

CALCULUS INSTRUCTORS' AND STUDENTS' DISCOURSES ON THE DERIVATIVE

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ABSTRACT

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Recently, there has been an increasing interest in collegiate mathematics education, especially teaching and learning calculus (e.g., Oehrtman, Carlson, & Thompson, 2008; Speer, Smith, & Horvath, 2010). Of many calculus concepts, the derivative is known as a difficult concept for students to understand because it involves various concepts such as ratio, limit, and function (e.g., Thompson, 1994) and it can be represented in multiple ways (e.g., Zandieh, 2000).

This study explored and compared university calculus instructors' and students' discourse on the derivative with the lens of a communicational approach to cognition (Sfard, 2008). Specifically, it examined how they describe a) the concept of derivative, b) the relationships between a function, the derivative function, and the derivative at a point, and c) the derivative of a function as another function. The data for this study were collected from the three calculus classes during six weeks of derivative lessons at a large Midwestern university. The study used mixed methods including classroom observations, student survey, and student and instructor interviews. Surveys were scored and used for selecting the interview participants. Classes and interviews were videotaped and transcribed. Transcripts of the instructors' and students' discourse were coded for discussion topic, including the derivative of a function ($f'(x)$), the derivative at a point ($f'(a)$), the relationships among $f(x)$, $f'(x)$, and $f'(a)$, and $f'(x)$ as a function.

Analysis of instructors' discourse shows that they explicitly addressed what $f'(a)$ and $f'(x)$ represent in relation to $f(x)$ by stating them as the slope of $f(x)$ or describing the behavior of $f(x)$ based on their signs. They addressed the relationship between $f(x)$ and $f'(x)$ with

differentiation rules, and the relationship between $f(x)$ and $f'(a)$ by addressing the property of the derivative function at a point where $f(x)$ has extreme values. How $f'(x)$ and $f'(a)$ are related was also explicitly addressed with the substitution method. However, the instructors did not explicitly address the relationship between $f'(x)$ and $f'(a)$ in most cases. First, they tended to use the word "derivative" without specifying it as the derivative function or the derivative at a point. This use of the word was mostly identified when they used $f'(a)$ as a representative of $f'(x)$ on an interval, which might have confused students not only about what the word, "derivative" referred to but also what $f'(a)$ and $f'(x)$ represent in terms of $f(x)$. Second, they also did not specifically address that $f'(a)$ is a number, a value of $f'(x)$ that is a function. Although instructors stated once or twice the aspect that $f'(x)$ is a function, they used this aspect mostly without mentioning.

Analysis of students' discourses showed that they explained more consistently and correctly the aspects of the derivative addressed explicitly in their classrooms than those addressed implicitly. In other words, they explained the relationships between $f(x)$ and $f'(a)$, and between $f(x)$ and $f'(x)$ better than the relationship between $f'(x)$ and $f'(a)$, and $f'(x)$ as a function. Though most students performed well in problems involving the first two relationships, some of them showed a mixed notion of the derivative as a point-specific value and a function on an interval without distinguishing these two aspects; the most common incorrect description of the derivative was the tangent line at a point, which was a point-specific object but a function.

The results of this study indicate that when instructors were clear and unambiguous about mathematical aspects of the derivative, students' thinking on these aspects were closer to what is considered as true in mathematical community. This suggests that using exact mathematical terms and discussing the mathematical aspects that students have trouble explaining or using may provide better opportunity for students to learn the concept of a derivative.

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To my amazing parents, Nok-Soon Kang and Hae-Kwon Park

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CHAPTER 1: INTRODUCTION AND RATIONALE

There has been increasing interest in mathematics education at the college level. Many mathematics education research papers on students' thinking about concepts in calculus such as limit, derivative, and integration have been published recently (e.g., Artigue, Batanero, & Kent, 2007; Oehrtman, Carlson, & Thompson, 2008). Of these concepts, the derivative seems to be one of the most complex concepts for students to understand because the definition of the derivative builds on mathematical concepts such as ratio, limit, function, and tangency, and the derivative can be represented in various ways such as the slope of the tangent line on the graph, an instantaneous rate of change in a physical situation, and with a form of Leibniz's notation. More specifically, students' thinking about the derivative is based on their thinking about other mathematical concepts which have been formed already from their prior mathematics learning. However, their thinking about those concepts may not always coincide with what the professional mathematics community may consider as the definitions or properties of those concepts. Many studies report these discrepancies as students' *misunderstandings* about those concepts (e.g., Carlson, Smith, & Persson, 2003; Ferrini-Mundy, & Graham, 1994; Hahkioniemi, 2005). Additionally, since the derivative can be represented in various ways, students have to make sense of these various representations and the relationships among them to use the concept of the derivative flexibly in different situations. Existing studies have shown that students have different levels of understanding across different representations of the derivative and thus, have difficulty connecting different representations of the derivative (e.g. Santos & Thomas, 2002; Zandieh, 2000).

The existing literature about students' thinking on the mathematical concepts related to the derivative and different representations of the derivative are primarily based on test results

(e.g., Monk, 1994; Orton, 1983) and interview(s) with students using tasks involving the derivative (e.g., Ferrini-Mundy & Graham, 1994; Zandieh, 2000). The concepts of the derivative of a function and the derivative at a point are derived from the function, and one word, “derivative,” can be used for the both concepts. Therefore, how students are taught and how they learn about uses of the word and the concepts in relation to the function are important to explore. However, none of the existing studies addressed whether and how students used the word “derivative” systemically while they are describing the concept of the derivative in relation to the original function or solving problems involving the derivative. Moreover, no study addressed how the concept of the derivative is taught in calculus classes.

Purpose of the Study

The purpose of this study is to address how the derivative is taught and learned in university level calculus courses. More specifically, the thinking of instructors and their students about the derivative in relation to the original function and the derivative as a function was investigated and compared through the lens of the communicational approach to cognition. In this approach, thinking is considered as communication with oneself; therefore, one’s thinking about a mathematical object is investigated through his or her mathematical discourse including what he or she said or did to communicate about the object with oneself or others (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; Sfard and Lavie, 2005; Sfard, 2008). Therefore, this study extends the previous work on how the derivative is learned by examining how the derivative is taught in calculus classrooms by reporting on classroom and instructors’ discourse on the derivative as well as students’ discourse.

Problem Statement

Calculus instructors and students occasionally use the word, “derivative,” without

specifying it as derivative of a function or derivative at a point. Listeners may conceive the word, “derivative,” differently from what the speaker intended, and this discrepancy may happen in communications about the derivative between instructors and students. Although I could not find evidence of the discrepancy between instructors’ and students’ use of the word “derivative,” in the existing studies, my pilot study (Park, 2008) revealed students’ ambiguous use of the word, “derivative.” One of the participants in the study used “derivative” as “the derivative function at a point” with which he referred to the tangent line to the graph of a function. Another participant changed what “derivative” referred to from “the derivative at a point” and “the derivative of a function” while describing various relationships between them, and most of his uses did not coincide with the relationships described in the textbook. Students may experience an ambiguous use of the word “derivative” without specifying it as “the derivative of a function” and “the derivative at a point” in their calculus classroom; their instructors may have used the word, “derivative,” ambiguously while they explain these concepts, and the relationships between them and to the original function.

The above hypothetical mismatch between instructors’ and students’ uses of the word “derivative”, which is supported by anecdotal evidence from my teaching experience and observing other instructors in a calculus tutoring center, raises some questions about teaching and learning the concept of derivative. The first question would be which of “the derivative at a point” and “the derivative of a function” is first defined and how the other one is built up from the first one when an instructor introduced the concept, and how the instructor addresses the difference between those two concepts. If the instructor uses one word, “derivative,” for both concepts without distinguishing them, he or she may not be able to address important aspects of these two concepts explicitly: the derivative of a function as another function, which is an

dynamic object which changes as the independent variable changes, and the “derivative at a point” as a static object, which is a point-specific value, thus a number.

The second question would be how an instructor addresses the relationship between a function and its derivative. This relationship can be discussed as two relationships: between a function and the derivative at a point, and between a function and the derivative of a function (Thomas, Weir, Hass, & Giordano, 2005). The former relationship could be addressed by describing the local property of a function around a point, which is captured by the derivative at a point, for example, as the slope of the tangent line. On the other hand, the latter can be addressed by describing the global behavior of a function, which the derivative of the function describes over the domain. These two connected but different relationships may not be clearly addressed with the ambiguous use of the word, “derivative”.

The third question would be how an instructor addressed the relationship between the derivative of a function and the derivative at a point. If an instructor primarily uses one word, “derivative,” for both concepts, this relationship may not be discussed explicitly in the class. For example, while graphing the derivative of a function by plotting points based on the values of the derivative at several points and connecting them, the instructor may use the word, “the derivative,” for both the points and the graph itself. This graphing method combined with the instructor’s ambiguous use of the word “derivative” may not only confuse students about what the word refers to but also make it hard for them to see the difference between the derivative at a point as a point-specific value and the derivative of a function as another function defined on an interval. In other words, this way of graphing the derivative of a function may lead students to conceive of this graphing process as applying the steps procedurally without making sense of the underlying mathematics; the derivative function is a function which consists of values of the

derivative at all the points of the domain where the function is differentiable.

All three questions listed above are related to the derivative of a function as another function whose value at a specific point is the derivative at a point (Thomas et al., 2005, p. 147). The derivative of a function as another function is imbedded in the derivative of a function as a dynamic object, which describes global changes of a function on the domain and in the graphing method in which the derivative of a function consists of the derivative at several points. Therefore, the last question, which can be addressed in the instructors' discourses, would investigate how they addressed the derivative of a function as another function.

Additionally, my earlier study (Park, 2008) identified that students also used one word, "derivative," for the derivative of a function and the derivative at a point while they solved problems involving the derivative and justified their answers during the interview. This ambiguous use of the word "derivative" without specifying its referent may affect ways that students approach questions involving the derivative and their justification for solutions. Using "derivative" for both the derivative of a function and the derivative at a point allows students to change the meaning of the word "derivative" from derivative at a point to derivative of a function, or vice versa. This relationship between the derivative at a point and the derivative of a function is closely linked to the derivative at a point is a value of the derivative function that is another function. Therefore, the four questions listed above also can be addressed in students' discourses on the derivative to find out their thinking about the derivative. Finally, the analyses of students' and instructors' discourses on the derivative may reveal similarities and differences between these two discourses which may explain some aspects of students' *misunderstandings* about the derivative.

Research Questions

Thus, this study addressed the following questions about the discourse of calculus instructors and their students:

1. How do instructors introduce, describe, or define the concepts of the derivative at a point and the derivative of a function?
2. To what extent do instructors address the relationships between a function, the derivative at a point, and the derivative of a function?
3. To what extent do instructors address the derivative of a function as another function?
4. How do students describe or define the concepts of the derivative at a point and the derivative of a function?
5. How do students describe the relationships between a function, the derivative at a point, and the derivative of a function?
6. How do students use the derivative of a function as another function?

Data were collected during observations of three college calculus classes and during interviews with the instructors and a sample of students from those classes. Data from the classroom observations and interviews with instructors were used to answer Questions 1 – 3; data from interviews with students were used to answer Questions 4 – 6.

To answer these questions, I focused on key characteristics of instructors and students' discourses on the derivative using a communicational approach to cognition introduced by Sfard (2008). Sfard's framework consists of four elements of mathematical discourse: word use, visual mediators, endorsed narratives, and routines. Using these four properties as the theoretical framework, instructors' and students' uses of *key words* such as "function," "derivative,"

“derivative of a function,” and “derivative at a point” were closely examined. If a routine, which is a repetitive pattern in discourses, was identified, it was tabulated with what instructors or students said (excerpts) and what they did (*visual mediators* such as writing equations, and drawing figures and graphs) during the class or interviews, which provided the evidence of the identified routine. *Endorsed narratives*, which are statements accepted as true by instructors or students were also identified and included in the analysis. Each property is addressed in more detail in Chapter 2.

Significance of the Study

This study may contribute to the field of mathematics education in the following ways. First, by exploring the use of words in discourses about a mathematical object, this study may reveal the role of the mathematical words and terms in students’ learning. The importance of the word use in children’s thinking about the early mathematical concepts has been reported in several studies (Fuson and Kwon, 1992; Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Sfard and Lavie, 2005; Sfard, 2008). However, no similar studies have been conducted regarding advanced mathematical concepts such as derivative. This study may reveal the role of mathematical terms related to the derivative in student’s thinking on the derivative. Using one word, “derivative,” without specifying it as the derivative of a function or the derivative at a point, may play an important role when students try to make sense of these concepts and their relationships and use them in problem solving situations.

Second, this study may also help us have a better understanding of how the derivative is taught in calculus classrooms. Although there have been studies about calculus curriculum materials (Ferrini-Mundy & Graham, 1991; Wu, 1997; Hurley, Koehn, and Ganter, 1999), no study addressed what is happening in high school or university calculus classrooms.

Mathematics educators have emphasized the differences between intended and enacted curricula and importance of exploring the latter in relation to students' learning (Newton, 2008; Stein, Remillard, & Smith, 2007, p. 337). The analysis of the classroom discourse in this study would reveal various aspects of derivative lessons including how instructors address the key topics in the derivative unit, and whether and how students participate in the classroom discourse.

Third, this study explores both instructors and students' discourses on the derivative. Analyzing both discourses can reveal the discrepancy between instructors' and students' use of the words related to the derivative, which can lead to conflicts in communication between them. Therefore, the results of examining the leading characteristics of instructors and their students' discourses, and the classroom discourse (which is supposed to be the combination of the first two), can provide possible explanations about students' *misconceptions* of the derivative, and thus lead to suggestions to help students overcome some phenomena which are known as misconceptions of the derivative. If instructors' vague use of words influences students' discourses on the derivative, instructors should consider specifying the exact meaning of mathematical words while communicating about the derivative with students.

Overview of Dissertation

In this chapter, I have set up the research questions as well as the rationale for this study. Chapter 2 is a review of related literature that reports findings from existing studies about students' thinking about the derivative. It also addresses limitations in those studies and explains an alternative framework through which I explored calculus instructors' and students' thinking about the derivative.

Chapter 3 describes the research design and methods. It consists of an overview of this mixed method study which included surveys, classroom observations, and interviews, and a

rationale for the methods used. This chapter also describes the process of data collection, the survey instrument, the method used to recruit participants for interviews, information about the research site and participants, and a description of the methods used to analyze data.

Chapter 4, 5, and 6 present the findings from the discourse analyses. Chapter 4 includes three sections that report the findings from the three instructors' classroom discourses about how they defined the derivative of a function and the derivative at a point, and to what extent they addressed the relationships between a function, the derivative of a function, and the derivative at a point, and the derivative of a function as another function. At the end of the chapter, the three sets of classroom discourses are summarized and used to answer the first three research questions.

Chapter 5 presents the findings from interviews with the instructors, in which instructors were asked to solve problems as if they were explaining them to their students. The instructors' discourses are reported problem by problem and the summary and the discussion provide additional information about the second and third research questions.

Chapter 6 consists of three sections of findings from interviews with students from the three classes, in which the students are asked questions eliciting descriptions or definitions of the derivative, and for solutions to the same problems that their instructors solved. It follows the same structure of Chapter 5, and provides responses to Research Questions 4 – 6.

Chapter 7 provides a brief overview of this study, summary and discussion on the findings by comparing the findings of the classroom, student, and instructor discourses. It also includes the future research questions and the implications of this study about how to improve teaching and learning of the concept of the derivative.

CHAPTER 2: THEORETICAL BACKGROUND

There has been substantial research related to students' thinking about the derivative (Tall, 1986; Thompson, 1994; Zandieh, 2000). The existing literature can be divided into two categories: students' thinking on mathematical objects included in the concept of the derivative and representations of the derivative. These two categories are inspired by Zandieh's framework (2000, p. 106, See Table 2.1). The first section of this chapter reviews the findings of existing studies using the two categories. The second section addresses the limitations of these studies in terms of how they addressed the derivative in relation to the original function and the derivative of a function as another function. The third section presents a communicational approach to cognition, which provides the rationale for the research design and the theoretical framework of this study. Finally, the findings from a pilot study about students' discourse on the derivative are summarized.

Previous Research on Students' Thinking about the Derivative

This section reviews existing studies about students' thinking about the derivative utilizing two categories: students' thinking about a) the mathematical concepts related to the derivative such as ratio, limit, function, and tangency; and b) the various representations of the derivative such as the symbolic, graphical, physical, and algebraic representations.

Findings of existing studies in each of the categories follow.

Students' Thinking about Mathematical Concepts related to the Derivative

The definition of the derivative, $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, consists of three mathematical concepts that have been covered in previous mathematics courses: function, ratio, and limit (Zandieh, 2000). Therefore, students' thinking about these concepts becomes the basis for learning the concept of the derivative. Moreover, since the derivative is interpreted as a slope of

a tangent line to the original function, students' thinking about tangency may also be related to their realization¹ of the derivative especially in graphical situations. Although there is a huge amount of research on the concepts of limit, ratio, function, and tangency (e.g., Tall and Vinner, 1981; Thompson, 1994), I will focus on findings about students' thinking about these concepts in the context of the derivative.

Ratio

Several studies found various aspects of students' thinking about ratio or the rate of change which might affect their thinking about the derivative. Because the derivative can be interpreted as the instantaneous rate of change (IRC) of a function, it is important to explore a) how students realize the average rate of change (ARC) of a function, b) how they make connections between ARC and IRC, and c) how they use the concept of rate of change to realize the derivative.

First, studies found that students have a strong procedure-based concept of ARC, yet have trouble realizing ARC as a ratio of the difference between y values to the difference between the corresponding x values. In Orton's (1983) study, many students failed to find ARC of a nonlinear function, $y = 3x^2 + 1$, which was given as the graph. Out of 110 students, 60 failed to find the ARC from $x = a$ to $a + h$. This difficulty may cause problems because this calculation is a process of finding the derivative of the function at $x = a$ using the definition,

$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$. Another study done by Hauger (1998) found that students' thinking

¹ One "realizes" a mathematical word or symbol with accessible objects. For example, one can realize a word, "function," with a graph or table (Sfard, 2008, p. 154). Here, "realization" is used instead of "understanding," considering the difficulty in articulating criteria for judgment of whether someone understands mathematical objects (p. 33).

about the rate of change is closely connected to the procedure of calculating ARC with x and y coordinates of the given points. According to Hauger (1998), this procedure-based concept of the rate of change cannot help students deepen their understanding of the connection between ARC and IRC because it is hard to relate this procedure to the IRC over infinitesimal intervals.

Second, Hauger's (1998) argument about students' difficulties connecting ARC and IRC are supported by Orton's (1983) study. Many students in his study did not appreciate the limit process of obtaining IRC at a point from the sequence of ARC values in graphical situations because they recognized ARC and IRC as separate ratios.

Last, Thomson (1994) showed that a relational understanding of ARC helps students understand the derivative and theorems in calculus. He argued that an operational understanding of ARC is crucial for students to make sense of the Fundamental Theorem of Calculus (FTC) as well as the Mean Value Theorem (MVT). He explained the operational understanding of ARC as knowing that "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, we would end up with the same amount of change in the dependent quantity as actually occurred" (p. 165). For example, traveling at an average speed of 30km/hr means that one would travel exactly the same distance in the same amount of time if one repeats the trip at a constant rate of 30km/hr. With operational understanding, he interpreted the MVT as follows: any differentiable function has its ARC over a given interval, and this ARC is equal to some instantaneous rate of change of the function within the interval. He argued that these interpretations of the rate of change and MVT help students make sense of FTC. From the pre-interviews with students, he found that they did not have the operational understanding of ARC and understood Riemann sum as a fixed quantity. However, after participating in his calculus lessons focusing on the operational understanding of ARC, which is embedded in the derivative,

MVT, and FTC, the students showed improvement in explaining calculus concepts including integral (Thompson, 1994).

Limit

Since the derivative is defined as the limit of the ratio of the change in y over the change in x , students' thinking about the limit may affect their thinking about the derivative. Existing studies showed that students' realizations of the limit are related to their difficulties appreciating a tangent line, whose slope is the derivative in a graphical situation, as a limit of secant lines. The most popular realization of the limit was that a sequence could approach its limit as close as possible but never get there, e.g., the sequence, 0.9, 0.99, 0.999, 0.9999, ... approaches but never gets to 1 (Tall & Vinner, 1981). By applying the same logic, students concluded that the sequence of secant lines approaching to a tangent line could never get to the tangent line; for example, "The tangent line can't be a limit because a part of the chord already past it" (Tall, 1986, p. 2). Students also determined the limit of secant lines of a curve incorrectly by conceiving of those lines as chords of a circle; they said that the limit was zero or a point (Tall, 1986) or did not exist because the length of the chord approaches zero as two points get closer, for example, "The lines get smaller," or "It disappears" (Orton, 1983, p. 237). The authors pointed out that these realizations hamper students' understanding of the limiting process to obtain the tangent line from several secant lines when the definition of derivative is explained graphically (Tall, 1986; Orton, 1983).

Function

Because taking the derivative of a function can be considered as an action operating on the function (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997), and the derivative of a function is another function, students' thinking about a function should be taken into account

when their thinking about the derivative is investigated. Of various aspects of students' thinking about a function, their thinking about the co-varying nature of a function, the derivative of a function as another function, and their tendency to find an algebraic representation of a function seem to be closely related to their realization of the derivative.

Regarding the co-varying nature of a function, Thompson (1994) reported a common students' image that a function has two sides, a short expression on the left and a long expression on the right, which are separated by an equal sign (p. 268). He argued that this image can lead students to consider a function as one thing changing rather than as a relation of co-varying quantities. This image might also hamper them to interpret the behavior of a composite function appropriately. In Santos and Thomas's (2003) research, a student, who calculated the first and second derivatives of a function, $f(x)$, correctly but could not acknowledge the difference between $f''(x)$ and $f'(f'(x))$, exemplifies a case of students' lack of understanding about the co-varying nature of a function. To him, taking the derivative of a function twice was the same as composing the derivative function with itself as $f'(f'(x))$, and this is perhaps a result of his concept that a function was varying only based on the independent variable, x .

One of the important aspects of the concept of the derivative is that the derivative of a function is another function. Monk (1994) addressed this aspect of the derivative by defining two types of understandings of a function: "pointwise understanding" and "across-time understanding". The former describes the value of the function at a point whereas the latter refers to a function as a dynamic quantity which changes its value over an interval as the independent variable changes. He gave four problems involving these two types of understanding, and one of the problems involves the derivative at a point and the derivative of a function. He compared the percentage of students' correct answers for each of "pointwise" and "across-time" parts of the

fours problems, and concluded that “pointwise understanding” of a function is prerequisite to “across-time understanding” of a function and that students have fragmented realizations of the derivative rather than realizing the derivative of a function as another function (Monk, 1994).

Given that a function is a dynamic object which changes with respect to its independent variable, identifying the independent variable of the derivative of a function can be considered as one aspect of realizing the derivative as a function. An existing study shows students’ difficulty appreciating the independent variable of the derivative of a function. Carlson, Oehrtman, and Thompson (2008) observed that many calculus students could not explain the difference between the rates of change of the volumes of a sphere and a cube because they lacked an understanding of variables of functions. In the case of a sphere, the surface area is the derivative of its volume when both the surface and volume are considered as functions of the radius of the sphere. However, in the case of a cube, one cannot obtain the surface area by taking the derivative of its volume. This difference comes from the difference in the nature of independent variables, the radius of a sphere and the side of a cube, which were not appreciated by many calculus students (Carlson et al., 2008).

Several studies found that students’ tendency to find algebraic representations of function is related to their thinking about differentiability. Based on their thinking that a function should have one equation, students tend to think that a piecewise function is not differentiable at a point where it changes its equation because it has two derivatives at the point (e.g., Ferrini-Mundi & Graham, 1994). Students’ preference for algebraic representations is also reported to be an important factor when they determine the differentiability of a function given by a graph. To determine if a curve represents a differentiable function, a student in Ferrini-Mundi and Graham’s (1994) study first tried to find an equation for the curve, on which she could apply

derivative formulas. If she could not find an equation, or she found two equations (e.g., a piecewise function), she determined that the given graph did not represent a differentiable function.

Several studies reported students' lack of understanding of the relationship between continuity and differentiability (Selden , Mason, et al., 1989, 1994; Selden, Selden, et al., 2000; Viholanen, 2006). This student difficulty was identified when they tried to calculate the derivative of a piecewise function at the point where the function changes its equation. Especially, their tendency of using an algebraic approach to determine the differentiability of a piecewise function seemed to lead to incorrect interpretations of continuity. Viholanen (2006) found that many students tended to calculate the derivative of equations on both sides at the point where the function changes, and to decide that the function was differentiable if the values were the same. This method leads to a conclusion that a discontinuous function can be differentiable.

Tangency

Tangency is not a mathematical object included in the definition of the derivative. Studies, however, have reported that students' experiences with circles and tangent lines are closely related to their thinking about the tangent line to the graph of a function, whose slope is the derivative. Many students said that a tangent line to the curve should intersect the curve only once at the tangency point (Biza, Christou, & Zachariades, 2006). This phenomenon was found in a problem-solving situation in which the graph of the tangent line to a function was given. Students responded that the given line was not tangent to the graph of the function because the line intersected the graph at other points aside from a tangency point when the line was extended (Biza et al., 2006).

Students also describe a tangent line in various ways, some of which are not accepted by

professional mathematics community. For example, in a study done by Biza et al. (2006), students mentioned that drawing more than one tangent line at a point is possible, and a tangent line exists at a cusp. To them, existence of more than one tangent line at one point and a tangent line at a cusp may imply that its derivative function has two values at one point, and y value of the derivative function exists at the point where the original function is not differentiable.

In summary, existing studies have shown that students' thinking about the ratio, limit, function and tangency that they previously learned is related to their thinking about the derivative. In the next section, what is known about how different ways of representing the derivative affect students' thinking about the concept will be addressed.

Students' Thinking about Representations of the Derivative

The derivative can be represented in various ways – symbolic, graphical, physical, and algebraic. In this paper, symbolic representations refer to notations, $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ and $\frac{dy}{dx}$. Graphical representations of the derivative refer to the graph of tangent line to the graph of a function, whose slope represents the derivative at a point, and the graph the derivative function at an x - y coordinate plan. Physical representations of the derivative refer to the applications in physical situations such as the velocity of a moving object. Algebraic representations are used for algebraic expressions as products of applying the derivative formulas to equations of functions. Existing studies have shown that students have different levels of understanding when the derivative is represented in different ways (e.g., Zandieh, 2000). In this section, how students make sense of symbolic, graphical, physical, and algebraic representations of the derivative and how they explain relationships among these representations will be discussed.

Symbolic Representations

There have been studies about how students describe the symbols dx , dy , and dy/dx , and the relationship between the symbols $\Delta y/\Delta x$, and dy/dx . First, it is revealed that students have difficulties making sense of dx and dy although most students interpret Δx and Δy correctly as the changes in x and y , respectively. They tended to interpret dx as the differential of x , the rate of change of x , an amount of x , or x -increment, without mentioning an independent variable (Orton, 1983).

Many studies showed that students recognized the symbol, dy/dx , as a signal of taking the derivative of a given function without distinguishing the independent variable (x) and dependent variables (y), which are embedded in the symbol. For example, in Santos and Thomas' (2003) study, a student calculated $dm/dq = 6m - 7/3m^2$ when the given function was $q = 3m^2 - 7/m^3$ by treating the dependent variable m in dm/dq as the independent variable instead of q . Moreover, students could not appreciate the equivalence between the two symbols $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$.

Thomas (2002) reported that many students interpreted the former as differentiating y twice, but they were not able to see the latter in this way.

Graphical Representations

One of the major students' confusions about the graphical representation of the derivative is their assumption that the graphs of a function and its derivative resemble each other (e.g., Nemirovsky & Rubin, 1992). In Nemirovsky and Rubin's (1992) study, when asked to graph the derivative function, many students drew a similar graph to the original function. They tended to match overall characteristics of the two graphs rather than interpreting their relationships: the derivative as the slope of the tangent line to the graph of the function and the

original function as the accumulated rate of change of the derivative. The authors argued that this tendency was likely due to the plausibility of the physical contexts (e.g., if an object has positive and increasing velocity, its distance also increases as time goes by) rather than students' inability to see the difference between a function and its derivative (Nemirovsky & Rubin, 1992). The authors suggested that employing examples from various situations can help students understand this relationship better (Nemirovsky & Rubin, 1992).

Physical Representations

In physical situations, the first and second derivatives can be interpreted as the velocity and acceleration of a moving object, respectively. In existing studies, students' weak understanding of the derivative in physical contexts was identified in their interpretations of the distance, velocity, and acceleration. For example, in Bezuidenhout's (1998) study, when a distance function, $S(t)$, was given, half of the students interpreted $S'(80) = 1.15$ incorrectly: "For the velocity of 80 km/h, the deceleration is 1.15m/s," "The rate of change of the distance of a car at 80km is 1.15km/h," and "At 80km/h, the change in time is 1.15 seconds" (p. 395). During the interview, he found that the students could not provide correct units or interpretations of these quantities in the given physical context. This lack of understanding of the rate of change in physical contexts also seems related to their confusion regarding velocity and acceleration.

Algebraic Representations

In existing studies, many researchers found that students prefer algebraic representations of the derivative over other representations, and their dominant image of the derivative is a set of derivative formulas (e.g., Asiala, et al., 1997). Studies also found students' misconceptions and errors when they applied the formulas (e.g., Hirst, 2002).

Students' preference for algebraic representations was identified, even when they worked

with questions in which a function or a derivative was given as a graph, and algebraic representations were not helpful to solve the problem (Asiala et al., 1997). Asiala et al. (1997) conducted interviews to explore students' mental constructions of the graphical representation of the derivative and found a strong tendency to use algebraic approaches. When graphs of a curve and its tangent line were given (Figure 1), many students approached the problem algebraically. For example, to calculate $f'(5)$, many students found the equation of the tangent line and differentiated it instead of calculating the slope based on the two given points. Then, when asked to explain the relationship between the tangent line and the curve, they tried to integrate the slope, which was a constant, to find the equation of the curve.

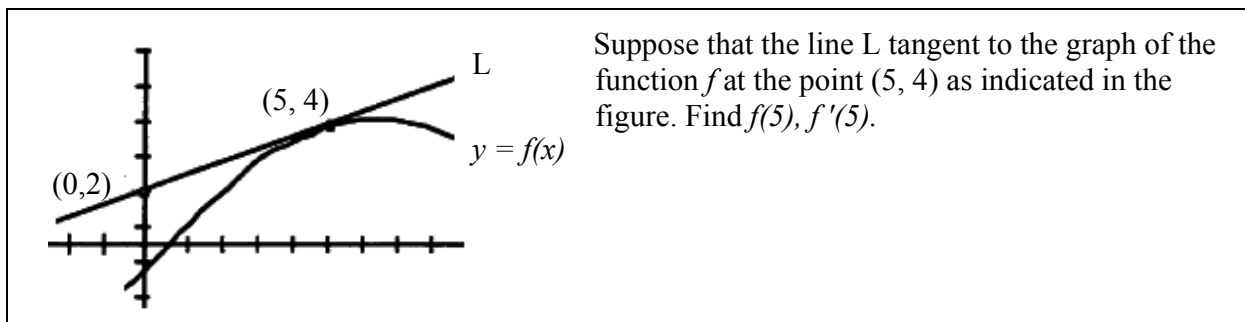


Figure 1. Problem 6 (Asiala et al., 1997, p. 404)

Despite their preference for algebraic representations, students showed incorrect use of algebraic representations of the derivative when they applied derivative formulas to equations of functions. Studies have shown that most prominent procedural errors made by students came from extrapolation – applying derivative rules based on external similarity of functions (Hirst, 2002). In other words, when asked to find the derivative of a composite function $y = f(g(x))$, many students obtained $y = f'(g(x))$ ignoring the chain rule. Similarly, many students applied the power rule $((x^n)') = nx^{n-1}$ to non-polynomial functions (e.g., exponential or trigonometry functions) and obtained $(e^x)' = xe^{x-1}$, $(x^x)' = xx^{x-1}$, and $[(\sin x)^x]' = x(\sin x)^{x-1}$ (Hirst, 2002).

Students' Thinking about the Relationships among Representations

Students' thinking about relationships among representations reported that a) a student can have a different level of thinking depending on how the derivative is represented, and b) students in different disciplines can develop different representations of the derivative as their dominant image of the derivative.

Different Levels of Understanding across Representations

Existing studies report that students showed different levels of thinking about the derivative depending on how the derivative is represented and how they explain the relationships between different representations of the derivative. These studies employed hierarchical frameworks presented as matrices with columns of several levels of understanding and rows of different representations. Zandieh (2000) used a matrix-form framework with various representations of the derivative, such as graphical (slope), verbal (ratio and rate), physical (velocity), and symbolic (difference quotient) (Table 1). For columns, she specified the definition of the derivative, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, with three hierarchical concept layers: the ratio between the change in y and the change in x , the limit over this ratio, and the derivative function as the limit of infinitely many x values in the domain of a function.

Table 1. Zandieh's (2000) Framework for the Derivative (p. 106)

Contexts	Graphical	Verbal	Paradigmatic/Physical	Symbolic	Other
Process-Object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

With this framework, Zandieh (2000) analyzed nine high school calculus students' responses to the question, "What is a derivative?" Her results showed that a student had a

different level of thinking about the derivative in different contexts. One student provided detailed explanations of all three concept layers of the symbolic representation but could not explain any layer of graphical representations; he just mentioned the derivative as the slope of a tangent line of a function. Another student explained graphical and symbolic representations of the derivative and their relationship, but did not mention IRC (verbal) or physical examples. Zandieh (2000) interviewed the students several times while the derivative was taught in their class and reported the changes in their level of thinking.

Hakkioniemi (2005) discussed two levels of understanding of the derivative based on strengths of the connections students make among different representations. Students who have a weak understanding of connections of representations are able to “predict, identify, or produce the counterpart of a given external representation” (p. 2). However, when an action is given to the original representation, they cannot “predict, identify, or produce” the result of the action in the counterpart because this mental activity involves a strong understanding of the link between the representations (p. 2). In his study, three out of five high school students showed a weak understanding; they were able to explain each representation of the derivative separately but could not describe how phases of taking the derivative in symbolic representations are related to their counterparts in graphical representations. However, one student, who showed a strong

understanding of this connection, explained the equivalence between $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$ and $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$ using the concept of average rate of change in graphical representations.

Santos and Thomas (2002) also compared interviews with two students who showed different levels of understanding of the derivative: Steven with a procedure-oriented

understanding, and James with an object-oriented understanding.² During the interview, Steven who consistently used the algebraic representations of the derivative in the procedure of applying the derivative formulas could not make any connections to other representations of the derivative when prompted by an interviewer. However, James was able to explain how function, ratio, and limit layers in the definition of the derivative, $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, are related to the gradients of the graph of the function.

Different Realizations of the Derivative in Different Disciplines

There have been studies about the effect of departmental goals for teaching calculus in students' thinking about the derivative. Bingolbali and Monaghan (2004 & 2008) found that emphasis on different aspects of the derivative in two departments, mathematics (M) and mechanical engineering (ME) affects students' developing different dominant "concept images" of the derivative. They employed a socio-cultural perspective to investigate the difference between ME and M students' concept images of the derivative. They found that ME students tended to think about the derivative as the rate of change whereas M students showed a tendency to interpret the derivative as a slope of the tangent line. Moreover, whereas there was no difference between M and ME students' performance of achievement on problems involving the derivative as the rate of change and as the slope of tangent line in the pretest which was held in the beginning of the semester of the first calculus course, there was significant difference between the students' achievement in the posttest, which contained the same problems as the pretest and was administered at the end of the semester. In the posttest, ME students

² Santos and Thomas (2002) described five dimensions of knowing, in order, from the low to high levels: procedure-oriented knowing, process-oriented knowing, object-oriented knowing, concept-oriented knowing, and versatile knowing.

outperformed M students on problems involving rate of change as an interpretation of the derivative while the M students outperformed ME students on problems involving slope of the tangent line as an interpretation of the derivative. Based on interviews with students, the authors concluded that this difference came from students' affiliation and positional identity related to their major, including the difference between calculus courses that they took (Bingolbali & Monaghan, 2004 & 2008).

Limitations of Existing Studies

As discussed in the previous sections, existing studies have revealed various aspects of the students' thinking about the derivative in terms of their thinking about a) mathematical objects that are included in or related to the definition of the derivative, and b) representations of the derivative and their relationships. However, there are limitations associated with the previous studies on learning and teaching about the derivative.

First, despite various findings about students' thinking about the derivative, existing studies did not explore students' thinking about the derivative in relation to the function. Although there were some studies about students' thinking on a function which may affect their thinking on the derivative, they did not directly address how students make sense of the derivative in relation to the original function. For example, in existing research, students' difficulty interpreting the instantaneous velocity or acceleration and providing the correct units (Bezuidenhout, 1998) was classified as a result of their *misconception* of the derivative. However, this difficulty can be addressed in terms of how students explain the first and second derivatives in terms of the original function which is given as a position function. Given that the definition of the derivative of a function and the derivative at a point are derived from the original function (Thomas et al., 2005), how students described the derivative in terms of the function and how

they used the descriptions in problem solving situations would be important to explore. This description can be extended to a discussion on the derivative of higher degree.

Second, as mentioned in Chapter 1, calculus instructors and students sometimes used the word, “derivative,” without specifying it as the derivative of a function or the derivative at a point. They may use the terms “the derivative,” $f'(x)$, and “the derivative at a point,” $f'(a)$ in a way parallel to the terms “a function,” $f(x)$, and “a function at a point,” $f(a)$. Consistent use of this parallel structure of terms can imply that the speakers who use these terms understand that “the derivative” as a function, and “the derivative at a point” is a value at a point, $x = a$, thus a number. For example, a student’s difficulty determining differentiability of a piecewise function (Ferrini-Mundy & Graham, 1994) can be also addressed focusing on how she explained the derivative of a function and the derivative at a point. The results showing that she determined the piecewise function is not differentiable may come from her confusion between the derivative of a function, which has two equations on both sides of the point where it changes its equation, and the derivative at a point, which is a value at the point, which may or may not be the same from the both sides. Students’ trouble explaining the equivalence between two notations for the second derivative, $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$ (Santos & Thomas, 2003) and interpreting the velocity and acceleration (Bezuidenhout, 1998) may be related to if and how they conceive of the derivative as a function, because the derivative function is another function that has its own derivative defined as the second derivative. However, none of these previous studies about students’ thinking about the derivative explored how calculus students or instructors use of these terms with a systematic analysis.

Third, although the existing studies help mathematics educators understand better

students' thinking about the derivative, they do not provide information about how to *improve* students' understanding about the derivative. Although I found one study in which the author conducted teaching experiment with his students (Thompson, 1994), the goal for the experiment was to improve the students' understanding about the integration in general, the fundamental theorem of calculus in particular, not to help students better understand the derivative. This limitation may come from the lack of information about the kinds of opportunities students might have while learning in these classes. This opportunity can be explored by analyzing curriculum material or the classroom discourses when the derivative is taught.

In summary, the previous studies on teaching and learning of the derivative have limitations, in their topics, by not addressing the derivative in relation to the original function, in data, by not including how the derivative is taught to students (Speer, Smith, & Horvath, 2010), and in its methodology, by not investigating the use of the word, "derivative" in a systematic way. Therefore, this study extends the previous work on the students' thinking about the derivative by including the information about how the derivative is taught and learned in the classroom with a new perspective—a communicational approach to cognition. In this study, instructors' and students' discourses on the derivative in the classrooms and interview settings are explored. More specifically, how they addressed the concept of the derivative of a function and the derivative at a point, the relationships between a function, the derivative of a function, and the derivative at a point, and the derivative as a function will be examined through the lens of the communicational approach, which will describe in the next section.

Communicational Approach to Cognition

A communicational approach to cognition is based on the assumption that thinking is communication with oneself, an "individualized version of interpersonal communication" (Sfard,

2008, p. xvii). According to this assumption, communication and cognition are not separated but viewed as “two facets of the same phenomenon” of communicating (Sfard, 2008, p. xvii). In this approach, mathematics is conceived as a form of discourse about mathematical objects such as numbers, symbols and geometrical shapes. These objects have developed through the process of objectification. For example, in beginning stages of numerical thinking, children use a number word as an adjective to describe a set of discrete objects such as 10 cookies. As they grew older, they began acknowledging the *ten-ness* of various sets including ten concrete objects and finally used the word “ten” as a noun, a mathematical entity without attaching it to any other noun (Sfard, 2008). In these beginning stages, a tendency to imitate what adults do plays an important role although the children could not describe the reason for performing an action such as counting discrete objects when they are asked to compare two sets of them. Imitation is a starting point of recognizing the sameness of sets of the same number of objects and finally leads children to use the number words as objects of discourse (Sfard, 2008). A similar phenomenon can be found when students learn the derivative. Before objectifying the derivative as an operation on a set of differentiable functions, students may use a tangent line to a curve in problem solving situations in a way that is not considered correct in the professional mathematical community. They may also confuse the terms, the derivative of a function and the derivative at a point, when they describe the behavior of a function (Park, 2008). However, these two cases exemplify a process of objectifying the mathematical topic. By knowing what may lead students to such discourses on the derivative, we may expedite their process of objectifying the derivative. To explore the leading features of three instructors and their students’ discourses about the derivative, four properties of mathematical discourses—word use, visual mediator, narrative, and routine—will be examined. Details of each property follow.

Word Use

Words in mathematics mainly signify mathematical objects such as quantities, shapes, and relationships between objects. Because mathematics is viewed as a form of discourse, what a word refers to may differ across the users of the word. For example, children's use of number words (e.g., 'five are more than three') may not be the same as those used by adults (e.g., 'five is bigger than three') (Sfard and Lavie, 2005). Children's utterances show that they are not yet considering "three" and "five" as stand-alone objects but as adjectives that need nouns attached to them. Therefore a word can be accepted differently by listeners from what a speaker intended.

The word, "derivative," may also be used differently by instructors and students. For example, an instructor's utterance, "the derivative is increasing here," can be understood as "the derivative function is increasing here" but also as "the derivative at a point is increasing here" by students who consider the tangent line as "the derivative function at a point." Because it is possible that instructor and student word uses are different from each other, instructors cannot assume that students speak about mathematical concepts including the derivative in a way that is consistent with the way the professional mathematical community explains them. In this study, instances where instructors and students use the word, "derivative," will be closely examined focusing on the cases in which they described the concept or addressed the relationships between a function, the derivative at a point, and the derivative of a function.

Visual Mediator

Visual objects used as a means of communication are called visual mediators. Mathematical discourses tend to be mediated by various symbolic, iconic, and visual representations of mathematical objects. The visual mediator of a mathematical object can be a standard form such as its algebraic definitions using equations or can be created by individuals as

a form of a drawing on a blackboard, a concrete object such as marbles, or a gesture. For example, discourse on the slope of a linear function can be conventionally mediated by the equation, $y = mx + b$, a graphical image of a straight line, or a table of values. It also can be expressed by drawing several right triangles with different slopes or a gesture with an arm which represents the slope of a straight line. Similarly, discourse on the derivative can be visually mediated by the graph of a tangent line to a graph of a function or symbolically mediated by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ or } \frac{dy}{dx}.$$

It also can be expressed as the rate of change with a moving object such as a car or gears.

Routine

Based on the assumption that human activities including communication are rule-governed (Sfard, 2008), routines are defined as repetitive patterns in the discourse consisting of *when-* and *how-routines* (Sfard, 2008, p. 134). A *when-routine* is a metarule that determines situations where a performance is considered appropriate to start (“applicability condition”) and to finish (“closure”) by performers. *How-routine* is a metarule that determines patterns of discursive performance (“course of action”) (Sfard, 2008, p. 208). Routines can be found in use of words or mediators, or in the process of endorsing narratives. Because of the repetitive nature of routine, it can be represented in a table format. Table 2 is an example of a routine table that was taken from Sfard’s study (Sfard, 2008, p. 198). It presents a discussion between a teacher and a student about the biggest number between a teacher and a student. The routine, which is shown in the first column, consists of conjecture, test, and evaluation, which is an example of course of action. The rest of the columns include actual excerpts from the transcripts.

Table 2. *Routine and Excerpts from the Discussion on the Biggest Number*

Iteration	1	2	3	4
Conjecture Q1	[5] T: What is the biggest number you can think of?	[11] T: What is the biggest number?	[15] T: Can one arrive at the biggest number?	[19] T: What is the biggest number?
A1	[6] N: Million	[12] N: Two million	[16] N: Yes. [17] T: Let's assume google is the biggest number.	[20] N: There is no such number.
Test Q2	[7] T: What happens when we add one to million?	[13] T: And if we add one to two million?	[17] T: can we add one to google?	
A2	[8] N: Million and one	[14] N: It's more than two million.	[18] N: Yes, there are numbers bigger than google.	
Q3	[9] T: Is it bigger than million?			
A3	[10] N: Yes			
Evaluation to Q1	[11] T: So...	[15] T: So...	[19] T: So...	

Note. T and N refer to the teacher and Noa, a student.

Regarding the derivative, consider the case when an instructor solves several problems

that involve finding $f'(a)$ using the definition $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, when the equation of

$f(x)$ is given. If the instructor draws the graph of $f(x)$, calculates the equation of $f'(x)$

algebraically using the definition, and then finishes up the problem by drawing a tangent line to the graph of $f(x)$ at $x = a$ and comparing its slope to the value of $f'(a)$, this patterned procedure can be considered as his or her routine for finding the derivative at a point. The given problems can be seen as an applicability condition if the instructor solved all such problems in this way.

The last step of comparing the value with the slope can be viewed as a closure of this routine because the instructor finishes up the solution by using a graphical approach, as he had when he started the problems. The course of action following this procedure can be seen as a how-routine.

Endorsed Narrative

Narrative is defined as utterances about mathematical objects, their relationships, or

processes involving objects, which can be endorsed as a true statement or rejected as a false one (Sfard, 2008, p. 134). Endorsed narratives are narratives accepted as true by speakers.

Mathematically endorsed narratives can be found in definitions, axioms and theorems in the textbooks (Sfard, 2008, p. 223). Students' endorsed narratives do not always coincide with mathematicians' or their instructors' endorsed narratives. For example, an instructor can endorse a narrative about the derivative by stating "the derivative of a function is also a function." Students, however, may endorse the statement, "the tangent line is the derivative function" based on other narratives such as "the derivative is represented by a tangent line" and "a line is a linear function" (Park, 2008).

In this communicational approach, learning is viewed as gradual change in students' discourse that helps the students become participants in the mathematical community (Sfard, 2008). Therefore, how instructors and students speak about mathematical objects in terms of the four features of mathematical discourse, and how their discourses are similar to and different from each other, needs to be examined to help students learn effectively. Such an examination would be beneficial to both instructors and students. Instructors who have authority as mathematics experts in the classroom can make the gradual change in students' discourse more effective and accurate utilizing information about how students talk about the topic and how this can be continuously changed to coincide with discourses of the mathematical community.

In this study, classroom and interview data about three instructors and their students' discourses about the derivative are analyzed based on these four features—word use, visual mediators, routines, and endorsed narratives. The analysis focuses on their routines of the key word use, but endorsed narratives and visual mediators will also be taken into account in this analysis. I conducted a survey and interviews with calculus students to explore the key features

of their discourse on the derivative as a pilot study. A summary of the research design and findings of the pilot study follow.

Pilot Study

A pilot of this study was conducted at a university in the Midwest in spring and summer 2008 (Park, 2008). Seven survey questions about the relationships among a function, the derivative function, and the derivative at a point were developed (Appendix I). Most questions, except the last one, were included in the survey for the current study with some modifications. The survey was administered during the break time after the derivative unit for 14 minutes. Eighty four volunteer students from eight sections of Calculus I and Applied Calculus took the survey. Students' selections on multiple choice questions were tabulated, about 50 students who selected popular choices were contacted via email, and finally seven students were interviewed. During the interview, students were asked to justify their responses to the survey questions. The interview data was transcribed and analyzed to find key features of the students' discourses on the derivative.

Two main features in students' discourse on the derivative were identified. First, their use of word, "derivative," for both the derivative of a function and the derivative at a point was closely related to their endorsed narratives about the derivative. For example, Bob frequently used "derivative" without specifying its referent and came up with a new concept, "the derivative function at a point." He described that "the derivative function at a point" was a linear function represented by the tangent line of a given function at that point. Moreover, using "the derivative function at a point," he justified why a function increases if and only if its derivative function increases. Another participant, Ian, stated many possible relationships between a function and its derivative function: a) the derivative was positive if and only if the derivative increases, b) a

function increases if and only if its derivative function increases; and that a function is positive if and only if its derivative function is positive. Although these statements often conflicted with each other, Ian did not mention such conflicts at all during the interview. Instead, he changed his statement about the relationship between a function and its derivative function several times even in justification of one problem. The fact that he could use the word, “derivative,” for both “the derivative of a function” and “the derivative at a point” played a crucial role. Unlike Bob, Ian’s discourse was not consistent, and its repetitive patterns were hard to find; his discourse consisted of fragmented descriptions of the derivative, which sometimes conflicted with each other.

Second, students’ dominant concept of the derivative, either the derivative of a function or the derivative at a point, was closely related to their use of visual mediators of a function and the derivative (e.g., graphical or algebraic). This routine was identified in the interviews with Laura and Hannah, who also used the word, “derivative,” without specifying the derivative of a function or the derivative at a point. However, unlike Bob and Ian’s cases, it was often easy to infer which of the two concepts Laura and Hannah were referring to in the given context. Laura’s discourse on the derivative was mainly related to the output from the action of taking the derivative using the formulas. Her “derivative” mostly referred to the derivative of a function unless the question explicitly involved calculating the derivative at a point (e.g., calculate $C'(5)$). In most questions, she tried to convert given graphical representations of a function or the derivative function to equivalent algebraic representations, and then applied the derivative formula. On the other hand, Hannah approached most of the questions in the questionnaire graphically. Her discourse on the derivative was mainly about the process of graphing the derivative of a function when its original function was given. When a graph was not given, she generated a sample graph which satisfied the conditions stated in the question and worked with

the graph. Her description of “the derivative at a point” as “zero” was consistent during the interview because finding zero slopes was her first step of graphing the derivative of a function. She rarely approached questions algebraically.

The pilot study revealed some aspects of students’ discourses on the derivative. Based on these findings, I expand the scope of the study by including instructors’ discourse and classroom discourse, which is ostensibly a combination of students’ and instructors’ discourses. By examining three types of discourses on the derivative, I explore how the main characteristics of students’ discourses are similar to or different from their instructors’ discourses. The methodology I used to collect data for these discourses and to analyze them are discussed in the next chapter.

CHAPTER 3: METHODOLOGY

The purpose of this study is to explore and compare the discourses of three calculus instructors and their students on the derivative in classroom and interview settings. More specifically, this study investigates the discourses of three college-level calculus instructors and their students to address the following research questions:

1. How do instructors introduce, describe, or define the concepts of the derivative at a point and the derivative of a function?
2. To what extent do instructors address the relationships between a function, the derivative at a point, and the derivative of a function?
3. To what extent do instructors address the derivative of a function as another function?
4. How do students describe or define the concepts of the derivative at a point and the derivative of a function?
5. How do students describe the relationships between a function, the derivative at a point, and the derivative of a function?
6. How do students use the derivative of a function as another function?

To this end, I used a mixed design of qualitative and quantitative research methods consisting of classroom observations, a survey of students in the observed classrooms, and interviews with instructors and students. To select instructors, I observed several sections of the calculus courses in two colleges at the university with instructors' permission, focusing on how instructors and students' discourses vary from section to section. Then, I chose three instructors who had different teaching styles and were willing to participate in the study. Lessons about the derivative were video- and audio-taped. After the derivative unit was over, the survey, which includes

questions about students' mathematical background and mathematic problems involving the derivative, was taken by the students in the classroom. Based on their written answers on the survey, I purposefully recruited 12 students for interviews in order to further investigate their solution processes and justifications on the survey questions. How I selected instructors and students are explained in detail later in this chapter. After interviewing the students, I interviewed the instructors about how they would explain mathematical problems in the survey to their students. This methodology allowed me to collect data for the instructors and their students' discourses on the derivative in the classroom and interview settings. To analyze these discourses, I used the framework described in the communicational approach to cognition (Sfard, 2008). In this chapter, I will describe each part of the research design and discuss how I used the framework to analyze the data with examples.

Classroom Observations

The purpose of observing the lessons in the derivative unit was to collect data about instructors and students' discourses when the derivative was introduced and the relationships between a function, the derivative of a function, and the derivative at a point were addressed. This section includes information about the research site, courses, sections, instructors, and data recording.

Research Site

This study was conducted in calculus classrooms at a university in the Midwest. At the time of the study, the university had approximately 47,000 undergraduate, graduate, and professional students, of which around 10 % were international students.

At the university, calculus courses are offered by the mathematics department and a residential college. The mathematics department offers calculus series including four calculus

courses: *Calculus I*, *Calculus II*, *Multivariable Calculus*, and *Differential Equations*. The residential undergraduate college at the university devoted to studying natural science offers *Calculus I*, *Calculus II*, and *Calculus III (Multivariable Calculus)*. This study was conducted focusing on some sections of *Calculus I* courses. At the time of this study, a total of 1,000 students were enrolled in 40 sections of *Calculus I* offered by the mathematics department. In the residential college, a total of 296 students were enrolled in 14 sections of *Calculus I*.

Courses

Two of the three sections that I studied were offered by the mathematics department, which I will refer to as MATH A. The other section was offered by the residential college at the university, which I will refer to as MATH B. MATH A is required for students who are natural science or engineering majors at the university, and MATH B is a requirement for all students in the residential college. Both MATH A and MATH B are mostly taught by faculty members; however, graduate assistants who have completed a master's degree in mathematics or passed the qualifying exams in the mathematics department are also qualified to teach.

In all MATH A sections, *Thomas' Calculus* (Thomas, Weir, Hass, & Giordano, 2005) was used as the textbook. MATH B sections used *University Calculus* (Hass, Weir, & Thomas, 2008). These two books are written by the same author team, and both include two chapters for the derivative units with the same titles: "Differentiation" and "Application of Derivatives". However, sections were slightly different in these two textbooks; *University Calculus* (Hass et al., 2008) contained more content than *Thomas' Calculus* (Thomas et al., 2005). Table 3 provides a comparison of these two chapters.

Table 3. *Content of Derivative Unit in Thomas' Calculus and University Calculus*

Textbooks			
	Thomas' Calculus	University Calculus	
Chapter	Section Title		
Differentiation	The derivative as a function	Tangents and Derivative at a point	
	Differentiation rules	The derivative as a function	
	Derivative as a rate of change	Differentiation Rules	
	Derivatives of Trigonometric Functions	Derivative as a Rate of change	
	Chain Rule and Parametric Equations	Derivatives of Trigonometric Functions	
	Implicit Differentiation	Exponential Functions	
	Related Rates	The Chain Rule	
	Linearization and Differentials	Implicit Differentiation	
	Extreme Value of Functions		Inverse Function and Their Derivatives
			Logarithmic Functions
		Inverse Trigonometric Functions	
		Related Rates	
		Linearization and Differentials	
Application of Derivative	The Mean Value Theorem	The Mean Value Theorem	
	Monotonic Functions and the First Derivative Test	Monotonic Functions and the First Derivative Test	
	Concavity and Curve Sketching	Concavity and Curve Sketching	
	Applied Optimization Problems	Applied Optimization Problems	
	Intermediate Forms and L'Hôpital's Rule	Intermediate Forms and L'Hôpital's Rule	
	Newton's Method	Newton's Method	
	Antiderivatives	Hyperbolic Functions	

Sections

The three selected sections were taught by Tyler, Alan, and Ian.³ Tyler and Alan were selected because they were teaching the same course, MATH A, but showed different teaching styles: traditional lecture style using blackboard explanation vs. PowerPoint presentation and small group work, respectively. Ian was contacted because he was an instructor of MATH B, which was offered by the residential college and used a different textbook. Ian's lessons were also lecture-based.

³ These are pseudonyms of the instructors.

All three classes consisted of 30 - 40 students. In the MATH A classroom, students were seated in seven lines of individual desks facing the blackboard, and in the MATH B classroom, students were seated at long tables parallel to the board.

Instructors

Alan and Tyler each taught one section of MATH A, and Ian taught one section of MATH B. Alan and Tyler were graduate teaching assistants. Alan was a third year doctoral student in a mathematics education program and completed a master's degree in mathematics at the university where I conducted this study. Tyler was a fifth year doctoral student and Ph. D candidate in the mathematics department. Ian was a mathematics instructor who completed his Ph. D at another university; his major research interest was in Partial and Ordinary Differential Equations.

All the instructors had teaching experience in university level mathematics courses; Ian had taught calculus several times. Alan and Tyler were teaching MATH A for the first time although both had tutored students for this course and taught *Applied Calculus*, twice before. Tyler and Ian's lessons were lecture-based using a blackboard and chalk. Alan used PowerPoint presentations for most of the lessons and allocated time for small group activities four times during the derivative unit, although most students worked individually during this time. All sections met for 50 minutes on Monday, Wednesday, and Friday. Table 4 shows the information about the instructors and their sections.

Table 4. *Information about the Three Instructors and Their Classes*

Instructor Name	Years Taught Math	Years Taught Calculus I	Academic Status	Number of Students	Time for Class
Alan	4.5	0	Doctoral student	31	13:50-14:40
Tyler	4.5	0	Doctoral student	32	15:00-15:50
Ian	11	11	Instructor	35	12:40-1:30

Data Recording

All lessons taught by the three instructors were audio- and videotaped everyday for six weeks while the derivative unit was taught, except days when the students had an exam. The video camera was located in the back of the classrooms for Alan and Tyler's classes and on the right side of Ian's classroom. The voice recorder was placed on the table to the left of the blackboard. The camera focused on the instructor and the board. If students participated in the classroom discourse, their voices were recorded, but their faces were not in order to lessen interference with their behavior or response, and to expedite the permission process from the Human Research Protection Program at the university. Detailed field notes were taken during each lesson focusing on the writing on the board and discussion between the instructor and students.

Survey

The purpose of the survey was to collect information about students' mathematical background, and their thinking about the derivative and the relationships between a function, the derivative function, and the derivative at a point. I also used this information to select students for interviews. This section addresses participants, survey problems, procedure, and scoring.

Participants

The participants of the survey were 88 students enrolled in MATH A or MATH B class: 27 from Alan's, 29 from Ian's and, 32 from Tyler's classes. Students in MATH A were mostly from the college of natural science or engineering and students in MATH B were from the residential college devoted to studying natural science. Of the students who were enrolled in the three sections, 10 students (4 from Alan's, 3 from Ian's and, 3 from Tyler's classes) did not take the survey because they were absent on the day when the survey was conducted.

Survey Questions

The student survey consisted of two parts: students' background questions and mathematical problems involving the derivative (APPENDIX 2). The background questions asked about a) students' background, such as their year in college and major (questions 1 & 2), b) students' native language and language-specific terms for "derivative," "derivative of a function," and "derivative at a point" (questions 3 & 4), and (c) students' first exposure to the derivative before taking the current course (question 5) (Figure 2). Questions 3 and 4 were taken from Kim's (2009) study and modified slightly.

<p>Part I. Background</p> <ol style="list-style-type: none">1. Year: a) Freshman b) Sophomore c) Junior d) Senior2. Major (Specify): _____.3. Native Language: a) English b) Other (Specify): _____.4. Have you studied calculus in any language other than your native language? If yes, specify the languages in which you learned calculus: _____.5. Can you remember the first mathematics course in which you learned the word derivative? a) Yes b) No If yes, specify in which grade(s) and in which course/class you learned the word derivative? Grade(s): _____ Course: _____.
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Figure 2. Background Questions in the Survey

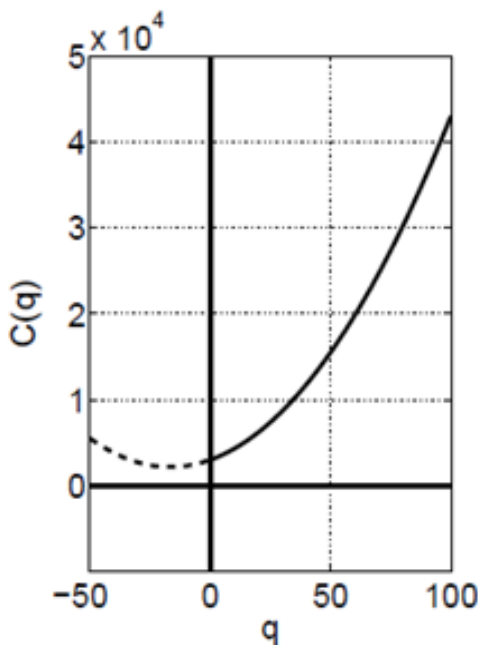
Questions about students' native language were added because they might have learned calculus in a language other than English, which may have two different words for the derivative of a function and the derivative at a point. These two terms may not be able to be combined into one word, "derivative" in the language.

Problems in the mathematics portion involved the relationships between a function, the derivative of a function, and the derivative at a point in their algebraic and graphical representations. I developed these questions based on exercise problems in calculus textbooks (Hughes-Hallett & Gleason, 2003; Thomas et al., 2005) and my lecture notes. Problems 4, 6, 7, 8 and 9 were selected from the Calculus Concept Inventory that was developed by Epstein (2006),

which included information that the items were reliable. All problems, except for three, were used in my pilot study (Park, 2008). Problems 2, 3, and 5 were added specifically for the current study. The problems were reviewed by professors in mathematics and mathematics education departments, calculus instructors and students who were not participants of this study. When the survey was administered, students were directed to “please answer the following problems and show your work”. It should be noted that I gathered the information about students’ written responses from the survey, and then further investigated the discourses behind the responses during the interviews. The rationale for each problem follows.

Question 1 involved the relationship between a function and the derivative at a point when the equation and graph of the function were given (Figure 3).

1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the equation is given by $C(q) = 3000 + 100q + 3q^2$, and the graph is given as follows.



- Find the value of $C(2)$.
- What is the unit of 2 in (a)?
- What is the unit of $C(2)$?
- What is the meaning of $C(2)$ in the problem context?
- Find the value of $C'(2)$.
- What is the unit for 2 in (e)?
- What is the unit of $C'(2)$?
- What is the meaning of $C'(2)$ in the problem context?

Figure 3. Problem 1

The correct answers on the parts are ‘3212,’ ‘miles,’ ‘dollars,’ ‘the total cost to produce 2 miles

of rope,’ ‘112,’ ‘miles,’ ‘dollars/mile,’ and ‘the rate at which the cost increases when the company produces 2 miles of rope,’ in turn. These parts were designed to find if students could calculate the values of the function and the derivative at the same point and give their values, units, and meanings using the problem context. By looking at whether students used the equation or the graph of a function, their preference for algebraic or graphical representations could also be investigated. Question 1 was slightly modified from the pilot study (Park, 2008); only parts e), g), and h) were asked in the previous study. However, based on the result that many participants in the pilot study confused the meaning of $C'(2)$ with $C(2)$, I added other parts. Also, this question was given as open-ended because students’ responses from the pilot study varied too much to make it multiple-choice.

Questions 2 and 3 involved relationships between and the derivative at a point and the derivative function in algebraic representation graphical representation, respectively (Figure 4).

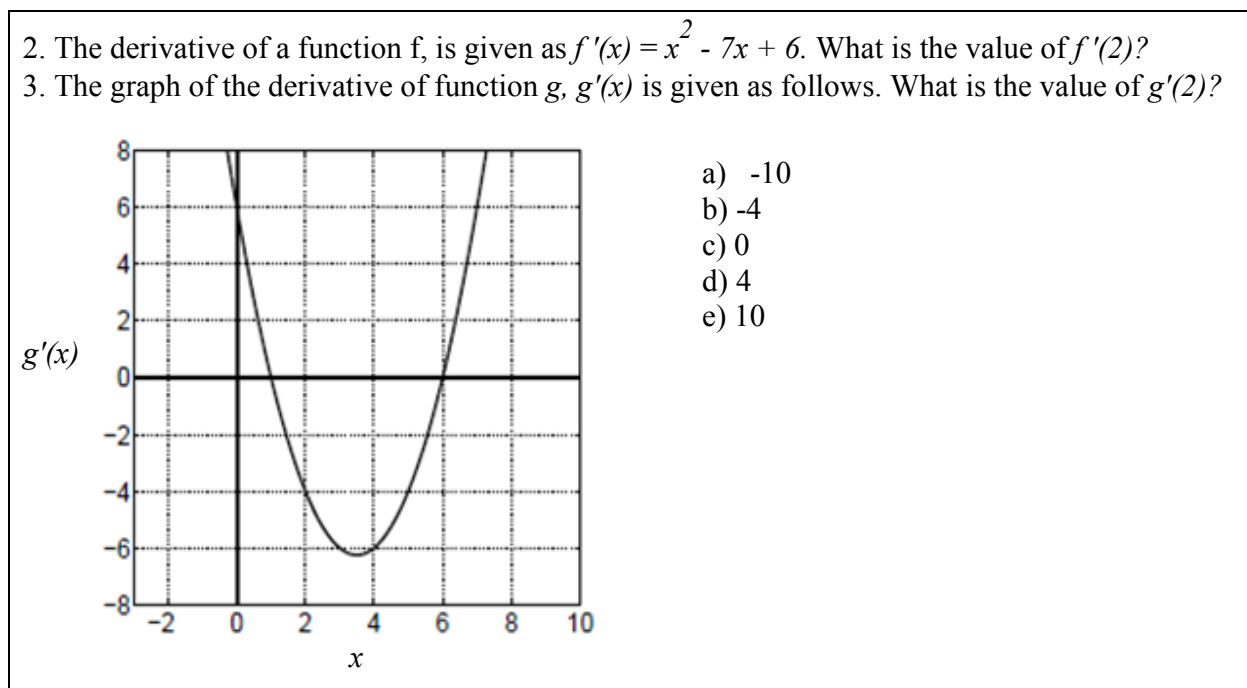


Figure 4. Problems 2 and 3

Questions 2 and 3 were added for this study because providing the same derivative of a function

with different representations might help obtain different reactions to the given representation from the students. The correct answers are -4 and b), respectively.

Problems 4 and 5 were about the relationship between a function and the derivative in graphical representations (Figure 5).

4. Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the first derivative, $f'(x)$?

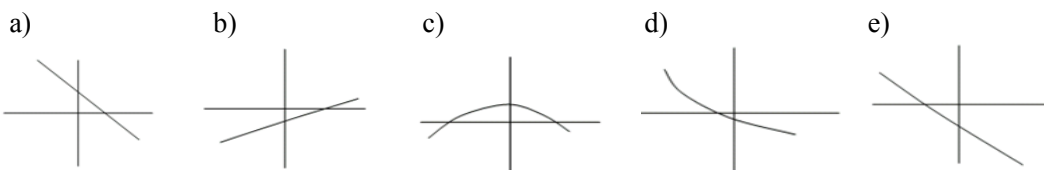
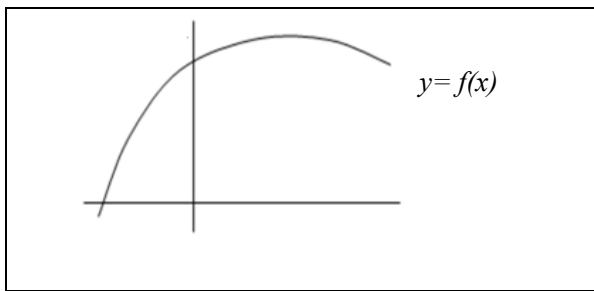


Figure 5. Problems 4 and 5

5. Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?

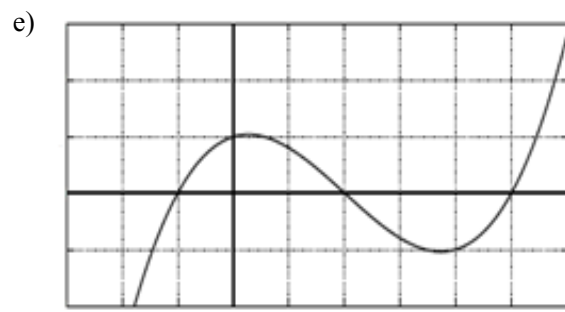
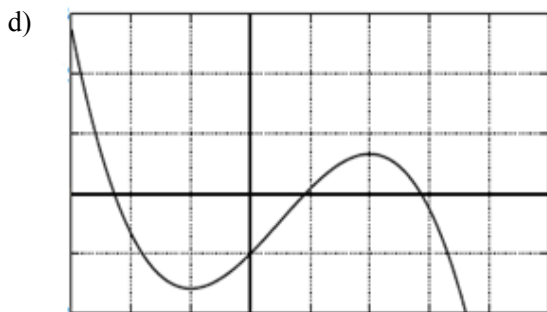
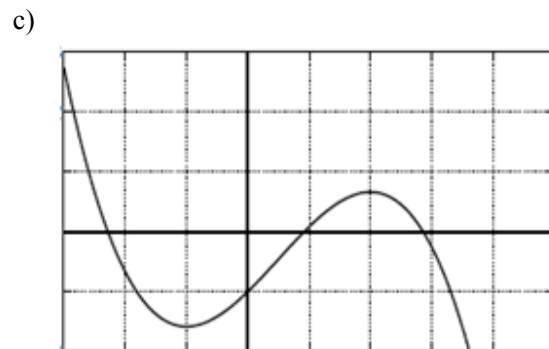
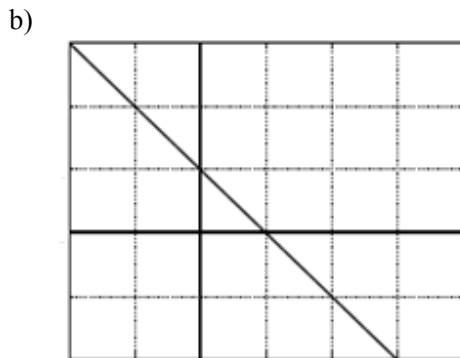
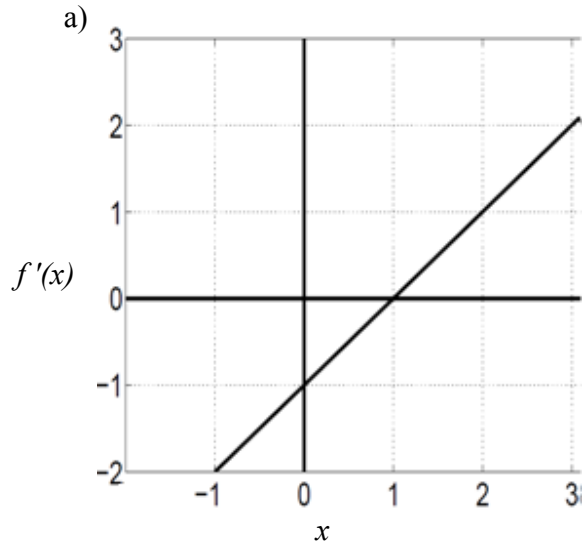
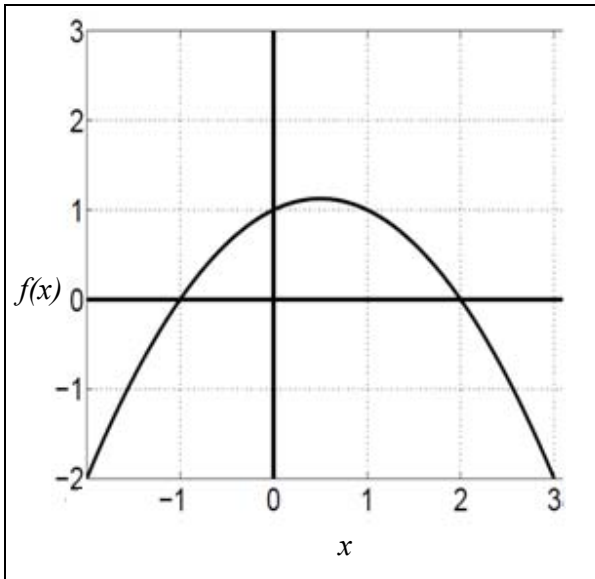


Figure 5. (Cont'd)

In Problem 4, which was also used in the pilot study, students were asked to find a graph of the derivative of a function, $f'(x)$, when the graph of the function, $f(x)$, was given. The correct answer is a). Problem 5 was added to the survey because students might answer differently when the graph of the derivative of a function was given, and the graph of the function should be found. With these problems, how students approach the relationship between a function and the derivative of a function in a graphical situation was further explored when they justified their responses during the interview. The correct answer is c).

Problems 6 and 7 were about the relationship between a function and the derivative of a function (Figure 6).

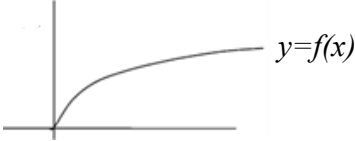
6. If a function is always positive, then what must be true about its derivative function?
- a) The derivative function is always positive.
 - b) The derivative function is never negative.
 - c) The derivative function is increasing.
 - d) The derivative function is decreasing.
 - e) You can't conclude anything the derivative function.
- Why?
7. The derivative of a function $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?
- a) The function $f(x)$ is positive on this interval.
 - b) The function $f(x)$ is negative on this interval.
 - c) The maximum value of the function $f(x)$ over the interval occurs at $x = 2$.
 - d) The maximum value of the function $f(x)$ over the interval occurs at $x = 3$.
 - e) We cannot tell any of the above.
- Why?

Figure 6. Problems 6 and 7

Problem 6 asked about the behavior of the derivative of a function when the function is positive, and the correct answer is e). Problem 7 asked about the behavior of the function when its derivative is negative, and its correct answer is c). In these problems, how students describe the behavior of functions using given information and which representation they use were further explored during the interviews.

Problem 8 was about the relationship between a function and its tangent line at a point (Figure 7).

8. The tangent line to this graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?



a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$ e) None of these

Figure 7. Problem 8

Although students could solve problem 8 without using knowledge about the derivative, I included this item because several students in my pilot interviews pointed out that the derivative function at $x = 1$ is given as $= \frac{1}{2}x + \frac{1}{2}$, which is not mathematically correct. Therefore, I decided to use this item to explore students' descriptions about the relationship between a function and the derivative at a point. The correct answer is c).

Problem 9 was designed to find out whether students could use the fact that the derivative at a point is the slope of the tangent line (Figure 8).

9. The derivative of a function is $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to f will be positive at $x = 0$.

a) a and b must both be positive numbers.
 b) a must be positive, while b can be any real number.
 c) a can be any real number, while b must be positive.
 d) a and b can be any real numbers.
 e) None of these

Why? _____.

Figure 8. Problem 9 in the Survey

Unlike problems 2 and 3, students were supposed to find the slope of the tangent line using the fact that the derivative at a point is a value of the derivative of a function at that point, which can be interpreted as the slope of the tangent line. How students perform in this problem, and

problems 2 and 3, would provide a good comparison because it may show how students use their knowledge about the derivative being the slope of a tangent line. How students describe the relationship between the derivative of a function and the derivative at a point was further explored during the interview. The correct answer is c).

Procedure

The survey was distributed to students at the end of the last class of the derivative unit. The students were given 15 minutes, and their written responses on the survey were collected. To motivate students to take the survey seriously, the instructors gave extra credit based on students' scores. The survey allowed them to earn a maximum of 10 extra credit points as part of 600 or 750 total points. To this end, I graded surveys and sent the scores to the instructors. I used the written responses to select students to interview.

Scoring

Students written responses on the survey questions were scored with two criteria: a) mathematical correctness and b) frequency. Scores based on mathematical correctness indicated students' achievement on these survey questions. Scores based on the frequency give information about how closely students' written answers match the answers written or chosen by the other students in the classroom. The details of these scoring methods and how I utilized these scores to recruit students for interviews follow.

Scoring Based on Mathematical Correctness

Each part of problem 1 was graded as 1 point except part h). For part d), the answer should be 'the total cost to set up a company and produce 2 miles of rope' or an equivalent form including 'cost' and 'point 2.' If an answer did not mention 'cost' or 'point 2', it received 0 points. I also coded their answers for further analysis; five codes were developed for part d):

Cost and Point (CP), Cost (C), Irrelevant context and Point (IP), Rate of change (R), and Irrelevant (I). The CP was the only correct answer; others received zero points.

Since part h) asked students to interpret the derivative at a point in the problem context, students gave various responses, some of which were partially correct; therefore, I assigned 2 points for part h). Answers graded with 2 points include: the rate of change or equivalent answers such as marginal cost, the rate of the cost, and the statement about $q = 2$ in any form such as ‘2 miles of rope.’ The code for these correct answers was RP, Rate of change at a specific Point. Partially correct answers, marked as 1 point, were a student’s answer missing a point-specific statement (R), and increment or change in the cost (I or CC). Codes for wrong answers were, Cost (C), Average rate of change for five units (AR), and Minimum or Maximum cost or quantity when minimum or maximum is attained (M). Table 5 shows the rubric for part h).

Table 5. *Rubric for Problem 1 Part h)*

Score	Code	Criteria
2	RP	(Instantaneous) Rate of change in cost at $q=5$
		Marginal cost at $q=5$
1	R	Increasing rate in cost or the rate of change of total cost or marginal cost without “at $q=5$ ” or without mentioning “cost”
	I	Increment in the cost from $q=4$ to $q=5$ (4) or from $q=5$ to $q=6$ (6)
	CCP	Change in Cost at $q=5$
0	AR	Average rate of change for 5 units or cost for one unit.
	C	Total cost for 5 units
	M	Maximum (minimum) cost or where Maximum or Minimum is attained
	B	Blank or irrelevant (e.g., production level)

Problems 2 and 3 were graded as 1 point because they involved a simple computation for the derivative and reading the y value from the graph, respectively. All other problems (problems 4 through 9) were graded as 2 points for each because these problems involved using the relationships between a function and the derivative, and between the derivative of a function and the derivative at a point rather than simple calculation. Therefore, the total score for the survey

was 23. Table 6 shows the mean and standard deviation of each of Alan, Tyler, and Ian's classes.

Table 6. Means and Standard Deviations of Survey Score Based on Correctness

Instructor	Course number	Number of Students	Mean	Standard Deviation
Alan	MATH A	27	10.1	4.6
Tyler	MATH A	29	14.1	4.4
Ian	MATH B	32	10.1	3.34

Scoring Based on Frequency

To find out common written answers on problem 1, I first coded students' written answers on parts d) and h) based on the rubrics. After coding the two parts, I calculated and then tabulated frequencies of students' answers or codes on each part of the problem. Frequencies of students' answers on other questions were tabulated in the same way.

After completing frequency tables of responses from the whole class for each part of problem 1 and the rest of the problems, I graded the students' responses on the survey based on the frequencies. I assigned 2 points for the most popular category for an open-ended item and the most popular response for a multiple choice item (say n students select that response). If there was a response selected by more than $n/2$ students, I assigned 1 point for the response. If there were two (or more) most popular responses (say m students select each of those choices), I assigned 2 points for each response, and 1 point for a response that more than $m/2$ students selected. A student whose answers coincided with the most popular answers on all the problems received 32 points. Table 7 shows the mean and standard deviation of each of the students in the three classes.

Table 7. Means and Standard Deviations of Survey Score Based on Frequency

Instructor	Course number	Number of Students	Mean	Standard Deviation
Alan	MATH A	27	22.6	4.0
Tyler	MATH A	29	22.63	6.4
Ian	MATH B	32	21.4	3.9

Student Interviews

The students were interviewed on completion of the derivative unit. The purpose of the student interview was: a) to collect data about students' discourse on the derivative because

students seldom participated in verbal discussion during most of the classes that I observed and b) to collect their discourses on the relationships among function, the derivative of a function, and the derivative at a point. In this section, each part of the student interview are discussed in detail: a) the process of recruiting interviewees, b) information about participants, c) design of interview, d) interview process, and e) data recording.

Recruitment of Students

To make the list of potential interviewees, I first chose students whose frequency scores were 16 or higher because I planned to interview students whose answers were close to the answers chosen by most of their class. Then, I sorted those students' scores based on mathematical correctness from highest to lowest, and divided them into four quarters, with the 25th quartile as the lower and the 75th as the upper. I contacted a student who obtained the highest scores for each interval since I planned to recruit a heterogeneous sample of students in terms of their achievement on the survey questions. If there were several students who obtained the top scores for each interval, I prioritized students who gained higher frequency scores. Students were contacted via email. If the first student on an interval did not want to be interviewed, I contacted the student with the second highest score on the interval. However, because of low response rate, I could not find a student from each of the four intervals of correctness scores for each class. For example, most of the students interviewed from Alan's class had high scores (above 16), most students interviewed from Ian's class had low scores (below 17), and there was a wide range of scores in interviewees from Tyler's class. However, frequency scores of all the students who I interviewed were higher than 16 out of 32.

Participants

From each section that I observed, I recruited four students. All the interviewees except one, were freshmen and in the departments of engineering, pre-med, or natural science. Only

two interviewees took calculus for the first time at university level, while the remaining 12 students had studied the derivative in Calculus or Advanced Placement (AP) Calculus courses in high schools. Table 8 shows the information of each participant.

Table 8. *Interviewed Participants*

Name	Gender	Major	Year	First Math Class Including Derivative	Instructor	Score based on Correctness ^a	Frequency Score ^b
Bill	M	Engineering	Freshman	Pre-calculus in HS	Alan	18	27
Joe	M	Civil Engineering	Freshman	AP calculus in HS	Alan	21	26
Cole	M	Pre-med	Freshman	Pre-calculus in HS	Alan	17	25
Zion	M	Chemical Engineering	Freshman	Pre-calculus in HS	Alan	17	28
Roy	M	Mathematics	Senior	Calculus I	Tyler	8	20
Neal	M	Computer Science	Freshman	Calculus in HS	Tyler	20	31
Liz	F	Med-Tech	Freshman	Calculus in HS	Tyler	11	22
Zack	M	Computer Science	Freshman	Pre-calculus in HS	Tyler	15	26
Sara	F	Biology	Freshman	Calculus I	Ian	8	17
Mary	F	Genomics and Molecular Genetics	Freshman	Survey Calculus in HS	Ian	13	21
Mona	F	Natural Science or Engineering	Freshmen	Pre-calculus in HS	Ian	13	24
Clio	F	Astrophysics	Freshman	Pre-calculus in HS	Ian	16	21

Note. In the table, AP, and HS refer to gender, Advanced Placement, and high school, respectively.

^a Maximum score = 23. ^b Maximum score = 32

Design of Interview

A task-based and semi-structured interview method was used in this study. Task-based interview has been widely used in studies about students' thinking about mathematical concepts (e.g., Ainley & Pratt, 2002; Goldin, 2000). Many researchers who conducted studies about students' thinking about the derivative also used tasks involving the derivative (e.g., Selden et al.,

1989 & 1994; Tall & Vinner, 1981 & 1983) and interviewed students to obtain more detailed information about their responses to tasks (Ferrini-Mundy et. al., 1994; Tall, 1985 & 1986). Since one of the purposes of this study is to explore students' thinking about the derivative, the task-based interview was a valid method to answer the research questions. Interviews were also semi-structured because I adapted the questions based on students' responses to a list of questions about tasks on the survey to probe their thinking further. Since students were asked to recall their solution process of mathematical tasks on the survey which they had completed before, stimulated recall was also used in the interviews (Mackey & Gass, 2005, p. 77)

Interview Procedure

An interview consisted of two parts: (a) warm-up questions about the meaning of the derivative, derivative of a function, derivative at a point, and the relationships among them and with the function (Figure 9); and (b) justification of the students' written answers to the survey questions.

- | |
|--|
| <p>Q1. What is the derivative? Can you make a sentence with the word, "derivative"?</p> <p>Q2. What is the derivative of a function?</p> <p>Q3. What is the derivative at a point?</p> <p>Q4. Is there any relationship between the last two terms?</p> <p>Q5. Is a function related to the derivative of a function or derivative at a point?</p> |
|--|

Figure 9. Warm-up Questions

Students were asked to answer the warm-up questions using their own words, not necessarily with mathematical terms. The first question was designed to find out which one, the derivative at a point or the derivative of a function, is more dominant in their description of the derivative; for example, a student's description could be close to either the derivative at a point (e.g., "the slope of a tangent line at a point") or on the derivative function (e.g., "it's like $2x$ when you have x^2 "). Before asking questions 2 and 3, I told students that there were two uses of

the word, “derivative,” the derivative of a function and the derivative at a point, and asked which one was closer to their description about the derivative. After a student chose one of the two concepts, I asked how they would describe the other one. Then, I asked question 4 about whether there was a relationship between the derivative of a function and the derivative at a point. If they said there was a relationship, I asked them to explain the relationship. The last warm-up question was designed to investigate how students internalize the meaning of the derivative in relation to the function. The main questions in the interview to elicit the students’ responses to each survey questions followed the warm-up questions. The interview protocol and the list of the questions that I asked students during the interview are included in Appendix 2.

Data Recording

Each of the face-to-face individual interviews was conducted for about an hour in my office at the university. All of the interviews were video- and audio-taped. The student was sitting at a desk answering my questions while I was standing next to the video camera. During the interview, I asked the students to reflect on how they solved the survey questions and write down their explanations if necessary. A colored pen and a piece of a blank paper were provided to encourage them to write down their explanations; a colored pen was used to differentiate their writings during the interview from those during the survey. When they did not remember why they selected a specific choice, however, I gave them some time to recall before they justified the answers. If students did not remember how they solved a question, I asked them to solve the problem again; they sometimes changed their answers during the interview. I tried to record their writing on the paper and note their gestures while they were justifying their answers.

Instructor Interviews

The purpose of interviewing the instructors was to explore their mathematical discourses

about the derivative on the same problems that were given to students. After finishing interviews with students, I interviewed the instructors individually for about 30 minutes in the same office where student interviews were conducted. In order to find out instructors' discourse patterns, narratives, use of words, and choice of visual mediators, they were asked to explain how they would solve the survey questions as if they had been explaining them to the students. A pen and a piece of blank paper were provided to encourage them to write down their explanations. The interviews were video- and audio-taped focusing on what they wrote and did during the interview.

Since one of the purposes of this study is to compare instructors and students' discourses on the derivative, I gave instructors the same survey questions as the students. However, I did not ask the warm-up questions because I collected information about how they addressed the definition of the derivative and relationships among a function, the derivative of a function, and the derivative at a point from the classroom observations. At the end of the interview, I asked instructors if they would use each survey question as an example while they teach the derivative in class.

Data Analysis

After collecting video clips from observation and interviews, I first transcribed parts where instructors and students addressed: the definitions of the derivative of a function and the derivative at a point; the relationships between a function, the derivative of a function, and the derivative at a point; and the derivative as a function. Then, I marked cases where instructors and students explicitly and implicitly addressed these topics and relationships, and I counted them. When a routine was identified, a table with the routine and excerpts that provided evidence of the routine was made. The detail of each stage of the data analysis is described as follows.

Transcribing

Before transcribing classroom discourses, I looked at all the videos of the derivative lessons, created rough transcripts for all lessons, and noted the parts where the instructor and students used the terms— “derivative,” “derivative of a function,” and “derivative at a point”— to describe the relationships among them or relationships with the function. Most parts of the video clips were transcribed, except instances when the instructors were showing calculations or reviewing other topics (e.g., the limit of a function or composite functions) without saying the words, *derivative* or *differentiation*. Detailed transcripts were made using five columns— “number of line,” “time,” “who said,” “what is said,” and “what is done.” “What is done” included all kinds of visual mediators such as gestures, and writing and drawings on the board (e.g., *drawing an increasing and concave-up curve with his finger* and *making an increasing line with his arm*). Transcripts were made with Excel and broken into lines when the speaker or topic changed. Excel allowed me to assign a number for each line of the transcript easily. Transcripts of the classroom discourse were supplemented by field notes when views of the board in the video clips were not clear.⁴ Pauses were recorded in parentheses with seconds. The same format was used to transcribe interviews. This transcript format provided information about the four components of mathematical discourses: word use, visual mediators, routines, and endorsed narratives. The excerpts in which students or instructors explained the derivative and its relationship with the original function were closely examined to explore: a) how concepts of the derivative at a point and the derivative of a function were introduced and/or described, b) to what extent the relationships between the derivative at a point and the derivative of a function, and

⁴ Since the figures directly taken from the video clips were too dark or unclear to insert in the document, the figures were reproduced based on the video clips and field notes.

between a function and its derivative were addressed, and c) to what extent the derivative of a function as another function was addressed.

Coding

After transcribing the video clips from classroom observations and interviews, I first analyzed the transcripts to find how the derivative of a function and the derivative at a point were defined. More specifically, which one of the two concepts, the derivative of a function or the derivative at a point was defined first, and then how the other concept was built up from the previously defined concept. Then, I marked the parts where the instructors addressed the relationships between a function, its derivative function, and the derivative at a point, and the derivative as a function. After grouping the excerpts for each of these four categories, I came up with a list of cases that belonged to each category.

The relationship between the derivative of a function and the derivative at a point was closely related to the use of the word, “derivative,” and could vary depending on which one was introduced first in the classroom discourse or was more dominant in students’ mathematical discourse. Therefore, cases about this relationship were further divided into three subcategories: a) the derivative at a point as a value of the derivative of a function at a specific point, b) transition from the derivative of a function to the derivative at a point, and c) transition from the derivative at a point to the derivative of a function. Table 9 shows some of the cases that I included in each of the four categories. In the table, $f(x)$, $f'(x)$, and $f'(a)$ refer to a function, the derivative of a function, and the derivative at a point, respectively.

Table 9. Cases Belonging to Each Coding Category

Categories	Cases	
Relationship between $f(x)$ and $f'(x)$	Graphing or describing $f(x)$ using the sign of $f'(x)$ on an interval	
	Proving the differentiation rules	
	Interpreting the chain rule as a product of rates of change	
	Finding anti-derivative algebraically or from graphs	
	Finding the concavity of $f(x)$ using whether $f'(x)$ increases or decreases	
Relationship between $f(x)$ and $f'(a)$	Graphing $f'(x)$ using values of the derivative at points on the domain	
	Determining the differentiability of $f(x)$ using $f'(a)$	
	Specifying $f'(a) = 0$ when $f(x)$ has an extreme value at $x = a$	
Relationship between $f'(a)$ as a value of $f'(x)$ and $f'(a)$	Mentioning that $f'(a)$ is a value of $f'(x)$ at $x=a$	
	Difference between non-differentiability of function and at a point	
	Interpreting a point on the graph of $f'(x)$ as the slope of a tangent line	
	Transition from $f'(a)$ to $f'(x)$	Mentioning that several values of the derivative at points form the derivative of a function
	Transition from $f'(x)$ to $f'(a)$	Finding the equation of $f'(x)$ and then substituting a number to evaluate $f'(a)$
$f'(x)$ as a function	Mentioning that $f'(x)$ is a function as a part of its definition.	
	Mentioning that $f'(x)$ is a function that one could graph.	
	Mentioning that $f'(x)$ is as a function that has its own derivative, $f''(x)$	

After coding the transcripts, I used Transana (Woods & Fassnacht, 2007) to group and count the excerpts that belonged to each case. Each excerpt was an *implicit* or *explicit* statement about one of the relationships or the derivative as function as a form of a preview, example, or summary. Instructors' or students' statements were classified as *explicit* when they stated and used one of the relationships. Cases in which they used one relationship without stating it were classified as *implicit*. For example, when defining the second derivative of a function, $f(x)$, as the derivative of the first derivative, $f'(x)$, they can state that $f'(x)$ itself is a function which deserves its own derivative. This case is counted as *explicit*. If they define $f''(x)$ without mentioning $f'(x)$ is a function, the case was counted as *implicit*. After analyzing their discourses, I created frequency tables to display how many times they discussed the relationships and the derivative as a function, and nature of the discussion using *E* for *explicit*, and *I* for *implicit*.

Routine Table

While analyzing the transcripts into the four categories described above, I was also able to identify the routines in the discourses. As discussed in the previous chapter, routine is defined as a repetitive pattern in the discourse and can be reported in table format (Sfard, 2008). A routine table includes several components: prompt, course of action, and closure along with actual discourses that provide evidence for the routine. For example, a prompt can be a specific type of problem, course of action can be a solution process of the problem, and the closure can be the process of checking answers in the problem context. Routine tables also include key words (representing mathematical objects), visual mediators (all other forms of discourse aside from verbal expression such as drawings and gestures), and endorsed narratives (the statement that is accepted as true by speakers). Key words or terms related to the derivative are “function,” “the derivative of a function,” and “the derivative at a point.” Examples of visual mediators are graphs of a function or the derivative function, and instructors’ or students’ gestures to represent the behavior of a function or the derivative function. An endorsed narrative can be a statement about a function or the derivative such as “A function is increasing where the derivative of a function is positive.” These three components of mathematical discourse supplement the routine. In this study, I identified a pattern in the discourse as a routine when it was iterated at least three times in the derivative unit. However, as exemplars, I reported one or two excerpts that have more detail and show the routine explicitly.

Table 10 is an example of a routine table from classroom discourse when Tyler explained the derivative at a point. This routine was identified several times with different functions, but I reported two examples which included greater details than others. The following routine table shows Tyler’s routine when he explained how to find the derivative at a point using the

definition of the derivative of a function.

Table 10. Tyler's Routine and Explanations of Finding the Derivative at a Point

Comp. of Routine	Action	Example 1 $f'(1)$ when $f(x) = x^2$	Example 2 $f'(1)$ when $f(x) = \sqrt{x+2}$
Prompt	Stating Goal of Activity	1T82. Let's compute the derivative of a function. 1T85. Let's try to figure out...what the derivative at 1.	2T28. Say our function is $f(x)$ is defined to be $f(x) = \sqrt{x+2}$ We want to find $f'(1)$.
Course of action	Mentioning Function	1T83. Let's look at... $y = x^2$... This is a nice smooth function.	2T30. I will illustrate this on the graph (Graphing $f(x) = \sqrt{x+2}$).
	Mentioning a Point	1T87a. Look at 1	2T37. What you need to do is you look at the point 1.
	Mentioning Another Point	1T87b. [Look at] some other point b . So, this is b .	2T38. Then, you are going to little bit to the right by some small amount h .
	Mentioning Y Values		2T39. This is $f(1+h)$.
	Finding Difference Quotient	1T88. We are computing the slope of this line... [which is] $f(b) - f(1)$ divide by $b - 1$	2T40. We are dividing this by h .
	Calculating the Limit	1T89. b goes to 1...You can just factor top out and you get $(b-1)x$ $(b+1)$ divided by $b-1$.	2T41. Take the limit of this as h goes to 0 (writing down $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$).
	Stating the Value of $f'(a)$	1T97. We get the limit as b goes to 1 of [the fraction]...you can just evaluate it by plugging in.	2T42. This is the general formula for any derivative at 1. 2T51. This is $\frac{1}{2\sqrt{3}}$.
Closure	Checking $f'(a)$ on Graph		2T52. The slope is less than 1 and positive so this matches with the graph

Note. Comp. in the first cell is appreciation for component.

As shown in Table 10, the first reports column reports the component of routines—*when-routine*, which are prompt and closure, and *how-routine*, which is a course of action. The second column shows instructors or students action in each step of the routine. The Other two columns show what the instructor or students said, which provide the evidences for this routine. Since a prompt or closure was not always identifiable in a routine, their cells were sometimes blank in those

cases or not included in the table. The first and second rows show cases—equations or graphs—that an instructor and/or students work with when the routine is identified. A line number in the table reports which day of the derivative unit the utterance was identified, who said it, the line number in the transcript, in order. For example, 1T82 represents Tyler’s or students’ 82nd utterance on Day 1. When a student participated, I used S. A line number was repeated in the table when the different parts of one sentence were used in more than one cell, I used lowercase letters with a line number (e.g., 1T871a and 1T87b). After reporting a routine table, I discussed to what extent an instructor or student addressed relationships between a function, the derivative function, and the derivative at a point, and how he or she used the word, “derivative” in this routine.

In summary, I analyzed the transcripts focusing on the parts where instructors and students: defined or described the derivative; addressed the relationships among a function, the derivative function and the derivative at a point; and explained the derivative as a function. In each case, I investigated whether the discussion was implicit or explicit and whether there was a pattern in the discourse. When a pattern was identified, a routine table was made. Therefore, in the next chapter, I report the results based on how instructors and students defined and/or described the derivative of a function and the derivative at a point, and the relationships between them and with the original functions. The evidence was provided in routine tables or with excerpts when the routine was not identified.

CHAPTER 4: RESULTS OF CLASSROOM DISCOURSE ANALYSIS

Classroom discourse is ideally a combination of the students and instructors' discourse. However, the data from classroom observation showed that the classroom discourse mainly consisted of the instructors' discourses. Therefore, this chapter reports how instructors addressed various aspects of the derivative while teaching the derivative to address the following questions:

1. How do instructors introduce, describe, or define the concepts of the derivative at a point and the derivative of a function?
2. To what extent do instructors address the relationships between a function, the derivative at a point, and the derivative of a function?
3. To what extent do instructors address the derivative of a function as another function?

The analyses of Tyler's, Alan's and Ian's classroom discourses are reported in turn in separate sections. In each section, I will address the research questions in varying order depending on the organization of each instructor's class. I separated the discussion on definitions and rules of the derivative, and the discussion on the application of the derivative. However, if an instructor extensively explained a topic while discussing definitions and rules and readdressed it in the application for the purpose of review, I put all discussions of the topic in the section of definitions and rules because reading all of the discussions about one topic makes it easier to identify a pattern of the discussions on the topic, if a pattern exists. For example, I put Alan's discussion on *differentiability of a function* in the sections of definitions and rules since he discussed the topic mostly there and revisited it once in the application. Although this format may make it hard to see the chronological order of topics addressed in class, the analysis focuses on how the definitions and the relationships described in the research questions were addressed

in the classroom discourse rather than reporting what happened in the classroom chronologically.

Discussions and explanations that are not related to the definitions and relationships stated in the research questions were not included in the results section. For example, I did not include discussions on L'Hopital's Rule because all the instructors explained the rule as a tool for finding special types of limits without connecting it to any of the items in the research questions. I also did not include the discussions on the limit or composite functions, which arose when some instructors reviewed these concepts without addressing the derivative.

When a repeated pattern, *routine*, was identified in the discussion of a concept, I made a routine table including the routine and excerpts from one or two episodes, which provide evidence for the routine, and reported how many times this routine was identified. When a concept was discussed once, I summarized the discussion with an excerpt from the transcript. At the end of each section, the summary addresses in which topics, to what extent, how many times each instructor addressed the concepts and relationships included in the three research questions. Then, a comparison between the three cases follows.

Tyler

Tyler spent 18 class periods on the derivative unit. His classes were lecture-based using a blackboard and chalk. The class mostly consisted of his exposition; discussions between him and students were rare. The transcripts of classroom discourses were broken down into lines when the speaker or topic changed. Of a total of 1122 lines of transcripts, 34 lines (3%) were said by a student or a group of students. In most cases, they simply answered Tyler's closed questions such as "What is the derivative of $\sin x$?" (16 out of 34). The second frequent categories, six for each, were students' answers to Tyler's open questions (e.g., "How do we find the maximum value of this volume?") and students' questions about Tyler's explanations (e.g., "Is that true for

all constant functions?). Three questions about calculation (e.g., “How did you get the third line from the second?”), two answers to yes-no questions (e.g., when Tyler asked “ok?”), and one non-mathematical question (e.g., will it be on the exam?) were also identified.

Table 11 shows the topics that Tyler addressed on each day of the derivative unit.

Table 11. *Topics Addressed by Tyler in the Derivative Unit*

Day	Topics
1	Definitions of the derivative at a point and derivative of a function
2	Differentiation rules
3	Definitions of the derivative at a point and derivative of a function, Differentiation rules
4	Higher derivatives, Position, Velocity, Acceleration, Derivative of trigonometric functions
5	Derivative of trigonometric functions, The chain rule
6	The chain rule, Implicit differentiation
7	Implicit differentiation, Related rate (with respect to time)
8	Related rate (with respect to time), Linearization
9	Related rate (with respect to time)
10	Review of the topics covered previously, Extreme values of functions
11	Extreme values of functions, Critical points, Rolle’s Theorem, Mean value theorem
12	Monotonic Functions and First derivative test
13	Concavity and curve sketching, Point of Inflection
14	Review of the quiz problems from Day 11, 12, and 13
15	Concavity and curve sketching, Applied Optimization Problems
16	Applied Optimization Problems
17	Applied Optimization Problems, L’Hôpital’s rule
18	Applied Optimization Problems, L’Hôpital’s rule, Newton’s Method

Definitions and Rules of the Derivative

On Day 1, Tyler started the derivative unit by introducing the derivative at a point with a graphical illustration of a function and the tangent line. He focused on a) the locality of the behavior of the original function, which is captured by the derivative at a point and b) the limit process used to obtain the slope of the tangent line from the slopes of the secant lines. Using this limit process, he defined the derivative of a function in two different but equivalent ways and evaluated the derivative of a specific function at a specific point, $f'(a)$, using these definitions.

On Day 2, Tyler graphed the derivative of a function based on several values of the derivative at

a point. Then, he explained the differentiability of a function based on the existence of the derivative at a point. On Day 3, he explained differentiation rules with algebraic proofs or graphical illustrations.

Definition of the Derivative at a Point: Local Behavior of a Function and Limiting Process

Tyler’s explanation about the derivative at a point consists of two parts: the derivative as a slope of a tangent line that captures the local property of a function and the limit process to obtain the slope of the tangent line from the slopes of the secant lines.

Local Behavior of a Function. Tyler began to describe the derivative with the graph of a function, $y = f(x)$ by saying, “We are interested in the behavior of the function at this point” (Day 1). Then, he drew a tangent line to the graph at a point and explained that the line only captures local behavior of the function near the point (Figure 10).

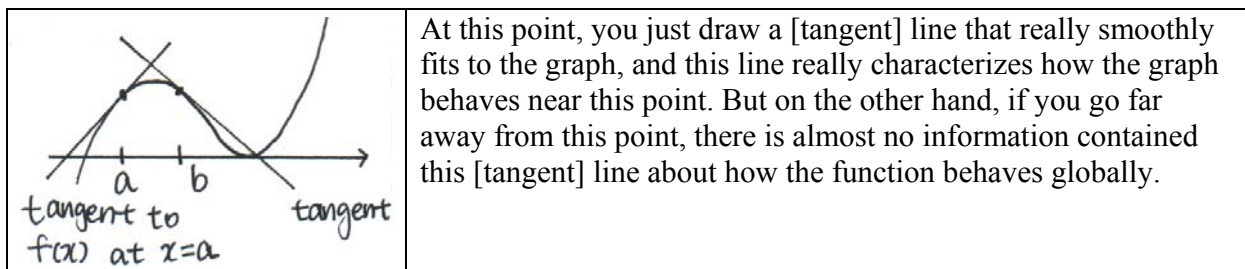


Figure 10. Tyler’s Explanation of the Derivative at a Point as Slope

Using the tangent line at a point, he wrote the first definition of the derivative:

The derivative of $f(x)$ at $x = a$ ($f'(a)$) is the slope of the tangent line. (Definition 1)

The local property of a function, which is captured by its tangent line, was addressed four more times later in the derivative unit when he explained the linearization of a function as the tangent line at a point on Day 8. On Day 18, he repeated the same explanation and mentioned the difference between a function and the tangent line by saying “the function and its linearization are not the same. So, we are a little off.”

Limiting Process. On Day 1, Tyler showed that the slope of the tangent line can be derived from the limit. He explained how to obtain the slope of a tangent line to $y = f(x)$,

$$f'(x) = \lim_{x \rightarrow b} \frac{f(x) - f(b)}{x - b} \text{ or } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ at a point } x = a, \text{ from the slopes of}$$

secant lines passing through $x=a$ and another point, $x = b$, moving to $x = a$ (Figure 11).

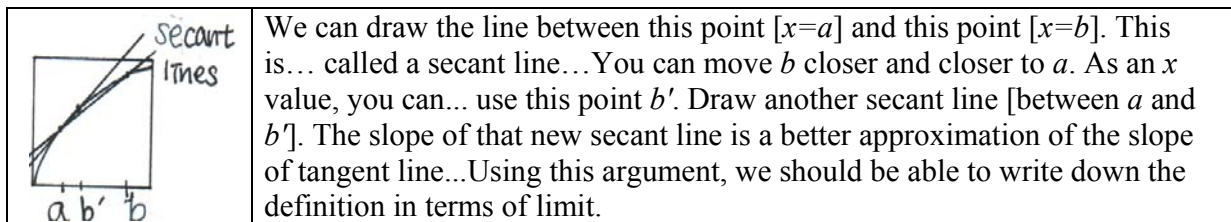


Figure 11. Tyler's Drawing and Explanation of Secant and Tangent lines

Transition from the Derivative at a Point to the Derivative of a Function

On Day 1, after explaining how to obtain the slope of the tangent line from slopes of the secant lines, Tyler gave students the limit definition of the derivative of a function without providing the definition of the derivative at a point. He started with the derivative at a point $f'(x_0)$ and suddenly changed his notation to $f'(x)$ by saying, "We want to figure out what is $f'(x_0)$. We have our point x_0 here. I don't like my notation very much. Let's try to compute $f'(x)$ because what always you want to do is to try to compute $f'(x)$ " (Day1). Because he did not state that $f'(x_0)$ is a specific value of $f'(x)$, or the independent variable x can be inferred from a point x_0 , the transition from a point to the independent variable was not clearly stated. This implicit transition was also identified on Day 4 when he defined "instantaneous velocity" as the limit of "average speed between [a] point and [another point on] the slightly right of it" as the second point approaches "closer and closer" to the first one, and said, "It's exactly same thing as finding the derivative at this point." Without stating that this explanation works at any point of time, he

introduced the definition of the velocity function of a variable t , $\lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$. He did not state that an instantaneous velocity is a value of the derivative of the position, or velocity is another function.

Definition of the Derivative of a Function: The Limit Definition

On Day 1, Tyler defined the derivative of a function, $f'(x)$, using the limit. He used a graphical illustration, in which x is a fixed point on the graph of $y = f(x)$ and another point, b , which is approaching x shown in Figure 12:

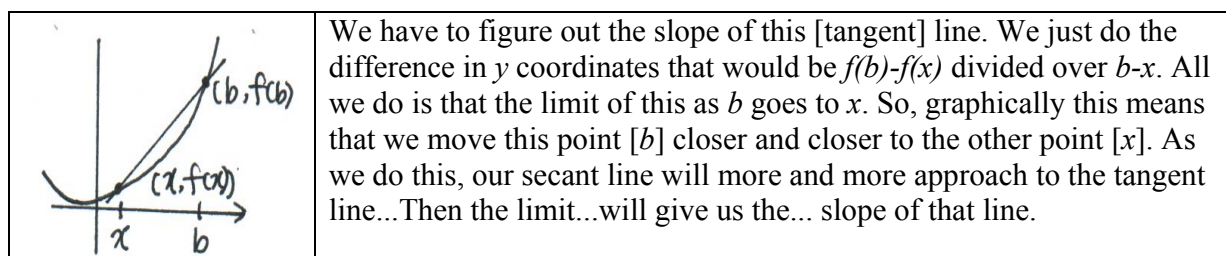


Figure 12. Tyler's Drawing of the First Limit Definition of the Derivative

Tyler, then, gave the definition of the derivative of a function using the limit.

$$f'(x) = \lim_{x \rightarrow b} \frac{f(x) - f(b)}{x - b} \quad (\text{Definition 2})$$

He solved one example of finding $f'(1)$ with a specific function using Definition 2, and introduced another limit definition by changing the point $(b, f(b))$ to $(x + h, f(x + h))$:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (\text{Definition 3})$$

Table 12 shows the routine of Tyler's explanations of Definitions 2 and 3. He stated the definition of $f'(x)$, mentioned a function $f(x)$, and two points on $f(x)$ at x and $x + h$, y values of these points, wrote down the difference quotient, $\frac{f(x + h) - f(x)}{h}$, applied the limit on the

quotient, and mentioned the definition of $f'(x)$ again.

Table 12. Tyler's Routine and Explanations of the Definition of $f'(x)$

Comp. of Actions Routine ^a		Definition 2	Definition 3
		$f'(x) = \lim_{x \rightarrow b} \frac{f(x) - f(b)}{x - b}$	$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
Prompt		1T56 ^b . We should be able to write down the definition in terms of limit.	1T100. The second definition... we do this same procedure.
Course of action	Mentioning function	1T57. So, we have our function.	
	Mentioning point	1T65. What you always want to do is to... try to compute $f'(x)$ (Changing x_0 to x)	1T101.[L]ook at...point x
	Mentioning another point	1T66. Just call this point b .	1T102. We go a little bit to the right or...the left, we compute the slope between x and $x+h$.
	Mentioning y values	1T71. This point has coordinates $(b, f(b))$ and this point has coordinates $(x, f(x))$.	1T109. This point has coordinates $(x+h, f(x+h))$, and this has $(x, f(x))$.
	Finding difference quotient	1T74. We just do the difference in y coordinates that would be $f(b) - f(x)$ divided over $b - x$.	1T109. The slope of this [secant] line...is...the difference quotient written as $f(x+h)-f(x)$ divided by h .
	Calculating the limit	1T75 All we do is [taking] the limit of this as b goes to x ... We move this point closer and closer to the other point. As we do this, our secant line will more and more approach to the tangent line.	
	Mentioning definition of $f'(x)$	1T78. The limit...will give us the actual slope of that line. This is one way of writing down the equation of tangent.	1T105. We can define the derivative of $[f]$ as the limit of this slope as h goes to 0.

Note. ^aComp. refers to component. ^bThe number in each cell, represent which Day of the derivative unit, who said, the line number in the transcript, in order. For example, 1T56 represents Tyler's and students' 56th utterance on Day 1. When a student participated, S was used. When one sentence was broken up into more than one parts, I put lower case letter at the end of the number. Tyler's actions are shown in parentheses.

On Day 1 and 2, Tyler repeated this routine three times to explain Definitions 2 and 3. He

also used the same routine twice later when he defined the velocity of a moving object from its position function and showed that the derivative of $f(x) = \sin x$ is $\cos x$ using Definition 3 on Day 4. However, he did not mention that $f'(x)$ is another function.

Transition from the Derivative Function to the Derivative at a Point

After giving Definitions 2 and 3, Tyler used them to evaluate the derivative of a specific function at a point. The same routine that he used to explain the definitions was identified. He solved two examples: finding $f'(1)$ when $f(x) = x^2$ using Definition 2 on Day 1 and finding $f'(1)$ when $f(x) = \sqrt{x+2}$ using Definition 3 on Day 2 (Table 13).

Table 13. *Tyler's Routine and Explanations of Finding the Derivative at a Point*

Comp. of Action Routine		Example 1	Example 2
		$f'(1)$ when $f(x)=x^2$	$f'(1)$ when $f(x) = \sqrt{x+2}$
Prompt		1T82. Let's compute the derivative of a function. 1T85. Let's try to figure out...what the derivative at 1.	2T28. Say our function is $f(x)$ is defined to be $f(x) = \sqrt{x+2}$ We want to find $f'(1)$.
Course of Action	Mentioning function	1T83. Let's look at... $y=x^2$...This is... nice smooth function.	2T30. I will illustrate this on the graph (Graphing $f(x) = \sqrt{x+2}$).
	Mentioning a point	1T87a. Look at 1	2T37. What you need to do is you look at the point 1,
	Mentioning another point	1T187 b. [Look at] some other point b . So, this is b .	2T38. then you are going to little bit to the right by some small amount h .
	Mentioning y values		2T39. This is $f(1+h)$.
	Finding the difference quotient	1T88. We are computing the slope of this line... [which is] $f(b)-f(1)$ divide by $b-1$	2T40. We are dividing this by h .
	Calculating the limit	1T89. b goes to 1...You can just factor top out and you get $(b-1)x$ $(b+1)$ divided by $b-1$.	2T41. Take the limit of this as h goes to 0 (writing $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$).
	Stating the value of $f'(a)$	1T97. We get the limit as b goes to 1 of [the fraction]...you can just evaluate it by plugging in.	2T42. This is the general formula for any derivative at 1. 2T51. This is $1/2\sqrt{3}$.
Closure		2T52. The slope is less than 1 and positive so this matches with the graph	

As shown in the first columns of Tables 12 and 13, the same routine was identified when Tyler explained the limit definition of $f'(x)$ and used it to calculate $f'(1)$. He also used the same graphical representations. However, he did not mention this equivalence between

$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ and $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$, or $f'(1)$ is a value of $f'(x)$. On day 2, he

addressed that the derivative can be defined at any point by saying, “If you start with some function, $f(x)$...your derivative is basically telling you, at any point of this function, how fast this function is increasing or decreasing,” but did not explicitly mention that the derivative of a function is another function, or the derivative at a point is a value of the derivative of a function. Although the students could have realized this relationship between the derivative of a function and the derivative at a point by noticing the same procedure used in both cases, the explicit explanation about the relationship was not given when the derivative of a function was defined and used to evaluate the derivative at a point. Later in class, he used Definition 3 to calculate the derivatives of various functions at a point; on Day 4, he found the derivative of $f(x) = \frac{3}{\sqrt{x}}$ at $x = 2$, and on Day 9, he found $f'(-1)$ when $f(x) = x + 1/x$. When solving these six examples, he did not give a definition of the derivative at a point, or mention the derivative of a function as another function in which one can substitute a number to calculate the derivative at a point.

Connection from Derivative at Point to Derivative of a Function: Derivative as Function

On Day 2, after explaining how to find $f'(1)$ using Definition 3, Tyler explained that one can apply the definition of $f'(1)$ at any point, thus one obtains a new function:

This (pointing to the solution process of evaluating $f'(1)$ when $f(x) = \sqrt{x+2}$ is like the way to define the derivative at one point...We can do simultaneously at any point of the graph, and then we get a new function. Starting with some function, $f(x)$, we get a new

function $f'(x)$. And we can graph this function as well.

In this excerpt, Tyler connected the derivative at a point to the derivative of a function by saying that one obtains a new function by applying the definition of the derivative “at any point.”

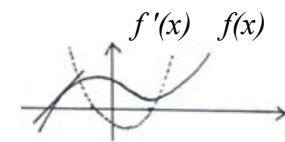
However, he did not make a transition from a point to the independent variable of $f'(x)$ explicit; he did not mention that a value, l in the excerpt, is an element of the domain which can be extended to any point where $f(x)$ is differentiable. Here, $f'(x)$ as a function was explicitly addressed, but the relationship between $f'(x)$ and $f'(l)$ was not.

Relationships between a Function and Derivative at a Point: Graphing the Derivative

After mentioning the derivative of a function, $f'(x)$, as another function (Day 2), Tyler graphed $f'(x)$ using the slope of the tangent line. He plotted points where $f(x)$ has horizontal tangency, found if $f'(x)$ is positive or negative between these points by describing the behavior of $f(x)$ or the tangent line, and completed the graph of $f'(x)$ by connecting the points considering its sign between them. Table 14 shows his explanation of the relationship between $f(x)$ and $f'(x)$ at a point of horizontal and negative tangency.

Table 14. Tyler's Routine, Explanations, and Drawings of Graphing $f'(x)$

Comp. of Routine	Action	Types of Tangency	
		Horizontal tangency	Negative tangency
Prompt		2T64. What's the derivative at this point?	2T68. Between those two [zero] points, are you expecting positive or negative derivative?
Course of Action	Finding $f'(a)$	2T65. Zero	2S70. Negative.
	Describing graph of $f(x)$		2T72. This corresponds to the fact that the function is decreasing.
	Drawing tangent		2T73. You are drawing a tangent at a point
	Finding the slope	2T66a. The slope at this point is zero.	2T74. That tangent will have a negative slope.
	Stating $f'(a)$	2T66b. The derivative is going to be zero.	(Plotting a point on the negative part)
Closure	Completing graph of $f'(x)$	2T83. (Drawing a dotted line) this is how the graph of $f'(x)$ looks like this.	



In Table 14, he addressed the relationship between a function, $f(x)$, and the derivative at a point, $f'(a)$ explicitly by stating and finding $f'(a)$ based on the behavior of $f(x)$ or tangent lines. However, he did not connect this relationship to the relationship that the derivative at several points form $f'(x)$. This routine's aim seems to be showing that $f'(x)$ is a function that can be graphed rather than describing how it can be built up from the behavior of $f(x)$.

On Day 4, Tyler used the same routine to graph the derivative of $f(x) = \sin x$ after showing the derivative of $\sin x$ is $\cos x$ algebraically. By plotting and connecting several points corresponding to the slopes of the tangent lines to $f(x) = \sin x$, he showed that this graph looks similar to $f'(x) = \cos x$. While graphing, instead of estimating the slope from the graph of $f(x) = \sin x$; instead, he stated the slope without explaining where it came from. For example, he plotted point $(0, 1)$ for the graph of $f'(x)$ by saying, "We already know at this point [$x = 0$], the slope is one". He also

said that the purpose of drawing this graph was to “convince [oneself] that actually this [$f'(x) = \cos x$] makes sense” (Day 4). He did not mention either building up a graph of the derivative of a function based on the several values of the derivative at a point, or the derivative of a function as another function.

Connection between a Function and the Derivative at a Point: Differentiability

On Day 2, after graphing the derivative function, Tyler explained differentiability. He first wrote down its definition as “a function $f(x)$ is differentiable at x , if... $f'(x)$ exists.” He did not distinguish the differentiability of a function at a point and on an interval in contrast to the definition in the textbook, which explained these two concepts differently: “If f' exists at a particular x , we say that f is differentiable (has a derivative) at x . If f' exists at every point in the domain of f , we call f differentiable” (Thomas et al., 2005). Instead of making this distinction, he mainly talked about the differentiability at a specific point. He then listed cases where a function fails to be differentiable. A group of his examples includes functions with discontinuity such as a jump or a vertical asymptote such as $y = 1/x$. The other group contains continuous functions with multiple tangent lines, such as $y = |x|$, or vertical tangency at a point, such as $y = x^{1/3}$. To explain differentiability, he a) illustrated a function, b) described its behavior, c) mentioned Definition 3, d) plotted a point of non-differentiability and another point near the point, e) drew a secant line, f) described the limit process, and g) made a general statement about differentiability. Table 15 shows the routine of his explanation about differentiability of $y = |x|$ at $x = 0$. Since the secant lines coincide with the graph of $y = |x|$, step f) was not addressed here.

Table 15. Tyler's Routine and Explanations of Differentiability of $f(x) = |x|$ at $x=0$

Comp. of Routine	Action	Example		
		Function with multiple slopes at a point ($y = x $)		
Prompt		2T116. Differentiability is a stronger condition than continuity. A function is continuous doesn't mean that it's differentiable.		
Course of action	Mentioning $f(x)$	2T118. We have the absolute value function (Graphing $y = x $).		
	Describing the behavior of $f(x)$	2T119. This function is continuous.		
		2T121. I know the limits always exist (Plotting the origin)		
	Stating definition of $f'(x)$	2T122. You only look only at the derivatives, look at this definition.		
	Mentioning a point	2T123. At this point [the origin] we can do either of two things.		
	Mentioning another point	2T124. Your h can come from the positive side.	2T126. If you are coming from the other side.	
		2T125a. Then, all your slopes coincide with the slope of this line.		
	Drawing secant lines	2T125b. I can actually say all the tangent, all the secant lines are this line (pointing to the right side of the graph)	2T127. You always get this line as secant line (pointing to the left-side).	
Finding limit	2T129a. So, the derivative isn't defined to this function.			
Closure	2T129b. So, I guess "not differentiable at corners" (writing down).			

This routine was identified four times. In all cases, Tyler explained that a function is not differentiable because its derivative at a specific point does not exist. This logic makes sense only if one already knew that the derivative at a point, $f'(a)$, is a value of the derivative of a function, $f'(x)$; however, he did not make it explicit that the function fails to be differentiable because it does not have a value $f'(a)$. In summary, he showed differentiability of a function based on differentiability at a point without making the relationship between $f'(x)$ and $f'(a)$ explicit. He distinguished the differentiability of a function and at a point later on Day 11 when

he explained the critical point of $f(x) = \begin{cases} -x & \text{if } -2 \leq x < -1 \\ x^2 & \text{if } -1 \leq x \leq 2 \end{cases}$. After graphing $f(x)$,

he addressed non-differentiability at a point explicitly (Figure 13).

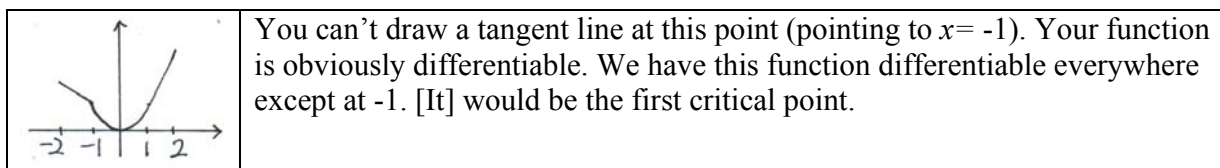


Figure 13. Tyler's Drawing and Explanation of Differentiability

In this example, he specified that the function is differentiable everywhere except a point where the function changes its equation. However, he did not explain the relationship between the non-differentiability at a point to the non-differentiability of a function.

Relationship between a Function and its Derivative Function: General Rules

On Day 2, after discussing differentiability, Tyler said that what he had covered so far was a “function at a point”, and was going to work on “general rules...of how to find the derivative”. He stated and justified differentiation rules for $f(x) = x^n$ and $f(x) = g(x) \pm h(x)$ with Definition 3, for $f(x) = cg(x)$ and $f(x) = mx + b$ with graphs, and for $f(x) = c$ with both. While proving a rule with Definition 3, he a) stated the rule, b) calculated the difference quotient, and c) took the limit on it. Table 16 shows his proof of $f'(x) = 0$ when $f(x) = c$.

Table 16. Tyler's Routine and Explanation of the Differentiation Rule for $f(x) = c$

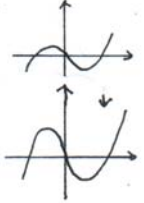

Comp. of Action		Function
Routine		$f(x) = c$
Prompt		2T153. What if $f(x)$ is just c ?
Course of Action	Stating definition 3	2T165. We also can use our limit definition to do this.
	Finding the quotient	2T167. $f'(x)$ is the limit as h goes to zero of $f(x + h) - f(x)$ over h .
	Limit	2T171. This function doesn't even depend on x , whatever you plug in there, you always get c . This is the limit as h goes to zero of $c - c$ over h
		2T173. This is just zero over h so...zero. We get that the limit is zero.

In this routine, Tyler did not discuss the derivative of a function, $f'(x)$, as another function or connect $f(x)$ to $f'(x)$. He referenced $f(x)$ once when he substituted $(x + h)$ for x in $f(x)$.

To justify a differentiation rule graphically, Tyler a) mentioned a function, $f(x)$, b) stated a definition of $f'(x)$, c) graphed $f(x)$, d) drew a tangent line, and e) mentioned that its slope is

$f'(x)$. Table 17 shows this routine and his explanation of $f'(x) = cg'(x)$.

Table 17. Tyler's Routine and Explanations of Differentiation Rule $f'(x) = cg'(x)$

Comp. of Routine	Action	Function
Prompt		$f(x) = cg(x)$
Prompt		2T189. What is the derivative? Say you have function f , you can get a form of some other function, g multiplied by a constant.
Course of action	Writing $f(x)$	2T191. c times some other function, if c is constant (Writing $f(x) = cg(x)$)
	Stating $f'(x)$	2T190. $f'(x)$ is actually going to be what exactly what you think... c times the derivative of the other function (Writing $f'(x) = c g'(x)$).
	Graphing $f(x)$	2T196. You have your function (Drawing a function)...We are multiplying this function by some constant...Depending on what c is...the function might be stretched or contracted. But in any case...your function $f(x)$ is just going to be some stretching of this function, it looks like this. 
Drawing tangent line	2T197. You are looking at the derivative at one point (Plotting a point on the graph)...What does this do to the tangent line? If you stretched tangent line,...when you [are] ever stretching, if you draw a little triangle, one side of the triangle is the same, and the other side of the triangle is stretched by this factor, c . 	
Stating slope	2T204. Overall your slope is gonna be multiplied by...number c .	

As shown in Table 17, he focused on how the behavior of the tangent line changed when a constant c was multiplied by a function $g(x)$ at a point to explain why the derivative of $cg(x)$ is $cg'(x)$. This use of a tangent line at a point to justify the derivative of a certain function was identified three times; he used it for the constant function, $f(x) = c$, whose derivative is $f'(x) = 0$ (Day 2), and for two functions differing by a constant, which have the same derivative (Day 11). In all cases, he did not mention that this mechanism works at any point on the graph, which leads to the concept of variable. In other words, he did not address the relationship between the derivative of a function and the derivative at a point or the derivative of a function as another function explicitly.

Tyler concluded the routines of proving differentiation rules by confirming the result as

the derivative or explaining how to apply the rules to functions. Tyler's closures of the routine when he used Definition 3 were identified in three cases. For $f(x) = x^n$, he concluded that "all this long computation tells us that if you want to figure out what the derivative of some power of x is"(Day 2). For the sum of two functions, he went over the rule again by saying, "This should make plausible that you can get the derivative of the sum just by adding the derivatives of the terms" (Day 2). For a linear function, he just said, "This is the proof" (Day 3). When he explained differentiation rules graphically, he stated that the derivative is the slope. For example, he said, "Overall, your slope is gonna be multiplied by this number c ," for $f'(x) = cg'(x)$ (Day 2), and "So, the derivative of a linear function is just the slope," for $f(x) = mx + b$ (Day 3). In any cases, he did not mention that the result from these procedures, which is the derivative of a function, is another function.

Applications of the Derivative

From Day 4, based on the definitions and rules of the derivative, Tyler covered the application of the derivative (Table 11). In this section, I will report the extent to which he addressed the relationships between a function, the derivative at a point and the derivative of a function, and the derivative of a function as another function.

Relationship between the derivative of a function and the derivative at a point

In Tyler's class, the relationship between the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, was addressed in two directions: a) transition from $f'(a)$ to $f'(x)$ extending properties of $f'(a)$ to $f'(x)$ and b) transition from $f'(x)$ to $f'(a)$ to find $f'(a)$ by substituting $x = a$ in $f(x)$. In the first three classes, he did not make a transition from Definition 1 of $f'(a)$ to Definition 2 or 3 of $f'(x)$. He addressed the other direction to find $f'(a)$ using Definition 2 and 3 to evaluate $f'(a)$, but did not explain $f'(a)$ as a value of $f'(x)$.

Transition from derivative function to derivative at a point: Evaluating. While discussing the application of the derivative, Tyler made the transition from the derivative of a function, $f'(x)$, to the derivative at a point, $f'(a)$, when he substituted $x = a$ in $f'(x)$ to find $f'(a)$. He mentioned and/or used this substitution 10 times in three examples. In the first example about the slope of the tangent line to $f(x) = \sin x + x^2$ at $x = \pi$, Tyler said that one has to find the derivative of a function before evaluating the derivative at a point: “You can’t find the derivative at a specific point without finding a general first. So, we are gonna have to look at $f(x)$ and derive it” (Day 4). Then, he found $f'(x)$ and substituted π for x . In the second example about the slope of a function at $x = 4$, Tyler explicitly said that the substitution connected the derivative of a function, $f'(x)$, and the derivative at a point, $f'(4)$: “The slope of this tangent line is just the derivative of this function at 4. We can compute the slope by just plugging in $x = 4$ into this [derivative] equation” (Day 6). In the last example about a constant function, $f(x)$, he made such a connection again by saying, “We know what $f'(c)$ is. $f'(x)$ is zero everywhere. In particular, wherever this c is, the derivative at this point is zero” (Day 11).

While evaluating the derivative at a point algebraically, Tyler mentioned and used substitution consistently. However, he did not explicitly mention that the derivative at a point, $f'(a)$, is a value of the derivative function, $f'(x)$. Rather, he implicitly addressed their relationship when the equation of $f'(x)$ is given and $f'(a)$ has to be found. He also did not mention that $f'(x)$ is a function in which one can substitute a number to find $f'(a)$.

Transition from derivative at a point to derivative function: extending. Transition from the derivative at a point, $f'(a)$, to the derivative of a function, $f'(x)$, was made in the section about curve sketching on Days 14 and 17. After dividing the domain of a function, $f(x)$, into several intervals based on the points where $f'(x) = 0$, he estimated the derivative at a point on an

interval using the slope of the tangent line to decide if $f(x)$ increases or decreases on the interval.

In other words, he decided the sign of $f'(x)$ on an interval based on the sign of the derivative at a point. He provided the rationale for this method:

To decide if it [a function] is increasing or decreasing, it really suffices to just plug in a single point into the derivative. I wanted to talk about the derivative being positive or negative... We can decide if $f'(x)$ is positive by just looking at a single point. The reason we can do this is because our function can't jump around. If it is continuous [and] it goes from positive to negative, it has to cross the x -axis. So, let's just compute $f'(0)$ and $f'(3)$ (Tyler, Day 14).

In this excerpt, his ambiguous use of the words, “derivative” and “function” was identified. First, he did not specify “derivative” as the derivative of a function, $f'(x)$, or the derivative at a point, $f'(a)$, which might confuse the students because he was using $f'(a)$ to determine the sign of $f'(x)$. What “function” and “it” referred to are also ambiguous; when he said, “Our function can't jump around,” “function” could refer to the original function, $f(x)$, or its derivative function, $f'(x)$. From the context, “it” in “If it is continuous [and] it goes from positive to negative, it has to cross the x -axis” seems to refer to $f'(x)$, but it was not explicitly specified. Although he used the facts that $f'(x)$ is a function that changes its sign at a point where it crosses the x axis, and $f'(a)$ as a specific value of $f'(x)$, he did not state these properties. Therefore, he addressed $f'(x)$ as a function and the relationship between $f'(x)$ and $f'(a)$ implicitly when he used $f'(a)$ to decide the sign of $f'(x)$. In later lessons, he used this method in five different examples without justifying it.

Relationship between a Function and the Derivative at a Point

When discussing applications of the derivative, Tyler addressed the relationship between a function, $f'(x)$, and the derivative at a point, $f'(a)$. He connected the behavior of $f(x)$ to $f'(a)$ by explaining local extreme values and velocity as the derivative function.

Velocity as the Derivative of the Position. Tyler addressed the relationship between a function and the derivative at a point on Day 4 when he explained velocity as “the derivative of a

position with respect to time,” which “contains the information about the direction.” After

writing down the definition of velocity as $f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$ and stating, “Velocity

$v(t)$ is the derivative of the position with respect to time,” he explained the direction of a moving object at a point using the sign of the velocity at the point three times when the graph of the position function was given (Figure 14):

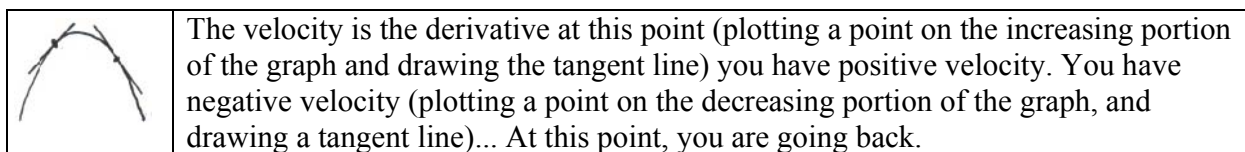


Figure 14. Tyler’s Graph and Explanation about Velocity

He also talked about the zero velocity at a point where a moving object changes its direction:

“You want to find this point where it [a ball] doesn’t move up or down...We want to find this place where...the velocity given by the tangent line at this point...is zero.”

In this episode, Tyler started with the definition of the derivative of a position function, $f(t)$, but primarily talked about velocity at a point (positive, negative, or zero) to describe the direction of a moving object at the point (going up, going down, or stopping) without referring back to the definition of $f'(t)$. In summary, the relationship between a function and the derivative at a point was explicitly addressed, but the relationship between the derivative of a function and the derivative at a point was not.

Local Minima and Maxima. Tyler addressed the relationship between a function and the derivative at a point when he explained local extreme values. He stated and used the derivative is zero at a point where a function has extreme values 10 times. On Day 13, he first mentioned horizontal tangency at those points: “If you have a local maximum, and your function is differentiable, then you...have a critical point, which... means that you have a horizontal tangent

...Whenever your function has a local maximum, we definitely need the function to be zero at this point” Here, he used the word, “function,” ambiguously; it can be inferred as “the derivative of a function” according to the context. He also provided the reason for $f'(x)$ being zero at those points using the change in sign of $f'(x)$: “The sign of $f'(x)$ changes...at the local maximum changes from being increasing to decreasing, so the derivative according to this changes from being positive to...negative” (Day 13). “Derivative” that he used without specifying its referent can be inferred as the derivative function because he was talking about its sign change. These inferences may not be clear to the students, but questions for clarifications or further discussion was not observed. In summary, he addressed the relationship between a function, $f(x)$, and the derivative at a point explicitly by stating that it is zero where $f(x)$ has local extremes. While explaining why $f'(x)$ is zero at those points, he did not state that $f'(x)$ is a function.

The Relationship between a Function and its Derivative Function

While discussing the application of the derivative, Tyler addressed the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, when he interpreted $f'(x)$ as a rate of change when explaining the chain rule, the related rate, and linearization, and b) explained the behavior of $f(x)$ based on the sign of $f'(x)$.

Derivative as a Rate of Change. Tyler used $f'(x)$ as a rate of change of $f(x)$, twice. On Day 5, he explained the change in $u = x^2$, as a product of the u' and the change in x :

Tyler: What’s the derivative of u with respect to x ?

Students: $2x$.

Tyler: What this is saying is that if I increase my x by a small amount, my u is going to increase by $2x$ times the small amount. Now, we know how u changes.

On Day 6, in an example involving volume, Tyler said “The rate at which the volume, V ,

changes is exactly the rate at which you are pumping the air in. This...is dV/dt . If you denote the time by t , we can...write $V'(t)$." While explaining these examples, he interpreted and used the derivative of a function as the rate of change of the function.

Behavior of a Function Based on the Sign of the Derivative. Tyler mainly addressed the relationship between a function, $f(x)$, and the derivative of a function, $f'(x)$, when he described the behavior of $f(x)$, such as increasing or decreasing and having extreme values, using signs of $f'(x)$ (33 out of 36 cases). On Day 12, he stated that an increasing function has a positive derivative: "[In] the parts where the function is increasing...if you look at the tangent line...we are gonna have a positive slope, so we are gonna have to have a positive derivative." Later in the lesson, he consistently connected whether $f(x)$ increases or decreases with the sign of $f'(x)$ (16 times). In most of cases, he used this connection to describe the behavior of a function near the point where it has an extreme value. For example, on Day 13, he said, "If you see that the sign of the derivative of f ...changes from negative to positive, you have your local minimum. Here's your critical points, horizontal slope. Your original function changes from decreasing to increasing."

As shown in the episode above, although he used changes in the sign of $f'(x)$ to describe the behavior of $f(x)$, he did not specifically use the term, "the derivative of a function"; instead, he simply said "derivative" without specifying its referent 29 times out of 36. For example, on Day 12 he said, "If the derivative is positive, then the function is increasing." In the rest of the cases, he used " f prime x " ($f'(x)$) for the derivative of a function f . This uses of the word, "derivative," without specifying its referent might have confused students since he explained that the derivative of a function is positive with a gesture of imitating the tangent line at a point with his hand with the word, "derivative." "Derivative" here might be interpreted as the derivative at a point rather than the derivative of a function. However, clarification on this use of the word was

not observed in the class. On Day 13, he used the relationship between $f(x)$ and $f'(x)$ to explain the concavity; he used $f'(x)$ as a function and $f''(x)$ as the derivative of $f'(x)$. Figure 15 shows his use of the slopes of tangent lines to a concave-down curve to show $f'(x)$ decreases, thus $f''(x) < 0$.

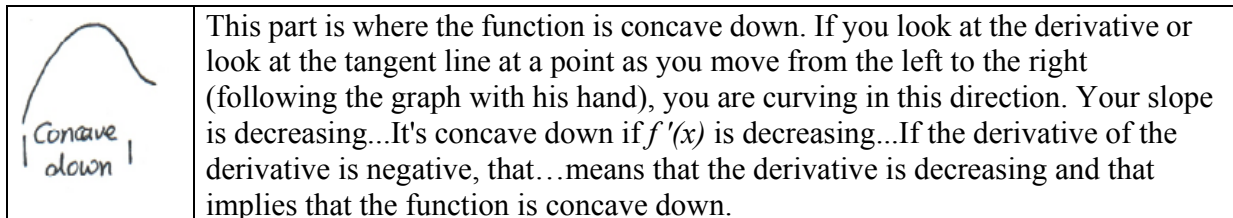


Figure 15. Tyler's Drawing and Explanation about a Concave-down Curve

In both cases above, the relationship between a function (the first derivative) and the derivative of a function (the second derivative) was clearly addressed. However, in the first episode involving a function and the derivative of a function, he did not specify the word, “derivative” as the derivative of a function or the derivative at a point although it can be inferred as the former. In the second episode while using the relationship between the first and second derivatives, $f'(x)$ and $f''(x)$, respectively, he did not mention that $f'(x)$ is a function, so one can apply the definition of the derivative to $f'(x)$ to find $f''(x)$.

The Derivative as a Function

As discussed earlier, Tyler addressed that the derivative of a function is another function by stating that one can obtain “a new function” by applying the definition of the derivative “at any point” (Day 2). While covering the application of the derivative, Tyler did not explicitly address the derivative as a function. However, he used it when defining the second and the third derivatives and mentioning the variable of differentiation. Given that having the independent variable is an aspect of a function, identifying the independent variable was considered as implicit discussion on the derivative as a function.

Second and Third Derivatives. On Day 4, Tyler introduced the second and third

derivatives as the derivative of the derivatives by saying “You can find the second derivative, which just means that you take the derivative of the derivative...You start with the second derivative and if you derive the second derivative, you get the third derivative, and then it continues like this.” Here, Tyler mentioned that one could take the derivative several times to obtain the second and third derivatives. However, he did not mention that the derivative is another function of which one can take the derivative. Similarly, on the same day when he defined the acceleration as the derivative of the velocity, which is the derivative of a position function, he did not mention that velocity is another function.

Tyler addressed the derivative as another function implicitly by mentioning the independent variable of the derivative function. On Day 5, he pointed out the independent variable, x , in the notation dy/dx : “[This notation] makes it clear what's the independent variable and what's the x value that you can change. This is...your x axis on the bottom. And on the top, this is what plays a role of the y -axis.” On Day 9, he also interpreted the notation da/dt as “how a is changing with respect to time.” In these quotes, he pointed out the variable of the derivative of a function in relation to the original function.

On Day 6, Tyler specified the independent variable of differentiation when explaining the rate of a changing quantity. For example, he specified the independent variable of a function and of implicit differentiation with a circle, $x^2 + y^2 = 25$ (Figure 16).

$x^2 + y^2 = 25$ <p style="margin-left: 20px;"> \uparrow \uparrow indep. dep. var. var. </p> $2x + (y^2)' = 0$ $2x + 2y \cdot y' = 0$	<p>We want to locally be able to say that y is a function of x. So, this means that x, this is the independent variable, y is somewhat a function of x. Even though we don't really know what the equation of y, we still know it's somewhat dependent on x. We just take this equation and take derivative on both sides. We are taking the derivative with respect to x, apply this to the equation.</p>
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Figure 16. Tyler's Drawing and Excerpt for Implicit Differentiation

In the excerpt, he specified the independent variable of the given function and the variable that one differentiates the function with respect to. However, he did not mention that x should be the same in both cases, or x is the independent variable of the derivative function. Later in the lesson, he used expressions, “taking (or finding) the derivative...with respect to” (11 times) or “deriving with respect to” (11 times). In summary, he mentioned the independent variable in the context of differentiating (22 times) more often than as the variable of the derivative of a function (4 times). Aside from cases when he explained the notation of dy/dx , he never addressed the independent variable of the derivative of a function that is another function.

Summary

To define the derivative of a function, $f'(x)$, and the derivative at a point, $f'(x_0)$, Tyler first introduced $f'(x_0)$ as the slope of the tangent line to the graph of a function, $f(x)$, (Definition 1) and moved to the limit definitions of $f'(x)$ (Definitions 2 and 3). In this transition, he did not address that explanations about the derivative at “a point x_0 ” can be extended to a variable x for $f'(x)$. Since he used the same routine to explain the definition of $f'(x)$ and to evaluate $f'(x)$ at a point, students might have noticed their relationship; however, it was not explicitly addressed. He did not mention that $f'(x_0)$ is a value of $f'(x)$.

Tyler addressed the relationships between a function, $f(x)$, the derivative function, $f'(x)$,

and the derivative at a point, $f'(a)$, while explaining various topics in the derivative unit. Table 18 shows in which topics, to what extent, and how many times (in parentheses) he addressed these relationships. When he mentioned and used relationships, the nature of the explanation was considered as “explicit” whereas when he used relationships without stating them, it was considered as “implicit”.

Table 18. Tyler’s Actions in Discussion of Relationships among $f(x)$, $f'(x)$, and $f'(a)$ on Topics and Nature of Discussion

Concepts being related	Topic	Action	Nature
$f(x)$ & $f'(x)$	Differentiation rules	Proving rules with the definition of $f'(x)$	E(3)
	$f'(x)$ as rate of change	Using $f'(x)$ as a rate to calculate change in $f(x)$	E(3)
	Graph of $f(x)$	Describing behavior of $f(x)$ based on the sign of $f'(x)$	E(16)
	Concavity	Discussing sign of $f''(x)$ using behavior of $f'(x)$	E(17)
$f(x)$ & $f'(a)$	Graph of $f'(x)$	Plotting several $f'(a)$ from several slopes	E(4)
	Differentiability	Discussing differentiability of $f(x)$ at a point	E(1)
	Differentiation rules	Showing the rules with tangent line at a point	E(3)
	Velocity	Describing the direction of a moving objects the velocity at a point	E (3)
	Local extremes	Mentioning $f'(x) = 0$ at local extremes of $f(x)$	E(10)
$f'(x)$ & $f'(a)$	Definition of $f'(x)$	Changing notations from x_0 to x	I(1)
	Definition of $f'(x)$	Using the same routine when defining $f'(x)$ and evaluating $f'(a)$ based on the definition of $f'(x)$	I(1)
	Definition of $f'(x)$	Finding the slope “at any point” to graph $f'(a)$	I (1)
	Graph of $f'(x)$	Connecting several values of $f'(a)$ to graph $f'(x)$	I(1)
		Identifying $f'(a)$ from the graph of $f'(x)$	I(3)
	Differentiability	Discussing existence of $f'(x)$ based on $f'(a)$	I(4)
	Differentiation rules	Showing the rules with tangent line at a point without mentioning $f'(a)$ as a value of $f'(x)$	I(3)
	Evaluating $f'(a)$	Substituting a value to $f'(x)$ to evaluate $f'(a)$	I(10)
	Using sign of $f'(a)$ for $f'(x)$	Explaining why the sign of $f'(a)$ can be used for the sign of $f'(x)$ on an interval	I(1)
	Velocity	Describing the direction of a moving objects the velocity at a point	I (3)

As shown in Table 18, in Tyler’s class the relationships between $f(x)$ and $f'(x)$, and between $f(x)$ and $f'(a)$ were addressed explicitly, but the relationship between $f'(x)$ and $f'(a)$ was not. The first

relationship was mostly addressed when he described the behavior of $f(x)$ based on signs of $f'(x)$. In all cases, he used the sign of $f'(a)$ instead of the of $f'(x)$ as indicator of whether $f(x)$ increases or decreases without addressing the relationship between $f'(x)$ and $f'(a)$. In other words, his statement “the derivative is positive if and only if the function is increasing” was used to describe the behavior of a $f(x)$ on an interval but primarily based on the sign of the derivative at a point not the sign of the derivative of a function over an interval. Therefore, he used the relationship that $f'(a)$ as a value of a function $f'(x)$, without mentioning it explicitly. Although he explained once why they can use $f'(a)$ for the sign on $f'(x)$ on an interval, he did not make it explicit that $f'(x)$ is a function which only changes its sign at critical points because of his ambiguous use of the word, “function.” A similar phenomenon was identified when he addressed the relationship between $f(x)$ and $f'(a)$ with differentiability and differentiation rules. He determined differentiability of $f(x)$ based on existence of $f'(a)$ without addressing the relationship between $f'(x)$ and $f'(a)$. He also justified three differentiation rules based on the behavior of a tangent line at a point without mentioning that the justification can be applied to all points on the domain. The relationship between $f'(x)$ and $f'(a)$ was mostly addressed when he substitute a number in $f'(x)$ to evaluate $f'(a)$; however, he never mentioned that $f'(a)$ is a value of the function, $f'(x)$; thus $f'(a)$ is a number.

Tyler also did not address that $f'(x)$ is a function which is formed by values of the derivative at several points. Aside from one case in which he graphed $f'(x)$, he never mentioned $f'(x)$ as a function although he used it several times, for example, when he defined $f'(x)$ or the second derivative as the derivative of $f'(x)$. Frequently, when he took the derivative of a function, he specified the variable with the phrase “with respect to” but he never identified it as the independent variable of the derivative of a function.

Alan

Alan spent 19 class periods on the derivative unit. In most classes, he used power-point slides. Of a total of 1192 lines of transcripts, 262 lines (22%) were said by a student or a group of students. In most cases (75 cases), they answered Alan's open questions (e.g., "What are the differences between these two notations?" or "How do we start this problem?"), and their answers to Alan's closed questions such as "What is the derivative of $\sin x$?" follows (70 cases). They also asked questions about Alan's explanation (e.g., "Wouldn't we have to take the derivative?") 41 times and questions about calculation 18 times. They also expressed that they understood Alan's explanation as a form of "Ok," "That makes sense," or "Thank you," 33 times. They completed Alan's sentence while he explained or solved problems (e.g., When Alan said, "We just found the critical points," a student continued, "Then, we have to plug them back in the original function.") 14 times. They also answered other students' questions before Alan did (When a student asked "How did you know what's in measurement we are gonna be? How does he know ft/sec?," another student answered, "Because it's the rate of change, and we have ft."). Six questions were non-mathematical (e.g., "Do we have to show all the work on the quiz?"). Table 19 shows the topics that Alan addressed on each day of the derivative unit.

Table 19. *Topics Addressed by Alan in the Derivative Unit*

Day	Topics addressed by Alan
1	Rate of change, the definition of the derivative of a function
2	Graphing derivative of a function, Differentiability, Differentiation rules
3	Higher derivatives, Position, Velocity, Acceleration
4	Derivative of trigonometry functions, The chain rule
5	The chain rule
6	Implicit Differentiation
7	Related rate (with respect to time)
8	Related rate (with respect to time), Linearization
9	Related rate (with respect to time), Linearization
10	Review for the topics covered previously
11	Extreme values of functions, Critical points
12	Rolle's Theorem, Mean value theorem
13	Monotonic Functions and First derivative test
14	Concavity and curve sketching, Point of Inflection
15	Concavity and curve sketching
16	Applied Optimization Problems
17	Applied Optimization Problems
18	Intermediate forms and L'hôpital's rule
19	Newton's Method

Definitions and Rules of the Derivative

On Day 1, Alan first defined the derivative of a function using the instantaneous rate of change. Then, he explained the difference between the derivative at a point and the derivative of a function, and their relationship. He also addressed the difference between differentiability of a function and differentiability at a point. He then graphed the derivative of a function that was given as a graph, and explained differentiation rules.

Review of Average and Instantaneous Rate of Change

On Day 1, Alan reviewed average and instantaneous rate of change with algebraic and graphical representations. He stated the average rate of change of a function f between x and $x +$

h , $\frac{f(x+h) - f(x)}{h}$ and by taking the limit as h approaches 0, connected it to the instantaneous

rate of change $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. He mentioned that the instantaneous rate of change is a point-specific value that represents “how things were changing at an instance”. Then, he illustrated the limit process with $y = 16t^2$ at $t = 0$ (Figure 17).

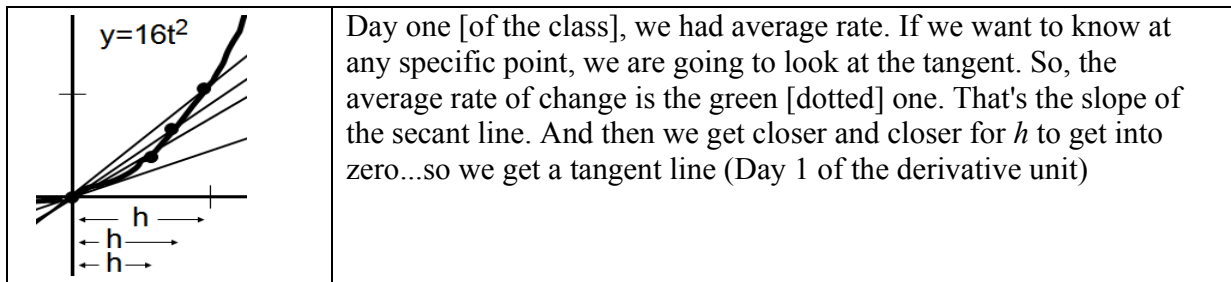


Figure 17. Alan’s Drawing and Explanations the Tangent Line of $y = 16t^2$ at $t = 0$

He also emphasized that the instantaneous rate of change is point-specific while showing algebraic calculation of the average rate of change of $y = 16t^2$ between t_0 and $t_0 + h$. He said, “ t_0 was a specific time that we want to know the speed of a rock.” Finally, he showed the slide of the instantaneous rate of change at x_0 , “ $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ ”, and said, “ x_0 was the exact time or point we cared about. The rate of change at that point.”

In all these four cases, he mentioned that the instantaneous rate of change of $f(x)$ is point-specific but did not state that it can be calculated at any point on the differentiable domain of $f(x)$, which may lead to the concept of the independent variable of $f'(x)$.

Definition of the Derivative of a Function

On Day 1, after giving the definition of the rate of change at a point, Alan started the derivative unit by defining the derivative of a function “with respect to the variable x ” as “the

function f' whose value at x is $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ provided the limit exists” and

reading it aloud. While reading it, Alan clearly mentioned that f' is another function.

Relationship between Derivative of a Function and Derivative at a Point: their Difference

After giving the definition of the derivative of a function, Alan discussed the difference between this definition and the definition of the rate of change at a point (Figure 18). This discussion explicitly addressed that x can be any point on the domain:

<p>What's the difference?</p> $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$	<p>Alan: What's the difference between these two things?</p> <p>Student 1: Before, we were getting a form at x_0. This [second] one is just x.</p> <p>Alan: So, we've gotten this x_0 and this doesn't have a naught. So, what does that mean? What is the naught doing?</p> <p>Student 2: That's just like a starting point?</p> <p>Alan: Ok, this is like a point. What's this x?</p> <p>Student 1: Any of x's?</p> <p>Alan: It could be any of x's. It could be a bunch of points or any of the points in that full range. We know where this function is defined on the domain. So, here [the first expression] is a slope of the tangent line at a single point, so it will give us a single value.</p>
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Figure 18. Alan’s Comparison between Instantaneous Rate of Change and $f'(x)$

In this excerpt, Alan explicitly mentioned the difference between the rate of change at a point, which is “the slope of the tangent line at a single point”, and the derivative of a function, which can be defined at “any of the points..where this function is defined on the domain.” He explained this difference further by comparing “slope of the tangent line at a point” and “slopes of the tangent lines at all the points on the domain” (Figure 19).

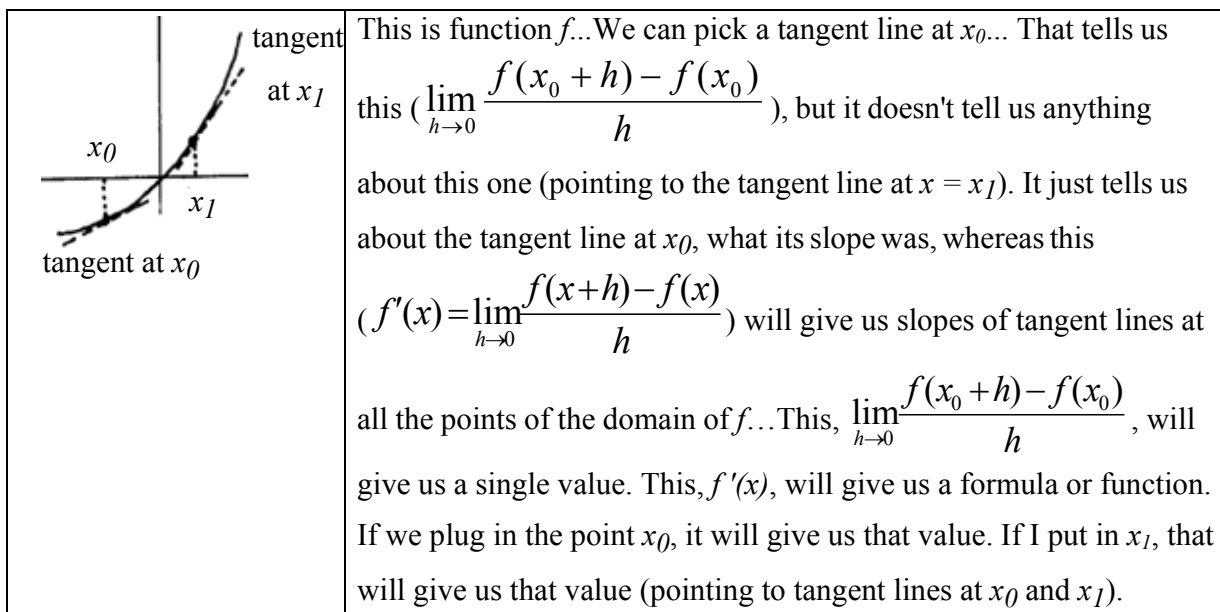


Figure 19. Alan's Explanation of Difference between Rate of Change and $f'(x)$

In this excerpt, he compared the slope of the tangent line at a point and $f'(x)$, "slopes of tangent lines at all the points." He also explicitly stated that $f'(x)$ is "a formula or function" in which one

can calculate a value of $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ by substituting a number x_0 . He, then gave an

example, "Find $k'(-1)$, $k'(1)$, and $k'(\sqrt{2})$ when $k(z) = \frac{1-z}{2z}$." He said, "We can do this one

$(\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h})$ three times or do this one $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, one time

and plug in these values." He found $k'(z)$ and substituted -1, 1, and $\sqrt{2}$ for z .

In this episode, Alan defined $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ as the instantaneous rate of change at a point x_0 and emphasized that it is a point-specific value. Then, he defined the derivative of a function $f'(x)$ as "a formula or function" in which one can substitute $x = x_0$ to

calculate $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$. Therefore, although he did not give a definition of the

derivative at a point or the notation, $f'(x_0)$, or said that the instantaneous rate of change is the same as the derivative at a point, he explicitly addressed the difference between $f'(x)$

and $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$, and their relationship by stating that one can find the former by

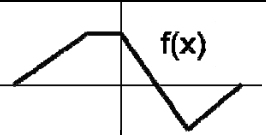
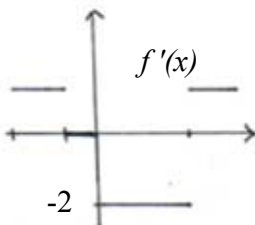
substituting x_0 in $f'(x)$. However, he did not mention the reverse, a transition from $f'(x_0)$ to $f'(x)$,

which can be addressed by saying, for example, the derivative at all points on the domain form $f'(x)$. Later, he addressed this transition graphically by plotting and connecting points to graph $f'(x)$ but without stating it. He stated that $f'(x)$ is a function three times.

Relationship between a function and the derivative: graphing the derivative of a function.

On Day 2, Alan addressed the relationship between a function and its derivative function when graphing derivatives of two functions given as graphs. For the function, $f(x)$, that was a piece-wise function consisting of constant and linear functions, he first said that “derivative is gonna be the slope of the tangent line.” Then, for each portion of the graph, he found the slope and placed its value for the graph of $f'(x)$. He repeated the procedure four times. Table 20 shows his discussion on the first two portions.

Table 20. Alan's Routine and Explanations of Derivative of a Piece-wise Function

Comp. of Action Routine	Segments in the Graph of $f(x)$		
	The First Segment	The Second Segment	
Course of action	Graphing $f(x)$	2A10. Here is that gonna be a positive or negative?	2A14. What's the slope of this tangent line?
	Mentioning Sign of the Slope	2S11. Positive. 2A12. So, all the tangent lines have positive slope.	2S15. Zero?
	Slope	2A13a. It's about one.	2A16. Zero.
	Segments on the graph of $f'(x)$	2A13b. From here [The right end point] and to here [The 1 st corner], we are gonna have the same slope (drawing $y = 1$)	2A17. Alan: How do I put that in my graph? (drawing a zero segment). I got zero.
			
			

For the first portion of the graph of $f(x)$, he showed that its tangent line coincided with $f(x)$ by making a gesture with his arm and saying, “Here, the tangent line is gonna be like this for all these points,” and connected it to a constant derivative by saying, “All the tangent lines have positive slope...the same slope” (Day 2). He drew the rest of the graph based on the sign of the tangent line without describing the behavior of tangent lines or $f(x)$.

In the graphing process, Alan focused on determining the sign of $f'(x)$ on an interval and placing the value on the x - y plane for the graph of $f'(x)$ instead of describing the behavior of $f(x)$ directly relating to $f'(x)$ (e.g., when $f(x)$ increases/decreases, $f'(x)$ is positive/negative). In other words, he addressed the relationship between $f(x)$ and $f'(x)$ through the relationship between slopes of the tangent lines and $f'(x)$, which might come from the characteristic of the example; the graph of $f(x)$ consisted of pieces of linear or constant functions whose tangent lines coincided with the graph of $f(x)$. This characteristic might also allow him to graph the derivative of a function without finding the derivative at a point, and thus without relating a function to the

derivative at a specific point.

While graphing the derivative of another function, $g(x)$ (Figure 20), Alan also used slopes of the tangent lines but used them differently because $g(x)$ does not have constant slopes. He described the behavior of the tangent lines as x increases:

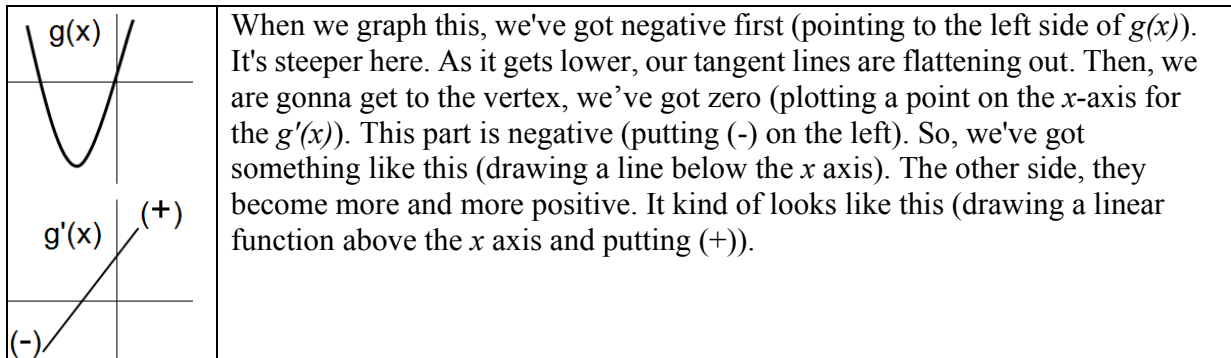


Figure 20. Alan's Graphs and Explanations of the Derivative Function

Similar to his explanation for $f(x)$, Alan did not describe the behavior of $g(x)$ directly while graphing $g'(x)$. Instead, he mostly described the behavior of the tangent line on the interval (e.g., “steeper” and “flattening out”) to find out behavior of the derivative of a function (e.g., “negative,” “getting close to zero,” and “more positive”). The relationship between a function and the derivative at a point was once addressed when he plotted the x intercept of $g'(x)$ at the vertex of $g(x)$. Here, he described $g(x)$ and $g'(x)$ as dynamic objects that change as x changes, rather than static object which focuses on a point (Monk, 1994).

In this episode, Alan used the word, “derivative,” twice when he said, “We [are] gonna graph the derivative function,” and “The derivative is the slope of the tangent line.” In the rest of the discussion, the word did not appear. This lack of use of “derivative” may come from the way he graphed the derivative; he first observed the behavior of tangent lines and graphed the derivative function instead of repeating the match between the slope of the tangent line at each point on a function and the point on its derivative function.

In summary, Alan graphed the derivative of a function, $f'(x)$, based on how the slopes of the tangent lines changes over an interval. In this procedure, he explicitly addressed the relationship between $f(x)$ and $f'(x)$ by describing $f'(x)$ as a dynamic object. Since he did not graph $f'(x)$ by plotting and connecting several values of the derivative at a point, he did not address the relationship between a function and the derivative at a point except one case of a vertex of $g(x)$. For the same reason, he did not mention the derivative at a point as a specific value of the derivative of a function; thus, he did not address the relationship between the derivative of a function and the derivative at a point explicitly.

Relationship between a Function and the Derivative at a Point: Differentiability

After Alan graphed the derivative of $f(x)$ (Table 20), a student asked whether he could connect pieces in the graph of $f'(x)$. Alan said that the class was going to talk about this issue related to the differentiability of a function. Then, he reviewed the definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ and discussed its right and left hand limits:}$$

Alan: Why and where does the derivative not exist?

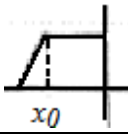
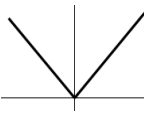
Student 1: In the corners.

Alan: That's one of the places. There's more. Think about the definition [of $f'(x)$], the limit has to exist as this interval goes zero. If the limit doesn't exist, the derivative doesn't exist. The right-hand and left-hand limits should be the same.

Then, he interpreted the right- and left-hand limits as the slopes of the tangent lines from the left and right sides to discuss where and why the derivative of four given functions do not exist. The first and last functions were given as graphs and the other two were $f(x)=|x|$ and $y = \sqrt{x}$. While discussing these examples, he repeated the following procedure: a) asking where the derivative

does not exist, b) specifying the point, c) finding the slopes of tangent lines from the left and right sides of the point, and d) stating that the two slopes are different. Table 21 shows this routine and the excerpts for the first two examples. x_0

Table 21. *Alan's Routine and Explanation of Differentiability*

Comp. of Action		Function	
Routine	Graph of $f(x)$		$f(x)= x $
prompt		2A41. Why and where does the derivative not exist?	 2A50. Where do you think the derivative might not exist?
Course of Action	Finding point Showing Slopes from left and right-hand sides	2S42. In the corners. 2A43. The left hand derivative is 1. This is the straight line. All the tangent lines are the same (writing $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = 1$). Where h [comes] from the right, we got zero (writing $\lim_{h^+ \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = 0$).	2S51. Zero? 2A52. Why do you say zero? 2A53a. The tangent lines are negative on that way (pointing to the left side of the graph) 2A53b. And positive on the other way (pointing to the right-hand side of the graph)
	Stating the Difference	2A45. So, here left hand and right limits are different	2A54. We've got different slopes from both directions.
Closure		2A46. The derivative won't exist there.	

He used the same method to discuss the non-differentiability of $y = \sqrt{x}$ at $x = 0$ (Day 2), and to explain differentiability of a function as a necessary condition for linearization (Day 8).

On Day 2, after discussing specific examples, Alan gave general cases “when the derivative does not exist” by showing the graphs of functions with a corner, cusp and point of vertical tangency (Figure 21), and explained that these functions are not differentiable at those points using the same argument he used for the previous examples. He described the behavior of the tangent lines and mentioned that the slopes from the left and right-hand sides of those points are different, or the function has “vertical tangency” at those points. He again mentioned that the

derivative does not exist at $x = a$, not the entire domain of a function by saying “those are three types, the derivative does not exist at that point.”

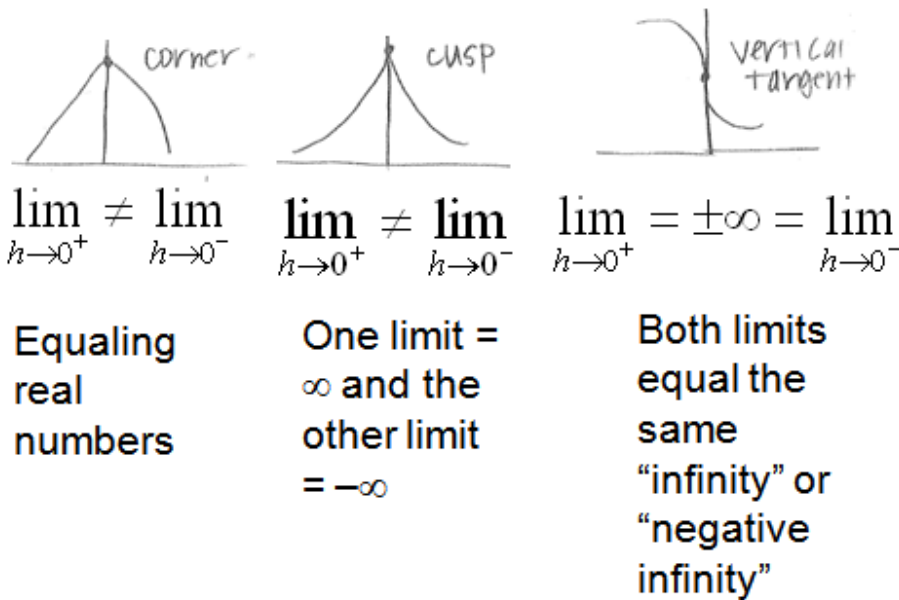


Figure 21. Alan's Examples of Non-differentiability at a Point

Therefore, while discussing differentiability, he addressed the relationship between a function and the derivative at a point by utilizing the behavior of a tangent line at a point.

Relationship between Derivative of a Function and Derivative at a Point: Differentiability

While discussing differentiability on Day 2, Alan also addressed the relationship between the derivative of a function and the derivative at a point by emphasizing the difference between the differentiability of a function at a point and on the entire domain. After explaining that $y = |x|$ does not have the derivative at $x = 0$, he first addressed this difference. He showed two statements about the differentiability on the slide: “ $f(x) = |x|$ does not have a derivative” and “ $f(x) = |x|$ does not have a derivative at the point $x = 0$,” and asked students, “Anybody...seeing anything different between these two statements?” After a student answered, “Everywhere else, the left-hand limit and right hand limit are the same,” Alan referred to the slopes of tangent line from the left and right derivatives at several points in the graph of $f(x) = |x|$ to confirm that the

second statement was correct. For $g(x) = \sqrt{x}$, he used the same argument that the right and left

derivatives are different, but he also utilized the definition, $g'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} =$

$\lim_{h \rightarrow 0} \frac{\sqrt{h}}{h}$, which is defined if $h > 0$. He emphasized non-differentiability at a point; for $f(x)$, he

said, “There is only no derivative at zero...Don't ever say $|x|$ doesn't have the derivative. Only at the origin, it doesn't have the derivative” and for $g(x)$, “It does not have the derivative at $x = 0$ ”

(Day 2).

Alan also examined differentiability of $g(x) = \sqrt{x}$ at $x = 0$ by finding its derivative function first, and then trying to substitute $x = 0$. By applying the definition of $g'(x)$, he obtained $g'(x) = 1/2\sqrt{x}$ and showed that substituting zero in this function is not possible: “When we plug in zero, we get zero in the denominator, so [it] is not defined” (Day 2). Here, he used the relationship that $g'(0)$ can be found by substituting $x = 0$ in $g'(x)$ without stating it.

Relationship between a Function and the Derivative of a Function: Differentiation Rules

On Day 2, after discussing differentiability, Alan explained differentiation rules. He addressed the rules for a constant function, constant multiple of a function, power function and product and quotient of two functions. For each case, he first stated a rule, justified it (only for the first two functions), and then applied it to examples. Figure 22 shows his justification of $f'(x) = 0$ when $f(x) = c$ with a graph:

$y = c$	Alan: If $f(x) = c$, the derivative is going to be equal to zero. Why is that? Student 1. Can we...just figure out a straight kind of line, which has slope of zero? Alan: The tangent line looks like this (pointing to the graph), which has slope of 0.
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Figure 22. Alan's Graph of $y = c$ and Explanation of its Derivative Function

For the constant multiple rule, $y' = cf'(x)$, he stated the rule and justified it with the definition of the derivative of a function. After he factored out constant c on the numerator, a student found that the remaining part is the derivative of f (Figure 23).

$\begin{aligned} \frac{d}{dx}(cf(x)) &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]}{h} \\ &= cf'(x) \end{aligned}$	<p>Alan: (after factoring out constant c) What is this? Student 1: This is the derivative. Alan: [This is] the derivative of f from the definition. So we get our constant multiple rule</p>
---	--

Figure 23. Alan’s Writing and Explanation of the Constant Multiple Rule.

While explaining the rules, Alan used the word, “derivative” several times (seven out of 12) without specifying it as the derivative of a function or the derivative at a point. According to the context, all of them can be inferred as the former. In six out of the seven specified cases, he used the derivative attaching to the original function such as “the derivative of u ”; there was one case when he said “the derivative function” (Alan, Day2).

In the discussion of differentiation rules, Alan addressed the relationship between a function and the derivative of a function by stating rules for five functions and justified two of them using the graph or the definition of the derivative of a function.

Application of the Derivative

From the third day of the derivative unit, Alan discussed the application of the derivative based on the definitions and rules. The extent to which Alan and his students addressed the relationships between a function, $f(x)$, the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, and $f'(x)$ as a function will be discussed in this section.

Relationship between a Function and the Derivative

Relationships between $f(x)$ and $f'(a)$, and between $f(x)$ and $f'(x)$ were intertwined and

imbedded in the discussion of various topics: a) the rate of change, b) zero derivative at a point where $f(x)$ has horizontal tangency, c) relationship between the behavior of $f(x)$ and signs of $f'(x)$, and d) anti-derivatives. Alan addressed the relationship between $f(x)$ and $f'(a)$ in a) and b), and the relationship between $f(x)$ and $f'(x)$ in all cases.

Derivative as Rate of Change. Alan described the derivative of a function as the rate of change of a function while explaining the chain rule and the rate of change of a quantity. On Day 4, he stated that the derivative is the rate of change and showed what makes changes in a composite function $y = f(g(x))$ with three gears (Figure 24):

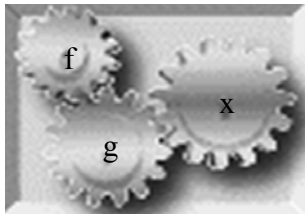
	<p>The derivative is telling us how things are changing. It's rate of change...we've got multiple things changing...We've got x is changing. x goes to 0, 1, 2, 3. It's changing constantly. And g, which is y value on that $g(x)$ graph, it's changing as x changes... As this is changing, it's causing our f to change. We can think about this like three gears. One of them is x. It's moving constantly. As it moves, it makes this one move, which is g. As our g is moving, that makes f also change.</p>
--	---

Figure 24. Alan's Explanation of Changes in a Composite Function

Then, he connected these changes to the rate of change, and then stated the chain rule as a product of the derivative of f evaluated at $g(x)$ and the derivative of $g(x)$ (Figure 25).

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$	<p>The chain rule is going to relate all these forms of changes. It's gonna help us figure out what big change is looking like. Here is the chain rule...the derivative of this f, this thing on the outside... with the input...g, times the derivative of that g. $\frac{dy}{dx}$ is equal to $\frac{dy}{du}$ times $\frac{du}{dx}$.</p>
---	--

Figure 25. Alan's Writing and Explanation about the Chain Rule

Interpreting the derivative of a function as a rate of change was also identified when Alan and students discussed the rate of change of a quantity changing with respect to time. On Day 7, when Alan showed that the derivative of the volume of the liquid in a cylinder, $V = \pi r^2 h$ (Figure

26) with respect to t is $\frac{dV}{dx} = \pi r^2 \frac{dh}{dt}$, a student asked why $\frac{dr}{dt}$ was not included in the

derivative. Alan answered using the rate of change of the radius:

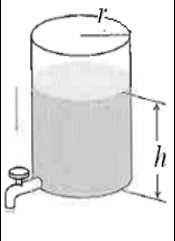
	<p>Student 1: Would we have to take the derivative of r too...because it is varying? Alan: Very good question? Why didn't I have to put in dr/dt in there? Student 2: Because the radius isn't changing. Alan: The radius is not changing. πr^2 is a constant because r doesn't change. So we don't have this (dr/dt)...no rate of change... If it wasn't constant, say we had a cone that changes the radius...we would have to do the product rule with r and h.</p>
---	---

Figure 26. Alan's Drawing and Explanation of the Rate of Change

In this excerpt, Alan explained why dr/dt was not multiplied based on the rate of change of the radius of a cylinder, r ; r is constant. He also compared this constant radius to the radius of a cone in which the radius changes according to the height. However, he did not say the rate of change of r is zero; instead, he said, "The radius is not changing at all." He addressed the relationship between a function and its derivative function implicitly here.

Alan and students also interpreted the derivative at a specific time as the rate of change of the original function. In Figure 27, when a student asked why ds/dt was not negative, Alan interpreted the sign of ds/dt as the indicator of the behavior of s (hypotenuse):

<p>A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?</p>	<p>Student 1: How come ds/dt wouldn't equal -20? You are saying that dy/dt is -60 if you are doing in terms of direction? Alan:...This [height, y] is a moving direction whereas this [hypotenuse, s] isn't moving. The question, is do you think this distance s is getting smaller or bigger?</p>
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$\frac{dy}{dt} = -60$

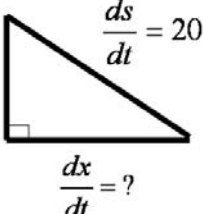


Figure 27. Alan's Discussion on an Example of Related Rates

In the excerpt, Alan addressed the difference between dy/dt and ds/dt ; dy/dt showed a moving direction and ds/dt was related to whether the hypotenuse s is bigger and smaller. Then, to show that ds/dt is positive, he drew six right triangles as time increases (Figure 28), measured s (hypotenuse) with a yard stick, and obtained 20, 17, 16, 19, 30 and 34. Then, he said, “It seems to be this [third] point where it decreases and starts increasing again.”



Figure 28. Alan's Six Right Triangles

Although he tried to answer why the derivative, ds/dt , is positive by showing that s increases, he did not specifically mention that the two statements explained the same phenomena or the point where $ds/dt = 20$ belongs to the portion where s is increasing. Therefore, the derivative at a point as rate of change of a function was implicitly addressed.

Alan also interpreted the sign of the derivative in another example involving the rate of water running into a container. He stated that the derivative is positive where the original function increases by saying, “It's coming in, so we are gonna have a positive rate [for the volume]. And dy/dt [y refers to the height of the water], is gonna be positive...because the water's filling up. It's rising, so we should get a positive value.”

Last, Alan used the derivative as the rate of change to estimate the change in a function. He said, “A small change in y is equal to the derivative $f'(x)dx$. It's derivative times a small change in x (writing $dy=f'(x)dx$),” and estimated the change in a function when the independent variable changes from x_0 to $x_0 + dx$ using this method (Day 9). He, however, did not explicitly address the derivative as a rate of change or why the change in a function is the product of the

derivative and the change in the independent variable.

In summary, the derivative of a function as the rate of change of the original function was explicitly addressed once when he explained the chain rule as the product of multiple nested rates of change. In the cases involving related rate problems, he explained the derivative using how the function changes as time changes, but did not explicitly state the derivative as the rate of change. Therefore, when using the derivative of a function as the rate of change of a function, he addressed their relationship implicitly in most cases.

Derivative Equal to Zero at Point of Horizontal Tangency. Alan used the relationship that the derivative is zero at points where the function has horizontal tangent line while graphing the derivative of $f(x) = \sin x$ on Day 3. He described the characteristics of those points and matched them with the x intercepts of $f'(x) = \cos x$ (Figure 29):

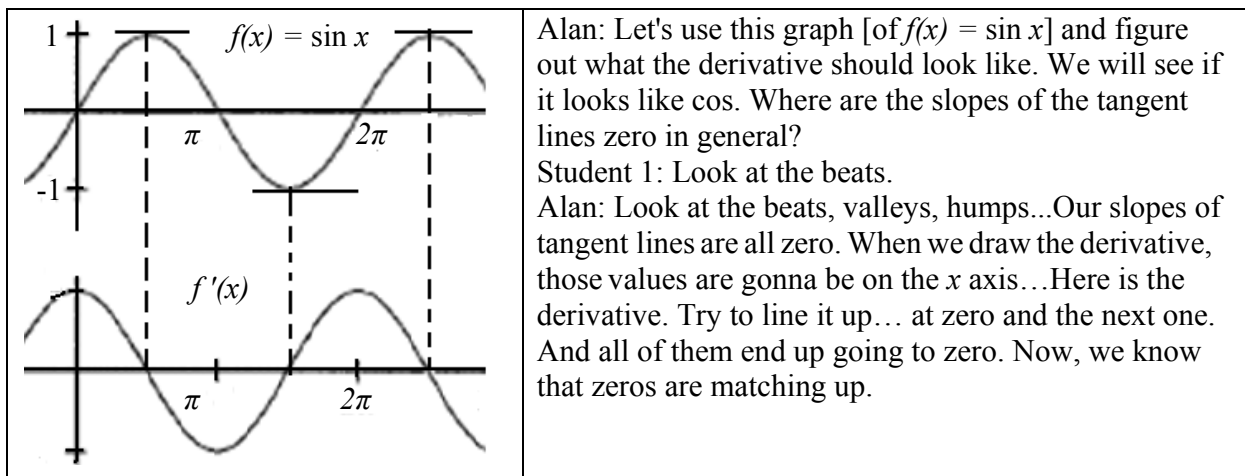


Figure 29. Alan's Graph and Explanation of the Derivative of $f(x) = \sin x$

In this excerpt, Alan mentioned that the function has zero slopes at beats, valleys and humps and “matched up” these zero slope and the x intercepts of $f'(x)$. When a student asked “Why are we matching up zeros?” Alan recapped the procedure of placing zero slopes on the x axis: “The slope here is zero. So when you graph it, the y' or the f' ...you got zero. Here is the zero when you are gonna cross the x axis (pointing to a x intercept of the graph of $f'(x)$).” Although he said,

“The slope is zero,” and “[The graph of $f'(x)$] is going to cross the x axis,” he did not provide the reasons for these statements in relation to $f(x)$. The reasons can be addressed, for example, by explaining that the derivative at a point is the slope of the tangent line of $f(x)$ and relating the behaviors of $f(x)$ and $f'(x)$. Therefore, the relationship between a function and the derivative at a point was addressed implicitly; it was used but not mentioned explicitly. Later, he used this relationship to find the points where a polynomial function has horizontal tangency (Day 10).

He also used the relationship that the derivative is zero at a point where a function has horizontal tangency while calculating extreme values of the function six times. He started with asking about the characteristics of the curve at a point where it has a maximum or minimum, mentioned that the slope is zero at the point, so is the derivative, and calculated the extreme value. Table 22 shows this routine when Alan discussed the maximum or minimum point of a function in terms of its derivative on Days 2 and 11.

Table 22. *Alan’s Routine and Explanation of Finding Extreme Values*

Comp. of Routine	Action	Example
	Height of a rock thrown up to the air, $s = 160t - 16t^2$	A function $y = 4 - x^2$
Prompt	2A69. How do we know this point, the peak?	11A11. what are all these [local extreme] points are in common?
Course of action	Finding Slope of Tangent Line	2S70. Is the derivative zero? 2A71. We've got a tangent line, which has slope of zero
	Mentioning Derivative Being Zero	11S12. Slopes are zero? 11A13. Slopes are zero at all those. 11S14. You set the derivative equal to zero. f' is gonna be $-2x$ so we want that to be equal to zero.
	Finding the Extreme Value	2A83. Now $s(5)$ is equal to max, which is 400 ft. 11A25. What's our y value there? 11S26. 4? 11A27. We got $y = 4 - 0^2 = 4$.
Closure		11A28. It's one possible candidate for a local extreme value.

While discussing extreme values, Alan once described the behavior of the rock, which a

given function modeled in the first example of Table 22: “As time goes, it's getting higher...and higher. Then, it reaches something where it reaches its maximum and then it starts fall[ing] back down.” However, he did not explicitly relate this behavior to its derivative function. This relationship was also not explicitly addressed in other examples. Referring to the maximum height of a moving object, he simply stated “the velocity is zero” as “the important feature” that indicates where the moving object has the maximum height (Day 7). In two other similar cases in which the equations of functions were given, he again stated that the derivative of a function is zero at a point where the function has its maximum or minimum: “ f has a local max and min value at interior point c ...and if f' , the derivative is defined, then $f'(c) = 0$,” (Day 11) and “The values that we get from $g'(x)$ are critical points, we plug those into the original equation as well. The ones...are local max and min” (Day 14). He continued to interpret a critical point where a function potentially turns around and as a general case of points where an extreme occurs.

In summary, to explain that the derivative is zero at points where the function has extreme values, Alan prompted students to think about the characteristics of those points or the tangent line at those points such as “beats, humps, valleys” or “turning around.” Since he emphasized horizontal tangency at those points and mentioned that the derivative is zero, he addressed the relationship between a function and the derivative at points where the function has local extremes. The relationship was also explicitly addressed later when he recapped how to use the first and second derivatives to find those points on Day 14.

Behavior of a Function Based on the Sign of the Derivative. Alan addressed the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, when he found the interval on which $f(x)$ increases or decreases based on the sign of $f'(x)$. Then, he used this relationship to graph a function, $f(x)$, when its equation was given by a) finding the critical points

of $f(x)$, b) dividing the domain of $f(x)$ based on critical points, c) finding out the sign of $f'(x)$ over each interval by looking at the slope of a tangent line or substituting a number in $f'(x)$, and d) graphing $f(x)$. The relationship between $f(x)$ and $f'(x)$ was addressed in c). This routine was identified twice when he graphed $g(x) = x^2\sqrt{5-x}$ on Day 14 and $h(x) = -x^3 + 2x^2$ on Day 15. For example, while graphing $h(x)$, he divided the domain into three parts using critical points, evaluated $h'(x)$ at a point on each interval, and said, “I pick my x value on each of these intervals...and then I am going to evaluate the derivative to see what numbers I get. I get these numbers, and so my signs are... negative, positive, and negative.” Then, he connected the signs to the behavior of $h(x)$ by saying that they inform “the behavior [of h] is decreasing, increasing, and then decreasing.” In this example, the sign of the derivative as the indicator of the behavior of the original function was explicitly addressed. However, he used the sign of the derivative at one point to determine the sign of the derivative function on an interval without addressing the relationship between the derivative of a function and the derivative at a point.

A similar routine was identified when Alan graphed the derivatives of three other functions given as graphs. The steps in the routine were identical to the previous one except the last step; it was ‘graphing the derivative of a function’ in this routine. The relationship between a function and its derivative function was again addressed in c). In the first example in which he was graphing the derivative of $y = \sin x$, he said, “Whenever we have a negative slope (pointing to a decreasing portion of $\sin x$), we are gonna be... below the x -axis (pointing to the negative portion of $\cos x$)” (Figure 30). In the second example (Figure 31), after labeling the point of the horizontal tangency as $x = 2$, he described the behavior of the function, $f(x)$, and connected it to the sign of $f'(x)$ (Day 14):

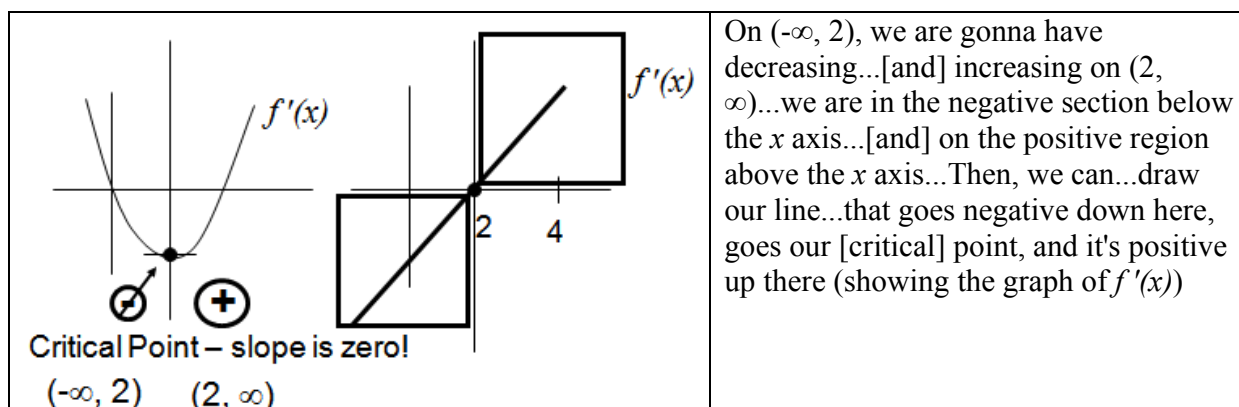


Figure 30. Alan's Drawing and Explanation of Graphing the Derivative Function

In the third example involving two critical points, a similar connection between the sign of the slope of the tangent line and the sign of the derivative was identified. In these three examples, he mostly stated the sign of the derivative without mentioning the behavior of the original function. In a few cases when he mentioned the function, he stated the sign of the slope of the tangent line instead of its behavior (e.g., “increasing”) without specifying “derivative” as the derivative of a function on an interval or the derivative at a point.

Alan also addressed the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, when discussing the behavior of $f(x)$ near an extreme point. For example, when a student explained the reason for the local minimum at $x = 0$ based on the behavior of $f(x)$ (Figure 31), Alan connected the behavior to the sign of $f'(x)$:

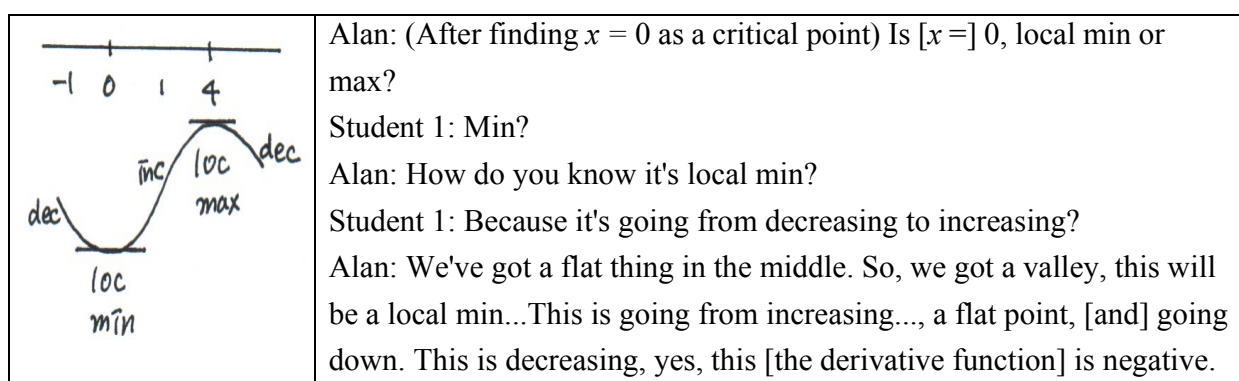


Figure 31. Alan's Drawing and Discussion of Local Extremes

Alan addressed the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, while explaining the concavity of $f(x)$. On Day 14, he described the slope of tangent lines to the graph of $f(x) = x^2$ to explain the sign of $f''(x)$ and the concavity (Figure 32):

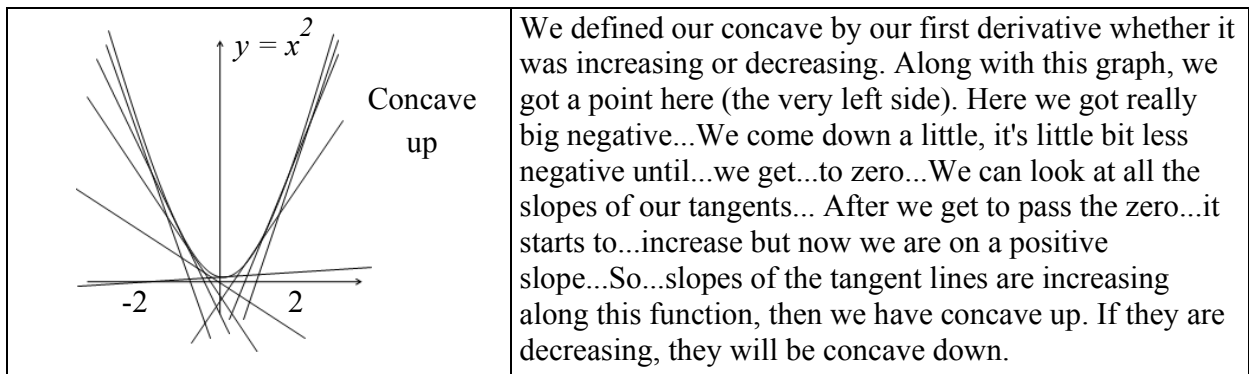


Figure 32. Alan's Graph of $y = x^2$ and Tangent Lines for Concavity

In this excerpt, he showed that the graph of $y = x^2$ is concave up by mentioning that “the slopes of all the tangents” are increasing “along this function $y = x^2$.” Then, he mentioned that the concavity is determined by whether the first derivative is increasing or decreasing, and connected the behavior of several tangent lines to “the second derivative [which] is... positive.” On the same day, the relationship between a function and the derivative of a function was addressed again when he graphed a function based on the conditions for its derivative function. In summary, while graphing a function or derivative function, he addressed the relationship between a function and the derivative of a function explicitly but not the relationship between the derivative of a function and the derivative at a point.

Anti-derivatives. The relationship between a function and its derivative function was addressed when Alan discussed anti-derivatives of $f'(x) = 0$, $f'(x) = 2x$, and $f'(x) = 2x - 1$. For the first two cases, he followed the procedure: a) asking students to think about functions that have the same derivative, b) giving the list of the functions, c) confirming that they have the

same derivative, and d) coming up with the algebraic expression for all the anti-derivatives with a constant c . The routine and excerpts are shown in Table 23.

Table 23. *Alan's Routine and Explanation of Finding Anti-derivatives*

Comp. of Routine	Action	Example	
		$f'(x)=0$	$f'(x)=2x$
Prompt		12A66. What kinds of functions have the derivative equal to zero? 12S67: Constants.	12A83. We've got $f(x)$ equals x squared...And, it's derivative is? 12S85. $2x$.
Course of action	Showing Functions	12A68. We have constants...This one is $f(x) = 39$. Here is another one and another one... (showing graphs of several constant functions)	12A87. x^2-30 , What's the derivative? 12S88. $2x$. 12A89a. If we have... x^2+17 , x^2+4 , x^2+300 , x^2-300 (showing graphs).
	Stating the Same Derivative	12A70. There are whole bunch of these and they have the same derivative, zero. What does that look like on the graph? (Graphing $y = 0$)	12A89b. They all have the same derivative. Although they are different functions, their derivative are the same.
	Formula of functions	12A80. If $f'(x)=0$ at each point x on the open interval, then $f(x)$ equals to c , constant for all x in (a, b)	12A92. We are gonna use this idea here... x^2+c .

After finding the anti-derivative of the two functions, Alan endorsed a narrative, “Functions at the same derivative...differ by a constant” with the statement “if $f'=g'$, if the derivatives are the same at each point on the open interval, then there exists a constant function, c such that $f(x) = g(x) + c$ for all x on (a, b) ” (Day 12).⁵ Then, he explained the process of finding anti-derivative as “go[ing] backwards” with the power rule, $(x^n)' = nx^{n-1}$: “To go backwards, we want to divide by this n and add one back up here. So, if I had nx^{n-1} , I am gonna divide out by this n and add by one back...then I get x^n ” (Day 12).

In the third example, $f'(x)=2x-1$, Alan included a constant c in the anti-derivative,

⁵ Endorsed narrative refers to the utterances or the sequence of the utterances that are accepted as true by speakers (Sfard, 2008, p. 134)

$f(x) = x^2 - x + c$, without listing functions that have the same derivative f' . He completed the discussion on the anti-derivative of a function with the explanation of how to find the value of c by substituting a given point on the original function.

To sum up, in the above discussion on the anti-derivative, Alan addressed the relationship between the derivative function and its original function algebraically while explaining the “go[ing] backwards” process of the power rule for differentiation and graphically while showing the graphs of the functions that have the same derivative to show that they only differ by a constant. The word, *derivative*, was used several times in the discussion without specification; all cases were inferred as the derivative of a function.

Relationship between the Derivative of a Function and the Derivative at a Point

While discussing the application of the derivative, Alan mostly addressed the relationship between the derivative of a function and the derivative at a point when substituting a value in the derivative of a function to evaluate the derivative at a point. Most examples were algebraic; only one example involved a graph.

Derivative at a Point as a Point on the Graph of the Derivative of a Function. This relationship was addressed once in the graphical situation to check if the values on the graph of the derivative of a function matched the slope of the original function at several points. Alan was showing the graph of $y = \cos x$ which is the derivative of $y = \sin x$. He mentioned several values of $y = \cos x$, and compared them to the behavior of the tangent line of $y = \sin x$ by making a gesture with his hand: “[At] zero, we get our cosine of zero as one. This is $\pi/2$ and this is negative 1 at π (making a gesture illustrating the tangent line to the graph of $y = \sin x$ at $0, \pi/2,$ and π), which is what cosine does. So, it makes some sense that this is gonna happen.” Since he did not directly mention that these slopes of the tangent lines are values of the derivative of a function, the relationship between the derivative at a point and the derivative of a function is addressed implicitly.

Evaluating the Derivative at a Point from the Derivative Function. The remaining cases

where Alan addressed the relationship between the derivative of a function and the derivative at a point were in algebraic situations involving evaluation of the derivative of a function to find the derivative at a point. This type of discussion was identified seven times. In all cases, after finding the equations of the derivative of implicit or explicit functions, Alan substituted a number in the equations. For example, he said, “You are evaluating this [derivative] function, we get the derivative at this point $x = 1$... we are putting 1 for our x 's” (Day 4). In other examples involving implicit differentiations, he said, “That gives us our implicit differentiation, we want to know the derivative at that exact point. So, we plug in our x and y values, we get 64,” and “(After simplifying dy/dx) we can plug in values to get the slope of the tangent line at this point” (Days 6 and 8).

Similar to the previous discussion on the relationship between the derivative of a function and the derivative at a point when Alan introduced these concepts, he addressed this relationship only in one direction: from the derivative of a function to the derivative at a point. He found the equation of the derivative of a function and then substituted a value in the derivative of a function to find the derivative at a specific point.

Derivative as a Function

Alan explicitly addressed the derivative of a function as another function once while defining the derivative of a function, $f'(x)$. Later in class, he used $f'(x)$ as a function without stating it when he graphed $f'(x)$, defined $f''(x)$, and applied the intermediate value theorem (IVT) to $f'(x)$. When he graphed $f'(x)$, he said, “We were gonna actually graph the derivative function. So, here is f and we want to draw a graph of f' ,” but he did not mention that $f'(x)$ is a function that one can graph (Day 2). When he defined the velocity as the derivative of the position, and acceleration as the derivative of velocity, he said, “Acceleration equals v' . So, we take the derivative of this,” and “We have to take the derivative of the first derivative” (Day 3). Here, he also did not mention that the velocity is another function. Last, he showed that the derivative function is zero where it changes the sign using IVT: “We know from the intermediate value

theorem, we have a positive slope, and then we have a negative slope...We will have zero slope in the middle” (Day 4). Since IVT can be applied to a continuous function, his explanation implies that the derivative function is a continuous function. However, he did not state this explicitly.

Alan also mentioned which independent variable that one should consider when taking the derivative of a function; he and his students specified the variable that they took “the derivative with respect to” while solving problems involving the chain rule, implicit differentiation, or related rates, which mostly involved more than one variable. Statements including “taking the derivative with respect to” were identified 14 times. He also mentioned the independent variable of the original function once before he specified which variable he took the derivative with respect to; he said “The derivative is gonna be 6y times the derivative of y...y equals some x’s...something of x” (Day 7). However, he did not explicitly discuss the independent variable of the derivative of a function.

Summary

Alan first defined $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ as the instantaneous rate of change at a point x_0 , and then the derivative of a function $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ as “a formula or function” in which one can substitute a value to calculate. Although he did not give a definition of the derivative at a point or the notation $f'(x_0)$, or mention that the instantaneous rate of change is the same as the derivative at a point, he explicitly addressed the difference between $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ and $f'(x)$, as a value and a function, respectively, and their relationship by mentioning the former is a specific case of $f'(x)$.

Alan also addressed the relationships between a function, $f(x)$, the derivative of a function,

$f'(x)$, and the derivative at a point, $f'(a)$. Table 24 shows in which topics, to what extent, how many times (in parentheses) he addressed these relationships.

Table 24. *Alan's Actions in Discussion of Relationships among $f(x)$, $f'(x)$, and $f'(a)$ on Topics and Nature of Discussion*

Concepts being related	Topics	Alan's Action	Nature
$f(x)$ & $f'(x)$	Graph of $f'(x)$	Graphing $f'(x)$ based on the sign of slopes	E(6)
	Differentiation rules	Proving rules with the definition of $f'(x)$	E(2)
		Proving rules with tangent lines	E(1)
		$f'(x)$ as rate of change	Stating chain rule as product of rates of change
	Graph of $f(x)$	Interpreting $f'(x)$ as related rate of an object	I(1), E(1)
		Using $f'(x)$ as a rate to calculate change in $f(x)$	I(1)
	Graph of $f(x)$	Describing behavior of $f(x)$ based on sign of $f'(x)$	E(7)
	Concavity	Discussing sign of $f''(x)$ using behavior of $f'(x)$	E(1)
	Anti-derivatives of a function	Finding anti-derivative from the graphs	E(2)
		Finding anti-derivative algebraically	E(2)
$f(x)$ & $f'(a)$	Local extremes	Mentioning $f'(x) = 0$ at local extremes of $f(x)$	E(9)
	Differentiability	Discussing differentiability of $f(x)$ at a point	E(7)
	Rate of change	$f'(a)$ as rate of change at a point	I(1)
$f'(x)$ & $f'(a)$	Definition of $f'(x)$	Difference between $f'(x)$ and $f'(a)$	E(1)
	Differentiability	Distinguishing differentiable on interval & at point	E(3)
	Evaluating $f'(a)$	Substituting a value to $f'(x)$ to evaluate $f'(a)$	I(1)
	Critical points	Matching points of horizontal tangency to $f'(x) = 0$	I(2)
	Graph of $f'(x)$	Identifying $f'(a)$ from the graph of $f'(x)$	I(3)
	Evaluating $f'(a)$	Substituting a value to $f'(x)$ to evaluate $f'(a)$	I(7)

As shown in Table 24, the relationships between $f(x)$ and $f'(x)$ were primarily addressed when Alan described the behavior of $f(x)$ (the first derivative) based on the sign of $f'(x)$ (the second derivative) or graphed $f'(x)$ based on the sign of the slopes of tangent lines to $f(x)$. In contrast to his discussion on the difference between $f'(x)$ and $f'(a)$ in their definitions, while discussing their relationship in graphs, he did not specify the word “derivative” as the derivative of a function or the derivative at a point. However, since his description of the derivative was mostly over an interval, his use of the word, “derivative,” could be inferred as the derivative of a function. Although this ambiguous use of the word, “derivative,” might come from the fact that many examples were involved constant or linear functions whose tangent lines coincided with

the functions, he could have specified the word, “derivative,” as the derivative of a function or the derivative at a point.

His emphasis on the difference between $f'(x)$ and $f'(a)$, which was present in their definitions, was observed again in his discussion of differentiability of a function over an interval and at a point. While discussing differentiability, he consistently emphasized that the differentiability at a point is different from differentiability over the domain of a function. He only discussed differentiability at a point and never used the term “differentiability” without specifying a point where the function is not differentiable.

Besides the two cases about the definition of the derivative and differentiability above, Alan mostly addressed the relationship between $f'(x)$ and $f'(a)$ with substitution: for example, while evaluating $f'(a)$ by substituting a number in $f'(x)$, reading a point for $f'(a)$ from the graph of $f'(x)$, and matching points where $f(x)$ has horizontal tangency to zeros on the graph of $f'(x)$. In these cases, he did not mention $f'(a)$ as a value of $f'(x)$.

Alan also did not explicitly address that $f'(x)$ is a function that has $f'(a)$ as its value at $x = a$. Aside from the three cases in which he mentioned that $f'(x)$ is a function, he used it without stating in various cases such as defining $f''(x)$ and $f'''(x)$, graphing $f'(x)$, and applying Intermediate Value Theorem to $f'(x)$. He frequently specified the variable when he took the derivative of a function with a phrase “with respect to” but never identified it as the independent variable of the derivative of a function.

Ian

Ian spent 19 class periods on the derivative unit. His students participated in discussion more frequently than other two instructors’ students in terms of the number of lines in the transcripts said by a student or several students. Such lines were 482 (20%) out of a total of 2486. More than half of the cases (257) were made when they answered to Ian’s closed questions such as “What is the derivative of e^x ?” or “Is the derivative positive or negative at this point?”

Answers to Ian's open questions such as "Why these two expressions are the same?" were made 76 times. They asked about Ian's explanation (e.g., "Does this work all the time?") 65 times and about calculation (e.g., "Isn't it -2 instead of 2?") 29 times. They also expressed that they understood Ian's explanation such as "That make sense" and "Oh, I didn't see the minus" 44 times. Six students' questions were not mathematical (e.g., "How much details we have to show on the exam?") Table 25 shows the topics that Ian addressed on each day of the derivative unit.

Table 25. *Topics Addressed by Ian in the Derivative Unit*

Day	Topics Addressed by Ian
1	Tangent line, Cases where tangent line does not exist. Definition of the derivative
2	Definitions of the derivative of a function at a point and the derivative of a function
3	Definitions of derivative at a point and derivative of a function, Differentiation rules
4	Derivative of trigonometry functions
5	Derivative of exponential functions, The chain rule
6	The chain rule, The second derivative
7	Implicit differentiation
8	Derivatives of inverse function and logarithmic function
9	Derivatives of inverse trigonometric functions
10	Review for the topics covered previously
11	Related rates of change with respect to time
12	Related rates of change with respect to time
13	Related rates of change with respect to time, Extreme values
14	Mean value theorem, First derivative test
15	Concavity and curve sketching, Asymptotes
16	Curve sketching
17	Curve sketching
18	Applied Optimization Problems
19	Applied Optimization Problems

Definitions and Rules of the Derivative

During the first four days of the derivative unit, Ian explained the slope of tangent line as the limit of slopes of secant lines, introduced the definitions of the derivative of a function and the derivative at a point, and then discussed the differentiation rules. While discussing these topics, he also addressed the relationship between a function, the derivative at a point, and the derivative of a function, and the derivative as a function.

Slope of Tangent Line as the Limit of the Slopes of Secant Lines

Ian started the derivative unit by illustrating the slope of the tangent line at a point $x = a$ as the limit of the slopes of secant lines between $x = a$ and $x = b$ as $x = b$ approaching $x = a$ (Figure 33).

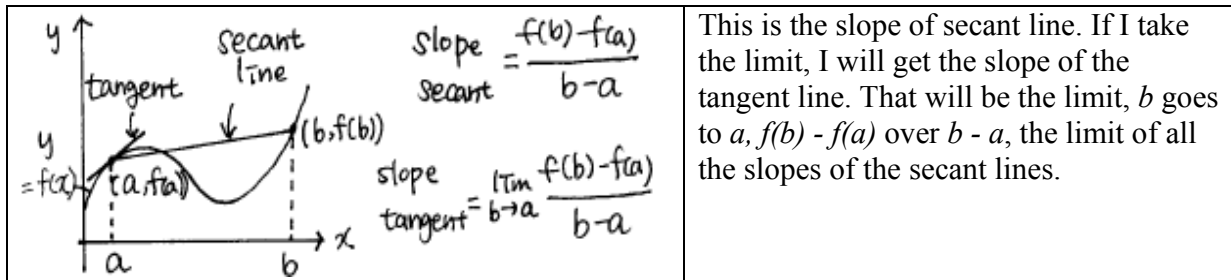


Figure 33. Ian's Drawing of the Tangent Line as the Limit of Secant Lines

Points where the Tangent Line does not Exist

After defining the slope of tangent line, Ian discussed cases where it does not exist. He gave four functions which are discontinuous with a hole and jump or undefined at a point, and then three continuous functions with multiple or vertical tangency (Figure 34).

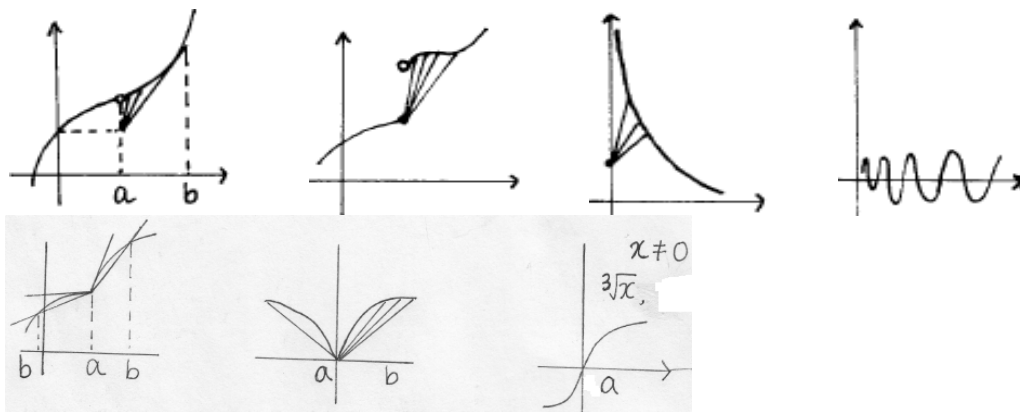


Figure 34. Ian's Graphs of with Discontinuity, Multiple or Vertical Tangency

For each of the seven functions, he followed the same steps: a) graphing a function, b) mentioning a point of no tangent line, c) drawing several secant lines, d) showing the limiting process of the secant lines, and e) concluding that the limit is vertical or has multiple values.

Table 26 shows how he explained the first and fifth examples on Figures 34.

Table 26. *Ian's Routine and Explanations of Functions with No Tangent Line*

Comp. of routine of course of action	Action	Example
	A function with a hole (The first graph on Figure 34)	A function with a corner (The fifth graph on Figure 34)
Course of action	Graphing $f(x)$	1I21a. The function has a hole.
	Mentioning Point of Non-differentiability	1I21b. If this is your a , how you would connect some point on the graph to some other point b . You see, there is no point here [at $x=a$], there is a hole.
	Drawing Secant Lines	1I21c. You have trouble with secant lines. There is no tangent line either. What do you think happens to the secant lines?
	Finding Limit [of slopes] of Secant Lines	1I21e. I will take this as b and let it closer and closer to a . If I take b even closer, that will be the new point. Do you see what happens to all these secant lines? What happens to those slopes?
	Showing Limit is Vertical or Multiple	1S21h. Becomes vertical? 1I21i. The line becomes pretty much vertical. The slope blows up. There is no finite limit to this expression.
Closure	Stating No Limit	1I22. The slope blows up. There is no finite limit to this expression.
		1I60a. This function is continuous. 1I60b. and has a corner at this particular point. 1I60c. This function exists at a , but if I take the point b . If I take this b closer to a (drawing arrows to a on the x axis), secant lines converge to this line, tangent line. But, the limit (of the slope) here is two side limit. We have to have the limit of both sides. 1I60d. It was left, ok, I will take b from this [right] part, then the secant line looks like this, and then move b close to a , then, secant lines look like this (drawing two more lines) and actually converges to this tangent line. 1I60e. The secant lines converge to different tangent lines on different sides. 1I60f. The tangent line does not exist. Two-side limit does not exist.

Since this discussion occurred before Ian defined the derivative, he did not use the word differentiability or address any relationships between a function and the derivative here.

Definition of the Derivative at a Point

After discussing cases in Figure 34, Ian rewrote the expression of the slope of tangent line,

$\lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$ as $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ using h as the length of the interval $[a, b]$. Then,

he emphasized that this expression depends on a , and thus as a function of a , and defined it as $f'(a)$, the derivative of a function $f(x)$ at the point, $x = a$ (Figure 35):

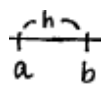
$\lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ $= f'(a)$	<p>This expression depends on a. If we change a, we will have a different limit. We will have different slopes of tangent line at different points on the curve. This expression is a function...of a. If you change a, it will change the value...Here is the definition of...f prime of a, the derivative of a function $f(x)$ at the point $x = a$.</p>
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Figure 35. Ian's Writing and Explanation of the Definition of $f'(a)$

Ian repeated the explanation that the value of $f'(a)$ depends on a when a student asked, "What do you mean if we change a , that changes the value?" He said, "Before this

$[f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}]$, I considered a to be fixed...But I can compute the slope of the

tangent line at any point...At different points, you will have different slopes." On Day 2, he again referred to this expression as "a new function [which] depends on a ."

In this episode, he explicitly addressed the transition from the derivative at a point to the derivative of a function and the derivative as another function three times. However, he did not introduce the notation $f'(x)$ involving the commonly used variable x ; instead, he used a in $f'(a)$ as a variable and a point at the same time. In fact, because he said that the value of $f'(a)$ changes as a changes, a was mostly used as the variable rather than a point aside from a case he specified it as a point. However, he did not explicitly address the other direction; the derivative at a point is a value of the derivative of a function.

While explaining the definitions, Ian explicitly used terms, "the derivative of a function" or "the derivative of a function at a " when he referred to the definition. Although he mentioned

that a is a point, and a can be changed, he did not distinguish the derivative of a function and the derivative at a point as a function and a number, respectively.

Relationship between a Function and the Derivative of a Function: Differentiation Rules

From Day 3, he derived differentiation rules using the definition of the derivative of a function (Figure 35). He first changed the variable from a to x in the definition, and wrote down

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

by saying, "The intention is to change this letter a to x , so that we

will have the same argument as x ." However, he did not say that x is the independent variable or repeat the previous discussion on a . Then, he used this definition to find the derivative of $f(x) = c$,

$$f(x) = ax + b, f(x) = x^2, f(x) = x^3, f(x) = x^n, f(x) = 1/x, f(x) = \sqrt{x} \text{ (Day 2), } f(x) = 1/\sqrt{x}, f(x) = |x|$$

(Day 3), and $f(x) = \sin x$ (Day 4). He gave sum, product, constant multiple, and quotient rules,

and the derivative of $f(x) = e^x$ without proofs. Table 27 shows how he justified the differentiation

rules for $y = c$ and $y = ax + b$.

Table 27. Ian's Routine and Explanations of Differentiation Rules for $f(x) = c$

and $f(x) = ax + b$

Comp. of Routine	Action	Functions	
		$f(x) = c$	$f(x) = ax + b$
Prompt	Mentioning the goal	2I12. We will compute the examples of the derivatives and...see this particular case.	2I44a. I want to compute the derivative of this function, $f'(x)$.
Course of action	Writing $f(x)$	2I15a. $f(x)$ is equal to a constant function c .	2I43. $f(x)$ is a linear function $ax+b$.
	Stating definition of $f'(x)$	2I15b. Let's just write down that particular fraction...That will be limit h goes to zero, a fraction [in the definition of $f'(x)$]	2I44b. I will use the definition $f'(x)$ [which] is the limit of a fraction $f(x+h) - f(x)$ over h as h goes to 0.
	Finding different quotient	2I16c. What's the value of the function at $x + h$? 2S17. $f(x+h)=c$. 2I19. Subtract the value of the function at x . What is the value of the function at x ? 2S20. c ? 2I21. Divide by h . This is my fraction for any x .	2I46. if I plug $(x+h)$ here, I will get $a(x+h)+b$...It's the value of the function at $x+h$... a gets to multiplied by whole $(x+h)$ This is b .
	Specify- ing 0/0 form	2I22. I cannot plug in $h=0$, here. I will have division by zero. This particular function here is undefined for $h=0$.	2I55. What I have here is limit 0 over 0... If I try to plug $h=0$ in the bottom, I will have division by 0.
	Finding limit	2I24. You know what that limit is? 2I25. Zero.	2I67. After this cancellation h over h . All we would have is constant a , not division by 0, So, the limit is a .
Closure	Stating the rule	2I32. We just started with the constant c . We just proved that its derivative is zero.	2I70. [The] derivative of this function [$y = ax + b$] is just a .

This routine was identified 10 times. On Days 2, 3, and 4, Ian addressed the relationship between $f(x)$ and $f'(x)$ explicitly by deriving differentiation rules 15 times with or without proofs using the definition of $f'(x)$. He then differentiated several functions using the rules.

Transition from the Derivative at a Point to the Derivative of a Function: Graphical Illustration of Differentiation Rules

For the differentiation rules for constant and linear functions, Ian also used their graphs

and made a transition from the derivative at a point to the derivative of a function. He drew secant lines passing through a point, showed that the limit is a tangent line, and used its slope to justify differentiation rule. Figure 36 shows his explanation for $y = c$:

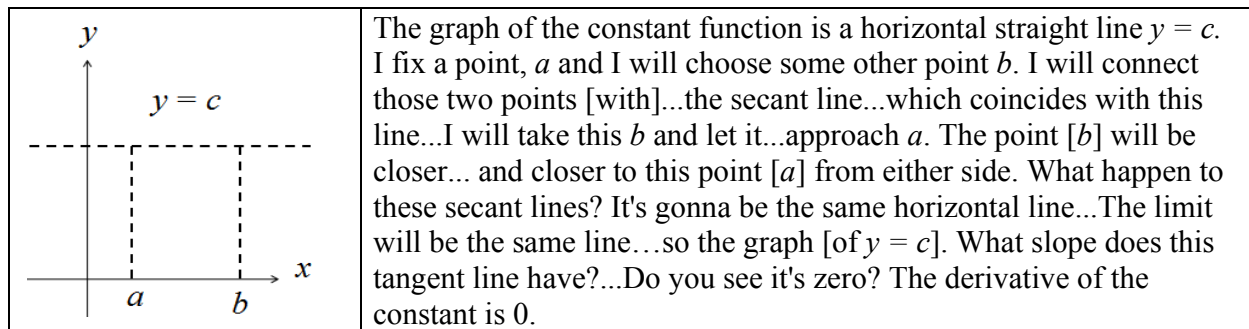


Figure 36. Ian's Graph and Explanation of the Derivative of $y = c$

Ian then, repeated this process for the differentiation rule for $f(x) = ax + b$. In these examples, he used the tangent line at a point to justify that $y' = 0$ for $y = c$ and $y' = a$ for $y = ax + b$. Therefore, he used the derivative at a point to justified the rules without saying that this explanation works for at any point on the differentiable domain, and thus without addressing the relationship between the derivative of a function and the derivative at a point. Later, he addressed this relationship when a student asked “Is $f'(x)$ is just a slope?” He answered by comparing linear and nonlinear functions: “For a straight or horizontal line, the tangent line is the same line...I have the same derivative everywhere...For non linear functions, I would have a certain dependence on x ...I will have a slope changed when x changes.” Here, by mentioning that the explanation can be applied at any point on the interval, not just a specific point, he addressed the relationship between the derivative at a point and the derivative of a function by making transition from the former to the latter. While making the transition, he distinguished “derivative function” from “derivative at a .”

Transition from the Derivative at a Point to the Derivative of a Function: Substitution

Ian addressed the other direction of transition, from the derivative of a function to the

derivative at a point implicitly on Day 2 while discussing the slope of the tangent line to a function $f(x) = x^2$ at $x = 1$ or $x = -1$ after finding that its derivative is $2x$ (Figure 37):

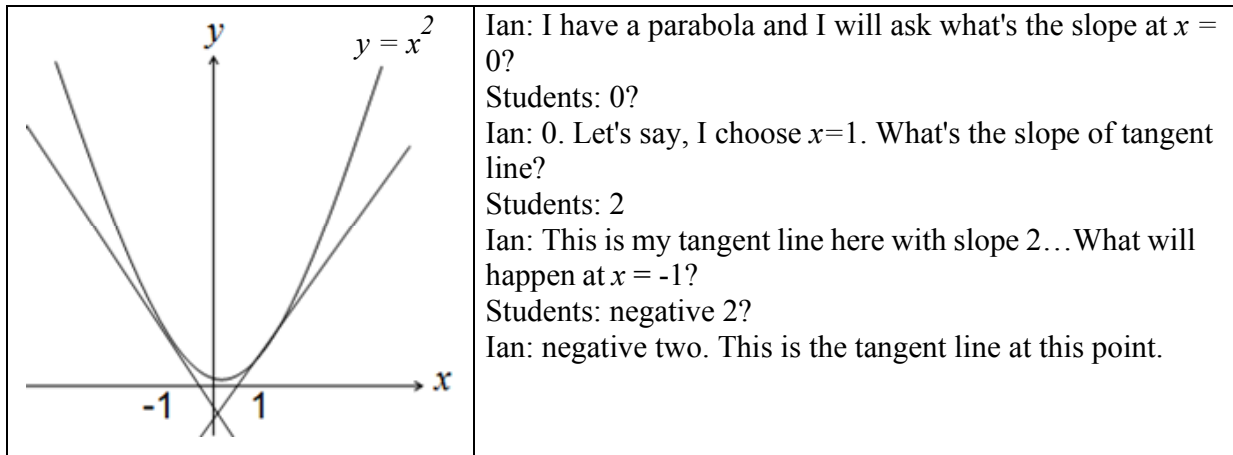


Figure 37. Ian's Drawing and Discussion of Tangent lines of $y = x^2$

Here, he gave the slope of the tangent lines at the two points without showing the process of substituting values in the derivative of a function. Later, he showed this process while explaining that the derivative of $f(x) = |x|$ does not exist at $x = 0$ on Day 3 (Figure 38):

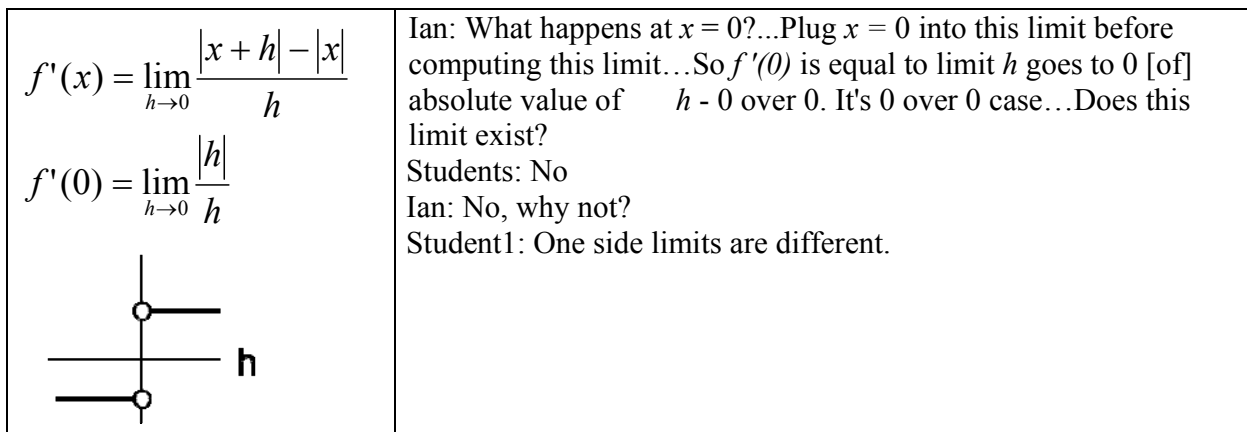


Figure 38. Ian's Writing and Discussion of the Derivative of $f(x) = |x|$ at $x = 0$

In this excerpt, he explicitly addressed that one could find out the derivative at a specific point by substituting the point in $f'(x)$. This relationship was identified one more time later in Day 5 when Ian explained the derivative of $y = a^x$ at $x = 0$ (Figure 39):

$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h}$	<p>Ian: The derivative of a to the power x is a to the power x times some number, which does not depend on x...So, next step I will look closely at this particular number...If I plug in 0, what's a to the power 0?</p>
$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$	<p>Students: one Ian: f' of 0 itself is this limit. So it happens to be this number, which is the slope of the tangent line to the graph of that function at $x=0$.</p>

Figure 39. Ian's Writing and Discussion about The Derivative of $y = a^x$ at $x = 0$

In the excerpt, Ian mentioned “plug[ging] in zero” in $f'(x)$ to calculate $f'(0)$, which he referred as

a number. He also explained that $\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$ is the slope of the tangent line to the graph of $y =$

a^x at $x = 0$. In summary, the relationship between the derivative of a function and the derivative

at a point was addressed explicitly twice and implicitly once.

The Derivative Function as a Function: Differentiation Rules

Ian addressed the derivative function as a function implicitly by explaining the dynamics of the derivative of a non-linear function. He said, “Its value will be changed when x changes.”

With a specific example, $f(x) = x^2$, he also said, “Do you see that this derivative depends on x ?

At different points, you will have different slopes” (Day 2).

Application of the Derivative

From Day 5, Ian explained the chain rule and implicit differentiation as parts of differentiation rules, and used them to compute the derivative of composite functions that have one or more variables. Then, he addressed the related rates and curve sketching.

The Relationship between a Function and the Derivative of Function

Ian addressed the relationship between a function and the derivative of a function while a) explaining the chain rule and implicit differentiation, b) comparing the equations of a function and the derivative of a function, c) explaining the behavior of a function based on the sign of the

derivative of a function, and d) mentioning differentiation rules.

Chain Rule and Implicit Differentiation. On Days 5, 6, and 7, Ian explained how to differentiate composite and implicit functions algebraically. He first showed a procedure of differentiating $y = (f \circ g)(x)$ by finding the derivatives of the “outside,” evaluating it at the “inside,” and multiplying the derivative of the “inside” (Day 5):

I have a composite function $(f \circ g)(x)$. If I knew how to differentiate f and g ... The derivative of this composite function, by the chain rule, is the derivative of the outside function $f'(g(x))$, just plug in $g(x)$, into the derivative of f . So, I need to calculate the derivative of f and plug $g(x)$... I also need to multiply by the derivative of what's inside. So, differentiate g and multiply it here. (Writing $g'(x)$).

After repeating this explanation for $y = (f \circ g \circ h)(x)$, he used the procedure to differentiate several composite functions. Table 28 shows this procedure with two examples; in the table, f and g were used as the “outside function” and “inside function,” respectively.

Table 28. *Ian's Routine and Explanations of the Chain Rule*

Comp. of Routine	Action	Cases	
		$y = e^{\sqrt{x}}$	$y = \sqrt{e^x}$
Course of action	Finding the Derivative of f	6I100. The derivative of the outside function is e to the some power, so e to whatever becomes e to what?	6I107. How would we differentiate this using what rule? $1/2\sqrt{x}$, That's the derivative of the outside function, square root, evaluated at?
	Evaluation at g	6S101. Square root of x .	6S108. e^x .
	Multiplying the Derivative of g	6T102. By the chain rule, times? 6S103. Square root of x .	6I109. And times? 6S110. e^x .

While differentiating composite functions using the process above, he often asked students, “What is the most outside function?” and “What is the derivative of inside?” (Days 5 and 6). However, he did not address the chain rule in any other contexts. For example, when

differentiating the equation of the diagonal of a rectangle whose sides x and y change with respect to time (Figure 40), he just included x' and y' in the derivative:

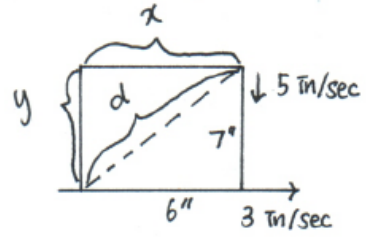
 <p> $x(t)^2 + y(t)^2 = d(t)^2$ $2xx' + 2yy' = 2dd'$ </p>	<p>Ian: I can use [the chain rule] to find...the relationship between rates of change of x, y and d. How I would do it? I just differentiate. This function of time equals that function of time. So what's the derivative of x^2?</p> <p>Students: $2x$?</p> <p>Ian: By the change rule, times dx/dt or x'. Plus, what's the derivative of y^2?</p> <p>Students: $2y$ times y'.</p>
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Figure 40. Ian's Writing and Explanation about Rate of Change

Although he later mentioned the units for xx' , yy' , and dd' are in^2/sec , he did not further explain the meaning of this unit in the context of the problem or mention the reason why x' and y' are multiplied; they contributed to the rate of change of $x^2(t)$ and $y^2(t)$.

Ian also addressed implicit differentiation algebraically. While differentiating an example, $x^2 + y^2 = 25$, Ian first identified the independent variable by saying "I am gonna assume that y is some function of x (writing $y(x)$)...The function $y(x)$ is unknown. I can rewrite this equation as $x^2 + y(x)^2 = 25$ " (Day 7). Then he differentiated it using the chain rule by repeating his previous explanation involving "inside" and "outside" functions:

Ian: How would you differentiate this (pointing to $y(x)^2$)?

Student 1: Chain rule?

Ian: The chain rule. It's a composite function... $y(x)$, some unknown function plugged into some other function in this case, taking squared...Derivative of x^2 is $2x$ plus derivative of something squared is 2 times something. I will write $2y$, for simplicity, I will skip that of

x in parenthesis. Times, by the chain rule, I have to differentiate whatever I have inside this parenthesis...it's the derivative y' , or dy/dx .

While explaining the chain rule and implicit differentiation, Ian addressed the relationship between $f(x)$ and $f'(x)$ algebraically without interpreting $f'(x)$ as a rate of change.

Comparison between Equations of a Function and the Derivative of a Function. Ian also addressed the relationship between a function and the derivative of a function by comparing their equations. He compared the function $y = a^x$ to its derivative function $y' = a^x \ln a$ by saying “the derivative of the [exponential] function is proportional to itself, and $y = e^x$ to its derivative e^x by mentioning “the derivative of the function is equal to itself.”

Ian also used the equations of the derivatives to show that two functions that differ by a constant have the same derivative. Using the identity, $\tan^2 x + 1 = \sec^2 x$, he first mentioned that the graphs of two functions are parallel by saying “I could prove that they [$\tan^2 x + 1$ and $\sec^2 x$] differ by a constant...Take one graph and slightly move it up by 1. And they will have the same slope at the same x ” and showed that the equations of the derivatives of $1 + \tan^2 x$ and $\sec^2 x$ are the same by applying differentiation rules.

In the three examples above, Ian addressed the relationship between a function and the derivative of a function algebraically by comparing their equations.

Behavior of a function based on the sign of its derivative function. Ian primarily addressed the relationship between a function and the derivative of a function when he explained the behavior of a function based on the sign of the derivative. He addressed this relationship a) with position, velocity, and acceleration, b) with the definition of the derivative of a function and a tangent line, c) in the first derivative test, d) in discussions on concavity with the first and

second derivatives and e) while graphing functions.

On Day 11, Ian first addressed the relationship between a function and the derivative of a function by defining the velocity as the derivative of a position function and explaining the velocity as a direction: positive (negative) velocity as an indicator of a moving object going to the right (left) on a horizontal line (Figure 41):

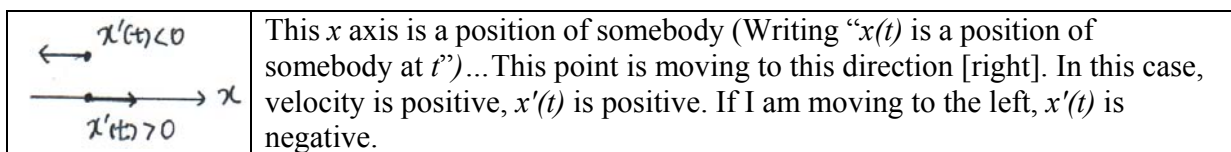


Figure 41. Ian's Drawing and Explanation of the Sign of Velocity as Direction

In the excerpt above, he mentioned that the direction is determined by the velocity "at a particular moment of time". In the next example (Figure 42), he extended the discussion on directions of an object using the sign of the derivative at a point to over an interval:

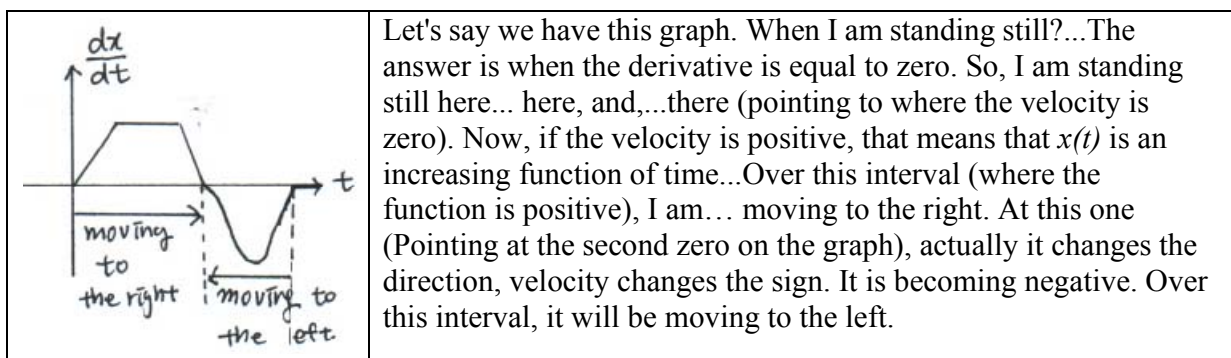


Figure 42. Ian's Graph and Explanation of the Velocity as Direction

In this excerpt, Ian explained the direction of the moving object based on the sign of the velocity over an interval. However, he did not explain why one can use the same argument on the sign of the derivative at a point and the derivative function over an interval, which can be addressed by mentioning the former as a value of the latter. Therefore, he addressed the relationship between a function and its derivative function without addressing the relationship between the derivative of a function and the derivative at a point. He defined acceleration as the derivative of velocity. The

relationships between position, velocity, and acceleration were addressed five times.

Ian next showed that $f'(x)$ is positive when $f(x)$ increases (Figure 43):

$f'(x) \nearrow \Rightarrow f'(x) > 0$ $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ $h > 0$	<p>$f(x)$ is a monotonic and increasing function...if I calculate the derivative by the definition of $f'(x)$...If the function is differentiable, this limit exists. If the function is going up that means for positive h, this $[f(x+h)]$ is greater than this $[f(x)]$. The top is positive and the bottom is positive. This fraction is positive. Therefore, the derivative has to be positive.</p>
---	---

Figure 43. Ian's Writing and Explanation of Derivative of an Increasing Function

He repeated this explanation with a negative h and stated that he could also show that $f'(x)$ is negative if $f(x)$ decreases. He also explained this relationship graphically (Figure 44):

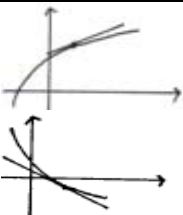
	<p>If the function is going up, the graph looks like this. I draw a tangent line. Do you see it has a positive slope? It goes up as well. If a function is going down like that, the tangent line...will have a negative slope. It will go down. [The] converse is also true. If the derivative is positive, that means the function is going up...The sign of the derivative determines whether the function goes up or goes down.</p>
--	---

Figure 44. Ian's Graphs of Tangent Lines to Increasing and Decreasing Functions

In this excerpt, he used the slope of the tangent line at a point as the indicator of if $f(x)$ increases or decreases on an interval. However, he did not mention that the derivative at a point, $f'(a)$, is a value of $f'(x)$, or show multiple tangents whose slopes are positive on an interval. Therefore, to address the relationship between $f(x)$ and $f'(x)$, he used $f'(a)$ without addressing the relationship between $f'(x)$ and $f'(a)$. He finally summarized this relationship as the first derivative test for the critical points with a graph (Figure 45).

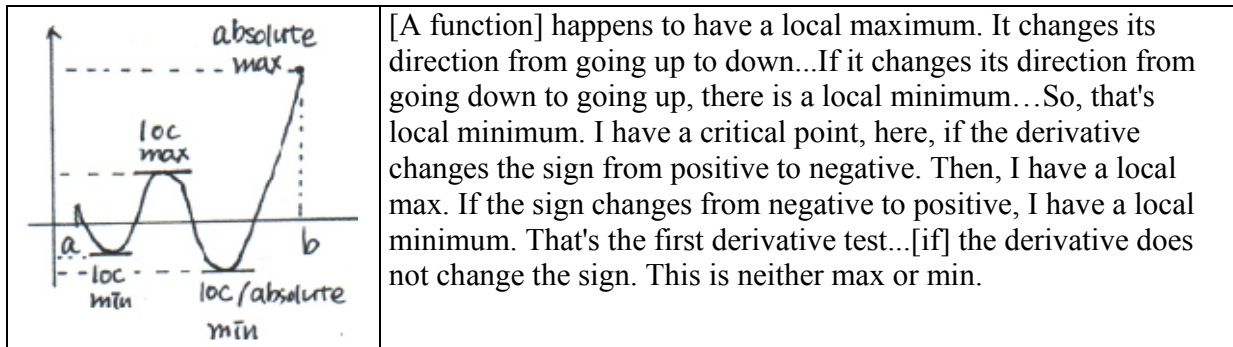


Figure 45. Ian's Graph and Explanations of Local Extreme Values

He applied the first derivative test to $y = |x|$ to show that it has a local minimum at $x = 0$.

On Day 15, Ian applied the relationship that if $f(x)$ increases/decreases, $f'(x)$ is positive/negative to the first and second derivatives to discuss concavity. He first said that $f''(x)$ is the derivative of $f'(x)$ and illustrated the case where $f'(x)$ increases (Figure 46):

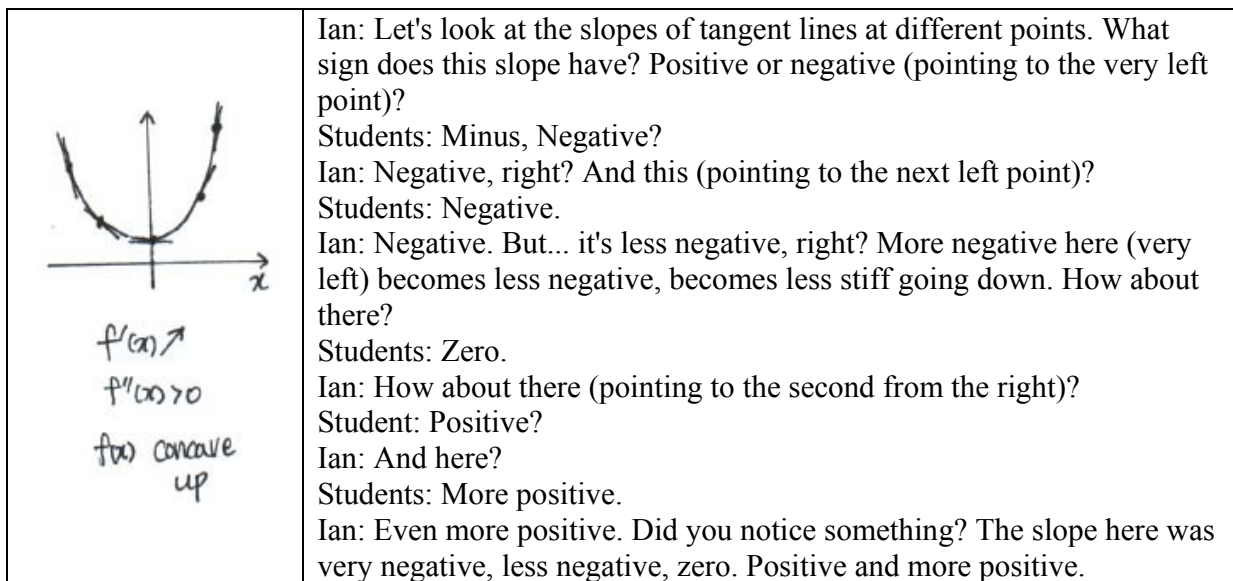


Figure 46. Ian's Graph and Explanations of an Increasing First Derivative

Then, he referred these slopes—"negative," "less negative," "zero," "positive," and "more positive"—as the first derivative and connected them to the sign of the second derivative:

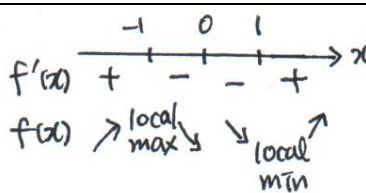
The slope, well, for a function like that, the derivative $f'(x)$ goes up. The second derivative in this case is positive. If the graph of the original function, $f(x)$, like this

continuous smooth graph (Figure 46), $f(x)$ is called concave up. The sign of the second derivative determines a certain property of the function, concavity.

Then, he repeated the same explanation for a concave-down curve. In this excerpt, he explicitly addressed the relationship between $f(x)$ and $f'(x)$ by connecting the behavior of $f'(x)$ to the sign of $f''(x)$. However, while explaining changes in slopes, he did not mention that $f'(x)$ is another function, whose behavior can be captured by its derivative, $f''(x)$.

After discussing the behavior of a function, $f(x)$, based on the signs of $f'(x)$ and $f''(x)$, Ian graphed functions given as equations by following steps: a) finding the domain, range, and points of discontinuity of $f(x)$, b) calculating $f'(x)$ and $f''(x)$, c) finding critical points, d) finding where $f(x)$ increases or decreases, e) finding where $f(x)$ is concave up or down, and inflection points, f) finding out asymptotes, and g) sketching the graph. He graphed four functions. The relationship between $f(x)$ and $f'(x)$ was identified in steps c) and d). Table 29 shows these steps when Ian drew the graph of $f(x) = x + 1/x$. He first found critical points, put them and points of discontinuity in a diagram, divided intervals, substituted a number on each interval to $f'(x)$, put the signs in the diagram, described the behavior of $f(x)$, and finally summarized the relationship between $f(x)$ and $f'(x)$.

Table 29. Ian's Routine and Explanation of Graphing $f(x) = x + 1/x$

Comp. of Routine	Action	Example $f(x) = x + 1/x$
Course of Action	Finding Critical Points	16I52. Step 3, critical points, the derivative is equal to zero, when? 16S53. x is equal to 1 or negative 1
	Drawing Sign Diagram	16I57. Now, in order to answer the step 4, what I strongly recommend to do is to draw the following diagram.
	Putting in Critical points	16I58. Basically, you draw this x -axis, and mark all your critical points, -1 and 1.
	Putting in Points of Discontinuity	16I59. But also, put all the points of discontinuity on this diagram. What are the points of discontinuity? 16S60. Zero? 16I61. Zero.
	Dividing Interval	16I64. So, ok, those two (-1 and 1) are critical points and this is a point of discontinuity The first derivative can only change the sign at those point. In between, interval here, here, and here, it should stay on the same sign.
	Finding Sign of $f'(x)$	16I67. So, all we need to do is to determine the sign of the first derivative on each of those intervals.
	Finding Numbers between Critical Points & Sign of $f'(x)$	16I68. Let's say, x is 1/2. We want to determine the sign of the derivative. 16S69. Negative? 16I70. Choose negative 1/2 16S71. Negative? 16I72. And choose 1 billion? 16S73. Positive.
	Completing Diagram	 <p>16I75. This diagram shows the sign of the first derivative from each interval.</p>
	Describing the Behavior of $f(x)$	16I77. We can make conclusion about my function $f(x)$. That goes up here, goes down, goes down, and goes up.
Closure	Relationship between $f(x)$ and $f'(x)$	16I78. This particular diagram, shows all the point of discontinuity, shows all the critical points, shows intervals where the function goes up, where it goes down.

Steps c) and d) of the routine were identified seven times while he graphed the several functions,

$$f(x) = x^3 - 3x, f(x) = 1 + 1/x, f(x) = |x|, y = x + 1/x, y = 1/1 + x^2, y = 1/1 - x^2, \text{ and } y = x/1 - x^2.$$

To graph the first three examples, he utilized the first derivative (Day 15) whereas he used both the first and second derivatives to graph the rest (Days 16 & 17).

As seen in Table 29, when Ian addressed the relationship that if a function, $f(x)$, increases/decreases, its derivative function, $f'(x)$, is positive/negative, he used a value of the derivative at a point, $f'(a)$. However, he did not provide all the reasons for this method; a) $f'(x)$ is a function, b) the derivative at a point, $f'(a)$, is the value of $f'(x)$ at $x = a$, and c) $f'(x)$ can only change its sign at critical points or points of discontinuity. He only addressed the last by saying, “The derivative can change the sign at the critical point... In between those [critical] points... it stays the same sign, either positive or negative” (Day 14). In summary, while graphing functions, he explicitly addressed the relationship between $f(x)$ and $f'(x)$ but not the relationship between $f'(x)$ and $f'(a)$, and $f'(x)$ as a function.

Differentiation Rules. Ian also addressed the relationship between a function and its derivative function by mentioning differentiation rules for linear, constant, trigonometric, power, product, and quotient functions. This relationship was mostly identified when Ian asked what the derivative of a given function is to students during calculation (e.g., he asked “What is the derivative of 5?” and students answered, “Zero”). There were 11 such cases. In other cases, he just applied the differentiation rules without asking or explaining.

Relationship between a Function and the Derivative at a Point

Compared to the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, the relationship between $f(x)$ and the derivative at a point, $f'(a)$, was limited to fewer cases. Ian addressed this relationship when he explained a) the rate of change of $f(x)$ at a point, and b) the derivative being zero at a point where a function has extreme values.

Rate of Change of a Function at a Point. Ian addressed the relationship between $f(x)$ and

$f'(a)$ while explaining $f'(a)$ as the rate of change of a moving object. For example, he calculated a rate of change of an area of a circle (its radius r is expanding at a rate of 1 inch/sec when r is 4 inches) using the derivative and specified its units (Table 30).

Table 30. *Ian's Routine and Interpretations of the Derivative at a Point*

Comp. of Routine	Action	Case
		Calculating A' when $A(t) = \pi r(t)^2$ and $r = 4$
Prompt	Asking units	11I120. What are the units of the rate of change?
Course of action	Specifying the unit	11S121. Inches per second? Inches per second squared? 11I123. [Pointing to $A'(4) = \pi \cdot 2 \cdot 4 \cdot 1 = 8\pi$] This is π , and 2 is just 2. 4 is 4 inches. And 1, the rate of change inch/second. What happens to the units? 11S24. inch^2/sec
	Interpreting Derivative at a Point	11I129. If I have an expanding disk, its radius r expanding at 1 inch per second and...the rate of area will be equal to $8\pi \text{ inch}^2/\text{sec}$...1 inch/sec is a positive rate of change, so my radius is expanding. As a result, I will get the positive rate of change of the area A . It means area is expanding, too.

In the excerpt above, Ian explicitly interpreted the result, 8π , as the rate at which the area of the circle is increasing. He also showed that the units for 8π are inch^2/sec from the calculation, which is aligned with his interpretation. He interpreted the derivative as the rate of change of a moving object and specified its units four more times in the lessons.

Derivative Being Zero at a Point Where a Function Has Extreme Values. Ian talked about the relationship between a function and the derivative at a point while discussing critical points and extreme values of a function. After defining a local minimum of $f(x)$ as “ c is a local minimum, if $f(c) \leq f(x)$ for x close to c ” and a local maximum similarly (Day 12), he discussed the derivative being zero at those points on the graph in Figure 45. He identified two local extreme points on the graph and asked students “What do you think the slope of the tangent line is gonna be?” and they answered, “Zero” (Day 12). He said that this zero slope “means the derivative of the function, $f'(x)$, is equal to...zero.” Then, he showed a case where a function has a

minimum at a point where its derivative does not exist with $y = |x|$ at $x = 0$. He concluded this discussion by defining a critical point as a point where “the derivative of a function is zero or is undefined” (Day 12). Using this definition, he found extreme values of functions eight times. Here, he explicitly addressed the relationship between the function and the derivative at a point where the function has an extreme value. However, he did not connect this explanation to the derivative as a function although he mentioned the change in sign of $f'(x)$ near these critical points.

Relationship between the Derivative of a Function and the Derivative at a Point

While discussing the application of the derivative, Ian addressed the relationship between the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, when he found the slope of tangent line and evaluated the equation of $f'(x)$ at a specific point.

On Day 7, when Ian calculated the slope of the tangent line of a given function, he implicitly mentioned the relationship between the derivative of a function and the derivative at a point by saying, “The slope, m , is equal to dy/dx [at] that particular point. dy/dx , it’s like the derivative” (Day 7). However, here, he did not explicitly mention that the derivative at a point being a value of the derivative of a function.

Ian used substitution consistently to evaluate the derivative function at a point in problems involving the rate of change or the slope of the tangent line at a point. He calculated the derivative function and substituted a number or a point. For example, he calculated the slope of the tangent line to a circle $x^2 + y^2 = 25$ at the point (3, 4) by finding $y' = -x/y$ and substituting (3, 4) in it. He said, “ y' is a certain expression. It depends on x and y . What's the derivative at this point? All you need to do is plug those x and y into y' . Then I will get y' at that point, (3, 4).” In this excerpt, he substituted a specific point to evaluate the derivative of the circle at the point.

However, he was not explicit about the relationship between the derivative of a function and the derivative at a point, the latter as a value of the former. He mentioned this “plug in” method while solving five problems.

Derivative as a Function

Ian addressed the derivative function, $f'(x)$, as a function explicitly and implicitly. He explicitly stated that $f'(x)$ is a function when he defined the second derivative, and addressed it implicitly by specifying the independent variable of the derivative function.

When defining $f''(x)$ as the derivative of $f'(x)$, Ian stated that $f'(x)$ is a function on Day 6. He said that “the derivative of [a] function, $f'(x)$, itself is a function... Since it's a function, it deserves the right to have its own derivative” and continued to define $f''(x)$, “Since it's a function, it deserves the right to have its own derivative... If I take this derivative, and differentiate it again, I will get another function. And that another function is called the second derivative of $f(x)$.” Here, he explicitly addressed that $f'(x)$ and $f''(x)$ are functions. He did not state that $f'(x)$ or $f''(x)$ is a function in the rest of the derivative unit.

Ian specified the independent variable of the derivative function especially while differentiating implicit or composite functions with respect to a variable other than x . On Day 5, he specified the variable y of function $f(y)$ when he introduced the notation df/dy and differentiated $f(y)$ with respect to y . He said, “If I differentiate $f(y)$, it's $f'(y)$. A different way to write it down is df/dy ... That is the derivative of function f with respect to its argument y .” In later classes, Ian mostly mentioned the independent variable of the derivative of a function when he differentiated implicit functions. The following excerpt shows how he interpreted the notation dy/dx when y was given as an implicit function:

The idea is differentiating this function implicitly. We are looking for dy/dx , the

derivative of y with respect to x . We are not finding how y depends on x explicitly. The idea is assuming that y is a function of x . I will differentiate this whole thing with respect to x . Whenever I have to differentiate y , I will just put dy/dx ... In implicit differentiation, you always expect to use the chain rule because what you will have is some unknown function y of x plugged in to some other functions.

In this excerpt, he not only interpreted dy/dx as “the derivative y with respect to x ” but also mentioned that y is “a function of x ”. Such clarification on the independent variable of implicit functions was made 21 times. He also specified quantities involved in related rates as “functions of time, t ” 19 times. After differentiating implicit functions and solving for y' , he also specified its independent variables. For example, when he recapped implicit differentiation, he said, “You just assume $y(x)$ is known function of x . Whenever you differentiate it, just use y' . Differentiate using the chain rule, and solve it for y' . Then you will get y' in terms of x and y ” (Day 7). Specification of the independent variable was mostly identified after he solved for y' . He mentioned that y' depends on either x or x and y depends on how y is defined 8 times during the derivative unit.

Summary

Ian defined the derivative at a point as $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ and then defined the derivative of a function using the same representation by mentioning it as “a function of a ”. Later, he changed a to x without mentioning that x is the independent variable of $f'(x)$. While defining $f'(a)$ he explicitly mentioned that it is “a function depending on a ” three times. Aside from referring to a as a point in the first definition, he did not address that the derivative at a point is the value of the derivative of a function at that point. Then, he addressed the relationships among a function, $f(x)$, the derivative function, $f'(x)$, and the derivative at a point

$f'(a)$ while explaining various topics. Table 31 shows in which topics, to what extent, and how many times (in parentheses) he addressed these relationships.

Table 31. Ian's Actions in Discussion of Relationships among $f(x)$, $f'(x)$, and $f'(a)$ on Topics and Nature of Discussion

Concepts Being Related	Topic	Action	Nature
$f(x)$ & $f'(x)$	Differentiation rules	Proving rules with the definition of $f'(x)$	E(10)
		Proving rules with tangent lines	E(1)
	Chain Rule & Implicit Differentiation	Explaining the chain rule and implicit differentiation algebraically using the rule	E(4)
	Equations of $f'(x)$	Comparing equations of $f(x)$ and $f'(x)$	E(3)
	Graph of $f(x)$ & $f'(x)$	Describing behavior of $f(x)$ or graphing $f'(x)$ based on sign of $f'(x)$	E(7)
	Behavior of $f(x)$	Proving if $f(x)$ increases, $f'(x)$ is positive using the definition of $f'(x)$	E(2)
	First derivative test	Describing $f'(x)$ near a critical point using $f'(x)$	E(1)
	Concavity	Discussing sign of $f''(x)$ using behavior of $f'(x)$	E(2)
	Differentiation rules	Mentioning rules while calculating $f'(x)$	E(11)
	$f(x)$ & $f'(a)$	Graph of $f(x)$	Describing behavior of $f(x)$ on an interval based on sign of $f'(a)$
Behavior of $f(x)$		Showing if $f(x)$ increases, $f'(a)$ is positive	E(2)
Rate of change		Interpreting $f'(a)$ as rate of change at a point	E(5)
Local extremes		Mentioning $f'(x) = 0$ at local extremes of $f(x)$	E(11)
$f'(x)$ & $f'(a)$	Definition of $f'(x)$	Mentioning that a can be any point	E(1)
	Differentiation rules	Mentioning $f'(x)$ of constant and linear functions is same everywhere	E(2)
	Definition of $f'(a)$	Substituting a to the definition of $f'(x)$ for $f'(a)$	E(2)
	Graph of $f(x)$	Determining sign of $f'(x)$ on interval using $f'(a)$	I (12)
	Evaluating $f'(a)$	Substituting a value to $f'(x)$ to evaluate $f'(a)$	E(5)

As shown in Table 31, the relationships between $f(x)$ and $f'(x)$ were primarily discussed when he proved differentiation rules with the definition of $f'(x)$ and used them to calculate $f'(x)$. He addressed these relationships algebraically when simplifying the difference quotient and taking the limit on it, or applying differentiation rules to the equation of the function. He also addressed the relationships graphically to describe the behavior of $f(x)$ based on the sign of $f'(x)$. He used signs of $f'(a)$ instead of signs of $f'(x)$ as the indicator of whether $f(x)$ increases or decreases on an interval without addressing the relationship between $f'(x)$ and $f'(a)$. In other words, in most

cases the statement “the derivative is positive if and only if the function is increasing” was used to describe the behavior of $f(x)$ on an interval but based on the sign of the derivative at a point. Therefore, in graphical situations, he used the relationship between $f'(x)$ and $f'(a)$, which is $f'(a)$ as a value of $f'(x)$ without mentioning it explicitly. However, when he justified differentiation rules for constant or linear functions with graphs, he addressed the relationship between $f'(x)$ and $f'(a)$ explicitly by mentioning that the derivative function is same at any point.

The relationship between $f(x)$ and $f'(a)$ was mostly addressed when he mentioned that $f'(x) = 0$ at a point where $f(x)$ has a local extremes; he, however, did not connect the derivative being zero to the behavior of $f(x)$ near these points. The relationship between $f'(x)$ and $f'(a)$ was mostly addressed when he substituted a number in $f'(x)$ to evaluate $f'(a)$; however, he never mentioned that $f'(a)$ is a value of $f'(x)$; thus $f'(a)$ is a number.

The derivative of a function, $f'(x)$, as a function was addressed explicitly several times in Ian’s classroom. First, while he defined the derivative at a point, $f'(a)$, and $f'(x)$, he mentioned that a point can be any point on the domain, and the definition of $f'(x)$ will give different slopes of the tangent line at different points, thus it is a function. He also said that $f'(x)$ is “a function which deserves its own derivative” when he defined the second and third derivatives. When he compared the derivatives of linear and nonlinear functions, he addressed $f'(x)$ as a function implicitly by mentioning that the value of $f'(x)$ changes as x changes. He specified the variable when he differentiated a function with a phrase “with respect to” but never identified it as its independent variable of the derivative function.

Discussion of the Research Questions

This section addresses the three research questions stated in the beginning of this chapter.

Definition of the Derivative

The three instructors introduced the definition of the derivative differently. Although all of them introduced the derivative at a point $f'(a)$ and then the derivative of a function $f'(x)$, they

built up $f'(x)$ from $f'(a)$ differently. Tyler first defined the derivative at a point x_0 as the slope of tangent line at x_0 and then moved to the derivative of a function, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$,

by changing the point x_0 to the variable x without addressing why it was possible. He also did not mention that $f'(x)$ is a function or $f'(x_0)$ is a specific value of $f'(x)$ at a point; thus, it is a number.

Alan first defined the rate of change at x_0 , $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$ and then the derivative of a

function, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. Although he did not give a definition of the derivative

at a point or the notation $f'(x_0)$, or mention that the instantaneous rate of change is the same as

the derivative at a point, he explicitly addressed the difference between $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$

and $f'(x)$: the former as a value and the latter as a function. He then substituted a value, $x = x_0$, in

$f'(x)$ to calculate $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$. Ian first defined the derivative at a point,

$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ and then the derivative of a function using the same representation

by mentioning it as “a function of a .” Although he defined these two concepts using the same notation, he did not mention that the derivative at a point is a specific value of the derivative of a function. Later, he changed a to x without mentioning that x is the independent variable of $f'(x)$.

In summary, while defining the derivative of a function and the derivative at a point, Ian mentioned the difference and relationship between them, Alan and Ian mentioned that the derivative of a function is a function. Tyler did not address any of them.

Relationships between $f(x)$, $f'(x)$, and $f'(a)$

As shown in Tables 18, 24, and 31 although there were some differences, in general the instructors addressed the relationships between a function, $f(x)$, and its derivative function, $f'(x)$, and between $f(x)$ and the derivative at a point $f'(a)$ explicitly but the relationship between $f'(x)$ and $f'(a)$ implicitly. First, they mainly addressed the relationship between $f(x)$ and $f'(x)$ when they proved or applied differentiation rules, and they connected the behavior of $f(x)$ to the sign of $f'(x)$. When they proved differentiation rules with the definition of $f'(x)$ and applied them to calculate $f'(x)$, they discussed $f'(x)$ directly without mentioning $f'(a)$. However, when instructors described the behavior of $f(x)$ (increasing/decreasing) to the sign of $f'(x)$ (negative/positive) on an interval, they substituted a point a on an interval in $f'(x)$ and used the sign of $f'(a)$ to determine the sign of $f'(x)$ on that interval without addressing the reason for substitution. A possible explanation lies in the relationship that $f'(a)$ is a value of $f'(x)$, and $f'(x)$ is a function that changes its sign only at critical points. Ian and Tyler mentioned that $f'(x)$ changes its sign at critical points. Tyler explained why, but the explanation was not explicit because he used the word, “function,” ambiguously without specifying it as “the derivative function” or “the original function” and did not include the word, “derivative,” in the explanation. Therefore in most cases, instructors’ statement about $f(x)$ and $f'(x)$ such as “a function is increasing when the derivative is positive” refers to the behavior of a function over an interval, but it was based on the sign of derivative at a point.

A similar use of $f'(a)$ for $f'(x)$ was also identified when instructors addressed the relationship between $f(x)$ and $f'(a)$ with the differentiability of $f(x)$. Tyler discussed differentiability of $f(x)$ base on existence of the derivative at a point, $f'(a)$; however, he did not explicitly distinguish differentiability of a function at a point and over an interval. After showing

that a function does not have the derivative at a point, he just concluded that this function is not differentiable. Here, again, he used the relationship between $f'(x)$ and $f'(a)$ without mentioning it explicitly. Unlike Tyler, Alan explicitly addressed the difference between differentiability of a function at a point and over an interval, which is aligned with his discussion of the definitions of $f'(x)$ and $f'(x_0)$. Since Ian addressed it using the existence of a tangent line at a point of a function before he defined the derivative, he did not address any relationships involving the derivative in the discussion.

The relationship between $f(x)$ and $f'(a)$ was also addressed when the instructors mentioned and used the derivative being zero at a point where the original function has a horizontal tangent line. They explicitly related the characteristics of $f(x)$ such as “beats, valleys, humps” or “turn-around” to the derivative (the slope of tangent line) being zero at the point. However, when most of them used this relationship, the derivative being zero at a point where $f(x)$ has a horizontal tangent line to graph $f'(x)$, they again made the relationship between $f'(x)$ and $f'(a)$ implicit. All of them plotted the x intercepts of the graph of $f'(x)$ where $f(x)$ has a horizontal tangent line without addressing that why they “match[ed] up” these points. None of them explicitly mentioned that $f'(a)$ is a specific value of $f'(x)$, for example, the derivative at several points form a function, $f'(x)$.

The relationship between $f'(x)$ and $f'(a)$ was most visible when the instructors found $f'(a)$ by substituting a number in $f'(x)$. While evaluating, they explained the reason for the substitution; a need for finding the rate of change, the slope of tangent line, or the derivative “at a specific point.” Therefore, the transition from $f'(x)$ to $f'(a)$ was explicitly made several times via substitution. However, they did not make the transition from $f'(a)$ to $f'(x)$ explicit. When graphing the derivative of a function by plotting several values of the derivative at a point, none

of instructors explained these several values form a function $f'(x)$. The instructors may assume that the relationship between $f'(x)$ and $f'(a)$ is obvious to students because they have used the notations— $f(x)$ for a function and $f(a)$ for its value—while they find $f(a)$ by substituting $x = a$ in $f(x)$. The relationship between $f'(x)$ and $f'(a)$ is equivalent to the relationship between $f(x)$ and $f(a)$ except the prime symbol in the notations. However, since explanation about $f'(x)$ and $f'(a)$ involves various other concepts such as limits, ratio, and tangency, which might distract students from identifying the equivalence, seeing this simple relationship between $f'(x)$ and $f'(a)$ might not be easy to students.

Derivative as a Function

The instructors' implicit discussions on $f'(x)$ and $f'(a)$ seems to connected to their discussions of $f'(x)$ as a function. Each instructor stated that $f'(x)$ is a function during the derivative unit with different topics; Tyler mentioned it once when he graphed $f'(x)$ by saying it is a function that one can “graph”, Alan mentioned it three times when he read aloud the definition of $f'(x)$ from the textbook and compared the definitions of $f'(x)$ and $f'(a)$, and Ian mentioned it twice while defining $f'(x)$ and $f''(x)$ as the derivative of $f'(x)$ by saying that “it is a function” which “deserves its own derivative.” Aside from those cases, they used $f'(x)$ as a function without stating it. However, $f'(x)$ as a function was imbedded in the relationships between $f(x)$, $f'(x)$, and $f'(a)$ and in the solution processes of many problems such as applying theorems about a function to the derivative of a function and graphing the derivative of a function. If they mentioned that the derivative of a function is a function while explaining those relationships and solving these problems, the relationship and solution processes may become much more explicit by providing the rationale, which is that the derivative of a function is another function whose value at a specific point is the derivative at a point.

The instructors' ambiguous use of the word "derivative" without specifying it as the derivative of a function, $f'(x)$, or the derivative at a point, $f'(a)$, also seems to make $f'(x)$ as a function implicit. Especially, their explanation about the relationship between $f(x)$ and $f'(x)$ by using the sign of $f'(a)$, without mentioning the relationship between $f'(x)$ and $f'(a)$ or $f'(x)$ as a function, may confuse students about what the instructors referred to with the word, "derivative". Since the instructors primarily used the slope of tangent line at a point rather than describing the change in behavior of slopes over an interval, students may connect the behavior of $f(x)$ to $f'(a)$ rather than $f'(x)$. This realization of the "derivative" might be problematic since it limits the derivative only as a point-specific value not as a dynamic quantity that changes its value on an interval (Monk, 1994). On the other hand, since the relationship between $f(x)$ and $f'(x)$ was mostly addressed when the instructors applied differentiation rules, students may have developed two disconnected realization of the derivative; the derivative at a point as the slope of tangent line on the graph of $f(x)$ and the derivative of a function as the result of applying differentiation rules.

In summary, the three instructor explicitly addressed the relationship between $f(x)$ and $f'(x)$, and between $f'(x)$ and $f'(a)$, but the relationship between $f'(x)$ and $f'(a)$; and $f'(x)$ as a function implicitly. Explicit discussion on these issues would provide a better opportunity for students to learn important aspects of the derivative of a function and the derivative at a point, which is imbedded in other topics and relationships in the derivative unit, in their calculus courses, and to better prepare them before taking higher mathematics involving the concept of the derivative.

CHAPTER 5: RESULTS OF INSTRUCTOR INTERVIEW ANALYSIS

This chapter reports the results of individual interviews with instructors about the derivative. During the interview, I asked instructors to solve the survey problems as if they were explaining them to their students and also whether they would use the problems in their class. I provided a copy of the survey as well as blank papers for the instructors to use when they needed extra space for writing or graphing. Their answers to all problems were correct. The instructors' discourses were examined in terms of aspects of the derivative used and how they were used to address the following questions:

1. To what extent do instructors address the relationships between a function, the derivative at a point, and the derivative of a function?
2. To what extent do instructors address the derivative of a function as another function?

Because the interviews were not a part of lessons, I report the ways in which instructors applied these topics and relationships in each survey problem, rather than the way the instructors introduced and discussed these ideas with students. For the same reason, instructors' uses of the relationships and concepts were primarily reported with excerpts rather than routine tables. Routines, repetitive patterns, were not easy to find during the interviews, possibly because the interviews were conducted only once for less than an hour. This chapter addressed a) the instructors' mathematical background focusing on their native language, b) their explanations of each survey problem, and c) how they addressed the relationships and concepts in the research questions in their explanations.

Instructors' Backgrounds

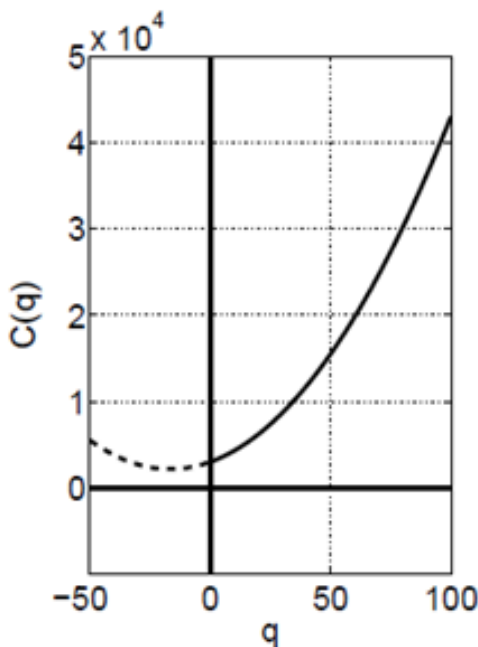
The three instructors learned mathematics in different languages from elementary school

through university. Tyler learned mathematics in German. In German, the terms for ‘derivative,’ ‘the derivative of a function,’ and ‘the derivative at a point’ are ‘ableitung,’ ‘ableitung einer funktion,’ and ‘ableitung an einem punkt,” respectively. Ian learned mathematics in Russian. The terms ‘derivative,’ ‘the derivative of a function’ and ‘the derivative at a point’ in Russian are ‘производное,’ ‘производной в точке,’ and ‘производной функции,’ respectively. Alan is a native English speaker. Because the German, Russian, and English terms work similarly, I assumed that Tyler, Ian, and Alan’s native languages influenced their uses of these terms in their discourse on the derivative in similar ways.

Problem 1

Problem 1 asks instructors to calculate and interpret values of a function and the derivative of a function at a point when the function was given (Figure 47).

1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the equation is given by $C(q) = 3000 + 100q + 3q^2$, and the graph is given as follows.



- i) Find the value of $C(2)$.
- j) What is the unit of 2 in (a)?
- k) What is the unit of $C(2)$?
- l) What is the meaning of $C(2)$ in the problem context?
- m) Find the value of $C'(2)$.
- n) What is the unit for 2 in (e)?
- o) What is the unit of $C'(2)$?
- p) What is the meaning of $C'(2)$ in the problem context?

Figure 47. Problem 1

All instructors' answers on parts a) through e) were the same. They calculated $C(2) = 3212$ by substituting 2 in $C(q)$, used the problem statement to find the units of 2 as "miles" and $C(2)$ as "dollars," and interpret $C(2)$ as "the cost for producing 2 miles of rope" or an equivalent response. In part e), they found $C'(q)$ using differentiation rules, substituted 2 for q , and obtained $C'(2) = 112$ using the relationship that $C'(2)$ is a value of $C'(q)$ at $q = 2$. They gave the units for 2 as "miles" and $C'(2)$ as "dollars per mile."

The instructors' answers on part h) were slightly different. Tyler explained the units for 2 in $C'(2)$ by saying, "We are interested in how the cost changes at, near the point where you are producing 2 miles of rope." Then, he explained the units of $C'(2)$ using its definition involving the limit as shown in Figure 48:

$C'(2) = \lim_{h \rightarrow 0} \frac{C(2+h) - C(2)}{h}$ <p style="text-align: center;">\$ miles</p>	<p>Look at the definition of the derivative...the limit as h goes to zero, $C(2+h)-C(2)$ over h. We'd plug in smaller and smaller values for h, here. To figure out the units, I can just look at the units of the denominator and the numerator and the units of $C'(2)$ are going to be the quotients of the units. The cost is given in dollars...h is given in miles. That's the delta in miles, like the additional amount of rope that you produce starting from 2 miles. The units of this expression is dollars/miles.</p>
--	--

Figure 48. Tyler's Writing and Explanation on Part g) in Problem 1.

In the excerpt, Tyler primarily used the definition of the derivative at a point to interpret $C'(2)$ in the problem context where $C(q)$ was defined as a cost function. He found the units of $C'(2)$ using its definition in which the numerator is the change in rope and the denominator is the change in cost, respectively. To answer part h), he also used the definition. He interpreted the relationship between the denominator and numerator by saying, "We are increasing the production of rope by this small amount h , and we are interested in how much is the cost increasing" and concluded that "the meaning of this would be the increase in the cost of rope. And it's per mile."

While solving problem 1, Tyler use the word, "derivative," in the two phrases "the

derivative of this function” and “the derivative at 2”; besides these two cases, he used the word without specifying it as the derivative of a function or the derivative at a point. For example, when he said “look at the units of the derivative,” the word, “derivative,” can be interpreted as either $C'(2)$ or $C'(q)$. He said that he would not use this problem in his class, because “this course is more concerned with the calculus side...how to compute stuff.”

Alan gave the correct units for 2 and $C'(2)$ without further explanation, and interpreted $C'(2)$ as “marginal cost, which is the cost of making one more miles beyond 2 miles...if we are doing linearization.” He did not use the word, “derivative” in problem 1. He said he would use this problem when introducing the derivative and differentiation rules, but would “not worry about units so much.” He said “the derivative as rate of change,” as “meaning of $C'(2)$ ” would be good for students to think about.

Ian also gave the units for 2 and $C'(2)$ without an explanation. He interpreted $C'(2)$ as “the rate of change in cost”. He used the word, “derivative” once in the phrase, “the value of the derivative at a point” when he pointed to $C'(2)$. He said that he would use this problem in relation to a rate of change.

Problem 2

Problems 2 asked students to find $f'(2)$ when the equation $f'(x)$ is given (Figure 49).

2. The derivative of a function f , is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

Figure 49. Problems 2

In problem 2, Tyler first stated, “we want to find the derivative of a function at a specific point. In order to do this, we always need to figure out what the derivative is in general.” Then, he mentioned, “We are given the derivative and evaluate the derivative at a point,” and calculated $f'(2) = -4$ by “plug[ging] 2 in” $f'(x)$. Here, he used the relationship that $f'(2)$ is a value of $f'(x)$.

In problem 2, he used the word, “derivative” without specifying it as the derivative of a function or the derivative at a point except a phrase, “the derivative of a function at a specific point”.

From the context, all the instances of “derivative” can be inferred as the derivative of a function.

He said that this problem is “too simple to use.”

Alan’s explanations coincided with Tyler’s. Alan said that he used problem 2 when reviewing “evaluating a function” and “reading off the graph” without mentioning the derivative.

Ian’s explanations also coincided with other instructors’ explanations. However, unlike the other two instructors, Ian added that “the derivative is just a function on its own, so the value at that point [$x = 2$] will be, take the value [$x = 2$] and plug into the derivative.” When he used the word, “derivative,” without specifying the point, the word referred to “the derivative of a function.” He said that the problem could be used “in the beginning of the derivatives” but it is a “pre-calculus thing to plug in points in an equation.”

Problem 3

Problems 3 asked students to find $g'(2)$ when the graph of $g'(x)$ was given (Figure 50).

3. The graph of the derivative of function g , $g'(x)$ is given as follows. What is the value of $g'(2)$?

- a) -10
- b) -4
- c) 0
- d) 4
- e) 10

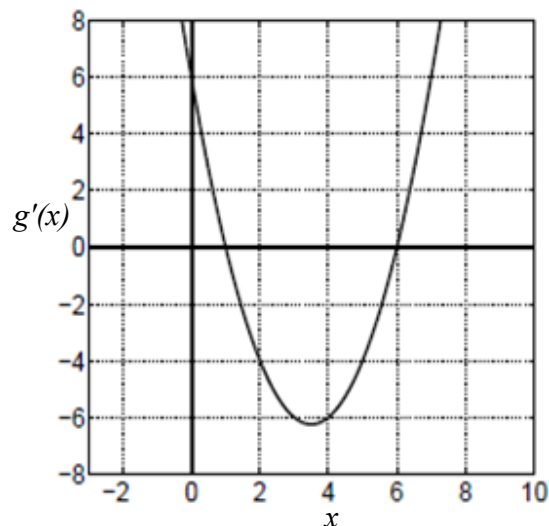


Figure 50. Problems 3

All the three instructors read the value of $g'(2)$ from the graph of $g'(x)$ by applying the same relationship between $g'(2)$ and $g'(x)$. For example, Alan said, “We just need to find where $x = 2$ is at because we've got $g'(x)$...read off the graph, our value which is -4. This is just remembering how to read off the graph.” Tyler’s and Ian’s explanations are equivalent to Alan’s explanation. Ian mentioned that “the derivative is just a function on its own,” and “if I use a different x , I would have to draw a different vertical line.”

Alan did not use the word, “derivative” in this problem. Tyler used the word without specifying it as the derivative of a function or the derivative at a point except one phrase, “the derivative of a function at a specific point”. From the context, all instances of “derivative” can be inferred as the derivative of a function. Ian also used the word, “derivative,” without specifying it to refer to “the derivative of a function.”

When I asked if they would use problem 3 in their class, Alan said he used these kinds of problems when reviewing “evaluating a function” and “reading off the graph” without talking about the derivative. Tyler said that it is “too simple to use.” Ian said that he could use it “in the beginning of the derivatives,” but it is a “pre-calculus thing.”

Problem 4

Problem 4 asked students to graph $f'(x)$ when the graph of $f(x)$ is given (Figure 51).

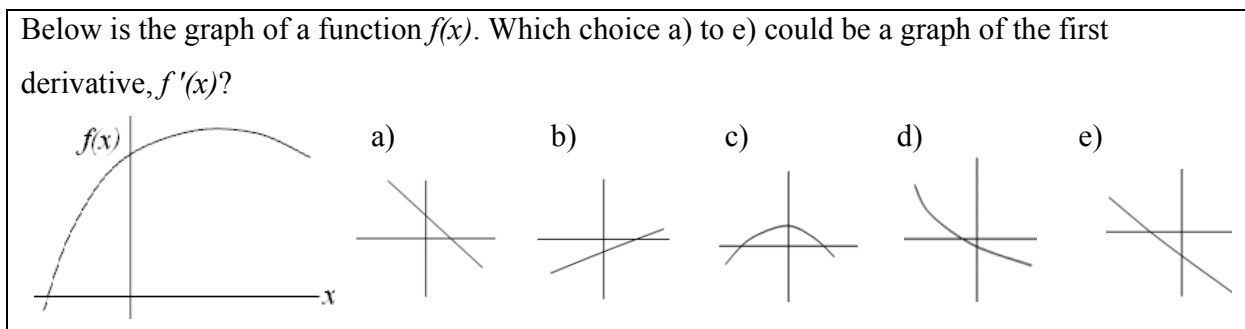
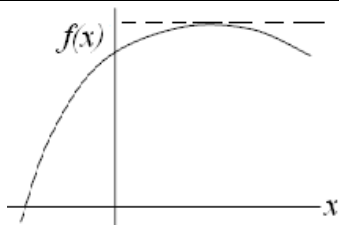
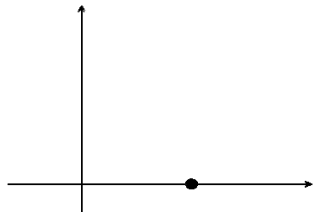
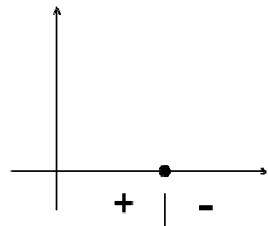
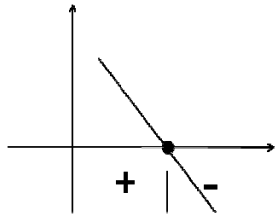


Figure 51. Problem 4

To find the graph of $f'(x)$, Tyler first mentioned the need to find out the sign of the

derivative by saying, “The most important things that you want to look at in the function [are] where is the derivative positive, where is the derivative negative, where is the derivative zero.” Then, he: a) found where the function has a zero slope; b) placed zero on the x - y coordinate plane for the graph of $f'(x)$; c) found the sign of the derivative; and d) chose the answer a). Table 32 shows his excerpts and drawings for each step.

Table 32. *Tyler’s Course of Action, Explanations, and Drawings on Problem 4*

Course of Action	Explanation	Drawing
Finding zero slope	So, we are looking at a graph. And we are first figuring out the points we have the zero slope. So, that's the local maximum there.	
Placing zero for $f'(x)$	This is going to correspond to the zero of the derivative, that's gonna be happening at right here.	
Finding the sign of $f'(x)$	Then, the derivative is positive to the left of this point because your function is increasing, and the derivative is negative to the right because after the maximum, the function is decreasing.	
Choosing the answer	So, we are gonna have just pick a function, one of the functions among the possible answers that does this.	

In the first step, Tyler used the relationship that the derivative is zero at a point where $f(x)$ has a maximum. In the next step, he applied the relationship between the derivative at a point and the derivative of a function by placing this zero slope on the x - y plane for the graph of $f'(x)$

in the second step. In the third step, he used the relationship between $f(x)$ and $f'(x)$: $f'(x)$ is negative/positive when $f(x)$ decreases/increases. Finally, by drawing a line passing through the point on the x axis to complete the graph of $f'(x)$, he used the relationship between the derivative at a point and the derivative of a function: the values of the derivative at several points form the derivative of a function.

In this excerpt, Tyler primarily used the word, “derivative,” without specifying it as the derivative of a function or the derivative at a point. All cases can be inferred as the former since he talked about zeros of the derivative or the sign of the derivative on an interval. He said that he would use this example when introducing the derivative.

Alan chose a) to compare the slope to the graph of f and the graphs in the choices instead of graphing the derivative of f . He a) found a point where the graph of f has a horizontal tangent, b) compared it with zeros in the graphs of f' in the choices, c) found where the positive or negative slopes are in f ; d) compared them to the sign of the graphs of f' , e) checked a point where the graphs of f' have a horizontal tangent; and f) compared it to where the concavity of f changes (Table 33).

Table 33. Alan's Course of Action, Explanations, and Drawings on Problem 4

Course of Action	Explanations	Drawings
Finding zero slope in $f(x)$	Looking at f to find your horizontal tangent. (Drawing a tangent line at the peak)	
Identifying zeros for $f'(x)$	My f' will be zero at that point...that kind of crosses zero little bit to the right of the x -axis of the origin. So, d) doesn't cross the x -axis over here. It doesn't touch zero. e) is also out...All these [a), b), & c)] cross the zero.	
Matching signs of the slope of $f(x)$ and $f'(x)$	So, we look here [f]. On the left of the zero, we have the positive slope (drawing the tangent line increasing), so, we want to make sure that we are on the left side. So, b) is not a candidate for that because it is in the negative region. And then, we are left with a) and c).	
Finding zero slope in $f'(x)$	In c), we've got the first derivative has zero tangent, which would mean that the concave is changing there.	
Finding inflection points of f	We don't see the concave change if we look back on the original function, so it can't be c). So, that's gonna be a)	

In Table 33, Tyler first addressed the relationship that $f'(a) = 0$ when $f(x)$ has horizontal tangent. Then, he addressed the relationship between $f'(a)$ and $f'(x)$ comparing zero slopes of f with zeros of $f'(x)$. Third, he addressed the relationships between $f(x)$ and $f'(x)$ by comparing the sign of the slope of $f(x)$ with the sign of $f'(x)$, and between $f(x)$ and $f'(a)$ by identifying where $f'(x)$ has zero slope as the inflection point of $f(x)$. He added that c) is not the answer because “In c), we have another zero [but] there is not another critical point on [f].” In this excerpt, Alan used the word, “derivative,” without specifying it as the derivative of a function or the derivative at a point; most uses can be inferred as the derivative of a function. He said that he assigned a similar problem as homework.

Ian solved problem graphically and then algebraically. First, he described the graph of $f(x)$ as a parabola and stated that its derivative function is linear. Then, he created a graph of $f'(x)$ by finding the x -intercept for the derivative of a function using the relationship that $f'(x) = 0$ where $f(x)$ has a maximum, connecting the concavity of $f(x)$ to the behavior of $f'(x)$, and completed the graph of $f'(x)$ (Table 34):

Table 34. Ian's Course of Action, Explanations, and Drawings on Problem 4

Course of Action	Excerpt	Drawing
Describing the shape of $f(x)$	This [pointing to $f(x)$] looks like a parabola	
Stating that $f'(x)$ is linear	The derivative of a quadratic function would be a linear function, so it's gonna be a straight line.	
Finding x intercept of $f'(x)$ using maximum of $f(x)$	That straight line will have zero at the point where the parabola has the maximum.	
Finding behavior of $f'(x)$ using concavity of $f(x)$	Since the parabola...is concave down, the slope is negative on this line going through the point.	

He also added that $f'(x)$ does not have to be “a straight line, if this [$f(x)$] is not a parabola.”

As shown in the excerpt, in the first and second steps, he used the relationship between a function, $f(x)$, and its derivative of a function, $f'(x)$, by saying that the derivative of a quadratic function is linear. In the third step, he used another relationship that the derivative is zero where the function has an extreme value. Lastly, he used another relationship between $f(x)$ and $f'(x)$: if $f(x)$ is concave up, $f'(x)$ is decreasing (“the slope [of the derivative] is negative”), which he explained further, “The function is concave down, so the second derivative is negative. So, the first derivative should go down.”

Ian also mentioned another strategy involving a differentiation rule: “It [$f(x)$] is a parabola, branches down, so the coefficient of the highest degree term would be negative. And then differentiate it. The slope of the straight line [$f'(x)$] would be negative.” Here, he used the relationship that the derivative of a quadratic function is linear.

Ian used the word, “derivative,” without specifying it as the derivative function or the derivative at a point excepting one phrase, “the derivative of a quadratic function.” All other cases can be inferred as the derivative of a function. He said that he would use the problem “in differentiating polynomials, the first derivative test, and concavity.”

Problem 5

Problem 5 asked instructors to graph $f(x)$ when the graph of $f'(x)$ is given (Figure 52).

Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?

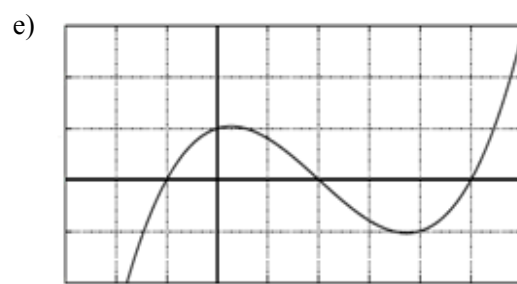
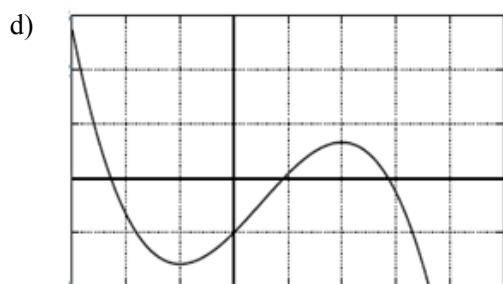
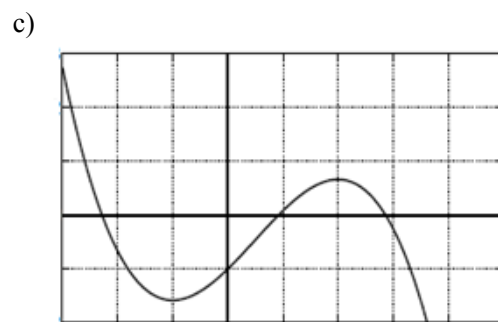
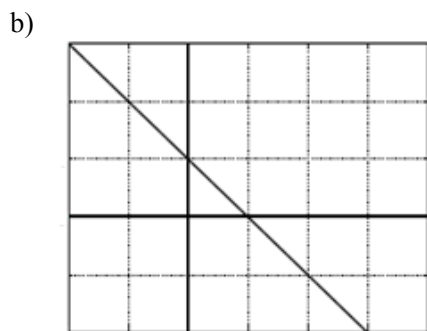
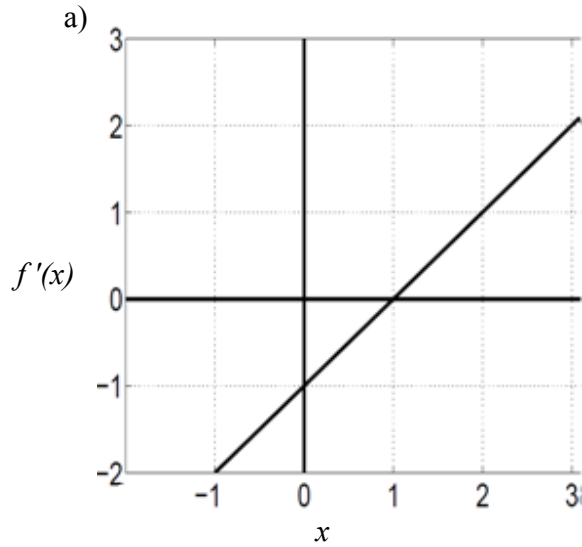
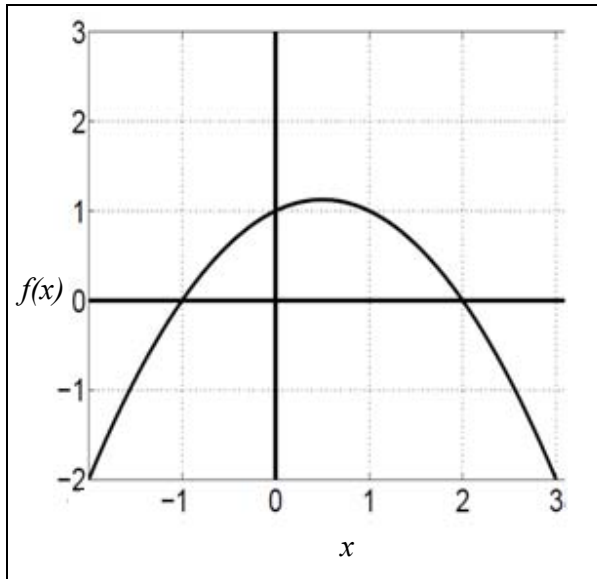
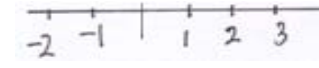
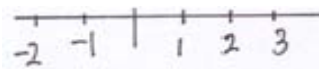
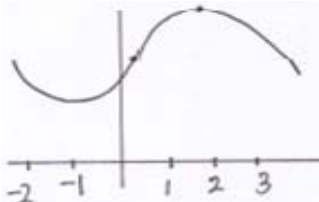


Figure 52. Problem 5

Tyler identified problem 5 as “opposite” of problem 4 and suggested two methods:

graphing the original function or going through the choices to rule out false solutions. For the first method, he listed “important things to know” such as “where is $f(x)$ increasing and decreasing, or ...concave up and concave down.” To graph $f(x)$, he: a) stated that $f(x)$ increases when $f'(x)$ is positive; b) identified intervals where $f'(x)$ is positive or negative; c) decided where $f(x)$ increases or decreases; d) stated that $f(x)$ is concave up when $f'(x)$ increases; e) identified intervals where $f'(x)$ increases or decreases; f) determined the concavity of $f(x)$; and g) completed the graph of $f'(x)$ (Table 35).

Table 35. Tyler’s Procedure, Explanations, and Drawing in Problem 5

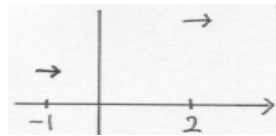

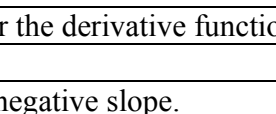
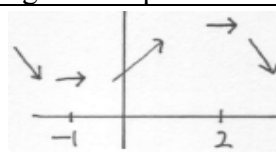
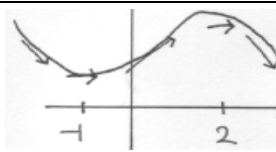
Course of Action	Explanations	Drawing
Relating f & f'	T51a. the original function is increasing whenever the derivative is positive	
Identifying the Sign of $f'(x)$	T51b. The derivative is positive...between - 1 and 2. T51e. The derivative as seen in a graph, it's negative.	
Describing Behavior of $f(x)$	T51c. The original function is increasing in between.	
	T51d. For the rests of the graph, the original function will be decreasing.	decr incr decr
Relating Concavity of f & f'	T54a. the original function is going to be concave up whenever the derivative is increasing	
Identifying Behavior of $f'(x)$	T54b. The derivative is increasing...between -2 and 0.5.	
	T54d. Between 0.5 and 3[the derivative] is [decreasing]	
Deciding Concavity of $f(x)$	T54c. I guess -2 to 0.5 is concave up.	
	T54e. The function is concave down [on [-2, 0.5]].	concave up concave down
Completing the Graph of $f(x)$	T55. We should be decreasing and concave up up to -1. After that, we should still be concave up, but not increasing. Then, we should reach an inflection point, 1/2, which is where it turns to be concave down, but we are still increasing. Then, we have local max at 2. After that, it goes down and still concave down.	
		decr incr decr concave up concave down

In Table 35, he stated two relationships between $f(x)$ and $f'(x)$: if $f'(x)$ is positive/negative, $f(x)$ increases/decreases: and when $f'(x)$ increases/decreases, $f(x)$ is concave up/down. Tyler, then

solved the problem again by getting rid of false choices based on the points where $f(x)$ in each choice has “a horizontal slope.” He said that these points “will give you zeros” for $f'(x)$, and “Only c) has zero slope at -1 and 2.” Here, he used the relationship that $f'(a) = 0$ when $f(x)$ has a horizontal tangent line at $x = a$. In this problem, he used the word, “derivative,” without specifying it as $f'(x)$ and $f'(a)$ to refer to $f'(x)$. At the end, he said that he would not use this problem because it involves anti-derivative.

Alan chose c) by comparing the points where the graph of “ f' crosses the x axis” and where the graphs of “ f have a horizontal tangent.” He first ruled out a) and b) because “they are linear functions, so they don’t have those flat points”. Of the rest, he ruled out c) and d) because they “do not have horizontal tangent at -1 or 2.” by using the relationship that $f'(a) = 0$ when $f(x)$ has a horizontal tangent line at $x = a$. When asked to solve the problem not using choices, he graphed f by identifying points where $f'(x) = 0$, drawing horizontal segments, checking the sign of f' , drawing arrows for the slope of f , and connecting the points and arrows (Table 36):

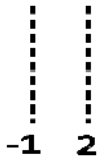
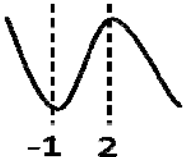
Table 36. Alan’s Procedure, Explanations, and Drawings in Problem 5

Course of Action	Explanations	
Finding zeros of $f'(x)$	A74. We’ve got those zeros at -2 and 1.	
Drawing horizontal segments	A75. We are gonna have flat points at -1...at 2 (drawing two arrows)	
Checking sign of f'	A76a. We’ve got negative values, left of $x = -1$ for the derivative function. A77a. That’s gonna be a positive slope. A78a. Afterwards, we are in the negative region, negative slope.	
Drawing arrows	A76b. We are gonna have negative slope, so we are just gonna be coming down (drawing ↘). A77b. It’s gonna be coming up (drawing ↗). A78b. We have it coming down (drawing ↘).	
Connecting points & arrows	A80. The behavior of the original function, I connect the arrows...and dots.	

In this excerpt, Alan again used the relationship that $f'(a) = 0$ when $f(x)$ has extreme value at $x = a$. He also used the relationship that $f'(x)$ is the slope of $f(x)$. He used the word, “derivative” once to refer to the derivative of a function, and mainly used “ f prime.” He said he would use this problem without choices to “make students think more about what they need to attend” to in order to graph $f(x)$ based on the behavior of $f'(x)$.

Ian also chose c) by identifying the critical points of $f(x)$ from the graph of $f'(x)$ and its sign. He ruled out a) and b) because those would be the result of “differentiating that [derivative] function.” Then, he graphed $f(x)$ by finding zeros in $f'(x)$, matching them to extreme values of $f(x)$, describing the behavior of $f(x)$, and justifying it with the sign of $f'(x)$, and then he chose c) which coincided with his graph of $f(x)$ (Table 37):

Table 37. Ian’s Explanations and Drawings for Graphing of $f(x)$ in Problem 5

Steps	Excerpt	
Finding zeros of $f'(x)$	It looks like a critical point should be at -1 and 2.	
Plotting max & min points on $f(x)$	I could just match all the maximum and minimum to zeros of that quadratic function.	
Describing the behavior of $f(x)$	The function goes down before -1 and goes up between -1 and 2. And that goes down after 2.	
Checking sign of f'	Between -1 and 2, the graph of this parabola is above the x axis...so the original function should go up...I would immediately see those critical points or minimum, and that c) is the only choice which would fit.	

In Table 37, Ian used the relationship between a function and the derivative at a point by matching zeros of $f'(x)$ with extreme values of $f(x)$, and the sign of $f'(x)$ to determine the behavior of $f(x)$. In this problem, he used the word, “derivative,” twice for the derivative function. He said he would use this problem in curve sketching or the first derivative test.

Problem 6

Problem 6 asks about the behavior of the derivative of a function when the original function is given as positive (Figure 53).

If a function is always positive, then what must be true about its derivative function?

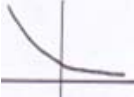
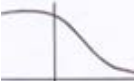
- a) The derivative function is always positive.
- b) The derivative function is never negative.
- c) The derivative function is increasing.
- d) The derivative function is decreasing.
- e) You can't conclude anything the derivative function.

Why? _____

Figure 53. Problem 6

After reading the problem, Tyler chose e) and said, “If your function is positive, you can’t say anything about the derivative.” Then, he justified it by mentioning shifting a graph, “Suppose you have a function that’s not positive, it’s negative somewhere... You can always shift it up.” Then, he provided two examples for a), b), c) and d) (Table 38).

Table 38. Tyler’s Explanations and Drawings for Each Choices of Problem 6

Choice	Explanations	Drawings
a)	Part a), my counter example is like this function e^{-x} (Drawing a curve)	
b)	This [curve] is also an counter example for b).	
c) & d)	We can just be concave down somewhere and concave up somewhere else (Drawing a curve). Then, that will kill both c) and d).	

As shown in Table 38, he did not explain why the first graph is the counter example for parts a) and b). For part d), he explained why the first graph is the counter example of “the derivative is decreasing” by stating, “the function is concave up, so the derivative is going to be increasing,” but he did not repeat the same explanation for part c). Therefore, in problem 6, Tyler used the relationships that when a function increases (or decreases), its derivative function is positive (or negative) and that a function is concave up when its derivative is increasing but only explain the second one explicitly. In this problem, all his uses of the word, “derivative,” can be inferred as

the derivative of a function. He added that he would use this problem in his class “to get people a feel for what the derivative does, what it does not...in the beginning of chapter 3.”

In problem 6, Alan chose e) by saying, “[the choice stating] ‘A functions is always positive’ means that it’s above the x axis, and it can be anything...Then, we don’t know anything about the derivative of the function.” He then mentioned the definition of the derivative of a

function, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, which involves a limit, to explain that $f(x)$ being

positive does not inform about the behavior of $f'(x)$:

The derivative isn’t really tied to the value being positive. It’s what’s the behavior of it, not its value. And then, that connects to the idea that the derivative is a limit, the formal definition is using the limit, where we care about what the limit was, not the actual function value. So, there was no guarantee.

Here, he used the derivative as an indicator of the behavior of the function, not its value. In problem 6, he used “derivative” as the derivative of a function. He said that he would talk about “the derivative being a formal definition” rather than using this problem.

In problem 6, Ian chose e) by eliminating incorrect choices by interpreting what the derivative of a function, which is given in each choice, informs about the function given as positive (Table 39):

Table 39. *Ian’s Explanations about Choices of Problem 6*

Choice	Explanations
a)	‘The derivative of a function is always positive’, it cannot be true...It tells me that the function is going up. Here, we don't have the function that goes up.
b)	‘Never negative’. The function never goes down. It may go down...Positivity of a function does not mean it cannot go down.
c)	‘The derivative of a function is increasing.’ That means that the function is concave up.
d)	‘The derivative function is decreasing.’ Then, the original function (can be) concave down. But, it could be concave up.

In Table 39, he ruled out a) and b) based on the relationship that if function, $f(x)$, increases/decreases, its derivative function, $f'(x)$ is positive/negative. For c) and d), he used the relationship that $f(x)$ is concave up/down when $f'(x)$ increases/decreases. Then, he concluded, “Always positive does not mean anything about it [the derivative function].” In this problem, He mainly used the word, “derivative” for “the derivative of a function” without specifying it. He said that he would use this problem to connect a function and its derivative” in curve sketching or for the first derivative test for critical points.

Problem 7

Problem 7 asked about the behavior of a function when the sign of its derivative function was given as negative on an interval (Figure 54).

The derivative of a function, $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?

- a) The function $f(x)$ is positive on this interval.
- b) The function $f(x)$ is negative on this interval.
- c) The maximum value of the function $f(x)$ over the interval occurs at $x = 2$.
- d) The maximum value of the function $f(x)$ over the interval occurs at $x = 3$.
- e) We cannot tell any of the above.

Figure 54. Problem 7

All the instructors chose c) using the relationship that $f(x)$ decreases when $f'(x)$ is negative, $f(x)$ decreases. For example, Tyler stated that the problem “tells us...the function is decreasing” and ruled out a) and b) without explaining why. Then, he drew a decreasing curve on the interval $[2, 3]$ and stated that the maximum occurs at $x=2$ (Figure 55):

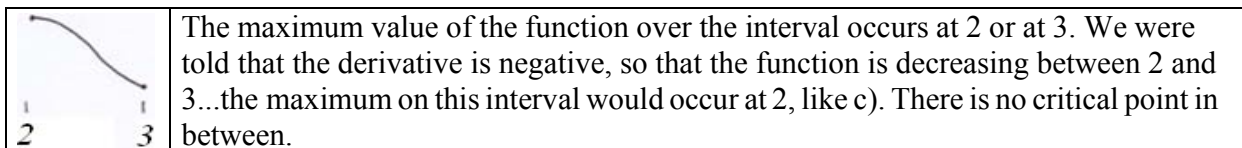


Figure 55. Tyler’s Explanations and Drawing on Problem 7

In the excerpt, he mentioned and used the relationship that $f(x)$ decreases when $f'(x)$ is negative.

He also pointed out that $f(x)$ does not have a critical point on the interval.

Tyler used the word, “derivative” for the derivative of a function consistently. He said that he would not use this problem because this course focuses on “dealing with equations and finding the derivatives using the formulas rather than doing stuff on graphs.” However, the problem statement does not necessarily involve the graphs directly.

Alan also applied the same relationship that Tyler used and created several graphs of a function whose derivative is negative on the interval $[2, 3]$ (Figure 56):

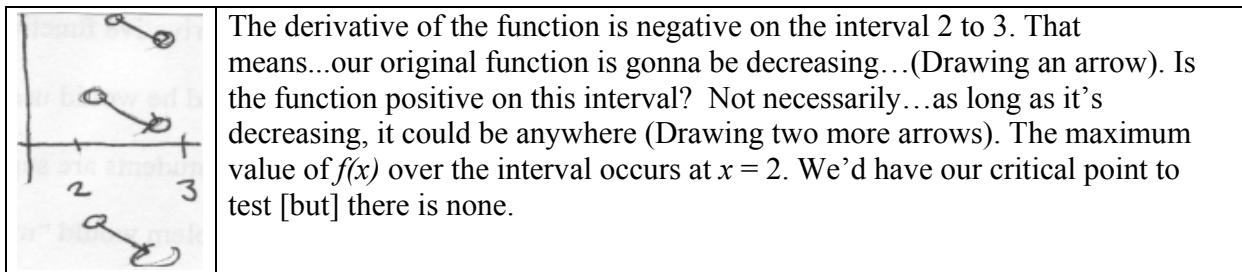


Figure 56. Alan's Drawings and Explanations on Problem 7

Like Tyler, Alan mentioned that there is no critical point. He used the word, “derivative,” once in the phase, “the derivative of a function” in the first sentence in the excerpt. He said that he would use this problem relating extreme values because “it could be a good way to talk about absolute max and min without actually being able to compute the values.”

Ian first stated the relationship that “[If] the derivative is negative... f has to go down within those two points,” and eliminated incorrect choices (Table 40):

Table 40. Ian's Explanations about Choices of Problem 7

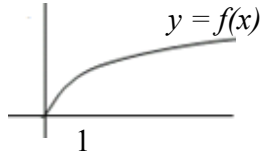
Choice	Explanations
a) & b)	The original function just goes down. It could be positive or negative or neither.
c)	c), the maximum value over the interval occurs at $x = 2$. I mean the absolute maximum over that interval... Assume that the function goes down from 2 to 3. There are no critical point between 2 and 3. So, that means that the absolute maximum has to occur at one of the end points. So, the maximum occurs at 2. This is true.
d)	The maximum occurs at 3. This is not true.

Additionally, Ian emphasized nonexistence of critical points on the interval, “the important part here is that there is no critical point in between 2 and 3. It could be an example of Mean Value Theorem, I guess, to prove that there are no critical points.” He used the word, “derivative” once to refer to the derivative of a function. He said that he would use Problem 7 because “it could be a good way to talk about absolute max and min without...being able to compute the values.”

Problem 8

Problem 8 is about the relationship between a function and its tangent line at a point (Figure 57).

The tangent line to this graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$.
 Which of the following statements is true and why?

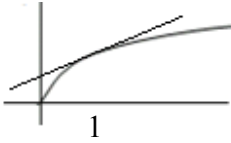


a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$ e) None of these.

Figure 57. Problem 8

Tyler drew the tangent line, and then checked if each choice is true (Table 41):

Table 41. Tyler’s Explanations and Drawings on Problem 8

Choice	Explanations and Drawings	
a)	$f(x)$ is not a linear function, so this [choice a] is clearly false.	
b)	For part b), we should actually draw the tangent line at $x=1$... We see that graph-wise $f(x)$ is completely below the tangent line. So... $f(x)$ is less than or equal to the equation of the tangent line, $1/2x + 1/2$. This goes exact opposite way than b) does. b) is false.	
c)	But c) is true.	
d)	d) is the same as a). It's still false. So, the graph we are given is not a linear function.	

In Table 41, for a) and d), Tyler said that the graph of $f(x)$ is not linear but the equations in a) and d) are. For choices b) and c), he compared the graphs of $f(x)$ and the tangent line.

Alan’s solution was equivalent to Tyler’s solution: eliminating incorrect choices by comparing graphs of $f(x)$ with the tangent line, or $f(x)$ with $y = x$. Ian also chose c) by comparing

graphs of $f(x)$ and its tangent line at $x = 1$ but did not check other choices.

The instructors did not use the concept of the derivative in this problem. They all said that they would not use this problem for different reasons. Tyler mentioned the goal of *Calculus I*, which is “not too concerned with graphs,” and Ian said that this problem is about the convexity of a curve, which is beyond the level of mathematics that Calculus I covered. Alan said, “Inequality is important, but I don’t know what would be the payoff for the course and the material that I am trying to get them to understand.” He added that students “try to generalize from one specific case,” and problem 8 “is not so generalizable.”

Problem 9

Problem 9 asked students to use the fact that the derivative at a point is the slope of the tangent line when the equation of the derivative of a function is given (Figure 58).

The derivative of a function is $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to f will be positive at $x = 0$.

- a) a and b must both be positive numbers.
- b) a must be positive, while b can be any real number.
- c) a can be any real number, while b must be positive.
- d) a and b can be any real numbers.
- e) None of these

Figure 58. Problem 9

All instructors solved problem 9 the same way by using the fact that the slope of the tangent line of a function f at $x = 0$ is $f'(0)$. They substituted 0 in $f'(x)$, and obtained b , and concluded that b has to be positive whereas a can be any value. For example, Ian said, “If I plug in $x=0$ into the derivative, I will get the value of the derivative equals b at $x = 0$. The slope is positive, that means that b has to be positive at $x = 0$. It doesn't matter what a is.” Here, he used the fact that the derivative at a point is the slope of tangent line of a function and applied the relationship between the derivative of a function and the derivative at a point using substitution.

The other two instructors gave the same explanation.

While solving Problem 9, Tyler used the word, “derivative,” in the phrase, “the derivative at zero,” for the derivative at a point, Alan used the word once to refer to the derivative of a function, and Ian also used it once for the derivative of a function.

When I asked if they would use this problem, Tyler said no, saying, “Not sure what this is trying to accomplish...The only content they need to realize is what the slope of the tangent line is and how this is represented in it, using the formula.” Alan said he would use it as homework because “students have to make a connection that the slope of the tangent line is the derivative” and he “would want them to think about the problem at first before” he solves it. Ian said, “This is...a meta question. It's not just making a conclusion about the function given the derivative. The derivative has certain parameters,” and he would use this problem as an extra credit problem.

Discussion of the Research Questions

This section addresses the research questions that stated in the beginning of this chapter. Table 42 summarizes instructors' use of the relationships among a function $f(x)$, its derivative function, $f'(x)$, and the derivative at a point, $f'(a)$ in their solution to survey problems.

Table 42. *The Relationships that Instructors Used to Solve Survey Problems*

Concepts being related	Action	Problems in which the relationship was used		
		Tyler	Alan	Ian
$f(x)$ and $f'(x)$	Applying differentiation rule	1e	1e	1e, 4, 5
	Stating (using) that the signs of $f'(x)$ the slope of $f(x)$ are the same.		4, 5, 9	4, 9
	Stating that $f'(x)$ is negative/positive when $f(x)$ decreases/increases.	4, 5, 7	7	5, 6, 7
	Stating that $f(x)$ is concave up/down when $f'(x)$ increases/decreases.	5, 6		4, 6
	Stating that $f'(x)$ decreases when $f''(x)$ is negative			4
	Stating that $f'(x)$ is the same as the slope of $f(x)$	9		
$f'(x)$ and $f'(a)$	Substituting a number in $f'(x)$ to evaluate $f'(a)$	1f, 2, 9	1f, 2, 9	1f, 2, 9
	Reading the value for $f'(a)$ from the graph of $f'(x)$	3	3	3
	Placing (comparing) zero slope for the graph of $f'(x)$	4	4	
	Connecting several values of $f'(a)$ to graph $f'(x)$	4		
$f(x)$ and $f'(a)$	Stating $f'(a)$ as the rate of change of $f(x)$ at $x = a$	1g & h	1g & h	1g & h
	Relating the inflection point of $f(x)$ to zero slope of $f'(x)$.		4	
	Stating (identifying) $f'(a) = 0$ when $f(x)$ has an extreme value at $x = a$	4, 5	4, 5	4, 5

Regarding the relationship between $f(x)$ and $f'(x)$, all instructors applied the differentiation rules to find $f'(x)$ when the equation of $f(x)$ was given. Ian also used them when the graphs of $f(x)$ and $f'(x)$ were given. All of them also stated that $f'(x)$ is the same as the slope of $f(x)$ or their signs are the same when they graphed $f(x)$ and $f'(x)$, or find the condition for the slope of tangent line using the equation of $f'(x)$. Although they all stated and used the relationship that $f'(x)$ is negative/positive when $f(x)$ decreases/increases, Tyler and Ian applied it in several problems whereas Alan used it in only one problem. When Alan graphed $f'(x)$, he used the sign of the slope of tangent line to $f(x)$ at several points more often than the sign of $f'(x)$ on an interval. Tyler and Ian also stated and used that $f(x)$ is concave up/down on an interval where $f'(x)$ increases/decreases when they graphed $f(x)$ or $f'(x)$. Ian justified this relationship using the second derivative, $f''(x)$; $f''(x)$ is negative on an interval where $f'(x)$ decreases.

The instructors mainly used the relationship between $f'(x)$ and $f'(a)$ — $f'(a)$ is a value of

$f'(x)$ at $x = a$ —when they substituted $x = a$ in $f'(x)$ or read the y value of the graph of $f'(x)$ at $x = a$. Only Ian mentioned that he could substitute $x = a$ in $f'(x)$ because it is a function. Tyler and Alan also used the relationship between $f'(x)$ and $f'(a)$ when they placed the points where $f(x)$ has the zero slope as the x intercepts of $f'(x)$ or comparing such points with the x intercepts. Tyler connected the values of derivative at all the points on the domain to complete the graph of $f'(x)$. Here, he used the relationship $f'(x)$ and $f'(a)$, in which the several values of $f'(a)$ form $f'(x)$ without mentioning it explicitly.

The instructors mostly used the relationships between $f(x)$ and $f'(a)$ when the instructors interpreted $f'(a)$ in relation to $f(x)$ as the rate of change of $f(x)$ at $x = a$ or the additional increase between $x = a$ and $a + 1$ and when they stated or identified that $f'(a) = 0$ where $f(x)$ has an extreme value at $x = a$. Alan mentioned a change of concavity of $f(x)$ where $f'(x)$ has zero slopes by applying the relationship between the concavity of $f(x)$ and $f'(x)$ at a point in contrast to the way Ian and Tyler applied the relationship on an interval.

The instructors mostly used the word, “derivative,” without specifying it as the derivative of a function or the derivative at a point. The word was mainly used to refer to the former. For the latter, they specified the point. This use of “derivative” is consistent with the word, “function,” when it used in “function” and “the value of function at a point”.

Additionally, I asked instructors if they would use the survey problems in their classes. Although this question is not directly related to the research questions, I assumed that their answers to this question would give information about how they see the survey problems. Tyler said he would not use most problems because they do not fit with the goal of the course, “finding the derivative using the formulas.” Alan said that he would use most problems because these problems can give students opportunities to think about what the derivative means to the original

function without calculations. Ian also said that he would use all the problems and stated in which parts of which lessons he could use them.

In summary, the extent to which the instructors addressed relationships between a function and its derivative was consistent with their classroom discourses. They showed tendencies to state and use the relationships between $f(x)$ and $f'(x)$, and between $f(x)$ and $f'(a)$ explicitly, while using the relationship between $f'(x)$ and $f'(a)$ without stating it. The instructors seemed to consider the last relationship would be straightforward for students; when asked if they would use problems 2 and 3, which are mainly about this relationship, only one instructor, Ian, mentioned it when he evaluated $f'(x)$ or read the y value of $f'(x)$ at $x = a$ to find $f'(a)$. Tyler said no because “they are too simple to use,” while Alan and Ian said “yes” but considered them best used as review problems for how to evaluate a function. However, this simple relationship between $f'(x)$ and $f'(a)$ would make sense when one realized $f'(x)$ as a function, which consists of several values of the derivative at several points. This underlying mathematics, which may not be obvious to students, seems to be considered as given in the instructors’ discourses.

CHAPTER 6: ANALYSIS OF STUDENT INTERVIEW DISCOURSE

This chapter reports the results of interviews with students about the derivative. As described in Chapter 3, two types of students' scores were calculated based on the mathematical correctness and the frequencies of answers after the survey. Using both scores, I contacted students to recruit 12 of them, four from each of the three classes, whose answers were somewhat heterogeneous in their achievement on the survey and close to the answers chosen by other students in the class. As was seen in Table 6 and 7, each of the classes had different degrees of variability in these two types of scores. It should be pointed out that the sample of interview participants may not be representative of each class, as Alan's students were all above the class mean (See Table 8). Ten students were interviewed during the semester immediately after the derivative unit was over, and two students, Clio and Cole, were interviewed after the semester was over because of low response rate. During one-hour individual interviews, I asked them to explain the derivative and their solution processes. Their discourses were examined in terms of the aspects of the derivative described and used, and how they were used in order to address the following questions:

1. How do students describe or define the concepts of the derivative at a point and the derivative of a function?
2. How do students describe the relationships between a function, the derivative at a point, and the derivative of a function?
3. How do students use the derivative of a function as another function?

The interviewees are grouped by their instructors—Tyler, Alan, and Ian—to facilitate discussion about a similarities and differences between student and instructor discourses, and thus students' *misconceptions* of the derivative. Regarding the first research question, the first section of this

chapter reports how each group of students explained the concepts of the derivative function and the derivative at a point; I list the explanations from the least to most mathematically correct. Regarding the second and third research questions, I describe students' solution processes problem by problem and how they described and utilized the concepts and relationships in the research questions. Then I addressed the research questions by reorganizing students' responses on each problem with respect to a) their own description of the derivative, b) their uses of the relationships between a function, its derivative function, and the derivative at a point, and c) the derivative as a function.⁶

Students' Descriptions of the Derivative

This section addresses the first research question regarding students' descriptions of the concept of the derivative. Their descriptions include their explanations about the derivative of a function and the derivative at a point, as well as the relationship between them and to the original function. During the interview, I first asked students what the derivative is to find their dominant concept of the derivative. After they answered, I asked them if their explanation was closer to the derivative of a function ($f'(x)$) or the derivative at a point ($f'(a)$). After they chose one, I asked them to explain the other. Then, I asked about the relationships between $f(x)$, $f'(x)$, and $f'(a)$. The details of students' description of the concept and relationships are as follows.

Tyler's Students

Collectively, the four students from Tyler's class described the derivative in various ways: velocity, extension and contraction of a function, the slope, and the change.

⁶ For, simplicity I will use short forms— $f(x)$, $f'(x)$, $f'(a)$ —for a function, the derivative of a function, and the derivative at a point, respectively when they are applicable in this chapter.

Roy – The Derivative as Expansion or Contraction of $f(x)$

In the beginning of the interview, Roy could not explain what the derivative was. However, while solving problems, he consistently interpreted the derivative as “expansion or contraction of the original function.” When asked which of the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, is closer to this interpretation, he chose the former by saying, “The expansion or contraction [works] over the entire function,” and added, “At an individual point...I can’t see how the derivative would work.” Although I prompted him several times, he could not explain what the derivative at a point is. Using this expansion-contraction idea, he argued that the graphs a function and the derivative of a function go in “the same direction”; both increase or decrease on the same interval.

Liz – The Derivative as Velocity

Liz mainly interpreted the derivative as the velocity (“How fast it [height of ‘ladder’ or ‘water’] is changing is the derivative”) but also mentioned differentiation rules (“if you have x^2 , and the derivative is $2xdx$ ”), and slope (e.g., “if you find the derivative at that point, then it gives you the slope at that moment”). She classified the first two interpretations as $f'(x)$, “it may be [the derivative of a] function cause it’s like the entirety of it” and identified the third interpretation as $f'(a)$. When asked to explain if and how “entirety” is related to “the slope at the point,” she said, “If you are moving...and...changing the velocity, then the change in velocity over the entire time is related to your velocity at a certain point.” She added that the velocity at a point is “the slope at the point.” To my question, “How is the derivative related to the original function?” she answered using velocity, “I think the original function [as]...the distance you are going, and the derivative [as]...how fast you are going.” She consistently used the interpretation of the derivative as velocity throughout the interview.

Neal – The Derivative as Slope

Neal explained the derivative as “the slope at any point where the direction of the tangent line is headed.” He added that the slope can be found at any point by saying, “The derivative would be graphical indication of every single point throughout a graph.” Then, he explained the derivative of a function as “the graph of the derivative.” When asked if the derivative at a point, $f'(a)$, and the derivative of a function, $f'(x)$, are related, he said yes and explained that $f'(x)$ is a continuous collection of $f'(a)$: “There is a tangent at every single point... On the derivative function, those points sync up to every single one of the other points. So, it’s not necessarily the derivative at many points. [It] is just the derivative of the function as a whole graph, the derivative graph.” In relation to the original function, Neal said, “The derivative at a point will be the exact slope at that point on the original function whereas the derivative of the function will tell you if the function is decreasing or increasing.” He further explained that if the derivative of a function on an interval or the derivative at a point is negative /positive, the original function is decreasing/increasing on the interval or at the point.

Zack – The Derivative as an Indicator of Function Behavior

Zack first explained the derivative as the slope, “The slope of the function tells us what the function is doing...and how a function is going to behave.” He added that one can find “another coordinate on the function” using the derivative, and identified this explanation as $f'(x)$. Then, he compared $f'(a)$ and $f'(x)$ by interpreting them as “the slope at that point” and “the slope of the function at all the points,” respectively. He used this interpretation to graph $f'(x)$ by relating the sign of $f'(a)$ to the behavior of $f'(x)$ (Figure 59):

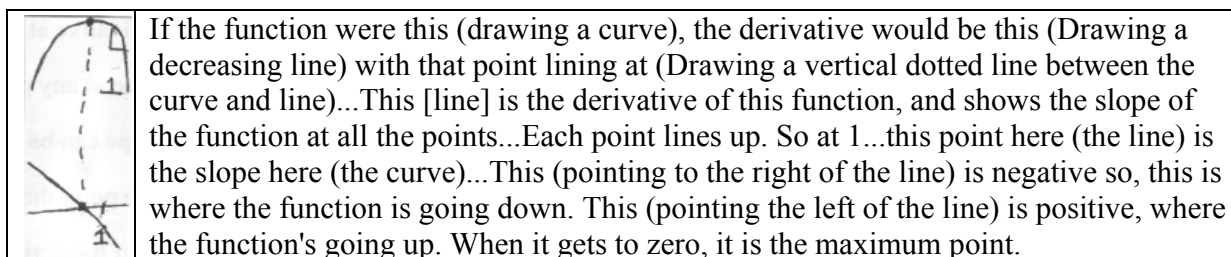


Figure 59. Zack's Graphs and Explanations of a Function and its Derivative Function

Zack explained the relationship between $f(x)$ and $f'(a)$ by interpreting $f'(a)$ as the slope of $f(x)$ at the point. Then, he explained $f'(x)$ as an indication of the behavior of $f(x)$.

Alan's Students

All four students from Alan's class described the derivative as the slope of a function, and two of them added the rate of change to their descriptions.

Bill– The derivative as Differentiation Rules

Bill explained the derivative as a new equation that one could “derive...from the original equation” and graphed the derivative of $C(q)$ in problem 1 (Figure 60):

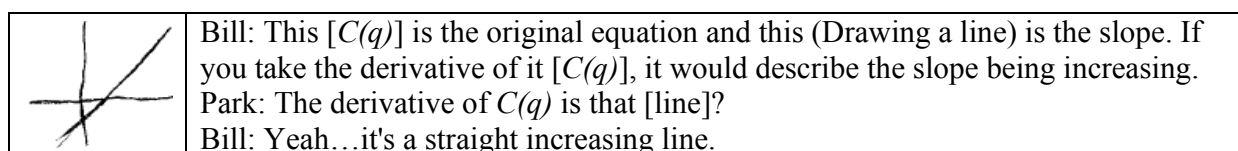


Figure 60. Bill's Graph and Explanation about the Derivative of a Function

However, he could not explain the graphing process and repeated, “I really don't know.” Then, when I asked whether the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, was closer to his explanation, he chose $f'(x)$ and said, “I don't know what the derivative at a point is.” Then, when I asked if and how $f(x)$ and $f'(x)$ are related, he said yes “because the derivative is from the original function...it tells us something about the original function. I am not sure. We really haven't gone over the meaning. We were just using it.” Then, I said that although he could not elaborate the meaning of the concept, he may still have “some image about it [the derivative]

when [he] work on it (Bill was nodding),” and asked him again to explain his “images about the derivative.” He said, “It simplifies functions... We can maybe find the variable easier... like x ” using algebraic notations. Here, he seemed to recall differentiation rules for power functions.

Zion– The Derivative as Slope and Rate of Change

Zion first explained the derivative as “the slope of the first graph” and connected it to the trend of $f(x)$: “The graph of the derivative will show you the slope of the function ... we take the derivatives for functions to figure out trends.” Then, he gave examples of “trends” in a physical situation: “An object is changing the position, and the derivative of that will be the velocity function. That would be how fast it’s going. The derivative of that is acceleration.” He identified this explanation as closer to $f'(x)$ than $f'(a)$. He explained $f'(a)$ as “[the slope] at that point, just one spot,” and compared it to $f'(x)$, which was “the slope of the function at all the point.”

Joe– The Derivative as Slope

Joe defined the derivative as “the slope of a function at a certain point.” He drew a curve and its tangent line, and said the tangent line is the slope at a point (Figure 61).

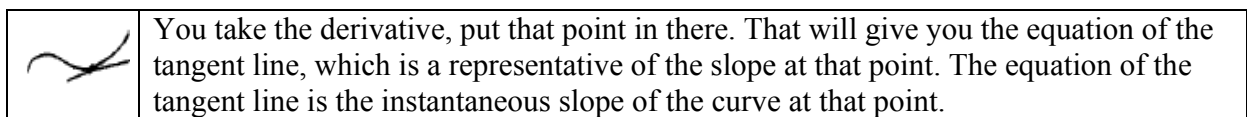


Figure 61. Joe’s Graph and Explanations of the Derivative at a Point

When I asked “The equation of the tangent line is the slope?” Joe said yes and used this mathematically incorrect idea to justify his answers in the survey problems later.

Joe also used the slope to explain the derivative of a function (Figure 62).

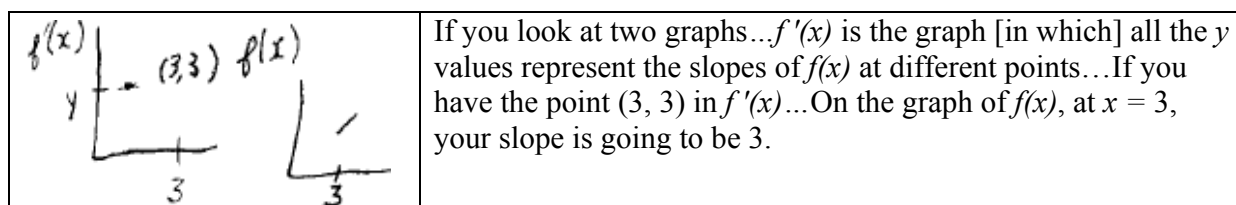


Figure 62. Joe's Graphs and Explanations of $f'(3)$ as the Slope of $f(x)$ at $x = 3$

In this excerpt, he described the derivative function, $f'(x)$, and the derivative at a point, $f'(a)$, as the slopes of $f(x)$ at all points and the slope at a point, respectively, and explained $f'(a)$ is a point on the graph of $f'(x)$. He also used this relationship to graph $f'(x)$ by “establish[ing] your slope on $f(x)$ at a point...translat[ing] the slope into a y value...of $f'(x)$,” and connecting those y values. He explained their relationship by graphing $f'(x)$ based on several values of $f'(a)$.

Cole– The Derivative as Rate of Change and Slope

Cole explained the derivative as the rate of change such as “how fast something moves,” and mentioned that this could be applied to both $f'(x)$ and $f'(a)$ by saying, “the derivative of a function can differ at different points.” He then explained their relationship: “If you take the derivative of a function, it’s gonna be a new function. If you plug in a certain point into that, it will give you the distance traveled at that time.” He also explained $f'(a)$ as “the slope of the tangent line at that point,” and added that the slope can also be used for $f'(x)$ because one “can take the derivative at multiple points and see how the slopes compare over the whole function, and [one] can...sketch the graph.” He used this argument to describe the behavior of $f(x)$:

The derivative of a function tells us the behavior of the slopes like, it’s going up or down. If it’s going up, that would mean that the derivative has a positive number. If you draw a tangent line on the original graph, and it’s going up, that would mean that the derivative at that point...would be positive as well.

Here, Cole used “a function is increasing” and “the slope is increasing” interchangeably.

Although he corrected them to “the slope is positive when the function is increasing,” he used “the slope is increasing” later again when he described an increasing function.

Ian’s Students

Three students from Ian’s class, Sara, Mary, and Mona, described the derivative as the slope of a function whereas Clio described it as a process of differentiating functions.

Clio– The Derivative as Differentiation Rules

When I asked what the derivative is, Clio said, “[It is] a way to figure out different equations to derive other equations from another using the rules...the derivative of a function [is] what the ending derivative like, what the ending result of the function is” with an example, $2x$ as the derivative of x^2 . She identified that this explanation is closer to the derivative of a function, $f'(x)$, and explained the derivative at a point, $f'(2)$, using substitution, “The derivative of x^2 is $2x$... You could plug in 2 afterwards. Then, you get 4.” Then, she confused $f'(2)$ with $f(2)$ and said another way of calculating $f'(2)$ is substituting $x = 2$ in the original function x^2 . However, she obtained different numbers for $f'(2)$ and $f(2)$ when $f(x) = x^3$, she said the same number, 4 in the previous example “might have been just coincident.” Her confusion between $f'(a)$ and $f(a)$ was also identified when I asked how $f(x)$ is related to $f'(a)$. She said, “If you had x^2 as your function, the derivative at a point could tell you what the coordinates are at the given point. That is supposed to be on the function, just plug it in.” In this excerpt, she may have used the word, “function,” for “the derivative of a function.” However, later in the interview, she tried to calculate $f'(a)$ by substituting a number in $f(x)$ again. For what $f'(x)$ tells about $f(x)$, she repeated the “way” of applying differentiation rules to $f(x)$. She consistently approached problems algebraically by finding the equations for given graphs rather than using the graphs.

Sara– The Derivative as Slope

Sara explained the derivative as “the slope of the tangent line,” said this can be applied to both the derivative at a point, $f'(a)$, and the derivative of a function, $f'(x)$. She distinguished $f'(a)$, “the slope of the tangent line at that point,” from $f'(x)$, “the derivative of a curve is like [at] all those points.” While explaining $f'(x)$, she repeated, “The more tangent lines you have, that gets like a better estimation,” twice. This statement could be interpreted in two ways: the more tangent lines, thus more values of $f'(a)$, can give a better estimation of the graph of $f'(x)$, or more secant lines can give a better estimation of the tangent line at a point. When I asked what she meant by “better estimation”, she repeated the same explanation rather than answering the question. The meaning of the statement was also not specified later; she did not explain the graphing process of $f'(x)$ further, use the terminology such as the secant lines, or draw them on the graph. She also did not use either interpretation while solving problems. About the relationship between $f'(a)$ and $f'(x)$, Sara indicated that $f'(a)$ is a value of $f'(x)$ by identifying $f'(a)$ as “one of the slope along the whole curve at that point.” When I asked what $f'(a)$ or $f'(x)$ informs about $f(x)$, she said that they informs “how the function changes over time,” and distinguished $f'(a)$ and $f'(x)$ by explaining $f'(x)$ as “how it changes at that instantaneous moment” and $f'(a)$ as “the slope so all those points and all these instantaneous times.”

Mary– The Derivative as Differentiation Rules and Slope

Mary described the derivative as “the process...the short cut...taking the exponent and bringing it to the front and subtract 1.” She said that this description was closer to the derivative function, $f'(x)$, and explained the derivative at a point, $f'(a)$ as “the slope of the function at that particular point.” She also said that $f'(x)$ and $f'(a)$ are “related” because one has “to find the derivative [function] to know what the slope is at that point.”

Mona– The Derivative as the Indicator of Function Behavior

Mona explained the derivative as “the instantaneous slope at the point,” identified this explanation as the derivative at a point, $f'(a)$. She explained the derivative function, $f'(x)$ by saying, “If you are finding the derivative of a function, you are finding pretty much the slope of the whole line.” She then stated that the derivative at several points form the derivative function, and it is another function: “You are...finding the slope at each of those points of the function and creating another graph or another function.” About the relationship between a function and its derivative, she explained the latter as the indicator of the behavior of the former by saying, “The derivative is the slope of the original function...It tells us when it [the function] is increasing and decreasing. The graph of the derivative...tells you when the original function has an increasing slope or decreasing slope, when it [the derivative] is positive and negative.” She added that the derivative at a point “gives you the maximums and minimums of the function”

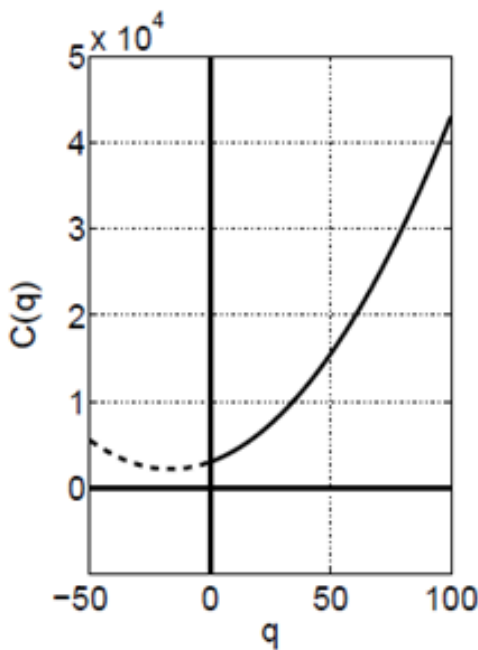
Students’ Problem Solving Processes

This section reports how students used a) the relationships between a function, the derivative function, and the derivative at a point, and b) the derivative as a function in their solution processes. How they solved each problem using these relationships and concepts will be demonstrated problem by problem. It was not plausible to organize this section by their use of each of the relationships and the concepts because a) they are interrelated and b) students often used several in one problem. For each problem, a common solution method is demonstrated, and then other methods used by each group of students, follow. These demonstrations report how students’ own concepts of the derivative and the relationships between a function, the derivative function, and the derivative at a point were used. The correct answers are included in Chapter 3.

Problem 1

Problem 1 asked students to calculate the values of a function and the derivative at a point, and interpret these values in the problem context (Figure 63).

2. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the equation is given by $C(q) = 3000 + 100q + 3q^2$, and the graph is given as follows.



- Find the value of $C(2)$.
- What is the unit of 2 in (a)?
- What is the unit of $C(2)$?
- What is the meaning of $C(2)$ in the problem context?
- Find the value of $C'(2)$.
- What is the unit for 2 in (e)?
- What is the unit of $C'(2)$?
- What is the meaning of $C'(2)$ in the problem context?

Figure 63. Problem 1

All 12 students correctly answered parts a) through e). They found $C(2) = 3212$ by substituting 2 in $C(q)$, and used the problem statement to find the units of 2 as “miles” and $C(2)$ as “dollars,” and interpret $C(2)$ as “the cost for producing 2 miles of rope” or equivalently. In part e), they found $C'(q)$ using the relationship between a function and its derivative function—differentiation rules—substituted 2 for q , and obtained $C'(2) = 112$ using another relationship that the derivative at a point, $C'(2)$, is a value of the derivative function, $C'(q)$, at $q = 2$. The students answered differently on the remaining parts. In part f), they explained whether 2 in $C(2)$ and $C'(2)$ are the same quantity with the same units or whether the two functions, $C(q)$ and $C'(q)$, have the same

independent variable. Their descriptions of the relationship between a function and its derivative, and their independent variables were explored in relation to the mathematical fact that the derivative function is another function. In parts g) and h), they interpreted the derivative at a point, $C'(2)$, in relation to the function, $C(q)$. In these parts, how they distinguished $C(2)$ from $C'(2)$ was explored as was their interpretations of $C'(2)$. The details are as follows.

Tyler's Students

Tyler's students' answers on parts f), g), and h) are shown Table 43.

Table 43. *Tyler's Students' Answers to Problem 1 Parts f), g), and h)*

Part	Interviewer's answers			
	Roy	Liz	Neal	Zack
f)	miles	miles	miles (change in mile)	miles
g)	dollars	miles/second (dollars/ time interval)	dollars (change in dollars)	dollars
h)	Cost producing 2 miles of rope after the first derivative	How fast the rope is made (how fast the cost is changing)	The change in cost of rope at 2 miles of rope	How a change in production would impact cost

Note. When they changed answers during the interview, I put them in parentheses.

Roy said that “miles” is the units of 2 “just like in b),” and both $C(q)$ and $C'(q)$ “are the functions of q .” Then, he gave “dollars” as the units of $C'(2)$ by interpreting it as “the cost of producing 2 miles of rope after you...derived $C(q)$.” He specified the meaning of “deriving $C'(q)$ ” as “[how] the cost of the factory would change or...reduce?” but could not explain further.

Liz also said that for the units of 2 in $C'(2)$ is “miles,” and the units for 2 in b) and f) are the same because “ q is always the miles of rope.” She interpreted $C'(2)$ as “a speed, how fast you are producing...You can produce like 2 miles per second” although the variable q was not given as time. When I reminded her that $C(q)$ is the cost, she changed her answer to “the change in cost over the time interval” and gave its units as “dollars/time interval.” She explained $C'(2)$ more, “the change in cost, it costs so many dollars to produce q miles of rope. Then, at another point,

the cost is less or more,” and specified “another point” as “the next time interval” not “ $q = 3$.”

Neal first gave “miles” as the unit of 2 in $C'(2)$ and changed it to “change in miles” as saying, “On the original function $[C(q)]$, you are gonna go from 500 miles of rope to 501 miles of rope...whereas in this one $[C'(q)]$, you have change of 1,” and specified 2 in $C(2)$ as “the actual number of miles of rope” and 2 in $C'(2)$ as “the change in miles.” He added that the change in q would be arbitrary as long as “2 miles is an end point.” He also changed the units for $C'(2)$ from “dollars” to “change in dollars,” and said that these two “adjustments are indiscriminant”

Zack gave “miles” to the unit of 2 in $C'(2)$ and said that it is the same as “2 miles” in $C(2)$. He interpreted $C'(2)$ as “how much more cost would be added for that one more unit...If we were to go to 3... $C'(2)$ would be the cost that we would have to add on...to get to 3” and gave “dollars” as its unit. When I asked if $C'(2)$ equals $C(3) - C(2)$, he said “yes.” However, after calculating $C(2) + C'(2) = 3324$ and $C(3) = 3327$, he said that the numbers differed “cause it $[C(q)]$ is curved.”

Alan’s students

Alan’s students’ answers on the parts f), g), and h) are shown Table 44.

Table 44. *Alan’s Students’ Answers to Problem 1 Parts f), g), and h)*

Part	Interviewer’s answers			
	Bill	Zion	Joe	Cole
f)	miles	miles	miles	miles
g)	dollars/mile	dollars/mile	dollars/mile	dollars/mile
h)	How much money a rope cost per mile	Rate of cost	When producing 2 miles of rope, it costs \$112/mile	The rate of price change of 2 miles of rope

Three students, all but Zion, said that 2 in $C(2)$ and 2 in $C'(2)$ are the same, or $C(q)$ and $C'(q)$ have the same variable. For the units of 2 in $C'(2)$, Zion “was debating between ‘rate of miles’ and ‘miles’...and not confident” about a function and its derivative function— $C(q)$ and $C'(q)$ —having the same variable. To part h), all students wrote “the rate of change” or equivalent

answers but interpreted them differently. Bill interpreted $C'(2)$ as “how much money it would cost per mile...at 2 miles,” but could not explain how it is different from $C(2)$ except saying, “it [$C'(2)$] is per miles.” The other three students correctly explained $C'(2)$ in relation to $C(q)$ and the differences between $C'(2)$ and $C(2)$. For example, Zion said, “It’s not the cost. It’s...how fast the cost is gonna go up.” Cole added that $C'(2)$ is “the cost to make another mile of rope,” but the next point “doesn't have to be 3. It can be any number greater than 2.”

Ian’s Students

Ian’s students’ answers on the parts f), g), and h) are shown Table 45.

Table 45. *Ian’s Students’ Answers to Problem 1 Parts f), g), and h)*

Part	Interviewer’s answers			
	Clio	Sara	Mary	Mona
f)	Miles	Miles	miles	miles
g)	dollars/mile	dollars ² (dollars)	dollars	dollars
h)	How many dollars/mile, there are in 2 miles of rope	If two more miles are used how does the cost in dollars change (how the money changes when you go up a mile in rope)	Marginal cost (cost to produce the last mile of rope)	Minimum cost of setting up a company and producing 2 miles of rope (Slope of $C(q)$ at 2)

For part f), they all gave ‘miles.’ Clio, Mary, and Mona said that $C(q)$ and $C'(q)$ have the independent variable, q . Mary said, “You’re still finding it [$C'(q)$] in relation to miles of rope. It’s the same values for both functions.” Mona and Sara gave equivalent answers.

Clio gave “dollars/mile” as the units of $C'(2)$ by saying, “Since you are doing the derivative... You’d have to divide by whatever the x is,” and interpreted $C'(2)$ as “how many dollars per mile there are in 2 miles of rope.” However, she changed it, saying, “It’d [$C'(2)$] just be the cost of the 2 miles of rope,” which was the same as her interpretation of $C(2)$. Then, she again changed it to “how much it costs per mile for the 2 miles.” When I asked if $C'(2)$ equals $C(2)/2$, she said yes but could not explain the difference between $C'(2) = 112$ and $C(2)/2 = 1606$.

Sara first gave “dollar²” as the units of $C'(2)$ because she “was thinking about anti-derivative” and changed it to “dollars,” because $C'(2)$ is “the amount of money it takes to produce...2 miles of rope.” When I asked if $C(2)$ and $C'(2)$ are the same, she changed it to “ $C'(2)$,” the units are still dollars. But it's talking about...slope...how the money changes...at 2. If you go up to $q = 3$, the y value would change by 112.” She interpreted $C'(2)$ as the slope and the increase in y values without recognizing its units correctly.

Mary gave “dollars” as the units of $C'(2)$ by interpreting it as “the marginal cost” and a change in the cost between $C(1)$ and $C(2)$: “dollars...to produce one more mile of rope from 1,” When I asked if $C'(2)$ equals $C(2) - C(1)$, she said yes without calculating the values. Mona also gave “dollars” as the units of $C'(2)$ but interpreted it as “minimum cost of setting up a company and producing 2 miles of rope” because “taking the derivatives... gives you max or min.” Then, she mentioned differentiation rules but could not make use of it in explaining $C'(2)$. When I reminded her that she said that the derivative is the slope earlier, she said that $C'(2)$ is “the slope of the original function at 2” and is “not the minimum cost...because [it] will be when the derivative is equal to zero.” However, she could not interpret “the slope” in the problem context.

Problem 2

Problems 2 asked students to find $f'(2)$ when the equation $f'(x)$ is given (Figure 64).

2. The derivative of a function f , is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

Figure 64. Problem 2

This problem was designed to explore students’ uses of the relationship between the derivative at a point and the derivative function—the former as a point-specific value of the latter— in an algebraic situation. In this problem, all 12 students gave the correct answer, -4 by substituting 2 in $f'(x)$. Additionally, I asked students, “What does the number, -4 tell about $f(x)$?” to explore

how they describe the relationship between the derivative at a point and the original function.

Students gave different descriptions of $f'(2)$ in relation to $f(x)$ as follows.

Tyler's Students

Tyler's students—Liz, Neal, and Zack—correctly interpreted $f'(2) = -4$ as the rate of change. Liz interpreted it as “the velocity at 2” and Neal and Zack as “the slope of the function at $x = 2$.” Neal also explained the behavior of a function, “If you have a negative slope, then the line is heading downwards.” In contrast, Roy said, “-4 tells us the tangent line” because “tangent lines are increasing/decreasing” as the function increases/decreases.

Alan's Students

Three of Alan's students, all but Bill, interpreted $f'(2) = -4$ as “the slope of $g(x)$ at $x = 2$,” and Cole further explained, “ $g(x)$ is going down at $x = 2$.” Bill said, “It is the value of a certain x and the derivative. I don't think we can relate that to the original equation.”

Ian's Students

Sara, Mona, and Mary correctly interpreted $f'(2) = -4$ as the slope. Sara visualized the slope, 4, with the tangent line to a curve. Later, she changed 4 to -4 (Figure 65).

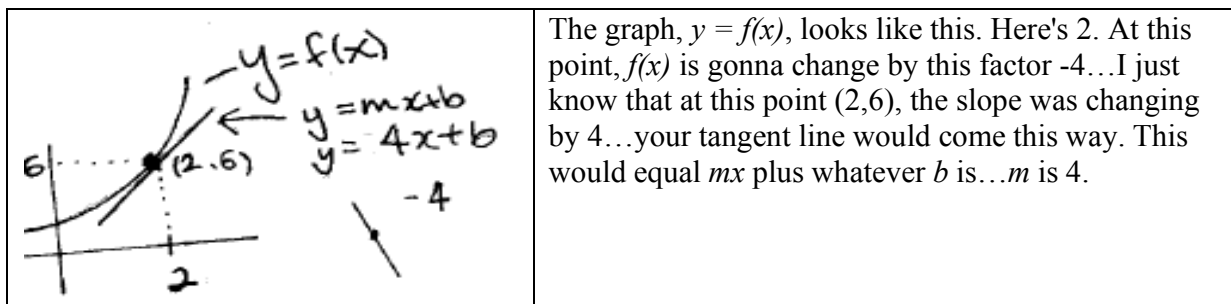


Figure 65. Sara's Drawing and Explanation about the Derivative at a Point

Mona also mentioned, “The function at 2 has a negative slope, -4, so it is decreasing.” Mary also interpreted $f'(2) = -4$ as the slope at $x = 2$ but said that $f'(2)$ only informs the sign of the slope: “It's just the negative slope, not necessarily -4”. When I asked Clio what $f'(2)$ informs about $f(x)$,

she explained how to calculate $f(2)$ from $f'(2)$: “Take the anti-derivative of -4 to get back to the original function...That would be $-4x$. If you plug in 2 , you get y .” Her explanation implies that she sees $f'(2)$ as a constant function not as a value.

Problem 3

Problems 3 asked students to find $g'(2)$ when the graph of $g'(x)$ is given (Figure 66).

The graph of the derivative of function g , $g'(x)$ is given as follows. What is the value of $g'(2)$?

- a) -10
- b) -4
- c) 0
- d) 4
- e) 10

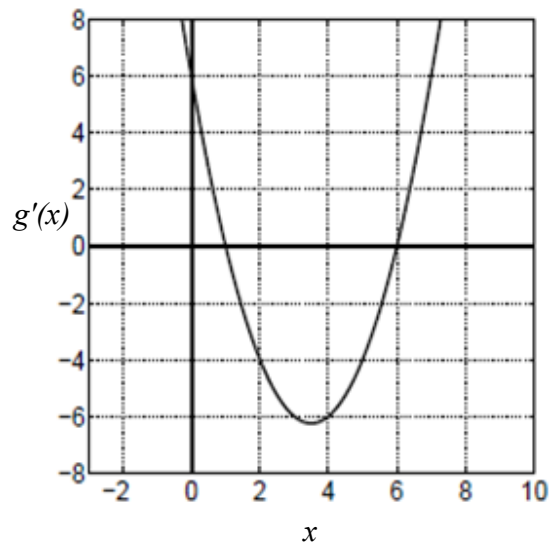


Figure 66. Problem 3

This problem is identical to Problem 2, but given in a graphical situation. All 12 students chose the correct answer, b), by reading the y value of $g'(x)$ at $x = 2$ based on the relationship that $g'(2)$ is the value of $g'(x)$ at $x = 2$. I also asked students the same question about what the number, -4 informs about the function, $g(x)$. Roy, Tyler’s student, could not give any explanation, and 11 other students said that the same explanation they gave in Problem 2 would be applied. Clio, from Ian’s class, gave the second solution; she found the equation, x^2 , for the graph, substituted 2 , and obtained 4 . However, she chose -4 as her final answer by saying “I got positive 4 because the graph is moved to the right and down [from the graph of x^2]...If I did that, that [answer] would

be different.” This solution shows her inability to represent graphs of functions using equations.

Problem 4

Problem 4 asked students to graph $f'(x)$ when the graph of $f(x)$ is given (Figure 67).

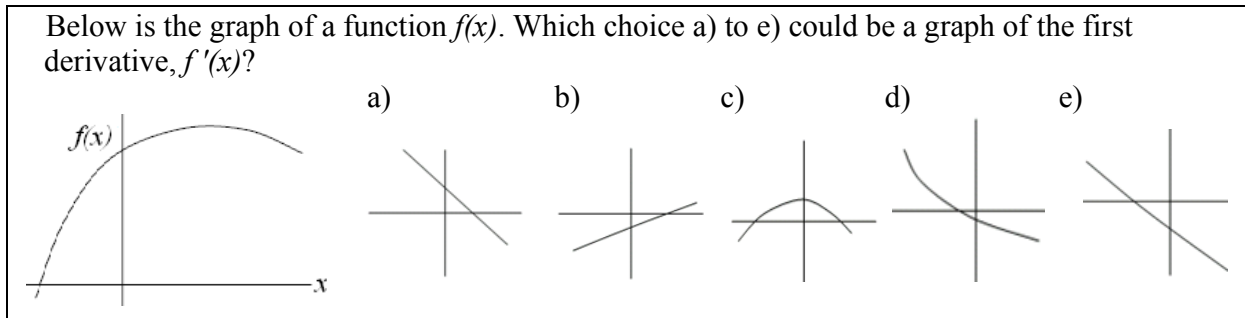


Figure 67. Problem 4

This problem was designed to explore students’ use of the relationship between a function and the derivative. More specifically, it investigated their uses of the derivative function or the derivative at a point in the process of graphing the derivative of a function when the original function is given as a graph. Their graphing processes showed whether they mainly used the derivative at several points, for example, slopes of the tangent lines at those points, or the behavior of the original function, $f(x)$ over interval to construct the derivative function. Of 12 interview participants, 11 chose a) and one f), originally, but one student changed his answer from a) to c). Their justifications for these answers are as follows.

Tyler’s Students

All interviewees from Tyler’s class chose a), but Roy changed it to c) during the interview. He interpreted $f'(x)$ as “contraction or extension” of $f(x)$, which is “just getting bigger keeping a lot of the same properties covering a greater...or...smaller amount of space.” He specified “properties” as “the way it curves.” He said, “c) also curves just on a smaller scale [than $f(x)$].”

The other three students chose a) based on the behavior of $f(x)$. Liz used the concavity of

$f(x)$ by saying, “The graph, $f(x)$, is concave down. So, the derivative, the slope would be going down.” When asked for the reason, she used velocity as an example, “You are thinking of a car moving. If it’s concave down, ...it’s decelerating...the velocity must also be decreasing.” Here, she used the relationship that $f(x)$ is concave down when $f'(x)$ decreases, by interpreting it as velocity. When asked why d) or e) was not the answer, she confused a value of a function, $f(a)$, with the derivative at a point, $f'(a)$, by mentioning that $f'(x)$ should have a positive y intercept, “because the graph [of $f(x)$]...passes through above the x axis. In d) and e), it [$f'(x)$] passes through the below the x -axis.” She also said that graphs of derivative functions are linear: “All the examples that we looked at for the derivative functions...were a straight line...or zigzag. I don’t think it curves.” These two explanations are incorrect. Since she drew a sign diagram during the survey, I asked her to explain the diagram. She connected the + to – signs to the behavior of $f(x)$ (Figure 68).

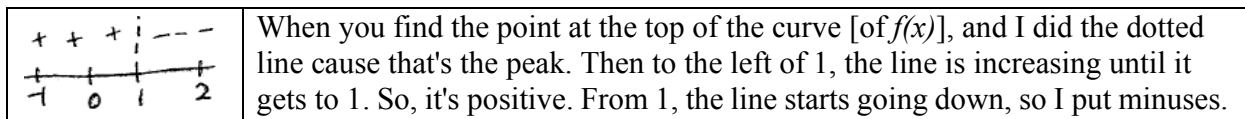


Figure 68. Liz’s Diagram and Explanation about $f(x)$ of Problem 4

In this excerpt, she did not identify that the + and - symbols are the signs of the derivative, or the derivative is zero at the point where $f(x)$ has a maximum. However, when asked to explain the meaning of the signs, she interpreted them as directions, “Like positive speed. If you are driving a car, you are moving forward. If you are moving back, it’s negative.”

Neal explained that points on the graph of $f'(x)$ show slopes of $f(x)$ (Figure 69):

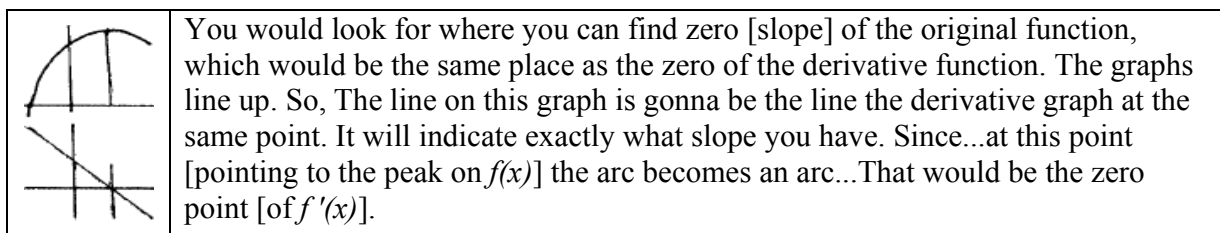


Figure 69. Neal's Drawing and Explanations of Problem 4

Neal used the relationship that $f'(a) = 0$ where $f(x)$ has a horizontal tangent at $x = a$. He also used the relationship that $f'(a)$ is a value of $f'(x)$ by placing the zero on the graph of $f'(x)$. Then, he related the behavior of $f(x)$ to the sign of $f'(x)$ by saying, "Prior to that zero, it was the positive derivative function, and the function was increasing. So that matches up. Beyond zero, it becomes negative and the function was decreasing at that point." Zack applied the same relationships but in a different order: he connected the sign of $f'(x)$ to the behavior of $f(x)$, identified extreme values of $f(x)$ and matched them to x intercepts of $f'(x)$.

Alan's Students

All students from Alan's class chose a) by interpreting $f'(x)$ and $f'(a)$ as the slope of $f(x)$ on an interval and at a point, respectively. For example, Bill compared the graph in a) with the sign of the slope of $f(x)$: "For the graph of the derivative, I chose a) because this has a decreasing slope. And where the slope goes from positive to negative (pointing to the peak of $f(x)$) is where a) crosses the x axis at 0." He further explained "a decreasing slope" by describing the changes in the slope over the interval, "It's getting less and less positive until it reaches negative." Here, he applied the relationship between $f(x)$ and $f'(x)$: $f'(x)$ is positive/negative when $f(x)$ increases/decreases. He also used the relationship between $f(x)$ and $f'(a)$ by matching the maximum of $f(x)$ to the x intercept of $f'(x)$. By describing the change in the slope continuously on the interval instead of calculating slopes at discrete points, he implicitly addressed $f'(x)$ as another function. Other students also graphed $f'(x)$ using the slope of $f(x)$. Like Bill, Cody and Zion

started on the left side of the interval and matched the change in the slopes with the graph in a), whereas Joe identified the point where $f'(x) = 0$, and then checked the slope on both sides of the point.

Ian's Students

Clio, Mary and Mona chose a), and Sara chose f). Mary and Mona described the slope of $f(x)$ on the interval and connected it to the sign $f'(x)$ in a). For example, Mona said, "The slope of the function... starts off positive, so the derivative has to be above the x axis ... it's getting zero slope, so the derivative becomes zero... $f(x)$ starts getting a negative slope, so the derivative has to go below the x axis." Here, she applied the relationships that $f'(x)$ is positive/negative when $f(x)$ increases/decreases and that $f'(x) = 0$ where $f(x)$ has a maximum. Mary gave an equivalent explanation. Mary and Mona said that they chose a straight line because $f(x)$ looks like a quadratic function, but $f'(x)$ does not have to be a straight line.

Clio supported a) by identifying the graph of $f(x)$ as "a parabola" with an "even" exponent and saying that its derivative is "a straight line that has to be above the x axis" when x is negative. She said that "the even exponent" makes "all negative x 's... positive." However, her argument with "even exponents" contradicts her choice a) in which $f'(x)$ of positive x values are negative. Moreover, she used the values of $f(x)$ as if they were those of $f'(x)$. These explanations show her inability to model a graph as an equation. She also incorrectly interpreted the tangent line at $x = a$ as "the derivative of a function at a point," and said, using those "tangent lines," one could create the graph of $f(x)$ but not $f'(x)$. Here, she recognized the derivative at a point as a linear function rather than a value.

Sara incorrectly used $-x^2$ as the equation of the graph of $f(x)$, found its derivative, $-2x$, and chose f) 'none of these' by saying, "It has to go through the origin." She also said that although

$f(x)$ had a y intercept, it would become zero while differentiating it. When I asked her if she could use her previous interpretation of the derivative as the slope in this problem, she said no because she did “not understand enough to use it”.

Problem 5

Problem 5 asked students to graph the original function given the graph of its derivative function (Figure 70).

Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?

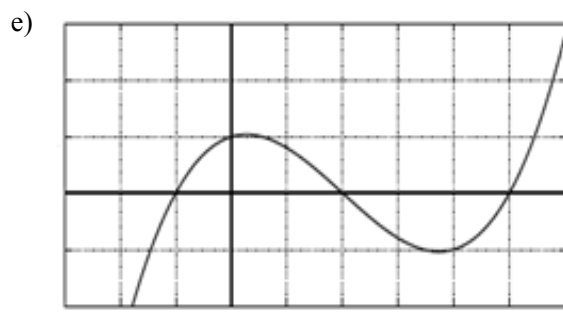
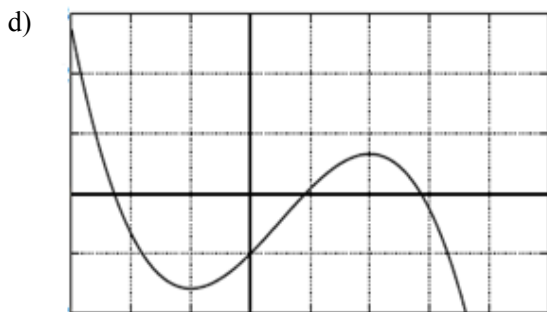
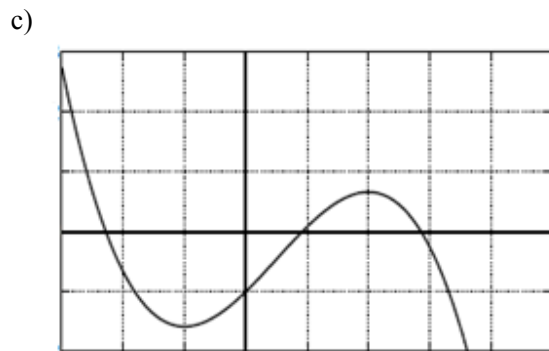
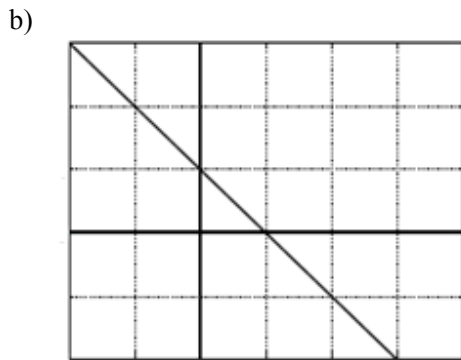
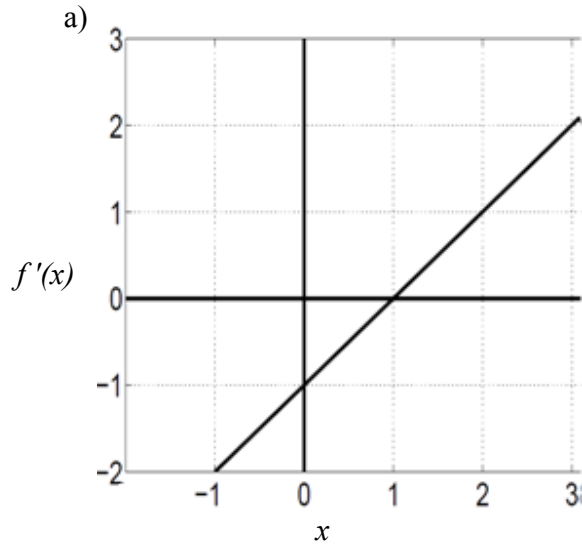
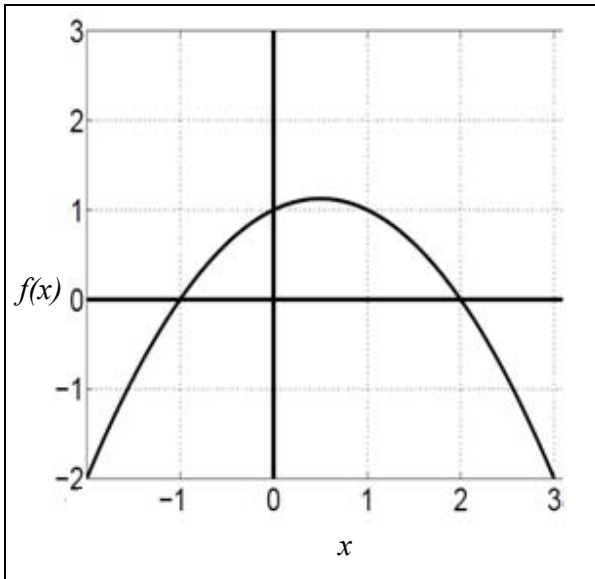


Figure 70. Problem 5

This problem explored students' use of the same relationship in Problem 2 in a graphical situation when the graph of the derivative of a function was given instead of the graph of the original function. In other words, students are supposed to create the graph of the original function based on its relationship to the derivative function. Students' graphing procedures in this problem provided a good comparison to those in the previous problem in terms of how they used the given information about the derivative function or the derivative at a point to graph the original function. During the survey, nine students chose c), two f), and one e). During the interview, nine chose c), one f), and two d) since some students changed the answers. Three students, Liz from Tyler's class, Mary and Mona from Ian's class misread the problem and found the graph of the derivative of $f'(x)$ on the survey, but solved it again during the interview.

Tyler's Students

Three interviewees, all but Roy, chose c). Roy first chose e) because $f'(x)$ is "contraction or expansion" of $f(x)$, and "they have the similar traits." Then, he changed it to d) because e) "is almost the same" to $f'(x)$ but "the first picture [$f'(x)$] could be a contraction of d)." Here, he argued that graphs of $f(x)$ and $f'(x)$ look similar not the same. Although he mentioned that c) could be the answer because it "has a similar way of curving," he finally chose d) because it "looks more like an expansion."

Liz chose c) based on the position of the inflection point of $f(x)$: "On $f'(x)$, the point at which the velocity goes from increasing to decreasing is at 0.5. That would coincide with the inflection point in c)." Here, she correctly matched the inflection point of $f(x)$ with the point where $f'(x)$ changes its behavior from increasing to decreasing. She also matched the local extreme values of $f(x)$ with the x intercept of $f'(x)$ by saying, " $f'(x)$ graph passes the x axis at -1 and at 2, and in c), the local max and local min at those points." However, she could not explain

the relationship between the behavior of $f(x)$ and $f'(x)$ in c) on the other part of the interval. Here, she used point-specific property of the derivative without recognizing $f'(x)$ as an indicator of the behavior of $f(x)$ on an interval.

Neal and Zack chose c) by interpreting $f'(x)$ as the slope of $f(x)$. Neal identified zeros of $f'(x)$ at -1 and at 2 and interpreted them as “the points where the slope changes.” He then matched the signs of $f'(x)$ near those points with the behavior of $f(x)$: “At -1 on c), it changes from a negative slope to a positive slope. It makes sense because the derivative was negative before the zero, and positive after, decreasing [and] increasing relationship there...at 2, exact same relationship, just opposite.” Here, he used $f'(x)$ and $f'(a)$ as the indicator of the behavior of $f(x)$ on an interval and at a point, respectively. Zack used the same relationships between $f(x)$ and $f'(x)$, between $f(x)$ and $f'(a)$, but in a different order.

Alan's Students

Zion, Joe, and Cole chose c), whereas Bill chose d). The three students justified c) based on the two relationships among a function, $f(x)$, the derivative function, $f'(x)$, and the derivative at a point, $f'(a)$: $f(x)$ increases/decreases when $f'(x)$ is positive/negative, and $f'(a) = 0$ when $f(x)$ has extreme values at $x = a$. For example, Cole first explored the sign of $f'(x)$ to find the behavior of $f(x)$: “The derivative is negative from negative infinity all the way up to -1, and positive from -1 to 2...The original function would be decreasing and then increasing.” He ruled out d) and e) “because they start off increasing, and we want to start off decreasing”. Then, he compared zeros in $f'(x)$ with the “peaks” in $f(x)$ and signs of $f'(x)$ with the behavior of $f(x)$: “It would be c)...The original function has a valley at $x = -1$ which corresponds to [where] $f'(x)$...crosses the x axis. Then, the original function increases again, because the derivative is positive so the slope is positive.” He gave the same explanation around $x = 2$. Joe and Zion gave similar explanations.

Bill chose c), and changed it to d) without explaining why. He justified d) by matching its behavior with the behavior of $f'(x)$: “This [$f'(x)$]...is increasing less and less. And it switches to positive and increases...[and] d) is increasing and getting less and less. At 0.5, it [$f'(x)$] switches to decreasing. It gets more and more negative.” Using the same argument, he said e) could also be the answer because maximum values of $f(x)$ and $f'(x)$: “e) fits...because it does not go above 1”. Therefore, he incorrectly solved problem 5, based on the similarity in the graphs of $f(x)$ and $f'(x)$ including their extreme values.

Ian's Students

Clio, Mary, and Mona chose c), and Sara chose f). Mary and Mona supported c) by interpreting the sign of $f'(x)$ as the sign of the slope of $f(x)$. For example, Mary said, “From -2 to -1, the derivative is negative, so the slope is negative...The slope should be 0 at -1 because the derivative...is 0 at -1, which means the slope of the original graph would be zero.” She also compared the sign of $f'(x)$ and slope of $f(x)$ in c) on the other part of the interval. Here, she used two relationships: when $f'(x)$ is positive/negative, $f(x)$ increases/decreases; $f'(a) = 0$ when $f(x)$ has an extreme value at $x = a$.

Clio and Sara solved problem 5 by finding the equation for the graph of $f'(x)$. Clio described the graph as “a negative parabola, so it'd be $-x^2$,” integrated it, and obtained $-x^3/3$. She chose c) because “it would be going...downwards slope rather than upward” due to $-1/3$. Although she integrated x^2 correctly, her explanation was not applicable to all the quadratic functions, and her example, $-x^2$, did not model the graph of $f'(x)$ correctly.

Sara found an incorrect equation, x^2 , for $f'(x)$, integrated it, and obtained $f(x) = 1/3x^3$ without adding a constant. She said none of the graphs looks like $y = 1/3x^3$. Later, she said that

the y intercept of $f(x)$ would be negative because that of $f'(x)$ is negative; this explanation implies her confusion between the values of $f(x)$ and $f'(x)$.

Problem 6

Problem 6 asks about the behavior of the derivative of a function when the original function is given as positive (Figure 71).

<p>If a function is always positive, then what must be true about its derivative function?</p> <p>a) The derivative function is always positive.</p> <p>b) The derivative function is never negative.</p> <p>c) The derivative function is increasing.</p> <p>d) The derivative function is decreasing.</p> <p>e) You can't conclude anything the derivative function.</p> <p>Why? _____.</p>

Figure 71. Problem 6

This problem was designed to explore students' uses of the relationship between a function and its derivative function to find the characteristics of the derivative of a positive function. Some students used the relationship correctly and answered that the sign of the original function could not inform about the behavior of its derivative function. Some students, however, described and used incorrect relationships. On the survey, nine students chose e), two c), and one a). During the interview, 10 students chose e), one c), and one a) since some students changed their answers.

Tyler's Students

Tyler's three students—Liz, Neal, and Zack—chose e). Roy chose c). He tried to use his description of $f'(x)$ as a contraction or expansion of $f(x)$ (Figure 72):

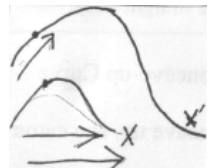
	<p>I am thinking...just the same way as contraction and expansion...cause this line is moving this way (Drawing a curve)...And so, the derivative doesn't change that the line is continuing to move that way (drawing another curve below). At this point (plotting a point on the upper curve), it's moving up, and increasing (plotting a point on the curve below).</p>
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Figure 72. Roy's Graphs and Explanations about Contraction and Expansion

Here, he said that $f(x)$ and $f'(x)$ move the same direction, which was not always true on his

graphs, and stated, “If a function is increasing, the derivative is increasing, too.” He, however, said that he could not use his contraction and expansion idea to support the statement because the idea “doesn’t involve whether the derivative...or the function is positive or negative.” However, he said, “c)...is the way that has stuck in my head.”

Liz drew a concave-up curve and its derivative function to support c) (Figure 73).

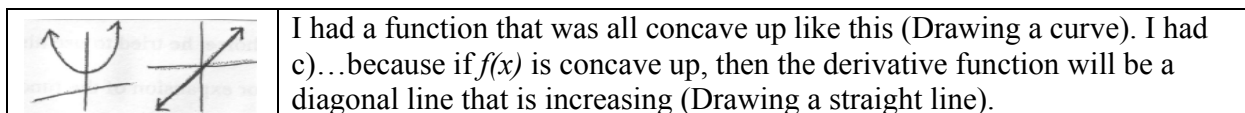


Figure 73. Liz’s Drawing and Explanation about a Concave-up Curve

When I asked if a function that is positive is always concave up, she came up with constant functions as other examples and confirmed e) as the answer. While explaining that the derivative of a constant function is zero, she said that she “won’t really be able to draw a line cause its derivative is zero”. Therefore, she used the relationship between a function and the derivative of a function algebraically applying the differentiation rule for a constant function, but did not see the derivative function, $y = 0$, as another function.

Neal and Zack chose e) by mentioning that a function being positive does not inform its behavior, thus it cannot inform the behavior of the derivative function. For example, Neal drew a positive function that behaves randomly (Figure 74).

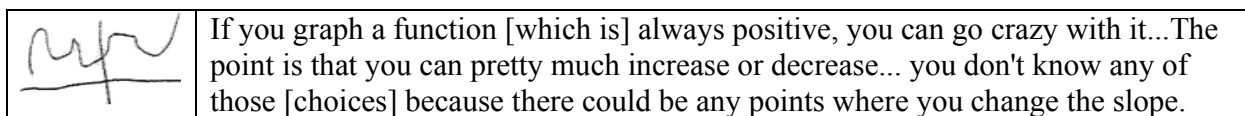


Figure 74. Neal’s Drawing and Explanations of Problem 6

Here, he used the relationship between a function, $f(x)$, and its derivative function, $f'(x)$, to determine the behavior of $f'(x)$ when $f(x)$ is positive. Zack gave a similar explanation with a similar graph. It should be noted that Neal could not apply this relationship to the first and the second derivatives: when $f'(x)$ increases, $f''(x)$ is positive. When I asked what choice c) ‘the

derivative function is increasing' informs about $f(x)$, he first said that $f''(x)$ is the derivative of $f'(x)$, and "You could not tell if the second derivative is positive or the graph of $f(x)$ is concave up. I always use double derivative to determine."

Alan's Students

Alan's three students—Zion, Joe, and Cole—chose e). Bill originally chose c), changed it to a) and then to e). Without explaining c), he justified a) by saying, "If a function is always positive, then the derivative would be always positive...because the derivative is describing the original function." When I asked what aspect of the original function the derivative function describes, he said, "How it increases and decreases." When I asked how this answer related to a function being "positive" he could not answer. Then, he changed the answer to e) and justified it based on the derivative as the slope:

The function can be positive even if the derivative is negative when the slope turns down.

So I guess a) and b) won't be true...c) and d) are not necessarily true...Like here [the graph of $f'(x)$ in problem 5] the derivative can be decreasing or increasing.

In this problem, Bill stated an incorrect relationship that the signs of $f(x)$ and $f'(x)$ were the same, and changed it to a correct relationship in which $f'(x)$ describes the behavior of $f(x)$.

Zion, Joe, and Cole used the derivative as the slope to justify e), by saying that $f(x)$ being positive does not inform anything about $f'(x)$. For example, Zion graphed a function that is always positive and its derivative function, and ruled out the choices (Figure 75):

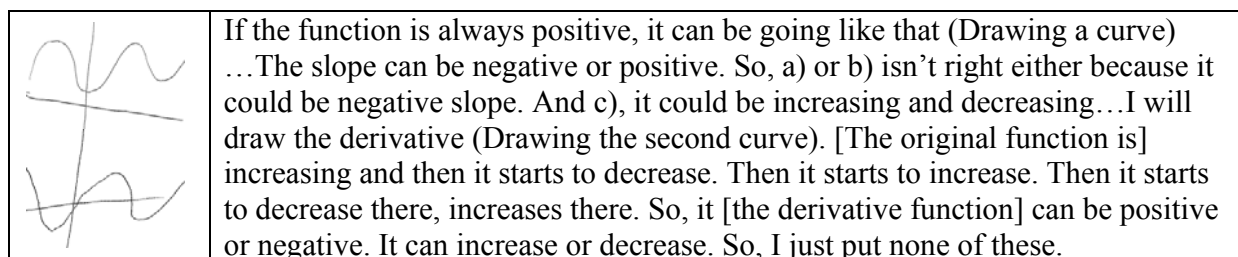


Figure 75. Zion's Graphs and Explanations of Problem 6

In this excerpt, Zion used the relationship that $f'(x)$ is positive/negative if $f(x)$ increases/decreases by interpreting $f'(x)$ as the slope of $f(x)$. Joe and Cole used the same method. It should be noted that Joe incorrectly used the words, “slope” and “derivative,” twice; when he described the behavior of $f(x)$ where it increases, he said that “the derivative” or “slope” increases instead of positive (negative when it decreases). Since he previously said that the tangent line is ‘the representative of the slope,’ he may have considered ‘the slope is increasing’ and ‘the tangent line is increasing’ as interchangeable. When I asked if “the slope is decreasing” and “the function is decreasing’ are the same, he admitted the first statement was wrong and changed it to “the slope is negative.” However, his ambiguous uses of “slope” and “derivative” were identified later again. It seems that he has a strong connection between the derivative at a point, $f'(a)$, and the slope at the point, but a weak concept of the derivative function, $f'(x)$, as the slope at any point on the domain. His word use shows his inconsistent application of $f'(x)$ as the slope when explaining the behavior of the slope over the domain. This may imply his point-specific concept of the derivative or his inability to appreciate $f'(x)$ as a function and $f'(a)$ as a number.

Ian’s Students

Clio chose a), and Sara, Mary, and Mona chose e). They supported e) by using the derivative function, $f'(x)$, as the slope of the function, $f(x)$. Mary said, “If the values of the function are always positive, the derivative function wouldn’t necessarily be increasing or decreasing...[or] positive or negative because it doesn’t say anything about the function’s slope.”

Like Joe, she stated, “If the values [of $f(x)$] are always increasing...the derivative is increasing.”

Although she accepted this statement as wrong, it was identified again later.

Mona gave a similar explanation but she explicitly stated the relationships that the derivative of an increasing/decreasing function is positive/negative, and the derivative of a concave up/down

curve increases/decreases. Sara also used the derivative as slope and a differentiation rule. She graphed a quadratic function for a positive function and ruled out a) and b). She then drew a constant function and ruled out c) and d) (Figure 76).



	<p>If the function is above the x-axis...it would be like quadratic. From this example, the derivative could be negative (drawing a tangent line). So, a) and b) are out.</p>
	<p>The derivative, you can't say it's always increasing because you could have a function like...$f(x) = 6$...[and] its derivative would be zero...it's not increasing or decreasing.</p>

Figure 76. Sara's Graphs and Explanation of Problem 6

She then said, "It must have some kind of relationship... They [choices] just do not work."

Clio supported a) by finding an algebraic expression for a positive function and differentiating it. She first wrote x^2 and found its derivative, $2x$. When I asked if " $2x$ " is always positive, she said, "Maybe...not in all cases, out of couple, that would be positive." Then, she changed the example to x^3 , found its derivative, $3x^2$, and said, "That would be still squared, so, that would make it actually positive." However, x^3 is not always positive.

Problem 7

Problem 7 asks about the behavior of $f(x)$ when $f'(x)$ is negative (Figure 77).

<p>The derivative of a function, $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?</p> <p>a) The function $f(x)$ is positive on this interval. b) The function $f(x)$ is negative on this interval. c) The maximum value of the function $f(x)$ over the interval occurs at $x=2$. d) The maximum value of the function $f(x)$ over the interval occurs at $x=3$. e) We cannot tell any of the above.</p>
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Figure 77. Problem 7

This problem explored students' use of the same relationship in Problem 6 when a condition for the derivative of a function on a specific interval was given instead of a condition for the original function. Some students used the relationship correctly to describe the behavior of a function

whose derivative function is negative on the given interval. Some students, however, described and used incorrect relationships which were often inconsistent with their explanations in the previous problem. In this problem, all 12 students chose c), but one student changed it to e) during the interview. The details of their explanations are as follows.

Tyler's Students

Three of the Tyler' students chose c), and Roy changed his answer several times. Roy chose c) based on the change of the derivative function that he graphed (Figure 78).

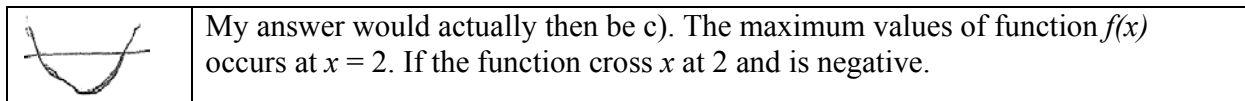


Figure 78. Roy's Graph and Explanation for Part c) of Problem 7

He then changed it to b) "because...the function $f(x)$ is negative on that same interval" where $f'(x)$ is negative. He, however, changed the answer again to e) "We cannot tell any of the above" by saying, "That would be something based on the expansion and contraction idea. That would be like case by case." When I asked him what we can say about $f(x)$ when $f'(x)$ increases, he said, "it would still be e)." Here, he did not use the two statements, " $f'(x)$ is a contraction or expansion of $f(x)$ " and " $f(x)$ and $f'(x)$ go the same direction" interchangeably like he did in problem 6.

Liz also drew a concave-up curve for the derivative function (Figure 79).

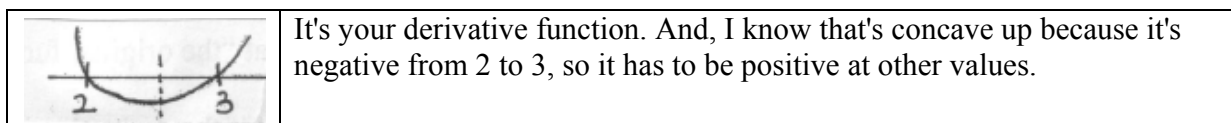


Figure 79. Liz's Graphs and Explanation of Problem 7

She then ruled out a) and b) by saying, "For a function...to be purely positive...it would have to be either a straight line like $y = 2$ or completely concave up above the x axis. That doesn't seem to correspond with the derivative given." She repeated a similar explanation for b). She then said that "the derivative equals zero" at a point where $f(x)$ has local extreme values, and

$f(x)$ has a maximum at $x = 3$ “since 3 is the point where the derivative of the function is increasing” but could not explain why. She just mentioned that it is “a rule” that she could “not quite remember why.”

When I asked her what a negative derivative informs about the original function, she said, “The original function is decreasing.” However, while explaining it, she confused negative velocity with negative acceleration by saying, “If the speed is negative, a car is slowing down.” She finally drew a sign diagram and chose c) based on the relationship that $f(x)$ increases/decreases when $f'(x)$ is positive/negative (Figure 80).

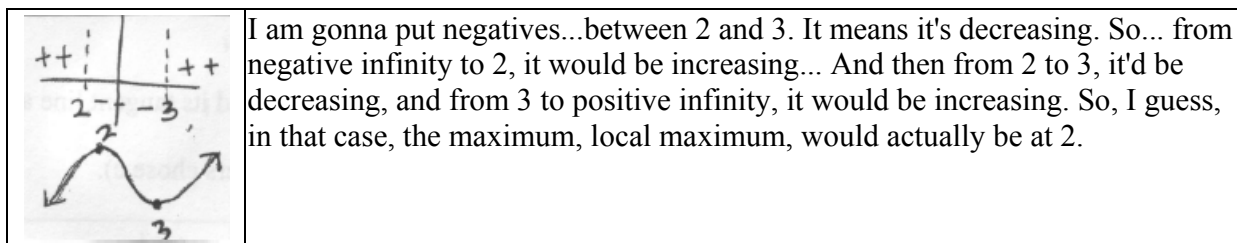


Figure 80. Liz’s Sign Diagram and Graph of Problem 7

Neal first explained that $f'(x)$ is the slope of $f(x)$ which does not inform its value, and ruled out a) and b), “The derivative of a function is negative from 2 to 3. It doesn’t necessarily tell you that the function $f(x)$ is positive on the interval. It doesn’t tell you about a) or b) because that would regard the slope.” Then, he related the derivative being negative to the behavior of the original function by saying, “If it’s negative, that only means to the original function is decreasing,” and confirmed that “the maximum value would occur at 2.” Here, he applied the relationship that $f(x)$ decreases if $f'(x)$ is negative.

Zack also gave a similar explanation using the same relationship (Figure 81):

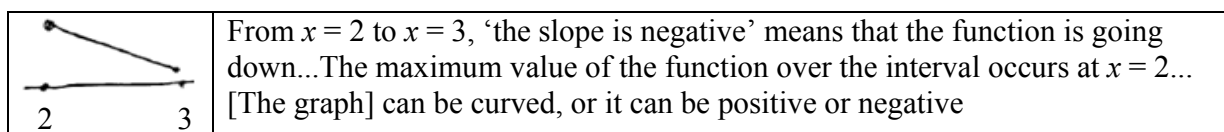


Figure 81. Zack’s Graph and Explanation of Problem 7

Alan's Students

All four students from Alan's class chose c) by interpreting the derivative of a function, $f'(x)$, as the slope of the function, $f(x)$: if $f'(x)$ is negative, $f(x)$ is decreasing, and confirmed that $f(x)$ has its maximum at $x = 2$. Figure 82 shows Cole's explanations.

	The derivative is negative, that just means that the slope is negative. That has nothing to do with whether or not $f(x)$ is positive or negative, you can rule out a) and b). It would be c) because if the derivative is negative, that would mean that the slope is negative. The highest point would have to be at 2. Because it's just decreasing from 2 to 3. So, at 3, it would be a minimum, and at 2 it would be a maximum, so c) is the right answer.
--	---

Figure 82. Cole's Graph and Explanations of Problem 7

In the excerpt, he mainly used the relationship between $f(x)$ and $f'(x)$: $f(x)$ decreases when $f'(x)$ is negative. He also added that the graph of $f(x)$ can be anywhere on the plane "as long as it's decreasing." Bill, Joe, and Zion gave equivalent explanations on this problem. It should be noted that Zion incorrectly used the two statements, "the slope is decreasing" and "the derivative function is negative," interchangeably while explaining the problem. He said, "The derivative function is negative, that means the slope is decreasing of the first one [$f(x)$]...It would be a negative slope...-4. It's going down by 4." When I asked if "the slope is negative" and "the slope is decreasing" are the same, he said, "No, not really" but could not elaborate their difference. He then said, "I am gonna stick with 'the function is decreasing' and 'the slope is negative.'" However, he used "slope" incorrectly again, which may imply his confusion between the slope at a point and the slope on an interval.

Ian's Students

All students from Ian's class chose c) based on the same reason that Alan's students provided: if the derivative of a function $f'(x)$ is negative, the function, $f(x)$ is decreasing, so the maximum of $f(x)$ happens at $x = 2$. Figure 83 shows Mary's answer.


	If the derivative of the function is negative from 2 to 3, then the slope of the function $f(x)$ is negative...the value at the $x = 2$ has to be the maximum value of that interval.
---	---

Figure 83. Mary's Graph and Explanations of Problem 7

Other students gave similar solutions and mentioned that the graph of $f(x)$ can be curved as long as it decreases. While explaining, Sara and Mona said, "the slope is decreasing" instead of "the derivative function is negative" for the part where $f(x)$ decreases. When I asked if the statements are the same, Sara said yes without further explanation. Mona said no and explained why with an increasing curve whose derivative decreases (Figure 84):


	Technically not because you could have a really fast increasing slope (drawing a curve and drawing a tangent line on the right side), and then you can say the slope is decreasing (drawing another tangent line). So, it means that the slope is negative.
---	---

Figure 84. Mona's Graph and Explanation about a Derivative that Decreases

Here, she used the relationship that the derivative of a concave-down curve is decreasing.

Problem 8

Problem 8 asks about the relationship between a function and its tangent line at a point (Figure 85).

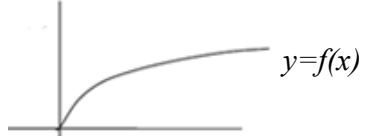
<p>9. The tangent line to this graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?</p> <p>a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$ e) None of these</p>	
--	---

Figure 85. Problem 8

As discussed in Chapter 3, this problem did not involve the concept of the derivative, and thus students can solve the problem without using their knowledge of the derivative. This problem, however, was added because the derivative at a point can be interpreted as the slope of the tangent line of the original function at the point and because one of the students' incorrect

descriptions of the derivative at a point in my earlier study (Park, 2008) was the tangent line. Some students' incorrect descriptions and uses of the relationship between a function and the derivative at a point were found in this problem. On the survey, 10 students chose c), one chose e), and one chose a). During the interview, one student, who originally chose a), changed the answer to c). The details of their justifications on these choices are as follows.

Tyler's Students

Of the four students from Tyler's class, Roy, Neal, and Zack chose c) and justified it based on the positions of the graphs of $f(x)$ and the tangent line at $x = 1$; they mentioned that the tangent line is always above the graph of $f(x)$. Figure 86 shows Zack's answer:

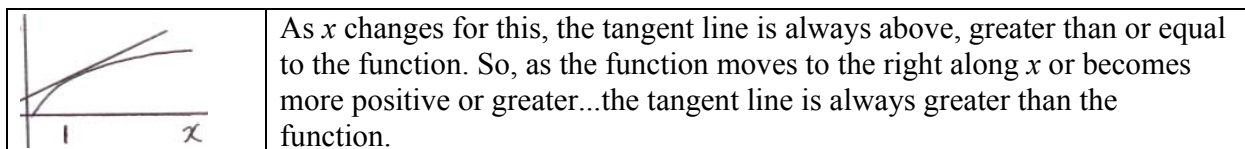


Figure 86. Zack's Drawing and Explanation of Problem 8

In contrast, Liz tried to use the derivative to support e). First, she argued that the anti-derivative of the equation of the tangent line is the original function (Figure 87):

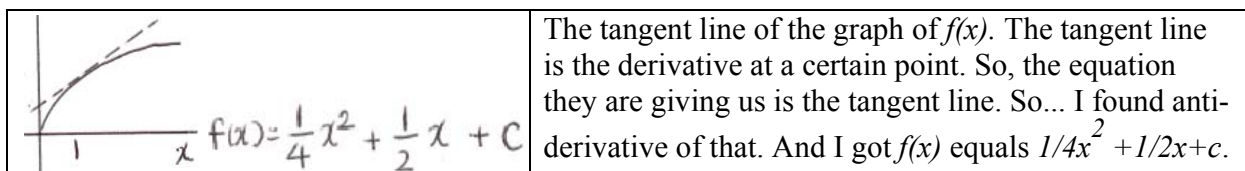


Figure 87. Liz's Graph, Equation, and Explanation of Problem 8

Using the equation, $f(x) = 1/4x^2 + 1/2x + c$, she ruled out a) and d) by stating that it is not the same as the equations in a) and d). For b) and c), she then evaluated $f(x) = 1/4x^2 + 1/2x + c$ at $x = 1$, $1/2 \times 1 + 1/2 \times 1 + c$, and the expression b) and c) at $x = 1$, $1/2 \times 1 + 1/2 = 1$, and said, "I couldn't figure out whether it was greater than...or less than or equal to 1 because didn't know what c was." Then, when I reminded her that she mentioned the derivative at a point as "the

slope of a function” earlier, she said, “The tangent line gives you the slope at one point,” and incorrectly calculated the slope at $x = 1$ by substituting $x = 1$ in the equation of the tangent line. Later, she admitted that her explanation with the anti-derivative was wrong because “the tangent line at point $x=1$...is not the derivative of $f(x)$. I feel like that’s the derivative at that point” but could not explain further. However, interpreting the derivative at a point as the tangent line is also not mathematically correct.

Alan’s Students

Cole and Zion chose c) and gave the same explanations as Zack’s (Figure 86). Joe and Bill chose e). Joe interpreted $f'(1)$ as the tangent line at $x=1$ and wrote “ $f'(1) = 1/2x + 1/2$.” He said that because “it is only relevant at the point $x = 1$,” he “can’t really relate the function for a [tangent] line to the entire function of $f(x)$.” could not be compared with the graph of $f(x)$. His incorrect interpretation of $f'(1)$ as the tangent line at $x = 1$ implies his inability to conceive of the derivative at a point as a constant. Then, for the slope at $x = 1$, he found two values, 1 and $1/2$, by substituting $x = 1$ in the equation of the tangent line and reading the slope from $y = 1/2 x + 1/2$, respectively. When I asked which is correct, he chose $1/2$ but said that he was still debating between the two slopes because “the tangent line is a representative of the slope at this point.” Here, he used inconsistent relationships between $f(x)$, $f'(x)$, and $f'(1)$ by incorrectly evaluating the equation of tangent line at 1 to calculate $f'(1)$ and interpreting $f'(1)$ as the tangent line at $x = 2$ and its slope simultaneously.

Bill drew the line $y = 1/2 x + 1/2$ and compared it with the graph of $f(x)$ (Figure 88):

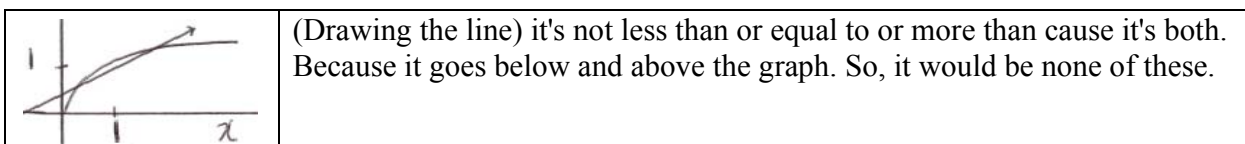


Figure 88. Bill’s Graph and Explanations of Problem 8

In Figure 88, the line he drew was not tangent to the graph of $f'(x)$. When I asked him what the tangent line means, he said, “I forgot...all I remember is tan is sin over cos.”

Ian's students

All students chose c). Three, all but Clio, gave the same explanations as Zack's (Figure 86). Clio incorrectly compared the slopes of $f(x)$ and the tangent line instead of their values: “The tangent line would be greater than $f(x)$ because... $f(x)$ has less of the slope...At the beginning, it [$f(x)$] would be faster, but overall since [the slope of] the tangent line...is always constant, that would be greater than $f(x)$.”

Problem 9

Problem 9 asked students to use the fact that the derivative at a point is the slope of the tangent line when the equation of the derivative of a function is given (Figure 89).

The derivative of a function is $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to f will be positive at $x = 0$.

- a) a and b must both be positive numbers.
- b) a must be positive, while b can be any real number.
- c) a can be any real number, while b must be positive.
- d) a and b can be any real numbers.
- e) None of these

Figure 89. Problem 9

This problem explores students' descriptions and uses of the slope of the tangent line as the derivative at a point, and its relationship to the derivative of a function. Their solution processes provided a good comparison of those in problem 2 and 3. Whereas all interviewees answered correctly in those problems, only seven out of 12 originally gave the correct answers. After interviewer's questions that reminded students that they described the derivative as slope in previous problems and prompts that the problem statement contained a specific point, three more students chose the correct answers. During the survey, seven students chose c), two a), two b),

and one d). During the interview, 10 chose c), one a), and one b).

Tyler's Students

Three students from Tyler's class, all but Roy, chose c) for two reasons: the derivative of a function, $f'(x)$, represents the slope of the function, $f(x)$, and the derivative at a point, $f'(a)$ can be evaluated from $f'(x)$. For example, to make the slope of the tangent line at $x = 0$ positive, Neal substituted $x = 0$ in the equation of $f'(x)$, obtained b , and claimed that b should be positive for the slope of tangent line at $x = 0$ to be positive:

The value of a is going to be irrelevant because x is going to equal zero... b value is going to be the only number that...affects where y will be at $x = 0$. So, a can be any real number and b must be positive...If you make zero x for $f'(x)$, [the slope of] tangent line of the function must be positive by making b positive.

Roy changed his answer between a) and c) several times. First, he chose c) by substituting $x = 0$ in $f'(x)$, "If x equals 0, then a times 0 is 0. b has to be either zero or a positive number." When asked the reason for the substitution, he said, "This is asking about the slope of the tangent line at $x = 0$, and you are using the derivative of a function," but did not explicitly state that the slope is the derivative. However, he changed it to a) by interpreting $f'(x)$ as a contraction or expansion of $f(x)$. He said, "If $f'(x)$ is a contraction or expansion of $f(x)$, a and b have to be positive...if they are positive, that will keep the function increasing...[and] give you a positive slope." He also explained the case in which a and b are negative and said that $f'(x)$ "is mimicking" $f(x)$. Here, he correctly connected the behavior of $f(x)$ with the sign of the slope, but his argument for a and b was not correct or related to his contraction-expansion idea. When I reminded him that the question asks about the slope at $x = 0$, he changed back to c) by substituting $x = 0$ in $f'(x)$ again.

Alan's Students

Three of Alan's students chose c), and Cole changed his answer from b) to c). Joe interpreted "the slope of tangent line of $f(x)$ is positive at $x = 0$ " as " $f'(0)$ to be positive" and evaluated $f'(0) = "a \cdot 0 + b"$ at 0. He concluded that " a can be anything...cause it's timesing by 0. As long as b is positive, you can have...the positive slope at 0" based on the relationships that $f'(0)$ is the slope of $f(x)$ at $x = 0$, and $f'(0)$ is a value of $f'(x)$ at $x = 0$.

Cole chose b) ' a must be positive while b can be any real number' by incorrectly interpreting a as "the slope of $f'(x) = ax^2 + b$." and stating that it "has to be positive." He also incorrectly stated that "the tangent line is going to be positive." Although he changed it to "the slope of it is positive," he used the same statement again later in the interview. When I reminded him that the point $x = 0$ was given in the problem, he said, "Then, my answer was wrong," and changed the answer to c) based on the same procedure as Joe's.

Zion chose c) during the survey but repeatedly said that he did not remember why he chose c) during the interview. When I reminded him that he previously said, "The derivative is a slope", and I drew a curve and asked him how to find the slope on the curve at a point, he explained the procedure using the derivative of a function (Figure 90):

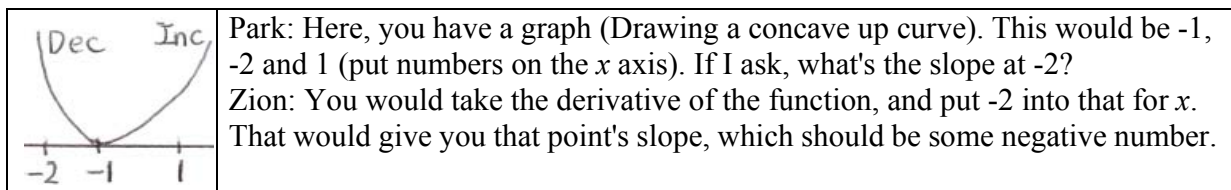


Figure 90. Park's Curve and Zion's Explanation about How to Find the Slope

In the excerpt, he correctly explained that $f'(a)$ is the slope of $f(x)$ at $x = a$, but could not apply it when the derivative function is given instead of the original function. In other words, he was not able to apply the relationship that $f'(a)$ is the value of $f'(x)$ at $x = a$.

Since Bill forgot the meaning of the tangent line, he could not justify his answer c) on problem 9. He said that he had never seen “problems like 8 and 9.”

Ian’s Students

Ian’s four students chose different answers for different reasons. They all changed their answers several times during the interview. While justifying their answers, Clio, Mary, and Mona mentioned that a in $f'(x) = ax^2 + b$ as the slope of the derivative of a function, and Sara integrated $f'(x) = ax^2 + b$ to find the original function $f(x)$.

Sara originally chose d). Without supporting d), she changed it to e), c), and b). First, she said because “the slope is positive at $x = 0$,” $f(x)$ “can go up at a curve or at a straight line” $x = 0$. Then, she integrated $f'(x) = ax^2 + b$, obtained $f(x) = 1/3ax^3 + bx$, evaluated it at $x = 0$, and chose e) ‘none of these’ as the answer because all terms are zero. However, she could not explain the substitution, and changed the answer to c) by substituting $x = 0$ in $f'(x)$ and obtaining b . Then, she integrated b , and obtained bx , and said, “I have to determine what x is.” After several failures of finding the sign of bx , she changed her answer to b) ‘ a must be positive, while b can be any real number’ by interpreting a in $f'(x) = ax^2 + b$ as the slope of $f'(x)$. Her explanation was not correct or consistent.

Clio also chose a) by interpreting a in $f'(x) = ax^2 + b$ as its slope. When I asked if she used “ $x = 0$ ” to solve the problem, she solved the problem again by substituting $x = 0$ in $f'(x)$ and obtaining b , and said that b is “the slope at $x = 0$ ” and should be positive. She added, “That whole equation [$f'(x) = ax^2 + b$] is your slope because that's the derivative.”

Mary’s original choice was b) but changed it to a), and then c). She chose b) by

interpreting a in $f'(x) = ax^2 + b$ as “the slope of the derivative.” However, she could not explain if “the slope of the tangent line” is the same as “the slope of the derivative.” He then changed the answer to a) by interpreting the whole equation, $f'(x) = ax^2 + b$, as slope: “The value of the derivative is positive, which makes the slope of the tangent line positive. If a and b ...are positive, the slope is always positive.” When I reminded her that the problem asks about the slope at $x = 0$, she changed it to c) by substituting $x = 0$ in $f'(x)$.

Mona’s first chose a), but changed it to b), and then c) by comparing ax^2 and b in $f'(x) = ax^2 + b$. She kept changing her explanation about which of ax^2 and b was bigger. Although her comparison and explanations are mathematically correct, she tried to make $f'(x)$ that she referred to as the slope, positive. When I reminded her that the problem was about the slope at $x = 0$, she solved the problem again by substituting at $x = 0$ in $f'(x)$, said, “ f' , the derivative is the slope of the original function and they want the slope of at the point 0 of the original function to be positive,” and changed the answer to c).

Discussion of the Research Questions

Previous sections reported how 12 interview participants explained their concepts of the derivative function, $f'(x)$, and the derivative at a point, $f'(a)$, and the relationships among $f(x)$, $f'(x)$, and $f'(a)$, and how they justified their answers on the survey using these concepts and relationships. This section addresses the three research questions in turn.

Description of the derivative at a point and the derivative of a function

This section summarizes students’ descriptions of the derivative, in terms of a) whether they chose the derivative of a function or the derivative at a point as their concept of the derivative, b) how they described those two concepts, c) how they explained the relationships

between them and to the original function. Table 46 shows which of $f'(x)$ or $f'(a)$ students chose as their concept of the derivative and their descriptions of $f'(x)$, and $f'(a)$.

Table 46. *Concept of Derivative Chosen by Tyler, Alan, and Ian's Students and Their Descriptions of $f'(x)$ and $f'(a)$*

Name	Choice	The derivative of a function, $f'(x)$	The derivative at a point, $f'(a)$
Tyler			
Roy	$f'(x)$	"Extension or contraction of" $f(x)$	Explanation was not provided
Liz	$f'(x)$	"How fast things change... Velocity over time"	"Velocity [or]... slope at a point"
Neal	$f'(a)$	"Graphical indication of every single point throughout a graph"	"Slope of a tangent line at a point" & "the direction the line is headed"
Zack	$f'(x)$	"Slope at all points" tells "what $f(x)$ is doing"	"Slope at a point"
Alan			
Bill	$f'(x)$	"You take it [$f(x)$] and derive the new equation... it describes... the slope"	Explanation was not provided
Zion	$f'(x)$	"Slope of the whole function at all the point"	"[Slope] at that single point."
Joe	$f'(a)$	"Slopes of the function at all points"	"Slope of a function at a point".
Cole	Both	"Rate of how fast something moves," & "Slope of the tangent line"	"it [these explanations] could be applied to both" $f'(x)$ and $f'(a)$
Ian			
Clio	$f'(x)$	"A way to... derive other equations from another using the rules"	"What the coordinates are at the given point... just plug it in"
Sara	Both	"The slope of the curve"	"Slope of the tangent line at a point"
Mary	$f'(x)$	"Short cuts" or "slope at all points"	"the slope of the function at a point"
Mona	$f'(a)$	"Slope of the whole line, another... function"	"Instantaneous slope at the point"

As shown in Table 46, of the 12 interviewees, seven (three from Tyler's, two from Alan's, and two from Ian's classes) chose the derivative function, $f'(x)$, as their concept of the derivative, three (one from each class) chose the derivative at a point, $f'(a)$, and two (one from each of Alan's and Ian's class) said that the same explanation can be applied to both $f'(x)$ on an interval and $f'(a)$ at a point. The slope is the most common interpretation of the derivative, and differentiation rules and the rate of change follows. Ten students used the same explanation for $f'(x)$ and $f'(a)$, two could not explain $f'(a)$, and one interpreted $f'(x)$ and $f'(a)$ differently.

Most students, nine out of 12, also explained the relationship between $f'(x)$ and $f'(a)$ correctly by interpreting $f'(a)$ as a point-specific value of $f'(x)$, for example, “slope at a point” vs. “the slope at all the points,” or “velocity at a point” vs. “the velocity over time.” Two of the students who gave the correct explanations—Cole and Clio—stated that $f'(a)$ can be found by “plug[ging]” $x = a$ in $f'(x)$. Regarding the incorrect descriptions of the relationship, two students—Roy and Bill—could not give an explanation of the relationship because they could not describe $f'(a)$. One student, Clio, gave an incorrect explanation that $f'(a)$ could be also obtained from $f(x)$ as well as correct explanation about substituting $x = a$ in $f'(x)$.

Based on their descriptions of the derivative, students explained what $f'(x)$ and $f'(a)$ inform about $f(x)$ by repeating their descriptions of $f'(x)$ and $f'(a)$ (Table 46), and only four students explained that $f'(x)$ and $f'(a)$ describe the behavior of $f(x)$ based on the signs of the $f'(x)$ and $f'(a)$: for example, “If $f'(x)$ on an interval is negative/positive, $f(x)$ decreases/increases on the interval,” or “If $f'(a)$ is negative/positive, $f(x)$ decreases/increases at the point.”

Relationships between a Function, the Derivative at a point, and the Derivative of a Function used in Problem Solving Situations

In this section, students’ explanations on the survey problems are reorganized to address the second research question regarding students’ uses of the relationships between a function, $f(x)$, the derivative of a function, $f'(x)$, and the derivative at a point, $f'(a)$, in problem solving situations. As mentioned earlier, interview participants were not representative of each class. Therefore, instead of reporting explanations of each group separately, this section describes all 12 students’ correct and incorrect explanations about the relationships.

Relationships between a Function and the Derivative of a Function. Students correctly explained the following relationships between a function, $f(x)$, and the derivative of a function,

$f'(x)$. The number of the students who used each relationship was reported in parentheses.

- $f'(x)$ can be obtained from $f(x)$ by applying the differentiation rule (12)
- $f(x)$ and $f'(x)$ have the same independent variable (10)
- $f'(x)$ is negative/positive when $f(x)$ decreases/increases (11)
- $f'(x)$ is the same as the slope of $f(x)$ (11)
- $f(x)$ is concave up/down when $f'(x)$ increases/decreases (2)

Most students correctly used the derivative algebraically using differentiation rules and graphically by describing the behavior of $f(x)$ in terms of sign of $f'(x)$. However, one student, Neal who used and explained the relationship correctly in a graphical situation, could not apply it to $f'(x)$ and $f''(x)$. Only a few students mentioned and used the concavity of a function in relation to the derivative. They also explained and used incorrect relationships between $f(x)$ and $f'(x)$:

- $f'(x)$ is a contraction or expansion of $f(x)$ (1)
- $f(x)$ and $f'(x)$ may have different independent variables (2)
- $f'(x)$ is increasing/decreasing when $f(x)$ is positive/negative (1)
- If $f(x)$ is positive/negative, $f'(x)$ is positive/negative (1)
- $f(x)$ and $f'(x)$ go the same direction at the same point or on the same interval (2)
- Graphs of $f'(x)$ are linear or zigzag not curved (1)
- $f'(x)$ [the slope] increases/decreases where $f(x)$ increases/decreases (5)
- $f'(x)$ and $f(x)$ have the same sign (1)

Students who described incorrect relationships used them inconsistently. For example, two students who mentioned that graphs of $f(x)$ and $f'(x)$ go in the same direction applied this relationship differently when $f(x)$ was given and when $f'(x)$ is given. One student, who stated that graphs of all derivative functions are linear or a piecewise linear in one problem, drew a concave

up curve for the derivative function in another problem. The most common incorrect description of the relationship between $f(x)$ and $f'(x)$ was that $f'(x)$ increases/decreases when $f(x)$ increases/decreases. Although most of students did not use this relationship to justify their answers, only two of them accepted that the statement was wrong and corrected it. Even after corrections, the statement was identified again several times.

Relationship between a Function and the Derivative at a Point. The interview participants correctly explained the following relationships between a function, $f(x)$, and the derivative of a function, $f'(a)$:

- $f'(a)$ is the same as the slope of $f(x)$ at $x = a$ (8)
- The sign of $f'(a)$ determines the direction of $f(x)$ at $x = a$ (5)
- $f'(a) = 0$ when $f(x)$ has an extreme at $x = a$ (9)
- $f(x)$ has an inflection point where $f'(x)$ changes the sign (1)
- $f'(a)$ is the rate of change of $f(x)$ at $x = a$ and approximate change of $f(x)$ between $x = a$ and $x = a + 1$ (3)
- The units of $f'(a)$ is the units of $f(x)$ divided by units of x (5)

Most students interpreted $f'(a)$ as an indicator of the behavior of $f(x)$ at $x = a$ by interpreting it as the slope. Based on this interpretation, they also explained that the derivative is zero at a point where the function changes its direction and used it to find the extreme values of $f(x)$. However, only a few students could distinguish the derivative at a point from the function at a point and recognize the units correctly.

Students also explained and used incorrect relationships between $f(x)$ and $f'(a)$:

- $f'(a)$ cannot be interpreted in the context of $f(x)$ (2)
- $f'(a)$ and $f(a)$ can be interpreted in the same way (1)

- $f'(a)$ is the change in $f(x)$ between $x = a$ and $x = a + 1$ (4)
- $f'(a)$ and $f(a)$ has the same unit (6)
- $f'(a)$ is the tangent line of $f(x)$ at $x = a$ (3)
- The slope of $f(x)$ at $x=a$ can be obtained by substituting $x = a$ in the equation of the tangent line at $x = a$ (1)
- $f(a)$ can be obtained by calculating the anti-derivative of $f'(a)$ at $x = a$ (2)
- Graph of $f(x)$ can be obtained from tangent lines (i.e., the derivative at points) (1)
- Stating that $f'(a)$ only informs the sign of the slope of $f(x)$ at $x = a$ (1)

About half of the students who showed explained the relationship between $f(x)$ and $f'(a)$ could not interpret $f'(a)$ as the rate of change. They interpreted it as the change in $f(x)$ between two consecutive x values. This answer is mathematically acceptable to some extent because $f'(a)$ can be used as the estimate for the change. However, the quantities and units of $f'(a)$ are different from this change in $f(x)$, but students could not appreciate these differences. In graphical situations, students incorrect interpretations of the relationship between $f(x)$ and $f'(a)$ seems closely related to the tangent line. Three students interpreted the tangent line at a point as the derivative at a point. Two of them integrated $f'(a)$ to calculate $f(a)$ or to find $f(x)$. These mathematically incorrect interpretations may come from their inability to conceive of $f'(a)$ as a value, not a linear function, or their routinized action of “plug in” numbers in the equation to calculate $f'(a)$.

Relationship between the Derivative at a Point and the Derivative of a Function. All students correctly used the relationship between $f'(x)$ and $f'(a)$ by interpreting $f'(a)$ as the value of $f'(x)$ at $x = a$ as follows:

- $f'(a)$ can be calculated by substituting $x = a$ in the equation of $f'(x)$ (12)

- $f'(a)$ is a y value on the graph of $f'(x)$ at $x = a$ (12)
- If $f'(a)$ is zero, it is the x intercept of the graph of $f'(x)$ (8)

All 12 students used the relationship between $f'(x)$ and $f'(a)$ correctly identifying $f'(a)$ as a value of $f'(x)$ at $x = a$ when the equation or graph of $f'(x)$ was given. They also correctly recognized this relationship when $f(x)$ has a local extreme value. However, nine students could not apply this relationship, $f'(a)$ as a value of $f'(x)$, or changed the answer several times when the problem is not designed for simple substitution. Although they mentioned and used the derivative at a point as the slope at the point earlier, they could not apply this relationship to $f'(x)$ to make the slope of the tangent line at a point positive. Therefore, students' performance on simple calculation of $f'(a)$ from $f'(x)$ does not always coincide with their conceptualization of the relationship between $f'(a)$ and $f'(x)$.

The Derivative of a Function as Another Function

This section addresses the third research question regarding *use* of the derivative as a function in problem solving processes. Only one student, Cole, mentioned that the derivative function, $f'(x)$, is a function when he explained what $f'(x)$ is. Although no student mentioned that $f'(x)$ is a function in their problem solving processes, some students used it, for example, when they described the behavior of $f'(x)$ over an interval or identified its units which described above.

Additionally, it is worth noting that some of the students incorrectly described or used the concept of a function while explaining the derivative as follows:

- A parabola can be represented by a quadratic function with a single term, ax^2 (2)
- An equation obtained from integration does not have a constant term (1)
- $y = 0$ cannot be graphed because it is a point (1)
- The relationship between two functions is determined by their slopes (1)

Two students, who mainly approached problem algebraically, could not find the exact equation of a function given as a graph or find the graph which the equation represents. They constantly applied integration formula, but one of them did not add a constant. One student said that the derivative of constant functions, $y = 0$, cannot be graphed because it is a point. One student did not know how to compare the graphs of two functions. One student did not remember the concept of tangent line, and several of them interpreted a in $f'(x) = ax^2 + b$ as the slope although the equation is not for a linear function. This concept of the slope seems to closely related to the slope-point form of a linear function, $y = mx + b$ and their concept of the derivative, which closely related to the slope. These incorrect notions of a function, tangent line, and slope seem to hamper their learning of the derivative; those students were not able to apply the concept of the derivative because of those incorrect concepts of function and tangent line. Therefore, students' thinking about the function and other related concepts seems to affect their thinking about the derivative in problem solving situations.

When comparing the instructor discourses (see Chapter 5) with the discourses of their students in this chapter, it can be noted that there are discrepancies between them in terms of descriptions and uses of the concept of the derivative as well as in the relationships between a function, the derivative function, and the derivative at a point. Some of these differences have been identified as students' *misconceptions* or instances of *lack of understandings* of the derivative in existing research. How instructors addressed the aspects of the derivative, of which students showed such incorrect descriptions and uses, will be discussed later in Chapter 7 to find possible explanations of students' *misconceptions* of the derivative.

CHAPTER 7: SUMMARY, DISCUSSION, AND CONCLUSIONS

As discussed in chapter 1, this study investigated the characteristics of calculus students and instructors' discourses on the derivative using the communicational approach to cognition. Although there have been a plethora of studies on students' thinking about the derivative, no study addressed this topic by focusing on the difference between the derivative of a function and the derivative at a point, and the relationships between them and to the original function. Moreover, no existing study explored how the derivative is taught in calculus classrooms.

In this study, calculus instructors' and students' discourses on the derivative were collected using mixed methods of classroom observations, surveys, and interviews. Three classes were videotaped and transcribed for six weeks and a survey was taken by students in the classes. Surveys were scored and used for selecting twelve student interviewees. Interviews with instructors and students were videotaped and transcribed. Transcripts were coded for topics including the derivative function, the derivative at a point, the relationships between them, and to the original function. Analysis of instructors' discourse explored how and to what extent they addressed each topic focusing on the key features of the discourse: word use, visual mediator, routine, and endorsed narratives. Cases in which they stated and used a relationship were classified as *explicit* whereas cases in which they used the relationship without stating it were classified as *implicit*. Analysis of students' discourse focuses on how they described and used each topic in problem solving situations. The coding scheme, which includes four categories of the relationships between a function, the derivative function, and the derivative at a point, and the derivative as a function (Table 9) was used as an analytical framework for when the instructors and students addressed the topic, and the four features of the mathematical discourse provided me a lens to look through how they addressed the topic in these instances.

The first section of this chapter addresses responses to the research questions about instructors' and students' discourses, which guided this dissertation. The second section compares instructors' and students' discourses. The third section provides implications of this study for the field of mathematics education, teaching, and assessment. The fourth and fifth sections discuss limitations of this study and the future research questions, respectively.

Summary of Responses to the Research Questions

This section addresses the six research questions by summarizing the main features of instructors' discourse on a) the definitions of the derivative function, $f'(x)$, and the derivative at a point, $f'(a)$; b) relationships among $f(x)$, $f'(x)$, and $f'(a)$; and c) $f'(x)$ as a function, and the main features of students' discourse on the corresponding topics. Discussions of instructors' discourse focus on to what extent and how they addressed these topics in class, and discussions of students' discourse focus on how they described and used these topics in problem solving situations.

Instructors' Discourse

Definitions of $f'(x)$ and $f'(a)$

While introducing the derivative, all the instructors defined $f'(a)$ and then $f'(x)$ by changing a point, a , to an independent variable, x . They addressed the relationship between $f'(a)$ and $f'(x)$ implicitly, at best. No instructors explained that the derivative at all the points form the derivative function, or $f'(a)$ is a specific value of $f'(x)$ that is a function though one instructor stated that a point where the derivative is defined can be expanded to an interval.

Relationships between $f(x)$, $f'(x)$, and $f'(a)$

The relationship between $f'(a)$ and $f'(x)$ was also implicitly addressed in later lessons. Instructors did not state that $f'(a)$ is a value of $f'(x)$ at $x = a$ while finding $f'(a)$ by substituting $x = a$ in $f'(x)$. They also determined the differentiability of a function, $f(x)$, based on the existence

of $f'(a)$ without stating the relationship between $f'(a)$ and $f'(x)$. This implicit approach was mostly found when they described the behavior of $f(x)$ on an interval based on the sign of $f'(a)$ instead of $f'(x)$ on an interval. In this description, they used the word, "derivative" without specifying it as "the derivative function" or "the derivative at a point," and used an arm gesture indicating the tangent line at a single point instead of several gestures for multiple tangent lines on the interval. These word use and gestures may confuse students about whether "derivative" referred to "derivative function" or "the derivative at a point" because instructors did not explicitly distinguish the two concepts or explain their relationship. In contrast, instructors explicitly addressed the relationships between $f'(a)$ and $f(x)$, which $f'(a)$ is the slope of $f(x)$ at a point, and used it in various situations such as graphing $f'(x)$ and computing the extreme values of $f(x)$. They also explicitly addressed the relationship between $f'(x)$ and $f(x)$ when stating and using differentiation rules and describing the behavior of $f(x)$ based on the sign of $f'(x)$.

Derivative Function as a Function

The instructors stated the property that the derivative function, $f'(x)$, is a function at least once in the beginning of the derivative unit, but implicitly addressed this property in most cases. At least two of them used it without mention while defining $f''(x)$ and $f'''(x)$ as derivatives of $f'(x)$, and applying theorems for a function to the derivative function. This property can also be addressed by discussing the independent variable (IV) of the derivative function, though no instructors explicitly stated that $f(x)$ and $f'(x)$ have the same IV. The discussion of the IV was limited to the procedure of differentiating implicit functions with multiple IVs.

Students' Discourse

Definitions of $f'(x)$ and $f'(a)$

While explaining their concept of the derivative, most students chose the derivative

function, $f'(x)$, as their dominant concept of the derivative, and then explained the derivative at a point, $f'(a)$, as the specific value of $f'(x)$. Most students described the derivative as the slope of a function on the graph, but less than half of them mentioned that the derivative at a point and the derivative function are indicators of the original function's behavior. Only one student gave an incorrect description of the derivative function as an “extension or contraction of” (stretching or shrinking of the graph of) $f(x)$. Two students did not provide any explanation about the derivative at a point because the derivative was only defined on an interval in their descriptions.

Relationships between $f(x)$, $f'(x)$, and $f'(a)$

Regarding the relationship between $f(x)$, $f'(a)$, and $f'(x)$, most students correctly used the differentiation rules and the substitution method to compute $f'(x)$ and $f'(a)$. However, most of them did not apply the substitution correctly beyond the simple computation. For example, in survey problem 9, which asks them to use the relationship that $f'(a)$ is a value of $f'(x)$ to make the slope at $x = a$ positive, most students, who specified the derivative as slope, answered incorrectly or changed their choices several times. This result shows that their ability to perform a simple calculation did not always show their conceptualization of $f'(a)$ as a specific value of $f'(x)$.

The Derivative Function as a Function

Many students did not correctly explain $f'(x)$ as a function and what $f'(x)$ represents in relation to the original function, $f(x)$. Five students incorrectly stated that $f(x)$ increases/decreases if and only if $f'(x)$ increases/decreases. Three of them described "derivative as a tangent line," which can link to their concept of $f'(x)$ involving the tangent line because $f(x)$ and its tangent line go in the same direction. This description shows their mixed notion of the derivative as a point-specific value and as a function on an interval. Students' difficulty appreciating $f'(x)$ as a function was also found when they described the IV of $f'(x)$. Most did not correctly identify the

IV; for example, one student stated that the IV of $f'(x)$ is time regardless of what $f(x)$ represents, and another student stated that the IV of $f'(x)$ is the rate of change of the IV of $f(x)$.

In summary, many students incorrectly described how $f'(a)$ and $f'(x)$ are related and what they represents in relation to $f(x)$. Such descriptions suggest that many students are unable to conceive of a) $f'(a)$ as a number, a specific value of $f'(x)$; and b) $f'(x)$ as a function consisting of the derivative as several points. Classroom discourse analysis showed that the instructors did not explicitly address these mathematical aspects while teaching. The next section addresses how instructors' and students' discourses are related by comparing to what extent instructors addressed various aspects of the derivative and whether students correctly described or used those aspects.

Discussion of Comparison between Instructors' and Students' Discourses

As mentioned previously, students tended to describe and use correctly the mathematical aspects that were addressed explicitly in class whereas they showed difficulties with the aspects addressed implicitly by their instructors. This section compares instructors' and students' discourses in a table format. Since not all the mathematical aspects of the derivative were identified in students' discourses, the tables include the aspects that are identified in at least three students' discourse. All instructors addressed each of these aspects entirely explicitly or implicitly. The first table shows the aspects that instructors addressed explicitly and students' descriptions or uses on the corresponding aspects. The second table includes the same information but about the aspects that instructors addressed implicitly. The first column in Table 47 shows the aspect that instructors explicitly addressed. The second and third columns show how many students described or used each aspect, and their descriptions or uses, respectively.

Comparison between Explicit Instructors' Discourse and Students' Performance

This section addresses the mathematical aspects of the derivative, which were addressed explicitly by instructors, and students' descriptions or uses of the aspects (see Table 47). The number of students for each aspect does not always add to twelve because not all the students described or used the aspect, and some of their descriptions or uses of the aspect overlapped.

Table 47. *Instructors' Explicit Discussions and Students' Descriptions or Uses of Mathematical Aspects of the Derivative with Frequencies*

Aspect	Number of Students	Student Descriptions or Uses
Defining $f'(a)$ as Slope of Tangent line	7	$f'(a)$ is the same as slope of $f(x)$ at $x = a$
	8	Using the slopes of $f(x)$ to graph $f'(x)$
	3	<i>Tangent line is the derivative at a point</i>
Describing $f(x)$ using $f'(a)$	5	The sign of $f'(a)$ indicates the behavior of $f(x)$ at $x = a$
Using $f'(a) = 0$ at Local Extremes of $f(x)$	9	$f'(a) = 0$ when $f(x)$ has an extreme at $x = a$
Discussing Differentiation Rules	12	$f'(x)$ can be obtained from $f(x)$ by applying the differentiation rule
Describing $f(x)$ Based on the sign of $f'(x)$	11	$f'(x)$ is negative/positive where $f(x)$ decreases/increases
	11	$f'(x)$ is the same as the slope of $f(x)$
	5	$f'(x)$ increases/decreases where $f(x)$ increases/decreases
Substituting $x = a$ in $f'(x)$ to Find $f'(a)$	12	$f'(a)$ can be found by substituting a in $f'(x)$
	12	$f'(a)$ is a y value on graph of $f'(x)$ at $x = a$

Note. Students' mathematically incorrect descriptions and uses are italicized.

Instructors explicitly addressed what $f'(a)$ and $f'(x)$ represent in relation to $f(x)$ by stating them as the slope of $f(x)$ or describing the behavior of $f(x)$ based on their signs. They addressed the relationship between $f(x)$ and $f'(x)$ with differentiation rules, and the relationship between $f(x)$ and $f'(a)$ by addressing the property of the derivative function at a point where $f(x)$ has extreme values. How $f'(x)$ and $f'(a)$ are related was explicitly addressed with the substitution method.

Most students correctly applied these explicit aspects in problem solving, which shows that the aspects of the instructors addressed explicitly coincide with those that students described or used correctly. However, three students incorrectly stated and used, "the derivative as a

tangent line," and five students stated or used " $f'(x)$ increases/decreases where $f(x)$ increases/decreases" or " $f(x)$ and $f'(x)$ go in the same direction." Three of the five students used the first statement to support the second because $f(x)$ and its tangent line go in the same direction. These incorrect notions seem related to students' concepts of the derivative as a point-specific value and as a function on an interval, and their instructors' implicit discussion on these two concepts and ambiguous use of the word, "derivative." These notions are addressed in detail in the following section.

Comparison between Implicit Instructors' Discourse and Students' Performance

This section addresses the aspects of the derivative, which were addressed implicitly by three instructors in the classroom, and students' descriptions and uses of the aspects (Table 48).

Table 48. *Instructors' Implicit Discussions and Students' Explanations or Uses of Mathematical Aspects of the Derivative with Frequencies*

Aspects	Number of Students	Student Explanations and Uses
Using one Word "derivative" for "derivative function" and "derivative at a point"	4	<i>"Derivative" at a point is a linear or constant function</i>
	3	<i>$f'(a)$ is the rate of change of $f(x)$ at $x = a$</i>
Interpreting $f'(a)$ as Rate of Change	5	The unit of $f'(a)$ is (unit of $f(x)$) / (units of x)
	4	<i>$f'(a)$ is change in $f(x)$ between $x = a$ and $a + 1$</i>
	6	<i>$f'(a)$ and $f(a)$ has the same unit</i>
	7	<i>$f'(x)$ determines the behavior of $f(x)$</i>
Interpreting a Point on Graph of $f'(x)$ as Slope of Tangent Line	7	<i>The slope of $f'(x)$ is the derivative at a point</i>
	3	<i>The derivative is tangent line at the point</i>
	3	<i>$f'(a)$ can be gained by substituting a value in the equation of the tangent line.</i>
	5	<i>$f'(x)$ is a linear or constant function</i>
	3	<i>The graph of $f'(x)$ is (piecewise) linear</i>
Identifying Independent Variable of $f'(x)$	5	<i>The independent variables of a function and its derivative function are different.</i>

Note. Students' mathematically incorrect descriptions and uses are italicized.

As shown in Table 48, when instructors implicitly addressed mathematical aspects of the derivative, many students incorrectly described or used those. Comparison between instructors'

and students' discourses in those aspects not only shows that students' descriptions and uses are not always mathematically correct while instructors' were correct, but also implies that there are other systemic differences in these two types of discourses. I organize these differences around three major themes: a) word use; b) explanations of the derivative; and c) the derivative as a function.

Word Use

The most prominent difference between instructors' and students' discourse was found in their use of the word, "derivative." Although both used the word without specifying it as "the derivative function" or "the derivative at a point," they used the word differently. Throughout the derivative unit, instructors used, "derivative" to refer to "the derivative function" in most cases; when they referred to "the derivative at a point," they specified the point. This way of using the word, "derivative," is consistent with the way the terms, "function" and "function at a point," are used. In contrast, students sometimes inconsistently used the word, "derivative." They used it to refer to "the derivative function," or "the derivative at a point," but sometimes changed its meaning frequently within a single interview question to support their incorrect concept such as the derivative as a tangent line or a constant function. Using the word, "derivative," without specifying its referent appeared to be related to the way students described the derivative as a point-specific value and as a function on an interval, which will be addressed in the next section.

Point-wise and Across-time Explanations of the Derivative

Monk (1994) defined two types of understandings: "pointwise understanding" as the value of a function at a point, and "across-time understanding" of a function as a dynamic quantity on an interval. The results of this current study show that students' explanations of the derivative included both types of understanding but not always in a mathematically accepted way.

One of the most prominent incorrect notions of “the derivative” was the tangent line at a point. This notion may come from students' descriptions of the derivative at a point, most of which include the tangent line, and the derivative function, which has variable x or is defined on an interval. The use of the word, “derivative” without specifying it as the derivative function or the derivative at a point also seems to play a part in forming this incorrect notion by allowing students to consider “derivative” as one object “pointwise” and “across-time,” simultaneously. Similarly, some students considered the derivative at a point as a constant function instead of a number. As shown in Table 48, the instructors implicitly addressed what $f'(x)$ and $f'(a)$ represents in terms of $f(x)$; they graphed $f'(x)$ using the slopes of tangent lines of $f(x)$ without stating that the slopes are the derivative at several points, and $f'(a)$ is a specific case of $f'(x)$.

Derivative Function as Function

Students incorrectly described 'the derivative' as a linear function represented by a tangent line or as a constant function of the derivative at a point. Some of them also stated that the derivative graphs are linear or piecewise linear. Such descriptions show that students have a concept of the derivative as a function but do not correctly conceive of what the derivative represents in relation to a function. While introducing or using this aspect of the derivative, instructors did not explicitly address what the derivative at a point, $f'(a)$, and the derivative function, $f'(x)$, represent in terms of the original function. As shown in Table 48, none of them explicitly explained $f'(a)$ as the rate of change of $f(x)$ while solving problems involving the related rates, or $f'(x)$ as the slope of $f(x)$ while graphing $f'(x)$.

In addition, given that a function is a dynamic object that changes as its independent variable (IV) changes, identifying the IV of the derivative of a function can be considered as one aspect of conceiving of the derivative as a function. The importance of recognizing the IV is also

argued in existing research (Carlson et al., 2008). However, most students did not correctly identify the IV of $f'(x)$ correctly or recognize that $f(x)$ and $f'(x)$ have the same IV. None of the instructors explicitly addressed these two aspects of the IV of $f'(x)$; the discussion was limited to the procedure of differentiating implicit functions with multiple variables.

In summary, students' incorrect concepts of the derivative were mostly found when they described the relationship between $f'(x)$ and $f'(a)$, which is $f'(x)$ as a function consisting of the values of the derivative at several points, and instructors tended to address this relationship implicitly. Moreover, when instructors use $f'(a)$ as a representative of $f'(x)$ on an interval to describe the property of $f(x)$ on an interval, their use of the word, "derivative" without specifying its referent makes implicit the relationship between $f'(x)$ and $f'(a)$, and what they represent in terms of $f(x)$. These implicit discussions may not only confuse students about what the word, "derivative" refers to but also allow them to have a mixed notion of "pointwise" and "across-time" understandings of the derivative without appreciating their difference or relationship.

With the current data, it is not possible to reveal the direct relationship between students' incorrect descriptions and uses of the derivative, and how their instructors taught the concept. Students' thinking of the derivative is also affected by various other factors aside from their instructors such as their thinking about other concepts that they previously learned, and the various representations of the derivative. The results of this study, however, suggest a way we can improve teaching and learning of calculus in general, and the derivative in particular. By using the mathematical terms correctly and providing explicit explanations of these relationships and the aspect of the derivative as function, calculus classes can provide better opportunities for students to learn the derivative. Modifying how the word, "derivative," is used and how its mathematical aspects are addressed may contribute to improving the explicitness of the classroom

discussion of the derivative. This is further elaborated in the following section.

Conclusion

This section addresses a) implications of this study in terms of how the results can contribute to the field of mathematics education; b) limitations of the study regarding the methodology and analysis; and c) research questions that can be further investigated in the future.

Implications

The results of this study can contribute to the current field of mathematics education research by pointing out the importance of how mathematical words or terms are used and the need for explicitness regarding certain aspects of the derivative. The classroom discourse analysis pointed out when and how instructors used the words and terms ambiguously and discussed some aspects of the derivative implicitly. This provided information about how to improve the teaching and learning calculus. The survey and interview data can also contribute to the development of performance assessment.

Word Use and Explanations of the Derivative in Learning Mathematics

This study contributes to the growing body of literature about the importance of mathematical word use in students' thinking by expanding such research to a more advanced topic: the derivative. There have been several studies about the role of the words for early mathematical concepts (e.g., Fuson and Kwon, 1992; Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Sfard and Lavie, 2005; Sfard, 2008). In addition, several comparative studies between Asian countries and the United States show that the way students read multi-digit numbers and fractions in different languages affects their performance on problems involving those concepts (Fuson and Kwon, 1992; Miura et al., 1999). Recently, children's use of number words was addressed by Sfard and Lavie (2005); their results show that children used the number

words as adjectives that need nouns attached to them not as stand-alone mathematical objects.

This study raises similar questions about the role of mathematical terminology in word use and notations as visual mediators in higher education in general, and in the teaching and learning of calculus in particular. The results show the importance of the use of the terms, “a function,” “the derivative,” “the derivative of a function,” and “the derivative at a point” in teaching and learning the derivative. More specifically, the use of the terms was related to the aspects of the derivative which the instructors addressed implicitly during instruction and statements involving a functions and their derivatives, which students believed as true (e.g., a function and its derivative function go in the same direction). This study, therefore, contributes to our understanding of instructors’ and students’ thinking about the derivative based on the key features of their discourses including their use of the terms.

Teaching

As discussed earlier, students explained more consistently and correctly the aspects of the derivative addressed explicitly in their classrooms than those addressed implicitly. These results imply that instructors' explicit discussion of mathematical aspects of the derivative including exact use of the mathematical terms would provide students better opportunities to learn. The purpose of instructors' exact uses of the mathematical terms is not only for teaching students to use the terms correctly; the results show that the use of the terms is closely connected to what learners considered as “mathematical fact.” Using the word, “derivative,” without specifying it as the derivative of a function or the derivative at a point allows students to change the concept from a function to a number, or vice versa, and to create incorrect concepts of the derivative. Some students described the derivative as a tangent line, as opposed to the slope of the tangent line, because it is a linear function with a variable and point-specific object, simultaneously.

Therefore, the instructors should carefully use the exact mathematical terms especially when they addressed the transition from one concept to the other and emphasize the derivative function, $f'(x)$, as a function and the derivative at a point, $f'(a)$, as a number representing a point-specific value of $f'(x)$. With the use of the exact and correct terms, instructors can also explicitly address the relationship between $f'(x)$ and $f'(a)$, and the derivative of a function as another function.

First, the relationship between $f'(x)$ and $f'(a)$, can be explicitly addressed by explaining that $f'(a)$ is a specific value of $f'(x)$. Reminding students that the function at a point, $f(a)$, is a value of the function, $f(x)$ would help students see the equivalence between the relationship between $f(x)$ and $f(a)$, and the relationship between $f'(x)$ and $f'(a)$. The equivalence can be addressed by reviewing the notations of the function at a point, $f(a)$, and the function, $f(x)$, and use of the letters a as a constant and x as a variable. Similarly, while graphing $f'(x)$, reminding students about the graphing process of $f(x)$ based on several values of the function would help them recognize the equivalence in the processes of graphing $f(x)$ and $f'(x)$. This equivalency would help students realize that $f'(x)$ consists of the values of the derivative at several points. When the instructors use $f'(a)$ as the representative of $f'(x)$ on an interval in explanations of differentiability and sign diagrams, it would be crucial to state that $f'(a)$ is a specific case of $f'(x)$ in order to address why the method at a specific point works at any point on the interval. When they prove the differentiation rules using the tangent line at a point on an interval, it is important to show that the proof is not point-specific. They can show that the same explanation works on the whole interval by showing the behavior of the tangent line at several points with multiple drawings or gestures.

Second, the derivative function as a function, can also be explicitly addressed by reminding students about the property of a function. Defining $f''(x)$ or $f'''(x)$, and applying

theorems for the function to $f'(x)$ are also good places to include instruction that $f'(x)$ is a function on which they can define its own derivative or apply the theorem for functions. Drawing parallels between the treatments of $f(x)$ and $f'(x)$ would help students see both of them as functions. The derivative as a function also can be addressed by emphasizing its dynamics as the independent variable changes and identifying its independent variable not only in the calculation process but also in context of the derivative function as a function. Regarding students' incorrect interpretations about what $f'(x)$ represents in terms of $f(x)$, the instructors may emphasize that each value of $f'(x)$ represents the slope of the tangent line, not the tangent line itself. They can show the process of plotting several points for $f'(x)$ based on the slopes of tangent lines of $f(x)$ at several points with dynamic geometry software. The different units of $f(x)$ and $f'(x)$ can be addressed by graphing $f'(x)$ and $f(x)$ on separate planes and explaining the y axes represent different quantities: a function value, and the rate of change of the function. Instructors can also address the units of $f'(x)$ algebraically with its limit definition. To compare the sign of $f'(x)$ and the behavior of $f(x)$ on the same interval, they can graph them on two transparent sheets and overlay them.

The suggestions, including the correct use of the terms and explicit discussions on the relationships between concepts, may not result in unnecessarily high student performance with the derivative. However, explicit discussions on mathematical aspects of the derivative would remind students about the properties of a function that they have learned previously and help them see the equivalence in the concepts of a function and the derivative, and thus help them understand the relationship between $f'(x)$ and $f'(a)$, and $f'(x)$ as a function. These discussions would provide students a better learning opportunity and help them expedite development of their mathematical discourse on the derivative closer to that of mathematicians.

Assessment

Analysis of students' interviews about their solution process on the survey problems showed that they have different reasons for the same choice. Students' justifications provided for processes used on assessments should be evaluated with a focus on the use of key terms such as "function," "derivative," "the derivative of a function," and "the derivative at a point." Students' discourses on the derivative can be evaluated as a form of performance assessment. While developing the items for these types of new assessments or selecting such items from an existing item pool, instructors should consider possible students' incorrect thinking about the derivative. More specifically, they need to select items whose distracters are all linked to common students' errors or multiple correct ways of thinking. Some of the survey items were more helpful to find out students' incorrect explanations and endorsed narratives about the derivative than other items. For example, problem 8 on the survey about the relationship between a function and its tangent line at a point is not directly related to the derivative. However, by interviewing students with this problem, I was able to gather information regarding their description and use of the derivative, some of which are not mathematically correct (e.g., describing "the derivative as the tangent line" and stating that "a function and the derivative go in the same direction"). Similarly, problem 9 involving a) the concept of the derivative as the slope and b) the relationship between the derivative function and the derivative at a point helped me to identify that most students, who correctly used these two aspects of the derivative before, did not apply them in this problem. Their responses showed that their ability to perform a simple computation to find $f'(a)$ using the equation of $f'(x)$ does not always show their conceptualization of $f'(a)$ as a specific value of $f'(x)$.

However, most instructors said that they would not use these items in the class because

they are "not generalizable," "beyond the level of calculus I," or "not the material covered in this course." If the instructors were provided various students' incorrect responses and explanations in these items, they might consider using them for reviews or the next semester. Instructors should be able to develop or select such items that have potential to reveal various aspects of the students' thinking of the derivative, not only their *misconceptions* but also how these incorrect concepts are used or repeated when students solve problems and justify their answers. Grading students' responses by identifying their mathematically incorrect concepts and a possible repetitive pattern of using such concepts in their answers would allow instructors to have a better understanding of students' thinking about the derivative. Information about students' incorrect concepts and uses of the derivative can be used for instructors to revise their assessment items. By integrating such information with the problems, instructors could construct problems to better explore the subtlety of students' thinking of the derivative for purposes of student feedback and modifying teaching practice.

Limitations of the Study

This study includes limitations with respect to the size of samples, variety of participants, the timing used to conduct the interviews during the semester, survey items, and the procedure of coding. The student participants interviewed in this study may not be a representative sample because they participated in this study on a voluntary basis, and they all came from only one university. Therefore, the findings from survey data, such as their answer patterns on each item, cannot be generalized beyond the group of students who participated in the survey. For the same reason, the findings from the analysis of classroom and interview data also cannot be generalized to extend to all calculus classrooms. The instructors who participated in this study were one visiting instructor and two graduate assistants. No professors volunteered for this study. As

described in the methodology, although I carefully designed the student interview recruiting process, I could not always find students who were representative of each classroom that I observed because of the students' low response rate.

As mentioned earlier, most of the survey items were selected from the calculus concept inventory, which was found to be reliable (Epstein, 2006). The items that I developed based on calculus textbooks or lecture notes were reviewed by professors in mathematics and mathematics education departments. However, I could not pilot those items with enough students to calculate the reliability because of limited time and resources; rather, I interviewed two students and two calculus instructors about those items before conducting the current study to check if they understood the items correctly.

The timing of the interviews with students may also be a limitation of this study. Although most students were interviewed during the semester immediately after the derivative unit, two interviews were conducted when the semester was over. The difference in timing may have influenced their discourses on the derivative because they studied integration as anti-differentiation by the time of the interview. However, exposure to integration may not be significant in relation to their use of integration, because one of the two students who mainly used integration of the tangent line and the derivative at a point was interviewed during the semester before her instructor had covered integration.

Coding procedure was also a limitation of this study. Although I used member checking (Lincoln & Guba, 1985) by reviewing the parts of the transcripts that I made with the instructors who participated in this study and a native English speaker who has taught calculus several times, the whole set of transcripts was not reviewed by any additional people. Also, I was the only coder who analyzed the discourses from the three data sources. Although I had the lists of cases

that I coded as each of the relationships between a function, its derivative function, and the derivative at a point and coded them twice to check intracoder reliability (Johnson & Christensen, 2004), another coder might code a single episode differently from what I have coded.

Future Research

Based on the results of this study, additional studies can be conducted. First, though this study provided possible explanations for students' thinking about the derivative, other important factors their thinking about other mathematical concepts, such as functions, can be explored. Second, analysis of curriculum materials can also be done in a follow-up study because a textbook is an important source of students' learning. Third, future research can also investigate instructors' assumptions and beliefs about mathematical aspects and students' knowledge about the derivative because it may play a role in their teaching, and thus students' learning opportunities in the classroom. Fourth, an intervention study can be conducted to evaluate the effectiveness of professional development using the materials developed based on the results of this study. Last, a comparison study between two language groups may also be warranted because one language may use similar terms for “the derivative of a function” and “the derivative at a point” just as in English, and another may have different terms for the two concepts. Further details on each of these future studies are discussed in the following sections.

Research on Students' Thinking of Function

As shown in existing studies (Chapter 2), students' thinking on mathematical concepts, such as the rate, limit, function, tangency, and slope, affects their thinking on the derivative (e.g., Carlson, Smith, & Persson, 2003; Ferrini-Mundy, & Graham, 1994). The results of the current study show that students' discourses on the derivative were also closely related to, or affected by, their thinking about a function. For example, most students did not know how to express the

relationship between two functions on the graphs using an inequality, and they considered that $y = 0$ was represented by the origin instead of the x -axis. These concepts affected their solution process of the problems involving the behavior of the derivative function.

Additionally, the results of students' interview analysis show that they described the derivative ambiguously and incorrectly as a mixture of their concept of the derivative function, which has a variable, and the derivative at a point, which is point-specific. Exploring their concepts of the function at a point and on the interval would be beneficial to find out possible reasons for such incorrect concepts of the derivative. It may come from their lack of "pointwise understanding" of a function at a point and "across-time understanding" of a function on an interval, or their incorrect thinking on other aspects of the derivative such as notation of $f'(a)$ and $f'(x)$ or the uses of letters a and x . For example, regarding the notations, $f(x)$, $f(a)$, $f'(a)$, and $f'(x)$ an analysis could be done with pre- and post-tests; a pretest would be administered before the derivative unit starts to find out how much students understand the concept of a function, $f(x)$, and the value of a function at a point, $f(a)$, and the post test would be administered after the derivative unit is over with the equivalent items on the pretest but with prime notation, thus, $f'(x)$ and $f'(a)$. If a student fail on the corresponding items on both tests, it may come from their lack of understanding of a function or notations, $f(x)$ and $f(a)$. If students succeed on the pretest and fails on the post test on the corresponding items, this implies that they have difficulty with the concept of the derivative rather than a function or its notation. In summary, students' "pointwise understanding" and "across-time understanding" of a function should be explored in relation to their thinking about the derivative at a point and the derivative of a function.

Research on Curriculum Materials

Many studies have addressed curriculum materials as a key factor of students thinking of

mathematical concepts (e.g., Fuson, Stigler, & Bartsch, 1988; Li, 2000; Senk & Thompson, 2003, p. x). The same research questions used in this study could guide a study exploring how calculus textbooks and students' thinking of the derivative are related. The following questions could be asked: Which concepts does the textbook define first, the derivative of a function or the derivative at a point? How does the textbook define the concept and build up one concept from the previous one? Does it make explicit the transition from one concept to the other? To what extent does the textbook address the relationship among a function, the derivative at a point and the derivative function, and the derivative function as another function? The results of textbook analysis may provide a good comparison to the instructor and classroom discourses and reveal possible reasons for the students' incorrect descriptions or uses of the derivative. It may reveal the differences in describing the concepts and the relationships among several textbooks.

Research on Instructors' Beliefs about Function and Derivative

Analysis of classroom discourse shows that the instructors did not explicitly address some mathematical aspects of the derivative. The implicitness of discussions does not seem to come from a lack of their content knowledge on these aspects because they correctly answered and explained the survey problems involving these aspects during the interview. However, they seemed to assume that some of these aspects are obvious to their students. For example, they said the problems about the relationship between $f'(x)$ and $f'(a)$ were “too simple” to use or good for “review about a function.”

Instructors' beliefs about their students' thinking of a function and the derivative may affect how they addressed the concepts during the class. Many studies addressed the relationship between the instructors' beliefs and practice (e.g., Aguirre & Speer, 2000; Cohen, 1990). Instructors' beliefs about the topic that they are teaching and about students' thinking or

preparation level on the topic may indicate how the instructors address the topic in the class.

During the interview with instructors, I could have asked additional questions to further explore their expectations about students' thinking on this relationship, but their beliefs or teaching goals for calculus, particularly about the derivative, were beyond the scope of this study. The reasons for implicitness in explanations about some aspects of the derivative could be addressed in a follow-up study. One method would be watching video clips with instructors, in which they addressed a certain aspect of the derivative implicitly, and discussing their beliefs or expectations about the aspect (e.g., Aguirre & Speer, 2000).

Research on Comparison between Two Languages

The results of this study show that both implicitness in the instructors' discourse on the derivative, and students' incorrect descriptions and uses of the derivative were closely related to their use of the word, "derivative," without specifying its referent. In some other languages, using one word, "derivative," for both "the derivative of a function" and "the derivative at a point" is not possible because the terms for these two concepts are not consistent. For example, in Korean the derivative at a point is "微分係數 (미분계수, Mi-Bun-Gye-Su, translated as the differential coefficient)," and the derivative function is "導関數 (도함수, Do-Ham-Su, translated as a path function)." Similarly, Japanese terms for these two terms are "微分係數(びぶんけいすう, Bi-Bun-Kei-Suu)," and "導関數(どうかんすう, Do-Kan-Suu)," respectively. These two terms in Korean and Japanese cannot be represented by one word, "derivative," because they do not share a common word unlike the English terms.

Research in Korean education has pointed out that inconsistency in mathematical terms related to the derivative in Korean is problematic (Han, 1998). Han (1998) said that this inconsistency might hamper students understanding of the relationship between the derivative at

a point and the derivative of a function. Similarly, a mathematics professor, who is originally from Japan but teaching in the United States, wrote to me about the difficulty of explaining the difference between these two concepts: “I have sometimes found it difficult to communicate the difference between ‘the derivative function’ and ‘the derivative at a point’ to my (American) students.” Therefore, how instructors teach the derivative function and the derivative at a point, and how students describe and use those concepts in other languages, such as Korean and Japanese, would provide more thorough information about the role of mathematical terms and uses in teaching and learning mathematics. This exploration may reveal the affordances and constraints in consistency versus the difference in terms for the two related but different mathematical concepts.

Research on Effectiveness of the Results: Experimental Study

An experimental study can be conducted using the results of this study to develop a workshop for calculus instructors as an experimental treatment. The experimental group instructors would participate in the workshop in which they are provided lessons and documents about the importance of using the exact terms, “the derivative function” and “the derivative at a point,” and addressing the derivative function as a function whenever they use it. The control group would consist of the instructors who do not participate in this workshop. After the derivative unit is over, two types of data can be collected: how instructors use the terms, “derivative,” “the derivative at a point,” and “the derivative of a function” during the class and how their students think about the derivative based on a test or interview. Instructors’ word use would reveal the effectiveness of the workshop by showing whether they adopted and utilized the knowledge about the importance in word use and used the exact terms during the class. The achievement of students of the experimental group instructors can be compared to that of control

group instructors based on their students' performance or answer patterns on the tests, or their descriptions and uses of the derivative in interviews. If I find a positive effect of the treatment, I would disseminate the workshop material by publishing it in journals that calculus instructors and course supervisors would read or presenting in conferences on research in undergraduate mathematics education.

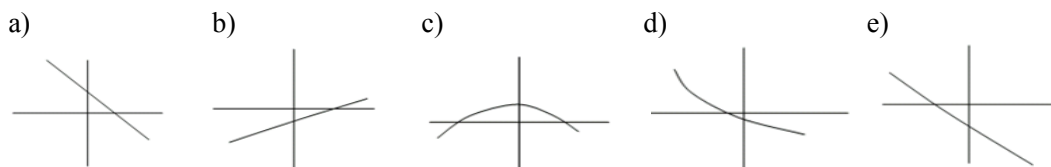
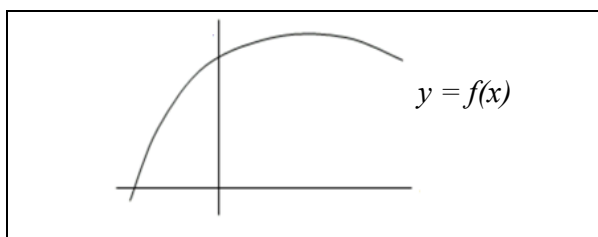
APPENDICES

APPENDIX A. Practicum Survey Questions

Please answer the following questions and show your work.

- $C(q)$ is the total cost (in dollars) required to set up a new factory and produce q units of a product. If the equation is given by $C(q) = 3000 + 100q + 3q^2 - 0.002q^3$,

 - Find the value of $C'(5)$.
 - What is the unit of $C'(5)$?
 - What is the meaning of $C'(5)$ in terms of the cost and units of the product?
- Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the first derivative, $f'(x)$?

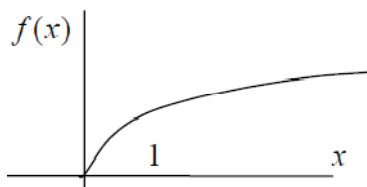


- If a function is always positive, then what must be true about its derivative?

 - The derivative is always positive.
 - The derivative is never negative.
 - The derivative is increasing.
 - The derivative is decreasing.
 - You can't conclude anything the derivative
- The derivative of a function is negative on the interval $x = 2$ to $x = 3$. Where on this interval does the original function have its maximum value?

- a) At $x=2$
- b) At $x=3$
- c) Somewhere between $x = 2$ and $x=3$
- d) It does not have a maximum.
- e) We cannot tell if it has a maximum or not.

5. The tangent line to the graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true everywhere on the graph?



- a) $\frac{1}{2}x + \frac{1}{2} = f(x)$
- b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$
- c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$
- d) $\frac{1}{2}x = \frac{1}{2}f(x)$
- e) None of these

6. The derivative of a function f is given by $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to the function f will be positive at $x = 0$.
- a) a and b must both be positive numbers.
 - b) a must be positive, while b can be any real number.
 - c) a can be any real number, while b must be positive.
 - d) a and b can be any real numbers.
 - e) None of these

7. The table shows the average cost per gallon of gasoline for different years in cents per gallon. For which span of years could be the cost of gasoline per gallon be a function that is increasing and concave down?

Year	1995	1996	1997	1998	1999	2000
Cost	112	120	122	101	121	144

- a) 1995-1997
- b) 1996-1998
- c) 1997-1999
- d) 1998-2000
- e) None of these

APPENDIX B. Students Survey Questions

Part I. Background

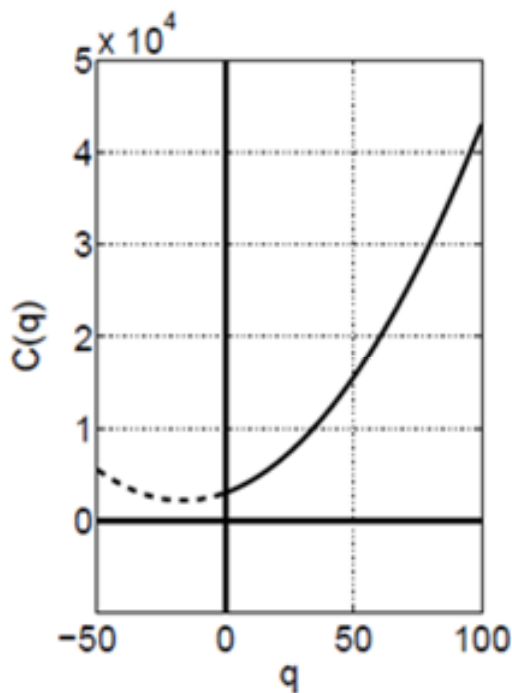
1. Year: a) Freshman b) Sophomore c) Junior d) Senior
2. Major (Specify): _____.
3. Native Language: a) English b) Other (Specify): _____.
4. Have you studied calculus in any language other than your native language?
If yes, specify the languages in which you learned calculus: _____.
5. Can you remember the first mathematics course in which you learned the word derivative?
a) Yes b) No

If yes, specify in which grade(s) and in which course/class you learned the word derivative? Grade(s): _____ Course: _____.

Part II. Mathematics

Please solve the following problems and show your work.

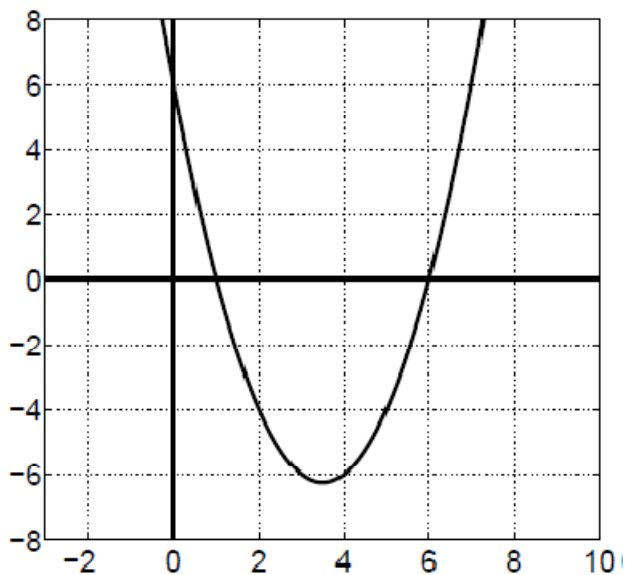
1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the cost satisfies the equation $C(q)=3000+100q+3q^2$, and the graph is given as follows.



- (a) Find the value of $C(2)$
- (b) What are the units of 2 in (a)?
- (c) What are the units of $C(2)$?
- (d) What is the meaning of $C(2)$ in the problem context?
- (e) Find the value of $C'(2)$.
- (f) What are the units for 2 in (e)?
- (g) What are the units of $C'(2)$?
- (h) What is the meaning of $C'(2)$ in the problem context?

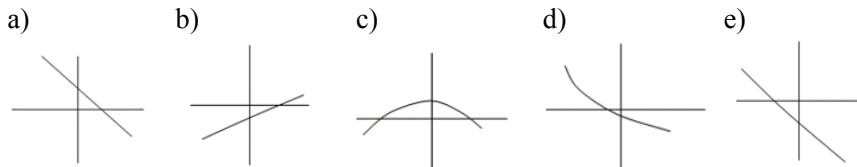
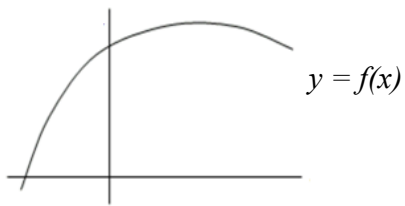
2. The derivative of a function f , is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

3. The graph of the derivative, $g'(x)$ of function g is given as follows. What is the value of $g'(2)$?

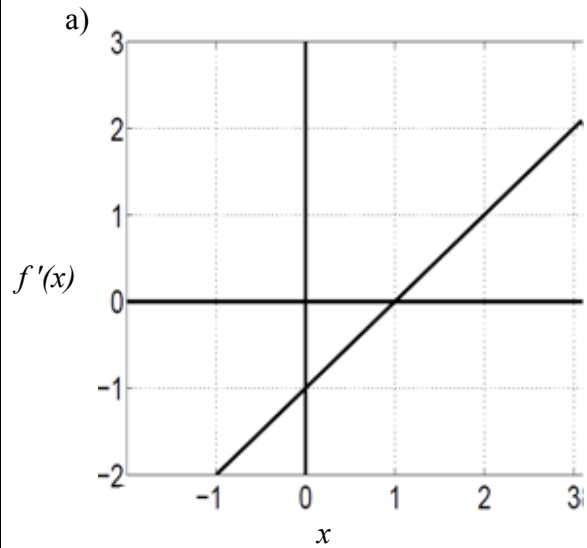
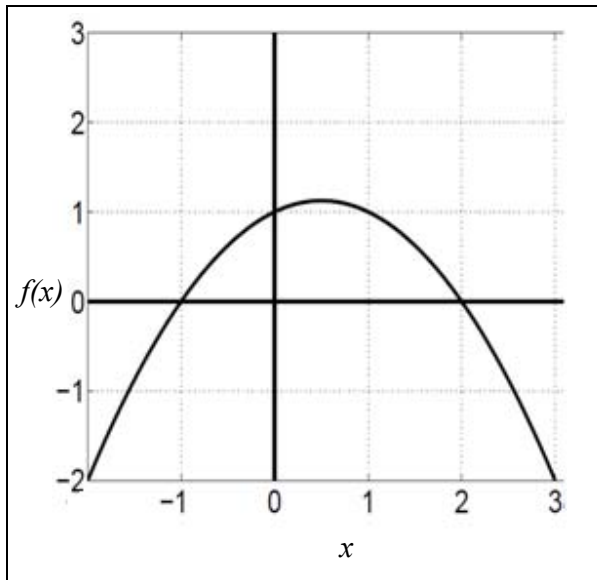


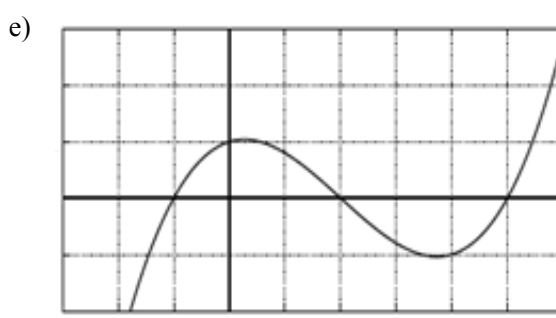
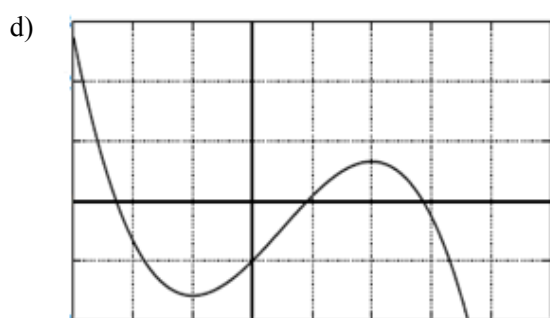
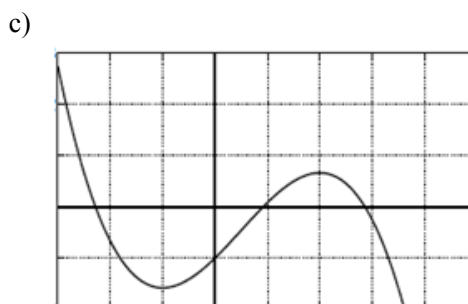
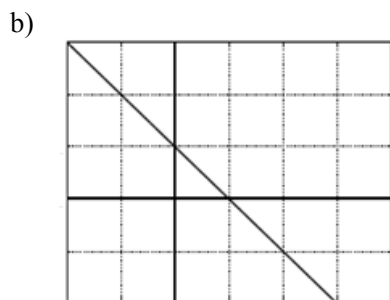
- a) -4
- b) -2
- c) 0
- d) 2
- e) 4

4. Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the first derivative, $f'(x)$?



5. Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?



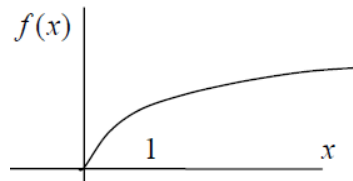


6. If a function is always positive, then what must be true about its derivative function?
- The derivative function is always positive.
 - The derivative function is never negative.
 - The derivative function is increasing.
 - The derivative function is decreasing.
 - You can't conclude anything about the derivative function.
7. The derivative of a function $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?
- The function $f(x)$ is positive on this interval.
 - The function $f(x)$ is negative on this interval.
 - The maximum value of the function $f(x)$ over the interval occurs at $x = 2$.

d) The maximum value of the function $f(x)$ over the interval occurs at $x = 3$.

e) We cannot tell any of the above.

8. The tangent line to this graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?



- a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$ e) None of these

9. The derivative of a function, f , is $f'(x) = ax^2 + b$. What is required of the values of a and b so

that the slope of the tangent line to the function f will be positive at $x = 0$.

- a) a and b must both be positive numbers.
b) a must be positive, while b can be any real number.
c) a can be any real number, while b must be positive.
d) a and b can be any real numbers.
e) None of these

Why?

APPENDIX C. Student Interview Protocol

First, the five warm-up questions will be asked.

Q1. What is the derivative? Can you make a sentence with the word, “derivative”?

You can explain your concept of the derivative with your own words. Your answer doesn't have to be mathematical. What it tells us?

There are more than one use of the word the derivative? Can you specify them?

If they do not come up with the derivative function and the derivative at a point, I will tell them these two and move to the next question.

Q2. What is the derivative function?

Can you give me an example of the derivative function with an kind of mathematical objects such as equation, graph, or table? Can you give me an example? You can either use algebra or draw something to support your idea.

Q3. What is the derivative at a point?

Use the same follow-up questions from Q2.

Q4. Is there any relationship between these two terms?

Case 1. When the student's answer is yes,

What is the relationship? Please describe their relationship. You may use your answers on the previous questions. You can use examples for example, equations or graphs.

Case 2. When the student's answer is no, there is no relationship between these two terms, or they are totally different concepts?

Can you explain why? You may use your answers on the previous questions. As you can see, we use the same term the derivative for these two concepts. What do you think about this?

Q5. How about function? Is function related to the derivative function and the derivative at a point?

If so, how are they related? You can use examples.

Secondly, students will be asked to explain their solution process and justify their answers. For example, “How did you solve this problem?”, “Could explain what you wrote (drew) here?”, and “In the process of explaining, I will ask some clarifying questions such as the derivative, what you just said, refer to the derivative function or the derivative at a point or does not matter?”

The followings are the list of example questions I would use as follow-up for some problems in the survey.

Problem 4

Question) Why you choose a)?

Anticipated answers) because it is decreasing.

Question) why it should be decreasing?

Anticipated answers) because the slope is decreasing.

Question) Why d) is not the answer?

Anticipated answers) it's curved

Question) Why it should be straight?

Problem 6

Case 1.

Anticipated interviewee's answers: either a) or b)

Follow-up questions: Why, is there any relationship between a function and its derivative?

Anticipated interviewee's answers: If its derivative is positive, the function is increasing.

Follow-up questions: Can you explain why using your definitions in warm-up questions?

Case 2.

Anticipated interviewee's answers: either c) or d)

Follow-up questions: Why?

Anticipated interviewee's answers: Because a function and its derivative act similarly.

Follow-up questions: Always?

Anticipated interviewee's answers: Yes / No

Follow-up questions:

If yes, consider the example of $y=-x$, what is its derivative? Do they act similarly?

If no, why you think this is a case of similarity of a function and its derivative. Or could you give me an example of a function which acts differently from its derivative?

Case 3.

Anticipated interviewee's answers: e)

Follow-up questions: Why? Can you explain this with your previous explanations?

Problem 7

Use the same questions for the previous problem, because this problem is about the relation between a function and its derivative, too.

Problem 8

Case 1.

Anticipated interviewee's answers: choose b), or c)

Follow-up questions: Why? Can you explain your answer with the graph of $f(x)$? Where would the tangent line be at?

Anticipated interviewee's answers: Draw the tangent line.

Follow-up questions: Can you explain how you got this inequality from this picture?

Anticipated interviewee's answers: The curve is below the line or above the line.

Case 2.

Anticipated interviewee's answers: choose a) or d)

Follow-up questions:

Are they same? Why?

Can you explain your answer with the graph of $f(x)$?

Where would the tangent line be at?

In the case of d), how did you get this equation? Where is this ' $\frac{1}{2}x$ ' from?

Anticipated interviewee's answers:

In the case of a), that's because the curve and the tangent line meet at this point.

In the case of d), that's because the curve and the tangent line meet at this point, and derivative at one point is the slope.

Follow-up questions: In this problem, should we look at the intersection point only? Why?

Anticipated interviewee's answers: Yes, because it is tangent line at one point. / No, I change the answer.

Follow-up questions: Then, we don't have to consider the other points beside the intersection point, right?

Anticipated interviewee's answers: No

Problem 9

Question) How did you get this answer?

Anticipated answer) a) or b) because "a" is the slope and "b" is the y-intercept.

Question) Why a is the slope and b is the y -intercept?

Anticipated answer) Because it has slope-intercept form.

Question) although it has x squared in this equation?

Anticipated interviewee's answers) Choose one of a , b , c , d , and e

Follow-up questions) Why did you choose that one? Can you explain with the graph of function $f(x)$? How did you guess the shape of the derivative? How did you find the x -intercept in the graph of the derivative? Can you explain how you use your definitions or explanations about the derivative at one point and the derivative function you gave in Task 1? Why are the graphs of a function and of its derivative function similar? Can you give me an example for the similarity?

APPENDIX D. Instructor Survey Questions

Part I. Background

1. Year(s) of teaching experience (Specify) _____.
2. Year(s) of teaching calculus (Specify) _____.
3. Native Language:
 - a) English
 - b) Other (Specify): _____.
4. Major (Specify) _____.
5. Have you studied and taught mathematics in any language other than English?
 - a) Yes
 - b) No

If yes, specify the languages in which you studied or taught mathematics: _____.

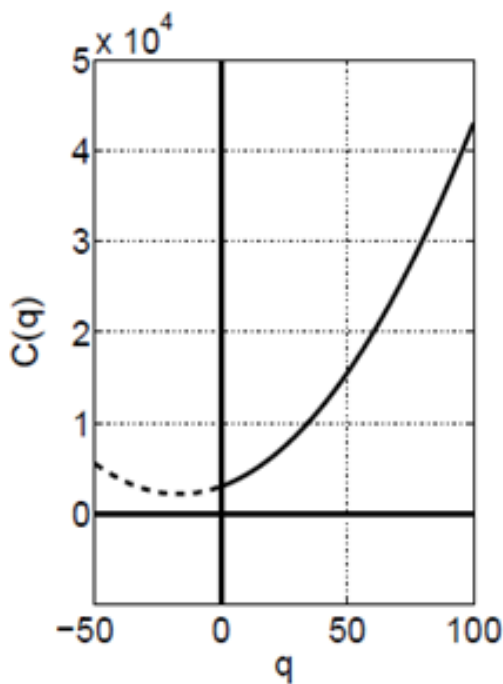
Does the language have different words for the derivative function and the derivative at a point?

- a) Yes
- b) No

Part 2. Mathematics

Please explain the following problems as if you solve them for students in calculus class.

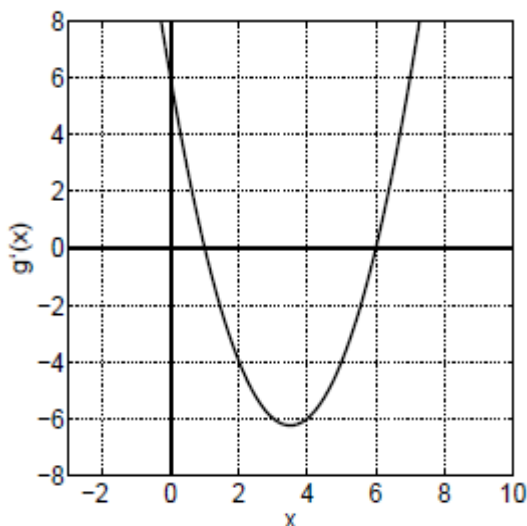
1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the cost satisfies the equation $C(q)=3000+100q+3q^2$, and the graph is given as follows.



- (a) Find the value of $C(2)$
- (b) What are the units of 2 in (a)?
- (c) What are the units of $C(2)$?
- (d) What is the meaning of $C(2)$ in the problem context?
- (e) Find the value of $C'(2)$.
- (f) What are the units for 2 in (e)?
- (g) What are the units of $C'(2)$?
- (h) What is the meaning of $C'(2)$ in the problem context?

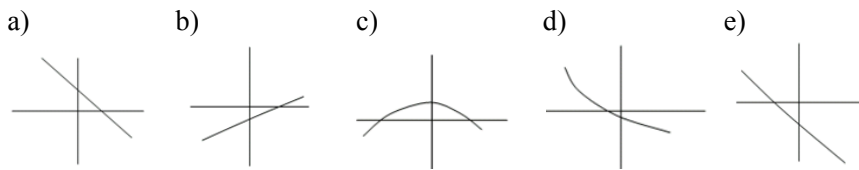
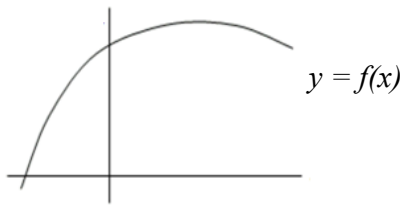
2. The derivative of a function f , is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

3. The graph of the derivative, $g'(x)$ of function g is given as follows. What is the value of $g'(2)$?

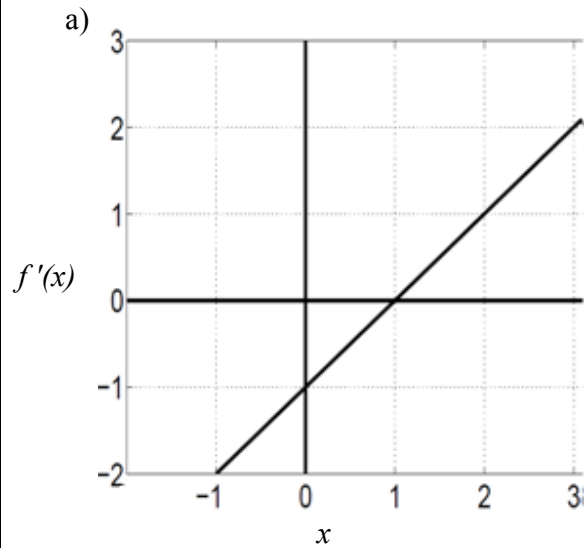
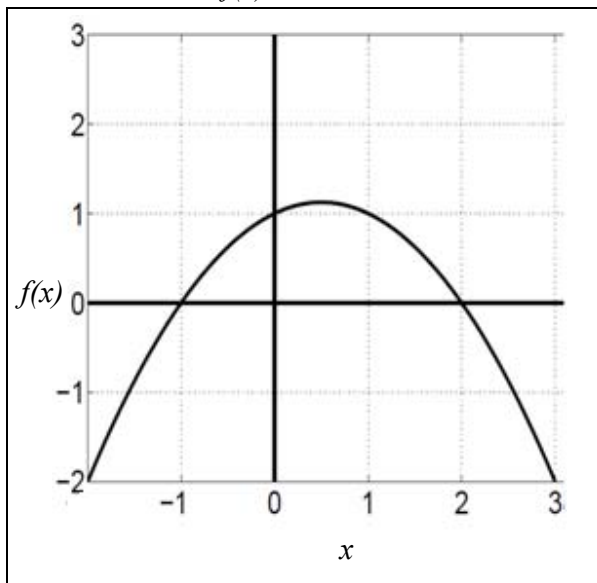


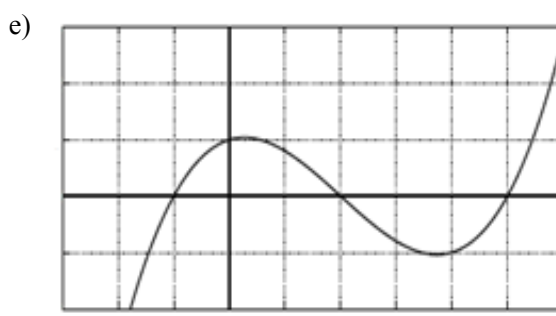
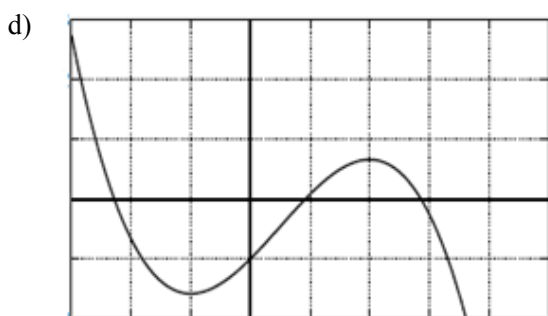
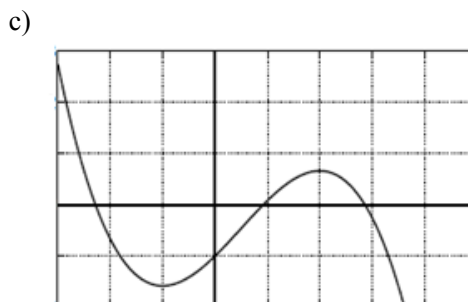
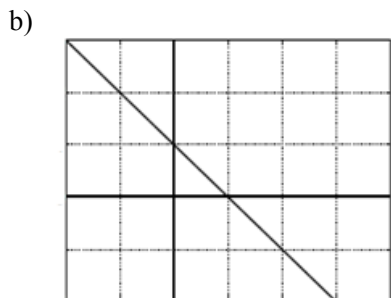
- a) -4
- b) -2
- c) 0
- d) 2
- e) 4

4. Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the first derivative, $f'(x)$?



5. Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?





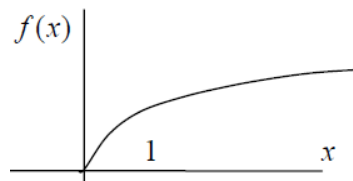
Why?

6. If a function is always positive, then what must be true about its derivative function?
- The derivative function is always positive.
 - The derivative function is never negative.
 - The derivative function is increasing.
 - The derivative function is decreasing.
 - You can't conclude anything about the derivative function.
7. The derivative of a function $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?
- The function $f(x)$ is positive on this interval.
 - The function $f(x)$ is negative on this interval.
 - The maximum value of the function $f(x)$ over the interval occurs at $x = 2$.

d) The maximum value of the function $f(x)$ over the interval occurs at $x = 3$.

e) We cannot tell any of the above.

8. The tangent line to this graph of $f(x)$ at $x = 1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?



- a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \leq f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \geq f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$ e) None of these

9. The derivative of a function, f , is $f'(x) = ax^2 + b$. What is required of the values of a and b so

that the slope of the tangent line to the function f will be positive at $x = 0$.

- a) a and b must both be positive numbers.
b) a must be positive, while b can be any real number.
c) a can be any real number, while b must be positive.
d) a and b can be any real numbers.
e) None of these

Why?

APPENDIX E. Instructor Interview Protocol

For each problem, following questions will be asked in order,

- a) How would you explain this question to students?
- b) In the process of explaining, I will ask some clarifying questions such as “the derivative”, refers to the derivative function, the derivative at a point, or there is no difference?”
- c) What do you think about the difficulty of this problem for students?
- d) What would make this problem hard (easy) for students to solve?

If they do not give detailed answers, I will ask

Because it is simple problem?

Because you covered similar problems during the class?

Because this problem is similar to exercise problems?

- e) What would be the most popular answer to students?
- f) Can you guess why students' choose this answer?

If it was incorrect choice, what would lead students to choose this answer rather than the correct one?

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