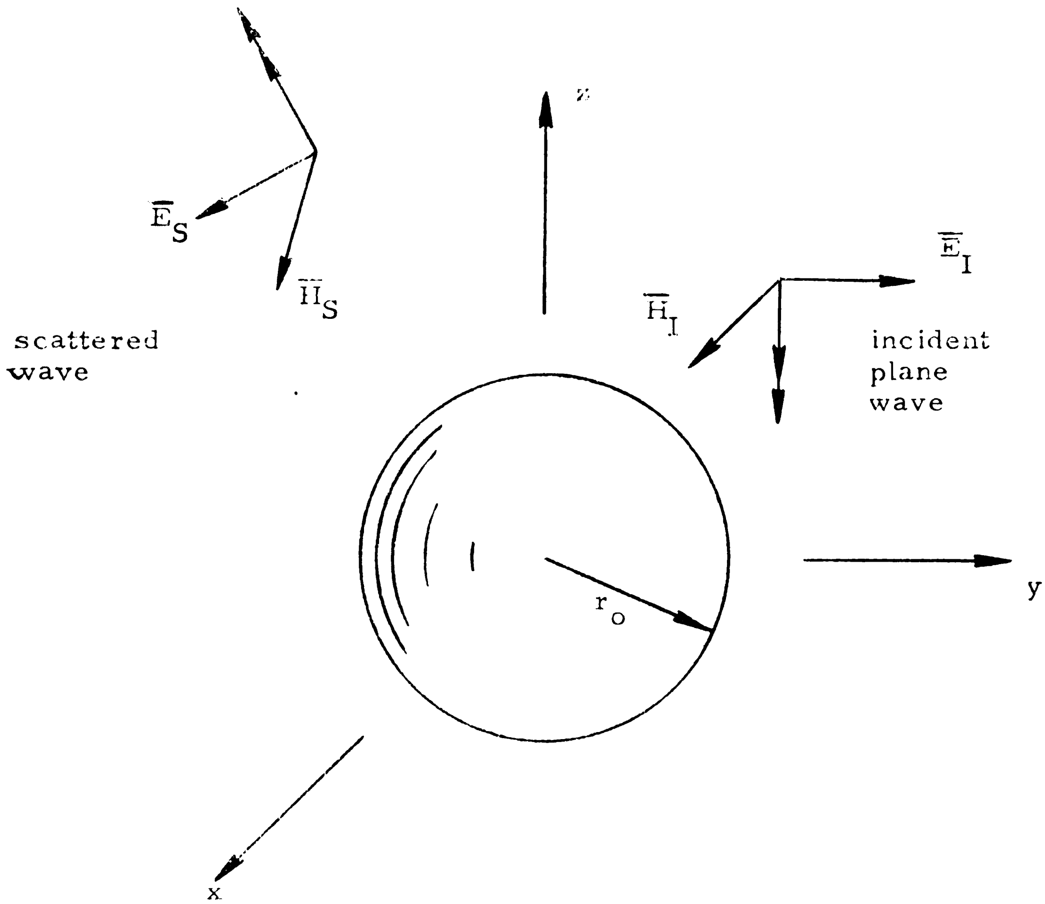

Figure 8. The unloaded sphere as a scatterer.

The radial components E_{Ir} and H_{Ir} of the incident field can be found from equations (130) as follows.

$$E_{Ir} = e^{jkr\cos\theta} \sin\theta \sin\phi$$

$$= -\frac{\sin\phi}{jkr} \frac{\partial}{\partial\theta}\left(e^{jkr\cos\theta}\right)$$

$$(132)$$

$$H_{Ir} = Y \cos\phi \sin\theta \, e^{jkr\cos\theta}$$

$$= -Y\frac{\cos\phi}{jkr} \frac{\partial}{\partial\theta}\left(e^{jkr\cos\theta}\right)$$

Using equation (131) in the expressions (132) gives the expanded form for the radial components.

$$E_{Ir} = \frac{\sin\phi}{jkr} \sum_{n=1}^{\infty} (j)^n (2n+1) j_n(kr) P_n^{\,1}(\cos\theta)$$

$$(133)$$

$$H_{Ir} = \frac{Y\cos\phi}{jkr} \sum_{n=1}^{\infty} (j)^n (2n+1) j_n(kr) P_n^{\,1}(\cos\theta)$$

In equations (133) the fact was used that

$$\frac{\partial}{\partial\theta} P_n(\cos\theta) = -P_n^{\,1}(\cos\theta) \qquad n \neq 0$$

By comparing equation (133) with the radial components of the general field components expressed in terms of the expansion coefficients A_{mn}, B_{mn}, C_{mn}, and D_{mn} in Appendix I equation (35), these coefficients can be determined for the incident field as follows. From Appendix I the radial components are

$$E_{Ir} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{n(n+1)}{r} z_n(kr) P_n^m(\cos\theta) \cdot \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi \right]$$

$$H_{Ir} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{n(n+1)}{r} z_n(kr) P_n^m(\cos\theta) \cdot \left[C_{mn} \cos m\phi + D_{mn} \sin m\phi \right] \tag{134}$$

The orthogonality of the ϕ and θ dependent parts show that the radial functions $z_n(kr)$ must be $j_n(kr)$ and the expansion coefficients must be

$$A_{mn} = D_{mn} = 0 \tag{135}$$

$$B_{mn} = \frac{C_{mn}}{Y} = \frac{\delta_{m1} j^{n-1} k (2n+1)}{n(n+1)} \tag{136}$$

Following the convention established in the previous chapters certain quantities will be grouped and defined as new functions as follows

$$V_n = n(n + 1)$$

$$W_{mn} = \frac{(2n + 1)(n - m)!}{(n + m)!}$$

$$\rho = kr$$

$$\xi_n(\rho) = \rho h_n^{(2)}(\rho)$$

$$\psi_n(\rho) = \rho j_n(\rho)$$

(137)

The coefficients (136) may be expressed in terms of definitions (137) to give

$$B_{mn} = C_{mn}/Y \quad = \delta_{m1}\, j^{(n-1)} k\, W_{1n}$$

Using the definitions (137) the components of the incident electric and magnetic fields are found to be

$$E_{Ir} = \frac{\sin\phi}{j\rho^2} \sum_{n=1}^{\infty} j^n W_{1n} V_n \psi_n(\rho)\, P_n^1(\cos\theta)$$

$$E_{I\theta} = \frac{\sin\phi}{j\rho} \sum_{n=1}^{\infty} j^n W_{1n}\left(\psi'_n(\rho) \frac{dP_n^1}{d\theta} + j\, \frac{\psi_n(\rho)\; P_n^1}{\sin\theta} \right)$$

$$E_{1\phi} = \frac{\cos\phi}{j\rho} \sum_{n=1}^{\infty} j^n W_{1n}\left(\frac{\psi'_n(\rho)\, P_n^{\,1}}{\sin\theta} + j\psi_n(\rho)\, \frac{dP_n^{\,1}}{d\theta} \right)$$

$$H_{Ir} = \frac{Y\cos\phi}{j\rho^2} \sum_{n=1}^{\infty} j^n W_{1n} V_n \psi_n(\rho)\, P_n^{\,1}(\cos\theta)$$

$$H_{1\theta} = \frac{Y\cos\phi}{j\rho} \sum_{n=1}^{\infty} j^n W_{1n}\left(\psi'_n(\rho)\, \frac{dP_n^{\,1}}{d\theta} + j\, \frac{\psi_n(\rho)\, P_n^{\,1}}{\sin\theta} \right)$$

$$H_{I\phi} = -\frac{Y\sin\phi}{j\rho} \sum_{n=1}^{\infty} j^n W_{1n}\left(\frac{\psi'_n(\rho)\, P_n^{\,1}}{\sin\theta} + j\psi_n(\rho)\, \frac{dP_n^{\,1}}{d\theta} \right)$$

(138)

6.2. The Scattered Field

The subscript S is used to indicate the scattered field and the boundary condition at $r = r_o$

$$E_{S\theta}(r_o, \theta, \phi) + E_{I\theta}(r_o, \theta, \phi) = 0$$

(139)

$$E_{S\phi}(r_o, \theta, \phi) + E_{I\phi}(r_o, \theta, \phi) = 0$$

is now used to find the unknown coefficients for this field. Since $\overline{E}_S$ must satisfy the radiation condition as $r \to \infty$ the radial functions $h_n^{(2)}(\rho)$ must replace the functions $z_n(\rho)$ of equations (35) in Appendix I. Using definitions (137) one obtains

$$E_{S\theta}(r_o, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{k}{\rho_o} \xi'_n(\rho_o) \frac{dP_n^m}{d\theta}$$

$$\cdot \left[A_{mn} \cos m\phi + B_{mn} \sin m\phi\right] + \frac{j\omega\mu\, m}{\rho_o \sin\theta} \xi_n(\rho_o) P_n^m$$

$$\cdot \left[C_{mn} \sin m\phi - D_{mn} \cos m\phi\right]$$

$$E_{S\phi}(r_o, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-mk}{\rho_o \sin\theta} \xi'_n(\rho_o) P_n^m$$

$$\cdot \left[A_{mn} \sin m\phi - B_{mn} \cos m\phi\right] + \frac{j\omega\mu}{\rho_o} \xi_n(\rho_o) \frac{dP_n^m}{d\theta}$$

$$\cdot \left[C_{mn} \cos m\phi + D_{mn} \sin m\phi\right]$$

(140)

Substitution of equations (138) and (140) into equation (139) and applying the orthogonal property of the ϕ dependent parts shows that all of the coefficients are zero except B_{1n} and C_{1n} $(n = 1, 2, \ldots)$ which must be found from the following equations which are obtained in the manner just described. For $\rho = \rho_o$

$$\sum_{n=1}^{\infty} \frac{dP_n^1}{d\theta} \left(j^n W_{1n} \psi'_n(\rho_o) + jk B_{1n} \xi'_n(\rho_o) \right)$$

$$+ \frac{j P_n^1}{\sin\theta} \left(j^n W_{1n} \psi_n(\rho_o) + j\omega\mu C_{1n} \xi_n(\rho_o) \right) = 0$$

$$\sum_{n=1}^{\infty} \frac{P_n^1}{\sin\theta} \left(j^n W_{1n} \psi'_n(\rho_o) + jk B_{1n} \xi'_n(\rho_o) \right)$$

$$+ j \frac{dP_n^1}{d\theta} \left(j^n W_{1n} \psi_n(\rho_o) + j\omega\mu C_{1n} \xi_n(\rho_o) \right) = 0 \tag{141}$$

The coefficient B_{1n} (and C_{1n}) is found by multiplying the first (second) equation of (141) by $P_{n'}^1$ and the second (first) by

$$\sin\theta \frac{dP_{n'}^1}{d\theta}$$

and adding the two equations thus obtained. Integration over the range $\theta = 0$ to π causes all terms in the sum to be zero except the n'th term due to the fact that

$$\int_0^{\pi} \left(\frac{dP_n^1}{d\theta} P_{n'}^1 + P_n^1 \frac{dP_{n'}^1}{d\theta} \right) d\theta = 0$$

$$\int_0^{\pi} \left(\frac{P_{n'}^1 P_n^1}{\sin^2\theta} + \frac{dP_{n'}^1}{d\theta} \frac{dP_n^1}{d\theta} \right) \sin\theta \, d\theta$$

$$= \delta_{nn'} \frac{2n^2(n+1)^2}{(2n+1)} = \frac{2\delta_{nn'} V_n}{W_{1n}} \tag{142}$$

One must conclude that the cofactor of the above expression must be zero if $n = n'$ to satisfy equation (141). Thus one arrives at the result,

$$\left(j^n W_{1n}\psi'_n(\rho_o) + jk\, B_{1n}\xi'_n(\rho_o)\right) = \left(j^n W_{1n}\psi_n(\rho_o) + j\omega\mu\, C_{1n}\xi_n(\rho_0)\right)$$

$$= 0$$

which gives

$$B_{1n} = \frac{j^{(n+1)}\, W_{1n}\psi'_n(\rho_o)}{k\ \xi'_n(\rho_o)}$$

$$C_{1n} = \frac{Y\, j^{(n+1)}\, W_{1n}\psi_n(\rho_o)}{k\ \xi_n(\rho_o)} \tag{143}$$

The scattered field components are as follows

$$E_{Sr} = \frac{\sin\phi}{\rho^2} \sum_{n=1}^{\infty} j^{(n+1)} \frac{V_n W_{1n}\psi'_n(\rho_o)\xi_n(\rho)\, P_n^1(\cos\theta)}{\xi'_n(\rho_o)}$$

$$E_{S\theta} = \frac{\sin\phi}{\rho} \sum_{n=1}^{\infty} j^{(n+1)}\, W_{1n}\left[\frac{\psi'_n(\rho_o)}{\xi'_n(\rho_o)}\ \xi'_n(\rho)\ \frac{dP_n^1}{d\theta}\right.$$

$$\left. + j\,\frac{\psi_n(\rho_o)\,\xi_n(\rho)\,P_n^1}{\xi_n(\rho_o)\ \sin\theta}\right]$$

$$E_{S\phi} = \frac{\cos\phi}{\rho} \sum_{n=1}^{\infty} j^{(n+1)} W_{1n} \left[\frac{\psi'_n(\rho_o)\xi'_n(\rho)\, P_n^1}{\xi'_n(\rho_o) \sin\theta} \right.$$

$$\left. + \frac{j\psi_n(\rho_o)}{\xi_n(\rho_o)} \quad \xi_n(\rho) \frac{dP_n^1}{d\theta} \right]$$

$$H_{Sr} = \frac{Y \cos\phi}{\rho^2} \sum_{n=1}^{\infty} j^{(n+1)} \frac{V_n W_{1n} \psi_n(\rho_o) \xi_n(\rho)\, P_n^1 (\cos\theta)}{\xi_n(\rho_o)}$$

$$H_{S\theta} = \frac{Y \cos\phi}{\rho} \sum_{n=1}^{\infty} j^{(n+1)} W_{1n} \left[\frac{\psi_n(\rho_o)}{\xi_n(\rho_o)} \quad \xi'_n(\rho) \frac{dP_n^1}{d\theta} \right.$$

$$\left. + j \frac{\psi_n'(\rho_o)\xi_n(\rho)\, P_n^1}{\xi'_n(\rho_o) \sin\theta} \right]$$

$$H_{S\phi} = - \frac{Y \sin\phi}{\rho} \sum_{n=1}^{\infty} j^{(n+1)} W_{1n} \left[\frac{\psi_n(\rho_o)\xi'_n(\rho)\, P_n^1}{\xi_n(\rho_o) \sin\theta} \right.$$

$$\left. + j \frac{\psi'_n(\rho_o)}{\xi'_n(\rho_o)} \quad \xi_n(\rho) \frac{dP_n^1}{d\theta} \right] \tag{144}$$

6.3. The Backscattered Field

The backscattered electric field intensity on the positive z axis as $r \to \infty$ is $\overline{E}_S(\infty, 0, \phi)$ and from equations (144) is seen to be

$$\overline{E}_S(\infty, 0, \phi) = \frac{e^{-j\rho}}{j2\rho} \left. \hat{y} \sum_{n=1}^{\infty} (-1)^n (2n+1) \left[\frac{\psi'_n(\rho_o)}{\xi'_n(\rho_o)} - \frac{\psi_n(\rho_o)}{\xi_n(\rho_o)} \right] \right|_{\rho \to \infty} \tag{145}$$

In equation (145) the facts used were

$$\left[\frac{dP_n^1}{d\theta}\right]_{\theta=0} = \left[\frac{P_n^1(\cos\theta)}{\sin\theta}\right]_{\theta=0} = \frac{n(n+1)}{2} \tag{146}$$

$$\lim_{\rho \to \infty} \frac{\xi'_n(\rho)}{j\rho} = -j \lim_{\rho \to \infty} \frac{\xi_n(\rho)}{j\rho} = (j)^n \left. \frac{e^{-j\rho}}{j\rho} \right|_{\rho \to \infty} \tag{147}$$

Equation (145) can be reduced still farther by using the fact that the Wronskian of the two functions $\xi_n(\rho)$ and $\psi_n(\rho)$ is

$$W(\xi_n, \psi_n) = \begin{vmatrix} \xi_n & \psi_n \\ \xi'_n & \psi'_n \end{vmatrix} = j \tag{148}$$

Using equation (148) in equation (145) gives

$$\overline{E}_S(\infty, 0, \phi) = \frac{\hat{y}}{2} \left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+1)}{\xi_n(\rho_o)\,\xi'_n(\rho_o)} \tag{149}$$

6.4. The Surface Current

In order to find the current $\overline{I}_S$ on the surface of the sphere it is necessary to examine the tangential $\overline{H}$ fields due to both the incident and the scattered fields.

$$\left.\begin{aligned} I_{S\phi} &= H_{S\theta} + H_{I\theta} \\ I_{S\theta} &= -H_{S\phi} - H_{I\phi} \end{aligned}\right\} \text{at } r = r_o \tag{150}$$

Substitution from equations (144) and (138) gives

$$I_{S\phi} = -\frac{Y\cos\phi}{\rho_o} \sum_{n=1}^{\infty} j^{(n+1)}\, W_{1n} \cdot \left[\frac{P_n^{\,1}}{\xi'_n(\rho_o)\sin\theta} + j\,\frac{1}{\xi_n(\rho_o)} \frac{dP_n^{\,1}}{d\theta} \right]$$

$$I_{S\theta} = -\frac{Y\sin\theta}{\rho_o} \sum_{n=1}^{\infty} j^{(n+1)}\, W_{1n} \left[\frac{1}{\xi_n(\rho_o)} \frac{dP_n^{\,1}}{d\theta} + j\,\frac{P_n^{\,1}}{\xi'_n(\rho_o)\sin\theta} \right] \tag{151}$$

CHAPTER VII

THE FIELDS RADIATED FROM AN APERTURE ON THE SPHERE

7.1. The Expansion Coefficients

Following the procedure outlined by Bailin and Silver[1] the fields due to an aperture source

$$\overline{E}_o = E_{o\theta}\,\hat{\theta} + E_{o\phi}\,\hat{\phi} \tag{152}$$

are expressed in terms of expansion coefficients as in Appendix I equation (35) but with $h_n^{(2)}(kr)$ replacing $z_\nu(kr)$ since the fields due to the aperture source must satisfy the radiation condition for large r.

The coefficients $A_{mn}, \ldots, D_{mn}$ for the radiation field $\overline{E}_R$ are found by applying the condition that at $r = r_o$

$$E_{R\theta} = \begin{cases} E_{o\theta} = & \text{on the aperture} \\ 0 = & \text{elsewhere} \end{cases}$$

$$E_{R\phi} = \begin{cases} E_{o\phi} & \text{on the aperture} \\ 0 & \text{elsewhere} \end{cases} \tag{153}$$

In order to take advantage of the orthogonal properties of the θ dependent parts one cannot apply directly the conditions (153) but rather these conditions are applied in the following identities.

$$\iint\limits_{\substack{\text{Active}\\\text{Area}}} \left[E_{o\theta}(\theta, \phi) \sin\theta \frac{dP_{n'}^{m'}}{d\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} m\phi \right.$$

$$\left. + m' E_{o\phi}(\theta, \phi) \; P_{n'}^{m'} \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} m'\phi \right] d\phi d\theta$$

$$= \int_0^{\pi} \int_0^{2\pi} \left[E_{R\theta}(r_o, \theta, \phi) \sin\theta \frac{dP_{n'}^{m'}}{d\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} m'\phi \right.$$

$$\left. + m' E_{R\phi}(r_o, \theta, \phi) \; P_{n'}^{m'} \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} m'\phi \right] d\phi d\theta \tag{154}$$

$$\iint\limits_{\substack{\text{Active}\\\text{Area}}} \left[E_{o\phi}(\theta, \phi) \sin\theta \frac{dP_{n'}^{m'}}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m'\phi \right.$$

$$\left. + \; m' E_{o\theta}(\theta, \phi) \; P_{n'}^{m'} \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} m'\phi \right] d\phi d\theta$$

$$= \int_0^{\pi} \int_0^{2\pi} \left[E_{R\phi}(r_o, \theta, \phi) \sin\theta \frac{dP_{n'}^{m'}}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m'\phi \right.$$

$$\left. + m' E_{R\theta}(r_o, \theta, \phi) \; P_{n'}^{m'} \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} m'\phi \right] d\phi d\theta \tag{155}$$

If the expanded expressions for $E_{R\theta}$ and $E_{R\phi}$ are used in equations (154) and (155) integration will give

$$\iint_{\substack{\text{Active}\\ \text{Area}}} \left[E_{o\theta} \sin\theta \, \frac{dP_{n'}^{m'}}{d\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} m'\phi + m' E_{o\phi} \, P_{n'}^{m'} \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} m'\phi \right] d\theta d\phi$$

$$= \frac{k\pi}{\rho_o} (1 + \delta_{om'}) \sum_{n=0}^{\infty} \begin{Bmatrix} B_{m'n} \\ A_{m'n} \end{Bmatrix} \xi'_n(\rho_o) \int_0^{\pi} \left[\frac{dP_{n'}^{m'}}{d\theta} \, \frac{dP_n^{m'}}{d\theta} + \frac{m'^2}{\sin^2\theta} P_n^{m'} \, P_{n'}^{m'} \right] \sin\theta d\theta \tag{156}$$

In obtaining equation (156) the following facts were used.

$$\int_0^{2\pi} \sin m\phi \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} m'\phi d\phi = \pi\delta_{mm'} \begin{Bmatrix} (1 - \delta_{om}) \\ 0 \end{Bmatrix} \tag{157}$$

$$\int_0^{2\pi} \cos m\phi \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m'\phi d\phi = \pi\delta_{mm'} \begin{Bmatrix} (1 + \delta_{om}) \\ 0 \end{Bmatrix} \tag{158}$$

$$m(1 - \delta_{om}) = m(1 + \delta_{om}) \tag{159}$$

$$m' \int_0^{\pi} \left(P_n^{m'} \frac{dP_{n'}^{m'}}{d\theta} + P_{n'}^{m'} \frac{dP_n^{m'}}{d\theta} \right) d\theta = 0 \tag{160}$$

Equation (156) can be reduced further by using the Legendre associated equation in the integrand on the right-hand side as follows.

$$\int_0^{\pi} \left(\frac{dP_{n'}^{m'}}{d\theta} \frac{dP_n^{m'}}{d\theta} + \frac{m'^2 P_n^{m'} P_{n'}^{m'}}{\sin^2 \theta} \right) \sin \theta d\theta$$

$$= n(n + 1) \int_0^{\pi} P_n^{m'} P_{n'}^{m'} \sin \theta \, d\theta$$

$$= \frac{2 \delta_{nn'} n(n + 1) (n + m')!}{(2n + 1)(n - m')!}$$

$$= \frac{2 \delta_{nn'} V_n}{W_{m'n}} \tag{161}$$

where V_n and W_{mn} are defined by equations (137).

After relation (161) is used in equation (156) one may solve for $\begin{Bmatrix} A_{mn'} \\ B_{mn'} \end{Bmatrix}$ and by a similar process equation (155) yields the expression for $\begin{Bmatrix} C_{mn} \\ D_{mn} \end{Bmatrix}$ and thus all the coefficients are found as follows.

$$\begin{Bmatrix} A_{mn} \\ B_{mn} \end{Bmatrix} = \frac{\rho_o W_{mn}}{2\pi(1+\delta_{om}) V_n k \xi'_n(\rho_o)} \iint\limits_{\text{Source}} \left[E_{o\theta} \frac{dP_n^m}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\phi \right.$$

$$\left. + \frac{mE_{o\phi} P_n^m}{\sin\theta} \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} m\phi \right] \sin\theta \, d\theta d\phi$$

$$\begin{Bmatrix} C_{mn} \\ D_{mn} \end{Bmatrix} = \frac{-j\, Y \rho_o W_{mn}}{2\pi(1+\delta_{om}) V_n k \xi_n(\rho_o)} \iint\limits_{\text{Source}} \left[E_{o\phi} \frac{dP_n^m}{d\theta} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\phi \right.$$

$$\left. + \frac{mE_{o\theta} P_n^m}{\sin\theta} \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} m\phi \right] \sin\theta \, d\theta d\phi \tag{162}$$

7.2. The Surface Currents and the Radiation Field

Using these expansion coefficients the components of the field due to a source placed on the surface of the sphere are found to be,

$$E_{Rr} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{mn}\,\rho_o \xi_n(\rho) P_n^m(\cos\theta)}{2\pi(1+\delta_{om})\rho^2\, \xi'_n(\rho_o)}$$

$$\cdot \iint_{\text{Source}} \left[E_{o\theta}(\theta',\phi') \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta' \cos m(\phi-\phi') \right.$$

$$\left. + \, m\, E_{o\phi}(\theta',\phi')\, P_n^m(\cos\theta') \sin m(\phi-\phi') \right] d\phi' d\theta' \tag{163}$$

$$E_{R\theta} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{mn}\,\rho_o}{2\pi(1+\delta_{om})\, V_n \rho} \left\{ \frac{\xi'_n(\rho)}{\xi'_n(\rho_o)} \quad \frac{dP_n^m(\cos\theta)}{d\theta} \right.$$

$$\cdot \iint_{\text{Source}} \left[E_{o\theta}(\theta',\phi') \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta' \cos m(\phi-\phi') \right.$$

$$\left. + \, m\, E_{o\phi}(\theta',\phi')\, P_n^m(\cos\theta') \sin m(\phi-\phi') \right] d\phi' d\theta'$$

$$+ \, m \frac{\xi_n(\rho)\, P_n^m(\cos\theta)}{\xi_n(\rho_o)\sin\theta} \cdot \iint_{\text{Source}} \left[E_{o\phi}(\theta',\phi') \frac{dP_n^m(\cos\theta')}{d\theta'} \right.$$

$$\left. \left. \cdot \sin\theta' \sin m(\phi-\phi') + m\, E_{o\theta}(\theta',\phi')\, P_n^m(\cos\theta') \cos m(\phi-\phi') \right] d\phi' d\theta' \right\} \tag{164}$$

$$E_{R\phi} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{mn}\,\rho_o}{2\pi(1+\delta_{om})\,V_n\rho} \left\{ \frac{\xi_n(\rho)}{\xi_n(\rho_o)} \quad \frac{dP_n^m(\cos\theta)}{d\theta} \right.$$

$$\cdot \iint_{\text{Source}} \left[E_{o\phi}(\theta',\phi')\sin\theta' \,\frac{dP_n^m(\cos\theta')}{d\theta'} \quad \cos m(\phi-\phi') \right.$$

$$\left. - mE_{o\theta}(\theta',\phi')P_n^m(\cos\theta')\sin m(\phi-\phi') \right] d\phi'\,d\theta'$$

$$- m\,\frac{\xi'_n(\rho)\;P_n^m(\cos\theta)}{\xi'_n(\rho_o)\;\sin\theta} \iint_{\text{Source}} \left[E_{o\theta}(\theta',\phi')\sin\theta' \,\frac{dP_n^m(\cos\theta')}{d\theta'} \right.$$

$$\left. \left. \cdot\ \sin m(\phi-\phi') - m\,E_{o\phi}(\theta',\phi')\,P_n^m(\cos\theta')\,\cos m(\phi-\phi') \right] d\phi'd\theta' \right\} \tag{165}$$

$$H_{Rr} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-jW_{mn}Y\rho_o\xi_n(\rho)\;P_n^m(\cos\theta)}{2\pi(1+\delta_{om})\,\rho^2\;\xi_n(\rho_o)}$$

$$\cdot \iint_{\text{Source}} \left[E_{o\phi}(\theta',\phi')\sin\theta' \,\frac{dP_n^m(\cos\theta')}{d\theta'} \quad \cos m(\phi-\phi') \right.$$

$$\left. - mE_{o\theta}(\theta',\phi')\;P_n^m(\cos\theta')\;\sin m(\phi-\phi') \right] d\phi'd\theta' \tag{166}$$

$$H_{R\theta} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-j\,Y\,W_{mn}\,\rho_o}{2\pi(1+\delta_{om})V_n\rho} \left\{ \frac{\xi'_n(\rho)}{\xi_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \right.$$

$$\cdot \iint_{\text{Source}} \left[E_{o\phi}(\theta',\phi') \sin\theta' \frac{dP_n^m(\cos\theta')}{d\theta'} \cos m(\phi-\phi') \right.$$

$$\left. - m\,E_{o\theta}(\theta',\phi')\,P_n^m(\cos\theta')\sin m(\phi-\phi') \right] d\phi' d\theta'$$

$$+ m \frac{\xi_n(\rho)\,P_n^m(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} \iint_{\text{Source}} \left[E_{o\theta}(\theta',\phi') \sin\theta' \frac{dP_n^m(\cos\theta')}{d\theta'} \right.$$

$$\left. \left. \cdot \sin m(\phi-\phi') - m\,E_{o\phi}(\theta',\phi')\,P_n^m(\cos\theta')\cos m(\phi-\phi') \right] d\phi' d\theta' \right\} \qquad (167)$$

$$H_{R\phi} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{jY\,W_{mn}\,\rho_o}{2\pi(1+\delta_{om})V_n\rho} \left\{ \frac{m\,\xi'_n(\rho)\;P_n^m(\cos\theta)}{\xi_n(\rho_o)\sin\theta} \right.$$

$$\cdot \iint_{\text{Source}} \left[E_{o\phi}(\theta',\phi') \sin\theta' \frac{dP_n^m(\cos\theta')}{d\theta'} \sin m(\phi-\phi') \right.$$

$$\left. + m\,E_{o\theta}(\theta',\phi')\,P_n^m(\cos\theta')\cos m(\phi-\phi') \right] d\phi' d\theta'$$

$$- \frac{\xi_n(\rho)}{\xi'_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \iint_{\text{Source}} \left[E_{o\theta}(\theta',\phi') \sin\theta' \frac{dP_n^m(\cos\theta')}{d\theta'} \right.$$

$$\left. \left. \cdot \cos m(\phi-\phi') + m\,E_{o\phi}(\theta',\phi')P_n^m(\cos\theta')\sin m(\phi-\phi') \right] d\phi' d\theta' \right\} \qquad (168)$$

The radiation field may be found by letting $\rho \to \infty$ in $E_{R\theta}(\infty, 0, \phi)$ and noting that

$$\frac{\xi_n(\rho)}{j\rho} = \frac{\xi'_n(\rho)}{\rho} = j^n \frac{e^{-j\rho}}{\rho} \quad \text{as} \quad \rho \to \infty$$

$$\frac{P_n^m(\cos\theta)}{\sin\theta} = \frac{d\,P_n^m(\cos\theta)}{d\theta} = \delta_{1m}\frac{V_n}{2} \quad \text{as} \quad \theta \to 0 \tag{160}$$

$$\overline{E}_R(\infty, 0, \phi) = \frac{e^{-j\rho}}{\rho}\Bigg|_{\rho\to\infty} \sum_{n=0}^{\infty} \frac{W_{1n}\rho_o j^n}{4\pi}$$

$$\cdot \Bigg\{ \frac{1}{\xi'_n(\rho_o)} \iint_{\text{Source}} \Bigg[E_{o\theta}(\theta', \phi') \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta'$$

$$\cdot (\hat{x}\cos\phi' + \hat{y}\sin\phi') - E_{o\phi}(\theta', \phi') P_n^1(\cos\theta')(\hat{x}\sin\phi' - \hat{y}\cos\phi') \Bigg] d\phi' d\theta'$$

$$- \frac{j}{\xi_n(\rho_o)} \iint_{\text{Source}} \Bigg[E_{o\phi}(\theta', \phi') \frac{dP_n^1(\cos\theta')}{d\theta} \sin\theta' (\hat{x}\sin\phi' - \hat{y}\cos\phi')$$

$$+ E_{o\theta}(\theta', \phi') P_n^1(\cos\theta') (\hat{x}\cos\phi' + \hat{y}\sin\phi') \Bigg] d\phi' d\theta' \Bigg\} \tag{170}$$

The surface currents on the sphere are expressed by letting $\rho = \rho_o$ in the tangential components of the magnetic field. Since there is no significant reduction in the equations by making $\rho = \rho_o$

the result will not be rewritten here but will be left to the next chapters where the shapes of the impedances to be placed on the surface of the sphere are specified.

CHAPTER VIII

THE WEDGE IMPEDANCE ON THE SPHERE

8.1. The Shape of the Impedance Area

In considering the problem of elimination of radar backscattering from a sphere by the placement of an impedance on the surface one must decide the shape, size and location of the impedance before much further progress can be made. There are an infinite number of choices that could be made and one must be guided by several factors in making this choice. These factors include such things as construction, realizability, and ease in computation. Since other information as to overall function and purpose of the object being treated must be known in order to validate a choice based on the first two factors this work will be mainly concerned with the last mentioned factor, namely, ease in computation. If the impedance is limited to that class which have such a shape as to be bounded by coordinate surfaces then the limits on the integrals of equations (163) to (170) can be constants rather than functions. Such shapes include circumferential slots, being bounded by surfaces of constant θ, wedges, being bounded by surfaces of constant ϕ, and patches being bounded by segments of both types. Liepa and Senior[2] have examined the circumferential slot impedance on the sphere and have validated the method by experimental verification, so this work will be limited to the investigation of the wedge and patch impedances in this and the next chapters respectively.

Since the wedge is only a special case of the more general patch impedance it should seem unnecessary to give it particular attention as in a separate chapter, but for the wedge of small width the analysis is sufficiently simplified as to merit special handling. Thus this chapter deals with only the narrow wedge being bordered by ϕ_1 and ϕ_2 where

$$kr_o(\phi_2 - \phi_1) = kr_o \Delta\phi \quad << 1 \tag{171}$$

8.2. Location of the Wedge Impedance

In order to eliminate the backscattering from an object the condition is imposed that

$$\overline{E}_S + \overline{E}_R = 0 \tag{172}$$

Examination of equations (149) and (170) shows that $\overline{E}_o$ must be chosen so that $\overline{E}_R = E_{Ry}\hat{y}$ which means that

$$\begin{aligned} \int_{\phi_1}^{\phi_2} E_{o\theta}(\theta', \phi') \cos \phi' d\phi' &= 0 \\ \int_{\phi_1}^{\phi_2} E_{o\phi}(\theta', \phi') \sin \phi' d\phi' &= 0 \end{aligned} \tag{173}$$

If $E_{o\theta}$ and $E_{o\phi}$ are both non-zero constants, equations (173) lead to the contradictory conclusion:

$$\phi_2 + \phi_1 = \pi$$

$$\phi_2 + \phi_1 = 0$$

This dilemna is easily resolved by making $\Delta\phi$ sufficiently small as indicated in statement (171) so that $E_{o\theta}$ may be considered to be zero, and requiring that $\phi_2 + \phi_1 = 0$.

The wedge impedance on the sphere is analogous to the radial slot impedance on the cone and the same problem appeared in the analysis of that case. In both cases it is necessary to center the impedance at $\phi = 0$ or π. This is the most logical location anyway since the incident $\overline{E}_I$ field being polarized in the $\hat{y}$ direction would tend to give maximum current across an impedance in that position as may be seen from equations (151). Now allowing for the approximation

$$\sin\frac{\Delta\phi}{2} \doteq \frac{\Delta\phi}{2} \quad \text{for small } \Delta\phi$$

the equation (170) may be reduced to give

$$\overline{E}_R = \frac{\hat{y}\,\Delta\phi\, E_{o\phi}\rho_o e^{-j\rho}}{4\pi\rho}\Bigg|_{\rho\to\infty} \sum_{n=1}^{\infty} W_{1n}\, j^n$$

$$\cdot\left[\frac{1}{\xi'_n(\rho_o)}\int_0^{\pi} P_n^{\,1}(\cos\theta')d\theta' + \frac{j}{\xi_n\rho_o}\right.$$

$$\left.\cdot\int_0^{\pi}\frac{dP_n^{\,1}(\cos\theta')}{d\theta'}\sin\theta' d\theta'\right] \tag{174}$$

Equation (174) can be reduced further by carrying out the indicated integration using the facts that

$$P_n^1(\cos\theta) = \frac{dP_n^0}{d\theta}(\cos\theta)$$

$$\int_{-1}^{1} P_n^0(x)\,dx = 0$$

$$P_n^0(1) = 1; \qquad P_n^0(-1) = (-1)^n$$

and integrating by parts. The result is

$$\overline{E}_R = \frac{\hat{y}\Delta\phi E_{o\phi}\rho_o e^{-j\rho}}{4\pi\rho}\Bigg|_{\rho\to\infty} \sum_{n=1}^{\infty} j^n W_{1n}$$

$$\cdot\left[\frac{1}{\xi'_n(\rho_o)}\left(1-(-1)^n\right) - \frac{j}{\xi_n(\rho_o)}\left(1+(-1)^n\right)\right] \tag{175}$$

Equation (175) can also be written

$$\overline{E}_R = \frac{j\hat{y}\Delta\phi E_{o\phi}\rho_o e^{-j\rho}}{2\pi\rho}\Bigg|_{\rho\to\infty} \sum_{n=1}^{\infty} (-1)^n\left(\frac{W_{1,2n+1}}{\xi'_{2n+1}(\rho_o)} - \frac{W_{1,2n}}{\xi_{2n}(\rho_o)}\right) \tag{176}$$

Substitution from equations (149) and (176) into condition (172) and solving for $E_{o\phi}$ gives the expression

$$E_{o\phi} = \frac{\sum_{n=1}^{\infty} j(-1)^n \frac{(2n+1)}{\xi_n(\rho_o)\,\xi'_n(\rho_o)}}{\rho_o \frac{\Delta\phi}{\pi} \sum_{n=1}^{\infty} (-1)^n \left(\frac{W_{1,\,2n}}{\xi_{2n}(\rho_o)} - \frac{W_{1,\,2n+1}}{\xi'_{2n+1}(\rho_o)} \right)} \tag{177}$$

Continuing the use of the notation adopted in earlier chapters the sought impedance may be given the expression

$$Z = \frac{M}{NY_S + MY_R} \tag{178}$$

with M and N being the numerator and denominator of the expression (177) for $E_{o\phi}$, Y_S being found from the expression (151) for the surface current of the unloaded sphere, and Y_R being found from expression (167) for the surface current of the sphere with the appropriate source $E_{o\phi}$ in the wedge area. Due to the orientation of the impedance only the ϕ component of the current is used in finding the admittances Y_S and Y_R and these currents I_ϕ come from the θ components of the magnetic fields at the surface of the sphere. The integrals to be evaluated in order to find Y_R are seen from equation (167) to be

$$\int_{-\frac{\Delta\phi}{2}}^{\frac{\Delta\phi}{2}} \cos m\phi'\, d\phi' \doteq \Delta\phi \tag{179}$$

$$\int_0^{\pi} P_n^m(\cos\theta')\, d\theta' \tag{180}$$

$$\int_0^{\pi} \frac{dP_n^m(\cos\theta')}{d\theta} \sin\theta' d\theta' \tag{181}$$

Thus the impedance is found by substitution of the following quantities into equation (178),

$$M = \sum_{n=1}^{\infty} j(-1)^n \frac{2n+1}{\xi_n(\rho_o)\,\xi'_n(\rho_o)} \tag{182}$$

$$N = \frac{\rho_o \Delta\phi}{\pi} \sum_{n=1}^{\infty} (-1)^n \left(\frac{W_{1,2n}}{\xi_{2n}(\rho_o)} - \frac{W_{1,2n+1}}{\xi'_{2n+1}(\rho_o)} \right) \tag{183}$$

$$Y_S = \frac{-Y}{\rho_o} \sum_{n=1}^{\infty} j^{(n+1)} W_{1n} \left[\frac{P_n^1(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} + j \frac{1}{\xi_n(\rho_o)} \frac{dP_n^1(\cos\theta)}{d\theta} \right] \tag{184}$$

$$Y_R = \frac{-jY\Delta\phi}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{W_{mn}}{(1+\delta_{om})V_n} \cdot \left\{ \frac{\xi'_n(\rho_o)}{\xi_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \left[\int_0^{\pi} \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta' d\theta' \right] - \frac{m^2 \xi_n(\rho_o) P_n^m(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} \left[\int_0^{\pi} P_n^m(\cos\theta')\, d\theta' \right] \right\} \tag{185}$$

CHAPTER IX

THE PATCH IMPEDANCE ON THE SPHERE

The analysis of the patch impedance on the surface of the sphere is analogous to the analysis of the patch impedance on the cone which was discussed in Chapter V. The problems encountered there are similar to those which will arise in this chapter. Such problems as that of complete cancellation of both $\hat{x}$ and $\hat{y}$ components of the radar echo, and that of obtaining an isotropic impedance will be handled in this chapter in the same manner as they were in Chapter V. In fact where the work is repetitious much of it will be omitted. As was mentioned in the previous chapter the patch area is bounded by coordinate curves on the sphere. Specifically it is described by the relations,

$$\phi_1 \leq \phi \leq \phi_2$$

$$\theta_1 \leq \theta \leq \theta_2$$

$$\rho = \rho_o$$

The electric field intensity in the patch area will be designated

$$\overline{E}_o = E_{o\theta} (\hat{\theta} + K\hat{\phi}) \qquad (186)$$

where again K is used to make the impedance isotropic and

$$K = \frac{E_{o\phi}}{E_{o\theta}} \qquad (187)$$

The four admittance symbols corresponding to those of section 5.1 equation (102) are

$$Y_{\theta\theta} = \frac{H_{R\theta\theta}}{E_{o\theta}}$$

$$Y_{\theta\phi} = \frac{H_{R\theta\phi}}{E_{o\theta}}$$

$$Y_{\phi\theta} = \frac{H_{R\phi\theta}}{E_{o\phi}} \tag{188}$$

$$Y_{\phi\phi} = \frac{H_{R\phi\phi}}{E_{o\phi}}$$

It will be remembered that the middle subscript on H in equations (188) indicates the sole cause for that component. For example $H_{R\theta\phi}$ is that part of the $\hat{\phi}$ component of $\overline{H}_R$ which is due solely to $E_{o\theta}$ and not to $E_{o\phi}$. One must also remember that these fields are to be found at $\rho = \rho_o$.

In terms of these admittances and the components of $\overline{I}_S$ obtained from equation (151), one is able to write the expression for the total surface current $\overline{I}_T$ on the loaded sphere as

$$\begin{aligned} \overline{I}_T &= \overline{I}_S + \overline{I}_R \\ &= E_{o\theta}\left[\left(I_{S\theta}/E_{o\theta} - Y_{\theta\phi} - K\,Y_{\phi\phi}\right)\hat{\theta} \right. \\ &\quad \left. + \left(I_{S\phi}/E_{o\theta} + Y_{\theta\theta} + K\,Y_{\phi\theta}\right)\hat{\phi}\right] \end{aligned} \tag{189}$$

There are two conditions which must be applied to provide that (1) the impedance is isotropic and that (2) the radar echo is eliminated. These conditions precisely stated are as follows.

$$\overline{E}_o \times \overline{I}_T = 0 \tag{190}$$

$$\overline{E}_R + \overline{E}_S = 0 \tag{191}$$

Applying these conditions simultaneously will give an expression for K independent of $\overline{E}_o$ analogous to equation (110) in section 5.1.

$$K = \frac{(C_1C_2+C_3C_4-Y_{\phi\theta}-Y_{\theta\phi}) \pm \sqrt{(C_1C_2+C_3C_4-Y_{\phi\theta}-Y_{\theta\phi})^2-4(C_1C_4+Y_{\phi\phi})(C_2C_3+Y_{\theta\theta})}}{2(C_1C_4+Y_{\phi\phi})} \tag{192}$$

Where

$$\begin{aligned} C_1 &= E_{R\phi\theta}(\infty, 0, \phi)/E_{o\phi} \\ C_2 &= I_{S\phi}/E_S(\infty, 0, \phi) \\ C_3 &= -E_{R\theta\theta}(\infty, 0, \phi)/E_{o\theta} \\ C_4 &= I_{S\theta}/E_S(\infty, 0, \phi) \end{aligned} \tag{193}$$

In equations (193) the three subscripts on E have the same meaning as that given to those on H of equation (188). It was found in section 5.3 that in order to eliminate both $\hat{x}$ and $\hat{y}$ components of the radar echo it was necessary to place two impedances symmetrically across the $\phi = \pi/2$ plane on the cone with the two electric fields $\overline{E}_{o1}$ and $\overline{E}_{o2}$ given in equations (126) and (127). In this section the same approach will be taken and the fields are

$$\overline{E}_o = \overline{E}_{o1} = E_{o\theta}(\hat{\theta} + K\hat{\phi}) \;;\quad \phi_1 \le \phi \le \phi_2$$

$$\overline{E}_o = \overline{E}_{o2} = E_{o\theta}(\hat{\theta} - K\hat{\phi}) \;;\quad \pi - \phi_2 \le \phi \le \pi - \phi_1 \tag{194}$$

$$\overline{E}_o = 0 \qquad\qquad \text{elsewhere}$$

These conditions make $\overline{E}_R(\infty, 0, \phi)$ have only a $\hat{y}$ component as may be seen from equation (170). Carrying out the integration of the ϕ' dependent parts of this equation involves two sets of integrals the first having limits ϕ_1 to ϕ_2 the second set having limits $\pi - \phi_2$ and $\pi - \phi_1$ and having a sign change if the integral contains $E_{o\phi}$ as a factor. As a result the expression for $\overline{E}_R(\infty, 0, \phi)$ becomes

$$\overline{E}_R(\infty, 0, \phi) = \frac{\hat{y}\, e^{-j\rho}}{2\pi\rho}\Bigg|_{\rho \to \infty} \left[E_{o\theta}(\cos\phi_1 - \cos\phi_2)S_3 + E_{o\phi}(\sin\phi_2 - \sin\phi_1)S_1\right] \tag{195}$$

where

$$S_1 = \sum_{n=1}^{\infty} W_{1n}\rho_o j^n \int_{\theta_1}^{\theta_2} \left[\frac{P_n^1(\cos\theta')}{\xi'_n(\rho_o)} + \frac{j}{\xi_n(\rho_o)} \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta'\right] d\theta'$$

$$S_3 = \sum_{n=1}^{\infty} W_{1n}\rho_o j^n \int_{\theta_1}^{\theta_2} \left[\frac{1}{\xi'_n(\rho_o)} \frac{dP_n^1(\cos\theta')}{d\theta'} - j\frac{P_n^1(\cos\theta')}{\xi_n(\rho_o)}\right] d\theta' \tag{196}$$

Equations (195) and (196) make it possible to identify the constant expressions for C_1 and C_3 as follows

$$C_1 = \left.\frac{e^{-j\rho}}{2\pi\rho}\right|_{\rho\to\infty} \cdot (\sin\phi_2 - \sin\phi_1)\, S_1$$

$$C_3 = \left.\frac{e^{-j\rho}}{2\pi\rho}\right|_{\rho\to\infty} \cdot (\cos\phi_2 - \cos\phi_1)\, S_3 \tag{197}$$

The quantities C_2 and C_4 are not constants but are dependent on both ϕ and θ and they are found from the equations (151) for the surface current and (149) for the back scattered field for the unloaded sphere.

$$C_2 = \frac{\dfrac{2Y\cos\phi}{\rho_o}\displaystyle\sum_{n=1}^{\infty} j^{(n+1)} W_{1n}\left[\frac{P_n^1(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} + \frac{j}{\xi_n(\rho_o)}\,\frac{dP_n^1(\cos\theta)}{d\theta}\right]}{\left.\dfrac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,(2n+1)}{\xi_n(\rho_o)\,\xi'_n(\rho_o)}}$$

$$C_4 = \frac{\dfrac{2Y\sin\phi}{\rho_o}\displaystyle\sum_{n=1}^{\infty} j^{(n+1)} W_{1n}\left[\frac{1}{\xi_n(\rho_o)}\,\frac{dP_n^1(\cos\theta)}{d\theta} + j\,\frac{P_n^1(\cos\theta)}{\xi'_n(\rho_o)\sin\theta}\right]}{\left.\dfrac{e^{-j\rho}}{\rho}\right|_{\rho\to\infty}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,(2n+1)}{\xi_n(\rho_o)\,\xi'_n(\rho_o)}} \tag{198}$$

In expressing the four admittance quantities the equations (167) and (168) are used and the integration over the source involves θ from θ_1 to θ_2 and two ranges for ϕ, namely ϕ_1 to ϕ_2 and $(\pi - \phi_2)$ to $(\pi - \phi_1)$. Since the symmetry across the $\phi = \pi/2$ plane causes certain terms to be zero in the integration for ϕ it is convenient to introduce two new symbols defined as follows

$$\delta_{m,ev} = \begin{Bmatrix} 1 & \text{if } m \text{ is} & \text{even} \\ 0 & & \text{odd} \end{Bmatrix}$$

$$\delta_{m,od} = \begin{Bmatrix} 1 & \text{if } m \text{ is} & \text{odd} \\ 0 & & \text{even} \end{Bmatrix} \tag{199}$$

In terms of these symbols the four admittances are found to be

$$Y_{\theta\theta} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-jYW_{mn}}{\pi(1+\delta_{om})V_n} \Big[\delta_{m,ev}(\sin m\phi)(\sin m\phi_2 - \sin m\phi_1)$$

$$+ \delta_{m,od}(\cos m\phi)(\cos m\phi_2 - \cos m\phi_1)\Big]$$

$$\cdot \left[\frac{\xi'_n(\rho_o)}{\xi_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \int_{\theta_1}^{\theta_2} P_n(\cos\theta')\, d\theta'\right.$$

$$\left. + \frac{\xi_n(\rho_o)\, P_n^m(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} \int_{\theta_1}^{\theta_2} \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta'\, d\theta'\right]$$

$$Y_{\phi\theta} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{jW_{mn}}{\pi(1+\delta_{om})V_n^m} \left[\delta_{m,ev}(\sin m\phi)(\cos m\phi_2 - \cos m\phi_1)\right.$$

$$\left. - \delta_{m,od}(\cos m\phi)(\sin m\phi_2 - \sin m\phi_1)\right]$$

$$\cdot \left[\frac{\xi'_n(\rho_o)}{\xi_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \int_{\theta_1}^{\theta_2} \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta' \, d\theta'\right.$$

$$\left. - \frac{m^2 \xi_n(\rho_o) P_n^m(\cos\theta)}{\xi'_n(\rho_o)\sin\theta} \int_{\theta_1}^{\theta_2} P_n^m(\cos\theta')\, d\theta'\right]$$

$$Y_{\theta\phi} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{jYW_{mn}}{\pi(1+\delta_{om})V_n{}^m} \left[\delta_{m,ev}(\cos m\phi)(\sin m\phi_2 - \sin m\phi_1)\right.$$

$$\left. - \delta_{m,od}(\sin m\phi)(\cos m\phi_2 - \cos m\phi_1)\right]$$

$$\cdot \left[\frac{m^2 \xi'_n(\rho_o) P_n^m(\cos\theta)}{\xi_n(\rho_o)\sin\theta} \int_{\theta_1}^{\theta_2} P_n^m(\cos\theta')\, d\theta'\right.$$

$$\left. - \frac{\xi_n(\rho_o)}{\xi'_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \int_{\theta_1}^{\theta_2} \frac{dP_n^m(\cos\theta')}{d\theta} \sin\theta' \, d\theta'\right]$$

$$Y_{\phi\phi} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{jYW_{mn}}{\pi(1+\delta_{om})V_n} \Big[\delta_{m,ev}(\cos m\phi)(\cos m\phi_2 - \cos m\phi_1)$$

$$+ \delta_{m,od}(\sin m\phi)(\sin m\phi_2 - \sin m\phi_1) \Big]$$

$$\cdot \left[\frac{\xi'_n(\rho_o) P_n^m(\cos\theta)}{\xi_n(\rho_o)\sin\theta} \int_{\theta_1}^{\theta_2} \frac{dP_n^m(\cos\theta')}{d\theta'} \sin\theta'\, d\theta' \right.$$

$$\left. - \frac{\xi_n(\rho_o)}{\xi'_n(\rho_o)} \frac{dP_n^m(\cos\theta)}{d\theta} \int_{\theta_1}^{\theta_2} P_n^m(\cos\theta')\, d\theta' \right] \tag{200}$$

The expression for the impedance may be found from these quantities to have the same form as it had in section 5.1 equation (113).

$$Z = \frac{1}{C_3C_4 - Y_{\theta\phi} - K(C_1C_4 + Y_{\phi\phi})} \tag{201}$$

It will be noticed that C_i never appears in Z or K except as a product C_iC_j where ij is an odd number. Examination of equations (197) and (198) will show that in this product the factor

$$\left. \frac{e^{-j\rho}}{\rho} \right|_{\rho \to \infty}$$

will cancel out so that Z is dependent only on the variables θ and φ.

It will be remembered that at the conclusion of Chapter V a discussion was offered concerning the possibility of using four patch impedances spaced $\pi/2$ apart around the z axis on the surface of the cone to eliminate the radar backscattering of arbitrary incident polarization. The same discussion would apply to the case of the sphere as also would the discussion about using small patch areas so as to allow the use of a constant impedance in the patch area. Since these considerations would only be a repetition and would offer nothing essentially new they will be omitted from this chapter.

The analysis of the sphere has two obvious advantages over that of the infinite cone. The first is seen in the fact that the values of the functions $P_n^m(\cos\theta)$ and $z_n(kr)$ are more easily found, and the second is in the fact that the sphere, being a finite structure offers more in the way of practical application.

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APPENDIX I

SPHERICAL MODE FUNCTIONS

1. The Radial Vector Potential for the TM Modes

The general magnetic and electric fields may be expressed in the spherical coordinate system in terms of two independent scalar potentials as follows.

From Maxwell's equations the electric field intensity in a homogeneous charge free medium is

$$\overline{E} = \frac{-j}{\omega\mu\epsilon} \nabla \times \overline{B}$$

Using the vector potential $\overline{A}$ defined in part by

$$\overline{B} = \nabla \times \overline{A}$$

gives

$$\overline{E} = \frac{-j}{\omega\mu\epsilon} \nabla \times \nabla \times \overline{A} \tag{1}$$

The condition is arbitrarily imposed on $\overline{A}$ that it is a radial vector

$$\overline{A} = A_r \hat{r}$$

The results of this limitation is dealt with later. The components of $\overline{E}$ in terms of A_r are then found by expansion of equation (1).

$$E_r = \frac{j}{\omega\mu\epsilon r^2 \sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial A_r}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 A_r}{\partial\phi^2} \right] \tag{2}$$

$$E_\theta = \frac{-j}{\omega \mu \epsilon r} \frac{\partial^2 A_r}{\partial r \partial \theta} \tag{3}$$

$$E_\phi = \frac{-j}{\omega \mu \epsilon r \sin\theta} \frac{\partial^2 A_r}{\partial r \partial \phi} \tag{4}$$

2. The Gauge Condition

An alternative expression also from Maxwell's equations for the electric field intensity is

$$\overline{E} = -j\omega \overline{A} - \nabla\Phi \tag{5}$$

The components of which are

$$E_r = -j\omega A_r - \frac{\partial \Phi}{\partial r} \tag{6}$$

$$E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \tag{7}$$

$$E_\phi = -\frac{1}{r \sin\theta} \frac{\partial \Phi}{\partial \phi} \tag{8}$$

The gauge condition compatible with equations (3), (4), (7), and (8), is

$$\Phi = \frac{j}{\omega \mu \epsilon} \frac{\partial A_r}{\partial r} \tag{9}$$

3. The Scalar Wave Equation

Equating the right-hand sides of equations (2) and (6) gives the differential equation for A_r

$$- j\omega A_r - \frac{\partial \Phi}{\partial r} = \frac{j}{\omega \mu \epsilon\, r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_r}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right]$$

which when rearranged and with Φ eliminated with the use of equation (9), becomes

$$\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_r}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right] + k^2 A_r = 0 \tag{10}$$

where $k = \omega\sqrt{\mu\,\epsilon}$

Now by letting $A_r = r\Pi$ where Π is a scalar function equation (10) becomes

$$r\frac{\partial^2 \Pi}{\partial r^2} + 2\frac{\partial \Pi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right)$$

$$+ \frac{1}{r\sin^2\theta}\frac{\partial^2\Pi}{\partial\phi^2} + k^2 r\,\Pi = 0 \tag{11}$$

If equation (11) is divided by r the first two terms are identical to the expansion of $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Pi}{\partial r}\right)$. Thus equation (11) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Pi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Pi}{\partial\phi^2} + k^2\Pi = 0 \tag{12}$$

Equation (12) is the scalar wave equation in spherical coordinates and is separable. Multiplication by r^2 and regrouping yields

$$\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Pi}{\partial r}\right) + k^2 r^2 \Pi\right] + \left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\ \frac{\partial \Pi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Pi}{\partial \phi^2}\right] = 0 \tag{13}$$

4. Separation of Variables

If Π is assumed to be the product of three functions $f_1(r)$, $f_2(\theta)$ and $f_3(\phi)$, each of which is dependent on only one coordinate then after division of equation (13) by Π the result is

$$\left[\frac{1}{f_1}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f_1}{\partial r}\right) + k^2 r^2\right] + \left[\frac{1}{\sin\theta\ f_2}\frac{\partial}{\partial \theta}\left(\sin\theta\ \frac{\partial f_2}{\partial \theta}\right) + \frac{1}{\sin^2\theta\ f_3}\frac{\partial^2 f_3}{\partial \phi^2}\right] = 0 \tag{14}$$

Separation constants ν and m are used as follows. The first bracket in equation (14) is the only part of the equation which has any r dependence, and it has only r dependence so it must be equal to a constant which for reasons to be seen later will be designated $\nu(\nu + 1)$.

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f_1}{\partial r}\right) + (k^2 r^2 - \nu(\nu+1))\ f_1 = 0 \tag{15}$$

Since only one variable is involved equation (15) may be rewritten using full derivatives.

$$\frac{d}{dr}\left[r^2 \frac{df_1}{dr}\right] + \left[k^2r^2 - \nu(\nu+1)\right]f_1 = 0 \qquad (16)$$

In order to obtain a differential equation of a familiar form from equation (16) the substitution

$$f_1 = \frac{Z(kr)}{\sqrt{kr}}$$

is made and after simplification the result is

$$r^2 \frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - (\nu + \tfrac{1}{2})^2\right]Z = 0 \qquad (17)$$

From equation (17) it is seen that Z must be a cylinder function of half order.

$$f_1(r) = \frac{1}{\sqrt{kr}} Z_{\nu+\frac{1}{2}}(kr)$$

Equation (16) is satisfied then by a spherical Bessel function defined as follows

$$z_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, Z_{n+\frac{1}{2}}(\rho) \qquad j_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{n+\frac{1}{2}}(\rho)$$

$$n_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, N_{n+\frac{1}{2}}(\rho) \qquad h_n^{(2)} = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{n+\frac{1}{2}}(\rho)$$

The function $f_1(r)$ may be expressed as

$$f_1(r) = z_\nu(kr) \qquad (18)$$

where f_1 is redefined to absorb a constant factor.

The remaining part of equation (14) gives

$$\left[\frac{\sin\theta}{f_2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f_2}{\partial\theta}\right)+\nu(\nu+1)\sin^2\theta\right]+\frac{1}{f_3}\frac{\partial^2 f_3}{\partial\phi^2}=0 \qquad (19)$$

Here again in equation (19) separation of variables is used with the separation constant m^2 which gives two differential equations.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df_2}{d\theta}\right)+\nu(\nu+1)f_2-\frac{m^2 f_2}{\sin^2\theta}=0 \qquad (20)$$

$$\frac{d^2 f_3}{d\phi^2}+m^2 f_3=0 \qquad (21)$$

Upon substitution of $\xi = \cos\theta$ equation (20) is

$$\left(1-\xi^2\right)\frac{d^2 f_2}{d\xi^2}-2\xi\frac{df_2}{d\xi}+\left(\nu(\nu+1)-\frac{m^2}{1-\xi^2}\right)f_2=0 \qquad (22)$$

The solutions of equation (21) are the family of sinusoidal functions. In order for f_3 to be single valued in ϕ the constant, m, must be an integer $m = 0, \pm 1, \pm 2$

$$f_3=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi \qquad (23)$$

The solutions to equation (22) are the associated Legendre functions

$$f_2 = P_\nu^m(\xi) \qquad (24)$$

The values of both m and ν are restricted to the positive numbers and zero. The constant m is an integer whereas the values of ν are to be determined by the bounderies of the region considered. For the conducting sphere the points $\xi = \pm 1$ corresponding to $\theta = 0$, 180° are both in the region and this requires ν to be an integer. For the cone the $\theta = 180^{\circ}$ point is excluded so ν will not be an integer. The possible values of ν will be determined by the fact that the tangential electric field on the surface of the cone is zero.

Combining the results of equations (18), (23), and (24) the solution to the differential equation (12) is found to be of the form

$$\Pi = z_{\nu}(kr)\, P_{\nu}^{m}(\cos\theta) \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} m\phi$$

The general solution is the infinite series summed over all possible values of m and ν

$$\Pi = \sum_{m} \sum_{\nu} z_{\nu}(kr)\, P_{\nu}^{m}(\cos\theta) \left[A_{m\nu} \cos m\phi + B_{m\nu} \sin m\phi \right] \tag{25}$$

5. Field Components for the TM Modes

Returning now to the problem of expressing the fields in terms of Π, the substitution is made

$$A_r = r\,\Pi \tag{26}$$

into equations (2), (3), and (4) and using equation (10) to simplify the expression for E_r the components of E are

$$E_r = \frac{-j}{\omega\mu\epsilon}\left[\frac{\partial^2 (r\Pi)}{\partial r^2} + k^2 r\Pi\right]$$

$$E_\theta = \frac{-j}{\omega\mu\epsilon r}\left[\frac{\partial^2 (r\Pi)}{\partial r\partial\theta}\right] \tag{27}$$

$$E_\phi = \frac{-j}{\omega\mu\epsilon r\sin\theta}\,\frac{\partial^2 (r\Pi)}{\partial r\partial\phi}$$

With the use of the relations

$$\overline{B} = \mu\overline{H} = \nabla \times \overline{A}$$

The components of $\overline{H}$ are

$$H_r = 0$$

$$H_\theta = \frac{1}{\mu r\sin\theta}\,\frac{\partial (r\Pi)}{\partial\phi} \tag{28}$$

$$H_\phi = -\frac{1}{\mu r}\,\frac{\partial (r\Pi)}{\partial\theta}$$

From the equations (28) it is seen that the radial component of magnetic field intensity is zero. This fact implies that the scalar potential Π gives only the TM modes and is the result of the condition imposed that

$$\overline{A} = A_r\,\hat{r}$$

6. The TE Modes

In order to have a completely general field there must be some non-zero expression for H_r. This may be obtained from another scalar potential.

Let $\overline{A}^*$ be defined by the relation

$$\overline{D}^* = \nabla \times \overline{A}^* \tag{29}$$

with the imposed condition

$$\overline{A}^* = A_r^* \, \hat{r} \tag{30}$$

Equations (29) and (30) make the radial component of the electric field intensity zero.

The potential $\overline{A}$ gives rise to the TM part of the field and $\overline{A}^*$ gives rise to the TE part. The total field is the sum of the two parts. Following a procedure for $\overline{A}^*$ analogous to that for $\overline{A}$ it is found that

$$\Pi^* = \sum_m \sum_{\nu'} z_{\nu'} (kr) \, P_{\nu'}^m (\cos\theta) \Big[C_{m\nu'} \cos m\phi$$

$$+ D_{m\nu'} \sin m\phi \Big] \tag{31}$$

The resulting field components are as follows

$$E_r^* = 0$$

$$E_\theta^* = \frac{1}{\epsilon r \sin\theta} \frac{\partial (r\Pi^*)}{\partial \phi}$$

$$E_\phi^* = - \frac{1}{\epsilon r} \frac{\partial (r\Pi^*)}{\partial \theta}$$

$$H_r^* = \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2 (r\Pi^*)}{\partial r^2} + k^2 r\Pi^*\right]$$

$$H_\theta^* = \frac{j}{\omega\mu\epsilon r}\left[\frac{\partial^2 (r\Pi^*)}{\partial r\,\partial\theta}\right]$$

$$H_\phi^* = \frac{j}{\omega\mu\epsilon r}\sin\theta\,\frac{\partial^2 (r\Pi^*)}{\partial r\,\partial\phi} \tag{32}$$

7. The General Field Components

The general fields are expressed as a linear combination of the two sets of equations. Since the equations are homogeneous it is convenient to absorb the constants $-\frac{j}{\omega\mu\epsilon}$ and $+\frac{j}{\omega\mu\epsilon}$ into the scalars Π and Π^* respectively and redefine them accordingly. The general field components are then

$$E_r = \frac{\partial^2 (r\Pi)}{\partial r^2} + k^2 r\Pi$$

$$E_\theta = \frac{1}{r}\frac{\partial^2 (r\Pi)}{\partial r\,\partial\theta} - \frac{j\omega\mu}{r\sin\theta}\frac{\partial (r\Pi^*)}{\partial\phi}$$

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2 (r\Pi)}{\partial r\,\partial\phi} + \frac{j\omega\mu}{r}\frac{\partial (r\Pi^*)}{\partial\theta}$$

$$H_r = \frac{\partial^2 (r\Pi^*)}{\partial r^2} + k^2 r\Pi^*$$

$$H_\theta = \frac{1}{r} \quad \frac{\partial^2 (r\Pi^*)}{\partial r\, \partial\theta} + \frac{j\omega\,\epsilon}{r \sin\theta} \quad \frac{\partial\, (r\Pi)}{\partial\,\phi}$$

$$H_\phi = \frac{1}{r \sin\theta} \quad \frac{\partial^2 (\, r\Pi^*)}{\partial r\, \partial\phi} - \frac{j\omega\,\epsilon}{r} \quad \frac{\partial\, (r\Pi)}{\partial\theta} \tag{33}$$

where

$$\Pi = \sum_m \sum_\nu z_\nu\,(kr)\, P_\nu^m(\cos\theta)\, \left[A_{m\nu}\cos m\phi + B_{m\nu}\sin m\phi\right] \tag{25}$$

and

$$\Pi^* = \sum_m \sum_{\nu'} z_{\nu'}\,(kr)\, P_{\nu'}^m\,(\cos\theta)\, \left[C_{m\nu'}\cos m\phi + D_{m\nu'}\sin m\phi\right] \tag{34}$$

With the substitution of equation (25), and (34) into equations (33) the general expressions for the field components with both TM and TE modes are as follows:

$$E_r = \sum_m \sum_\nu \frac{\nu(\nu+1)}{r}\; z_\nu(kr)\, P_\nu^m\,(\cos\theta)\, \left[A_{m\nu}\cos m\phi + B_{m\nu}\sin m\phi\right]$$

$$E_\theta = \sum_m \sum_\nu \frac{1}{r}\; \frac{d}{dr}\left[r\, z_\nu(kr)\right] \frac{d}{d\theta}\, P_\nu^m\, \left[A_{m\nu}\cos m\phi + B_{m\nu}\sin m\phi\right]$$

$$+\, j\,\omega\mu \sum_m \sum_{\nu'} \frac{m}{\sin\theta}\; z_{\nu'}\; P_{\nu'}^m\, \left[C_{m\nu'}\sin m\phi - D_{m\nu'}\cos m\phi\right]$$

$$E_\phi = \sum_m \sum_\nu \frac{-m}{r \sin\theta} \frac{d}{dr} \left[r\, z_\nu \right] P_\nu^m \left[A_{m\nu} \sin m\phi - B_{m\nu} \cos m\phi \right]$$

$$+ j\,\omega\mu \sum_m \sum_{\nu'} z_{\nu'} \frac{d}{d\theta} P_{\nu'}^m \left[C_{m\nu'} \cos m\phi + D_{m\nu'} \sin m\phi \right]$$

$$H_r = \sum_m \sum_{\nu'} \frac{\nu'(\nu'+1)}{r} z_{\nu'} P_{\nu'}^m \left[C_{m\nu'} \cos m\phi + D_{m\nu'} \sin m\phi \right]$$

$$H_\theta = \sum_m \sum_{\nu'} \frac{1}{r} \frac{d}{dr} \left[r\, z_{\nu'} \right] \frac{d}{d\theta} P_{\nu'}^m \left[C_{m\nu'} \cos m\phi + D_{m\nu'} \sin m\phi \right]$$

$$- j\,\omega\epsilon \sum_m \sum_\nu \frac{m}{\sin\theta} z_\nu P_\nu^m \left[A_{m\nu} \sin m\phi - B_{m\nu} \cos m\phi \right]$$

$$H_\phi = \sum_m \sum_{\nu'} \frac{-m}{r \sin\theta} \frac{d}{dr} \left[r\, z_{\nu'} \right] P_{\nu'}^m \left[C_{m\nu'} \sin m\phi - D_{m\nu'} \cos m\phi \right]$$

$$- j\omega\epsilon \sum_m \sum_\nu z_\nu \frac{d}{d\theta} P_\nu^m \left[A_{m\nu} \cos m\phi + B_{m\nu} \sin m\phi \right] \qquad (35)$$

In equations (35) use was made of the fact that

$$\frac{\partial^2(r\,\Pi)}{\partial r^2} + k^2\, r\,\Pi = \frac{\nu(\nu+1)\,\Pi}{r} \tag{36}$$

as may be seen from equation (15).

APPENDIX II

PROPERTIES OF THE ASSOCIATED LEGENDRE FUNCTIONS

In order to facilitate the following derivations the operator L_ν^m will be introduced as follows

$$L_\nu^m(f) \equiv \sin\theta f'' + \cos\theta f' + \nu(\nu+1)\sin\theta f - \frac{m^2}{\sin\theta} f \tag{37}$$

Thus if $\qquad L_\nu^m(f) \equiv 0$

then $\qquad f \equiv P_\nu^m(\cos\theta) \qquad (38)$

Another abbreviation will also be used as follows

$$V_i = \nu_i(\nu_i + 1)$$

$$V'_i = \nu'_i(\nu'_i + 1) \tag{39}$$

1. Proof of the Orthogonality of the Associated Legendre Functions in the Range $\theta = 0$ to θ_o

The relation to be proved is as follows

$$\int_0^{\theta_o} P_{\nu_i}^m \, P_{\nu_j}^m \sin\theta \, d\theta = \delta_{ij} \int_0^{\theta_o} \left[P_{\nu_i}^m\right]^2 \sin\theta \, d\theta \tag{40}$$

The symbol δ_{ij} is the Krondecker delta. Since $L_{\nu_i}^m (P_{\nu_i}^m) = 0$ the relation holds,

$$P_{\nu_i}^m L_{\nu_j}^{m'}\left(P_{\nu_j}^{m'}\right) - P_{\nu_j}^{m'} L_{\nu_i}^m\left(P_{\nu_i}^m\right) = 0 \tag{41}$$

Relation (41) can be put into the form

$$\frac{d}{d\theta}\left[\sin\theta\left(P_{\nu_i}^{m}\frac{dP_{\nu_j}^{m'}}{d\theta} - P_{\nu_j}^{m'}\frac{dP_{\nu_i}^{m}}{d\theta}\right)\right]$$

$$= (m'^2 - m^2)\frac{P_{\nu_i}^{m}P_{\nu_j}^{m'}}{\sin\theta} + (V_i - V_j)\,P_{\nu_i}^{m}P_{\nu_j}^{m'}\sin\theta \tag{42}$$

Integration of equation (42) over the range $\theta = 0$ to θ_o yields

$$\sin\theta\left(P_{\nu_i}^{m}\frac{dP_{\nu_j}^{m'}}{d\theta} - P_{\nu_j}^{m'}\frac{dP_{\nu_i}^{m}}{d\theta}\right)\Bigg|_0^{\theta_o}$$

$$= (m'^2 - m^2)\int_0^{\theta_o}\frac{P_{\nu_i}^{m}P_{\nu_j}^{m'}}{\sin\theta}\,d\theta + (V_i - V_j)\int_0^{\theta_o}P_{\nu_i}^{m}P_{\nu_j}^{m'}\sin\theta\,d\theta \tag{43}$$

If $m' = m$ the equation (43) becomes

$$\sin\theta_o\left(P_{\nu_i}^{m}\frac{dP_{\nu_j}^{m}}{d\theta} - P_{\nu_j}^{m}\frac{dP_{\nu_i}^{m}}{d\theta}\right)_{\theta_o}$$

$$= (V_i - V_j)\int_0^{\theta_o}P_{\nu_i}^{m}P_{\nu_j}^{m}\sin\theta\,d\theta \tag{44}$$

The orthogonality of the associated Legendre functions depends on the fact that the bracket on the left hand side is zero. If the boundary condition at $\theta = \theta_o$ requires that

$$P_{\nu_i}^m(\cos\theta_o) = \left.\frac{d\,P_{\nu'_i}^m(\cos\theta)}{d\theta}\right|_{\theta_o} = 0 \quad ; \begin{array}{l} i = 0, 1, 2, \ldots \\ m = 0, 1, 2, \ldots \end{array} \tag{45}$$

then the functions $P_{\nu_i}^m$ and $P_{\nu'_i}^m$ are orthogonal in the set over $i = 0, 1, 2, \ldots,$ and equation (40) holds.

Two further properties of the Legendre associated functions were used in section 2.5 to reduce the expression for K, they are shown below as equations (46) and (47).

$$\int_0^{\theta_o} \left[m^2 \frac{P_{\nu_i}^m P_{\nu_j}^m}{\sin^2\theta} + \frac{d\,P_{\nu_i}^m}{d\theta} \frac{d\,P_{\nu_j}^m}{d\theta} \right] \sin\theta\, d\theta$$

$$= \delta_{ij}\, V_i \int_0^{\theta_o} \left(P_{\nu_i}^m\right)^2 \sin\theta\, d\theta \tag{46}$$

$$\int_0^{\theta_o} \left[P_{\nu'_j}^m \frac{d\,P_{\nu_i}}{d\theta} + P_{\nu_i}^m \frac{d\,P_{\nu'_j}^m}{d\theta} \right] d\theta = 0 \tag{47}$$

Expansion of the equation $P_{\nu_i}^m\, L_{\nu_j}^m (P_{\nu_j}^m) = 0$ gives

$$\sin\theta\, P_{\nu_i}^{m} \frac{d^2 P_{\nu_j}^{m}}{d\theta^2} + \cos\theta \frac{d P_{\nu_j}^{m}}{d\theta} P_{\nu_i}^{m}$$

$$+ \quad V_j \sin\theta - \frac{m^2}{\sin\theta} \quad P_{\nu_i}^{m} P_{\nu_j}^{m} = 0 \tag{48}$$

Rearranging and combining terms in equation (48) yields

$$\frac{d}{d\theta}\left[\sin\theta\, P_{\nu_i}^{m} \frac{d P_{\nu_j}^{m}}{d\theta}\right] - \sin\theta \frac{d P_{\nu_i}^{m}}{d\theta} \frac{d P_{\nu_j}^{m}}{d\theta} = \left(\frac{m^2}{\sin\theta} - V_j\right) \sin\theta \tag{49}$$

Integration of equation (49) and application of either of the boundary conditions gives

$$\int_0^{\theta_o} \left[\frac{m^2 P_{\nu_i}^{m} P_{\nu_j}^{m}}{\sin^2\theta} + \frac{d P_{\nu_i}^{m}}{d\theta} \frac{d P_{\nu_j}^{m}}{d\theta}\right] \sin\theta\, d\theta$$

$$= V_j \int_0^{\theta_o} P_{\nu_i}^{m} P_{\nu_j}^{m} \sin\theta\, d\theta \tag{50}$$

Equation (50) could also have been obtained with V'_j replacing V_j. It has already been shown that if $i \neq j$ the right hand side of equation (50) is zero. Thus equation (50) becomes the desired result (46).

The left hand side of equation (47) may be rewritten as follows

$$\int_0^{\theta_o} \frac{d}{d\theta}\left(P_{\nu_i}^{m}\; P_{\nu'_j}^{m}\right) d\,\theta = P_{\nu_i}^{m}\; P_{\nu'_j}^{m}\Bigg|_{\theta=0}^{\theta_o} \qquad (51)$$

Letting $\theta = 0$ in the expression $\sin\theta\, L_{\nu_i}^{m}(P_{\nu_i}^{m}) = 0$ shows that

$$m^2\, P_{\nu_i}^{m}(\cos\theta)\Bigg|_{\theta=0} = 0 \qquad (52)$$

Boundary condition (45) with equation (52) which, incidentally, also holds for ν'_i, make the right hand side of equation (51) vanish and consequently equation (47) holds for $m \neq 0$.

2. Finding the Eigenpairs V_i and $P_{\nu_i}^{m}$

The boundary conditions (45) cause the set ν_i ; $i = 0, 1, 2, \ldots,$ to be different for each m, necessitating the use of a double subscript on ν as ν_{mi}. This means that one must find each ν_{mi} independently. The problem of finding the ν_{mi} and ν'_{mi} is of considerable difficulty. If in equation (37) $\nu = \nu_{m+1,\,i}$ and $(m + 1)$ is used for m while the function f is

$$f = \frac{m\cos\theta}{\sin\theta}\; P_{\nu_{m+1,\,i}}^{m} - \frac{d\,P_{\nu_{m+1,\,i}}^{m}}{d\theta}$$

where $P_{\nu_{m+1,\,i}}^{m}$ is the solution of $L_{\nu_{m+1,\,i}}^{m}(P) = 0$ then the result reduces identically to zero and in accordance with statement (38)

$$P_{\nu_{m+1,i}}^{m+1}(\cos\theta) = \frac{m\cos\theta}{\sin\theta} P_{\nu_{m+1,i}}^{m}(\cos\theta) - \frac{dP_{\nu_{m+1,i}}^{m}(\cos\theta)}{d\theta} \tag{53}$$

From equation (53) it is possible to obtain the familiar expression for $P_{\nu_{mi}}^{m}$

$$P_{\nu_{mi}}^{m} = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_{\nu_{mi}}^{o}$$

$$x = \cos\theta \tag{54}$$

This $P_{\nu_{mi}}^{m}$ must satisfy the boundary condition

$$P_{\nu_{mi}}^{m}(\cos\theta_o) = 0 \tag{55}$$

The differential equation for $P_{\nu_{mi}}^{o}$ is

$$L_{\nu_{mi}}^{o}\left(P_{\nu_{mi}}^{o}\right) = \sin\theta \frac{d^2 P_{\nu_{mi}}^{o}}{d\theta^2} + \cos\theta \frac{dP_{\nu_{mi}}^{o}}{d\theta}$$

$$+ \nu_{mi}(\nu_{mi} + 1)\sin\theta\, P_{\nu_{mi}}^{o} = 0 \tag{56}$$

Equation (56) may be written in terms of $x = \cos\theta$ as

$$(1 - x^2) \frac{d^2 P_{\nu_{mi}}^{o}(x)}{dx^2} - 2x \frac{d P_{\nu_{mi}}^{o}(x)}{dx} + \nu_{mi}(\nu_{mi} + 1) P_{\nu_{mi}}^{o}(x) = 0 \tag{57}$$

In order to find the proper values V_{mi}, which satisfy the boundary conditions of the problem the $V_m = \nu_m(\nu_m + 1)$ will be considered as a variable and $P_{\nu_m}^{o}(x)$ will be considered as a power series as follows.

$$P_{\nu_m}^{o}(x) = \sum_{j=0}^{\infty} a_j x^j \tag{58}$$

The standard procedure shows that

$$a_{j+2} = a_j \left(\frac{j}{j+2} - \frac{V_m}{(j+1)(j+2)} \right) \tag{59}$$

For each j it is possible to obtain a polynomial expression in powers of V_m for a_j with a_0 and a_1 arbitrary.

$$\begin{aligned} a_{2j} &= a_0 Q_{2j}(V_m) \\ a_{2j+1} &= a_1 Q_{2j+1}(V_m) \end{aligned} \tag{60}$$

Where $Q_k(V_m)$ is a polynomial of degree k - 1 or less.

At this point it is necessary to consider the boundary conditions. One was mentioned in equation (55) and according to equation (56) is satisfied if

$$\left.\frac{d^m P_{\nu'_m}^{\,o}}{dx^m}\right|_{x_o} = 0 \qquad (61)$$

Using the series expression for $P_{\nu'_m}^{\,o}$ from equation (58) this condition (61) may be written

$$\sum_{j=0}^{\infty} \frac{(m+j)!}{j!} \, a_{(j+m)} \, x_0^j = 0 \qquad (62)$$

Using the polynomial expressions (60) the equation (62) takes the following form.

$$a_0 \left(\sum_{j=0}^{\infty} b_{oj} V_m^j \right) + a_1 \left(\sum_{j=0}^{\infty} b_{1j} V_m^j \right) = 0 \qquad (63)$$

Condition (61) is not sufficient to fix the values of V_{mi} since both a_0 and a_1 are left arbitrary. Another boundary condition is needed to completely determine the roots V_{mi} of equation (63).

This other boundary condition applies at $\theta = 0$ and is expressed as follows for both ν_{mi} and ν'_{mi}

$$P_{\nu_{mi}}^{\,o}(1) = 1 \qquad (64)$$

The second boundary condition (64) may be treated in the same manner as was the first to give an expression similar to expression (63) as follows.

$$a_o \left(\sum_{j=0}^{\infty} c_{0j} V_m^{\ j} \right) + a_1 \left(\sum_{j=0}^{\infty} c_{1j} V_m^{\ j} \right) = 1 \tag{65}$$

Now the two equations (63) and (65) are sufficient to fix the values of V_{mi}, $i = 0, 1, 2, \ldots$, which are the values of the variable V_m such that the two equations are satisfied. With V_{mi} found the definition (39) for V_{mi} may be used to find the ν_{mi}. The V_{mi} are used in expression (60) to obtain the coefficients a_j which are used in turn to give $P_{\nu_{mi}}^{\ o}$ using equation (58). Finally $P_{\nu_{mi}}^{\ m}$ are obtained by substitution of $P_{\nu_{mi}}^{\ o}$ in equation (54). The same procedure must be followed for each ν_m and ν'_m, ($m = 0, 1, 2, \ldots$) with the latter having the boundary condition

$$\left. \frac{dP_{\nu'_{mi}}^{\ m}}{d\theta} \right|_{\theta = \theta_o} = 0$$

substituted for the condition (55) in the case of ν'_m.

The above discussion is not intended either to be complete or to imply the best method for finding the needed functions but simply to indicate some of the difficulties involved.

3. Continuity at the $\theta = 0$ axis.

Condition (64) insures that the field components such as those mentioned in equation (35) Appendix I are continuous and single valued at $\theta = 0$. On this axis the θ and ϕ components of $\overline{E}$ can be written

$$\left.\begin{aligned} E_\theta &= E_x \cos\phi + E_y \sin\phi \\ E_\phi &= -E_x \sin\phi + E_y \cos\phi \end{aligned}\right\} \quad \text{at } \theta = 0 \tag{66}$$

with E_x and E_y independent of ϕ. An examination of equations (35) in Appendix I will show that if equation (66) is to hold then

$$\left.\frac{dP_{\nu_i}^{m}}{d\theta}\right|_{\theta=0} = \left.\frac{P_{\nu_i}^{m}(\cos\theta)}{\sin\theta}\right|_{\theta=0} = 0 \quad \text{if } m \neq 1, \tag{67}$$

and if $m = 1$

$$\left.\frac{dP_{\nu_i}^{1}}{d\theta}\right|_{\theta=0} = \left.\frac{P_{\nu_i}^{1}(\cos\theta)}{\sin\theta}\right|_{\theta=0} \neq 0 \tag{68}$$

Consider the function $P_{\nu_{mi}}^{m+1}$ which satisfies the equation

$$L_{\nu_{mi}}^{m+1}\left(P_{\nu_{mi}}^{m+1}\right) = 0 \tag{69}$$

If equation (69) is multiplied by sin θ and θ is made zero it becomes

$$(m+1)^2 \left. P_{\nu_{mi}}^{m+1}\right|_{\theta=0} = 0 \tag{70}$$

Thus $P_{\nu_{mi}}^{m+1}(1) = 0$ for $m \geq 0$. If the following substitution (71) is made for $P_{\nu_m}^{m+1}$ in the left hand side of (69) expansion reveals it to be identically zero. According to statement (38) this proves the equation.

$$P_{\nu_{mi}}^{m+1} = \frac{m \cos \theta}{\sin \theta} P_{\nu_{mi}}^{m} - \frac{d P_{\nu_{mi}}^{m}}{d\theta} \tag{71}$$

By letting $\theta = 0$ in equation (71) while using equation (70) the left hand side is zero and it is seen that

$$m \lim_{\theta \to 0} \left[\frac{P_{\nu_{mi}}^{m} (\cos \theta)}{\sin \theta} \right] = \lim_{\theta \to 0} \left[\frac{d P_{\nu_{mi}}^{m} (\cos \theta)}{d\theta} \right] \tag{72}$$

Applying L' Hopital's Rule to the left hand side of equation (72) yields

$$(m - 1) \lim_{\theta \to 0} \left[\frac{d P_{\nu_{mi}}^{m} (\cos \theta)}{d\theta} \right] = 0 \tag{73}$$

Equations (72) and (73) combine to prove equation (67). Relation (68) remains yet to be proved. This can be done by using equation (53) and its first derivative both with m = 0 obtaining

$$\begin{gathered} \frac{d^2 P_{\nu_{1i}}^{o}}{d\theta^2} = - \frac{d P_{\nu_{1i}}^{1}}{d\theta} \\ \frac{d P_{\nu_{1i}}^{o}}{d\theta} = - P_{\nu_{1i}}^{1} \end{gathered} \tag{74}$$

If substitutions are made from equations (74) into $L_{\nu_{1i}}^{\,o}(P_{\nu_{1i}}^{\,o}) = 0$ one obtains after division by $\sin\theta$

$$\frac{d\,P_{\nu_{1i}}^{\,1}}{d\theta} + \frac{\cos\theta}{\sin\theta}\,P_{\nu_{1i}}^{\,1} = V_{1i}\,P_{\nu_{1i}}^{\,o} \tag{75}$$

Using equation (72) while letting $\theta = 0$ gives

$$\left.\frac{d\,P_{\nu_{1i}}^{\,1}}{d\theta}\right|_{\theta=0} = \left.\frac{P_{\nu_{1i}}^{\,1}}{\sin\theta}\right|_{\theta=0} = \frac{V_{1i}\,P_{\nu_{1i}}^{\,o}(1)}{2} \tag{76}$$

The boundary condition at $\theta = 0$

$$P_{\nu_{mi}}^{\,o}(1) = 1 \tag{64}$$

guarantees the validity of relation (68) and with the help of definition (39) one obtains the following useful result which also holds for ν'_{1i}

$$\left.\frac{d\,P_{\nu_{1i}}^{\,1}}{d\theta}\right|_{\theta=0} = \left.\frac{P_{\nu_{1i}}^{\,1}}{\sin\theta}\right|_{\theta=0} = \frac{\nu_{1i}\,(\nu_{1i}+1)}{2} \tag{77}$$