A STATE MODEL ANALYSIS OF ELECTRIC POWER SYSTEMS

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ABSTRACT

A STATE MODEL ANALYSIS OF ELECTRIC POWER SYSTEMS

by Albert L. Duke

Many of the electro-dynamic problems of electric power systems have been extensively investigated in the past primarily through the extensions of steady-state concepts, concepts which are applicable for linear conditions but not for the nonlinear conditions that often exist. No formal attempts have previously been made to analyze electric power systems in terms of the state models used effectively in nonlinear control and other system studies, nor to utilize the large body of theory developed for the effective study of such system models.

To generate the system state models used in this thesis it is convenient to express the component models in the state form. Such models are developed in terms of two sets of variables, consisting of 1) line-to-line voltages and two line currents and 2) voltage from one line to neutral and the neutral current. These variables are used to establish identical topological representations for both three and four wire components. The form of the model is then simplified by using a set of linear transformations of variables to define two sets of variables which are independent for balanced operating conditions and which can be associated with elements of a linear graph. The linear graph serves as a basis for selecting the variables to be used in the state vector and as a basis for formulating the state model of the system. In this application, the graph serves the same purpose in dynamic studies that symmetrical component sequence graphs serve in steady-state studies.

A systematic procedure is developed for modeling large electric power systems in terms of state models of subassemblies. Such a technique is essential to the study of large-scale systems. A typical system is utilized to exemplify the techniques involved and to illustrate the necessary detail.

The system solution is discussed and the stability concepts used in the power system industry are related to the mathematical concepts of stability and existence of solutions.

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by

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I INTRODUCTION

Historical Review

The development of the electric power system complex of today has taken place in large measure through the application of several distinctive but interrelated disciplines to solve the myriad problems of the industry. One of the first disciplines, the use of phasors to represent sinusoidally varying time functions, is perhaps the most widely known and commonly used type of analysis in power system engineering today. The use of phasors has served to greatly reduce the labor involved in manipulating the transcendental functions arising from the introduction of alternating current components. This type of analysis is a very direct and effective approach to the problems involving single-phase systems operating at steady-state conditions. The same approach is applied to the steady-state analysis of three-phase balanced systems when reduced to three equivalent single-phase systems and is effective in providing answers to a large class of problems.

The steady-state analysis of the unbalanced power system as presented by Fortescue [1] and further developed by Wagner and Evans [2], Clarke [3] and others [4], extends the use of phasors to unbalanced systems. The application of this type of analysis gives a particularly effective method, which is in widespread use, for designing protection against sustained system faults.

The more general dynamic or transient analysis problems can be classified into two general subclasses, the short-time or so-called electrical transient problem, and the longer-term dynamic problem, sometimes called the stability problem. These problems have until recent years been considered only in a piece-meal fashion.

The electric transient problem is almost always approximated by linear models, the rotating machinery in the system being considered as having constant velocities for the duration of the problem. Such a linear analysis also neglects the phenomena of hysteresis, saturation, and variation of resistance with frequency. In the past, attempts have been made to include such effects by the use of techniques similar to the "describing functions" discussed by Truxal [5 Chapter 10] and others, e.g., transient reactances and subtransient reactances.

The problem of "stability of power systems" is in reality more than one problem. The "transient stability" problem normally considered, is a true stability problem in the Liapunov [6] sense while the common "steady-state stability" problem defined by Crary [7 Section 25] and Kimbark [8 Chapter 1] is in reality not a stability problem but an existence problem. Until recently, analytical or graphical solutions to these two types of problems could only be obtained by making several rather gross approximations, some of which are: neglecting of damping factors in the dynamic equations, assuming constant rms voltages during mechanical oscillations, and isolation of machines by pairs. Examples of the graphical methods used include the "power circle diagrams" for studies of the existence problem and the "equal area method" for the stability problem [7], [8]. Prior to the advent of the large digital computer the analytical solutions were limited almost completely to the "swing equations" with either point-by-point or analog solutions. Systems considered were generally limited to one machine operating on an "infinite bus" or to two similar machines operating together. Extensions which were made to large systems were apparently based upon something similar to Bellman's "principal of wishful thinking" [9 page 7].

Recent Developments

Prior to the period of the early 1950's most studies involving larger systems composed of nonlinear elements were limited of necessity to the approximations discussed previously or to analog simulation devices such as the a.c. network analyzer. Subsequently, with the widespread use of large digital computers and the development of the electronic analog computer, investigations began to be undertaken on a theoretical basis. The first attempts to use numerical methods were extensions of the existing techniques to obtain machine solutions. Applications of electronic analog computers were on a somewhat more fundamental basis in that the differential equations of the components were considered to be solved rather than the system performance studied by comparison to analogous components.

The theoretical work in power systems that is necessary to take full advantage of machine techniques has been only recently begun. What might be called the first of this, by Lyon [10], was not actually directed at computer solutions but at transient studies of electrical machines. However, for the first time in a major study, the symmetrical component transformations were applied to instantaneous variables rather than to phasor variables. White and Woodson [11] along with Koenig and Blackwell [12 Chapters 11-14] extended these ideas using additional transformations and a more detailed analysis of several components, with a generalized modeling of the rotating machinery components. Koenig and Blackwell, along, with Gilchrist [13], using transformed instantaneous variables, introduced the multi-terminal component representation and began the theoretical development necessary to build a generalized system discipline highly suitable for computing-machine solutions. In introducing these concepts to the power system field, Koenig and Blackwell considered several types of polyphase systems such as single machines with known terminal conditions. Both linear and nonlinear representations of systems of two and three synchros and systems of two synchronous machines and other similar systems were considered by them with analytical solutions given in closed form for special conditions of operation. Gilchrist obtained numerical solutions to the more general nonlinear mathematical models describing the dynamics of two interconnected synchronous machines and one machine operating on an "infinite bus".

The system discipline, in which the component characteristics and the system topology are considered explicitly seems to offer the most logical means to build a discipline capable of being extended indefinitely both in system magnitude and in sophistication. Such a discipline, of course, must satisfy the correspondence principle, i.e. each of the previously mentioned disciplines must appear as special cases of the general discipline. It should be recognized at the outset, however, that, due to the nonlinearities of several of the components and the large numbers of components involved, there is no simple panacea which will supply the desired answers when the proper buttons are pushed and the crank turned. The large number of different disciplines, techniques and methods of solution can and should be brought under one central discipline.

General System Analysis

This thesis is considered to be an extension of the systems discipline as it applies to three-phase power systems. By means of the techniques developed here a capability is provided for mathematically modeling systems composed of large numbers of the type of nonlinear components normally used in three-phase systems. System models are established in such a form that disciplines developed in other areas can be brought to bear on the subject either directly or indirectly. This study is not to be thought of as a means to obtain a more accurate representation of certain components. No such attempt has been made. Rather, the purpose has been to apply the concepts and the advances made in the studies of systems during the past few years to the specific problems of electric power systems.

The critical factor in the use of the systems concept is the development of the component model. It would seem, with the many hundreds of studies made throughout this century on the subject of polyphase equipment, that a highly suitable model would have been developed. This has not been the case although some of the models developed were similar to the model developed here. The component models developed here and to an even greater extent the concepts involved in the development have appreciable significance.

An important property of balanced three-phase or, for that matter, balanced n-phase components is that the coefficient matrices of the component equations, whether algebraic or time varying, are cyclic symmetric [14]. Use of the symmetrical component transformation applied to the instantaneous phase variables takes advantage of the cyclic symmetric properties to diagonalize the coefficient matrices in the component models. If the phase connections were identical for all the components in a given electrical power system then without question this transformation would be effective for general system studies. The use of both delta and wye connections of the phase windings in the same system in large measure negates the advantages gained through the use of the symmetrical components variables. The two-phase or $\alpha\beta$ components of Clarke [3] obtained either from the measured phase variables or from the symmetrical components variables have the advantage of providing a model which is more suitable for computer solutions in that the entries of the coefficient matrices are real rather than complex numbers. The basic problem of the delta-wye interconnection nevertheless remains the same as for the symmetrical component model.

Other variables frequently used are the so-called rotating field or fb components of Y.H. Ku[15] which, like symmetrical components, lead to complex coefficient differential equations. The real coefficient counterpart of the fb components are the so-called cross-field or dq components extensively developed by R.H. Park [16]. The use of these components results in the removal of the effects of the machine rotation from the equations of themodel. From an alternative point of view, removing the rotational effects from the equations is the same as changing the coordinate system to a rotating reference frame, thereby simplifying the form of the model for the study of one machine and to a lesser degree for two machines. Considerable difficulty is encountered, however, in the analysis of systems involving several machines, each with a different frame of reference. The component model developed in this thesis retains the desirable properties of these transformations and also provides some additional properties which are necessary for general system use.

The variables used in modeling the terminal characteristics represent a set of measureable variables referred to as the x variables, consisting of one line-to-neutral and two line-to-line voltages along with two line currents and the neutral current. Application of a simplifying transformation of the measured variables provides two sets of variables, one called the scalar variables, very similar to the well known zero sequence variables and the other, called the vector variables, somewhat similar to the dq components. One difference between the dq components and the vector variables is that in this thesis the "nomial system frequency, ω, rather than the individual machine rotation is used as a common frame of reference for all machines of the system. The component models based on the x variables are particularly useful in steady-state studies. In dynamic studies where the only feasible means of solution is by computing machines a more suitable form of model is required. To effectively utilize the modern developments in system theory the component models are placed in a derivative explicit or state model form. Theorems developed by Wirth [17] for formulating state models are used here. The state models of power systems as established in this thesis also make it possible to apply the existence theorems of Ince [18], Murray and Miller [19] and Wirth, and bring the electric power system within the framework of the stability and optimization theorems presented by Liapunov, Bellman, Pontryagin and many others. In this thesis a preliminary investigation of the application of some of the existence and stability theorems is carried out, using a typical electric power system as an example.

II COMPONENT REPRESENTATION

The components to be discussed here are all in common use in three-phase systems. These components are usually distinguished by the presence of one or more three-phase ports having either three or four terminals each, as indicated in Figure 2 - 1 (a). As a convention, the terminals of each three-terminal port are lettered a, b and c and of each four-terminal port a, b, c and n, as indicated. Three-phase systems are formed by interconnecting the corresponding terminals of two or more ports; i.e., connecting a to a, b to b, c to c and, for four-terminal ports, connecting n to n.

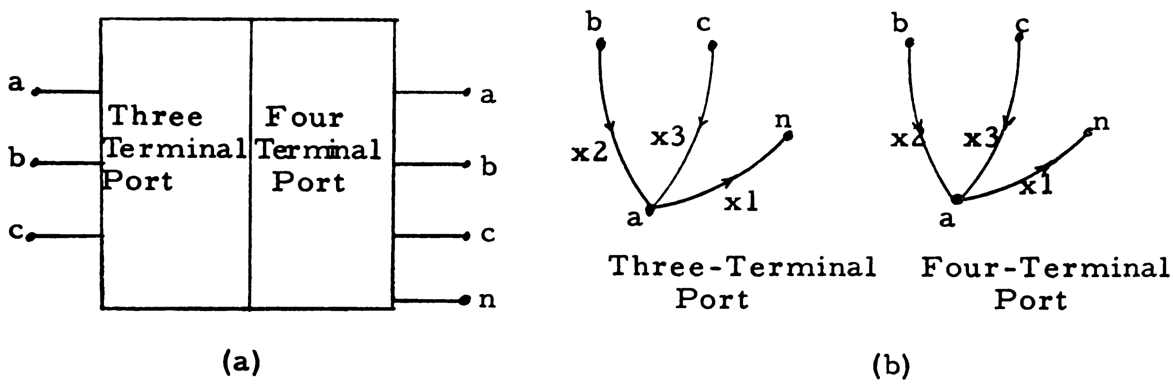


Figure 2-1. Representation of Three and Four Terminal Ports of Three-phase Components.

The component topological representation used in this thesis consists of a three-element terminal graph as shown in Figure 2 - 1(b) for both the three-terminal and the four-terminal ports.

The representation for each four-terminal port is established directly while the representation of each three-terminal port is established by the addition of the reference terminal n and the trivial element x1 which has the terminal equation $i_{x1} = 0$. This scheme presents each port as having an identical topological representation. The significance of this scheme becomes evident in the later development.

The characteristics of the components are modeled by associating with each port the vectors* of variables:

$$\underline{V}_x^{**} = \begin{Vmatrix} v_{x1} \\ v_{x2} \\ v_{x3} \end{Vmatrix} \qquad \text{and} \qquad \underline{I}_x = \begin{Vmatrix} i_{x1} \\ i_{x2} \\ i_{x3} \end{Vmatrix} \tag{2-1}$$

In general, if the vectors $\underline{V}_x$ and $\underline{I}_x$ are interrelated by a second or third order coefficient matrix of the form:

---

* The two vertical lines are used throughout the thesis to represent a vector or a matrix.

** The underscore is used to represent a vector variable or a matrix.

$$\underline{C}_2 = \begin{vmatrix} 2a_2 & a_2 \\ a_2 & 2a_2 \end{vmatrix} \quad \text{or} \quad \underline{C}_3 = \begin{vmatrix} a_1 & -a_2 & -a_2 \\ -a_2 & 2a_2 & a_2 \\ -a_2 & a_2 & 2a_2 \end{vmatrix} \qquad (2\text{-}2)$$

the component is said to be a balanced, algebraic component. The form of $\underline{C}_2$ and $\underline{C}_3$ suggests that a more convenient mathematical model of three-phase components of this type can be realized by applying a symmetric transformation of variables. This transformation is designed to diagonalize the coefficient matrices $\underline{C}_2$ and $\underline{C}_3$ and is obtained by taking $\underline{V}_T = \underline{T}\ \underline{V}_x$ and $\underline{I}_T = (\underline{T}^{-1})^T\ \underline{I}_x$* with

$$\underline{T} = \begin{vmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix} \quad \text{and} \quad (\underline{T}^{-1})^T = \begin{vmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{vmatrix} \qquad (2\text{-}3)$$

---

* The superscript T is used to represent the transpose of a vector or a matrix.

When the voltage vector at a four-terminal port is of the form:

$$\underline{V}_x = V \begin{vmatrix} \cos(\omega t-\theta) \\ \sqrt{3}\,\cos(\omega t-\theta - 150^{\circ}) \\ \sqrt{3}\,\cos(\omega t-\theta + 150^{\circ}) \end{vmatrix} \qquad (2\text{-}4)$$

the terminal voltages are said to be balanced with "carrier" frequency ω. The transformed port vector $\underline{V}_T$ is then of the form:

$$\underline{V}_T = \begin{vmatrix} v_{t1} \\ v_{t2} \\ v_{t3} \end{vmatrix} = \frac{\sqrt{3}}{\sqrt{2}}\,V \begin{vmatrix} 0 \\ \sin(\omega t-\theta) \\ -\cos(\omega t-\theta) \end{vmatrix} \qquad (2\text{-}5)$$

where it is evident that v_{t2} and v_{t3} form an orthogonal basis for the two dimensional vector space. The presence of the zero in the top position is used in the later development. For balanced systems the "carrier" is removed from the system model by means of a second transformation of variables established by taking $\underline{V}_m = \underline{M}\,\underline{V}_T$ and $\underline{I}_m = \underline{M}\,\underline{I}_T$ with

$$\underline{M} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Sin}\,\omega t & -\mathrm{Cos}\,\omega t \\ 0 & \mathrm{Cos}\,\omega t & \mathrm{Sin}\,\omega t \end{vmatrix} \tag{2-6}$$

where $\underline{M}$ is, of course, orthogonal. It is to be noted that $\underline{M}$ and $\underline{T}$ are non-singular for all t with determinants equal to unity. For the balanced terminal voltages the port vector $\underline{V}_m$ takes the form:

$$\underline{V}_m = \begin{vmatrix} v_{m1} \\ v_{m2} \\ v_{m3} \end{vmatrix} = \frac{\sqrt{3}}{\sqrt{2}}\, V \begin{vmatrix} 0 \\ \mathrm{Cos}\,\theta \\ -\mathrm{Sin}\,\theta \end{vmatrix} \tag{2-7}$$

Combining the two transformations given above:

$$\underline{V}_m = \underline{M}\,\underline{T}\,\underline{V}_x \text{ and } \underline{I}_m = (\underline{M}^{-1})^T (\underline{T}^{-1})^T \underline{I}_x \text{ or } \underline{I}_x = \underline{T}^T \underline{M}^T \underline{I}_m \tag{2-8}$$

with:

$$\underline{M}\,\underline{T} = \frac{\sqrt{2}}{\sqrt{3}} \begin{vmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \mathrm{Cos}\,(\omega t - 120^{o}) & \mathrm{Cos}\,(\omega t + 120^{o}) \\ 0 & -\mathrm{Sin}\,(\omega t - 120^{o}) & -\mathrm{Sin}\,(\omega t + 120^{o}) \end{vmatrix}$$

The question arises at this point, what is the effect of the transformations on the terminal graphs of the variables $\underline{V}_x$ and $\underline{I}_x$?

The oriented linear graph as used by Reed [20] and others for two terminal elements is based upon using an oriented line segment to represent a pair of measurements taken in a specified manner. Koenig [12] and others extended this approach to multiterminal components through the use of n-1 elements to represent 2(n-1) measurements on n-terminal components. This procedure can be further extended to identify the variables defined by the above transformations with a set of measurements. This identification is established by considering the measuring device shown in figure 2-2, which consists of d.c. amplifiers and idealized a.c. generators with each generator having two quadrature fields. The functioning of this device can be seen to be exactly that of the transformations $\underline{T}$ and $\underline{M}$. When suitable values are taken for the resistors shown, the potentials at the points j, k and l represent the voltage variables v_{t1}, v_{t2} and v_{t3} respectively. When the rotor-stator coupling coefficients of the a.c. generators have the proper values, the voltages represented as v_1, v_2 and v_3 are seen to be the variables resulting from the application of the transformation, M, to the vector of variables, V_T. Similar devices serve to identify the transformed currents and also the inverse relations giving the vectors of variables in terms of the transformed variables.

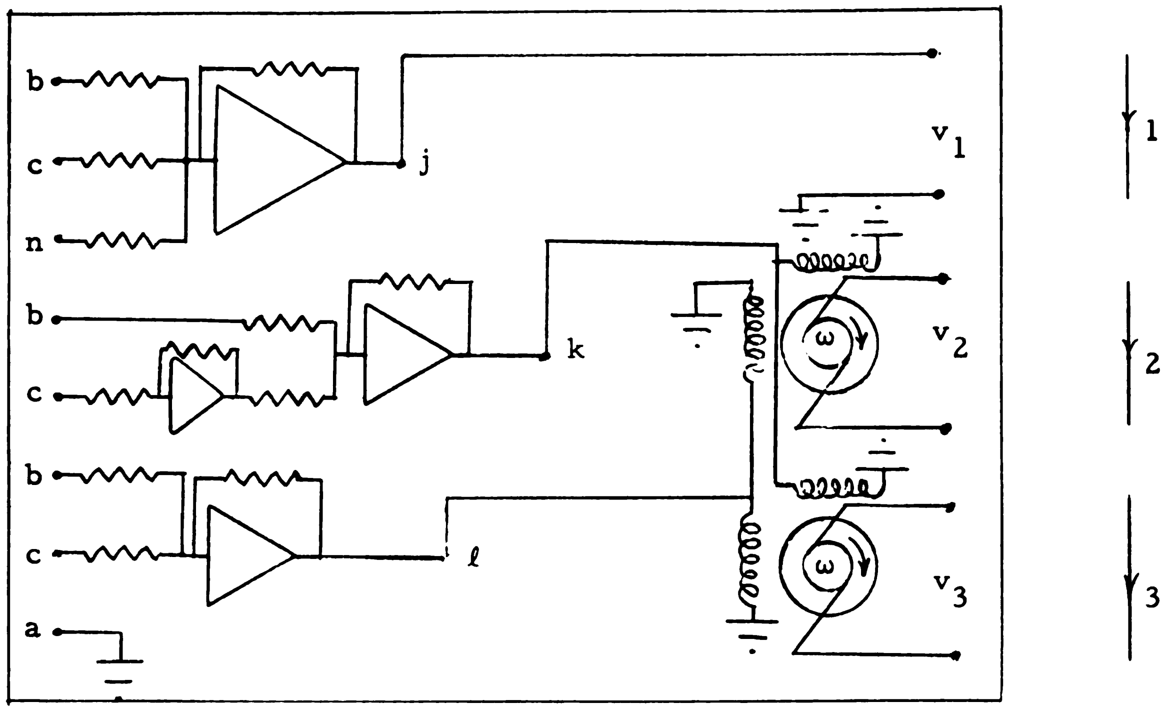


Figure 2-2. A Device for Measuring Transformed Voltages.

All component and system models and all analyses given in this thesis are presented in terms of the port vectors $\underline{V}_m$ and $\underline{I}_m$ or their subsets and the associated graphs defined by the type of instrumentation shown in the figure. The form of mathematical model relating these port vectors can be developed from models of the components such as those given by Koenig and Blackwell [12 Chapter 11], or similar forms wherein the phase windings are represented as three- or four-terminal ports.

The models so obtained can certainly be presented in any one of several forms each involving a degree of approximation. Any mathematical model must be balanced between the two extremes of being so simple as to produce no useful results and of being so complex as to be mathematically intractable. In general, the approximations used here closely parallel the assumptions made by other investigators.

The final form of the component model developed is the so-called state model which has the general form

$$\frac{d}{dt}\begin{vmatrix} \underline{X}(t) \\ \\ \underline{Y}(t) \end{vmatrix} = \begin{vmatrix} \underline{F}[\underline{X}(t),\ \underline{Y}(t),\ \underline{E}(t)] \\ \\ \underline{G}[\underline{X}(t),\ \underline{Y}(t),\ \underline{E}(t)] \end{vmatrix} \tag{2-10}$$

In the next section these state model forms are shown explicitly for the important example of a synchronous machine.

III SYNCHRONOUS MACHINE MODEL

The procedures used in establishing a model of a component of a three-phase power system, as well as some of the implications of the model itself can be further clarified by means of an example. The example of the synchronous machine used here represents an important nonlinear component in electric power systems. The stator is considered as having three isolated single-phase ports in one model and as a three-phase three-or-four terminal port in a second model.

A mathematical model of the synchronous machine with isolated phase windings may be established by considering it to be a five-port component as illustrated in figure 3-1.

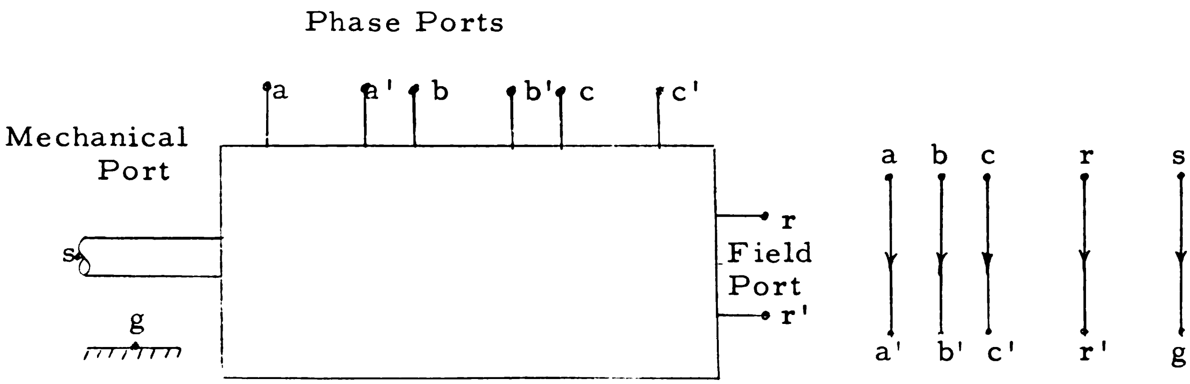


Figure 3-1. Representation of a Synchronous Machine as a Five-port Component.

The form of the equations usually used in modeling the terminal characteristics of the five-port synchronous machine without damper windings is developed elsewhere [11], [12] and is of the form

$$
\begin{vmatrix} v_a(t) \\ v_b(t) \\ v_c(t) \\ v_r(t) \end{vmatrix} = \begin{vmatrix} R_{aa}+\frac{d}{dt}L_{aa}(\phi) & \frac{d}{dt}L_{ab}(\phi) & \frac{d}{dt}L_{ac}(\phi) & \frac{d}{dt}L_{ar}(\phi) \\ \frac{d}{dt}L_{ab}(\phi) & R_{bb}+\frac{d}{dt}L_{bb}(\phi) & \frac{d}{dt}L_{bc}(\phi) & \frac{d}{dt}L_{br}(\phi) \\ \frac{d}{dt}L_{ac}(\phi) & \frac{d}{dt}L_{bc}(\phi) & R_{cc}+\frac{d}{dt}L_{cc}(\phi) & \frac{d}{dt}L_{cr}(\phi) \\ \frac{d}{dt}L_{ar}(\phi) & \frac{d}{dt}L_{br}(\phi) & \frac{d}{dt}L_{cr}(\phi) & R_{rr}+\frac{d}{dt}L_{rr} \end{vmatrix} \begin{vmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \\ i_r(t) \end{vmatrix}
$$

$$
\tau(t) = -\left| i_a(t)\ i_b(t)\ i_c(t) \right| \frac{d}{d\phi} \begin{vmatrix} L_{ar}(\phi) \\ L_{br}(\phi) \\ L_{cr}(\phi) \end{vmatrix} i_r(t) + (B(\phi) + J\frac{d}{dt})\,\dot{\phi}(t) \; ^{*} \qquad (3\text{-}1)
$$

The inductance coefficients in these equations, in general, are functions of shaft position as indicated and may also be functions of the currents, if the latter type of nonlinearity is to be included in the model. The coil inductances L_{aa}, L_{bb}, L_{cc}, L_{ab}, L_{ac}, L_{bc} and L_{rr} are periodic functions of shaft position and each can be represented in terms of Fourier Series. If the rotor and stator are cylindrical and concentric, i.e., if the slot and salient pole effects are neglected, then

---

* The overdot, e.g. $\dot{x}$, is used to represent the derivative with respect to time.

these coefficients can be considered independent of ϕ. Under the same conditions the coefficients $L_{ar}(\phi)$, $L_{br}(\phi)$ and $L_{cr}(\phi)$ are taken as:

$$\begin{vmatrix} L_{ar}(\phi) \\ \\ L_{br}(\phi) \\ \\ L_{cr}(\phi) \end{vmatrix} = \begin{vmatrix} L_{ar} \text{ Cos } \phi \\ \\ L_{br} \text{ Cos } (\phi - \theta_b) \\ \\ L_{cr} \text{ Cos } (\phi - \theta_c) \end{vmatrix} \qquad (3\text{-}2)$$

When stator windings are identical the third order coefficient matrix associated with the stator ports is cyclic symmetric, i.e. $L_{ab} = L_{ac} = L_{bc}$, $L_{aa} = L_{bb} = L_{cc}$ and $R_{aa} = R_{bb} = R_{cc}$ and the coupling coefficients are equated, i.e., $L_{ar} = L_{br} = L_{cr}$ and $\theta_b = -\theta_c = 120^o$. When the salient pole effects are included, an additional term is included in the series expansion of all inductance coefficients except L_{rr}.

Perhaps the most severe limitation of the model of equations (3-1) is the fact that the inductance coefficients are considered independent of the currents and an attempt to include these nonlinearities at this point leads to intractable mathematics. If saturation effects must be included in the model it is easier to do so at a later stage in the development.

The coefficient B in the torque equation is usually considered as a constant although there is no difficulty in so far as numerical solutions are concerned when the coefficient is considered to be a function of $\dot{\phi}$.

The form shown in equations (3-1) serves as a basis for deriving the general form of a more acceptable model. Connecting the stator windings of the machine as a four-terminal wye reduces the machine from a five-port component to a three-port component as shown in figure 3-2.

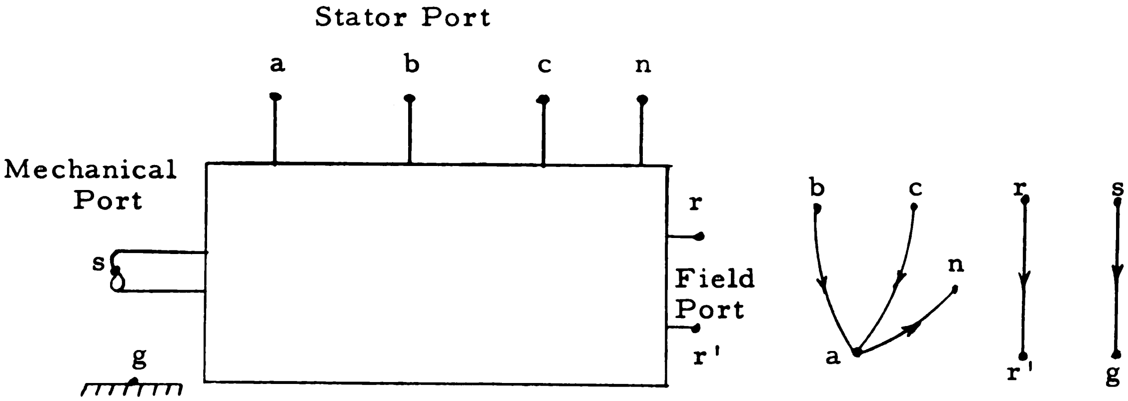


Figure 3-2. Representation of a Synchronous Machine as a Three-port Component

When the characteristics of the four-terminal three-phase port are modeled in terms of the vectors $\underline{V}_x$ and $\underline{i}_x$ defined in equation (2-1), the terminal equations as derived from the previous five-port model are

$$\begin{vmatrix} v_{x1}(t) \\ v_{x2}(t) \\ v_{x3}(t) \\ --- \\ v_r(t) \end{vmatrix} = \begin{vmatrix} Z_a & -Z & -Z & \frac{1}{\sqrt{3}} M(\emptyset) \\ -Z & 2Z & Z & M(\emptyset-150^\circ) \\ -Z & Z & 2Z & M(\emptyset+150^\circ) \\ \hline \frac{1}{\sqrt{3}} M(\emptyset) & M(\emptyset-150^\circ) & M(\emptyset+150^\circ) & Z_r \end{vmatrix} \begin{vmatrix} i_{x1}(t) \\ i_{x2}(t) \\ i_{x3}(t) \\ ----- \\ i_r(t) \end{vmatrix}$$

$$\tau_s(t) = \frac{P}{2} \left| i_{x1}(t) \; i_{x2}(t) \; i_{x3}(t) \right| \frac{d}{d\emptyset} L_{ar} \begin{vmatrix} \text{Cos } \emptyset \\ \sqrt{3}\text{Cos}(\emptyset-150) \\ \sqrt{3}\text{Cos}(\emptyset+150) \end{vmatrix} i_r(t) + T_m(\dot{\emptyset}) \tag{3-3}$$

where $Z_a = R_a + \frac{d}{dt} L_a$, $Z = R_a + \frac{d}{dt} (L_a - L_{ab}) = R + \frac{d}{dt} L$

$$Z_r = R_r + \frac{d}{dt} L_r \text{ , } M(\emptyset) = \sqrt{3} \frac{d}{dt} L_{ar} \text{ Cos } \emptyset \tag{3-4}$$

$$T_m(\dot{\emptyset}) = \frac{2}{P} (B + J \frac{d}{dt}) \dot{\emptyset}(t), \; P = \text{No. of poles.}$$

The form of the equations (3-3) as well as the values of the coefficients, can be determined directly from measurements. For example, if i_{x2}, i_{x3}, and i_r are set equal to zero then, by applying a suitable signal to the terminals represented by x1, the values of the coefficients R_a, L_a, L, R, and L_{ar} are determined from the measured relationships between suitable pairs of variables.

The three-terminal machine model can also be established either by derivation or by measurement. The relations between the coefficients of the three and four terminal ports, as well as the differences due to wyeor delta internal connections, are apparent when the models are derived from the five-port model of the previous section. The differences in the wye and delta configurations, as would be expected, are reflected in the values of the constant coefficients and in a shift of the rotational reference. Another difference, usually ignored or not recognized, is the presence of an auxilary equation of the form $\frac{d}{dt}$ i = Ki for the delta configuration. This term is insignificant to the external characteristics. The three-terminal model can be written as

$$i_{x1}(t) = 0$$

$$\begin{vmatrix} v_{x2}(t) \\ v_{x3}(t) \\ v_r(t) \end{vmatrix} \quad \begin{vmatrix} 2Z & Z & \sqrt{3}\,M(\phi-150) \\ Z & 2Z & \sqrt{3}\,M(\phi+150) \\ \sqrt{3}\,M(\phi-150) & \sqrt{3}\,M(\phi+150) & Z_r \end{vmatrix} \begin{vmatrix} i_{x2}(t) \\ i_{x3}(t) \\ i_r(t) \end{vmatrix}$$

$$\tau_s(t) = \frac{P}{2} \quad \left| i_{x2}(t) \; i_{x3}(t) \right| \frac{d}{d\phi} \; \sqrt{3} L_{ar} \begin{vmatrix} \text{Cos}(\phi-150) \\ \text{Cos}(\phi+150) \end{vmatrix} i_r(t) + \frac{2}{P}(B + J\frac{d}{dt}) \; \dot{\phi}(t)$$

The delta coefficients of equations (3-5) are related to the wye coefficients of equations (3-4) with

$$Z_\Delta = \frac{1}{3} Z_y, \; L_{ar\Delta} = \frac{1}{\sqrt{3}} L_{ar\,y}, \; \phi_\Delta = \phi_y - 90^\circ \qquad (3\text{-}5)$$

Note that the coefficient matrix of the above voltage-current equations is a sub-matrix of the corresponding four-terminal model.

The application of the transformations of variables to both the four-terminal and three-terminal model is greatly facilitated by compacting the notation. The four terminal model of equations (3-3) after applying the transformations $\underline{T}$ and $\underline{M}$, is written in matrix form as

$$\begin{vmatrix} \underline{V}_m(t) \\ \\ v_r(t) \end{vmatrix} = \begin{vmatrix} \underline{M}\,\underline{T}\,\underline{Z}_x\,\underline{T}^T\,\underline{M}^T & \underline{M}\,\underline{T}\frac{d}{dt}\underline{L}_{ar}(\phi) \\ \\ \frac{d}{dt}\,[\underline{L}_{ar}(\phi)]^T\,\underline{T}^T\underline{M}^T & Z_r \end{vmatrix} \begin{vmatrix} \underline{I}_m(t) \\ \\ i_r(t) \end{vmatrix} \tag{3-6}$$

$$\tau_s(t) = \frac{P}{2}\,[\underline{I}_m(t)]^T\,\underline{M}\,\underline{T}\,\frac{d}{d\phi}\;\underline{L}_{ar}(\phi)\;i_r(t) + T_m(\dot{\phi})$$

where

$$\underline{Z}_x = \begin{vmatrix} Z_a & -Z & -Z \\ \\ -Z & 2Z & Z \\ \\ -Z & Z & 2Z \end{vmatrix} \qquad \underline{L}_{ar}(\phi) = L_{ar} \begin{vmatrix} \text{Cos}\ \phi \\ \\ \sqrt{3}\ \text{Cos}(\phi - 150^\circ) \\ \\ \sqrt{3}\ \text{Cos}(\phi + 150^\circ) \end{vmatrix} \tag{3-7}$$

and $T_m(\dot{\phi}) = \frac{2}{P}\,(B + J\frac{d}{dt})\,\dot{\phi}(t)$

Since the product $\underline{T}\ \underline{Z}_x\ \underline{T}^T$ is diagonal and since the column matrix obtained from the product $\underline{T}\ \underline{L}_{ar}(\phi)$ has a zero in the top position, the top equation in the four-terminal model is independent of all remaining equations and is of the simple form $v_{m1}(t) = Z_1\, i_{m1}(t)$. The relation between the coefficient Z_1 and the coefficients in the five-port model is $Z_1 = R_a + \frac{d}{dt}(L_a + 2L_{ab})$. The remaining equations of the four-terminal model are of the same form as thoseobtained by application of the transformations of the variables to the three-terminal model of equations (3-5).

Since the rotational velocity of the machine is nearly equal to the nominal frequency of the system of which it is a part it is desirable to use the difference function $\alpha(t) = \phi(t) - \omega t$ as a variable in establishing the model rather than $\phi(t)$. The resulting equations then can be expressed as

$$v_1(t) = Z_1\, i_1(t)$$

$$\begin{vmatrix} \underline{V}(t) \\ \\ v_r(t) \end{vmatrix} = \begin{vmatrix} \underline{R} & -(\dot{\alpha}+\omega)\underline{L}_m(\alpha) \\ \\ -\dot{\alpha}\,[\underline{L}_m(\alpha)]^T & R_r \end{vmatrix} \begin{vmatrix} \underline{I}(t) \\ \\ i_r(t) \end{vmatrix} + \begin{vmatrix} \underline{L} & \underline{L}_m'(\alpha) \\ \\ [\underline{L}_m(\alpha)]^T & L_r \end{vmatrix} \frac{d}{dt} \begin{vmatrix} \underline{I}(t) \\ \\ i_r(t) \end{vmatrix}$$

$$\tau_s(t) = -\frac{P}{2}\ [\underline{I}(t)]^T\ \underline{L}_m(\alpha)\ i_r(t) + T_m(\dot{\alpha}) \qquad (3\text{-}8)$$

where:

$$\underline{R} = \begin{vmatrix} R & -\omega L \\ \omega L & R \end{vmatrix} \qquad \underline{L}_m(a) = L_m \begin{vmatrix} \text{Sin}a \\ -\text{Cos}a \end{vmatrix} \qquad \underline{L}_m'(a)^{*} = \frac{d}{da} \underline{L}_m(a) = L_m \begin{vmatrix} \text{Cos}a \\ \text{Sin}a \end{vmatrix}$$

$$\underline{L} = \begin{vmatrix} L & 0 \\ 0 & L \end{vmatrix} \qquad \underline{V}(t) = \begin{vmatrix} v_{m2}(t) \\ v_{m3}(t) \end{vmatrix} \qquad \underline{I}(t) = \begin{vmatrix} i_{m2}(t) \\ i_{m3}(t) \end{vmatrix}$$

$L_m = \dfrac{\sqrt{3}}{\sqrt{2}} L_{ar}$ (for wye machine) and $T_m(a) = \dfrac{P}{2} B\omega + \dfrac{P}{2}(B + J\dfrac{d}{dt})\dot{a}$

In realizing this result it is necessary first to expand the derivatives indicated in Eq. (3-6). This set of equations, (3-8), explicit in voltages and torques, represents a very useful model of a three-phase synchronous machine and is very similar to the equations used by other investigators. Specifically, if a and $\dot{a}$ are zero, the form of the

---

* The prime is used here to represent the derivative with respect to the argument of the function e.g. $f'(x) = \dfrac{d}{dx} f(x)$.

equations (3-8) reduces to that of the direct and quadrature component equations used extensively in power system studies. Essentially all theoretical work on power systems has been based on this type of model.

The state model form of equations (3-8) is obtained by solving for the derivatives. Although the only requirement for existence of the required inverse is the non-vanishing of the determinant $L(LL_r - L_m^2)$, it is noted that for practical machines the determinant is also positive. Since the only other leading principal minors L and L^2 are also positive the coefficient matrix is positive definite.

The resulting state model can be expressed in terms of matrix products or these multiplications can be executed and the resulting equations expressed functionally. The detailed model in the form of matrix products is

$$\frac{d}{dt} i_1 = \frac{1}{L_1} v_1 - \frac{R_1}{L_1} i_1 \qquad \text{or} \qquad i_1 = 0, \quad \frac{d}{dt} i_s = -\frac{R_3}{L_3} i_s$$

$$\frac{d}{dt}\begin{vmatrix} i_2(t) \\ i_3(t) \\ i_r(t) \end{vmatrix} = \frac{1}{LL_c^2}\begin{vmatrix} LL_r - L_m^2 \operatorname{Sin}a & L_m^2 \operatorname{Sin}a \operatorname{Cos}a & -LL_m \operatorname{Cos}a \\ L_m^2 \operatorname{Sin}a \operatorname{Cos}a & LL_r - L_m^2 \operatorname{Cos}^2 a & -LL_m \operatorname{Sin}a \\ -LL_m \operatorname{Cos}a & -LL_m \operatorname{Sin}a & L^2 \end{vmatrix}$$

$$
x \left| \begin{vmatrix} v_2(t) \\ v_3(t) \\ v_r(t) \end{vmatrix} - \begin{vmatrix} R & -\omega L & -L_m(\dot{a}+\omega)\operatorname{Sin} a \\ \omega L & R & L_m(\dot{a}+\omega)\operatorname{Cos} a \\ -L_m \dot{a} \operatorname{Sin} a & L_m \dot{a} \operatorname{Cos} a & R_r \end{vmatrix} \begin{vmatrix} i_2(t) \\ i_3(t) \\ i_r(t) \end{vmatrix} \right|
$$

$$
\frac{d}{dt}\dot{a} = \frac{P}{2J} \tau_s(t) - \frac{B\omega}{J} - \frac{B}{J} \dot{a} - \frac{P^2 L_m}{4J} \left| i_2(t) i_3(t) \right| \begin{vmatrix} -\operatorname{Sin} a \\ \operatorname{Cos} a \end{vmatrix} i_r(t)
$$

(3-9)

where $L_c^2 = L\, L_r - L_m^2$

When these multiplications are carried out and the notation condensed, the result is expressed as

$$
\frac{d}{dt} \begin{vmatrix} \underline{I}(t) \\ \\ i_r(t) \end{vmatrix} = \begin{vmatrix} \dfrac{-LL_r\underline{R} + \underline{L}_m(a)[\underline{L}_m(a)]^T \underline{R}}{L\, L^2} & \dfrac{-\omega \underline{L}_m(a)}{L} & \dfrac{R_r \underline{L}_m'(a)}{L_c^2} \\ \\ \dfrac{-[\underline{L}_m'(a)]^T \underline{R}}{L_c^2} & & \dfrac{-L\, R_r}{L_c^2} \end{vmatrix} \begin{vmatrix} \underline{I}(t) \\ \\ i_r(t) \end{vmatrix}
$$

$$
- \dot{\alpha} \begin{vmatrix} \dfrac{L L_r \underline{U} - \underline{L}_m(\alpha)[\underline{L}_m(\alpha)]^T}{L_c^2} & \dfrac{\underline{L}_m(\alpha)}{L} \\ \\ \dfrac{L}{L_c^2} \quad [\underline{L}_m(\alpha)]^T & 0 \end{vmatrix} \begin{vmatrix} \underline{I}(t) \\ \\ i_r(t) \end{vmatrix}
$$

$$
+ \begin{vmatrix} \dfrac{L L_r \underline{U} - \underline{L}_m(\alpha)\,[\underline{L}_m(\alpha)]^T}{L L_c^2} & \dfrac{\underline{L}'_m(\alpha)}{L_c^2} \\ \\ \dfrac{[\underline{L}'_m(\alpha)]^T}{L_c^2} & \dfrac{1}{L_c^2} \end{vmatrix} \begin{vmatrix} \underline{V}(t) \\ \\ v_r(t) \end{vmatrix}
$$

$$
\frac{d}{dt}\,\dot{\alpha}(t) = \frac{P}{2J}\;\tau_s(t) - \frac{B}{J}\,\omega - \frac{B}{J}\,\dot{\alpha}(t) + \frac{P^2}{2J}\;[\underline{I}(t)]^T\;\underline{L}_m(\alpha)\;i_r(t) \tag{3-10}
$$

Equations (3-9) and (3-10) can also be written in a more tractable functional notation.

$$
\frac{d}{dt} \begin{vmatrix} i_1(t) \\ \\ \underline{I}(t) \\ \\ i(t) \\ \\ \dot{\alpha}(t) \\ \\ \alpha(t) \end{vmatrix} = \begin{vmatrix} K_1\, i_1(t) + K_2\, v_1(t) \quad \text{or} \quad i_1(t) = 0 \\ \\ \underline{g}_1\,(\underline{I}(t),\; i(t),\; \dot{\alpha}(t),\; \alpha(t),\; v(t)\,) + \underline{f}_1(\alpha(t)\,)\,\underline{V}(t) \\ \\ g_2\,(\underline{I}(t),\; i(t)\; \dot{\alpha}(t)\; \alpha(t)\; v(t)\,) + \underline{f}_2\,(\alpha(t))\;\underline{V}(t) \\ \\ g_3\,(\underline{I}(t),\; i(t),\; \dot{\alpha}(t)\; \alpha(t)\,) + K_r\,\tau\,(t) \\ \\ \dot{\alpha}(t) \end{vmatrix} \tag{3-11}
$$

The form of the model of the three-phase synchronous machine (either delta or wye) developed here includes most of the major nonlinearities evident in the machine. The machine is modeled in terms of a set of variables related to the measured or x variables by a linear transformation. These variables can be measured directly using special instruments. The detailed form of the model in equations (3-9) shows that the equations are nonlinear in the speed variable $\dot{\alpha}$. If this variable is held constant the model consists of a set of linear first order differential equations and one algebraic equation. Even when the variable $\dot{\alpha}$ is considered to be time varying it is noted from equation (3-10) that the equations are linear in the variable V(t).

IV MODELS OF OTHER COMPONENTS

The procedure used to establish the state model for other three-phase components follows that given above for the synchronous machine, and can be summarized by the following three steps:

(1) Model the component in terms of the standarized variables $\underline{V}_x(t)$ and $\underline{I}_x(t)$ to realize a set of equations in the form

$$\underline{f}_1(\underline{V}_x) = \underline{f}_2(\underline{I}_x) \qquad (4\text{-}1)$$

(2) Transform the variables $\underline{V}_x(t)$ and $\underline{I}_x(t)$ to the more convenient vectors $\underline{V}_m(t)$ and $\underline{I}_m(t)$ to establish the model

$$\underline{f}_1(\underline{V}_m) = \underline{f}_2(\underline{I}_m) \qquad (4\text{-}2)$$

The vectors $\underline{V}_m$ and $\underline{I}_m$ are partitioned into subvectors referred to as the "scalar" variables

$$v(t) = v_m(t)$$

$$i(t) = i_m(t)$$

and the "vector" variables

$$\underline{V}(t) = \begin{vmatrix} v_{m2}(t) \\ \\ v_{m3}(t) \end{vmatrix} \qquad \underline{I}(t) = \begin{vmatrix} i_{m2}(t) \\ \\ i_{m3}(t) \end{vmatrix}$$

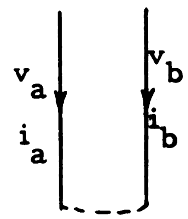
(3) Establish the state model form by solving for the derivative vector.

This solution involves the inversion of a coefficient matrix. The conditions for existence of this inverse place restrictions on the values of the coefficients of the component as already discussed for the case of the synchronous machine. Convenient forms of models thus established for several components, together with a short discussion of the limitations of these models, are found in the following paragraphs.

Three-phase Transformer Bank

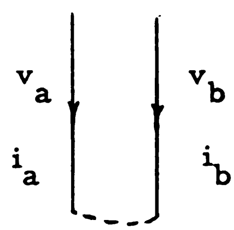
A model of a transformer, either single-phase or three phase, is frequently required to include the effects of the magnetizing currents. When these effects are included, the model may be established in a form explicit in the voltage, i.e. in terms of the open circuit parameters. At other times it may be permissable and desirable to neglect the magnetizing current. In the latter case the model cannot be established in a form explicit in the voltages of the windings, but must be modeled in the so-called h-parameters. This form of model can also be used to include the magnetizing effects by using the operator Z_m or $(R_m + L_m \frac{d}{dt})$ as illustrated in equations (4-4) rather than the form of equations(4-3). To show the techniques involved in modeling three-phase transformers and limitations of the models, consider first the single-phase transformers. Two of the basic forms of the models for single-phase transformers are

$$\begin{vmatrix} v_a(t) \\ i_b(t) \end{vmatrix} = \begin{vmatrix} Z_e & n \\ -n & 0 \end{vmatrix} \begin{vmatrix} i_a(t) \\ v_b(t) \end{vmatrix}$$



(4-3)

$$\begin{vmatrix} v_a(t) \\ Z_m v_b(t) \end{vmatrix} = \begin{vmatrix} Z_e & n \\ -nZ_m & 1 \end{vmatrix} \begin{vmatrix} i_a(t) \\ v_b(t) \end{vmatrix}$$



(4-4)

Where Z_e and Z_m represent the short and open-circuit parameters.

The two-winding transformer is obviously a four terminal component and should require a three element topological representation. However, the transformer in use is characterized by connections to the primary terminals and to the secondary. This use marks the transformer as a two-port isolation component or when one side of each winding is grounded it is a three terminal component. In either case the elements of the graph can be joined by the dashed lines shown above to obtain a three-terminal representation. Alternate forms of the transformer equations such as the open circuit form can also be used as a basic model but this frequently involves the problem of numerical accuracy. The former model uses the standard open and short circuit tests to establish the parameters, thus avoiding this problem.

In general it is convenient to normalize the transformer turns ratio. This can be done either by using the per unit system of measurements or by making a change of variables of the type $v'_b = nv_b$ and $i'_b = {}^1/_n \; i_b$.

Since the transformer is modeled as a three-terminal component with the turns ratio normalized, the model can be simplified by applying a tree transformation to the above graphs as indicated in figure 4-1.

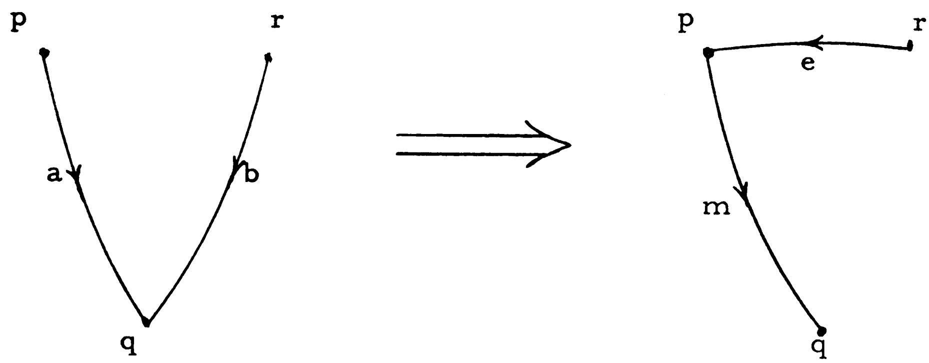
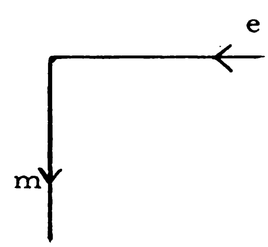


Figure 4-1. Application of a Tree Transformation

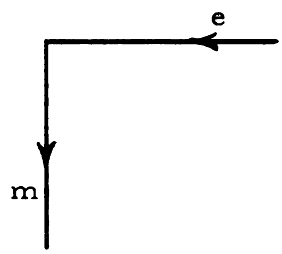
The resulting models of single-phase transformers are shown in equations (4-5) and (4-6). This form also provides better correlation between the linear model and the experimental variations observed due to the saturation effects of iron-core transformers.

$$\begin{vmatrix} v_e(t) \\ \\ i_m(t) \end{vmatrix} = \begin{vmatrix} Z_e \, i_e(t) \\ \\ 0 \end{vmatrix}$$



(4-5)

$$\begin{vmatrix} v_e(t) \\ \\ v_m(t) \end{vmatrix} = \begin{vmatrix} Z_e\, i_e(t) \\ \\ Z_m\, i_m(t) \end{vmatrix}$$

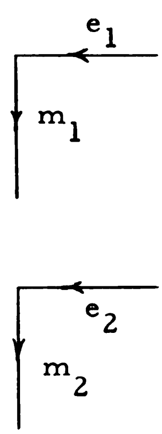


(4-6)

The technique described for modeling the single-phase transformer is readily extended to the three-phase transformer bank. This is accomplished by executing the first two steps of the procedure previously mentioned, followed by a tree transformation similiar to that used in the single-phase case. This tree transformation is applied to the vector variables and also to the scalar variables of each four-terminal port.

The model of the three-phase transformer bank thus established for the transformers connected wye-wye, with grounded neutrals and including magnetizing currents is

$$\left|\begin{array}{c} v_{e1}(t) \\ \hline v_{e2}(t) \\ v_{e3}(t) \end{array}\right| = \left|\begin{array}{c|cc} R_e + L_e \frac{d}{dt} & 0 & 0 \\ \hline 0 & R_e + L_e \frac{d}{dt} & -\omega L_e \\ 0 & \omega L_e & R_e + L_e \frac{d}{dt} \end{array}\right| \left|\begin{array}{c} i_{e1}(t) \\ \hline i_{e2}(t) \\ i_{e3}(t) \end{array}\right|$$



$$\left|\begin{array}{c} v_{m1}(t) \\ \hdashline v_{m2}(t) \\ v_{m3}(t) \end{array}\right| \quad \left|\begin{array}{c:cc} R_m + L_m\frac{d}{dt} & 0 & 0 \\ \hdashline 0 & R_m + L_m\frac{d}{dt} & -\omega L_2 \\ 0 & \omega L_m & R_m + L_m\frac{d}{dt} \end{array}\right| \left|\begin{array}{c} i_{m1}(t) \\ \hdashline i_{m2}(t) \\ i_{m3}(t) \end{array}\right| \quad \begin{array}{l} \leftarrow e_3 \\ \downarrow m_3 \end{array}$$

(4-7)

On separating the derivative terms and partitioning as indicated the model is

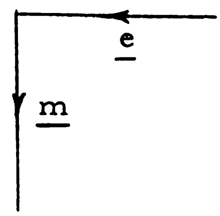
$$\begin{aligned} \underline{V}_e(t) &= \underline{R}_e \, I_e(t) + L_e \frac{d}{dt}\underline{I}_e(t) \\ \underline{V}_m(t) &= \underline{R}_m \underline{I}_m(t) + L_m \frac{d}{dt} \underline{I}_m(t) \end{aligned} \quad \begin{array}{l} \leftarrow \underline{e} \\ \downarrow \underline{m} \end{array}$$

$$\begin{aligned} v_e(t) &= R_e \, i_e(t) + L_e \frac{d}{dt} \, i_e(t) \\ v_m(t) &= R_m \, i_m(t) + L_m \frac{d}{dt} i_m(t) \end{aligned} \quad \begin{array}{l} \leftarrow e \\ \downarrow m \end{array}$$

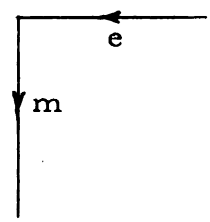
(4-8)

Solving the equations for the derivatives, the final model for the wye-wye transformer with grounded neutral is

$$\frac{d}{dt}\begin{vmatrix} \underline{I}_e(t) \\ \\ \underline{I}_m(t) \end{vmatrix} = \begin{vmatrix} -\frac{1}{L_e}\ \underline{R}_e \underline{I}_e(t) \\ \\ -\frac{1}{L_m}\ \underline{R}_m \underline{I}_m(t) \end{vmatrix} + \begin{vmatrix} \frac{1}{L_e}\ \underline{V}_e(t) \\ \\ \frac{1}{L_m}\ \underline{V}_m(t) \end{vmatrix}$$



$$\frac{d}{dt}\begin{vmatrix} i_e(t) \\ \\ i_m(t) \end{vmatrix} = \begin{vmatrix} -\frac{R_e}{L_e}\ i_e(t) \\ \\ -\frac{R_m}{L_m}\ i_m(t) \end{vmatrix} + \begin{vmatrix} \frac{1}{L_e}\ v_e(t) \\ \\ \frac{1}{L_m}\ v_m(t) \end{vmatrix}$$



(4-9)

The models for three-phase transformers connected delta-wye and delta-delta are similar to the models given above. The differences are 1) for the grounded wye-delta connection, the scalar equations become $i_b = 0$ and $v_a = Z_e\, i_a$ (or $v_a = Z_m\, i_a$ if the Y is ungrounded) and the matrix used to normalize the turns ratio takes the form shown by equations (4-10) rather than the previous form $\underline{N} = n\,\underline{U}$ and 2) for the delta-delta connection, the scalar equation reduces to $i_a = 0$ and $i_b = 0$ with $\underline{N}$ remaining as $\underline{N} = n\,\underline{U}$.

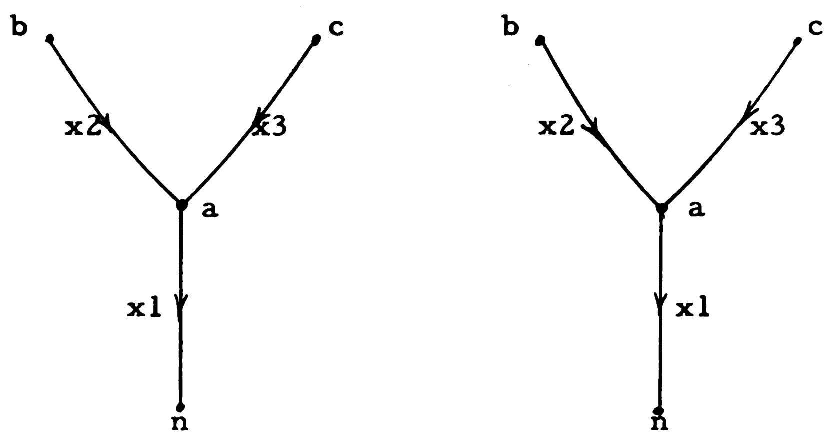
$$\underline{N} = \begin{vmatrix} 0 & \sqrt{3}\,n \\ \sqrt{3}\,n & 0 \end{vmatrix} \qquad (4\text{-}10)$$

Transmission Lines

Any one of several lumped parameter forms can be used as a model of a power transmission line. The specific form depends on the operating voltage level and the length of the line. One basic form from which other forms can be easily obtained is referred to as the "L" line or "L" section. This form considers all leakage effects, including resistive and capacitive leakage from line-to-line as well as line-to-ground, to be lumped at one end of the line. The technique for obtaining the state model of an "L" section is almost identical to that of the three-phase transformer of the previous section.

The basic form of the equations of the "L" section model can be obtained easily and accurately in terms of the measured variables. These equations for the grounded return transmission line are of the form

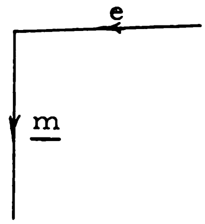
$$
\begin{vmatrix} v_{x1}(t) \\ v_{x2}(t) \\ v_{x3}(t) \\ ---- \\ i'_{x1}(t) \\ i'_{x2}(t) \\ i'_{x3}(t) \end{vmatrix}
=
\begin{vmatrix}
Z_a & -Z & -Z & 1 & 0 & 0 \\
-Z & 2Z & -Z & 0 & 1 & 0 \\
-Z & Z & 2Z & 0 & 0 & 1 \\
\multicolumn{6}{c}{----------------} \\
-1 & 0 & 0 & Y_o & \frac{1}{3}Y_o & \frac{1}{3}Y_o \\
0 & -1 & 0 & \frac{1}{3}Y_o & Y_1 & Y_2 \\
0 & 0 & -1 & \frac{1}{3}Y_o & Y_2 & Y_1
\end{vmatrix}
\begin{vmatrix} i_{x1}(t) \\ i_{x2}(t) \\ i_{x3}(t) \\ ---- \\ v'_{x1}(t) \\ i'_{x2}(t) \\ i'_{x3}(t) \end{vmatrix}
$$



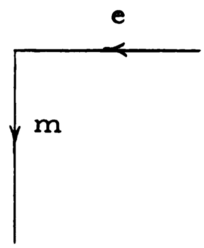
(4-11)

The model of the line in terms of the transformed variables can be expressed either as a two port vector model or as the following "three terminal" vector model.

$$\begin{vmatrix} \underline{V}_e(t) \\ \\ \underline{I}_m(t) \end{vmatrix} = \begin{vmatrix} \underline{Z}_e & \underline{0} \\ \\ \underline{0} & \underline{Y}_m \end{vmatrix} \cdot \begin{vmatrix} \underline{I}_e(t) \\ \\ \underline{V}_m(t) \end{vmatrix}$$



$$\begin{vmatrix} v_e(t) \\ \\ i_m(t) \end{vmatrix} = \begin{vmatrix} Z_o & 0 \\ \\ 0 & Y_o \end{vmatrix} \begin{vmatrix} i_e(t) \\ \\ v_m(t) \end{vmatrix}$$



(4-12)

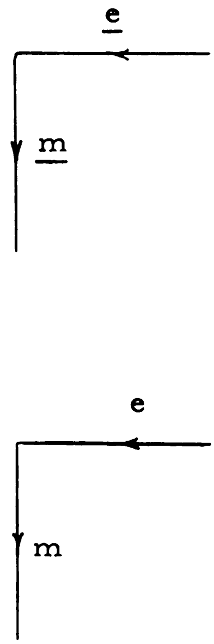
where $\underline{Z}_e = \underline{R}_e + \underline{L}_e \frac{d}{dt}$ and $\underline{Y}_m = \underline{G}_m + \underline{C}_m \frac{d}{dt}$ have the forms

$$\underline{Z}_e = \begin{vmatrix} R_e + L_e \frac{d}{dt} & -\omega L_e \\ \\ \omega L_e & R_e + L_e \frac{d}{dt} \end{vmatrix} \qquad \underline{Y}_m = \begin{vmatrix} g + c \frac{d}{dt} & -\omega c \\ \\ \omega c & g + c \frac{d}{dt} \end{vmatrix}$$

(4-13)

The state model is then

$$\frac{d}{dt}\begin{vmatrix} \underline{I}_e(t) \\ \underline{V}_m(t) \\ i_e(t) \\ v_m(t) \end{vmatrix} = \begin{vmatrix} -\frac{1}{L_e}\underline{R}_e\underline{I}_e(t) \\ -\frac{1}{c}\underline{G}_m\underline{V}_m(t) \\ -\frac{R}{L_e}\ i_e(t) \\ -\frac{g}{c}\ v_m(t) \end{vmatrix} + \begin{vmatrix} \frac{1}{L_e}\ \underline{V}_e(t) \\ \frac{1}{c}\ \underline{I}_m(t) \\ \frac{1}{L_e}\ v_e(t) \\ \frac{1}{c}\ i_m(t) \end{vmatrix}$$



(4-14)

This model can be easily reduced to the short line form by setting $\underline{I}_m$ and i_m to zero. To extend the model to the T or π models, two or more "L" sections are cascaded by considering them as components of a system, as discussed in a later section.

Induction Motor

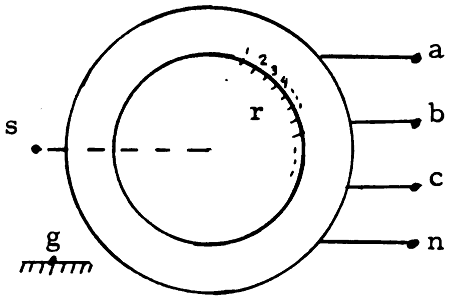
The three-phase induction motor model is obtained by considering the machine construction to differ from that of the synchronous machine only in the number of phases of the rotor and in the voltages applied to the rotor. The model for a wound rotor machine with three-phase windings on the rotor as well as the stator is determined by using the procedure already presented for the synchronous machine, with the stator and rotor each considered as three-phase ports. The model of a squirrel-cage induction motor is obtained by a slight variation of this procedure. The stator is considered as any other three-phase port and the rotor as an n-phase port. The differential equations of the machine model are obtained in terms of the stator x variables and the rotor phase variables. Since the rotor is to be short-circuited in use and not interconnected with other components, different transformations are used to simplify the differential equations. These transformations are essentially those used by Koenig and Blackwell [12chapter 12 and 13] for the n-phase rotor and are not considered further in this thesis except to note that an additional normalizing transformation is necessary in the squirrel-cage induction motor in order that the coefficients can be determined experimentally. The implications of the form of the model obtained are similar to those of the synchronous machine. It may be noted from the equations that, just as in the synchronous machine model, only the fundamental frequency effects are considered, thus neglecting the slot effects and the so-called "deep-bar effects". A more complete model for showing starting performance might use two sets of rotor bars, particularly if the deep-bar or double-cage type rotor is involved.

The experimental determination of the parameters of the model can be attained through steady-state operating tests. The standard no-load or synchronous test and blocked rotor test are sufficient.

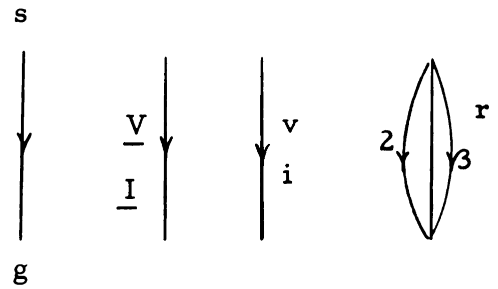
The resulting induction motor model is

Induction Motor

Schematic Diagram



Terminal Graph



State Model Terminal Equations for four-wire wye-wye connection

$$\frac{d}{dt}\, i_1(t) = \frac{1}{L_1}\, v_1(t) - \frac{R_1}{L_1}\, i_1(t)$$

$$\frac{d}{dt}\begin{vmatrix} \underline{I}_s(t) \\ \\ \underline{I}_r(t) \end{vmatrix} = \begin{vmatrix} \frac{1}{L_c^2}(L_r\underline{R} - \dot{\alpha} L_{sr}^2\,\underline{E}) & -\frac{1}{L_c^2}(\omega L_r L_{rs}\underline{U} + L_{rs}\underline{R}_r\underline{E}) \\ \\ \frac{1}{L_c^2}(L_{sr}\underline{R}\underline{E} - \dot{\alpha} L L_{sr}\underline{U}) & \frac{1}{L_c^2}(L\,\underline{R}_r - \omega L_{rs}^2\,\underline{U}) \end{vmatrix} \begin{vmatrix} \underline{I}_s(t) \\ \\ \underline{I}_r(t) \end{vmatrix}$$

$$+\begin{vmatrix} \frac{L_r}{L_o^2} & \underline{V}_s(t) \\ & \\ \frac{L_{sr}}{L_c^2} & \underline{V}_r(t) \end{vmatrix} \tag{4-15}$$

$$\frac{d}{dt}\dot{\alpha}(t) = \frac{1}{J}\ \tau_s(t) - \frac{B}{J}\ \omega - \frac{P}{2J}\ L_{sr}\ [\underline{I}_s(t)]^T\ \underline{I}_r(t) - \frac{B}{J}\ \dot{\alpha}_s(t)$$

where:

$$\underline{R} = \begin{vmatrix} R & -\omega L \\ & \\ \omega L & R \end{vmatrix} \qquad \underline{R}_r = \begin{vmatrix} R_r & \dot{\alpha} L_r \\ & \\ -\dot{\alpha} L_r & R_r \end{vmatrix} \qquad \underline{E} = \begin{vmatrix} 0 & -1 \\ & \\ 1 & 0 \end{vmatrix}$$

$$Z_1 = Z_s = R_1 + L_1 \frac{d}{dt} = R_A + (L_A + 2L_B) \frac{d}{dt},$$

$$Z = R + L \frac{d}{dt} = R_A + (L_A - L_{AB}) \frac{d}{dt}$$

$$L_c^2 = LL_r - L_{sr}^2 \qquad L_r = \frac{L}{L_r^+}, \qquad L_{sr} = \frac{L}{L_r^+}, \qquad L_{sr} = \frac{L}{L_r^+} \frac{\sqrt{3}\sqrt{n}}{2} L_{A1}$$

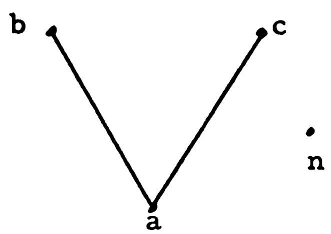
$$R_r = \frac{L}{L_r^+} \qquad R_r^+ = \frac{L}{L_r^+} R_1 \qquad B = \frac{2}{P}\ B_s \qquad J = \frac{2}{P}\ J_s \tag{4-16}$$

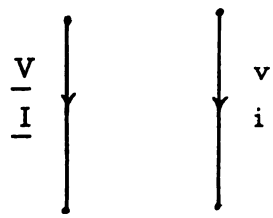
Fault Conditions

The various types of fault conditions, in fact any type of unbalanced fault or load, can be considered as simply another component with its own properties and characteristics. The terminal equations of the various conditions are conveniently written in terms of the x variables, and the direct application of the transformations gives the terminal equations in terms of the m variables. Typical fault conditions follow, where the lines between terminals indicate the fault connections.

Connections	Terminal Graphs	Terminal Equations

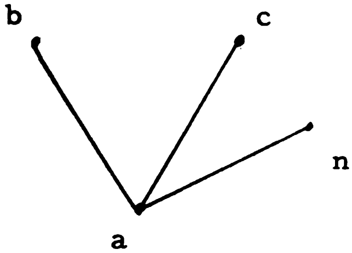
Three-phase

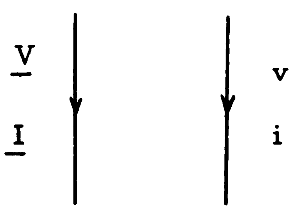




$$\begin{vmatrix} i(t) \\ \\ \\ \underline{V}(t) \end{vmatrix} = \begin{vmatrix} 0 \\ \\ \\ 0 \end{vmatrix} \qquad (4\text{-}17)$$

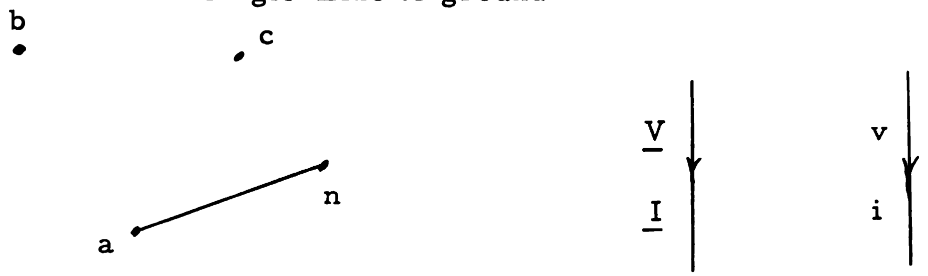
Three-phase to ground





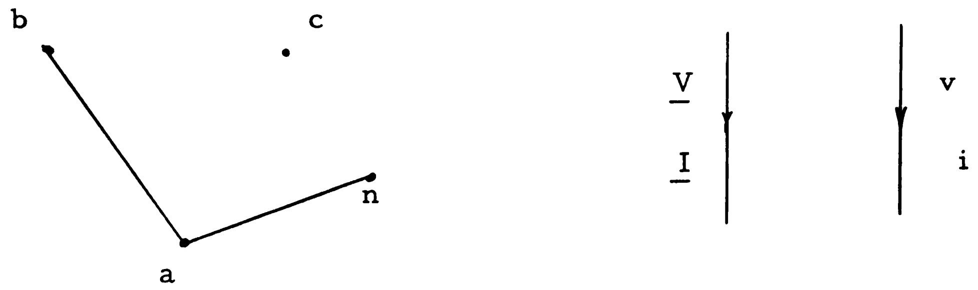
$$\begin{vmatrix} v(t) \\ \\ \\ \underline{V}(t) \end{vmatrix} = \begin{vmatrix} 0 \\ \\ \\ 0 \end{vmatrix} \qquad (4\text{-}18)$$

Single-Line to ground

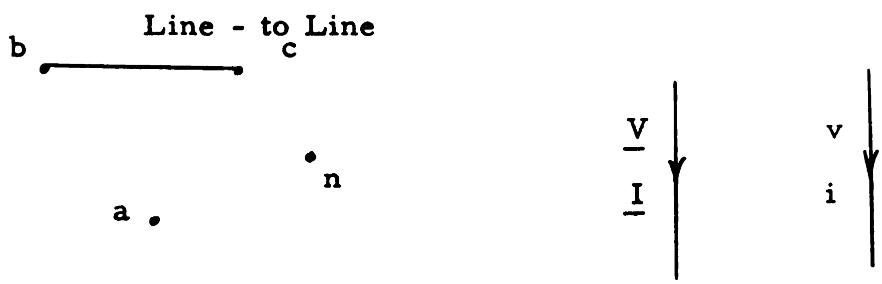


$$\begin{vmatrix} v_{m1}(t) \\ i_{m2}(t) \\ i_{m3}(t) \end{vmatrix} = \begin{vmatrix} 0 & -\sqrt{2}\,\mathrm{Cos}\,\omega t & \sqrt{2}\,\mathrm{Sin}\,\omega t \\ \sqrt{2}\,\mathrm{Cos}\,\omega t & 0 & 0 \\ -\sqrt{2}\,\mathrm{Sin}\,\omega t & 0 & 0 \end{vmatrix} \begin{vmatrix} i_{m}(t) \\ v_{m2}(t) \\ v_{m3}(t) \end{vmatrix} \qquad (4\text{-}19)$$

Double-Line to ground



$$\begin{vmatrix} i_{m1}(t) \\ \\ v_{m2}(t) \\ \\ v_{m3}(t) \end{vmatrix} = \begin{vmatrix} 0 & [\frac{1}{\sqrt{2}}(\sqrt{3}\,\text{Sin}\omega t + \text{Cos}\omega t)] & [\frac{1}{\sqrt{2}}(\sqrt{3}\,\text{Cos}\omega t - \text{Sin}\omega t)] \\ \\ \frac{\sqrt{3}}{\sqrt{2}}(\text{Sin}\omega t - \text{Cos}\omega t) & 0 & 0 \\ \\ \frac{1}{\sqrt{2}}(\text{Sin}\omega t + \text{Cos}\omega t) & 0 & 0 \end{vmatrix} \begin{vmatrix} v_{m1}(t) \\ \\ i_{m2}(t) \\ \\ i_{m3}(t) \end{vmatrix} \qquad (4\text{-}20)$$



$$\begin{vmatrix} 1 & 0 & 0 \\ \\ 0 & \text{Sin}\omega t & 0 \\ \\ 0 & 0 & \text{Sin}\omega t \end{vmatrix} \begin{vmatrix} i_{m1}(t) \\ \\ i_{m2}(t) \\ \\ i_{m3}(t) \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \\ 0 & 0 & -\text{Cos}\omega t \\ \\ 0 & \text{Cos}\omega t & 0 \end{vmatrix} \begin{vmatrix} v_{m1}(t) \\ \\ i_{m2}(t) \\ \\ v_{m3}(t) \end{vmatrix} \qquad (4\text{-}21)$$

It is to be noted from equations (4-19), (4-20) and (4-21) that the "carrier" frequency, ω, which was removed from the coefficient matrices under balanced conditions is not removed under unbalanced conditions.

V SYSTEM MODELS

System Topology

The compilation of the system graph for any given system follows directly from the component graphs and the interconnection pattern. With addition of trivial x1 elements as necessary, the system graph consists of three identical subgraphs. In the sequel it is convenient to refer to the graph in two separate subgraphs, 1) the "vector subgraph", which is composed of the subset of all x2 and x3 elements, and 2) the "scalar subgraph", which is composed of the subset of all x1 elements. Frequently the scalar subgraph is simplified by not showing the trivial x1 elements explicitly. If the system consists only of balanced components then the component equations of the x1 subset are linear in the x1 variables and independent of the x2 and x3 variables. These conditions imply that the system equations consist of two independent sets, with the set associated with x1 being a linear, homogeneous, autonomous set of equations. Typical subgraphs for the power system represented by figure 5-1 are given in figure 5-2 (a) and (b).

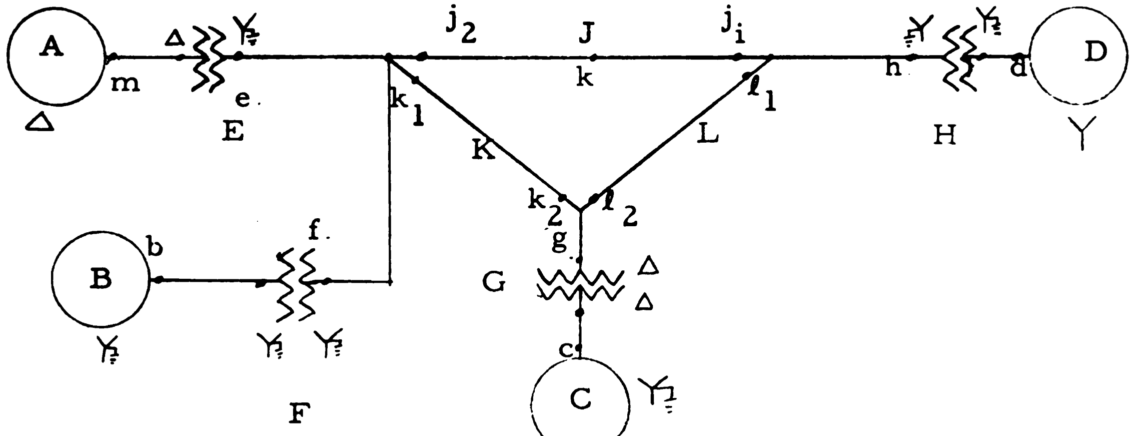
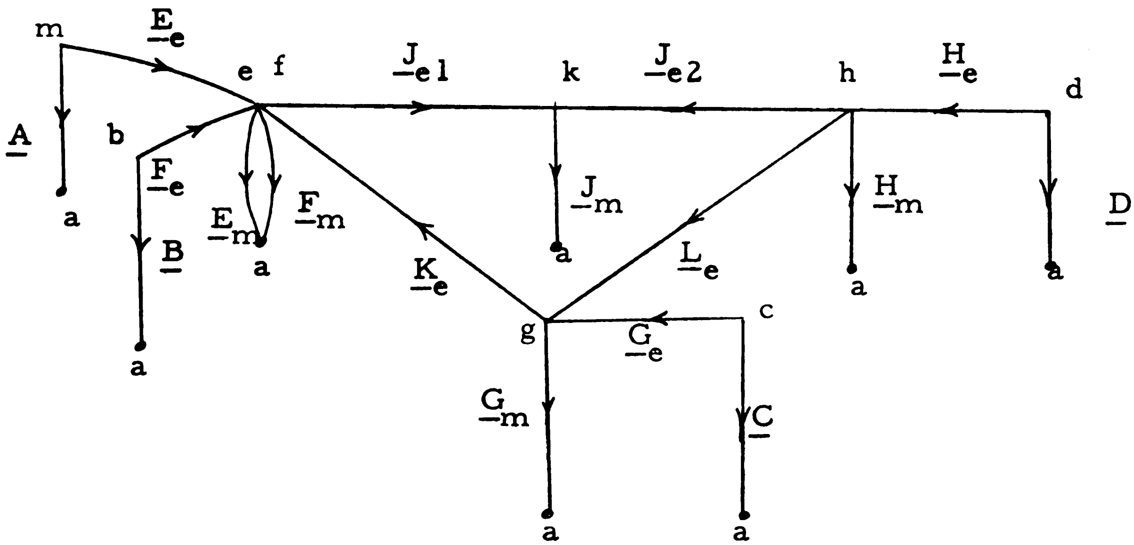
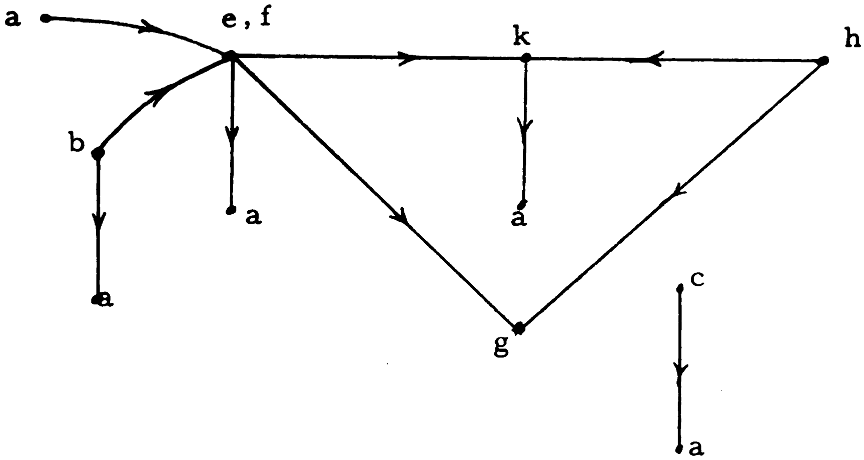


Figure 5-1. Single Line Diagram of a Typical Power System.



(a) Vector Subgraph



(b) Scalar Subgraph

Figure 5-2. System Graph of the System Represented by the Single Line Diagram of Figure 5-1.

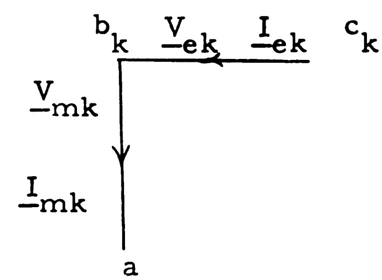
The vector subgraph, (5-2a), represents orthogonal components of the line-to-line voltages and line currents. The scalar subgraph, (5-2b), is similar to the zero phase sequence and represents the neutral voltages and ground currents.

The system model can be established from the component models and the system graph in a number of ways. It may frequently be desirable to combine some of the components into subassemblies.

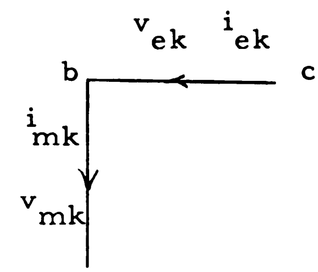
This is illustrated by the simple example of two identical transmission line "L" sections combined into one T section as follows.

The "L" section models for sections 1 and 2 are given by equation set(5-1) .

$$\frac{d}{dt}\begin{vmatrix} \underline{I}_{ek} \\ \\ \underline{V}_{mk} \end{vmatrix} = \begin{vmatrix} -\frac{1}{L_{ek}} & \underline{R}_{ek}\ \underline{I}_{ek} \\ \\ -\frac{1}{C_k} & \underline{G}_{mk}\ \underline{V}_{mk} \end{vmatrix} + \begin{vmatrix} \frac{1}{L_{ek}} & \underline{V}_{ek} \\ \\ \frac{1}{C_k} & \underline{I}_{mk} \end{vmatrix}$$



$$\frac{d}{dt}\begin{vmatrix} i_{ek} \\ \\ v_{mk} \end{vmatrix} = \begin{vmatrix} \frac{R_{ek}}{L_{ek}} & i_{ek} \\ \\ -\frac{g_k}{c_k} & v_{mk} \end{vmatrix} + \begin{vmatrix} \frac{1}{L_{ek}} & v_{ek} \\ \\ \frac{1}{C_k} & i_{mk} \end{vmatrix} \qquad (5\text{-}1)$$



for $k = 1, 2$

The graph of the system is given in figure 5-3,

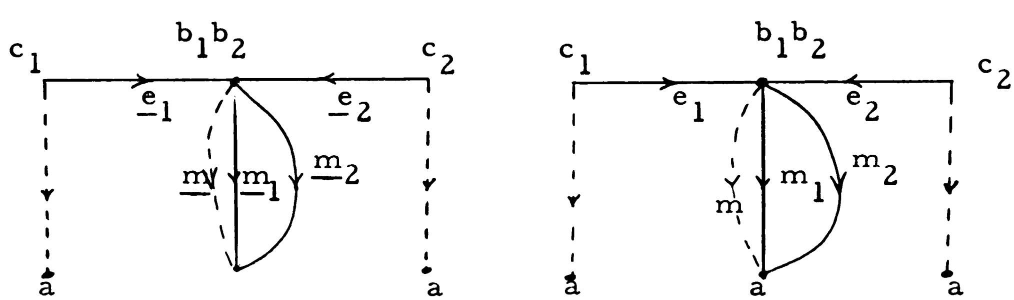


Figure 5-3. System Graph for Two L section Transmission Lines.

The graph gives the equations* $v_m = v_{m1} = v_{m2}$ $i_m = - m_1 - i_{m3}$ and $\underline{V}_m = \underline{V}_{m1} = \underline{V}_{m2}$, $\underline{I}_m = - \underline{I}_{m1} - \underline{I}_{m2}$. When these equations are substituted into the component equations and the results are simplified the model of the 'T'' subassembly is then

$$\frac{d}{dt}\left|\begin{matrix} \underline{I}_{e1} \\ \underline{I}_{e2} \\ \underline{V}_m \end{matrix}\right| = \left|\begin{matrix} -\frac{1}{L_e}\,\underline{R}_e\,\underline{I}_{e1} \\ -\frac{1}{L_e}\,\underline{R}_e\,\underline{I}_{e2} \\ \frac{1}{c}\,\underline{G}_m\,\underline{V}_m \end{matrix}\right| + \left|\begin{matrix} \frac{1}{L_e}\,\underline{V}_{e1} \\ \frac{1}{L_e}\,\underline{V}_{e2} \\ \frac{1}{c}\,\underline{I}_m \end{matrix}\right|$$

$$\frac{d}{dt}\left|\begin{matrix} i_{e1} \\ i_{e2} \\ v_m \end{matrix}\right| = \left|\begin{matrix} -\frac{R_e}{L_e}\,i_{e1} \\ -\frac{R_e}{L_e}\,i_{e2} \\ -\frac{g_m}{c_m}\,v_m \end{matrix}\right| + \left|\begin{matrix} \frac{1}{L_e}\,v_{e1} \\ \frac{1}{L_e}\,v_{e2} \\ \frac{1}{c}\,e_m \end{matrix}\right| \qquad (5\text{-}2)$$

with $R_e = R_{e1} = R_{e2}$ and $G_m = G_{m1} + G_{m2}$

---

* The argument (t) is not shown explicitly in the remainder of the thesis for the variables v, i, τ , $\dot{\alpha}$ and α except as necessary for clarity.

Formulation of System Models

A general method for obtaining the system model from the component model and the system graph is given by Wirth [17]. The five basic steps as given are listed for reference.

1) Select an appropriate forest of the system graph.

2) Write the fundamental circuit and cut-set equations in a form explicit in the chord across and branch through variables.

3) Substitute the fundamental circuit and cut-set equations into the component equations.

4) By elementary operations, eliminate the coefficient matrix multiplying the derivative vector.

5) Solve the algebraic equations for the variables not appearing in the derivative vector and substitute this result into the differential equations.

There are, of course, many forests that can be selected as indicated under 1). Selecting a forest is equivalent to selecting the state variables. In selecting a forest the general requirements as given by Wirth are observed, namely:

1) All the specified across drivers are made branches ;

2) As many as possible of the elements having equations explicit in the derivative of the across variable are branches;

3) As many as possible of the elements having equations explicit in the derivative of the through variables are chords;

4) All the through drivers are chords.

This requires that all specified voltage and rotational velocity drivers must be branches, all specified current and torque drivers must be chords. The elements in the terminal graphs used to model the fault conditions must be classified as part of the branch or chord system as follows.

1. Three-phase to ground fault
   - vector and scalar elements in the tree

2. Three-phase ungrounded fault
   - vector element in the tree and scalar element in the chord set.

3. Single-line to ground fault
   - vector element in the chord set and scalar element in the tree

4. Double-line to ground fault
   - vector element in the tree and scalar element in the chord set

5. Line to line ungrounded fault
   - scalar element in the chord set

Similar classifications apply, of course, to other unbalanced conditions. In most cases all of the elements with equations explicit in the derivatives of the across variables can be placed in a tree, however, more frequently than not all elements with equations explicit in the derivatives of the through variables cannot be placed in a chord set. In the partial graph of figure 5-2 the element J_m is placed in a tree but all other elements shown are explicit in current derivatives and hence should be in the chord set. No tree for which this is possible exists. A partial remedy can be found if the lines K and L can be represented by π sections, thereby establishing elements on each line for which the equations are explicit in the derivatives of the voltages. This reduces the number of branches which are explicit in the derivatives of currents. If this procedure is used indiscriminately, however, computer solutions may not exist due to the size of the numbers involved. One possible tree for the system of figure 5-2(a) is shown below, by the heavy lines. Line J is modeled as a "T" line and K and L as short lines. If an attempt is made to implement the remaining steps in the general formulation procedure of Wirth[17] it becomes immediately obvious that large blocks of equations must be manipulated analytically. (Approximately fifty differential equations would be involved for the system used as an example here). The difficulty of managing such large systems of equations is avoided by the application of a method which involves a repetitive series of operations applied to a smaller number of equations. This is an extremely desirable and perhaps necessary point in the consideration of large power systems and makes it possible to consider simultaneous formulation of different parts of the same system.

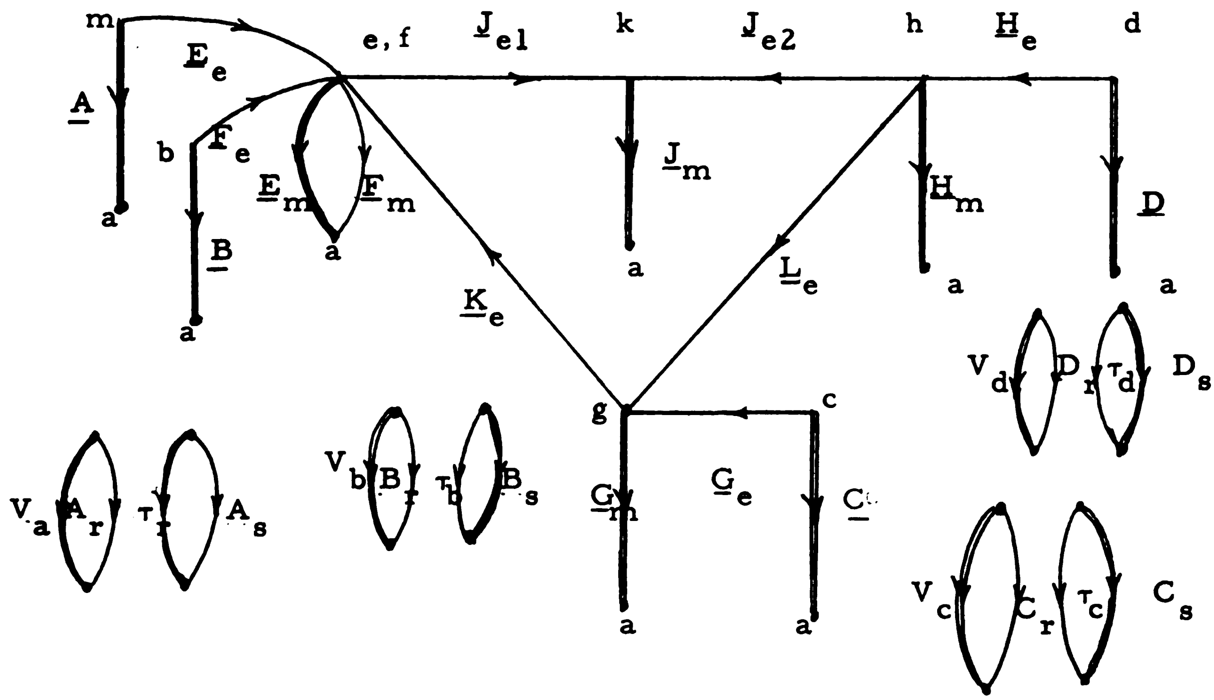


Figure 5-4. Vector Graph of the System of Figure 5-2(a) Showing One Possible Tree.

The technique used involves a separation of the system graph into convenient subgraphs with tie points or junction points retained as desired for present or future connection to other parts of the system. This procedure is similar to a technique described by Kron [21], [22] as "tearing". This is not to infer that the equations are to be solved in small groups. Such is, in general, not possible.

The technique can also be viewed as a procedure in which the equations are manipulated to obtain two groups, one of which is necessary for further use in the formulation process and is retained for additional manipulations. The other group of equations is not necessary for further formulation but is necessary for the solution phase and, consequently, must be retained.

VI EXAMPLE OF A SYSTEM MODEL

The concepts involved in the formulation process can, perhaps, best be presented by using the above system as an example. In order to provide as much simplicity as possible and still develop the desired concepts, certain conditions of operation of the system are considered.

1) The synchronous machines are considered to operate with constant field voltage and constant torque drivers. (The extension to different conditions, such as controlled torque or constant horsepower or other types of drivers, results in no more than the addition of other similar elements and equations).

2) No additional loads and no additional tie points are to be included. The extension to include additional tie points for future expansion is carried out by considering such tie points just as any other tie point until the equations are stored. An additional equation setting the appropriate currents to zero is simultaneously stored until such time as the future expansion is implemented.

System Classification

Since the system involved has no unbalanced components the scalar set of variable need not be considered. The system represented by the vector-graph can be conveniently classified into three general subsets as follows :

A ___ Transmission lines J, K and L

B ___ Transformer Banks E, F, G and H

C ___ Synchronous Machines A, B, C and D

Although the selection of the subgroups is to some extent arbitrary there are frequently some rather obvious considerations. For example, the elements representing nonlinear equations such as the synchronous machine are relegated to the last group in order to avoid the necessity of manipulating nonlinear equations at the early stages. Also, subgroups should be chosen, if possible, so that the tie point elements do not form a circuit or a cut-set. An example of such a cut-set appears in subgroup A when the leakage current of transmission line J is neglected, i.e., when element J_m is removed. When such is the case a transformer with its magnetizing current is included in this subassembly. The sleection of such assemblies is not necessary but if does simplify, somewhat, the formulation procedures.

Subgroup A - Transmission Lines

To continue the present example, consider the graph of the subassembly A as shown in figure 6-1.

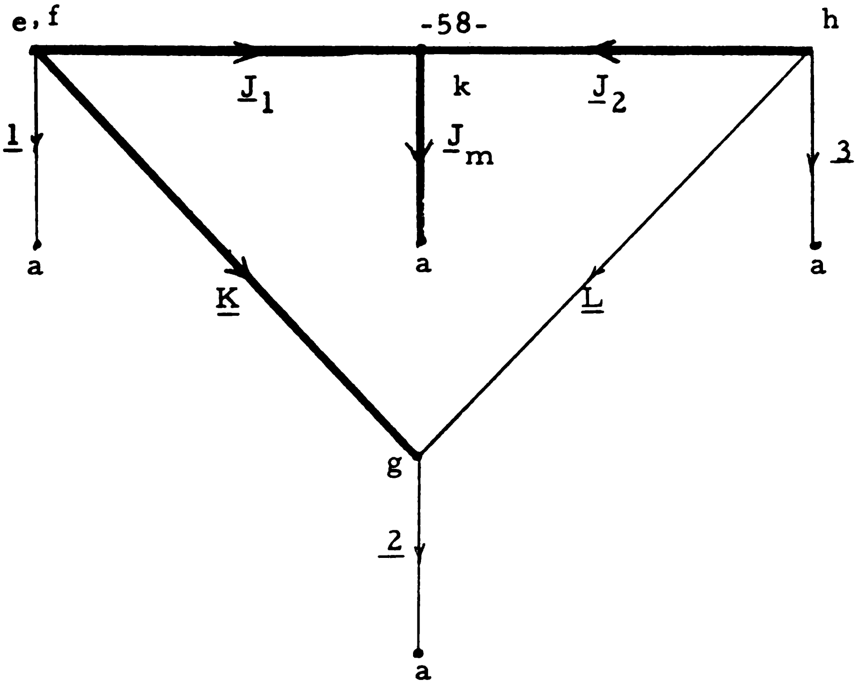


Figure 6-1 . System Subgraph of Subassembly

The elements 1, 2 and 3 are tie point elements of the subgraph with the remainder of the system and for formulation purposes are considered to be drivers. For convenience it is desirable that elements 1, 2 and 3 be considered as current drivers. If the three elements form a cutset, however, one of the three must be considered as a voltage driver.

The general procedure given by Wirth is followed in modeling the system of subassembly A. The component equations are

$$\text{Line J}\ \frac{d}{dt}\begin{vmatrix} \underline{I}_{j1} \\ \underline{I}_{j2} \\ \underline{V}_m \end{vmatrix} = \begin{vmatrix} -\frac{1}{L_e}\underline{R}_e\,\underline{I}_{j1} \\ -\frac{1}{L_e}\underline{R}_e\,\underline{I}_{j2} \\ \frac{1}{c}\,\underline{G}_m\,\underline{V}_m \end{vmatrix} + \begin{vmatrix} \frac{1}{L_e}\,\underline{V}_{j1} \\ \frac{1}{L_e}\,\underline{V}_{j2} \\ \frac{1}{c}\,\underline{I}_m \end{vmatrix} \qquad (6\text{-}1)$$

Line K $$\frac{d}{dt}\underline{I}_k = -\frac{1}{L_k}\underline{R}_k\underline{I}_k + \frac{1}{L_k}\underline{V}_k$$

Line L $$\frac{d}{dt}\underline{I}_L = -\frac{1}{L_L}\underline{R}_L\underline{I}_L + \frac{1}{L_L}\underline{V}_L$$

A tree is selected for the subgraph A with element J_m as a branch and with elements 1, 2, and 3 as chords. Elements J_1, J_2 and K are selected to complete thetree as shown by the heavy lines. The circuit and cutset equations can be written explicit in chord voltages and branch currents respectively, as follows

$$\begin{vmatrix} \underline{V}_1 \\ \underline{V}_2 \\ \underline{V}_3 \\ \underline{V}_L \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} \underline{V}_{j1} \\ \underline{V}_{j2} \\ \underline{V}_K \\ \underline{V}_m \end{vmatrix} \qquad \begin{vmatrix} \underline{I}_{j1} \\ \underline{I}_{j2} \\ \underline{I}_K \\ \underline{I}_m \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} \underline{I}_1 \\ \underline{I}_2 \\ \underline{I}_3 \\ \underline{I}_L \end{vmatrix}$$

(6-2)

These topological equations are substituted into the terminal equations and by row operations the equations are reduced to the set

$$
\frac{d}{dt}\left|\begin{array}{l}
\underline{I}_1 \\[1.5em]
\underline{I}_2 \\[1.5em]
\underline{I}_3 \\[1.5em]
\underline{I}_L \\[1.5em]
\underline{V}_m
\end{array}\right|
=
\left|\begin{array}{l}
\frac{1}{L_e}\underline{R}_e(-\underline{I}_1 - \underline{I}_2 - \underline{I}_L) \quad - \frac{1}{L_K}\underline{R}_K(-\underline{I}_2 - \underline{I}_L) \\[1.5em]
-\frac{1}{L_K}\underline{R}_K(\underline{I}_2 + \underline{I}_2) + \frac{1}{L_L}\underline{R}_L\,\underline{I}_L \\[1.5em]
\frac{1}{L_e}\underline{R}_e(-\underline{I}_3 + \underline{I}_L) - \frac{1}{L_L}\underline{R}_2\,\underline{I}_2 \\[1.5em]
-\frac{1}{L_L}\underline{R}_L\,\underline{I}_L \\[1.5em]
\frac{1}{c}\underline{G}_m\,\underline{V}_m
\end{array}\right|
$$

$$
+\left|\begin{array}{l}
\frac{1}{L_e}\underline{V}_{j1} + \frac{1}{L_K}\underline{V}_K \\[1.5em]
\frac{1}{L_K}\underline{V}_K - \frac{1}{L_L}(-\underline{V}_{j1} + \underline{V}_{j2} + \underline{V}_K) \\[1.5em]
\frac{1}{L_e}\underline{V}_{j2} + \frac{1}{L_L}(-\underline{V}_{j1} + \underline{V}_{j2} + \underline{V}_K) \\[1.5em]
\frac{1}{L_L}(-\underline{V}_{j1} + \underline{V}_{j2} + \underline{V}_K) \\[1.5em]
\frac{1}{c}(\underline{I}_1 + \underline{I}_2 + \underline{I}_3)
\end{array}\right| \qquad (6\text{-}3)
$$

At this point the importance of the previous restrictions on the selection of subassembly A becomes obvious. Since elements 1, 2, and 3 are chords but could also have been branches the top three equations of the set of circuit equations (not used as yet) can be solved for the three voltages $\underline{V}_{j1}$, $\underline{V}_{j2}$, and $\underline{V}_K$ in terms of $\underline{V}_1$, $\underline{V}_2$, $\underline{V}_3$ and $\underline{V}_m$ and the results used to eliminate these variables from the system equations. The system of equations obtained can be partitioned into the following two groups of equations which shall be referred to as A(1), A(2), A(3), and E(1) and E(2).

$$
\begin{array}{l} A(1) \\ \\ A(2) \\ \\ A(3) \end{array}
\quad \frac{d}{dt}
\begin{vmatrix} \underline{I}_1 \\ \\ \underline{I}_2 \\ \\ \underline{I}_3 \end{vmatrix}
=
\begin{vmatrix}
-\frac{\underline{R}_e}{L_e} & [\frac{\underline{R}_K}{L_K} - \frac{\underline{R}_e}{L_e}] & \underline{0} \\
\\
\underline{0} & \frac{\underline{R}_K}{L_K} & \underline{0} \\
\\
\underline{0} & \underline{0} & \frac{\underline{R}_e}{L_e}
\end{vmatrix}
\begin{vmatrix} \underline{I}_1 \\ \\ \underline{I}_2 \\ \\ \underline{I}_3 \end{vmatrix}
$$

$$
+
\begin{vmatrix}
\frac{\underline{U}}{L_e} + \frac{\underline{U}}{L_K} & -\frac{\underline{U}}{L_K} & \underline{0} \\
\frac{\underline{U}}{L_K} & [-\frac{\underline{U}}{L_K} - \frac{\underline{U}}{L_L}] & -\frac{\underline{U}}{L_L} \\
\underline{0} & \frac{\underline{U}}{L_L} & [\frac{\underline{U}}{L_L} + \frac{\underline{U}}{L_L}]
\end{vmatrix}
\begin{vmatrix} \underline{V}_1 \\ \underline{V}_2 \\ \underline{V}_3 \end{vmatrix}
+
\begin{vmatrix}
-[\frac{\underline{R}_e}{L_e}\ \frac{\underline{R}_K}{L_K}]\ \underline{I}_L - \frac{1}{L_e}\underline{V}_m \\
[\frac{\underline{R}_K}{L_K} + \frac{\underline{R}_L}{L_L}]\ \underline{I}_L \\
[\frac{\underline{R}_e}{L_e} - \frac{\underline{R}_L}{L_L}]\ \underline{I}_L - \frac{1}{L_e}\underline{V}_m
\end{vmatrix}
\tag{6-4}
$$

$$
\begin{array}{l} E(1) \\ \\ E(2) \end{array}
\quad \frac{d}{dt}
\begin{vmatrix} \underline{I}_L \\ \\ \underline{V}_m \end{vmatrix}
=
\begin{vmatrix} \frac{1}{L_L}\ \underline{R}_L\ \underline{I}_L \\ \\ \frac{1}{c}\ \underline{G}_m\ \underline{V}_m \end{vmatrix}
+
\begin{vmatrix} \frac{1}{L_L}(\underline{V}_3 - \underline{V}_2) \\ \\ \frac{1}{c}(\underline{I}_1 + \underline{I}_2 + \underline{I}_3) \end{vmatrix}
\tag{6-5}
$$

The last two equations E(1) and E(2) can be considered to be internal or auxiliary equations not needed in the further formulation but necessary in the complete solution of the problem. Equations A(1), A(2) and A(3) are retained with the corresponding graph elements 1, 2 and 3 as the model of the subsystem A.

Subgroup B - Transformer Banks

The subassembly B can be combined with the subsystem A as a complete unit or in groups of one or two transformers, i.e., each tie point considered separately. Following the latter procedure and starting with element 1, the subgraph A_1 of figure 6-2, is obtained. with the tree E_e, F_e and F_m selected by the same procedures as before. The terminal equations of the components of this graph can be written as follows using a more condensed form of functional notation for the equation A(1). The equations A(2) and A(3) are held in "temporary storage" as they are not needed at this point.

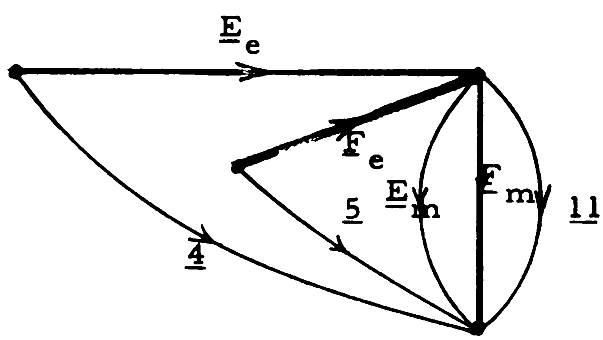


Figure 6-2. Subgraph A_1

Element 1 $\quad \frac{d}{dt}\underline{I}_1 = \underline{f}_1(\underline{I}_1, \underline{I}_2, \underline{I}_L, \underline{V}_1, \underline{V}_2, \underline{V}_m)$

Transformer

E

$$\frac{d}{dt}\underline{I}_{Em} = -\frac{1}{L_{Em}}\underline{R}_{Em}\,\underline{I}_{Em} + \frac{1}{L_{Em}}\underline{V}_{Em}$$

$$\frac{d}{dt}\underline{I}_{Ee} = -\frac{1}{L_{Ee}}\underline{R}_{Ee}\,\underline{I}_{Ee} + \frac{1}{L_{Ee}}\underline{V}_{Ee}$$

Transformer

F

$$\frac{d}{dt}\underline{I}_{Fm} = -\frac{1}{L_{Fm}}\underline{R}_{Fm}\,\underline{I}_{Fm} + \frac{1}{L_{Fm}}\underline{V}_{Fm}$$

$$\frac{d}{dt}\underline{I}_{Fe} = -\frac{1}{L_{Fe}}\underline{R}_{Fe}\,\underline{I}_{Fe} + \frac{1}{L_{Fe}}\underline{V}_{Fe}$$

(6-6)

The topological equations are obtained from the graph as before and substituted into the component equations and the resulting equations reduced to the two groups of equations

$$\begin{array}{ll} B(1) & \\ & \frac{d}{dt}\begin{vmatrix} \underline{I}_4 \\ \\ \underline{I}_5 \end{vmatrix} = \begin{vmatrix} \frac{1}{L_{Ee}}\underline{R}_{Ee}\,\underline{I}_4 + \frac{1}{L_{Ee}}\underline{V}_4 \\ \\ \frac{1}{L_{Fe}}\underline{R}_{Fe}\,\underline{I}_5 + \frac{1}{L_{Fe}}\underline{V}_5 \end{vmatrix} + \begin{vmatrix} -\frac{1}{L_{Ee}} & \underline{V}_1 \\ \\ -\frac{1}{L_{Fe}} & \underline{V}_1 \end{vmatrix} \\ B(2) & \end{array}$$

(6-7)

$$\begin{array}{ll} E(3) & \\ & \frac{d}{dt}\begin{vmatrix} \underline{I}_{Em} \\ \\ \underline{I}_1 \end{vmatrix} = \begin{vmatrix} \frac{-1}{L_{Em}}\underline{R}_{Em}\, I_{Em} + \frac{1}{L_{Em}}\underline{V}_1 \\ \\ f_1(\underline{I}_1, \underline{I}_2, \underline{V}_1, \underline{V}_2, \underline{I}_L, \underline{V}_m) \end{vmatrix} \\ E(4) & \end{array}$$

$$E(5) \qquad -\underline{V}_1 = \left[\frac{1}{L_{Fm}} - \frac{1}{L_{Fe}} - \frac{1}{L_{Ee}} - \frac{-1}{L_{Em}} - \frac{1}{L_e} - \frac{1}{L_L}\right]^{-1}\underline{f}_4(t)$$

(6-8)

where

$$\underline{f}_4(t) = \frac{1}{L_K}\underline{V}_2 + \frac{1}{L_e}\underline{V}_m + \frac{1}{L_{Ee}}\underline{V}_4 + \frac{1}{L_{Fe}}\underline{V}_5 + \left(\frac{R_e}{L_e} + \frac{R_{Fm}}{L_{Fm}}\right)\underline{I}_1$$

$$+ \left(\frac{R_e}{L_e} - \frac{R_K}{L_K}\right)\underline{I}_2 + \left(\frac{R_{Em}}{L_{Em}} + \frac{R_{Fm}}{L_{Fm}}\right)\underline{I}_{Em} - \left(\frac{R_K}{L_K} + \frac{R_e}{L_e}\right)\underline{I}_L$$

$$- \left(\frac{R_{Ee}}{L_{Ee}} + \frac{R_{Fm}}{L_{Fm}}\right)\underline{I}_4 - \left(\frac{R_{Fe}}{L_{Fe}} + \frac{R_{Fm}}{L_{Fm}}\right)\underline{I}_5 \qquad (6\text{-}9)$$

The equations E(3), E(4) and E(5) can be stored since they are considered as auxiliary equations. Equations B(1) and B(2) with corresponding graph elements 4 and 5 along with equations A(2) and A(3) with the corresponding elements 2 and 3 and, of course, the stored auxiliary equations E(1) --- E(6) constitute a state model of the combined subsystem containing transmission lines J, K, and L and transformers E and F.

When the above procedure is applied to the remaining parts of subassembly B, represented in figure 6-3, the resulting equations are the auxiliary equations E(6), E(7), E(8) and E(9) of set (6-10) and the equations B(3) and B(4) of set (6-12) which are retained for ports 6 and 7.

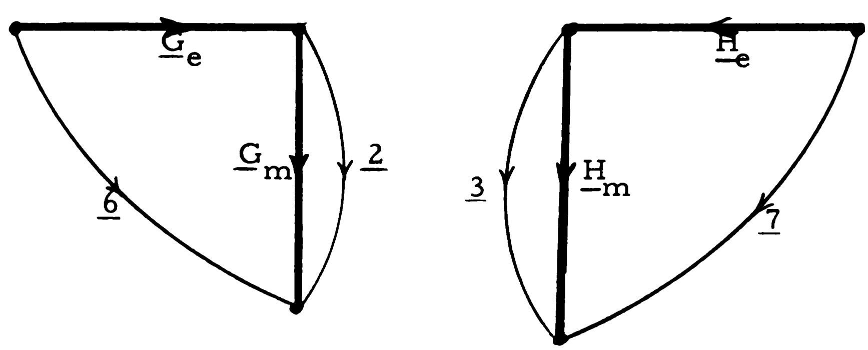


Figure 6-3 Subgraph A_2

$$\begin{array}{l} E(6) \\[2ex] E(7) \\[2ex] E(8) \\[2ex] E(9) \end{array} \quad \frac{d}{dt}\begin{vmatrix} \underline{0} \\[2ex] \underline{I}_2 \\[2ex] \underline{0} \\[2ex] \underline{I}_3 \end{vmatrix} = \begin{vmatrix} \underline{V}_2 + \left[\frac{-1}{L_{Gm}} - \frac{-1}{L_{Ge}} + \frac{1}{L_L} + \frac{1}{L_K}\right]^{-1} \underline{f}_5 \\[2ex] -\frac{R_K}{L_K}\underline{I}_2 - \frac{1}{L_K}\underline{V}_1 + \left(\frac{1}{L_K} - \frac{1}{L_L}\right)\underline{V}_2 + \frac{1}{L_L}\underline{V}_3 + \left(\frac{R_K}{L_K} - \frac{R_L}{L_L}\right)\underline{I}_L \\[2ex] \underline{V}_3 + \left[\frac{1}{L_e} + \frac{1}{L_L} + \frac{1}{L_{Hm}} + \frac{1}{L_{He}}\right]^{-1} \underline{f}_6 \\[2ex] -\frac{R_e}{L_e}\underline{I}_3 + \frac{1}{L_L}\underline{V}_2 + \left(\frac{1}{L_L} - \frac{1}{L_e}\right)\underline{V}_3 + \left(\frac{R_L}{L_L} - \frac{R_e}{L_e}\right)\underline{I}_L - \frac{1}{L_e}\underline{V}_m \end{vmatrix} \tag{6-10}$$

where

$$\underline{f}_5 = \frac{1}{L_K}\underline{V}_1 + \frac{1}{L_L}\underline{V}_3 + \frac{1}{L_{Ge}}\underline{V}_6 + \left(\frac{R_K}{L_K} - \frac{R_{Gm}}{L_{Gm}}\right)\underline{I}_2 - \left(\frac{R_{Ge}}{L_{Ge}}\frac{R_{Gm}}{L_{Gm}}\right)\underline{I}_6 - \left(\frac{R_L}{L_L} - \frac{R_K}{L_K}\right)\underline{I}_2$$

$$\underline{f}_6 = -\frac{-1}{L_e}\underline{V}_m - \frac{1}{L_L}\underline{V}_2 - \frac{1}{L_{He}}\underline{V}_7 + \left(\frac{R_{He}}{L_{He}} - \frac{R_{Hm}}{L_{Hm}}\right)\underline{I}_7 - \left(\frac{R_e}{L_e} - \frac{R_{Hm}}{L_{Hm}}\right)\underline{I}_3 - \left(\frac{R_L}{L_L} - \frac{R_e}{L_e}\right)\underline{I}_6 \tag{6-11}$$

$$\begin{array}{l} B(3) \\[2ex] B(4) \end{array} \quad \frac{d}{dt}\begin{vmatrix} \underline{I}_6 \\[2ex] \underline{I}_7 \end{vmatrix} = \begin{vmatrix} \frac{R_{Ge}}{L_{Ge}}\underline{I}_6 + \frac{1}{L_{Ge}}\underline{V}_6 & -\frac{1}{L_{Ge}}\underline{V}_2 \\[2ex] -\frac{R_{He}}{L_{He}}\underline{I}_7 + \frac{1}{L_{He}}\underline{V}_7 & -\frac{1}{L_{He}}\underline{V}_3 \end{vmatrix} \tag{6-12}$$

It is important to the later discussion to note that the relations represented by $\underline{f}_4$, $\underline{f}_5$ and $\underline{f}_6$ in equations (6-9) and (6-11) are linear in the variables $\underline{V}_1$, $\underline{V}_2$ and $\underline{V}_3$.

Subgroup C - Synchronous Machines

At this stage of the formulation process the subgroups A and B have been combined to form the model consisting of the four elements 4, 5, 6 and 7 and the associated equations B(1), B(2), B(3) and B(4) and also the stored equations E(1) --- E(9). The next stage of the process is the combination of this model with the model of the remaining subgroup C. The process is almost a repetition of the previous steps and hence is only discussed briefly.

The machines of subgroup C can also be considered one at a time. The equations of the first, machine A, are written in a condensed functional form and referenced as follows

$$\text{C(1)} \qquad \frac{d}{dt}\underline{I}_A = \underline{g}_{a1}(\underline{I}_A, i_a, \dot{\alpha}_a, \alpha_a, v_a) + \underline{f}_{a1}(\alpha)\,\underline{V}_A$$

$$\text{E(6)} \qquad \frac{d}{dt} i_a = g_{a2}(\underline{I}_A, i_a, \dot{\alpha}_a, \alpha_a, v_a) + \underline{f}_{a2}(\alpha)\,\underline{V}_A \qquad (6\text{-}13)$$

$$\text{E(7)} \qquad \frac{d}{dt}\dot{\alpha}_a = g_{a3}(i_a, \underline{I}_A, \dot{\alpha}_a, \alpha_a) + K_a\,\tau_a$$

$$\text{E(8)} \qquad \frac{d}{dt}\alpha_a = \dot{\alpha}_a$$

The last three equations E(6), E(7) and E(8) can be stored immediately since they are considered as auxiliary equations for the operating conditions of constant field voltage and constant torque drivers. If other operating conditions were to be considered then other component subgroups could be used to extend the system to include the field and mechanical systems.

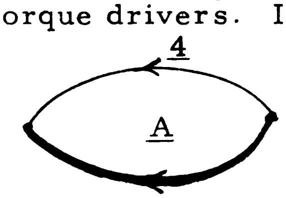


Figure 6-4.
Subgraph C_1

The first equation C(1) with the element A is considered to be the state model representation of Machine A. Combining this equation with Equation B(1) according to the rather trivial subgraph C_1 gives the two equations:

$$\text{E(10)} \qquad \frac{d}{dt}\underline{I}_4 = \underline{g}_{a1}(\underline{I}_4, i_a, \dot{\alpha}_a, \alpha_a, v_a) + \underline{f}_{a1}(\alpha)\,\underline{V}_4$$

$$\text{E(11)} \qquad 0 = \underline{V}_A - [\underline{f}_{a1}(\alpha) + \frac{\underline{U}}{L_{Ee}}]^{-1}[\frac{-1}{L_{Ee}}\underline{R}_{Ee}\underline{I}_4 - \underline{f}_{a1}(\underline{I}_4, i_a, \dot{\alpha}_a, \alpha_a, v_a)] \qquad (6\text{-}14)$$

These two equations can be stored and the process repeated for the other three machines to obtain the equations which, along with Equation E(1) - E(11), must be solved to determine the system operation. It is possible, although it may not be particularly advantageous, to reduce the number of equations in the set slightly by further analytical substitutions. An example of this is seen in equations E(9) and E(10) where equation E(10) could be used to eliminate $\underline{V}_A$ in equation E(9). It is noted here that all of the algebraic equations are linear in the variables for which solutions are necessary although some of the equations, e.g., E(5), E(6) and E(8), involve simultaneous solutions.

Model of Entire System

The resulting mathematical model of the electric power system represented by the vector graph of figure 5-4 consists of the following algebraic and first order differential equations sets, where the various vector and scalar functions g and f were defined previously.

$$0 = \underline{V}_1 + \left[\frac{1}{L_{Fm}} - \frac{1}{L_{Fe}} - \frac{1}{L_{Ee}} - \frac{1}{L_{Em}} - \frac{1}{L_e} - \frac{1}{L_L}\right]^{-1} \underline{f}_4(\underline{V}_2, \underline{V}_m, \underline{V}_4, \underline{V}_5, \underline{I}_1, \underline{I}_2, \underline{I}_{Em}, \underline{I}_2, \underline{I}_4, \underline{I}_5)$$

$$0 = \underline{V}_2 + \left[\frac{1}{L_{Gm}} - \frac{1}{L_{Ge}} - \frac{1}{L_L} - \frac{1}{L_K}\right]^{-1} \underline{f}_5(\underline{V}_1, \underline{V}_3, \underline{V}_6, \underline{I}_2, \underline{I}_6, \underline{I}_2)$$

$$0 = \underline{V}_3 + \left[\frac{1}{L_e} + \frac{1}{L_L} + \frac{1}{L_{Hm}} + \frac{1}{L_{He}}\right]^{-1} \underline{f}_6(\underline{V}_m, \underline{V}_2, \underline{V}_7, \underline{I}_7, \underline{I}_3, \underline{I}_6)$$

$$0 = \underline{V}_4 - \left[\underline{f}_{a1}(a) + \frac{\underline{U}}{L_{Ee}}\right]^{-1} \left[-\frac{1}{L_{Ee}} \underline{R}_{Ee} \underline{I}_4 - \underline{g}_{a1}(-\underline{I}_4, i_a, \dot{a}_a, a_a, v_a)\right]$$

$$0 = \underline{V}_5 - \left[\underline{f}_{b1}(a) + \frac{\underline{U}}{L_{Fe}}\right]^{-1} \left[\frac{1}{L_{Fe}} \underline{V}_1 - \frac{\underline{R}_{Fe}}{L_{Fe}} \underline{I}_5 - \underline{g}_{b1}(-\underline{I}_5, i_b, \dot{a}_b, a_b, v_b)\right]$$

$$0 = \underline{V}_6 - \left[\underline{f}_{c1}(a) + \frac{\underline{U}}{L_{Ge}}\right]^{-1} \left[\frac{1}{L_{Ge}} \underline{V}_2 - \frac{\underline{R}_{Ge}}{L_{Ge}} \underline{I}_6 + \underline{g}_{c1}(-\underline{I}_6, i_c, \dot{a}_c, a_c, v_c)\right]$$

$$0 = \underline{V}_7 - \left[\underline{f}_{d1}(a) + \frac{\underline{U}}{L_{Ge}}\right]^{-1} \left[\frac{1}{L_{He}} \underline{V}_3 - \frac{\underline{R}_{He}}{L_{He}} \underline{I}_7 + \underline{g}_{d1}(-\underline{I}_7, i_d, \dot{a}_d, a_d, v_d)\right]$$

(6-15)

$$\frac{d}{dt}\left|\begin{array}{l} \underline{I}_L \\ \underline{V}_m \\ \underline{I}_{Em} \\ \underline{I}_1 \\ \\ \underline{I}_2 \\ \underline{I}_3 \\ \underline{I}_4 \end{array}\right| = \left|\begin{array}{l} -\frac{1}{L_L}\underline{R}_L \underline{I}_L + \frac{1}{L_e}(\underline{V}_3 - \underline{V}_2) \\ \frac{1}{c}\underline{G}_m \underline{V}_m + \frac{1}{c}(\underline{I}_1 + \underline{I}_2 + \underline{I}_3) \\ -\frac{1}{L_{Em}}\underline{R}_{Em}\underline{I}_{Em} + \frac{1}{L_{Em}}\underline{V}_1 \\ -\frac{\underline{R}_e}{L_e}\underline{I}_1 + \left[\frac{\underline{R}_K}{L_K} - \frac{\underline{R}_e}{L_e}\right]\underline{I}_2 + \left[\frac{1}{L_e} + \frac{1}{L_K}\right]\underline{V}_1 - \frac{1}{L_K}\underline{V}_2 \\ \qquad - \left[\frac{\underline{R}_e}{L_e} + \frac{\underline{R}_K}{L_K}\right]\underline{I}_6 - \frac{1}{L_e}\underline{V}_m \\ -\frac{\underline{R}_K}{L_K}\underline{I}_2 - \frac{1}{L_K}\underline{V}_1 + \left[\frac{1}{L_K} - \frac{1}{L_L}\right]\underline{V}_2 + \frac{1}{L_L}\underline{V}_3 + \left[\frac{\underline{R}_K}{L_K} - \frac{\underline{R}_L}{L_L}\right]\underline{I}_L \\ -\frac{\underline{R}_e}{L_e}\underline{I}_3 + \frac{1}{L_L}\underline{V}_2 + \left[\frac{1}{L_L} - \frac{1}{L_e}\right]\underline{V}_3 + \left[\frac{\underline{R}_L}{L_L} - \frac{\underline{R}_e}{L_e}\right]\underline{I}_L - \frac{1}{L_e}\underline{V}_m \\ \underline{g}_{a1}(\underline{I}_4, i_a, \dot{a}_a, a_a, v_a) + \underline{f}_{a1}(a_a)\,\underline{V}_4 \end{array}\right|$$

$$
\frac{d}{dt}
\begin{bmatrix}
\underline{I}_5 \\
\underline{I}_6 \\
\underline{I}_7 \\
i_a \\
i_b \\
i_c \\
i_d \\
\dot{\alpha}_a \\
\dot{\alpha}_b \\
\dot{\alpha}_c \\
\dot{\alpha}_d \\
\alpha_a \\
\alpha_b \\
\alpha_c \\
\alpha_d
\end{bmatrix}
=
\begin{bmatrix}
\underline{g}_{b1}(\underline{I}_5, i_b, \dot{\alpha}_b, \alpha_b, v_b) + \underline{f}_{b1}(\alpha_b)\ \underline{V}_5 \\
\underline{g}_{c1}(\underline{I}_6, i_c, \dot{\alpha}_c, \alpha_c, v_c) + \underline{f}_{c1}(\alpha_c)\ \underline{V}_6 \\
\underline{g}_{d1}(\underline{I}_7, i_d, \dot{\alpha}_d, \alpha_d, v_d) + \underline{f}_{d1}(\alpha_a)\ \underline{V}_7 \\
g_{a2}(\underline{I}_4, i_a, \dot{\alpha}_a, \alpha_a, v_a) + \underline{f}_{a2}(\alpha)\ \underline{V}_4 \\
g_{b2}(\underline{I}_5, i_b, \dot{\alpha}_b, \alpha_b, v_b) + \underline{f}_{b2}(\alpha)\ \underline{V}_5 \\
g_{c2}(\underline{I}_6, i_c, \dot{\alpha}_c, \alpha_c, v_c) + \underline{f}_{c2}(\alpha)\ \underline{V}_6 \\
g_{d2}(\underline{I}_7, i_d, \dot{\alpha}_d, \alpha_d, v_d) + \underline{f}_{d2}(\alpha)\ \underline{V}_7 \\
g_{a3}(\underline{I}_4, i_a, \dot{\alpha}_a, \alpha_a) \quad + \quad K_a\, \tau_a \\
g_{b3}(\underline{I}_5, i_b, \dot{\alpha}_b, \alpha_b) \quad + \quad K_b\, \tau_b \\
g_{c3}(\underline{I}_6, i_c, \dot{\alpha}_c, \alpha_c) \quad + \quad K_c\, \tau_c \\
g_{d3}(\underline{I}_7, i_d, \dot{\alpha}_d, \alpha_d) \quad + \quad K_d\, \tau_d \\
\dot{\alpha}_a \\
\dot{\alpha}_b \\
\dot{\alpha}_c \\
\dot{\alpha}_d
\end{bmatrix}
\tag{6-16}
$$

The procedure used to formulate the system model of the example can be summed up by the following steps.

1) The system is divided into convenient subassemblies with at least one element included in the first sub-subassembly, if possible, so that the tie point elements do not form a cut-set.

2) A state model of the subassembly is obtained consisting of (a) a graph element for each tie point, representing a pair of vector variables, (b) a vector differential equation or terminal equation relating the graph variables, and (c) a vector of auxiliary differential equations to be stored until the complete system model is solved.

3) The state model of the subassembly is combined with those of other subassemblies until all components have been incorporated into the system.

The technique of formulation listed here is useful for many power systems problems, although in some problems it may not be possible to fully execute step two. For example, systems with tie points forming a cut-set involve an additional vector equation. In the example used here, if transmission line J were modeled as a short line then the first subassembly would produce a model of the form of equations (6-17) where the matrix A is singular.

$$\begin{aligned} \underline{Y}_1 &= \underline{A}\,\underline{Y}_2 \\ \dot{\underline{Y}}_2 &= \underline{f}(\underline{Y}_2\,\underline{X}_1) \end{aligned} \qquad (6\text{-}17)$$

Both equations are then considered to be a part of the state model of the subassembly.

VII EXISTENCE AND STABILITY OF SOLUTIONS

The final form of the state equations of the system, as given by equations (6-15) and (6-16) for example, can be represented in vector form as

$$(1) \quad 0 = \underline{F}_1 (\underline{X}_1(t), \underline{X}_2(t), \underline{E}_1(t))$$

$$(2) \quad \dot{\underline{X}}_1(t) = \underline{F}_2(\underline{X}_1(t), \underline{X}_2(t), \underline{E}_2(t)) \tag{7-1}$$

A numerical solution of these equations for a given set of initial conditions on X_1 is realized by the following steps.

1) The initial value of the vector, X_2, is determined from a solution of the algebraic equation given in equation (1) for $X_2(0)$, given $X_1(0)$ and $E_1(0)$.

2) The resulting vector, $X_2(0)$, is substituted into equation (2) and equation (2) is then integrated by some numerical technique such as the Runge-Kutta method [23 chapters 8, 9] over an increment of time, h, to obtain the vector, $X_1(h)$.

3) The vector, $X_1(h)$, becomes the new initial conditions vector and steps 1) and 2) are carried out recursively to realize the solution.

This procedure requires, of course, that the system of algebraic and the system of differential equations each in itself has a solution.

The existence of a solution to the system of algebraic equations is considered first. In the example developed in the previous section the algebraic equations are linear. This linear form is a direct consequence of the form of the component models, their interconnection pattern, and the technique for formulation of the system model. The mathematical models of the synchronous machines are the only nonlinear forms and these are nonlinear in the currents but not the voltages. These components were placed in the last subgroup of components to be incorporated in the system, thus utilizing the linear properties of the components.

Although the conditions have not been shown, in general, under which the resulting algebraic equations are linear in the variables represented in equations (1), the example used here is typical of a large class of power systems models.

Since the system algebraic equations are linear in the elements of $\underline{X}_2$, equations (1) can be expressed in the form

$$\underline{B}\ \underline{X}_2(t) = \underline{F}_3\ (\underline{X}_1(t),\ \underline{E}_1(t)\ ) \tag{7-2}$$

This equations then has a solution if $\underline{B}^{-1}$ exists. In the example of the previous section the required inverse is realized by inverting one sixth-order and four second-order matrices, the second order matrices having time varying entries. The existence of these inverses is difficult to determine except by numerical methods.

In the event that the algebraic equations are nonlinear, existence of the solution is determined by application of Theorem 4.3 given by Wirth [17 chapter 4]. Application of this theorem to the example of the previous section is given subsequently.

Existence and uniqueness criteria for a set of differential equations of the form represented in (7-1) are given by Murray and Miller [19 page 42]. The necessary and sufficient conditions for existence are that $\underline{F}_2$ must be defined and jointly continuous. Uniqueness of the solution is assured if $\underline{F}_2$ satisfies a Lipschitz condition. Again, these are difficult to apply. More useful conditions for the existence and uniqueness of a solution to the system models considered here are given by Wirth. In his Theorem 4.3 the hypotheses are given in terms of a set of conditions imposed on the components and the system topology. When these hypotheses are satisfied a solution to the set of differential and also the set of algebraic equations is assured. Application of this theorem to the example here illustrates the factors involved. The hypotheses concerning the components require the following.

1) When the component differential equations are expressed as in equation (7-3), $\underline{F}(\underline{Z})$ must satisfy a Lipschitz condition.

$$\frac{d}{dt} \begin{vmatrix} \underline{X}_1(t) \\ \\ \underline{Y}_2(t) \end{vmatrix} = \begin{vmatrix} \underline{F}_1(\underline{X}_1\ \underline{X}_2\ \ \underline{Y}_1\ \underline{Y}_2\ \ t) \\ \\ \underline{F}_2(\underline{X}_1\ \underline{X}_2\ \ \underline{Y}_1\ \underline{Y}_2\ \ t) \end{vmatrix} = \underline{F}(\underline{Z}) \qquad (7\text{-}3)$$

2) The matrix $[\frac{\partial \underline{F}}{\partial \underline{Y}_1} \quad \frac{\partial \underline{F}}{\partial \underline{X}_2}]$ must be defined and strictly positive definite with entries which are bounded and satisfy a Lipschitz condition.

3) The driving functions must be such that their derivatives exist and satisfy a Lipschitz condition.

All component models developed in this thesis satisfy the first condition. The matrix specified in the second condition is defined with bounded entries which satisfy a Lipschitz condition for all components. The parameters of the component must be such that the matrix is strictly positive definite. The third condition is a restriction on the driving functions and is, perhaps, too strong for a few conditions of interest such as step functions.

The hypothesis stating the restriction on the topology of the system requires that:

4) The across drivers can be placed in a tree and the through drivers in a chord set.

(Additional hypotheses are applicable to any algebraic component models that are of the perfect coupler or gyrator type).

The nonlinearities in the system model discussed above are introduced by the synchronous machines. If the speed of each synchronous machine is held constant then the system model is linear, and written as

$$\frac{d}{dt} \; \dot{\underline{X}}(t) = \underline{K} \; \dot{\underline{X}}(t) + \underline{E}(t) \qquad (7\text{-}4)$$

Where the matrix K is a constant matrix. The closed form solution in this case is

$$\underline{X}(t) = e^{\underline{B}t}\,\underline{X}(0) + \int_{o}^{t} e^{\underline{B}\tau}\,\underline{E}(\tau)\,d\tau \qquad (7\text{-}5)$$

The matrix $e^{\underline{B}t}$ can be evaluated analytically or numerically as the circumstances dictate .

Stability problems as encountered in electric power studies fall into one of two categories: The so-called steady-state stability and the transient stability. Steady-state stability studies are concerned with the problem of establishing the range in loads or other steady-state operating conditions under which all synchronous machines in the system remain in synchronism. This implies that the rotational variables $\dot{\alpha}_i$ of all the synchonous machines are zero. The system is said to be unstable in the sense used here, if for a given set of operating conditions some of the machines of the system are operating at other than synchronous speed, i.e., for at least one machine $\dot{\alpha} \neq 0$.

Transient or dynamic stability studies are concerned, first of all, with the question as to whether or not all machines will remain in synchronism when subjected to sudden loads and, secondly, as to how large these pertubations can be and how long any resulting oscillations are sustained.

These problems are all related to the mathematical concept of stability in the Liapunov sense, a definition of which is given by LaSalle and Lefschetz [6 chapter 2] and others [24].

The state model for power systems studies has been shown to be of the form

$$\frac{d}{dt}\begin{vmatrix} \underline{V} \\ \underline{I} \\ \dot{\underline{a}} \\ \underline{a} \end{vmatrix} = \begin{vmatrix} \underline{F}_4(\underline{V}, \underline{I}, \dot{\underline{a}}, \underline{a}) \\ \underline{F}_5(\underline{V}, \underline{I}, \dot{\underline{a}}, \underline{a}) \\ \underline{F}_6(\underline{V}, \underline{I}, \dot{\underline{a}}, \underline{a}) \\ \underline{a} \end{vmatrix} \tag{7-6}$$

In terms of this model, the concept of steady-state stability in power systems implies that the equilibrium point has the coordinates: $\underline{V} = \underline{K}_1$, $\underline{I} = \underline{K}_2$, $\dot{\underline{a}} = \underline{K}_3 = \underline{0}$, and $\underline{a} = \underline{K}_4$, where $\underline{K}_i$, $i = 1$ --- 4 is a constant. If the machines are not in synchronism then the coordinates of the equilibrium point are $\underline{V} = \underline{K}_1'$, $\underline{I} = \underline{K}_2'$, $\dot{\underline{a}} = \underline{K}_3' \neq \underline{0}$ and $\underline{a} = \underline{K}_4't + \underline{a}_o$, where again $\underline{K}_2'$ is a constant. The equilibrium points may or may not be stable in the Liapunov sense. This implies that the steady-state stability problem is not a stability problem according to Liapunov but rather a question of the existence of an equilibrium point satisfying certain restrictions. The system model thus serves as a formal and practical basis for investigating steady-state stability problems and does not require the solution to the differential equations.

The transient stability problem as encountered in power systems studies is precisely the Liapunov stability problem. The problem of establishing the size of a sudden load or other pertubation that can be tolerated by the system without causing loss of synchronism is referred to as the extent of stability in the Liapunov sense.

Many of the problems involving dynamic stability can be answered without realizing a solution to the system model, if a Liapunov function can be developed. Unfortunately, the determination of such a function appears formidable and there are, as yet, no systematic procedures for doing so.

VIII SUMMARY

The problems involved in electric power systems have been investigated extensively over a period of several years. This thesis mentions the more prominent disciplines which have developed in answer to specific problems. The use of these disciplines has resulted in solutions of a limited nature in that only special systems or restricted regions of operation have been considered. Only a few investigators have considered a generalized approach to the system involved. This thesis has continued the investigation of a generalized approach to the study of dynamic electric power systems.

Appropriate models of components have been developed which offer distinct advantages over other component models. These models of three-phase components are given in terms of two sets of variables. One set of variables is composed of two line-to-line voltages and two line currents. The other set is composed of the voltage from one line to neutral and the neutral current. These variables are used to establish identical topological representations of components for both three- and four-wire systems. The form of the component model has been simplified by the application of a set of linear transformations of variables. These sets of variables, referred to as the vector and scalar variables, are also established by measurements using special instruments, thus defining elements of a linear graph to identify the variables. The final form of the component model, referred to as the state model, is formed by solving for the derivative vector in the differential equations.

The synchronous machine model has been developed in some detail as a typical example of the component modeling procedures involved and also as an important component displaying extensive nonlinear effects. Other component models have been developed in order to be able to formulate models of typical systems and to gain appreciation of the factors involved in modeling components as well as an appreciation of the characteristics of the components themselves. The components selected for this purpose are components in common use in all power systems. Some of the components such as transmission lines and transformers involve two interrelated electrical ports, each to be connected into the power system. Others, such as the squirrel cage induction motor, have one port in the system with another port interrelated but isolated electrically from the system. Some fault conditions have been developed as representative of components with algebraic models. Perhaps the principle characteristic of this type of component is that the presence of such components in a system creates a dependence between the scalar variables and the vector variables.

The state model of the complete system has been established from the component models and the interconnection pattern of the system as represented by the system graph. A systematized formulation procedure was developed to be applied to many power systems composed of several components. This procedure, based heavily on the application of theorems developed by Wirth [17], involves the separation of the system into subassemblies consisting of some convenient subgroup of components. The first phase of the formulation procedure involves establishing a state model of each subassembly consisting of 1) tie point elements to be used in the system graph, 2) a set of terminal equations relating the variables represented by the tie point elements and 3) a set of auxiliary algebraic and differential equations relating other variables in the system to the state variables. These auxiliary equations may be present either due to the functional dependence of certain variables or to the desirability of using numerical techniques for large systems of algebraic equations. In the second phase of the formulation procedure the auxiliary equations of the subassemblies are stored, the subassemblies are grouped together into large subassemblies and a state model of the larger subassembly is established similarly. This type of concatenation is continued until all components have been assimilated into the system.

A typical system problem has been carried out as an example of the use of the state model form of the components and of the technique of formulation of large systems.

Considerable detail has been presented in this example in order that some appreciation of the applicability of the procedure can be obtained.

One of the major contributions of this thesis is the development of a procedure to formulate large dynamic system models. A systematic algorithm has been developed for generating a system graph from the component models of the system. This graph represents the dynamic variables of the system in a manner similar to themethod used to represent the steady-state symmetrical components variables by means of the sequence graphs.

The solution to the model of the example has been discussed. Some of the existence and stability properties of this model were examined with inferences extended to a large class of power systems problems. Identifications have been made between power systems stability concepts as used in the industry and concepts of the existence of solutions and stability of systems represented by differential equations. The direct application of the properties of the system model to problems of stability has been discussed briefly.

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