UNFOLDING OF A CLASS OF SINGULAR FREE BOUNDARIES FOR THE FOUR DIMENSIONAL AXI-SYMMETRIC OBSTACLE PROBLEM

By

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ABSTRACT

UNFOLDINGS OF A CLASS OF SINGULAR FREE BOUNDARIES FOR THE FOUR DIMENSIONAL AXI-SYMMETRIC OBSTACLE PROBLEM

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In this thesis we shall consider an elliptic free boundary value problem. We shall deal with the four dimensional axi-symmetric obstacle problem and shall consider a class of singularities of free boundary. Specifically, we study generic perturbations of the singularity $x = H(y) - \frac{2}{\pi} y \log y$, where $H(t)$ is a real analytic function about $t = 0$ with $H(0) = 0$ and $H'(0) \neq 0$ near the origin. This locus described by this formula represents the boundary of the region of contact between a membrane and a smooth rigid obstacle.

The point of view taken is that of generic bifurcation, where one scalar parameter (2K, the height of the membrane above the obstacle at the origin) and one functional parameter (ϕ) is present. Our prime interest is a description of the unfolding of such singularities, their normal forms, and generic conditions for unfoldings.

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TABLE OF CONTENTS

Chapter | Page

LIST OF FIGURES

Page

GLOSSARY OF NOTATIONS

R^n: Euclidean n-dimensional space, the product of n copies of the real line R.

Ω: An open, generally bounded and connected subset of R^n.

$\partial\Omega$: The boundary of Ω.

∇u: The vector $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$ if $u : R^n \to R$ has partial derivatives.

Δu: $\sum_{y=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$ if $u : R^n \to R$

u_x, u_y: The partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ if $u : R^2 \to R$.

$C_0^1(\Omega)$: The set of continuous functions with compact support in Ω.

$C^{m,\lambda}(\overline{\Omega})$: The functions m times continuously differentiable in $\overline{\Omega}$ whose mth derivatives satisfy a Hölder condition with exponent λ.

$C^m(\overline{\Omega})$: The functions m times continuously differentiable in $\overline{\Omega}$.

$H^{2,s}(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$: The usual Sobolev spaces of completion of $C^m(\overline{\Omega})$, $C^2(\overline{\Omega})$, $C_0^\infty(\Omega)$, respectively, each one with the corresponding norm.

1. INTRODUCTION

The minimal surface problem

1.1. Consider the classical minimal surface problem of finding a surface in three-dimensional space, spanning a fixed boundary loop, in such a way as to minimize the area of the surface (Plateau's problem) [18,16,6].

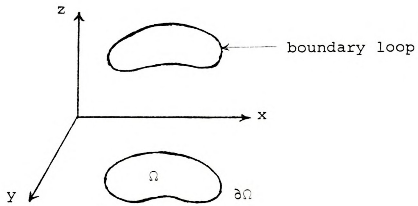

Figure 1.1

In particular in the non-parametric minimal surface problem [18,6,8], we suppose the surface and its boundary are graphs of functions u and g defined in subsets of R^2 as follows:

Given $\Omega \subset R^2$ a bounded domain with smooth boundary $\partial\Omega$, and $g : \partial\Omega \to R$, find

$u : \overline{\Omega} \to R$ such that $u = g$ on $\partial\Omega$ and u minimizes the area integral

$$A(u) = \iint_\Omega (1 + |\nabla u|^2)^{\frac{1}{2}}\, dx\, dy \qquad (1.1)$$

among such functions.

Suppose this problem has a solution u, sufficiently smooth. Let us derive the Euler equation satisfied by this solution. Fix $v \in C_0^1(\Omega)$; then for any $t \in R$

$$A(u + tv) \geq A(u)\ , \qquad (u + tv)\big|_{\partial\Omega} = g\ .$$

Hence

$$0 = \frac{d}{dt} A(u + tv)\Big|_{\epsilon=0} = \iint_\Omega (1 + |\nabla u|^2)^{-\frac{1}{2}} (u_x v_x + u_y v_y)\, dx\, dy$$

(Fréchet derivative) .

Integrating by parts and assuming u twice differentiable in Ω we get

$$0 = -\iint_\Omega \left[\left(\frac{u_x}{\sqrt{1 + |\nabla u|^2}} \right)_x + \left(\frac{u_y}{\sqrt{1 + |\nabla u|^2}} \right)_y \right] v\, dx\, dy$$

for any such v. Hence $u(x,y)$ satisfies the partial Differential Equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad \text{for } (x,y) \in \Omega ,$$

$$u = g \quad \text{on} \quad \partial\Omega . \tag{1.2}$$

The nonlinear equation (1.2) is called the minimal surface equation.

Existence, uniqueness, and regularity of the solution

It is known [18] that the above problem, in general, has no solution. The methods developed for the solution of the problem [18] are based on some further assumptions concerning the prescribed boundary conditions. One of these assumptions is introduced by the following definition.

Definition 1.1.1. Let C^* be the Jordan curve defined in the xyz-space by the equation $z = g(P)$, $P \in \partial\Omega$. Let P_1^*, P_2^*, P_3^* be three distinct points on C^*, and denote by θ the positive acute angle between the xy-plane and the plane passing through P_1^*, P_2^*, P_3^*. We shall say that the boundary condition $g(P)$ satisfies the three-point condition with the constant a if for all possible positions of the points P_1^*, P_2^*, P_3^* we have $\tan\theta \leq a$ for some fixed finite constant a.

Theorem 1.1.2. (existence [18]) Let there be given, on a convex Jordan curve $\partial\Omega$ in the xy-plane, a function $g(P)$ of the point P varying on $\partial\Omega$ which satisfies the three-point condition (Definition 1.1.1) with some constant a. Consider all the functions $z = u(x,y)$ which satisfy, in the Jordan region Ω, the Lipschitz condition and $u(P) = g(P)$ for any $P \in \partial\Omega$. Then there exists in this class a function $u_0(x,y)$ which minimizes the integral

$$A(u) = \iint_\Omega \left[1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy \qquad \#$$

Theorem 1.1.3 (regularity [18]). If $z = u_0(x,y)$ is a solution of the problem under the conditions of Theorem 1.1.2, then $u_0(x,y)$ is analytic and satisfies the minimal surface equation (1.2) [18]. #

Theorem 1.1.4 (uniqueness). The solution stated above is unique.

Proof: Suppose there are two such solutions w, v. The class of functions considered in Theorem 1.1.2 is clearly convex. The operator

$$A(u) = \iint_\Omega [1 + |\nabla u|^2]^{\frac{1}{2}} dx\,dy$$

is also convex because if, in general $f : R^2 \to R$ is convex, then

$$\iint_\Omega f(\nabla u)\, dx\,dy$$

is a convex function of u. Here f is the composition of two convex functions $| \, | : R^2 \to R$ and $\sqrt{1+x^2}$ from R to R. Let's define $B(t) = A(tw + (1-t)v)$, where w, v are the solutions. Since w, v are both minimizer of $A(u)$ and A is convex thus

$$A(w) = A(v) = A(tw + (1-t)v) \quad \text{constant for} \quad 0 \le t \le 1 .$$

Therefore $B(t)$ is constant for $0 \le t \le 1$. B is also a convex function:

$$B(\lambda t_1 + (1-\lambda)t_2)$$

$$= A(\lambda t_1 w + (1-\lambda)t_2 w + (1 - \lambda t_1 - (1-\lambda)t_2)v)$$

$$= A(\lambda(t_1 w + (1-t_1)v) + (1-\lambda)(t_2 w + (1-t_2)v)$$

$$\le \lambda B(t_1) + (1-\lambda)B(t_2) \quad \text{for} \quad 0 \le \lambda \le 1.$$

So $B : R \to R$ is a convex and therefore continuous function which is constant on $[0,1]$. Thus $B'(t) = B''(t) = 0$ for $t \in [0,1]$.

$$B(t) = \iint_\Omega f(t\nabla w + (1-t)\nabla v)\, dx\, dy ,$$

$$B'(t) = \iint_\Omega f'(t\nabla w + (1-t)\nabla v) \cdot \nabla(w-v)\, dx\, dy$$

$$B''(t) = \iint_\Omega [\nabla(w-v)]^T \cdot f''(t\nabla w + (1-t)\nabla v)$$

$$\cdot \nabla(w-v)\, dx\, dy \quad (f'' \text{ is the Hessian matrix}) .$$

Since f was convex, thus the integrand in B'' is positive semi-definite. Thus

$$B''(t) = 0 \Rightarrow [\nabla(w-v)]^T \cdot f''(t\nabla w + (1-t)\nabla v)\cdot\nabla(w-v) = 0$$

$$\text{for } 0 \leq t \leq 1 .$$

The computation also shows that the integrand is nonnegative as follows

$$B''(t) = \iint_\Omega \frac{(w_x-v_x)^2 + (w_y-v_y)^2 + [(w_x-v_x)(tw_y+(1-t)v_y)}{(1+|\nabla(tw+(1-t)v)|^2)^{3/2}}$$

$$\frac{+ (w_y-v_y)(tw_x+(1-t)v_x)]^2}{} dx\,dy = 0 .$$

Putting the integrand zero we get $w_x - v_x = 0 = w_y - v_y$ on the connected domain Ω. Thus $w(x,y) - u(x,y) =$ constant on Ω. Since $w = u = g$ on $\partial\Omega$ and the solutions are continuous on $\overline{\Omega}$, thus $w = v$ on Ω. This completes the proof of uniqueness. #

The obstacle problem

1.2. An "obstacle problem" is obtained by introducing a further constraint on the minimal surface, namely that, it lies above a given fixed surface (the obstacle) represented by the graph of some $\psi : \overline{\Omega} \to R$, where $\psi < g$ on $\partial\Omega$. In other words, given g and ψ, the solution

u minimizes the area (1.1) subject to the boundary condition $u|_{\partial\Omega} = g$ and the constraint $u \geq \psi$ in $\overline{\Omega}$ (see Figure 1.2).

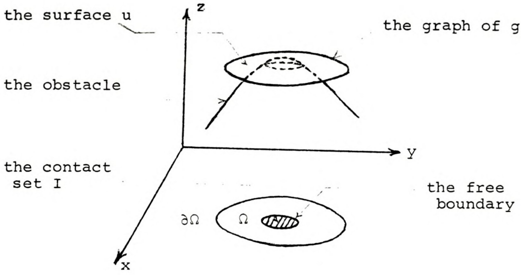

Figure 1.2

The existence of a solution $u \in H^{2,s}(\Omega) \cap C^{1,t}(\overline{\Omega})$, $1 \leq s < \infty$ and $0 < t < 1$, to this problem was proved by H. Lewy and G. Stampacchia [13] subject to the following conditions:

Ω is a convex domain with smooth boundary,

$\psi \in C^2(\overline{\Omega})$,

$g \equiv 0, \max_{\Omega} \psi > 0$.

The proof in fact covers the higher dimensional case where $\Omega \subset R^n$. It was shown in [1] with the above conditions that $u \in C^{1,1}(\Omega)$.

The most interesting questions here focus on the contact set and its boundary:

$$I = \{(x,y) \in \Omega \mid u(x,y) = \psi(x,y)\} ,$$

$$\Gamma = \partial I = \text{the free boundary} .$$

The boundary ∂I of the region of contact is called a free boundary as it is not given apriori, but rather depends on the solution, that is, it is a part of the solution to be found. Observe that the derivation of (1.2) is still valid (at least formally) in the complement $\Omega - I$ of the contact set, namely where $u > \psi$.

A fundamental result on the geometry of Γ, in the two dimensional cases was proved by Kinderlehrer [11]: if $\Omega \subset R^2$ is strictly convex with smooth boundary, if $g \equiv 0$, and if ψ is strictly concave and analytic, then Γ is an analytic Jordan curve enclosing the contact set I. More generally if the smoothness of the obstacle is relaxed to assume only $\psi \in C^{2,s}(\overline{\Omega})$ (keeping all other assumptions), then Γ is a Jordan curve admitting a $C^{1,t}$ parametrization for any $t < s$ (see [8]).

A main purpose of this thesis is to study different possible geometrical shapes Γ of free boundaries, for a related class of problems. In particular, more complicated shapes (cusps) will arise. We shall also study how the set Γ varies as the data of the problem (such as ψ and g) are allowed to change.

1.2.1. The Dirichlet Integral

If g, ψ and its gradiant $\nabla\psi$ are sufficiently small, we might expect the solution u and its gradient also to be sufficiently small. If this is so, then using the approximation $\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 + O(x^4)$, near $x = 0$, leads us to consider the Dirichlet integral

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$$

as an approximation for $A(u) - A(0)$. We may therefore consider the obstacle problem obtained from the one above by minimizing $J(u)$ instead of $A(u)$. The Euler equation of $J(u)$ is Laplace's equation $\nabla u = 0$, in contrast to the (nonlinear) minimal surface equation (1.2) associated with $A(u)$. In general we may consider such a problem in a space of any dimension, as follows:

> Given $\Omega \subset R^n$ a bounded domain with smooth boundary $\partial\Omega$,
>
> $g : \partial\Omega \to R$ sufficiently smooth
>
> $\psi : \overline{\Omega} \to R$ sufficiently smooth
>
> with $\psi < g$ on $\partial\Omega$,
>
> find (*) $u : \overline{\Omega} \to R$, $u = g$ on $\partial\Omega$, $u \geq \psi$ on $\overline{\Omega}$,
> such that u minimizes $J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$ among those functions satisfying (*).

We stress that the relation between the minimal surface problem (which minimizes $A(u)$) and the above Dirichlet problem which minimizes $J(u)$ (or energy) is purely formal: they are two different problems. We may consider the Dirichlet problem above as a model problem as it has many features in common with the minimal surface problem, yet is simpler in many respects. In particular the Euler equation $\nabla u = 0$ associated with the functional $J(u)$ is linear. In this thesis we study the Dirichlet integral.

1.3. Existence, uniqueness, regularity

More precisely, $J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx$, $x \in \Omega \subset R^n$, is minimized over the closed convex subset K of the Sobolev space $H^1(\Omega)$:

$$K = \{u \in H^1(\Omega) : u = g \text{ on } \partial\Omega,\ u \geq \psi \text{ a. e.}\} .$$

Since K is a convex closed set and J is a convex continuous functional, it is easy to show [8]

$$u \text{ minimizes } J \text{ over } K \Leftrightarrow DJ(u)(v-u) = \int_\Omega \nabla u \cdot \nabla(v-u) \geq 0, \quad \forall v \in K ,$$

where $DJ(u)$ denotes the Fréchet derivative.

Therefore, an equivalent formulation of the problem seeks $u \in K$ so that

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq 0, \quad \forall v \in K.$$

This new formulation is called a variational inequality.

If g and $\partial\Omega$ are sufficiently smooth, then without loss of generality we can assume $g \equiv 0$ in the problem as follows: simply replace u with $v = u - u_1$ and ψ with $a = \psi - u_1$, where u_1 is the solution to the boundary value problem $\nabla u_1 = 0$ in Ω, $u_1|_{\partial\Omega} = g$. (It is known [12,6] that the solution u_1 exists if g and $\partial\Omega$ are smooth enough). Since $w \in K$ implies $w - u = 0$ on $\partial\Omega$, we have

$$\begin{aligned} DJ(u)(w-u) &= \int_\Omega \nabla u \cdot \nabla(w-u)dx = \int_\Omega \nabla(u-u_1) \cdot \nabla(w-u)dx \\ &= \int_\Omega \nabla(u-u_1) \cdot \nabla((w-u_1)-(u-u_1))dx \\ &= \int_\Omega \nabla v \cdot \nabla(w_1 - v)dx , \end{aligned}$$

where $w_1 = (w - u_1) \in K_1 = \{v \in H_0^1(\Omega) : v \geq a \text{ a.e.}\}$.

Thus u solves the minimizing problem with data (g,ψ) if and only if v solves it with $(0,a)$.

Remark: Unless otherwise stated, from now on we shall assume $g \equiv 0$ on $\partial\Omega$. #

We may summarize the above results in the following precisely stated problems.

(1) Minimization problem: Given $\psi \in H^1(\Omega)$, with $\psi \leq 0$ on $\partial\Omega$, minimize

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx$$

among $u \in H_0^1(\Omega)$ with $u \geq \psi$ a.e. .

(2) Variational problem: Given $\psi \in H^1(\Omega)$, with $\psi \leq 0$ on $\partial\Omega$, find $u \in K$

$$K = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$$

so that $\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq 0, \quad \forall v \in K$.

Theorem 1.3.1. These two problems have unique solutions, and they are the same solutions. #

For the proof and more detail in this context see [8].

A basic regularity result is the following (we state without proof):

Theorem 1.3.2 (Brezis and Kinderlehrer [1]). If $\psi \in C^2(\overline{\Omega})$, and $\psi < 0$ on $\partial\Omega$, then the unique solution u to the above problems satisfies: $u \in C^1(\overline{\Omega})$ and Du is uniformly Lipschitz in $\overline{\Omega}$. #

As before the contact set and free boundary

$$I = \{x \in \Omega \mid u(x) = \psi(x)\}, \quad \Gamma = \partial I$$

are compact subsets of Ω under the assumptions of this theorem. On $\Omega - I$ the derivation of the Euler equation is valid and so $\Delta u = 0$ there. It is also the case that $I \subseteq \{x \in \Omega : \Delta\psi \leq 0\}$ because $\Delta u \leq 0$ in Ω (by applying the variational inequality, see [8,20]). Finally, because u is C^1, it follows immediately that $\nabla(u - \psi) = 0$ on Γ. See [8] for details. Conversely we have the following result (see [2]).

Proposition 1.3.3. Suppose $I \subset \Omega$ is a compact set with boundary Γ which is piecewise smooth in the following sense: there is a finite subset $F \subseteq \Gamma$ such that if $x \in \Gamma - F$, then near x, Γ is an embedded C^{n-1} manifold. Suppose $u \in C^1(\overline{\Omega})$ and satisfies:

$$\Delta u = 0 \text{ and } u \geq \psi \text{ on } \Omega - I$$

$$u = 0 \text{ on } \partial\Omega$$

$$u = \psi \text{ and } \Delta\psi \leq 0 \text{ on } I .$$

Then u is the solution of the obstacle problem.

Outline of Proof: We show $\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq 0$ for every $v \in K$. It is enough to consider $v \in C^1(\overline{\Omega}) \cap K$, as this set is dense in K. Let $B_r = \bigcup_{x \in F} B(x,r)$, where $B(x,r)$ is the r-ball about x. Let also $E = \Omega - I$ and set $\Omega_r = \Omega - B_r$, $E_r = E - B_r$, and $I_r = I - B_r$.

By Sard's Theorem [4], for almost every r, the $(n-1)$-manifold ∂B_r and $\Gamma - F$ intersect transversally; we choose r so that this is the case. We may apply Green's Theorem (since ∂B_r and $\Gamma - F$ are transverse) to obtain

$$\int_{\Omega_r} \nabla u \cdot \nabla(v-u)dx$$

$$= \int_{E_r} \nabla u \cdot \nabla(v-u)dx + \int_{I_r} \nabla\psi \cdot \nabla(v-u)dx$$

$$= \int_{\partial E_r} \frac{\partial u}{\partial n}(v-u) + \int_{\partial I_r} \frac{\partial\psi}{\partial n}(v-u) - \int_{I_r}(\nabla\psi)(v-u)dx$$

As $r \to 0$, the integrals over Ω_r and I_r tend to those over Ω and I.

The integrals over ∂E_r and ∂I_r can be divided into two parts: those over $\Gamma - B_r$ and those over the spheres ∂B_r. The integrals over $\Gamma - B_r$ cancel as $\frac{\partial u}{\partial n} = \frac{\partial\psi}{\partial n}$ there, and the integrals over ∂E_r and ∂I_r are taken with opposite orientation. The contribution of the integrals over ∂B_r tends to zero as $r \to 0$, since the integrands are bounded and measure of ∂B_r tends to zero. Hence we get the result. #

Some Examples

Example 1. $n = 1$: A string fixed at endpoints $(a,0)$, $(b,0)$, where $\Omega = (a,b)$.

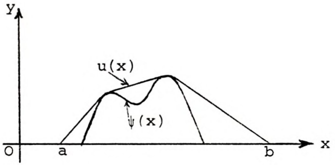

Figure 1.3

It is clear that on the intervals, where $u(x) > \psi(x)$, we have $u''(x) = 0$. So $u(x)$ is linear. It is also clear that $u''(x) = \psi''(x) \leq 0$ on intervals where $u(x) = \psi(x)$. In general $u \notin C^2((a,b))$ as u'' typically has jump discontinuities: at the endpoints of the intervals where $u(x) = \psi(x)$, $u''(x)$ jumps from $\psi''(x)$ to zero. See Figure 1.3.

Example 2. $n = 2$: Suppose $\Omega = \{(x,y) \in R^2 \mid x^2 + y^2 < 1\}$, $g \equiv 0$ on $\partial\Omega$, and $\psi(x,y) = -\frac{1}{2}(x^2 + y^2) + c$, where $0 < c < \frac{1}{2}$. So $\psi < 0$ on $\partial\Omega$. The radial symmetry of the obstacle and the invariance of the Dirichlet integral under rigid rotations implies the unique solution $u(x)$ also is radially symmetric.

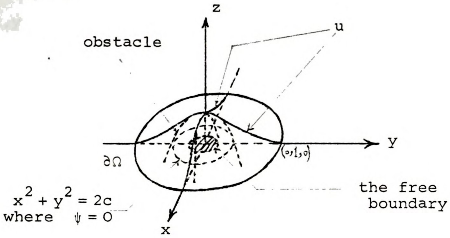

Figure 1.4

As a candidate for the contact set, consider the disk

$$I = \{(x,y) \in R^2 \mid x^2 + y^2 \leq r_0^2 < 1\}$$

of radius r_0, for some r_0. If this is the contact set, then in $\Omega - I$, $u(x)$ satisfies the Dirichlet problem

$$\Delta u = 0 \qquad \text{for} \quad r_0^2 \leq x^2 + y^2 \leq 1$$

$$u = 0 \qquad \text{for} \quad x^2 + y^2 = 1$$

$$u = c - \frac{1}{2} r_0^2 \quad \text{for} \quad x^2 + y^2 = r_0^2$$

and moreover $\nabla u = \nabla \psi$ on Γ, hence

$$\frac{\partial u}{\partial r} = -r_0 \quad \text{for} \quad x^2 + y^2 = r_0^2 , \qquad (1)$$

where $\frac{\partial u}{\partial r}$ is the outward radial derivative. The unique solution of this Dirichlet problem is (with $r^2 = x^2 + y^2$) $u(x,y) = A \log r$, $r_0 \leq r \leq 1$, where

$$A = (c - \tfrac{1}{2} r_0^2)/\log r_0 \qquad (2)$$

In addition (1) holds also if and only if $\frac{A}{r_0} = -r_0$, or

$$A = -r_0^2 \qquad (3)$$

There exists r_0 and A satisfying (2) and (3) if and only if

$$c = -\frac{r_0^2}{2} \log \left(\frac{r_0^2}{e}\right)$$

It is an easy exercise to show that for any $c \in (0, \frac{1}{2})$, there is a unique $r_0 \in (0,1)$ satisfying this relation. We claim this unique r_0 and $A = -r_0^2$ give the solution

$$u(x,y) = \begin{cases} \psi(x,y), & 0 \leq r \leq r_0 \\ A \log r, & r_0 \leq r \leq 1 \end{cases}$$

of the problem. This follows from Proposition 1.3.3 once we show $u > \psi$ on $\Omega - I$, that is, $f(r) = -r_0^2 \log r + \frac{1}{2} r^2 - c > 0$, for $r \in (r_0,1)$. But this is easy as $f(r_0) = 0$ and $f'(r) > 0$ on $(r_0,1)$.

Again note u is not C^2, as $\frac{d^2u}{dr^2}$ undergoes a jump at $r = r_0$ from

$$\frac{d^2\psi}{dr^2} = -1 \quad \text{to} \quad \frac{d^2}{dr^2}(A \log r)\Big|_{r=r_0} = 1 .$$

The anolog of this problem in dimensions $n \geq 3$ may be considered in a similar fashion. The contact set is again a disk of radius r_0, where r_0 depends on n. In $\Omega - 1$, u has the form $u = Ar^{-n+2} - A$.

Example 3. Lewy and Stampacchia [12] show that if $\Omega \subset R^2$ is convex and ψ is strictly concave and analytic with $\psi < 0$ on $\partial\Omega$, then Γ is an analytic Jordan curve, $\Omega - I$ is homeomorphic to an annulus, and I is simply connected. This result can be generalized for the case of obstacle $\psi \in C^2(\overline{\Omega})$. See [8].

Some additional properties of the solution

1. If $\psi < 0$ in $\partial\Omega$ and $\Delta\psi < 0$ in Ω, then $\Omega - I$ is connected.

Proof: This can be shown by strong maximum and minimum principle: (Let $\Delta w \geq 0$ (≤ 0) in A, a bounded domain, and suppose there exists a point $y \in A$ for which $w(y) = \sup_A w(\inf_A w)$. Then w is constant in A.) Suppose $\Omega - I$ is not connected, then it has some component $\mathcal{O}$ whose boundary is part of the boundary of I. Then $u - \psi$ is zero on $\partial\mathcal{O}$ and $u - \psi > 0$, $\Delta(u - \psi) > 0$ in $\mathcal{O}$, which contradicts the strong maximum principle.

2. Under quite general conditions [3,9,10,12] $\Gamma = \partial I$ consists of smoothly parametrized arcs, possibly with cusps (in fact analytically parametrized arcs if ψ is analytic). In particular if $\mathcal{O}$ is a neighborhood of a point $(x_0, y_0) \in \Omega \subset R^2$ and $\mathcal{O} \cap \Gamma$ is a Jordan arc through (x_0, y_0), and ψ is analytic, then the arc $\mathcal{O} \cap \Gamma$ admits an analytic parametrization (possibly with cusps). See [8] for more details.

3. If $\psi_1 < \psi_2$, then $0 < u_2 - u_1 < \sup(\psi_2 - \psi_1)$.

<u>Proof</u>: Let $K_1 = \{w \in H_0^1(\Omega) : w \geq \psi_1\}$ and $K_2 = \{w \in H_0^1(\Omega) : w \geq \psi_2\}$; and put $u_3 = \min(u_1, u_2)$, $v = \max(u_1, u_2)$. Then by the variational inequality formulation $\int_\Omega \nabla u_1 \cdot \nabla(u_3 - u_1)dx \geq 0$, $\int_\Omega \nabla u_2 \cdot \nabla(v - u_2)dx \geq 0$, where $u_3 \in K_1$ and $v \in K_2$. Since $v - u_2 = -(u_3 - u_1)$, thus $\int_\Omega \nabla u_2 \cdot \nabla(v - u_2)dx \geq 0 \Leftrightarrow \int_\Omega \nabla u_2 \cdot \nabla(u_3 - u_1)dx \leq 0$. Suntracting the first inequality above from the last one yields

$$\int_\Omega \nabla(u_2 - u_1) \cdot \nabla(u_3 - u_1)dx \leq 0 \qquad \text{or}$$

$$\int_{\{u_2<u_1\}} \nabla(u_2 - u_1) \cdot \nabla(u_2 - u_1)dx \leq 0 \, .$$

The last inequality is obtained by Theorem A.1. in [8]. Then C^1 property of $u_2 - u_1$ with $\int_{\{u_2<u_1\}} |\nabla(u_2 - u_1)|^2 dx \leq 0$ imply that $u_2 \geq u_1$ on Ω.

To establish the second part of the inequality put $\sup(\psi_2 - \psi_1) = c$, and $\psi_1 \leq \psi_2 \leq \psi_1 + c$. Also put $u_3 = \min(u_1 + c, u_2)$, $v = \max(u_1 + c, u_2)$. We want to show $u_2 \leq u_1 + c$. We have $\int_\Omega \nabla u_1 \cdot \nabla(v - u_1 - c)dx = \int_\Omega \nabla u_1 \cdot \nabla(v - u_1)dx \geq 0$. Since $v = u_1 + c + u_2 - u_3$, thus $\int_\Omega \nabla u_1 \cdot \nabla(u_3 - u_2)dx \leq 0$. Subtracting $\int_\Omega \nabla u_2 \cdot \nabla(u_3 - u_2) \geq 0$ from the last inequality we get

$$\int_{\{u_1+c<u_2\}} \nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)dx \leq 0 .$$

C^1 property of $u_1 + c$ and u_2 implies $u_2 \leq u_1 + c$ on Ω.

<u>Remark:</u> If $\psi_1 \leq \psi_2$, with the same boundary condition, then I_1 is not necessarily contained in I_2. Consider the following example with $n = 1$: See Figure 1.5.

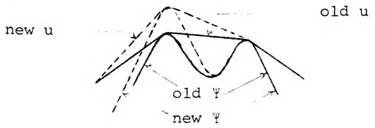

Figure 1.5

Here $I_2 - I_1$ or $I_1 - I_2$ may be non-empty sets.

Example 4. Schaeffer gave [20] two examples, for $n = 2$, for which the contact sets I_j $(j = 1,2)$ are bounded by curves with cusps as in Figures 1.6, 1.7 respectively.

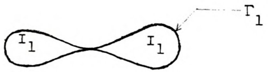

Figure 1.6

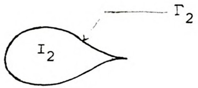

Figure 1.7

The curve Γ_1 is the analytic image of a circle, with non-vanishing derivative everywhere. In particular, near the double point, Γ_1 consists of two analytic arcs, tangent to each other without crossing, and with different curvatures at the point of tangency. The curve Γ_2 is analytic away from the cusp, near which it has the form $y^2 \sim Kx^5$ for some $K \neq 0$.

Consider first the analytic function $B(t) = (t + t^{-1})/2 + sP_j(t)(t - t^{-1})/2$, where s is a real parameter and $P_j(t)$ is chosen as

In example 1, $P_1(t) = t^2 + 2 + t^{-2}$

In example 2, $P_2(t) = (t - 2 + t^{-1})^2$.

For $s = 0$, observe that B is a conformal map of $\{t : |t| > 1\}$ onto the plane cut along the real axis from -1 to 1. By choosing $s > 0$ small, we may arrange that, in either example 1 or example 2, B maps the circles $|t| = 1$ and $|t| = 2$ onto Γ_j and a Jordan curve C_j which contains Γ_j in its interior, respectively. Let I_j be the (closed) region inside Γ_j and let Ω_j be the (open) region inside C_j. Schaeffer shows that B maps the annulus $A = \{t : 1 < |t| < 2\}$ onto $\Omega_j - I_j$ in a one-to-one manner. He proves that with the obstacle

$$\psi(x,y) = -\frac{1}{2}(x^2 + y^2) ,$$

The obstacle problem has its unique solution in Ω_j with contact set I_j and the free boundary Γ_j (for sufficiently small $s > 0$). This is accomplished by considering an equivalent (non-singular) problem in the disk $|t| < 2$ via the mapping B.

Example 5. In [20] Schaeffer gives an example with a C^∞ obstacle for which I has infinitely many connected components in the neighborhood of some point.

Local solution to the obstacle problem

1.4. The obstacle problem discussed in last sections has a local formulation as a differential equation which we define it more precisely now.

Definition 1.4.1. A local solution in an open set $\mathcal{O} \subseteq \Omega \subset \mathbb{R}^n$ is a $\mathcal{C}^1$ function $u : \mathcal{O} \to \mathbb{R}$ satisfying

$$u(x) \geq \psi(x) \quad \text{on} \quad \mathcal{O}$$

$$\Delta u(x) = 0 \quad \text{if} \quad u(x) > \psi(x)$$

$$\Delta \psi(x) \leq 0 \quad \text{if} \quad u(x) = \psi(x) \ . \qquad \#$$

If u is a local solution in $\mathcal{O}$, then the contact set in $\mathcal{O}$ is the set $I = \{x \in \mathcal{O} \mid u(x) = \psi(x)\}$, and the free boundary in $\mathcal{O}$ is $\Gamma = \partial I \cap \mathcal{O}$. Observe that $\nabla(u - \psi) = 0$ on I.

Now assume in the neighborhood of some point, Γ is an $(n-1)$-dimensional manifold (arc if $n = 2$, surface if $n = 3$) which is analytic with I lying on one side of Γ. Then on a one sided neighborhood V of Γ, contained in $\mathcal{O} - I$, we have $\Delta u = 0$, $u > \psi$, and on Γ we have $u = \psi$, $\nabla(u - \psi) = 0$. In fact the initial boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on} \quad V \\ u = \psi, \ \nabla(u - \psi) = 0 & \text{on} \quad \Gamma \end{cases}$$

has a unique solution u near Γ in $\mathcal{O} - I$ by the Cauchy-Kowalewski Theorem. In fact an explicit formula for u can be given when $n = 2$ (see [20]) and we shall present the method later on.

The higher dimensional cases are much more difficult. However, for n even, if the solution has enough symmetry, results can be obtained. We shall study the case $n = 4$ for a class of axisymmetric problems.

Example 6. Kinderlehrer and Nirenberg [10,8] gave an example of a local solution in which the free boundary has a cusp and away from it the free boundary is analytic. They showed for $m \geq 1$ odd the free boundary can never have a cusp of the form

$$(y - y_0)^2 \sim K(x - x_0)^{2m+1}, \quad K \neq 0$$

near some (x_0, y_0) in two dimensional case $(\Omega \subset R^2)$, whereas for m even such a cusp can occur. (In fact, the case $m = 1$ was first noted by Schaeffer: Example 4). Their proof involves first straightening out the cusp to a line segment by means of a conformal mapping; then an analysis of several terms of the Taylor series of u near (x_0, y_0), based on the equation $\Delta u = 0$ governing u in $\Omega - I$, gives a contradiction to $u \geq \psi$.

Further specialization

We shall study local solutions in the neighborhood of a point, say the origin. We assume the nondegeneracy condition $\Delta\psi(0) < 0$ (if $\Delta\psi(0) > 0$, then $\Gamma = \emptyset$ near 0). Then, as we see from the Definition 1.4.1, upon adding the same harmonic function h to a local solution u and obstacle ψ and multiplying by a positive constant c we get another local solution $u_1 = c(u+h)$ for the new obstacle $\psi_1 = c(\psi + h)$. In particular a quadratic polynomial can always be chosen for $h(x)$ to give

$$\psi(x) = -\frac{1}{2}|x|^2 + O(|x|^3), \quad x \in R^n .$$

We choose $\psi(x) = -\frac{1}{2}|x|^2$ again as a model problem near the origin and consider only this obstacle from now on. Deleting the higher-order terms in ψ should not affect the resulting theory significantly, although this has yet to be established.

Changing Boundary conditions

1.5. Schaeffer [20,21] has studied how the set I changes as the data ψ and g vary. He proved a theorem that if ψ and g are C^∞ with $\psi < g$ on $\partial\Omega$, Γ a C^∞ curve, and $\Delta\psi < 0$ on I, then if ψ_1, g_1 are sufficiently near ψ, g in the C^∞ topology, the free boundary of Γ_1 of the corresponding problem will be a

C_-^∞ curve near Γ in the C^∞ topology (in normal coordinates about Γ).

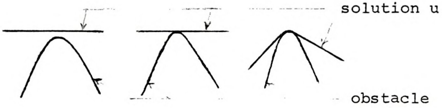

Figure 1.8

Schaeffer has also pointed out the need for a generic theory of such variations of I and ∂I. Mallet-Paret and Chow declared that such a theory presumably could take the form of a bifurcation theory or unfolding theory for the singularities of ∂I. Such a theory was described in the case $n = 2$ [14,5,15]. A significant point here is that the unfoldings one encounters are not generic in the sense of singularity theory as developed by Thom Mather, Arnold and others; only very special types of singularities can occur. This is seen for example from the result of Kinderlehrer and Nirenberg [10,8] described above in Example 6.

Mallet-Paret made a detailed study of a class of singularities of a free boundary, and their bifurcation, in the two dimensional case, as the data of the system varies parametrically. Typically cusps on the free boundary become smooth, and islands may appear as in Figure 1.9.

singular free boundary after perturbation

Figure 1.9

In this thesis we shall study the unfoldings (bifurcations) of a class of singularities of the four dimensional axi-symmetric obstacle problem. This case is the next simplest after the two-dimensional case, as harmonic functions admit a particularly nice representation. Let $u(x) = v(x_1, \sqrt{x_2^2 + \cdots + x_n^2})$, where v is defined in a region in R^2. Putting $y = \sqrt{x_2^2 + \cdots + x_n^2}$, for $n = 2$ and 4 we have locally

$\Delta u = 0$ if and only if

$$\begin{cases} v(x_1,y) = \operatorname{Im} f(x_1 + iy), & f(z) \text{ analytic } (n = 2) \\ v(x_1,y) = \frac{1}{y} \operatorname{Im} f(x_1 + iy), & f(z) \text{ analytic } (n = 4) . \end{cases}$$

There is no such formula in three dimensional space. In general there are anologs of the above $(n = 2,4)$ cases for even (but not odd) dimensions.

2. FORMULATION OF A PROBLEM

Local solution to the axi-symmetric obstacle problem in R^4

2.1. In this chapter we consider the axi-symmetric obstacle problem in R^4 $(n = 4)$ with the obstacle $\theta(x_1,x_2,x_3,x_4) = -\frac{1}{2} \sum_{j=1}^{4} x_j^2$, and we study the local solutions w defined near the origin. By axi-symmetric we mean

$$w(x_1,x_2,x_3,x_4) = v(x,y) ,$$

where $x = x_1$, $y = \sqrt{x_2^2 + x_3^2 + x_4^2}$. Note then $\theta(x_1,x_2,x_3,x_4) = -\frac{1}{2}(x^2+y^2) = \varphi(x,y)$. We shall formulate a problem of unfolding a singular free boundary passing through O in this chapter. But we need to state and prove a proposition and a lemma first.

Proposition 2.2. w is a local solution of the axi-symmetric obstacle problem with obstacle θ in a region for which $y > 0$ if and only if $u = yv$ is a local solution to the obstacle problem (for $n = 2$) with obstacle $\psi(x,y) = y\varphi(x,y)$ there.

Proof: We check the conditions of definition of local solution stated in Chapter 1. Clearly $w \geq \theta$ in the region if and only if $yv \geq \psi$ there. Also it is easy to check that $\Delta_{R^4} w = 0$ when $w > \theta$ if and only if $\Delta_{R^2}(yv) = 0$ when $yv > y\varphi$ (of course in the region $y > 0$) because

$$\Delta_{R^4} w = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{2}{y}\frac{\partial v}{\partial y} = \frac{1}{y}\Delta_{R^2}(yv) .$$

Now, it is only left to check the C^1 condition. Assume $v(x,y)$ is C^1 in a region for which $y > 0$, then its composition with $y = \sqrt{x_2^2 + x_3^2 + x_4^2}$, i.e., $w(x_1,x_2,x_3,x_4)$ is also C^1 there.

Conversely assume $w(x_1,x_2,x_3,x_4)$ is C^1 in the region, then $w(x_1,x_2,0,0) = v(x,y)$, where $x = x_1$ and $y = x_2$, is also C^1 there. Here is the proof complete because v is C^1 in the region if and only if $u = yv$ is C^1 there. #

The last proposition shows that in a simply connected domain in the upper half-plane in the noncontact set $v(z) = \frac{1}{y}\,\mathrm{Im} f(z)$, where $z = x + iy$, for an analytic function f.

Remark 1: If w is an axi-symmetric local solution with the above obstacle, then we can extend the function v to be even in y. We shall use the extended even

function v and the odd one $u = yv$ defined in a whole neighborhood of the origin in (x,y)-plane later on.

Remark 2: Following Remark 1, sometimes, we shall need to use a modification of Proposition 2.2 concerning the x-axis $(y = 0)$ entering the region (recall $y > 0$ in the Proposition). We claim here that w is C^1 in a region for which $y \geq 0$ if and only if $v(x,y)$ is C^1 there. To show this let $w(x_1,x_2,x_3,x_4)$ be C^1 in the region $y \geq 0$. Then $w(x_1,x_2,0,0) = v(x,y)$, where $x = x_1$ and $y = x_2$, is C^1 there.

Conversely if $v(x,y)$ is C^1 in the region, then $w(x_1,x_2,x_3,x_4)$ is continuous there by simply being a composite function of

$$w(x_1,x_2,x_3,x_4) = v(x_1,y)\ , \quad y = \sqrt{x_2^2 + x_3^2 + x_4^2}\ .$$

Moreover since $\frac{\partial w}{\partial x_j} = \frac{\partial v}{\partial y}\frac{x_j}{y}$, $j = 2,3,4$, clearly these partial derivatives of w are continuous when $y \neq 0$. Since v is an even function of y, thus $\frac{\partial v}{\partial y}$ is an odd function of y. If we now let y approach to zero, then $\lim\limits_{(x,y)\to(0,0)} \frac{\partial w}{\partial x_j} = 0$ because $\frac{\partial v}{\partial y}(x,0) = \lim\limits_{y\to 0} \frac{\partial v}{\partial y}(x,y)$ and $\frac{x_j}{y}$ remains bounded (between zero and one). Hence $w(x_1,x_2,x_3,x_4)$ is C^1 in the region for which $y \geq 0$. #

2.3. Local solution formula

Let us consider an analytic arc $\Gamma_1 : x = a(y)$ through (x_0, y_0), where (x_0, y_0) is fixed near the origin with $y_0 > 0$ and $x_0 = a(y_0)$. We shall construct a local solution in a neighborhood of (x_0, y_0), in the region $y > 0$, for the above axi-symmetric problem, with free boundary Γ_1 and contact set I lying on one side of Γ_1. Let us define the conformal mapping L_1 for t complex near y_0 as $z = L_1(t) = a(t) + it$, where $z_0 = x_0 + iy_0 \in \Gamma_1$. See Figure 2.1.

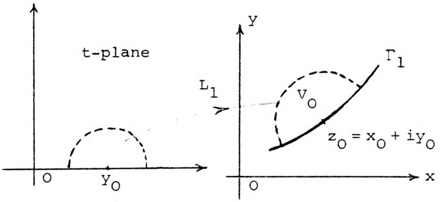

Figure 2.1

Lemma 2.4. For a sufficiently small neighborhood V of z_0, L_1^{-1} exists as a conformal mapping and $L_1^{-1}(x + iy) = y$ for $x + iy \in \Gamma_1$. Let V_0 be a sufficiently small one-sided neighborhood of Γ_1 about

z_0, and let $\psi(x,y) = -\frac{1}{2}\, y(x^2+y^2)$. Then $u = yv$ is a local solution to the obstacle problem, in the region $y > 0$ in a neighborhood of z_0, with obstacle ψ and the noncontact set in V_0, the free boundary Γ_1, and the contact set in $V - V_0$ if and only if

$$\begin{cases} u(x,y) = \psi(x,y), \quad \text{for} \quad (x,y) \in (V - V_0) \\ u(x,y) = u(z) = \psi(z_0) + \operatorname{Re} \int_{z_0}^{z} \frac{i}{2}[q^2 + 4(L_1^{-1}(q))^2]\,dq\,, \\ \qquad\qquad\qquad\qquad \text{for} \quad z \in V_0 \cup \Gamma_1\,. \end{cases}$$

<u>Proof</u>: Since $L_1'(t) = a'(t) + i \neq 0$ for t (as complex) near y_0 in the t-plane, then L_1 is a conformal mapping near y_0 taking the real axis (near y_0) to the arc Γ_1. Therefore L_1^{-1} exists and it is conformal in a small enough neighborhood V of z_0 and $L_1^{-1}(x+iy) = y$ on Γ_1. L_1 also maps the upper or lower half (the other half) of a neighborhood of y_0 conformally onto a one-sided neighborhood V_0 (the other one-sided neighborhood) of Γ_1 about z_0. The proof of the first part is complete here.

<u>Necessary part</u>: With obstacle ψ we have

$$\psi_x - i\psi_y = -xy + i(\tfrac{1}{2}\,x^2 + \tfrac{3}{2}\,y^2) = \tfrac{i}{2}\,x(x-iy) - \tfrac{3}{2}\,y(x-iy)$$

$$= \tfrac{i}{2}(x-iy)(x+3\,iy) = \tfrac{i}{2}(z-2i\,y)(Z+2i\,y)$$

$$= \tfrac{i}{2}(z^2+4y^2)\ .$$

For z near z_0 function $f(z) = \frac{i}{2}[z^2 + 4(L_1^{-1}(z))^2]$ is holomorphic. Note $u_x - iu_y$ is holomorphic in V_0 because u is harmonic there, and it is also continuous on $\overline{V}_0$ because u is C^1 in V. For $z \in \Gamma_1$ we have $\nabla u = \nabla\psi$. Therefore $u_x - iu_y = \frac{i}{2}[z^2 + 4y^2] = \frac{i}{2}[z^2 + 4(L_1^{-1}(z))^2] = f(z)$, for $z \in \Gamma_1$. Hence $u_x - iu_y = f(z)$, for $z \in V_0$ [8,19] because both analytic functions on V_0 agree on Γ_1. Thus $u_x - iu_y$ is well-defined and unique. Moreover $u_x - iu_y = f(z)$ implies $u(z) = \text{Re } g(z)$ [19] for an appropriate analytic function on V_0 with $g'(z) = f(z)$ by the Cauchy-Riemann equations. This means $u(z) = C + \text{Re}\int_{z_0}^{z} f(q)dq$. Then $u = \psi$ on Γ_1 implies $u(z_0) = C = \psi(z_0)$. Hence $u(z) = \psi(z_0) + \text{Re}\int_{z_0}^{z} f(q)dq$ for $z \in V_0 \cup \Gamma_1$; and $u = \psi$ on $(V - V_0)$. Here is the necessary part proved.

<u>Sufficient part</u>: Conversely assume u is given by the above formula on V. Then clearly u is harmonic in V_0, and it is C^1 in $\overline{V}_0$. Also for $z \in \Gamma_1$, if we choose the path of integration on Γ_1 and $q = \xi + i\eta$, we shall have

$$\begin{aligned} u(z) &= \psi(z_0) + \text{Re}\int_{z_0}^{z} \frac{i}{2}[\xi^2 + 3\eta^2 + 2i\xi\eta](d\xi + id\eta) \\ &= \psi(z_0) - \frac{1}{2}\int_{z_0}^{z} (\xi^2 + 3\eta^2)d\eta + 2\xi\eta d\xi \\ &= \psi(z) . \end{aligned}$$

Thus u is continuous on V. Now we want to show $\nabla u = \nabla \psi$ on Γ_1. For $z \in \Gamma_1$ we have

$$\nabla u = u_x + iu_y = \overline{u_x - iu_y} = \overline{\frac{d}{dz} \int_{z_0}^{z} \frac{i}{2}[q^2 + 4(L_1^{-1}(q))^2]dq}$$

$$= \overline{\frac{i}{2}[z^2 + 4y^2]} = \overline{\frac{i}{2}(x^2 + 3y^2 + 2ixy)}$$

$$= \nabla \psi \ .$$

Thus u is C^1 in V, in particular along Γ_1. To complete the proof we need to show $u > \psi$ in V_0. To show this observe that on Γ_1 we have $u - \psi = 0$, $\frac{\partial(u-\psi)}{\partial n} = 0$, and $\frac{\partial^2(u-\psi)}{\partial n^2} = \Delta(u - \psi) = -\Delta\psi = 4y > 0$, where n is the unit (outer) normal to Γ_1. Thus $u > \psi$ near Γ_1 in V_0 in the region $y > 0$. #

Remark. Instead of defining $u = \psi$ on $V - V_0 = I$, if we extend the integral formula for u to be defined in a full neighborhood of z_0, then $u > \psi$ holds on both sides of Γ_1 with $u = \psi$ on Γ_1. We also observe in the proof that $\nabla(u - \psi) = 0$ in the closure of the noncontact set if and only if $(L^{-1}(z))^2 = (\text{Im } z)^2$. #

Now we are going to start the main purpose of this chapter by constructing a local solution near 0 with a singular free boundary in Example 2.5. In Section 2.7 later on we shall discuss heuristically unfolding of the above free boundary by some perturbation of Example 2.5.

A class of examples

2.5. We want to construct a local solution to the above axi-symmetric problem whose free boundary possesses a cusp at the origin. Singular free boundaries for axi-symmetric obstacle problem are only in the form

$y^2 = Kx^{4n+1}$, or $z = H(t) - \frac{2}{\pi} t^{2n+1} \log|t| \pm it^{2n+1}$

t real near zero

Figure 2.2-a

or isolated points [17], where $n \geq 0$ is an integer, $H(t)$ is a real analytic function near $t = 0$ with $H(0) = 0$, $H'(0) > 0$; and let $\Gamma = \Gamma_1 \cup \Gamma_2$ be the union of the four arcs, near the origin in the z-plane given parametrically by $z = H(t) - \frac{2}{\pi} t \log |t| \pm it$, $0 \leq |t| < r_0$. See Figure 2.2. We assume symmetry of $v = \frac{1}{y} u$ as before about the x-axis, namely that v is even in y. In this example the contact set I will be the shaded region as shown in the Figure 2.2

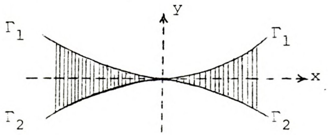

Figure 2.2-b

Following the local solution formula given in the Lemma 2.4, let us first define

$$L(t) = H(t) - \frac{2}{\pi} t \log t + it , \quad t \notin \{i\sigma : \sigma < 0\}$$

$$(\text{i.e.,} \ -\frac{\pi}{2} \leq \arg t < \frac{3\pi}{2}) ,$$

where $\log t$ has the branch which is real for $t > 0$; and $L(t)$ for $t > 0$ represents the right hand part of Γ_1. We proceed in five steps as follows:

(i) We are going to show that L maps the real axis near 0 onto Γ_1 and an upper one-sided neighborhood of the real axis about 0 conformally onto an upper one-sided neighborhood of Γ_1 about 0. To see this let t move along the real axis from 0 to $r > 0$. By the branch we chose L maps $[0,r]$ into the right hand part of Γ_1 in a one-to-one manner. Let t move from $t = r$ along $|t| = r$ in the upper half plane to $t = -r$, then back again to $t = 0$ along the negative axis (see Figure 2.3).

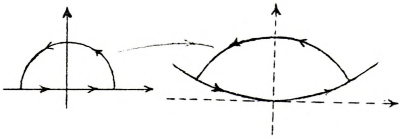

Figure 2.3

Continue the function L analytically along this closed path γ_r and its interior G_r. Clearly L is continuous on $\overline{G}_r$ and analytic on $\overline{G}_r - \{0\}$. Along the negative part of the real axis in the t-plane L has the image:

$$L(t) = H(t) - \frac{2}{\pi} t \log |t| - it, \quad -r \leq t \leq 0, \quad \text{Arg } t = \pi .$$

This image is exactly the left hand part of Γ_1 (above the x-axis) near the origin. We now want to show that the image of $\gamma_r \cap \{t : |t| = r\}$, for r small enough, stays away from the origin in the upper half-plane. Let $z = L(t)$, $t = re^{i\theta}$, $0 \leq \theta \leq \pi$, for r small enough. Then

$$\begin{aligned} \text{Im } z &= \text{Im}[t(H'(0) - \frac{2}{\pi} \log r + i(1 - \frac{2\theta}{\pi})] + O(r^2) \\ &= r[(1 - \frac{2\theta}{\pi})\cos \theta + (H'(0) - \frac{2}{\pi} \log r)\sin \theta] + O(r^2) \\ &= f(\theta; r) > 0 . \end{aligned}$$

This shows that for r small enough, the image of $\gamma_r \cap \{t : |t| = r\}$ stays away from the origin in the upper half-plane. Observe also that

$$\begin{aligned} \frac{d}{d\theta}(\text{Re } z) = r[(-H'(0) + \frac{2}{\pi} \log er)\sin \theta \\ + (\frac{2}{\pi} \theta - 1)\cos \theta] + O(r^2) < 0 . \end{aligned}$$

Thus $\text{Re } z$ decreases as θ varies from 0 to π.

Finally

$$\frac{d^2}{d\theta^2}(\operatorname{Im} z) = r[-(1-\frac{2\theta}{\pi})\cos\theta$$

$$+ (-H'(0)+\frac{2}{\pi}\log re^2)\sin\theta] + O(r^2) < 0 .$$

This shows $\operatorname{Im} z$ has only one critical point, a maximum point, at a point θ_0 near $\frac{\pi}{2}$, and $\operatorname{Im} z$ is increasing on $[0,\theta_0]$ and decreasing on $[\theta_0,\pi]$. Thus $\operatorname{Im} z$ takes its minimum values at $\theta = 0,\pi$. Hence L_1 is one-to-one on γ_r, and $L(\gamma_r)$ is a Jordan curve.

Consider the topological degree (or winding number) $d(L,G_r,z)$ for any $z \notin L(\gamma_r)$. This equals one in the region bounded by $L(\gamma_r)$ (the Jordan curve), and is zero in the unbounded component of $\mathbb{C} - L(\gamma_r)$. Because L is analytic, it follows that L maps G_r conformally onto the region bounded by the Jordan curve $L(\gamma_r)$.

Hence L^{-1} conformally maps $L(G_r)$ onto G_r. Moreover L^{-1} is analytic at each point of the closure $\overline{L(G_r)}$ except 0, where it is continuous

(ii) Near any $z_0 \in \Gamma_1 - \{0\}$ we have, by Lemma 2.4, a local solution

$$\begin{cases} u = \psi(z_0) - \frac{1}{2}\operatorname{Im}\int_{z_0}^{z} [q^2 + 4(L^{-1}(q))^2]dq , & z \text{ above } \Gamma_1 - \{0\} \\ u = \psi(z), & z \text{ beneath } \Gamma_1 - \{0\} . \end{cases}$$

The overlapping solutions along $\Gamma_1 - \{0\}$ agree and they are independent of z_0 on each side of $\Gamma_1 - \{0\}$. This is because if z_1, z_2 are two such initial points on the right hand part, for example, then the difference of

$$u(z) = \psi(z_1) - \frac{1}{2}\,\mathrm{Im}\int_{z_1}^{z} [q^2 + 4(L^{-1}(q))^2]dq$$

$$u(z) = \psi(z_2) - \frac{1}{2}\,\mathrm{Im}\int_{z_2}^{z} [q^2 + 4(L^{-1}(q))^2]dq$$

is $\psi(z_1) - \psi(z_2) - \frac{1}{2}\,\mathrm{Im}\int_{z_1}^{z_2} [q^2 + 4(L^{-1}(q))^2]dq = 0$, by taking the last integration path on $\Gamma_1 - \{0\}$. Therefore by taking zero as the limit case of z_0 in the formula we get

$$u(z) = -\frac{1}{2}\,\mathrm{Im}\int_{0}^{z} [q^2 + 4(L^{-1}(q))^2]dq .$$

This formula can be used for u in the noncontact set corresponding to both right hand and left hand part of $\Gamma_1 - \{0\}$. See Figure 2.4 (shaded region). For emphasis

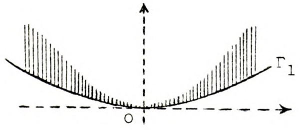

Figure 2.4

we may here check that $u = \psi$ (by taking integration path along $\Gamma_1 - \{0\}$) and $\nabla u = \nabla\psi$ on $\Gamma_1 - \{0\}$ by observing the fact that $[L^{-1}(z)]^2 = (\text{Im } z)^2$ there. Also $\frac{\partial^2(u-\psi)}{\partial n^2} = \Delta(u-\psi) = -\Delta\psi = \psi y > 0$ on $\Gamma_1 - \{0\}$, where n is the unit outer normal to $\Gamma - \{0\}$ at any $z \in \Gamma_1 - \{0\}$.

This shows $u - \psi > 0$ on the noncontact sets corresponding to the left hand and right hand parts of $\Gamma_1 - \{0\}$. Note u given above is defined in $\overline{L(G_r)}$ and it is the imaginary part of an analytic function in $\overline{L(G_r)} - \{0\}$. Thus $\Delta u = 0$ in $L(G_r)$. Define $u = \psi$ on Γ_1 and between Γ_1 and real axis. To show now that u is a local solution in the upper half of z-plane, near the origin, we need to show u is C^1 there and $u > \psi$ in $L(G_r)$.

(iii) At this step we want to show $u > \psi$ in $L(G_r)$

$$u(z) - \psi(z) = -\frac{1}{2}\text{ Im}\int_0^z [q^2 + 4(L^{-1}(q))^2]dq + \frac{1}{2}\, y(x^2+y^2)$$

$$= -\frac{1}{2}\text{ Im}\int_0^t [L^2(s) + 4s^2]L'(s)ds$$

$$+ \frac{1}{2}\,(\text{Im } L(t))[(\text{Im } L(t))^2 + (\text{Re } L(t))^2]$$

$$= -\frac{1}{2}\text{ Im}[\tfrac{1}{3}(L(t))^3 + 4t^2L(t) - 8\int_0^t sL(s)ds]$$

$$+ \frac{1}{2}(\text{Im } L(t))[(\text{Im } L(t))^2 + (\text{Re } L(t))^2]$$

$$= \frac{2}{3}(\mathrm{Im}\ L(t))^3 - 2\ \mathrm{Im}(t^2L(t)) + 4\ \mathrm{Im}\int_0^t sL(s)ds$$

$$= -\frac{16}{3\pi^3} r^3\sin^3\theta\ \log^3 r$$

$$+ \frac{8}{\pi^2} r^3\sin^2\theta\ \log^2 r((1-\frac{2\theta}{\pi})\cos\theta + H'(0)\sin\theta)$$

$$+ O(r^3 \log r \sin^2\theta)$$

$$= F(r,\theta) ,$$

where $t = re^{i\theta}$ and capital "O" is used here uniformly near $(r,\theta) = (0,0)$.

For θ away from zero or π, certainly $u(z) - \psi(z) > 0$ for r sufficiently small. For θ near 0 or π, we divide by $\sin^2\theta$:

$$\frac{F(r,\theta)}{\sin^2\theta} = \frac{8}{\pi^2} r^3(\log r)^2((1-\frac{2\theta}{\pi})\cos\theta + H'(0)\sin\theta)$$

$$- \frac{16}{3\pi^3} r^3(\log r)^3\sin\theta + O(r^3\log r) .$$

This shows that for θ near 0, π we have $\frac{F(r,\theta)}{\sin^2\theta} > 0$ if r is small enough. But we claim that there is a neighborhood of $(r,\theta) = (0,0)$ in which $\frac{F(r,\theta)}{\sin^2\theta} > 0$. To show this we do as follows

$$\frac{F(r,\theta)}{r^3(\log r)^2\sin^2\theta} = (1-\frac{2\theta}{\pi})\cos\theta + H'(0)\sin\theta$$

$$- (\log r)\sin\theta + O(\frac{1}{\log r})$$

$$G(r,\theta) = (1 - \frac{2\theta}{\pi})\cos\theta + H'(0)\sin\theta - (\log r)\sin\theta \geq c > 0$$

$$\text{for} \begin{cases} 0 \leq \theta \leq \pi \\ 0 < r \leq r_0 \end{cases}, \quad \text{for some fixed (constant) } r_0 .$$

We can assume r_0 is so small that $O(\frac{1}{\log r}) \geq -\frac{c}{2}$. Then $\frac{F(r,\theta)}{r^3(\log r)^2\sin^2\theta} > \frac{c}{2}$. This shows that $u - \psi > 0$ in the whole region $L(G_r)$. In the next step we want to discuss C^1 condition of the local solution.

(iv) In order to show that u is a local solution in the upper half of z-plane near the origin, it is now left only to show that u is C^1 there. By what we explained in the first two steps it is clear that u is C^1 in $z \neq 0$ there. It is also clear that u is continuous at the origin. So the only thing left is to show that the partial derivatives of u are continuous at $z = 0$. To do this it is enough to show that

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in\overline{L(G_r)}}} \frac{\partial u}{\partial x} = \frac{\partial \psi}{\partial x}(0,0) , \qquad \lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in\overline{L(G_r)}}} \frac{\partial u}{\partial y} = \frac{\partial \psi}{\partial y}(0,0) .$$

By looking at the formula for u defined on $\overline{L(G_r)}$, we have

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial x}\, \mathrm{Im} \int_0^z [q^2 + 4(L^{-1}(q))^2]dq$$

$$= -\frac{1}{2}\, \mathrm{Im}[z^2 + 4(L^{-1}(z))^2] \ ,$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\partial}{\partial y}\, \mathrm{Im} \int_0^z [q^2 + 4(L^{-1}(q))^2]dq$$

$$= -\frac{1}{2}\, \mathrm{Re}[z^2 + 4(L^{-1}(z))^2] \ .$$

Thus

$$\lim_{(x,y)\to(0,0)} \frac{\partial u}{\partial x} = 0 = \frac{\partial \psi}{\partial x}(0,0) \ ,$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial u}{\partial y} = 0 = \frac{\partial \psi}{\partial y}(0,0) \ .$$

In the next step we complete the process of constructing a local solution to the axi-symmetric problem promised at the beginning of Section 2.5.

(v) In order to show that $v = \frac{1}{y}u$ is a local solution to the axi-symmetric problem it is only left to show that v is C^1 at the origin. Since u is C^1 at the origin by step (iv), thus $v = \frac{1}{y}u$ is C^0 there. Thus we only need to show the partial derivatives of v are continuous at the origin. As we argued in last step we only need to show:

and thereafter

$$\lim_{(x,y)\to(0,0)} \frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial x}(0,0) \quad ,$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial y}(0,0) \quad .$$

$$\frac{\partial v}{\partial y} = -\frac{1}{2}\,\frac{y\ \mathrm{Re}[z^2+4(L^{-1}(z))^2]-\mathrm{Im}\int_0^z [q^2+4(L^{-1}(q))^2]dq}{y^2}$$

$$= \frac{1}{3}\,y - 2\,\frac{y\ \mathrm{Re}\ t^2 - \mathrm{Im}\int_0^t s^2 L'(s)ds}{y^2} \quad .$$

Hence

$$\lim_{z\to 0}\frac{\partial v}{\partial y} = -2\lim_{t\to 0}\left(\frac{\mathrm{Re}\,t^2}{\mathrm{Im}\ L(t)} - \frac{\mathrm{Im}\int_0^t s^2L'(s)ds}{[\mathrm{Im}\ L(t)]^2}\right) =$$

$$-2\lim_{r\to 0}\left[\frac{r\cos 2\theta}{-\frac{2}{\pi}\log r\ \sin\theta + H'(0)\sin\theta + (1-\frac{2\theta}{\pi})\cos\theta + O(r)} - \right.$$

$$\frac{-\frac{2}{3\pi}r(\sin 3\theta)\log r - \frac{4}{9\pi}r\sin 3\theta - \frac{2\theta}{3\pi}r\cos 3\theta}{(-\frac{2}{\pi}\log r\ \sin\theta + H'(0)\sin\theta}$$

$$\left.\frac{+\frac{r}{3}\cos 3\theta + \frac{r}{3}H'(0)\sin 3\theta + O(r^2)}{+(1-\frac{2\theta}{\pi})\cos\theta + O(r))^2}\right] = 0$$

because we've shown in step (i) that the minimum of the function $f(\theta) = -\frac{2}{\pi}\log r\ \sin\theta + H'(0)\sin\theta + (1-\frac{2\theta}{\pi})\cos\theta$ (>0) occurs at $\theta = 0$ or π with $f(0) = f(\pi) = 1$,

and therefore for $r \leq r_0$, r_0 small enough,

$$f(\theta) + O(r) > \frac{1}{2} .$$

Thus $\lim_{(x,y)\to(0,0)} \frac{\partial v}{\partial y} = 0 = \frac{\partial \varphi}{\partial y}(0,0)$. Also

$$\frac{\partial v}{\partial x} = -\frac{1}{2y} \operatorname{Im}[z^2 + 4(L^{-1}(z))^2] = x - \frac{2 \operatorname{Im}(t^2)}{\operatorname{Im} L(t)} .$$

Similarly we can show

$$\lim_{(x,y)\to(0,0)} \frac{\partial v}{\partial x} = \lim_{r\to 0} \frac{-2 \operatorname{Im}(t^2)}{\operatorname{Im} L(t)} = 0 = \frac{\partial \varphi}{\partial x}(0,0) .$$

This completes step (v), hence v is a local solution to the axi-symmetric problem near the origin of the z-plane. The shaded region is the contact set; Γ_1, Γ_2 are the free boundaries, and the rest of the neighborhood of the origin is the noncontact set in the z-plane. See Figure 2.5.

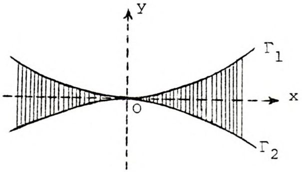

Figure 2.5

Remark 2.6. Simple calculation shows that Γ_1 is symmetric with respect to the y-axis if and only if H is odd. In particular then $L(i\sigma)$ is pure imaginary for $\sigma > 0$. Putting $a(t) = L(t) - it = H(t) - \frac{2}{\pi} t \log t$ and assuming H odd, we get $L(t) = a(t) + it$, which for $t > 0$ parametrizes the right hand part of Γ_1. The analytic continuation of this equation along γ_r yields at $-r < 0$ the equation

$$L(-r) = a(-r) - ir = -H(r) + r \log r + ir = -a(r) + ir ,$$

which shows that following the analytic continuation of a along γ_r from r to -r gives

$$a(-r) = -a(r) - 2i(-r) . \qquad (*)$$

Moreover the equation of the right hand part of the free boundary Γ_1

$$x = a(y) = H(y) - \frac{2}{\pi} y \log y$$

extended for complex variable implies

$$a''(t) = H''(t) - \frac{2}{\pi t} = L''(t) \qquad (**)$$

is single-valued, in fact $a''(t)$ is meromorphic in a neighborhood of the origin. See (**).

We shall show later on that the analogue of (*), (**) will also happen for the perturbed problem where the cusp is unfolded to a smooth free boundary.

Perturbation of Example 2.5: Heuristics

2.7. Let us now physically pull the obstacle down a little bit, keeping symmetry of the problem with respect to both axes. We then expect a new local solution surface v over φ near $(0,0)$ and a new free boundary. The new contact set should be near the old one. We might anticipate, for example, that the new free boundary would consist of two analytic curves Γ_1 and Γ_2, in the right and left half plane, as shown in Figure 2.6 (recall that the axi-symmetric solution $v(x,y)$ is extended to be even in y). We want the right hand part of the free

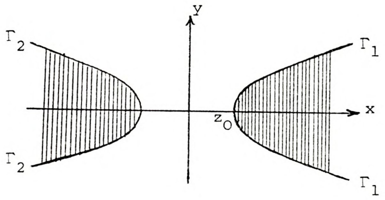

Figure 2.6

boundary Γ_1 of this local solution to be represented by $x = a(y)$, an analytic even function near zero, passing through $z_0 = x_0 = a(0)$ with the region to the right of Γ_1 lying in the contact set. Γ_1 has the equation $z = L(t) = a(t) + it$, for t real.

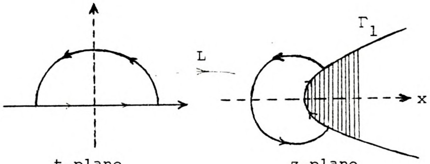

Figure 2.7

Let us now first construct the local solution near $z_0 = a(0)$. We extend L again on the complex numbers. Since $L'(0) \neq 0$, $L^{-1}(z)$ is defined as a conformal map from a neighborhood of z_0 to a neighborhood of $t = 0$, taking the contact set (to the right of Γ_1) to $\operatorname{Im} t < 0$, and the non-contact set (to the left of Γ_1) to $\operatorname{Im} t > 0$.

By the formula given in Section 2.3 for $u = yv$, on the left hand side of Γ_1 we have (note $\psi(z_0) = 0$)

$$v(x,y) = v(z) = -\frac{1}{2y}\,\mathrm{Im}\int_{z_0}^{z} [q^2 + 4(L^{-1}(q))^2]dq\,,$$

$$\text{for}\quad y = \mathrm{Im}\, z \neq 0\,.$$

Observe that the above formula for $v(x,y)$ defines a real analytic function, even when $y = 0$. Moreover this function is even in y. This is because $L^{-1}(q)$ is imaginary for q real, the integral above is analytic function which is real for z real. Therefore $\mathrm{Im}\int_{z_0}^{z} [q^2 + 4(L^{-1}(q))^2]dq$ is real analytic and is zero when $y = \mathrm{Im}\, z = 0$, and in fact is odd in y by the reflection principle. When $y = 0$ this formula yields

$$\begin{aligned} v(x_1,0) &= \lim_{y\to 0} v(x_1,y) \\ &= -\frac{1}{2}\Big[\frac{\partial}{\partial y}\,\mathrm{Im}\int_{z_0}^{z} [q^2 + 4(L^{-1}(q))^2]dq\Big]_{z=x_1} \\ &= -\frac{1}{2}\,[\mathrm{Re}(z^2 + 4(L^{-1}(z))^2]_{z=x_1} \\ &= -\frac{1}{2}[x_1^2 + 4(L^{-1}(x_1))^2] \\ &= \varphi(x_1,0) - 2(L^{-1}(x_1))^2\,. \qquad (1) \end{aligned}$$

On the right hand side of Γ_1 we have $v(z) = \varphi(z)$.

From the past theory, 2.4, 2.5, we have that $v(x,y)$ so defined is C^1 across the free boundary (even at $y = 0$),

with $\Delta(yv) = 0$ to the left of Γ_1. To check that v is a local solution near z_0, all there remains is to show $v > \varphi$ near z_0, to the left of Γ_1 and that $\Delta_{R^4} W = 0$ in the non-contact set. We show $\frac{\partial^2(v-\varphi)}{\partial n^2} = 4$ at any point of Γ_1 where n the normal to Γ_1 directed away from the contact set. We have

$$\frac{\partial^2(v-\varphi)}{\partial n^2} = \frac{\partial^2}{\partial n^2}\left(\frac{u-\psi}{y}\right) = \frac{1}{y}\frac{\partial^2(u-\psi)}{\partial n^2} \quad (\text{since } \nabla u = \nabla\psi,\ u = \psi)$$

$$= \frac{1}{y}\Delta(u-\psi) = -\frac{1}{y}\Delta\psi = 4, \quad \text{for } y \neq 0 .$$

This is also true for $y = 0$ by continuity. To see $\Delta_{R^4} W = 0$ observe that w is analytic in noncontact set since $v(x,y)$ is analytic and even in y, and hence v is an analytic function to x and y^2. Also observe that $\Delta_{R^4} W = 0$ if $y \neq 0$. Hence by continuity $\Delta_{R^4} W = 0$ if $y = 0$ also. Here is the construction of the local solution near z_0 complete. Recall that in 2.7 we assumed the local solution to the axi-symmetric problem exists near the origin of the z-plane. This local solution agrees with the one constructed near z_0.

2.7.1. Further Assumptions. We are assuming the perturbed local solution has a free boundary as shown in Figure 2.5. In particular assume $v - \varphi > 0$ on the interval $(-z_0, z_0)$ on the real axis (i.e., this interval is in the non-contact set). Assume further that the only

local maximum of $v(x,0) - \varphi(x,0)$ in this interval is at $x = 0$, that $\frac{\partial}{\partial x}(v(x,0) - \varphi(x,0)) \neq 0$ for $x \neq 0$, $x \in (-z_0, z_0)$, and $\frac{\partial^2}{\partial x^2}(v(0,0) - \varphi(0,0)) < 0$. See Figure 2.8, where $-K = -\frac{1}{2}(v(0,0) - \varphi(0,0))$, $Y = -\frac{1}{2}(v(x,0) - \varphi(x,0))$.

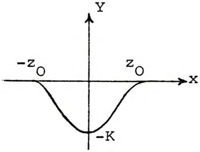

Figure 2.8

From (1) it follows that $(L^{-1}(x))^2$ has an analytic continuation as an even function of x throughout $[-z_0, z_0]$. Thus $Y = (L^{-1}(x))^z$ in Figure 2.8.

2.7.2. Heuristic Derivation of Explicit Formula For Function a.

First Step. We see from 2.7.1 that L^{-1} maps $[0, z_0]$ monotonically onto the segment $[i\sqrt{K}, 0]$ in the complex plane and is analytic there with nonzero derivative in $(0, z_0]$. Thus $L(t)$ (and $a(t) = L(t) - it$)

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have analytic continuations along $[0, i\sqrt{K})$. Indeed, the part of the graph in Figure 2.7 for $x > 0$ is described by

$$x = L(\sqrt{Y}) = a(\sqrt{Y}) + i\sqrt{Y}\,, \quad \text{where} \quad \sqrt{Y} \in [0, i\sqrt{K}] \,.$$

The graph for $x < 0$ is, by symmetry $x = -L(\sqrt{Y})$. At $t = i\sqrt{K}$, the Riemann surface has a branch point with two sheets, i.e. L is an analytic function of $(t - i\sqrt{K})^{\frac{1}{2}}$ there. This is because $\frac{\partial^2(v-\varphi)}{\partial x^2}(0,0) \neq 0$. As $L(\sqrt{Y})$ and $-L(\sqrt{Y})$ are analytic continuations of each other near $i\sqrt{K}$, it follows that $(L(t))^2$ is analytic on $[0, i\sqrt{K}]$. In fact $(L(t))^2$ has a simple zero at $i\sqrt{K}$.

Consider $t \in [-i\sqrt{K}, 0]$. $a(t)$ is real analytic, so $a(t)$, $L(t)$ have analytic continuation along this closed segment and

$$a(t) = \overline{a(\bar{t})}\,,$$

$$L(t) = a(t) + it = \overline{a(\bar{t})} + \overline{i\bar{t}} + 2\,it$$

$$= \overline{L(\bar{t})} + 2\,it\,.$$

Thus $(L(t) - 2\,it)^2$ is analytic on $[-i\sqrt{K}, 0]$ with a simple zero at $-i\sqrt{K}$. Consider $(L''(t))^2 = ((L(t) - 2\,it)'')^2 = (a''(t))^2$ which is analytic on $(-i\sqrt{K}, i\sqrt{K})$ as $L(t) \neq 0$

there. Near $i\sqrt{K}$, $L(t) = (t - i\sqrt{K})^{\frac{1}{2}}M(t)$, where $M(i\sqrt{K}) \neq 0$, M is analytic. Hence $(L''(t))^2$ is meromorphic near $i\sqrt{K}$ with a pole of order exactly three. The same thing is true at $-i\sqrt{K}$ with $L(t) - 2it$. Hence

$$(a''(t))^2 = \frac{h(t)}{(t^2+K)^3}$$

where h is real analytic, non-zero on $[-i\sqrt{K}, i\sqrt{K}]$, and even in t.

For clarity we display the analytic continuations of $a(\sqrt{Y})$ and $L(\sqrt{Y})$. See Figure 2.9. The symmetric

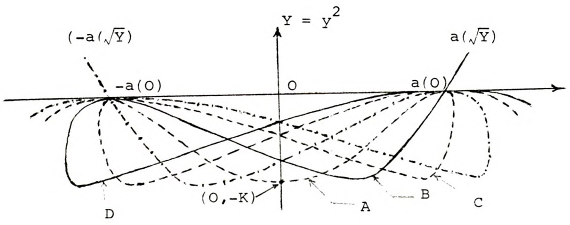

Figure 2.9

path passing through $(0,-K)$ is the graph A of

$$x = \pm L(\sqrt{Y}), \quad \sqrt{Y} \in [0, i\sqrt{K}] \qquad (*)$$

shown in Figure 2.7. Near $Y = 0$ we have $L(\sqrt{Y}) = a(\sqrt{Y}) + i\sqrt{Y}$, where $a(\sqrt{Y})$ is analytic function of Y because $a(t)$ is an even function of t. Therefore the graph of $x = a(\sqrt{Y}) = L(\sqrt{Y}) - i\sqrt{Y}$ cross the horizontal axis at $x = a(0)$, as shown by the solid line B in Figure 2.8. Indeed, the dashed graph A (*) may be displaced to the right to obtain the solid graph B

$$x = \pm L(\sqrt{Y}) - i\sqrt{Y} > \pm L(\sqrt{Y}) .$$

Near $(a(0),0)$ the graph A has the form $x = L(\sqrt{Y}) = a(\sqrt{Y}) + i\sqrt{Y}$, so may be continued as a smooth curve (as $a(\sqrt{Y})$ is analytic in Y) to the graph of

$$x = a(\sqrt{Y}) - i\sqrt{Y}$$

denoted C. Similarly, near $(-a(0),0)$ the graph B has the form

$$x = -L(\sqrt{Y}) - i\sqrt{Y} = -a(\sqrt{Y}) - 2i\sqrt{Y}$$

so may be continued to D: $x = -a(\sqrt{Y}) + 2i\sqrt{Y}$. Continuing in this fashion yields all graphs

$$(\#) \qquad x = \pm a(\sqrt{Y}) + Ni\sqrt{Y} = \pm a(t) + N\,it,$$

$$N = 0, \pm 1, \pm 2, \ldots$$

as shown in Figure 2.8. We may regard x as a function of t, where t lies on a Riemann surface with infinitely many sheets (#).

However, from (#) we see $\left(\frac{d^2x}{dt^2}\right)^2 = (a''(t))^2$ is single valued as shown above. This last result and the equation of path D are analogue of the Remark 2.6.

<u>Second step</u>. Let us calculate the above $h(t)$ for the unperturbed problem of Example 2.5, where $K = 0$. We have $t^2(a''(t))^2 = (tH''(t) - \frac{2}{\pi})^2$. Hence

$$h(t) = t^6(a''(t))^2 = \left(\frac{2}{\pi}\right)^2 t^4 + O(t^6) \qquad (**)$$

is real analytic and even with a fourth order zero at $t = 0$.

In the perturbed problem we have $(t^2+K)^3(a''(t))^2 = h(t,K)$, where the left hand side is some special perturbation of the left hand side of (**). Since the current problem is a perturbation of 2.5, we expect that $h : \mathbb{R}^2 \to \mathbb{R}$ must, near $(0,0)$, be a real analytic function even in t near the function $h(t,0) = \left(\frac{2}{\pi}\right)^2 t^4 + O(t^6)$ in some sense. Motivated by the Weierstrass Preparation Theorem [7] we might expect a family $h(t,K) = \left(\frac{2}{\pi}\right)^2 (t^4 + 2A(K)t^2 + B(K))\phi_1(t^2,K)$ of perturbations, where ϕ_1, A, B are real functions with ϕ_1 analytic in t^2 satisfying $A(0) = 0 = B(0)$, $\phi_1(0,0) = 1$. Thus

$$(a''(t))^2 = (\frac{2}{\pi})^2 \frac{t^4+2At^2+B}{(t^2+K)^3} \phi^2(t^2,K) ,$$

where $K, A, B \in \mathbb{R}$ are near zero, $K > 0$, $B > 0$ and $\phi(0,K)$ is near 1. Let

$$f(t) = \frac{2}{\pi} \frac{\sqrt{t^4+2At^2+B}}{(t^2+K)^{3/2}} \phi(t^2,K) . \qquad (***)$$

Description of a problem

2.8. In the heuristic derivation of the above formula we regarded $A(K)$, $B(K)$, and $\phi(t^2,K)$ as depending on the perturbation parameter K. For $K = 0$, an explicit local solution with $A(0) = B(0) = 0$, $\phi(t^2,0) = \phi_0(t^2)$ was constructed, where $\phi_0(0) = 1$ (Example 2.5).

Now we adopt a somewhat different view, that of multi-parameter bifurcation theory. We regard A, B, K and the function $\phi(\cdot)$ themselves as independent parameters. They vary in the parameter space as $(K,A,B,\phi) \in R^3 X Z$, in a neighborhood of $(0,0,0,\phi_0)$, where Z is a Banach space of functions ϕ defined as follows.

Definition 2.8.1. Fix $\rho > 0$ and let Z be the Banach space of functions $\phi(\tau)$ analytic on the open disk $|\tau| < \rho^2$, continuous on the closed disk $|\tau| \leq \rho^2$, such

that $\phi(\tau)$ is real for τ real, with the norm

$$\|\phi\| = \sup|\phi(\tau)| , \quad |\tau| \le \rho^2 . \qquad \#$$

<u>Remark</u>: Fix $\phi_0 \in Z$ (for some ρ) with $\phi_0(0) = 1$. Consider $f_0(y) = -\frac{2}{\pi}\frac{\phi_0(y^2)}{y}$ (the f corresponding to $K = A = B = 0$ and ϕ_0 given in (***)). This gives a family of curves $\Gamma_1 : x = a(y)$, $a''(y) = f_0(y)$. In fact $a''(y) = -\frac{2}{\pi}(\frac{1}{y} + \frac{\phi_0(y^2)-1}{y}) = -\frac{2}{\pi}(\frac{1}{y} + \psi_0(y))$, where ψ_0 is odd. Integrating twice we get $a(y) = D + Cy - \frac{2}{\pi}(y \log y - y) - \frac{2}{\pi}\psi_1(y)$, where ψ_1 is odd with cubic leading term. Comparing with Example 2.5 we get $D = 0$, $H(t) = (C + \frac{2}{\pi})y - \frac{2}{\pi}\psi_1(y)$. Hence given ϕ_0 we have a typical local solution as constructed in Example 2.5, which is not unique by presence of the constant C in the formula H above.

<u>Problem 2.8.2</u>. Fix $\mathcal{O}$, an open disk about the origin in $z = x + iy$ plane, small enough (say $\overline{\mathcal{O}} \subseteq$ domain of the local solution in Example 2.5). Fix $z_0 = x_0 + iy_0$, $x_0 > 0$, $y_0 > 0$, on the unperturbed curve (Γ_1 in Example 2.5). Consider A, B, K near 0, ϕ near ϕ_0 in Z, and $f(y)$ with the branch of f for which $f(y_0) < 0$, and $(a(y_0), x_0)$ near z_0, $a'(y_0)$ near $\frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\Big|_{y=y_0}$.

(1) Find necessary and sufficient condition that there exists a symmetric local solution in $\mathcal{O}$ with part of the free boundary

$$\Gamma_1 \begin{cases} x = a(y) \\ a''(y) = f(y) \end{cases}$$

near y_0 with $a'(y_0)$, $a(y_0)$ as stated above.

(2) Are the constants of integration unique? In other words, is the local solution unique for the given $f(y)$ above? #

We will devote chapter 5 to handle this problem. But first on the third and fourth chapters we develop some theory needed to go over chapter five. For this purpose we put an end to the chapter 2 and we start chapter 3 now.

3. A NECESSARY AND SUFFICIENT CONDITION FOR A LOCAL SOLUTION NEAR THE ORIGIN

The main purpose of this chapter is stated in problem 3.2 below. But first, using Proposition 2.2 and the Remarks 1, 2 after, we can restate the definition of local solution of the axi-symmetric obstacle problem as follows:

An equivalent definition of a local solution of axi-symmetric problem in R^4

Definition 3.1. By a local solution $(u,\mathcal{O})$ of the 4-dimensional axi-symmetric obstacle problem, symmetric with respect to x-axis, on a fixed open disk $\mathcal{O} \subset R^2$ centered at the origin, we mean $u : \mathcal{O} \to R$ such that

(i) $u(x,-y) = -u(x,y)$

(ii) $\frac{1}{y} u(x,y) = v(x,y)$ is C^1 in $\mathcal{O}$

(iii) $\frac{1}{y} u(x,y) = v(x,y) \geq \varphi(x,y)$ in $\mathcal{O}$,

where $\varphi(x,y) = -\frac{1}{2}(x^2+y^2)$

(iv) if $I = \{(x,y) \in \mathcal{O} \mid \frac{1}{y} u(x,y) = \varphi(x,y)\}$,

then $\Delta u = 0$ in $\mathcal{O} - I$. #

Remark. In part (iv), $\Delta u = 0$ in $(\mathcal{O} - I)$ is equivalent to $\Delta_{R^4} W = 0$ there. To see this let $\Delta_{R^4} W = 0$ in $(\mathcal{O} - I)$, then $u(x,y) = yw(x,y,0,0)$ implies

$$\frac{\partial^2 u}{\partial x^2} = y\,\frac{\partial^2 w}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial y^2} = 2\,\frac{\partial w}{\partial y} + y\,\frac{\partial^2 w}{\partial y^2}.$$

If $y = 0$, then $\Delta u = 2\frac{\partial w}{\partial y} = 0$ because $\frac{\partial w}{\partial y}$ is an odd real analytic function of y. If $y \neq 0$, then

$$\Delta u = y\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{2}{y}\frac{\partial v}{\partial y}\right) = 0 .$$

Conversely if $\Delta u = 0$ in $(\mathcal{O} - I)$, then as in (the end of) 2.7 we can show $\Delta_{R^4} W = 0$.

Let us now consider (as before) an analytic arc $\Gamma_1 : x = a_1(y)$, near $(x_0,y_0) \in \mathcal{O}$, where $y_0 > 0$, $x_0 = a_1(y_0)$ and $a_1(y)$ is real analytic near y_0.

Problem 3.2. What are the necessary and sufficient conditions for $(u,\mathcal{O})$ to be a local solution to the obstacle problem (in the sense of Definition 3.1) with Γ_1 as a part of the free boundary and the contact set I to one side of Γ_1? #

We shall obtain the solution to this problem step by step throughout this Chapter. Let us start by considering the complex form of

$$\Gamma_1 : z = L_1(t) = a_1(t) + it, \quad t \in R, \quad \text{near} \quad y_0 .$$

Extend L_1 to be defined for complex t near y_0 as in 2.3, 2.4. As in Lemma 2.4 we can construct a local solution u_0 in a neighborhood of (x_0, y_0), with free boundary Γ_1, and noncontact set lying on one side of Γ_1. Let $v_0 \subset \mathcal{O}$ be a one-sided neighborhood of z_0 on the side of Γ_1 in which the noncontact set lies. For

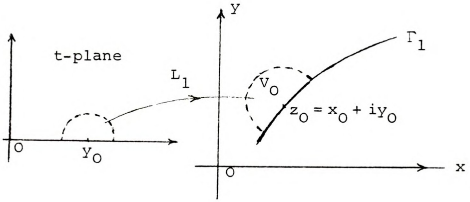

Figure 3.1

the remainder of this chapter, we deal with this local solution (u_0, v_0) arising from the given arc Γ_1 as above. We shall try to extend this to a local solution on the disk $\mathcal{O}$. Before doing this, we need to state and prove the following lemma.

Lemma 3.3. Assume $V(\subseteq \mathcal{O})$ containing V_0 is an open connected set. Then $\Delta u = 0$ in V with $u = u_0$ in V_0 if and only if $(L_1^{-1})^2$ defined in V_0 has a (single-valued) analytic continuation in V and $\int_c (L_1^{-1}(q))^2 dq$ is real for every closed path $c \subset V$. Moreover we have then

$$u(z) = \psi(z_0) + \mathrm{Re} \int_{z_0}^{z} \tfrac{i}{2}[q^2 + 4(L_1^{-1}(q))^2]dq, \quad z \in V$$

which is independent of the path of integration in V.

Proof: First assume u is harmonic in V, and equals u_0 in V_0. Then $u_x - iu_y$ is holomorphic in V. In V_0 we have that

$$\overline{\nabla u} = u_x - iu_y = \tfrac{i}{2}[z^2 + 4(L_1^{-1}(z))^2] = f(z)$$

is analytic, so f and $(L_1^{-1})^2$ have analytic continuation in V. Hence the integral representation of u remains valid:

$$u(z) = \psi(z_0) + \mathrm{Re} \int_{z_0}^{z} f(q)dq$$
$$= \psi(z_0) - \tfrac{1}{2} \mathrm{Im} \int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq, \quad z \in V$$

as both sides of this equation are harmonic functions agreeing on V_0. If z can be reached from z_0 by different paths γ_1, γ_2 in V, then

$\mathrm{Re} \int_{\gamma_1} f(q)dq = \mathrm{Re} \int_{\gamma_2} f(q)dq$ because u is single-valued.

Thus $\mathrm{Re} \int_{\gamma_1 - \gamma_2} f(q)dq = 0$, which implies

$\mathrm{Re} \oint_{\gamma_1 - \gamma_2} 2i(L_1^{-1}(q))^2 dq = 0$, or equivalently

$\oint_{\gamma_1 - \gamma_2} [L_1^{-1}(q)]^2 dq$ is real. This implies $\int_C (L_1^{-1}(q))^2 dq$ is real for any closed path $C \subset V$. Conversely let $(L_1^{-1})^2$ have an analytic continuation from V_0 to V, and put

$$u(z) = \psi(z_0) + \mathrm{Re} \int_{z_0}^{z} f(q)dq ,$$

where the integral is along some path from z_0 to z in V. We assume $\oint_C (L_1^{-1}(q))^2 dq$ is real, and therefore $\oint_C f(q)dq$ is imaginary for any closed path $C \subset V$. This guarantees $u(z) = \psi(z_0) + \mathrm{Re} \int_{z_0}^{z} f(q)dq$ to be single-valued in V and independent of the path of integration there. Hence u is harmonic in V and $u = u_0$ in V_0. This completes the proof. #

Let us now begin extending u_0 to a local solution of the obstacle problem in the sense of Definition 3.1. The first step is defining a set containing V_0 on which u_0 is extended as a harmonic function u_1 with the property $u_1 > \psi$.

Definition 3.4. Consider (continuous) paths $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) \in V_0$ such that

(1) u_0 can be harmonically extended along the path, i.e., there exist $0 = t_0 < t_1 < \cdots < t_N = 1$, and harmonic functions $u_{0,k}$ defined on some open disks $D_k \subseteq \mathcal{O}$ centered at $\gamma(t_k)$ with $u_{0,k}(z) = u_{0,k+1}(z)$ on $D_k \cap D_{k+1} \neq \emptyset$ and $u_0 = u_{0,0}$ on D_0. See Figure 3.2

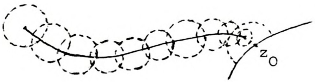

Figure 3.2

(Note this extension may not be single-valued on $\cup D_k$); and

(2) $u_{0,k}(z) > \psi(z)$ on D_k, where $\psi(z) = -\frac{1}{2} y(x^2 + y^2)$. Let $V_1' = \{\gamma(1) \mid \gamma$ is a path as above with properties stated in (1) and (2)}. #

Remark 1. V_1' is clearly an open connected set containing V_0. Following the proof of Lemma 3.3, we see that condition (1) above is equivalent to analytically continuing $(L_1^{-1})^2$ from $\gamma(0)$ to $\gamma(1)$ (along γ) with the disks D_k. Let $u_1(z)$ denote the corresponding harmonic continuation of $u_0(z)$.

Remark 2. u_1 as well as $(L_1^{-1})^2$ extended above may not be single-valued. By Lemma 3.3 u_1 is single-valued harmonic in V_1' if and only if $(L_1^{-1})^2$ has a single-valued analytic continuation (from V_0) to V_1' with $\oint_C [L_1^{-1}(q)]^2 dq$ = real for any closed path $C \subset V_1'$.

Remark 3. $\partial V_1'$ is a disjoint union of two subsets P and Q defined as follows:

$$P = \{z \in \partial V_1' \mid \text{there is no } \gamma : [0,1] \to \overline{\mathcal{O}} \text{ with } \gamma(1) = z,\ \gamma(0) \in V_0, \text{ and } \gamma([0,1)) \subset \mathcal{O}, \text{ having properties (1) and (2) as in Definition 3.4}\}, \quad Q = (\partial V_1') - P$$

observe $Q \subseteq (\partial V_1') \cap \partial\mathcal{O}$. We call P the Natural Boundary of V_1' and it is closed. #

Let us now extend (u_0, V_0) anti-symmetrically to its reflection in the lower half plane.

Definition 3.5. Let $V_2' = \{z : \overline{z} \in V_1'\}$,
$V_{00} = \{z : \overline{z} \in V_0\}$, $W_1 = V_1' \cap \{(x,y) : y \geq 0\}$,
$W_2 = V_2' \cap \{(x,y) : y < 0\}$, $W = W_1 \cup W_2$,
$J = \overline{W}_1 \cap \{(x,y) : y = 0\} = \overline{W}_1 \cap \overline{W}_2$, $\Gamma_2 = \{z : \overline{z} \in \Gamma_1\}$,
$u_{00}(x,y) = -u_0(x,-y)$ for $(x,y) \in V_{00}$.

Remark 1. $W \subseteq \mathcal{O}$ is open, and for (Γ_2, V_{00}) the maps L_2, L_2^{-1} can be defined as L_1, L_1^{-1} for (Γ_1, V_0), namely, $L_2(t) = a(t) - it$ for t near y_0. Moreover on V_{00} u_{00} can be constructed similarly as we did for u_0 and a modification of Lemma 2.4 ($u_{00} < \psi$ instead of $u_0 > \psi$) is valid.

Remark 2. Note even if W_1 is not connected, $u_1(z)$ can still be defined as before (by means of integration along the paths which extend into the lower half plane). However we easily see that $W_1 = V_1' \subset \{(x,y) : y > 0\}$, $W_2 = V_2' \subset \{(x,y) : y < 0\}$, and W_1, W_2 are connected if u_0 extends to a local solution u in $\mathcal{O}$, since $u(x,0) = \psi(x,0) = 0$.

Remark 3. ∂W_1 is a disjoint union of three subsets: $\nu_1 = P \cap \partial W_1$, $V_1' \cap \partial W_1 \subset J$, and $Q \cap \partial W_1$. Set $\nu_2 = \{z : \overline{z} \in \nu_1\}$, and $\nu = \nu_1 \cup \nu_2$. If we have $W_1 = V_1' \subset \{(x,y) : y > 0\}$, then the second subset of ∂W_1 above is empty.

Lemma 3.6. For $z = L_1(t) \in V_0$ we have $L_2^{-1}(\overline{z}) = -\overline{L_1^{-1}(z)}$ and $[L_2^{-1}(\overline{z})]^2 = \overline{[L_1^{-1}(z)]^2}$.

Proof: Γ_1, Γ_2 have equations $z = L_1(t) = a(t) + it$ for t real near $y_0 > 0$ and $z = L_2(t) = a(-t) + it$ for t real near $-y_0 < 0$, respectively. Therefore, for $t \in L_1^{-1}(V_0)$

$$\overline{L_1(t)} = \overline{a(t)} - i\overline{t} = a(\overline{t}) - i\overline{t} = L_2(-\overline{t}) \ .$$

Putting $z = L_1(t)$ above we get $L_2^{-1}(\overline{z}) = -\overline{t} = -\overline{L_1^{-1}(z)}$ for $z \in V_0$. Hence

$$[L_2^{-1}(\overline{z})]^2 = \overline{[L_1^{-1}(z)]}^2 \quad \text{for} \quad z \in V_0 \ . \qquad \#$$

We are now going to state two hypotheses and then prove they are equivalent.

Hypothesis 3.7.

The functions u_0 and u_{00} in V_0 and V_{00} respectively have a common single-valued harmonic extension u to w, which is C^1 in $\overline{W}$; and $u = \psi$ and $\nabla u = \nabla\psi$ on $(\overline{\nu - J})$. Observe that $\overline{\nu - J} \subset \partial W$ is the closure of the set of Natural Boundary points which do not lie on the x-axis. Note by this Hypothesis we have $u_2(x,y) \overset{D}{=} u(x,y) = -u(x,-y) = -u_1(x,-y)$ for $(x,y) \in W_2$ well-defined as a harmonic extension of u_{00} in W_2 for which $u_2 < \psi_2$. #

Hypothesis 3.8.

1) $(L_1^{-1})^2$ defined in V_0 has a single-valued analytic continuation in W_1 and $\oint_C (L_1^{-1}(q))^2 dq$ is real for any closed path $C \subset W_1$ (which imply similar properties for $(L_2^{-1})^2$ in W_2).

2) $(L_1^{-1})^2$ is continuous on $\overline{W}_1$, $[L_1^{-1}(z)]^2 = y^2$ for $z = x + iy \in (\overline{\nu - J})$, $[L_1^{-1}(z)]^2 = [L_1^{-1}(z)]^2 = [L_2^{-1}(z)]^2$ is real for $z \in J$, and $\lim_{\substack{z \to z_* \\ z \in W_1^*}} \operatorname{Im} \int_{z_0}^{z} [L_1^{-1}(q)]^2 dq$ exists for any $z_* \in \partial W_1$ and equals $\frac{1}{3}(y_*^3 - y_0^3)$ for any $z_* \in (\overline{\nu_1 - J})$, where the integration path is in $W_1 \cup \{z_0\}$. #

<u>Proposition 3.9</u>. Hypothesis 3.7 holds if and only if Hypothesis 3.8 holds, and in such a case we have $V_1' = W_1 \subset \{(x,y) : y > 0\}$, $V_2' = W_2 \subset \{(x,y) : y < 0\}$, with W_1 and W_2 open and connected.

<u>Note</u>: Under either one of the above Hypotheses if $J - (\overline{\nu_1 - J}) \neq \phi$, which is a countable union of disjoint open intervals on the real axis, then along any path from z_0 to $\overline{z}_0$ in $W \cup (J - (\overline{\nu - J}))$ passing through a point z in $J - (\overline{\nu_1 - J})$ we shall have

$$-\frac{2}{3} y_0^3 = \operatorname{Im} \int_{z_0}^{\overline{z}_0} (L^{-1}(q))^2 dq ,$$

where $(L^{-1})^2$ shows, in this case, both $(L^{-1})^2$ and $(L_2^{-1})^2$ because these two are the analytic continuation of each other in $W \cup (J - (\overline{\nu - J}))$ by the Reflection principle. Instead of the latter integral condition we can impose an equivalent condition:

$$-\frac{y_0^3}{3} = \operatorname{Im} \int_{z_0}^{z} (L^{-1}(q))^2 dq .$$

Proof: Assume Hypothesis 3.7 hold. Let us first show the second part (the set equation). If $J = \phi$, then clearly we have

$$W_1 = V_1' \subset \{(x,y) : y > 0\}, \quad W_2 = V_2' \subset \{(x,y) : y < 0\} .$$

If $J \neq \phi$, then u being C^1 in $\overline{W}$ and $u(z) \geq \psi(z)$ in W_1 imply $u \geq 0$ on J (continuity of u is enough here). Similarly $u(z) \leq \psi(z)$ in W_2 imply $u \leq 0$ on J. Thus $u \equiv 0$ on J. Hence $V_1' = W_1 \subset \{(x,y) : y > 0\}$, $V_2' = W_2 \subset \{(x,y) : y < 0\}$. These show W_1 and W_2 are open and connected.

Let us now show the first part. The Hypothesis 3.7 is assumed, thus, by Lemma 3.3, $(L_1^{-1})^2$ has a single valued analytic continuation in W_1 and $\oint_C (L_1^{-1}(q))^2 dq$ is real for any closed path $C \subset W_1$. Since u is C^1 in $\overline{W}$, thus

$$u(z) = \psi(z_0) + \mathrm{Re} \int f(q)dq$$

$$\Rightarrow \nabla u = \overline{f(z)} = -\frac{i}{2}[\overline{z}^2 + 4\overline{(L_1^{-1}(z))^2}], \quad \text{for } z \in W_1 ,$$

$$u(z) = \psi(\overline{z}_0) + \mathrm{Re} \int_{\overline{z}_0}^{z} f(q)dq$$

$$\Rightarrow \nabla u = \overline{f(z)} = -\frac{i}{2}[\overline{z}^2 + 4\overline{(L_2^{-1}(z))^2}], \quad \text{for } z \in W_2 ,$$

imply $(L_1^{-1})^2$, $(L_2^{-1})^2$ are continuous on $\overline{W}_1$, $\overline{W}_2$ respectively.

If $J \neq \phi$, then putting the above values of ∇u equal on J we get:

$$(L_1^{-1}(z))^2 = (L_2^{-1}(z))^2 \quad \text{for} \quad z \in J = \overline{W_1} \cap \overline{W}_2 .$$

Using Lemma 3.6 we can conclude $[L_2^{-1}(\bar{z})]^2 = \overline{[L_1^{-1}(z)]}^2$ for $z \in J$ because $\overline{[L_2^{-1}(\bar{z})]}^2 - [L_1^{-1}(z)]^2$ is an analytic function which is identically zero on V_0, and therefore it is zero on W_1. Comparing the last two equations we get $[L_1^{-1}(z)]^2 = \overline{[L_1^{-1}(z)]}^2$ for $z \in J$. Hence $[L_1^{-1}(z)]^2 = [L_2^{-1}(z)]^2$ is real for $z \in J$. For any $z \in W$ we have

$$\begin{aligned}
\nabla u - \nabla\psi &= u_x + iu_y - \nabla\psi = \overline{u_x - iu_y} - \nabla\psi \\
&= -\frac{i}{2}[\bar{z}^2 + 4\overline{(L_j^{-1}(z))}^2] - \nabla\psi \\
&= 2i[y^2 - \overline{(L_j^{-1}(z))}^2] .
\end{aligned}$$

Since u is C^1 on $\overline{W}$ and consequently $(L_j^{-1})^2$ is continuous on $\overline{W}_j$, thus

$$\nabla u(z) - \nabla\psi(z) = 2i[y^2 - (L_j^{-1}(z))^2], \quad \text{for} \quad z \in \partial w.$$

In particular for $z = x + iy \in (\overline{\nu - J})$, where $\nabla u - \nabla\psi = 0$, we have

$$[L_j^{-1}(z)]^2 = y^2 = (\operatorname{Im} z)^2 .$$

For any $z_* \in \partial W_1$,

$$u(z_*) = \lim_{\substack{z \to z_* \\ z \in W_1}} u(z)$$

$$= \psi(z_0) - \frac{1}{2} \lim_{\substack{z \to z_* \\ z \in W_1}} \operatorname{Im} \int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq$$

exists, that is, $\lim_{\substack{z \to z_* \\ z \in W_1}} \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq$ exists. In particular for $z_* \in (\overline{\nu - J})$, where $u(z_*) = \psi(z_*)$, we have then

$$u(z_*) = \lim_{\substack{z \to z_* \\ z \in W_1}} u(z) = \psi(z_*)$$

$$\Leftrightarrow \psi(z_0) - \psi(z_*) = \frac{1}{2} \lim_{\substack{z \to z_* \\ z \in W_1}} \operatorname{Im} \int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq$$

$$\Leftrightarrow \frac{1}{3}(y_*^3 - y_0^3) = \lim_{\substack{z \to z_* \\ z \in W_1}} \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq$$

Note that the last limit equals $\operatorname{Im} \int_{z_0}^{z_*} (L_1^{-1}(q))^2 dq$ if z_*, z_0 can be joined in $W_1 \cup \{z_0, z_*\}$ by a path with only endpoints z_0, z_* in ∂W_1. In this case then for $z \in W_1$,

$$u(z) = \psi(z_0) - \frac{1}{2}\,\mathrm{Im}\int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq$$

$$= \psi(z_0) - \frac{1}{2}\,\mathrm{Im}\int_{z_0}^{z_*} [q^2 + 4(L_1^{-1}(q))^2]dq$$

$$- \frac{1}{2}\,\mathrm{Im}\int_{z_*}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq$$

$$= \psi(z_*) - \frac{1}{2}\,\mathrm{Im}\int_{z_*}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq ,$$

which will be used later on. Note $\overline{W}_1$ may not be path connected even if W_1 is. If $x \in J - (\overline{\nu_1 - J})$, then there exists a disk $D(x;r)$ with $D(x;r) \cap (\nu_1 - J) = \phi$. Thus $(D(x;r) - [x-r, x+r]) \subset W$. In fact z can be joined to z_0 by a path in W_1. For any $z \in [J - (\overline{\nu_1 - J})]$ (which is a countable union of disjoint open intervals) we have

$$u_1(z) = u_2(z) = 0 \Leftrightarrow \psi(z_0) - \frac{1}{2}\,\mathrm{Im}\int_{z_0}^{z} [q^2 + r(L_1^{-1}(q))^2]dq$$

$$= \psi(\overline{z}_0) - \frac{1}{2}\,\mathrm{Im}\int_{\overline{z}_0}^{z} [q^2 + r(L_2^{-1}(q))^2]dq$$

$$\Leftrightarrow \psi(z_0) - \psi(\overline{z}_0) = \frac{1}{2}\,\mathrm{Im}\int_{z_0}^{\overline{z}_0} [q^2 + 4(L^{-1}(q))^2]dq$$

$$\Leftrightarrow \mathrm{Im}\int_{z_0}^{\overline{z}_0} (L^{-1}(q))^2 dq = -\frac{2}{3}\,y_0^3 ,$$

where $(L^{-1})^2$ represents both $(L_1^{-1})^2$, $(L_2^{-1})^2$ because these are analytic continuation of one another by the reflection principle (recall they are real-valued on x-axis).

Observe $u(z) = 0$ implies (as above)

$\mathrm{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq = -\frac{1}{3} y_0^3, \quad \forall\ z \in J - (\overline{\nu_1 - J})$. Here is the proof of the note also complete.

Conversely assume Hypothesis 3.8 holds. (1) in 3.8, by Lemma 3.3, implies u is single-valued harmonic extension of u_0 (defined in V_0) at least in the connected component of W_1 containing V_0.

If $J = \phi$, then clearly $W_1 = V_1' \subset \{(x,y) : y > 0\}$ and $W_2 = V_2' \subset \{(x,y) : y < 0\}$. In this case (1) in 3.8 implies that u (defined on W) is single-valued harmonic in W by Lemma 3.3. Since $\lim\limits_{\substack{z \to z_* \\ z \in W}} \mathrm{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq$,

$z_* \in \partial W_1$ exists and equals to $\frac{1}{3}(y_*^3 - y_0^3)$ if $z_* \in (\overline{\nu_1 - J})$, thus u is continuous on $\overline{W}_1$ and $u = \psi$ on $(\overline{\nu_1 - J})$. For a detail see the proof of "only if" part. Since $(L_1^{-1})^2$ is continuous on $\overline{W}$, thus u is C^1 on $\overline{W}$. $[L_1^{-1}(z)]^2 = y^2$ for $z \in (\overline{\nu_1 - J})$ implies $\nabla u = \nabla \psi$ there (again for a detail look at the "only if" part). Similarly we can argue in $\overline{W}_2$, $(L_2^{-1})^2$, ... in the lower half of z-plane.

If $J \neq \phi$, then we have three subcases

$$J - (\overline{\nu_1 - J}) = \phi, \quad \begin{cases} \neq \phi \\ \quad\quad , \\ \neq J \end{cases} \quad = J .$$

Consider the subcase $J - (\overline{\nu_1 - J}) = \phi$. Then all points of J are the limit points of $\nu_1 - J$. In this subcase

$$W_1 = V_1' \subset \{(x,y) : y > 0\}, \quad W_2 = V_2' \subset \{(x,y) : y < 0\}$$

(1) in 3.8 again implies, by Lemma 3.3, that u is a single-valued harmonic extension of u_0 defined in V_0. The first integral condition in (2) of 3.8 implies

$$u(z_*) = \lim_{\substack{z \to z_* \\ z \in W}} u(z) = \psi(z_*) = 0 , \qquad \forall \; z_* \in J .$$

The rest of the proof goes as in the case $J = \phi$.

Let us now consider the subcase $\phi \subsetneqq J - (\overline{\nu_1 - J}) \subsetneqq J$. Then (1) in 3.8 again implies, by Lemma 3.3, that u is the single-valued harmonic extension of u_0 at least in the component of W_1 containing V_0 (similarly we can argue about the lower half of the z-plane). Note $J - (\overline{\nu_1 - J})$ are those points of J which are not limit points of the part of ν_1 above (outside of) the real axis. Thus for any small enough disk, $D(z_*;r)$, neighborhood of any $z_* \in J - (\overline{\nu_1 - J})$ we have $D(z_*;r) \cap \{(x,y) : y(-1)^j < 0\} \subset W_j$. In fact $J - (\overline{\nu_1 - J})$ is open and it is a countable union of disjoint open intervals on the real axis. Let (a,b) be the maximal interval containing z_* and contained in $\mathring{J} - (\overline{\nu_1 - J})$. Any such point z_* in this open interval can be joined

to z_0 by a path in W_1 with only endpoints z_*, $z_0 \in \partial W_1$. Clearly a or $b \in (\overline{\nu_1 - J})$. By the first integral condition in (2) of (3.8) we get

$\lim_{\substack{z \to a (\text{or } b) \\ z \in W_1}} u(z) = \psi(a)$ (or $\psi(b)$) $= 0$ (for detail look at "only if" part). Suppose now $(a_1, b_1) \subset W_1 \subsetneqq V_1'$ with $u(a_1) = \psi(a_1) = 0$ or $u(b_1) = \psi(b_1) = 0$. Then

$$u(z_*) = \psi(z_0) - \frac{1}{2} \operatorname{Im} \int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2] dq$$

$- \frac{1}{2} \operatorname{Im} \int_{z}^{z_*} [q^2 + 4(L_1^{-1}(q))^2] dq$, for $z_* \in (a_1, b_1)$, where $z = a_1$ or $z = b_1$ according to whether $\psi(a_1) = u(a_1)$ or $\psi(b_1) = u(b_1)$ respectively. Thus

$u(z_*) = u(z) - \frac{1}{2} \operatorname{Im} \int_{z}^{z_*} [q^2 + 4(L_1^{-1}(q))^2] dq$. Therefore

$$u(z_*) = -\frac{1}{2} \operatorname{Im} \int_{z}^{z_*} [q^2 + 4(L_1^{-1}(q))^2] dq .$$

On the other hand $(L_1^{-1})^2$ is real valued on J, thus $u(z_*) = 0$, $\forall\ z_* \in (a_1, b_1)$. This contradicts $W_1 \subsetneqq V_1'$. Hence such an (a_1, b_1) does not exist and $W_1 = V_1'$. Hence we conclude $W_1 = V_1' \subset \{(x,y) : y > 0\}$, and by symmetry $W_2 = V_2' \subset \{(x,y) : y < 0\}$. Since the limit in (2) of 3.8 exists, and equals to $\frac{1}{3}(y_*^3 - y_0^3)$ for $z_* \in (\overline{\nu_1 - J})$, thus $u(z_*) = \lim_{z \to z_*} u(z)$, $z_* \in \partial W$, exists and can be defined, and equals to $\psi(z_*)$ for $z_* \in (\overline{\nu_1 - J})$. Since $(L^{-1})^2$ is continuous on $\overline{W}_1$, thus ∇u is continuous

on $\overline{W}_1$ (for detail look at "only if" part) and therefore u is C^1 there. Moreover $u \equiv 0$ on J. Also $[L_1^{-1}(z)]^2 = y^2$ for $z \in \overline{(\nu_1 - J)}$ implies $\nabla u = \nabla\psi$ on $\overline{(\nu_1 - J)}$.

Let us finally consider the subcase $J - \overline{(\nu_1 - J)} = J$, where $\overline{(\nu_1 - J)} \cap J = \phi$. Then any point $z \in J$ can be joined to z_0 (and $\overline{z}_0$) by a path in W_1 (and in W_2) with only endpoints z_0, z perhaps in ∂W_1 (and $\overline{z}_0$, $z \in \partial W_2$ perhaps). Let us show now $J \subset \partial W_1$. Since $(L^{-1})^2$ is continuous on $\overline{W}$, thus for $z \in J$ we have

$$u_1(z) = \psi(z_0) - \frac{1}{2}\,\mathrm{Im}\int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2]dq \geq \psi(z) = 0$$

$$u_2(z) = \psi(\overline{z}_0) - \frac{1}{2}\,\mathrm{Im}\int_{\overline{z}_0}^{z} [q^2 + 4(L_2^{-1}(q))^2]dq \leq \psi(z) = 0 ,$$

that is, $-\frac{1}{3}\,y_0^3 \geq \mathrm{Im}\int_{z_0}^{z} (L_1^{-1}(q))^2 dq$ and $\frac{1}{3}\,y_0^3 \geq \mathrm{Im}\int_{\overline{z}_0}^{z} (L_2^{-1}(q))^2 dq$. Since $(L_1^{-1})^2$, $(L_2^{-1})^2$ are analytic continuation of one another by the Reflection principle (use Lemma 3.6 and (2) in 3.8), thus $\mathrm{Im}\int_{\overline{z}_0}^{z} (L_2^{-1}(q))^2 dq = \mathrm{Im}\int_{\overline{z}_0}^{z} \overline{(L_1^{-1}(\overline{q}))^2}dq = \mathrm{Im}\,\overline{\int_{z_0}^{z} (L_1^{-1}(q))^2 dq} = -\mathrm{Im}\int_{z_0}^{z} (L_1^{-1}(q))^2 dq$. Hence $-\frac{1}{3}\,y_0^3 \geq \mathrm{Im}\int_{z_0}^{z} (L^{-1}(q))^2 dq$ and $-\frac{1}{3}\,y_0^3 \leq \mathrm{Im}\int_{z_0}^{z} (L_1^{-1}(q))^2 dq$ which imply $u_1(z) = 0$

and $u_2(z) = 0$. That is $u(J) = 0$ and $W_1 = V_1' \subset \{(x,y) : y > 0\}$, $W_2 = V_2' \subset \{(x,y) : y < 0\}$. Condition (1) of 3.8 again implies of course, by Lemma 3.3, that u is a single-valued harmonic extension of u_0 in W_1, W_2, and in $W \cup J$. The rest of the proof can be stated as before. #

Definition 3.10. Assume either one of Hypothesis 3.7 or 3.8. Now let us extend the function u from the set W to the set $\mathcal{O}$ as

$$u(z) = \begin{cases} u(z), & \text{if } z \in W \\ \psi(z), & \text{if } z \in \sigma - W \end{cases} .$$

Theorem 3.11. $(u,\mathcal{O})$ in Definition 3.10 is a local solution to the obstacle problem in the sense of Definition 3.1 if and only if

1) $\lim\limits_{\substack{z \to x_1 \\ z \in W_1}} \frac{1}{y}[\frac{1}{3} y_0^3 + \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq]$ exists and equals to $(L_1^{-1}(x_1))^2$ for any $x_1 \in J$,

2) $\lim\limits_{\substack{z \to x_1 \\ z \in W_1}} \frac{1}{y^2}[-y(\overline{L_1^{-1}(z)})^2 + \frac{1}{3} y_0^3 + \operatorname{Im} \int_{z_0}^{z} (L^{-1}(q))^2]$ exists and equals to $i[(L_1^{-1}(x_1))^2]'$, $\forall\ x_1 \in J$.

Note: Since $(\frac{1}{y} u - \varphi)|_x = -2(L^{-1}(x))^2 \geq 0$, $\nabla(\frac{1}{y} u - \varphi)|_x = -2[(L^{-1}(x))^2]'$ at any $x \in J$, thus any possible real zero of $(L^{-1})^2$ in $J - (\overline{\nu - J})$ is not

simple. These zeros are isolated in $J - (\overline{\nu - J})$ and they form a set S so that $S \cup (\overline{\nu - J})$ establishes the free boundary ∂I for the obstacle problem.

Proof: Assume u satisfies the requirement of Definition 3.1. Then $v = \frac{1}{y} u$ is C^1 in σ, in particular, it is C^1 on the real axis. If $J = \phi$, then nothing needs to be proved. Assume therefore $J \neq \phi$. For any $z \in W_1$ we have then

$$u(z) = \psi(z_0) - \frac{1}{2} \operatorname{Im} \int_{z_0}^{z} [q^2 + 4(L_1^{-1}(q))^2] dq .$$

Hence

$$\frac{1}{y} u(z) = -\frac{1}{2} x^2 + \frac{1}{6} y^2 - \frac{2}{y}[\frac{1}{3} y_0^3 + \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq] ,$$

$$\text{for } z \in W_1 . \qquad (*)$$

Furthermore for any $z \in W_1$ we have

$$\nabla(\frac{1}{y} u(z)) = \frac{1}{y} u_x + i(\frac{1}{y} u_y - \frac{1}{y^2} u) = \frac{1}{y}(u_x + iu_y) - \frac{i}{y^2} u$$

$$= -x + \frac{i}{2} y - \frac{i}{2y} x^2 - \frac{2i}{y}[\overline{L_1^{-1}(z)}]^2 + \frac{i}{2y} x^2 - \frac{i}{6} y$$

$$+ \frac{2i}{y^2}[\frac{1}{3} y_0^3 + \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq]$$

$$= -x + \frac{i}{3} y + \frac{2i}{y^2}[-y(\overline{L_1^{-1}(z)})^2 + \frac{1}{3} y_0^3$$

$$+ \operatorname{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq] . \qquad (**)$$

J may consist of two classes of points, one which are limit points of $\nu - J$ (the part of ν outside of J) and the others are the interior points of $J - (\overline{\nu - J})$. We treat them in different ways.

First let x_1 be a limit point of $\nu - J$, i.e., in the first class. Since $\frac{1}{y} u(z)$ is continuous on $\overline{W}_1$, thus by (*) we have $\lim_{\substack{z \to x_1 \\ z \in \overline{W}_1}} \frac{1}{y}[\frac{1}{3} y_0^3 + \text{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq]$

exists, where $z = x + iy$. To find this limit let $z_* \to x_1$, where $z_* \in (\nu - J)$, then

$$\lim_{\substack{z \to z_* \\ z \in W_1}} [\frac{1}{3} y_0^3 + \text{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq] = \frac{1}{3} y_0^3 + \frac{1}{3}(y_*^3 - y_0^3) = \frac{1}{3} y_*^3$$

and

$$\lim_{\substack{z_* \to x_1 \\ z_* \in \overline{W}_1}} \frac{1}{y_*}[\frac{1}{3} y_0^3 + \text{Im} \int_{z_0}^{z_*} (L_1^{-1}(q))^2 dq] = \lim_{y_* \to 0} \frac{1}{3} y_*^2 = 0$$

(there are such z_*).

Or we can argue as follows: $\lim_{z \to x_1} \frac{1}{y} u(z) =$

$\lim_{z_* \to x_1} \frac{1}{y_*} u(z_*) = \lim_{z_* \to x_1} \frac{1}{y_*} \psi(z_*) = -\frac{x_1^2}{2}$. Thus by (*)

$\lim_{\substack{z \to x_1 \\ z \in W_1^1}} \frac{1}{y}[\frac{1}{3} y_0^3 + \text{Im} \int_{z_0}^{z} (L_1^{-1}(q))^2 dq] = 0$, which proves

condition (1). (Note for such an x_1 we have $[L_1^{-1}(x_1)]^2 = 0$). Since $\frac{1}{y}u(z)$ is C^1 on the real axis also, thus by (**)

$$\lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \frac{1}{y^2}[-y(\overline{L_1^{-1}(z)})^2 + \frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{z}(L_1^{-1}(q))^2dq]$$

exists. Recall that $(L_1^{-1})^2$ is continuous on $\overline{W}_1$ and equals to y^2 on $(\overline{\nu - J})$, and $\lim_{\substack{z\to z_*\\ z\in W_1}} \operatorname{Im}\int_{z_0}^{z}(L_1^{-1}(q))^2dq = \frac{y_*^3 - y_0^3}{3}$ for $z_* \in \nu_1 - J$. To compute the above limit we can make these substitutions to get it equal to

$$\lim_{y_*\to 0} -\frac{2}{3}y_* = 0 \quad \text{(which proves condition (2))}$$

Or we can argue as saying $\lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \nabla(\frac{1}{y}u(z)) = \lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \nabla\varphi = -x_1$

and consequently by (**):

$$\lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \frac{1}{y^2}[-y(\overline{L_1^{-1}(z)})^2 + \frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{z}(L_1^{-1}(q))^2dq] = 0 ,$$

which proves (2) .

Now let x_1 be an interior point of $J - (\overline{\nu - J})$, i.e., in the second class. Then any point $z \in D(x_1;r)$,

for sufficiently small radius r, can be joined to z_0 or $\overline{z}_0$ in W, where only endpoints may not be in W. Recall that in this case $(L_1^{-1})^2$, $(L_2^{-1})^2$ are analytic continuation of one another across the x-axis and we may call both as $(L^{-1})^2$. Again since $\frac{1}{y}u(z)$ is continuous on $\overline{W}_1$, thus by (*)

$$\lim_{\substack{z\to x_1 \\ z\in\overline{W}_1}} \frac{1}{y}\left[\frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{z} (L^{-1}(q))^2 dq\right]$$

exists for any such x_1. To compute this limit we can do as follows

$$\lim_{\substack{z\to x_1 \\ z\in\overline{W}_1}} \frac{1}{y}\left[\frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{z} (L^{-1}(q))^2 dq\right]$$

$$= \lim_{\substack{z\to x_1 \\ z\in\overline{W}_1}} \frac{1}{y}\left[\frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{x} (L^{-1}(q))^2 dq + \operatorname{Im}\int_{x}^{z} (L^{-1}(q))^2 dq\right]$$

$$= \lim_{\substack{z\to x_1 \\ z\in\overline{W}_1}} \frac{1}{y}\operatorname{Im}\int_{x}^{z} (L^{-1}(q))^2 dq \quad \text{exists,}$$

where $z = x + iy$, $\frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{x} (L^{-1}(q))^2 dq = 0$.

(The latter equation was shown at the end of first part, "only if", of the proof of 3.9.) Moreover

$$\operatorname{Im}\int_x^{\bar z}(L^{-1}(q))^2dq = \operatorname{Im}\int_x^{z}(L^{-1}(\bar q))^2d\bar q = \operatorname{Im}\int_x^{z}\overline{(L^{-1}(q))}^2d\bar q =$$

$$\operatorname{Im}\overline{\int_x^{z}(L^{-1}(q))^2dq},$$ which shows $\operatorname{Im}\int_x^{z}(L^{-1}(q))^2dq$ is an odd function, real analytic vanishing for $y = 0$.

Thus $\frac{1}{y}\operatorname{Im}\int_x^{z}(L^{-1}(q))^2dq$ is a real analytic function even in y. Hence

$$\lim_{\substack{z\to x_1\\ z\in W_1^1}} \frac{1}{y}\left[\frac{1}{3}y_0^3 + \operatorname{Im}\int_{z_0}^{z}(L^{-1}(q))^2dq\right]$$

$$= \frac{\partial}{\partial y}\operatorname{Im}\int_{z_0}^{z}(L^{-1}(q))^2dq\Big|_{x_1}$$

$$= \operatorname{Re}(L^{-1}(z))^2\Big|_{x_1} = [L^{-1}(x_1)]^2 .$$

Since $\frac{1}{y}u(z)$ is C^1, i.e., $\nabla(\frac{1}{y}u)$ is C^0 in $\overline{W}$, thus by (**) we have

$$\lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \frac{1}{y^2}\left[-y\overline{(L^{-1}(z))}^2 + \frac{1}{3}\operatorname{Im}\int_{z_0}^{z}(L^{-1}(q))^2dq\right]$$

$$= \lim_{\substack{z\to x_1\\ z\in\overline{W}_1}} \frac{1}{y^2}\left[-y\overline{(L_1^{-1}(z))}^2 + \operatorname{Im}\int_{x_1}^{z}(L^{-1}(q))^2dq\right]$$

exists. To compute the value of this limit, since we have real analytic function (which vanishes at $y = 0$) divided

by y^2 involved, thus it is enough to consider $z \to x_1$ vertically, i.e. $z = x_1 + iy$. Then by L'Hopital's Rule

$$\lim_{\substack{z\to x_1 \\ z\in\overline{W}_1}} \frac{1}{y^2}[-y\overline{(L^{-1}(z))}^2 + \mathrm{Im} \int_{x_1}^{z} (L^{-1}(q))^2 dq]$$

$$= \lim_{\substack{z\to x_1 \\ z\in W_1^1}} \frac{-\overline{(L^{-1}(z))}^2 - y\frac{\partial}{\partial y}[\overline{(L^{-1}(z))^2}] + \mathrm{Re}(L^{-1}(z))^2}{2y}$$

$$= \lim_{\substack{z\to x_1 \\ z\in W_1^1}} \frac{i\ \mathrm{Im}(L^{-1}(z))^2 - y\frac{\partial}{\partial y}\overline{(L^{-1}(z))}^2}{2y}$$

$$= \frac{i}{2}\ \mathrm{Re}[(L^{-1}(x_1))^2]' + \frac{i}{2}\ \mathrm{Re}[\overline{(L^{-1}(x_1))}^2]'$$

$$= i[(L^{-1}(x_1))^2]' ,$$

where the last equality is because $(L^{-1})^2$ is real on J. This proves condition (2). We discover also that $(\frac{1}{y}u(z) - \varphi(z)|_x = -2(L^{-1}(x))^2$, for any $x \in J$, including the points of $J \cap \overline{(\nu - J)}$ because $(L^{-1}(z))^2 = (\mathrm{Im}\ z)^2$ for these points. The rest of the note is clear now.

Conversely assume conditions (1), (2) hold. If $J = \phi$, then we are done. So let $J \neq \phi$. Using conditions (1), (2), equations (*), (**) imply that u is C^1 on the real axis.

<u>Proposition 3.12</u>. The local solution $(u,\mathcal{O})$ is symmetric with respect to y-axis if and only if

$$[L^{-1}(z)]^2 = \overline{[L^{-1}(-\bar{z})]^2} .$$

<u>Proof</u>: Assume $(u,\mathcal{O})$ is symmetric with respect to y-axis: $u(-x,y) = u(x,y)$. Then $\frac{\partial u}{\partial x}(x,y) = -\frac{\partial u}{\partial x}(-x,y)$, i.e., u_x is odd with respect to x while u is even. Moreover $\frac{\partial u}{\partial y}(x,y) = \frac{\partial u}{\partial y}(-x,y)$. <u>Hence $u_x(-x,y) + iu_y(-x,y)$</u> $=$ $-u_x(x,y) + iu_y(x,y)$, that is, $u_x(-\bar{z}) - iu_y(-\bar{z}) =$ $-(u_x(z) - iu_y(z))$; where $u(z) =$ $\psi(z_0) + \text{Re}\int_{z_0}^{z} \frac{i}{2}[q^2 + 4(L^{-1}(q))^2]dq$. Applying Cauchy-Riemann equations we get $\overline{\frac{i}{2}[(-\bar{z})^2 + 4(L^{-1}(-\bar{z}))^2]} =$ $-\frac{i}{2}[z^2 + 4(L^{-1}(z))^2]$. Thus $-2i\overline{(L^{-1}(-\bar{z}))^2} = -2i(L^{-1}(z))^2$, that is, $\overline{[L^{-1}(-\bar{z})]^2} = [L^{-1}(z)]^2$. In particular if z is imaginary, then $-\bar{z} = z$ and therefore $[L^{-1}(z)]^2$ is real. Conversely assume now $\overline{[L^{-1}(-\bar{z})]^2} = [L^{-1}(z)]^2$. First notice that $u(-\bar{z}) = u(z)$ if and only if

$$-\frac{1}{2}\ \text{Im}\int_{z_0}^{z} [q^2 + 4(L^{-1}(q))^2]dq = -\frac{1}{2}\ \text{Im}\int_{z_0}^{-\bar{z}} [q^2 + 4(L^{-1}(q))^2]dq.$$

This is true if and only if $\text{Im}\int_{z}^{\bar{z}} [q^2 + 4(L^{-1}(q))^2]dq = 0$, if and only if $\text{Im}\int_{z}^{-\bar{z}} (L^{-1}(q))^2dq = 0$. Let us show now the last equation: $\text{Im}\int_{z}^{-\bar{z}} (L^{-1}(q))^2dq =$ $\text{Im}[\int_{z}^{c} (L^{-1}(q))^2dq + \int_{c}^{-\bar{z}} (L^{-1}(q))^2dq]$, where the path from z to $-\bar{z}$ is symmetric with respect to y-axis crossing

y-axis at c. Hence

$$\operatorname{Im} \int_z^{-\bar{z}} (L^{-1}(q))^2 dq$$

$$= \operatorname{Im}[\int_z^c (L^{-1}(q))^2 dq + \int_c^z (L^{-1}(-\bar{q}))^2 (-d\bar{q})]$$

$$= \operatorname{Im}[\int_z^c (L^{-1}(q))^2 dq + \int_z^c \overline{(L^{-1}(q))}^2 d\bar{q}]$$

$$= \int_z^c \operatorname{Im}[(L^{-1}(q))^2 dq] + \int_z^c \operatorname{Im}[\overline{(L^{-1}(q))}^2 d\bar{q}]$$

$$= 0 .$$

This proves therefore $u(-\bar{z}) = u(z)$. Here is the proof complete.

4. AN ANALYTIC CONTINUATION TO THE FREE BOUNDARY

Introduction

In this chapter we want to make some observations on the conditions stated in Chapter three as necessary and sufficient for a local solution u to the obstacle problem to exist. In particular we want to see what we can conclude about the maps L and a from the conditions imposed on $(L^{-1})^2$. We shall discover, roughly speaking, that the free boundary of the local solution everywhere has the equation $x = a(y)$ or $x = a(-y) - 2iy$, where a here is some analytic continuation of the original a inside $\tilde{W} = W \cup [J - (\overline{\nu - J})] = (\mathcal{O} - I) \cup S$. Since $(L^{-1})^2$ is analytic single-valued in $\tilde{W}$, thus in a neighborhood of any point z_1 in this set we have

$$[L^{-1}(z)]^2 = (z - z_1)^m \sum_{j=0}^{\infty} \alpha_j (z - z_1)^j ,$$

$$\text{where } \alpha_0 \neq 0, \quad m \geq 0 \tag{4.1}$$

Proposition 4.1. If we let $t^2 = [L^{-1}(z)]^2$ for z in the above neighborhood, then z can be written as

a power series in $[t^2 - (L^{-1}(z_1))^2]^{\frac{1}{p}}$, for some integer $p \geq 1$, t^2 in a small enough neighborhood of $(L_1^{-1}(z_1))^2$.

<u>Proof</u>: By (4.1) above we have

$t^2 = (z - z_1)^m \sum_{j=0}^{\infty} \alpha_j (z - z_1)^j$. Consider two cases.

<u>The case $m \geq 1$</u>: Taking m-th root of both sides we get $t^{\frac{2}{m}} = (z - z_1)g(z)$, where g is analytic near z_1 with $g(z_1) \neq 0$. By invoking the inverse function Theorem we get $z - z_1 = \theta(t^{\frac{2}{m}})$, where $\theta(\tau)$ is analytic in a small enough neighborhood of zero. Here is the proof of this case complete.

<u>The case $m = 0$</u>: Let $t^2 - \alpha_0 = \sum_{j=r}^{\infty} \alpha_j (z - z_1)^j = (z - z_1)^r \sum_{j=r}^{\infty} \alpha_j (z - z_1)^{j-r}$, where α_r is the first non-zero coefficient among $\alpha_1, \alpha_2, \alpha_3, \ldots$. Taking r-th root of both sides we get $(t^2 - \alpha_0)^{\frac{1}{r}} = (z - z_1)g(z)$, where g is analytic near z_1 with $g(z_1) \neq 0$. Again by invoking the inverse function Theorem we get $z - z_1 = \omega((t^2 - \alpha_0)^{\frac{1}{r}})$, where $\omega(\tau)$ is analytic in a sufficiently small neighborhood of zero. Thus $z = z_1 + \omega((t^2 - \alpha_0)^{\frac{1}{r}})$, and the proof is complete. #

<u>Corollary 4.2</u>. If z_1 is a simple zero of $(L^{-1})^2$ in $\tilde{W}$, then z_1 is a branch point of L^{-1}, and L is a single valued analytic function in a neighborhood of

$L^{-1}(z_1) = 0$ $(z = L(t) = \theta(t^2)$ by the notation of the proposition). If z_1 is a second order zero of $(L^{-1})^2$ in $\tilde{W}$, then $(L^{-1})^2$ has two analytic square roots in a sufficiently small neighborhood of z_1 each of which is invertible with analytic inverse L $(z = L(t) = \theta(t)$ by the proposition notation).

Proof: It is an immediate consequence of $m = 1$, $m = 2$, respectively, in the Proposition 4.1.

Definition 4.3. By (4.1) (power series) we can write

$$L^{-1}(z) = (z - z_1)^{\frac{m}{2}} \sum_{j=0}^{\infty} \alpha_j' (z - z_1)^j, \quad m \geq 0, \quad \alpha_0' \neq 0$$

which is double valued with a branch point at z_1 if m is odd, and consists of two analytic functions (with opposite signs) if m is even. Define
$A = \{z \in \tilde{W} \mid z$ is a zero of $(L^{-1})'\}$,
$B = \{z \in \tilde{W} \mid z$ is a simple zero of $(L^{-1})^2\}$. Clearly point of $A \cup B$ are isolated in $\tilde{W}$ and z_0 is not a limit point of $A \cup B$. In particular $\tilde{W} \cup \{z_0\}$ is path connected.

Lemma 4.4. Along any path $\gamma : [0,1] \to \tilde{W} \cup \{z_0\}$, where $\gamma(0) = z_0 \in V_0 \cup \Gamma_1$ and $\gamma(t) \notin A \cup B$, for any t, L^{-1} has an analytic continuation and is invertible locally with analytic inverse L. Moreover there exists a path $\sigma : [0,1] \to \mathbb{C}$ such that $\sigma(0) = L^{-1}(z_0) = y_0$

and $L(\sigma(t)) = \gamma(t)$; further L is analytic continuation along σ, of the original L.

Proof: Consider $z_1 = \gamma(t_1)$ for some $t_1 \in [0,1]$. Because $z_1 \notin A \cup B$, either $[L^{-1}(z_1)]^2 \neq 0$ and $(L^{-1})'(z_1) \neq 0$, or else $[L^{-1}(z_1)]^2 = 0$ and $m = 2$ in (4.1). In either case, $(L^{-1})^2$ has two square roots $\pm L^{-1}$ which are analytic near z_1 and which satisfy $(L^{-1})'(z_1) \neq 0$.

The fact that $\pm L^{-1}$ are analytic (locally) implies the original L^{-1} has an analytic continuation along the path γ. This is locally invertible because $(L^{-1})'(\gamma(t)) \neq 0$ for this analytic continuation. The path σ is given by $\sigma(t) = L^{-1}(\gamma(t))$. #

Remark 4.5. Note that because L^{-1} is double valued in $\tilde{W}$, there may be two paths $\gamma_1(t)$ and $\gamma_2(t)$ with the same endpoint $\gamma_1(1) = \gamma_2(1)$, which gives rise to different continuations of L^{-1}: $L^{-1}(\gamma_1(1)) = -L^{-1}(\gamma_2(1))$.

Given a path γ_1 there exists such a path γ_2 if and only if $(L^{-1})^2$ has an odd order zero in $\tilde{W}$. For example we can modify γ_1 as shown in Figure 4.1, where z is an odd order zero of $(L^{-1})^2$.

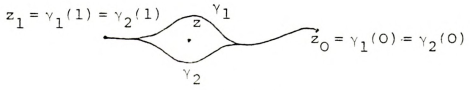

Figure 4.1

Theorem 4.6. For any $z_* \in \overline{(\nu - J)} \cup S = \Gamma$, the free boundary (established in Theorem 3.11) we have $z_* = x_* + iy_* = L(\pm y_*) = a(\pm y_*) + i(\pm y_*)$ in the following sense: For any $z_j \to z_*$ in $\tilde{W} - (A \cup B)$ there exists sequence $t_j \to y_*$ or $-y_*$, and branches (i.e. analytic continuations) L_j of L near t_j such that

$$z_j = L_j(t_j) \to z_* .$$

Proof: Let z_* belong to the free boundary Γ. Since $\tilde{W} - (A \cup B)$ is open and path connected, thus for any sequence $\{z_j\} \subset \tilde{W} - (A \cup B)$ with $z_j \to z_*$, each z_j can be joined to z_0 (or $\overline{z_0}$) by a path γ_j in $\tilde{W} - (A \cup B)$. By Lemma 4.4 the original map L^{-1} has a (unique) analytic continuation L_j^{-1} along γ_j which is locally invertible with analytic inverse, say L_j, along σ_j, where $L_j(\sigma_j(s)) = \gamma_j(s)$. On the other hand $z_j \to z_*$ implies $(L^{-1}(z_j))^2 \to y_*^2$, where $y_*^2 = [L^{-1}(z_*)]^2$, by continuity of $(L^{-1})^2$ in $\overline{W}$. Putting $t_j = L_j^{-1}(z_j)$, we get $z_j = L_j(t_j)$ and $t_j^2 \to y_*^2$, where t_j is the endpoint of σ_j in t-plane. If there is no odd order zero of $(L^{-1})^2$ in $\tilde{W}$, then $L_j^{-1} = L^{-1}$, $\forall\, j$, and either $t_j \to y_*$ or $t_j \to -y_*$ (one of these cases only). If there is an odd order zero of $(L^{-1})^2$ in $\tilde{W}$, then by Remark 4.5 we can assume either $t_j \to y_*$ or $t_j \to -y_*$ (each one is possible).

This shows that for any sequence $z_j \to z_* \in \Gamma$ in $\tilde{W}$, there exists $\{t_j\}$ $t_j \to y_*$ or $-y_*$ and branches L_j of L near t_j such that

$$z_j = L_j(t_j) \to z_* . \qquad \#$$

Remark 4.7. In the cases we will consider the branches L_j are in fact all the same branch of L, in a neighborhood of $\pm y_*$. Thus we can write $z_* = L(\pm y_*)$, or equivalently

$$x_* = a(y_*) \quad \text{or} \quad x_* = a(-y_*) - 2iy_* .$$

In case $(L^{-1})^2$ has an odd order zero in $\tilde{W}$, we may assume in fact $t_j \to y_*$, so we can write

$$x_* = a(y_*) \quad \text{for} \quad x_* + iy_* \in \Gamma . \qquad \#$$

5. A CLASS OF UNFOLDING OF THE SINGULARITY

In this chapter we use the theory developed in Chapters 3 and 4 to study a class of problems derived at the end of Chapter 2 (in 2.8.2). As before, we consider here free boundary curves $x = a(y)$ near a point $z_0 = x_0 + iy_0$ fixed on the unperturbed free boundary curve in $\mathcal{O}$ with $x_0 > 0$, $y_0 > 0$, where $f(y_0) < 0$ and $a''(y) = f(y)$ near y_0 as stated in Problem 2.8.2. Following Section 2.7 we want to treat a special case of Problem 2.8.2 here:

<u>Problem 5.1</u>. Study Problem 2.8.2 in the special case when $K > 0$. #

<u>Lemma 5.2</u>. Let I_0 be a compact segment (e.g. interval), and L be analytic in an open neighborhood of I_0. Let $L'(z) \neq 0$, $\forall\, z \in I_0$, and also let L be one-to-one on I_0. Then there exists a neighborhood of I_0 on which L is a conformal map.

<u>Proof</u>: Suppose not. Then there are sequences $\{a_n\}$, $\{b_n\}$, $a_n \to I_0$, $b_n \to I_0$, with $L(a_n) = L(b_n)$. Then there are subsequences $a_{n_k} \to a$, $b_{n_k} \to b$, where $a, b \in I_0$.

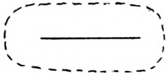

Figure 5.1

If $a \neq b$, then we get a contradiction to one-to-one assumption because $L(a) = L(b)$. If $a = b$, then

$$L'(a) = \lim_{k\to\infty} \frac{L(b_{n_k}) - L(a_{n_k})}{b_{n_k} - a_{n_k}} = 0 \text{ because } L(b_{n_k}) = L(a_{n_k}),$$

$\forall\ k \in \mathbb{N}$. Here we get a contradiction to the assumption $f'(t) \neq 0$ on I_0. This completes the proof. #

Necessary conditions in Problem 5.1

5.3. We want to find necessary conditions that there exist a local solution symmetric with respect to both axes in $\mathcal{O}$, with part of the free boundary

$$\Gamma_1 : x = a(y), \quad a''(y) = f(y), \quad \text{near } y_0 .$$

Here $(a(y_0), y_0)$ is near $z_0 = (x_0, y_0)$ fixed on the unperturbed free boundary curve $x_0 > 0$, $y_0 > 0$, and $a'(y_0)$ is near $\frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\big|_{y=y_0}$, $f(y_0) < 0$,

$$f(y) = -\frac{2}{\pi} \frac{\sqrt{y^4 + 2Ay^2 + B}}{(y^2 + K)^{3/2}} \phi(y^2), \quad \phi \text{ near } \phi_0 \text{ in } Z;$$

$$A, B, K \text{ near } 0, \ K > 0 .$$

We begin by building up the set v_0 on one side of Γ_1 (over) as in Lemma 2.4. Consider the complex form of Γ_1:

$$z = L(t) = a(t) + it, \quad t \in R \text{ near } y_0 .$$

Extend L to be defined for complex t near y_0 again as in the past. On an interval about y_0 (on the real axis) L is one-to-one, $L' \neq 0$, thus by Lemma 5.2. L^{-1} exists and is a conformal map in a neighborhood of $L(y_0)$. As long as f remains real on the real axis to the left of y_0, or equivalently $y^4 + 2Ay^2 + B$ remains non-negative there, we can extend Γ_1, L analytically and the local solution also nearby Γ_1 as in Section 2.7 by Lemma 5.2.

Lemma 5.4. The case where $t^4 + 2At^2 + B$ has a simple root $t^2 = \beta^2 \in (0, y_0^2)$ cannot happen.

Proof: Suppose it can. The proof proceeds in three steps.

First step: Without loss of generality we can assume $t^4 + 2At^2 + B > 0$ for $t \in (\beta, y_0)$. The relation

$$a'(y_2) - a'(y_1) = -\frac{2}{\pi}\int_{y_1}^{y_2} \frac{\sqrt{t^4 + 2At^2 + B}}{(t^2+K)^{3/2}}\,\phi(t^2)\,dt > 0 \quad \text{for}$$

$\beta \leq y_2 < y_1 \leq y_0$ shows that $a'(y_2) > a'(y_1)$ for the above y_1, y_2. This shows $a'(y) > 0$ for $\beta < y < y_0$ and $a(\beta) < x_0$. We now want to show $a(\beta) \geq 0$ and

conclude that $a(\beta) + i\beta \in \mathcal{O}$. Suppose $a(\beta) < 0$. Then $a(\beta) + i\beta$ is in the second quadrant. We can extend Γ_1 as the graph of function a as long as it remains inside $\mathcal{O}$. We can also extend L nearby $(\beta, y_0]$. The local solution built up near $a(y_0) + iy_0$ can then be extended near the new Γ_1 as shown in Figure 5.2 with

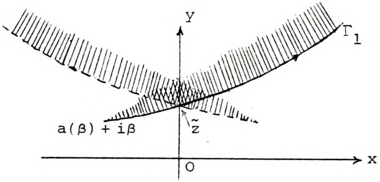

Figure 5.2

the shaded region as noncontact set. By symmetry with respect to the y-axis we have the same construction starting in second quadrant with a free boundary Γ_2 intersecting Γ_1 at $\tilde{z}$ on the y-axis. Here we get a contradiction because above Γ_1 near $\tilde{z}$ we have $u > \psi$, and on Γ_2 in the noncontact set to the left of $\tilde{z}$ we have $u = \psi$. This shows $a(\beta) \geq 0$.

Since we assume the existence of a local solution of the perturbed problem in $\mathcal{O}$, thus the shaded region is in the set W defined in Chapter 3. We are first going to show that any small circle about $a(\beta) + i\beta$ does not lie entirely in $\overline{W}$. Hence such a circle must intersect the free boundary somewhere other than on Γ_1.

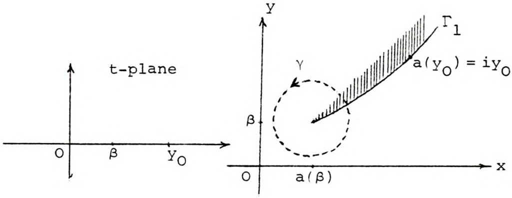

Figure 5.3

Since β is a simple zero of $t^4 + 2At^2 + B$, it follows $a''(t) = -\frac{2}{\pi}(t-\beta)^{\frac{1}{2}} \sum_{j=0}^{\infty} a_j(t-\beta)^j$ near $t = \beta$.

Integrating twice we get $a(t) = a(\beta) + \alpha'_{-1}(t-\beta) - \frac{2}{\pi}(t-\beta)^{\frac{5}{2}} \sum_{j=0}^{\infty} \alpha'_j (t-\beta)^j$, where α'_{-1}, α'_j are real, and $L(t) = L(\beta) + (\alpha'_{-1} + i)(t-\beta) - \frac{2}{\pi}(t-\beta)^{\frac{5}{2}} \sum_{j=0}^{\infty} \alpha'_j (t-\beta)^j$, for t near β. Putting $(t-\beta)^{\frac{1}{2}} = \xi$ and $z = L(t)$ we get for an analytic μ

$$z - L(\beta) = \xi^2[(\alpha'_{-1} + i) - \frac{2}{\pi}\xi^3\mu(\xi^2)] ,$$

$$\mu(0) = \alpha'_0 > 0, \quad \alpha'_{-1} + i \neq 0 .$$

Then $(z - L(\beta))^{\frac{1}{2}} = \xi(C_0 + C_3\xi^3 + O(\xi^4))$, where $C_0 \neq 0$, $C_3 \neq 0$, and inverting this function gives

$$\xi = (z - L(\beta))^{\frac{1}{2}}[b_0 + b_3(z - L(\beta))^{\frac{3}{2}} + O((z - L(\beta))^2)] ,$$

where $b_0 \neq 0$, $b_3 \neq 0$. Hence

$$(t - \beta) = (z - L(\beta))[b_0^2 + 2b_3b_0(z - L(\beta))^{\frac{3}{2}} + O((z - L(\beta))^2)] , \tag{*}$$

This shows upon calculating $t^2 = (L^{-1}(z))^2$ that

$$(L^{-1}(z))^2 = \beta^2 + 2\beta b_0^2(z - L(\beta)) + b_0^4(z - L(\beta))^2 + 4\beta b_0 b_3(z - L(\beta))^{\frac{5}{2}} + O((z - L(\beta))^{\frac{7}{2}}) .$$

Since $(L^{-1})^2$ must be continuous on $\overline{W}$ (by Theorem 3.11), and since $\beta b_0 b_3 \neq 0$, the formula above shows that no closed path (e.g. circle) about $L(\beta)$, close enough to this point, can be entirely in $\overline{W}$. Such a closed path therefore must intersect the free boundary somewhere other than on Γ_1. See Figure 5.3. If we evaluate L at $y > \beta$, $y_1 = \beta + (y-\beta)e^{2\pi i}$, which are in two different Riemann sheets, then $L(y_1) - L(y) = \frac{4}{\pi}(y-\beta)^{\frac{5}{2}} \sum_{j=0}^{\infty} \alpha_j'(y-\beta)^j > 0$ for $(y-\beta)$ small enough, which shows that L is locally one-to-one from a cut neighborhood of β (the interval (β, y) deleted) into a cut neighborhood of $L(\beta)$ (a cusp deleted) as shown in Figure 5.4. In

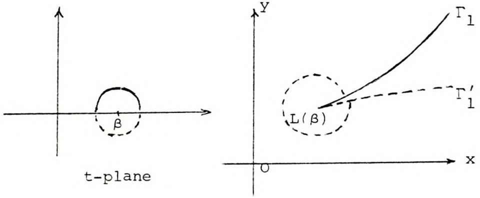

Figure 5.4

fact if y moves to β, then the image of y moves to $L(\beta)$ on Γ_1 and the image of y_1 moves to $L(\beta)$ on some path Γ_1'. The paths Γ_1 and Γ_1' are tangent at $L(\beta)$ by (*). Note $[L^{-1}(z_*)]^2 = (\text{Im } z_*)^2$ for a point on the free boundary. Thus $L^{-1}(z_*) = y_*$ near β here.

If there were a segment with left endpoint β (on the real axis) mapped onto a new part of the free boundary passing through $L(\beta)$, then $L'' = a''$ would have to be real there which is not true. Thus Γ_1, Γ_1' establish the boundary of W_1 near $L(\beta)$. Γ_1' is the new part of the free boundary, and it is easy to show directly $u - \psi \geq 0$ in the cusp deleted neighborhood of $L(\beta)$ with equality occuring only on Γ_1, Γ_1' (one can also refer here to the Kinderlehrer and Nirenberg cusp result [10]). This shows $a(\beta) = 0$ cannot happen by symmetry with respect to the y-axis if we argue as in Figure 5.2. Then we can extend the local solution near Γ_1' also as we did for Γ_1.

Second Step. In this step we want to end up with a contradiction which proves the lemma. Note this Lemma involves two cases according to whether there is another root, say α, $0 < \alpha < \beta$, of $t^4 + 2At^2 + B$ or not. Because of these singularities we may consider cuts along the x-axis as in Figure 5.5. In the upper

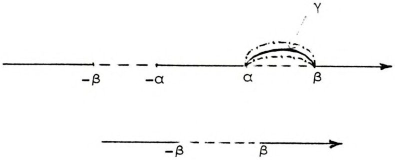

Figure 5.5

figure there is another simple root α, where in the lower one there is not any. Consider the analytic continuation in the upper half plane of the branch of f we were using for definition of Γ_1 on the right hand side of β. Note

$$\begin{aligned} a(t) &= a(\beta) + \int_\beta^t a'(\tau)d\tau \\ &= a(\beta) + [\tau a'(\tau)]_\beta^t - \int_\beta^t \tau f(\tau)d\tau \\ &= a(\beta) + a'(\beta)(t-\beta) + \int_\beta^t (t-\zeta)f(\zeta)d\zeta \ . \end{aligned}$$

Since the zeros of L' are isolated and $L'(t) = a'(\beta) + i + \int_\beta^t f(\zeta)d\zeta \neq 0$ for $-\beta \leq t \leq \beta$, thus we can join β to α (or β to $-\alpha$, or β to $-\beta$

in the second case in Figure 5.5) by a path γ staying away from the zeros of L' with

$$\text{Im}\,\gamma(s) > 0 \quad \text{for} \quad 0 < s < 1, \quad \gamma(0) = \beta,$$

$$\gamma(1) = \alpha \quad (\text{or} \quad \gamma(1) = -\alpha, \quad \text{or} \quad \gamma(1) = -\beta) \ ,$$

so that L is invertible locally in a neighborhood of any point $\gamma(s)$, $0 < s < 1$, with analytic inverse L^{-1}. So we have an open connected region containing $\gamma((0,1))$ in the upper half plane in which L is locally invertible. L is continuous on the closure of this region.

The part of γ near β is mapped by L in W_1. Thus either there is a point on γ, $\gamma(\hat{s})$, $0 < \hat{s} < 1$, which is mapped by L to a point on the free boundary or all points $\gamma(s)$, $0 < s < 1$ are mapped in $\tilde{W}$ (assume the image stays inside $\mathcal{O}$ in this case, we will show this in the third step). If there is such a point $\gamma(\hat{s})$, then

$$[L^{-1}(L(\gamma(\hat{s})))]^2 = [\text{Im}\,L(\gamma(\hat{s}))]^2 \ ,$$

$$L^{-1}(L(\gamma(\hat{s}))) = \gamma(\hat{s}) \ ,$$

which is not true no matter which one of pairs $\{\beta,\alpha\}$, $\{\beta,-\alpha\}$, $\{\beta,-\beta\}$ is joined by γ. Therefore there is no such a $\gamma(\hat{s})$, and all $\gamma(s)$, $0 < s < 1$, are mapped in $\tilde{W}$. At the end of this step we will show that $L(\alpha)$

(or $L(-\alpha)$, or $L(-\beta)$ cannot be mapped inside $\tilde{W}$ and they must be mapped into the free boundary by L. Assume this now. Hence

$$\alpha = L^{-1}(L(\alpha)) = \pm \operatorname{Im} L(\alpha) = \pm [\alpha + \frac{1}{i}\int_{\beta}^{\alpha}(\alpha - \zeta)f(\zeta)d\zeta$$

(note $\alpha(\beta)$, $\alpha'(\beta) > 0$) *

$$-\alpha = L^{-1}(L(-\alpha)) = \pm \operatorname{Im} L(-\alpha)$$

$$= \pm [-\alpha + \frac{1}{i}\int_{\beta}^{+\alpha}(-\alpha - \zeta)f(\zeta)d\zeta] \qquad *'$$

$$-\beta = L^{-1}(L(-\beta)) = \pm \operatorname{Im} L(-\beta)$$

$$= \pm [-\beta + \frac{1}{i}\int_{\beta}^{-\beta}(-\beta - \zeta)f(\zeta)d\zeta]$$

(there is no α in this case) **

Positive sign case in any one of these leads to a contradiction immediately (in *, *' leads to $\alpha = \beta$ and in ** leads to $-\beta = \beta$) because f has fixed sign there. In *, *' negative sign case implies

$2\alpha = \frac{1}{i}\int_{\alpha}^{\beta}(\alpha - \zeta)f(\zeta)d\zeta$, $+2\alpha = \frac{1}{i}\int_{\alpha}^{\beta}(+\alpha + \zeta)f(\zeta)d\zeta$,

respectively, where $\frac{1}{i}f(\zeta)$ is negative. These two results cannot be true simultaneously. Hence the case of having two roots $0 < \alpha < \beta$ for $t^4 + 2At^2 + B$ is ruled out. Now assume this polynomial does not have any zero in $(0,\beta)$.

Then ** implies $2\beta = \frac{1}{i}\int_{-\beta}^{\beta}(\beta+\zeta)f(\zeta)d\zeta$ which is not again possible because $\frac{1}{i}f(\zeta) \leq 0$. This contradiction proves the Lemma if we can resolve the two questions which are left open. We conclude this step by answering the second question:

Claim: $L(\alpha) \notin \tilde{W}$, $L(-\alpha) \notin \tilde{W}$, $L(-\beta) \in \tilde{W}$.

Proof of the claim: We only prove the first one and one can argue similarly for the others. Suppose $L(\alpha) \in \tilde{W}$, then $(L^{-1})^2$ has to be analytic at $L(\alpha)$ (note $(L^{-1})^2$ here is defined in the noncontact set) by results of Chapter 3. To check this observe that

$$L''(t) = -\frac{2}{\pi}\frac{\sqrt{(t^2-\beta^2)(t^2-\alpha^2)}}{(t^2+K)^{3/2}}\phi(t^2) = -\frac{2}{\pi}(t-\alpha)^{\frac{1}{2}}g_1(t),$$

where g_1 is analytic near α and $g(\alpha) = \frac{\sqrt{2\alpha(\alpha^2-\beta^2)}}{(\alpha^2+K)^{3/2}}\phi(\alpha^2)$.

Thus $z = L(t) = A + B(t-\alpha) - \frac{2}{\pi}(t-\alpha)^{\frac{5}{2}}g_2(t)$, where $g_2(\alpha) = \frac{4}{15}g_1(\alpha)$ and g_2 is analytic at α. If $B \neq 0$, then $z - A = L(t) - L(\alpha) = B(t-\alpha)[1 - \frac{2}{\pi}(t-\alpha)^{\frac{3}{2}}g_3(t)]$. Putting $(t-\alpha)^{\frac{1}{2}} = \xi$, taking square root from both sides, and applying the Inverse mapping theorem we can find $(t-\alpha)^{\frac{1}{2}}$ in terms of $(z-A)^{\frac{1}{2}}$. Squaring both sides then we get $t-\alpha = \frac{(z-A)}{B}[1 + \frac{2}{\pi}\beta_0(\frac{z-A}{B})^{\frac{3}{2}} + \cdots$ (higher order terms)$] = \frac{z-A}{B} + \frac{2}{\pi}\beta_0(\frac{z-A}{B})^{\frac{5}{2}} + \cdots$, where

$\beta_0 \neq 0$. Then $t^2 = (L^{-1}(z))^2 = [\alpha + \frac{z-A}{B} + \frac{2}{\pi}\beta_0(\frac{z-A}{B})^{\frac{5}{2}} + \cdots]^2$ will not be analytic at $L(\alpha)$ which is a contradiction to $L(\alpha) \in \tilde{W}$. If $B = 0$, then $(z-A)^{\frac{2}{5}} = (t-\alpha)g_4(t)$, where g_4 is analytic at α with $g_4(\alpha) \neq 0$. Again by applying the Inverse Function Theorem as above we get $t - \alpha = (z-A)^{\frac{2}{5}}[\gamma_0 + \gamma_1(z-A)^{\frac{2}{5}} + \cdots]$, where $\gamma_0 \neq 0$. This again shows that $t^2 = [L^{-1}(z)]^2$ is not analytic at $A = L(\alpha)$. This contradicts $L(\alpha) \in \tilde{W}$ again. So $L(\alpha) \notin \tilde{W}$.

Third step. To conclude the proof of the Lemma we only need to show that the image of the path γ in the above three situations (γ joining pairs $\{\beta,\alpha\}$, $\{\beta,-\alpha\}$, and $\{\beta,-\beta\}$) remains inside $\mathcal{O}$. This was the first question left open in the second step. To do this we first show that $\operatorname{Im} L(\gamma) \leq \beta$, $\operatorname{Re} L(\gamma) \leq a(\beta)$ if we let γ be the limit case of the paths (chosen before) coincided with the real axis as shown in Figure 5.6.

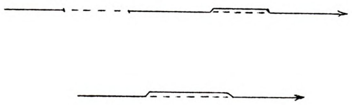

Figure 5.6

Clearly $\text{Im } L(t) = t + \text{Im} \int_{\beta}^{t} (t-\zeta)f(\zeta)d\zeta < t$, for $t < \beta$, for all three pairs and $\text{Re } L(t) = \alpha(\beta) + \alpha'(\beta)(t-\beta) < \alpha(\beta)$, for the pairs $\{\beta,\alpha\}$, $\{\beta,-\beta\}$, and $\text{Re } L(t) = \alpha(\beta) + \alpha'(\beta)(t-\beta) + \int_{\alpha}^{t} (t-\zeta)f(\zeta)d\zeta < a(\beta)$, for the pair $\{\beta,-\alpha\}$, where $t < \beta$ in any case above. So $L(\gamma)$ remains to the left of $x = a(\beta)$ and beneath $y = \beta$ in any case.

Let us now show that $L(\gamma)$ does not intersect the x-axis. Suppose it does, say at $L(\gamma(\hat{s}))$. Then by Theorem 3.11 and the fact that $(\frac{1}{y} u - \varphi)\Big|_{x} = -2(L^{-1}(x))^2 \geq 0$ on J, we must have $L^{-1}[L(\gamma(\hat{s}))]$ imaginary. This cannot happen obviously if γ is joining the pair $\{\beta,\alpha\}$. To show it cannot occur in the other two cases we do as follows. If $L^{-1}[L(\gamma(\hat{s}))]$ is imaginary we can assume the path γ intersects the imaginary axis only at 0. See Figure 5.7. Then $L(0) = a(\beta) - \beta a'(\beta) - \int_{\beta}^{0} \zeta\, f(\zeta)$

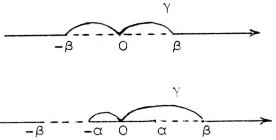

Figure 5.7

which is supposed to be real. But this is not real if γ is joining any one of the pairs $\{\beta,-\alpha\}$, $\{\beta,-\beta\}$ shown in Figure 5.7. This contradiction results that $L(\gamma)$ does not intersect x-axis in any case.

Now in order to show that $L(\gamma)$ remains inside $\mathcal{O}$, it is enough to show that it does not intersect y-axis. Suppose it does. Then by symmetry with respect to y-axis we have $\frac{\partial u}{\partial x}(0,y) = -2\ \mathrm{Im}(L^{-1}(iy))^2 = 0$ (on y-axis) by Proposition 3.12. This shows then $L^{-1}(iy)$ is either real or imaginary. For the pair $\{\beta,\alpha\}$ we can take the path γ in such a way that it does not intersect the x-axis except at endpoints. This rules out the case $\{\beta,\alpha\}$. For the other pairs $\{\beta,-\alpha\}$, $\{\beta,-\beta\}$ we can assume γ intersects x-axis and y-axis only at 0 as in Figure 5.7. Then $L^{-1}(iy) = 0$, and therefore $L(0) = a(\beta) - \beta a'(\beta) + \int_0^\beta \zeta\ f(\zeta) = iy$, for $y > 0$. The latter result is not true because $\frac{1}{i}\ f(\zeta) < 0$ and $a(\beta) - \beta a'(\beta)$ is real. Thus $L(\gamma)$ cannot intersect y-axis either. Hereby we showed $L(\gamma)$ remains inside $\mathcal{O}$ and in the first quadrant. Here are the third step and the proof of the Lemma both complete. #

Lemma 5.4 proved that $t^4 + 2At^2 + B$ cannot have a simple root $0 < \beta < y_0$. Thus it has either a double root α, $0 < \alpha < y_0$, or no root in $(0,y_0)$. Next Lemma shows that the latter case cannot happen either.

Lemma 5.5. The case $t^4 + 2At^2 + B > 0$, for $0 < t^2 < y_0^2$, cannot happen.

Proof: Since $a'(y) - a'(y_0) = \int_{y_0}^{y} f(\zeta)d\zeta > 0$ for $0 \le y < y_0$, thus $a'(y)$ is increasing to the left of y_0 but it is always finite. Moreover a defined by

$$a(y) = a(y_0) + a'(y_0)(y - y_0) + \int_{y_0}^{y} (y - \zeta)f(\zeta)d\zeta ,$$

$$\text{for } 0 \le y < y_0$$

is finite, decreasing as y moves to the left of y_0. Therefore Γ_1 intersects x-axis transversally (or y-axis non-horizontally). This provides a contradiction as follows. The local solution built up near $a(y_0) + iy_0$, with Γ_1 as a piece of free boundary, is continued to the left with noncontact set over Γ_1. By symmetry

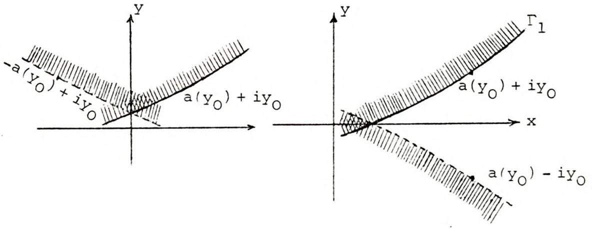

Figure 5.8

with respect to x-axis (or y-axis) we must have $u < \psi$ on the part of Γ_1 below x-axis (or $u > \psi$ on the right hand side of y-axis) as shown in Figure 5.8. On the other hand we must have $u = \psi$ there. Here is the proof complete. #

Corollary 5.6. The only possible case for f is $f(t) = \frac{2}{\pi} \frac{\alpha^2 - t^2}{(t^2 + K)^{3/2}} \phi(t^2)$, where $0 < \alpha^2 < y_0^2$. Moreover $a'(0) = 0$, $a(0) > 0$, $a'(y_0) = \int_0^{y_0} f(\zeta) d\zeta$.

Proof: The first part is an immediate conclusion of Lemmas 5.4, 5.5. If $a'(0) \neq 0$ (note $a'(0)$ is finite), then we get a contradiction as in Figure 5.8 in Lemma 5.5. This result yields $a'(y_0) = \int_0^{y_0} f(\zeta) d\zeta$. We can now build up the local solution near $a(0)$ with the noncontact set to the left of Γ_1 as in Section 2.7. If $a(0) < 0$, since $a'(y) = \int_0^{y} f(\zeta) d\zeta$ shows that the tangent to the curve Γ_1 never becomes horizontal, thus at the intersection point of Γ_1 with y-axis we get a contradiction, by symmetry with respect to y-axis, similar to the one we obtained at $a(0)$ in Figure 5.8. See Figure 5.9. Hence $a(0) \geq 0$. If $a(0) = 0$, then similar argument leads to a contradiction. Nearby the origin to the left of Γ_1 we have $u > \psi$; on the other hand on the part of the free boundary symmetric to Γ_1

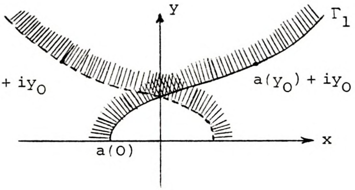

Figure 5.9

with respect to y-axis we must have $u = \psi$. This contradiction shows $a(0) \neq 0$. Hence we conclude $a(0) > 0$ and the proof is complete. #

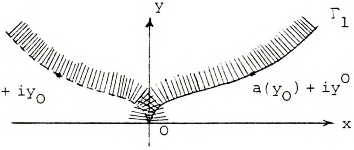

Figure 5.10

Remark 5.7. We can build up the local solution near $a(0)$ with the noncontact set on the left hand side of Γ_1 in the first and forth quadrants as in Section 2.7. Then $\Gamma_1 : z = L(t) = a(t) + it$, t near zero, is symmetric with respect to x-axis, and

$$u(z) = -\frac{1}{2} \operatorname{Im} \int_{a(0)}^{z} [\zeta^2 + 4(L^{-1}(\zeta))^2] d\zeta, \quad \text{for } z \text{ near}$$

$a(0)$ to the left of Γ_1, where L^{-1} is a conformal map in a uniform neighborhood of Γ_1 near $a(0)$, and $\frac{1}{y} u - \varphi = -2(L^{-1}(x))^2 \geq 0$ on the real axis to the left of $a(0)$, where is contained in J. #

Remark 5.8. Since u is a local solution to the problem, thus by Theorem 3.11 the map $(L^{-1})^2$ must have analytic continuation to the left of $a(0)$ on the real axis and it must be negative-valued until it becomes zero again (by symmetry with respect to y-axis) and we get a new point of the free boundary on the x-axis. Since $(L^{-1})^2 \neq 0$ there, thus $(L^{-1})^2$ has analytic continuation if and only if L^{-1} has, as long as $(L^{-1})^2$ is nonzero. #

For the time being we want to check the necessary conditions for L (obtained already) to fulfill these requirements, so that by these conditions we can conversely construct a local solution near $a(0)$ continuable to the left to get the local solution u eventually.

Lemma 5.9. L^{-1} in 5.7 does have analytic continuation to the left of $a(0)$ on the real axis at least on $(L(i\sqrt{K}), a(0))$ and $(L^{-1})^2$ is negative-valued. $L(i\sqrt{K})$ is the first critical point of $(L^{-1})^2$ to the left of $a(0)$. Moreover, the value of $(L^{-1})^2$ is $(i\sqrt{K})^2 = -K$. Further $\frac{1}{i} L^{-1}(z)$ is indeed strictly increasing as $z < a(0)$ decreases.

Proof: Note $a'(t) = \int_0^t f(\tau)d\tau$,

$a(t) = a(0) + \int_0^t (t-\tau)f(\tau)d\tau$, $L(t) = a(t) + it$.

$L'(0) = i \neq 0$ and L is conformal in a neighborhood of 0 taking the real axis to Γ_1 and the imaginary axis to the real axis to the left of $a(0)$ by $z = L(t) = a(t) + it$ as shown in Figure 5.11 (one can check the formulas above to see there).

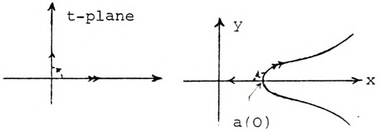

Figure 5.11

Furthermore $L'(t) = i + \int_0^t f(\zeta)d\zeta$ shows that $\frac{1}{i} L'(i\sigma) > 0$ for $0 < \sigma < \sqrt{K}$ and it becomes unbounded as $\sigma \to (\sqrt{K})^-$. Hence L is real and strictly decreasing on $[0, i\sqrt{K}]$, which can be seen from $L(t) = a(0) + \int_0^t (t-\zeta)f(\zeta)d\zeta + it$ also. Applying Lemma 5.2 for L on a sequence of intervals $[0, i(\sqrt{K} - \varepsilon_n)]$, where $\varepsilon_n \to 0$, we get L a conformal mapping on a neighborhood of each $[0, i(\sqrt{K} - \varepsilon_n)]$ not containing $i\sqrt{K}$. Hence L^{-1} is a conformal mapping in a neighborhood of $(L(i\sqrt{K}), a(0)]$ not containing $L(i\sqrt{K})$. Moreover L^{-1} takes this part of the real axis to the upper part of the imaginary axis and $i^{-1}L^{-1}$ is strictly increasing to the left $a(0)$ on the closed interval $[L(i\sqrt{K}), a(0)]$ (because of having similar property for L). Let $\hat{x} < a(0)$ be the first critical point of $(L^{-1})^2$ to the left of $a(0)$. Since $\frac{d}{dx}(L^{-1}(x))^2|_{\hat{x}} =$ $2L^{-1}(x)\frac{d}{dx}L^{-1}(x)|_{\hat{x}} = 0 \Leftrightarrow \frac{d}{dx}L^{-1}(x)|_{\hat{x}} = 0$, thus $L'(L^{-1}(\hat{x}))$ is infinity and by definition of L' we conclude that $L^{-1}(\hat{x}) = i\sqrt{K}$. Thus $L(i\sqrt{K}) = \hat{x}$. #

<u>Definition 5.10</u>. For the singularities $\pm i\sqrt{K}$, we assume the imaginary axis in t-plane is cut from $i\sqrt{K}$ to ∞ (upward) and from $-i\sqrt{K}$ to ∞ (downward). Define $m(t)$, for t near zero, in the lower Riemann sheet as analytic continuation of $L(t)$ along the paths of the

form $\mathcal{C}$, where $L(t) = a(t) + it = a(0) + \int_0^t (t - \zeta) f(\zeta) d\zeta + it$.
Note $a'(i\sqrt{K})$ is not defined (infinity) but $a(i\sqrt{K})$ is defined as a finite number. Also $m(t) = b(t) + it$, where b is the analytic continuation of a along the paths of type $\mathcal{C}$ (see Figure 5.12). #

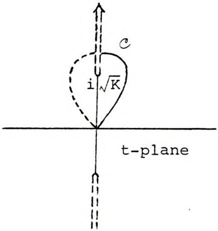

Figure 5.12

Remark 5.11. Recall that during move from $a(0)$ to the left, we are looking for necessary conditions for having local solution u so that they are going to be sufficient at least for constructing a local solution near $a(0)$ continuable to the left. At this stage in order to have a local solution u it is necessary to have $(L^{-1})^2$ negative-valued on the real axis near $L(i\sqrt{K})$ and L^{-1} must be analytic in a full neighborhood

of this point. Let us now check these requirements for our map L^{-1}.

Lemma 5.12. L^{-1} is analytic in a full neighborhood of $L(i\sqrt{K})$ with $(L^{-1})'(L(i\sqrt{K})) = 0$. L^{-1} is imaginary-valued on the real axis part of the neighborhood and $\frac{1}{i} L^{-1}$ attains a local maximum $\sqrt{K}$ at $L(i\sqrt{K})$. To the left of $L(i\sqrt{K})$, L^{-1} is invertible with inverse m as defined in 5.10.

Proof: In a sufficiently small neighborhood of $i\sqrt{K}$ we have

$$L''(t) = \frac{2}{\pi} \frac{\alpha^2 - t^2}{(t^2 + K)^{3/2}} \phi(t^2) = (t - i\sqrt{K})^{-\frac{3}{2}} \sum_{j=0}^{\infty} A_j (t - i\sqrt{K})^j,$$

$$A_0 \neq 0, \quad t \text{ near } i\sqrt{K} ,$$

and $L(t) - L(i\sqrt{K}) = A'_{-1}(t - i\sqrt{K}) + (t - i\sqrt{K})^{\frac{1}{2}} \sum_{j=0}^{\infty} A'_j (t - i\sqrt{K})^j$, $A'_0 \neq 0$ (by integrating twice). Putting $(t - i\sqrt{K})^{\frac{1}{2}} = \xi$, $z = L(t)$ we get $z - L(i\sqrt{K}) = A'_{-1}\xi^2 + \sum_{j=0}^{\infty} A'_j \, \xi^{2j+1} = \xi g(\xi)$, where $g(0) \neq 0$ and g is analytic near zero. By invoking the Inverse Function Theorem we get $\xi = \theta(z - L(i\sqrt{K}))$, where $\theta(\tau)$ is analytic near zero with $\theta(0) = 0$, $\theta'(0) \neq 0$. Hence $t - i\sqrt{K} = \theta^2(z - L(i\sqrt{K}))$, and $t = L^{-1}(z) = i\sqrt{K} + \theta^2(z - L(i\sqrt{K}))$ is analytic in a full neighborhood of $L(i\sqrt{K})$. Since $\theta(0) = 0$, $\theta'(0) \neq 0$, thus the latter equation of $L^{-1}(z)$ shows

$(L^{-1})'(L(i\sqrt{K})) = 0$. Since L^{-1} is imaginary-valued to the right of $L(i\sqrt{K})$, thus in the expansion of L^{-1} in power series of $(z - L(i\sqrt{K}))$ all coefficients are imaginary and L^{-1} is imaginary-valued to the left of $L(i\sqrt{K})$ also. By looking at the formula for L^{-1} we see that if $z - L(i\sqrt{K})$ changes its argument by π, then $t - i\sqrt{K}$ will change its argument by almost 2π if $|t - i\sqrt{K}|$ is small enough. In particular by starting from $z_1 > L(i\sqrt{K})$ and ending with $z_2 < L(i\sqrt{K})$ (see Figure 5.13), $t - i\sqrt{K}$ will change its argument by exactly 2π by the above argument, that is, $t = L^{-1}(z)$ starts from $t_1 = L^{-1}(z_1)$ on the imaginary axis and ends up with $t_2 = L^{-1}(z_2)$ on the imaginary axis again. Note t_1 is in the upper Riemann sheet and t_2 in the lower one. We observe that $\frac{1}{i} L^{-1}$ attains a local maximum $\sqrt{K}$ at $L(i\sqrt{K})$. The last part of the Lemma is clear.

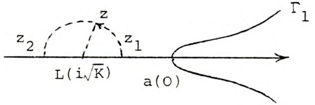

Figure 5.13

<u>Lemma 5.13</u>. $\frac{1}{i} L^{-1}$ is strictly decreasing as $z < L(i\sqrt{K})$ decreases, with inverse m, and it vanishes at $m(0) = b(0)$ with $(\frac{1}{i} L^{-1})'(b(0)) \neq 0$. b defined in 5.10 has the equation $b(t) = \mu + \lambda t - a(t)$, where $\lambda = \int_{\mathcal{C}} f(\zeta)d\zeta$ is pure imaginary, μ is real, and $\frac{1}{i}\lambda + 1 \leq 0$.

<u>Proof</u>: If $(\frac{1}{i} L^{-1})'$ vanishes at some point $\overline{x} \in (b(0), L(i\sqrt{K}))$, then $m'(t) = L + i - a'(t)$ becomes infinity at some point $i\sigma$, $0 < \sigma < \sqrt{K}$. This result is not true because m' becomes infinity only at $\pm i\sqrt{K}$. Hence $\frac{1}{i} L^{-1}$ is strictly decreasing as $z < L(i\sqrt{K})$ decreases with inverse m, strictly decreasing as t falls below $i\sqrt{K}$ on the imaginary axis. By symmetry with respect to y-axis L^{-1} must vanish at some point and it cannot have minimum before that. This implies m is strictly increasing on $[0, i\sqrt{K}]$. Again $\frac{1}{i} L^{-1}$ cannot vanish at point $m(0)$ with derivative zero there by $m'(0)$ being finite. To show the formula for b observe that $m'' = b''$ is also an analytic continuation of $L'' = a''$ along the paths of type $\mathcal{C}$ (referring to Definition 5.10). Since along $\mathcal{C}$ the argument of $(t - i\sqrt{K})^{\frac{3}{2}}$ is changed by 3π, thus $b''(t) = -a''(t)$ for t near 0. By integrating twice we get $b(t) = \mu + \lambda t - a(t)$ and $m(t) = \mu + (\lambda + i)t - a(t)$, where

$\lambda = b'(0) = \int_{\mathcal{C}} f(\zeta)d\zeta$. $m(0) = \mu - a(0)$ and $a(0)$ are real, therefore μ is real. Since m is real-valued on $[0, i\sqrt{K}]$, and $a(t)$, μ are real there, thus λ must be imaginary there. Moreover m being invertible on this interval implies

$$\frac{1}{i}\lambda + 1 \leq 0 .$$

We also have the following relations

$$\begin{aligned} m(i\sqrt{K}) &= L(i\sqrt{K}) \Rightarrow b(i\sqrt{K}) = a(i\sqrt{K}) \Rightarrow \lambda i\sqrt{K} \\ &= 2a(i\sqrt{K}) - \mu \Rightarrow \lambda i\sqrt{K} \\ &= 2a(0) + 2\int_0^{i\sqrt{K}} (i\sqrt{K} - \zeta)f(\zeta)d\zeta - \mu \end{aligned}$$

and $b(0) = a(0) - \int_{\mathcal{C}} \zeta f(\zeta)d\zeta$.

<u>Corollary 5.13</u>. $\frac{1}{y}u - \varphi$ attains its absolute maximum $(2k)$ on $[b(0), a(0)]$ at $L(i\sqrt{K})$, whereas it vanishes at $a(0)$, $b(0)$ and strictly monotone on each side of $L(i\sqrt{K})$.

<u>Remark 5.14</u>. By symmetry with respect to y-axis we must have $-a(0) \leq b(0) < a(0)$, and this condition will be fulfilled if μ (the constant of integration) ischosen real and appropriately. We have thus far imposed the following conditions also: $\lambda = \int_{\mathcal{C}} f(\zeta)d\zeta$

imaginary, $\mu = -\lambda i\sqrt{K} + 2a(0) + 2\int_0^{i\sqrt{K}} (i\sqrt{K} - \zeta)f(\zeta)d\zeta$,

and $\frac{1}{i}(\lambda + i) \leq 0$. In the next Lemma we show that equality case is not necessary here.

<u>Lemma 5.15</u>. The point $m(0) = b(0) = \mu - a(0)$ belongs to the free boundary and this implies the necessary condition $\lambda + i \neq 0$, i.e., $\frac{1}{i}(\lambda + i) < 0$.

<u>Proof</u>: Since $\frac{1}{y} u - \varphi = -2(L^{-1})^2$ on J, vanishes at $m(0)$, thus $m(0) \in \overline{(\nu - J)} \cup s$, the free boundary. To show $\lambda + i \neq 0$, we check $\nabla(\frac{1}{y} u - \varphi) = 0$ at $m(0) = b(0)$. Suppose $\lambda + i = 0$. Then $m(t) - \mu + a(0) = -\lambda_1 t^2 - \lambda_2 t^4 - \ldots$, where $\lambda_1 = a''(0) > 0$, i.e.,

$$z - \mu + a(0) = -\lambda_1 s(1 + \frac{\lambda_2}{\lambda_1} s + \frac{\lambda_3}{\lambda_1} s^2 + \cdots)$$ if we put

$t^2 = s$. Since $\frac{dz}{ds}\Big|_{s=0} = -\lambda_1 \neq 0$, thus by the Inverse mapping theorem we have $-\lambda_1 t^2 = (z - \mu + a(0)) + \frac{\lambda_2}{\lambda_1^2}(z - \mu + a(0))^2 + o((z - \mu + a(0))^2) = \theta(z - \mu + a(0))$,

where θ is an analytic function with $\theta(0) = 0$, $\theta'(0) \neq 0$. We discover that $(L^{-1})^2$ is analytic at $\mu - a(0)$ but $[(L^{-1}(x))^2]'\Big|_{\mu - a(0)} \neq 0$. Thus $\nabla(\frac{1}{y} u - \varphi)(m(0)) \neq 0$, which violates a necessary condition of Theorem 3.11. Hence $\lambda + i = 0$ is impossible. #

Lemma 5.16. Either $b(0)$ is an isolated point of the free boundary or $\lambda = -2i$.

Proof: We discovered so far that $m(t) - (\mu - a(0)) = z - b(0) = (\lambda + i)t - \lambda_1 t^2 - \lambda_2 t^4 - \cdots$ for t near zero, where $\lambda + i \neq 0$ by Lemma 5.15. This means that the inverse of the function m exists and is analytic in a full neighborhood of $b(0)$. The inverse, which is the analytic continuation of L^{-1}, can still be called L^{-1}:

$$t = \frac{z-b(0)}{\lambda+i}\left[1 + \frac{\lambda_1}{\lambda+i}t + \frac{\lambda_2}{\lambda+i}t^3 + \cdots + \left(\frac{\lambda_1}{\lambda+i}t + \frac{\lambda_2}{\lambda+i}t^3 + \cdots\right)^2 + \cdots\right],$$

i.e.,

$$t = L^{-1}(z) = \frac{z-b(0)}{\lambda + i}\left[1 + \frac{\lambda_1}{\lambda+i}\left(\frac{z-b(0)}{\lambda + i} + \frac{\lambda_1 t}{\lambda+i}\frac{z-z_0}{\lambda+i} + \cdots\right) + \frac{\lambda_1^2}{(\lambda+i)^2}\left(\frac{z-z_0}{\lambda+i}\right)^2 + \cdots\right],$$

where the rest of the terms are of order greater than three. Hence

$$t = L^{-1}(z) = \frac{z-b(0)}{\lambda + i} + \frac{\lambda_1(z-b(0))^2}{(\lambda + i)^3} + \frac{2\lambda_1^2(z-b(0))^3}{(\lambda + i)^5} + \cdots .$$

Let's investigate $\frac{1}{y}u(z) - \varphi(z)$ near $b(0) = m(0)$. For every z near $b(0)$ in W we have

$$\frac{1}{y}u(z) - \varphi(z) = \frac{2}{3}y^2 - \frac{2}{y}\operatorname{Im}\int_{a(0)}^{z}[L^{-1}(\zeta)]^2 d\zeta ,$$

where the integration is along a path in $\tilde{W} \cup \{a(0)\}$ from $a(0)$ to z. Since $(L^{-1})^2$ is real valued on $[b(0),a(0)]$, thus $\frac{1}{y} u(z) - \varphi(z) = \frac{2}{3} y^2 - \frac{2}{y} \operatorname{Im} \int_{b(0)}^{z} [L^{-1}(\zeta)]^2 d\zeta$, where the path of integration is in $\tilde{W} \cup [b(0)]$ (remember $\tilde{W} = W \cup [J - (\overline{\nu - J})]$). Let's now consider a disk neighborhood V of $b(0)$ in which $L^{-1}(z)$ is analytic and can be expanned in power series of $z - b(0)$. In the connected component C of open set $V \cap \tilde{W}$, which contains $V \cap [b(0),a(0)]$, we can join every z to a point in $V \cap [b(0),a(0)]$ by a path entirely inside the component. Thus every point z in the component can be joined to $b(0)$ by a path in $C \cup \{b(0)\} \subset V$. Since $(L^{-1})^2$ is analytic in V, thus we can redefine

$$\frac{1}{y} u(z) - \varphi(z) = \frac{2}{3} y^2 - \frac{2}{y} \operatorname{Im} \int_{b(0)}^{z} [L^{-1}(\zeta)]^2 d\zeta ,$$

integration on the ray path. Now

$$\int_{b(0)}^{z} [L^{-1}(\zeta)]^2 d\zeta = \int_{b(0)}^{z} \left[\frac{(\zeta-b(0))^2}{(\lambda + i)^2} + \frac{2\lambda_1(\zeta-b(0))^3}{(\lambda + i)^4} + \cdots\right] d\zeta$$

$$= \frac{1}{3} \frac{(z-b(0))^3}{(\lambda + i)^2} + \frac{\lambda_1}{2} \frac{(z-b(0))^4}{(\lambda + i)^4} + \cdots ,$$

and

$$\operatorname{Im} \int_{z_0}^{z} (L^{-1}(\zeta))^2 d\zeta = \frac{1}{(\lambda+i)^2}\left[(x - b(0))^2 - \frac{1}{3} y^3\right]$$

$$+ \frac{\lambda_1}{2(\lambda+i)^2}\left[4(x - b(0))^3 y - 4(x - b(0))y^3\right] + \cdots ,$$

where the higher order terms are ignored. Note all terms on the right hand side of the latter equation have y factor because we've taken the imaginary part of the right hand side of the former equation. Thus for $z \in C$ we have

$$-\frac{2}{y}\,\mathrm{Im}\int_{z_0}^{z}(L^{-1}(\zeta))^2 d\zeta$$

$$= \frac{-2}{(\lambda+i)^2}\left((x-b(0))^2-\frac{1}{3}y^2\right)$$

$$-\frac{\lambda_1}{(\lambda+i)^4}\left[4(x-b(0))^3-4(x-b(0))y^2\right]+\cdots ,$$

and $\frac{1}{y}u(z)-\varphi(z) = \frac{-2}{(\lambda+i)^2}(x-b(0))^2+\frac{2}{3}y^2\left(1+\frac{1}{(\lambda+i)^2}\right)$

$-\frac{4\lambda_1}{(\lambda+i)^4}\left[(x-b(0))^3-(x-b(0))y^2\right]+\cdots$, where the higher order terms are ignored. The right hand sides are real analytic.

Let's now define $\omega(x,y)$ as a function of x with y as a fixed parameter as follows:

$$\omega(x,y) = \frac{-2}{(\lambda+i)^2}(x-b(0))^2+\frac{2}{3}y^2\left(1+\frac{1}{(\lambda+i)^2}\right) .$$

At this point we are going to show $1+\frac{1}{(\lambda+i)^2}\geq 0$. Suppose not. Then

$$\omega-\frac{2}{3}y^2\left(1+\frac{1}{(\lambda+i)^2}\right) = \frac{-2}{(\lambda+i)^2}(x-b(0))^2 ,$$

for fixed small $|y|\neq 0$.

This shows a parabola with negative minimum $\frac{2}{3}y^2(1+\frac{1}{(\lambda+i)^2})$ which intersects the horizontal x-axis transversally (look at Figure 5.14). Since $\omega(x,y)$ has the leading

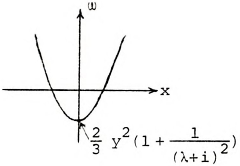

Figure 5.14

terms of $\frac{1}{y}u(x,y)-\varphi(x,y)$ near $b(0)$ and

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{y}u(x,y)-\varphi(x,y)\right)\Big|_{(b(0),0)} > 0 ,$$

bifurcation theory in this case [4] says that $\frac{1}{y}u(x,y)-\varphi(x,y)$ behaves similarly as $\omega(x,y)$ does, in the following sense, for small fixed perturbation of y from zero. $\frac{1}{y}u(x,y)-\varphi(x,y)$ as a function of x with parameter y, for any small fixed $|y| \neq 0$, intersects the xy-plane transversally and it has a negative minimum. But we know that if $\frac{1}{y}u(x,y)-\varphi(x,y)$ vanishes for any fixed small perturbation of y from zero, then it must have even multiple root with respect to x, i.e., $\frac{1}{y}u(x,y)-\varphi(x,y)$ must touch the horizontal plane

tangentially, not transversally. Hence $1+\frac{1}{(\lambda+i)^2} \geq 0$, i.e., $1 \geq \frac{1}{(i^{-1}\lambda+1)^2}$, that is, $(i^{-1}\lambda+1)^2 \geq 1$ which is equivalent to: $i^{-1}\lambda+1 \geq 1$ or $i^{-1}\lambda+1 \leq -1$. We already knew that $i^{-1}\lambda < -1$ (by Lemma 5.15), thus the case $i^{-1}\lambda \geq 0$ can't happen. Therefore $i^{-1}\lambda \leq -2$.

If $\frac{1}{i}\lambda < -2$, i.e. $1+\frac{1}{(\lambda+i)^2} > 0$, then we'll have an isolated point $b(0)$ of the free boundary because $\frac{1}{y}u(z)-\varphi(z) > 0$ for $z \in V-\{b(0)\}$, where V is a small enough neighborhood of $b(0)$. We'll show that this case cannot happen either. Finally if $1+\frac{1}{(\lambda+i)^2} = 0$, then $i^{-1}\lambda = -2$ will be the only possible case which works.

Lemma 5.17. The case $\frac{1}{i}\lambda < -2$, which says $b(0)$ is an isolated point of the free boundary, can't happen.

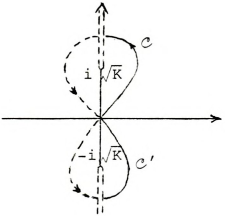

Figure 5.15

Proof: Suppose it can. Then $(L^{-1})^2$ has to have analytic continuation along the real axis to the left of

$b(0)$ and it has to be real-valued on the real axis. Again

$$\frac{1}{y} u(z) - \varphi(z) = -2[L^{-1}(z)]^2 > 0 ,$$

$$\text{for } z = x < b(0) \text{ near } b(0) .$$

Since $\lambda + i \neq 0$, thus L^{-1} exists and it is analytic in a neighborhood of $b(0)$ and is continued analytically to the left of $b(0)$ because $(L^{-1})^2$ is nonzero there. Since L^{-1} is pure imaginary-valued on the real axis to the left of $b(0)$, thus L^{-1} has the image on the lower half of the pure imaginary axis because if it had the image on the upper half of the pure imaginary axis, then m would be doublevalued on the upper part of the pure imaginary axis near zero which is not true because m is one-to-one near 0. Thus $\frac{1}{i} L^{-1}(x) < 0$ for $x < b(0)$. But then $m(t) = \mu + (\lambda + i)t - a(t)$ implies

$$m'(t) = (\lambda + 2i) - a'(t) - i = i(i^{-1}\lambda + 2) - i(\frac{1}{i} a'(t) + 1) ,$$

which shows $\frac{1}{i} m'(t) < 0$ for $\frac{1}{i} t < 0$ and $|t|$ sufficiently small, while $t \to -i\sqrt{K}$ on the pure imaginary axis implies $\frac{1}{i} m'(t) \to \infty$. Thus $m'(t)$ must vanish somewhere between 0 and $-i\sqrt{K}$ on the pure-imaginary axis. But this implies, then, m is not invertible there which is not true because L^{-1} exists and is analytic on the segment "$[-i\sqrt{K},0]$" and $(L^{-1})^2$ takes its minimum not before $-i\sqrt{K}$. Hence we get contradiction and we conclude

that $i^{-1}\lambda < -2$ cannot happen. Here is the proof complete. #

Our investigation and search on the real axis is still going on in $\mathcal{O}$ as long as we haven't reached to the part of the free boundary on the left symmetric to the earlier one with respect to y-axis. At this stage we realize that $i^{-1}\lambda = -2$ is the only possible case left. Then $\lambda = -2i$ and the midpoint of

$$L(t) = a(t) + it, \quad m(t) = \mu - a(t) - it,$$

$$\text{for} \quad t = i\sigma, \quad \sqrt{K} > \sigma > 0$$

is $\frac{1}{2}(L(t) + m(t)) = \frac{\mu}{2}$ independent of σ. In particular as we've already also seen $b(0) = \mu - a(0)$, i.e. $\frac{\mu}{2} = \frac{1}{2}(a(0) + b(0))$. By the next theorem and symmetry with respect to y-axis we'll discover the midpoint $\frac{\mu}{2} = 0$.

<u>Theorem 5.18</u>. There is a new part Γ_2 of the free boundary passing through $b(0)$ which is the mirror reflection of the old one (passing through $a(0)$) with respect to the vertical line $x = \frac{\mu}{2} = L(i\sqrt{K})$. Thereby $\mu = 0$ and $b(0) = -a(0)$.

<u>Proof</u>: Since $\lambda = -2i$, thus $m(t) = \mu - a(t) - it$. This formula, for t real, determines a curve passing through $m(0) = b(0) = \mu - a(0)$ (as a vertex which we already know belongs to the free boundary) concave to the left and symmetric with respect to x-axis because a is even.

m which was the analytic continuation of L along the paths of type $\mathcal{C}$ determines the upper (lower) half of the new curve for negative (positive) t, and clearly for $|t|$ small enough but $\frac{1}{i} t > 0$ we have

$$m(t) > \mu - a(0) = b(0) = m(0) .$$

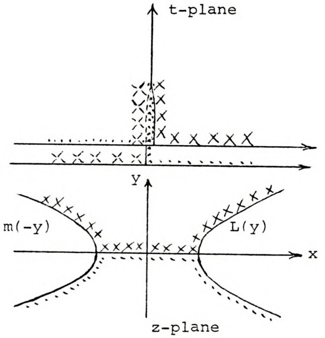

Figure 5.16

Let's try to show that the new curve is the left part of the free boundary. We have

$$m'(t) = -a'(t) - i \neq 0, \quad m \text{ is one-to-one,}$$

$$\text{for } t \text{ real .}$$

Thus by Lemma 5.2 there is a strip neighborhood of any segment of the real axis, centered at 0, in which m is a conformal mapping taking the upper (lower) part of the neighborhood to the right (left) hand side of the new curve. The inverse of m is L^{-1}. Along the new curve (the image of the real-axis under m) we have $L^{-1}(z) = -\mathrm{Im}\, z = -y$. For z on the new curve and taking the integration path on this curve we get

$$[L^{-1}(z)]^2 - y^2 = 0 \Leftrightarrow \nabla\left(\frac{1}{y}\, u(z) - \varphi(z)\right) = 0 ,$$

$$\frac{1}{y}\, u(z) - \varphi(z) = \frac{2}{3}\, y^2 - \frac{2}{y}\, \mathrm{Im} \int_{b(0)}^{z} [L^{-1}(\zeta)]^2 d\zeta$$

$$= \frac{2}{3}\, y^2 - \frac{2}{y}\, \mathrm{Im} \int_{b(0)}^{z} (\mathrm{Im}\, \zeta)^2 dz = 0 ,$$

where the first equality is shown by computations in the proof of Theorem 3.9. Moreover

$$\frac{\partial^2 (\frac{1}{y}\, u-\varphi)}{\partial n^2} = \frac{\partial^2}{\partial n^2}\left(\frac{u-\psi}{y}\right) = \frac{1}{y}\, \frac{\partial^2 (u-\psi)}{\partial n^2} = \frac{1}{y}\, \Delta(u - \psi) = 4 > 0$$

on the new curve. The latter three results about the new curve passing through $b(0) = m(0)$ imply that this new curve is part of the free boundary. This new free boundary is symmetric to the earlier one with respect to the vertical line $x = \frac{\mu}{2}$. Since we have symmetry with respect to both axis, thus $\frac{\mu}{2} = 0$, i.e., $\mu = 0$, $B(0) = -a(0)$, and $m(t) = -a(t) - it$. Moreover

$$L(i\sqrt{K}) = m(i\sqrt{K}) \Rightarrow a(i\sqrt{K}) + i(i\sqrt{K})$$

$$= \mu - a(i\sqrt{K}) - i(i\sqrt{K}) \Rightarrow \mu = 2a(i\sqrt{K}) + 2i(i\sqrt{K})$$

$$\Rightarrow L(i\sqrt{K}) = a(i\sqrt{K}) + i(i\sqrt{K}) = \frac{\mu}{2} = 0$$

which was expected by the symmetry with respect to y-axis. Hence the result. #

<u>Remark 5.19</u>. If we started by considering a piece of arc on the left (new) free boundary, which specifically starts with the branch of a'' negative at 0; $m(t) = -a(t) - it$, and assumed the noncontact set on the right, then we could make a similar study and by analytic continuation of m^{-1} we could reach to the right free boundary: $L(t) = a(t) + it$. In both cases cross signs (x) are mapped into cross signs and dot signs are mapped into dot signs in Figure 5.16. The signed area in the z-plane are contained in $\tilde{W}$. We've already seen that $\zeta^4 + 2A\zeta^2 + B$ cannot have a simple root $i\sigma$, $0 \leq \sigma \leq \sqrt{K}$, and it can't have a double root at 0, $i\sqrt{K}$.

<u>Remark 5.20.</u> (a list of necessary conditions for Problem 5.1). So far we have derived some necessary conditions for problem 5.1 as follows:

(i) $f(t) = \frac{2}{\pi} \frac{\alpha^2 - t^2}{(t^2+K)^{3/2}} \phi(t^2)$, where $A = -\alpha^2$, $B = \alpha^4$,

(ii) $a'(y_0) = \int_0^{y_0} f(\zeta)d\zeta$,

(iii) $\int_{\mathcal{C}} f(\zeta)d\zeta = -2i$,

(iv) $a(0) = \frac{1}{2}\int_{\mathcal{C}} \zeta f(\zeta)d\zeta > 0$

(v) $a(0) = a(y_0) - y_0\, a'(y_0) + \int_0^{y_0} \zeta f(\zeta)$

(this is obtained from $a(y_0) = a(0) + \int_0^{y_0}(y_0 - \zeta)f(\zeta)d\zeta$) .

Note using (iii) we can show that (iv) is equivalent to $a(0) = \sqrt{K} - \int_0^{i\sqrt{K}} (i\sqrt{K} - \zeta)f(\zeta)d\zeta$.

We realized that there is only one constant of integration arbitrary and the other one is imposed by the symmetry (e.g. by $a'(0) = 0$). Recall that in the unperturbed problem case also, given ϕ_0, the local solution was not unique by depending on one parameter c as stated in the Remark following Definition 2.8.1. #

Lemma 5.21. Except perhaps near the semicircle the mapping L is conformal in a semi-circular domain T bounded by γ consisting of segments $[0,y_0]$, $[0,i\sqrt{K}]$ in the upper Riemann sheet, segments $[-y_0,0]$, $[0,i\sqrt{K}]$ in the lower one, and the semi-circle centered at 0 (in both sheets) and radius y_0 started from y_0 in the upper Riemann sheet and ended with $-y_0$ in the lower one. The image will be bounded from below by $[-a(0),a(0)]$, Γ_1, Γ_2.

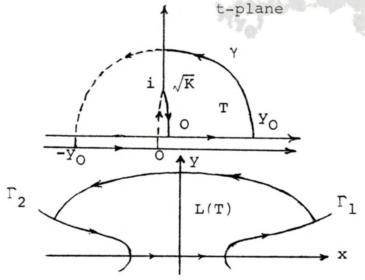

Figure 5.17

<u>Proof</u>: The mapping L is analytic in T, continuous up to the boundary. L takes the lower part of the boundary of T to $[-a(0),a(0)]$ and $(\Gamma_1 \cup \Gamma_2) \cap \{(x,y) : y > 0\}$. Recall that by assumptions of problem 5.1, $a(y_0)$, $a'(y_0)$ are near x_0, $\frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\Big|_{y_0}$ (in unperturbed problem) respectively. For any t on the semicircle we have

for unperturbed case:

$$L_u(t) = it + a_u(y_0) + (t - y_0)a_u'(y_0) + \int_{y_0}^{t} (t - \zeta)f_0(\zeta)d\zeta$$

$$= it + x_0 + (t - y_0)\left[\frac{d}{dt}(H(t) - \frac{2}{\pi} t \log t)\right]_{t=y_0}$$

$$- \frac{2}{\pi} \int_{y_0}^{t} (t - \zeta) \frac{\phi_0(\zeta^2)}{\zeta} d\zeta$$

for perturbed case:

$$L(t) = it + a(y_0) + (t - y_0)a'(y_0) + \int_{y_0}^{t} (t - \zeta)f(\zeta)d\zeta$$

$$= it + a(y_0) + (t - y_0)a'(y_0)$$

$$+ \frac{2}{\pi} \int_{y_0}^{t} (t - \zeta) \frac{\alpha^2 - \zeta^2}{(t^2+K)^{3/2}} \phi(\zeta^2)d\zeta \ ,$$

where the path of integration can be chosen on γ starting from 0 on the positive direction. Since K, α^2 are small, in comparison with $y_0 = |t| = |\zeta|$, thus the integrands are in both cases uniformly close; and therefore $L_u(t)$, $L(t)$ are also uniformly close, for all t on the semicircle. But the image of the semicircle is a simple path in the unperturbed case as we showed in Example 2.5 (Figure 2.3). Hence the image of the semicircle under L is uniformly close to the above simple path.

Now consider the topological degree (or winding number) $d(L,T,Z)$ for any $Z \notin L(\gamma)$. This is zero in the unbounded component of $\mathbb{C} - L(\gamma)$ and it is one in the major bounded component of $\mathbb{C} - L(\gamma)$ which is bounded by Γ_1, Γ_2, $[-a(0),a(0)]$ from below (note there may be small bounded components bounded only by the image of the semicircle under the map L). Hence L is a conformal mapping in the preimage T_0 of the above major bounded component of $\mathbb{C} - L(\gamma)$. (One can show directly, as in Example 2.5, that $L(\gamma)$ is a simple curve here also). #

We can argue similarly about the other semi-circular domain which is mapped below Γ_1, $[-a(0),a(0)]$, Γ_2 by the analytic continuation of L. Hence we have the following:

Theorem 5.22. L^{-1} is a conformal mapping in a neighborhood of the origin, more specifically in $L(T_0) \cup [-a(0),a(0)] \cup \overline{L(T_0)} \cup \Gamma_1 \cup \Gamma_2$, where by bar here we mean the complex conjugate. We can assume the disk $\mathcal{O}$ is inside this neighborhood. Moreover the free boundary consists of $\Gamma_1 \cup \Gamma_2$ only.

Proof: The first part is clear from the above argument. The last part is simply because $L^{-1}(z) = \pm \operatorname{Im} z$ can happen only on $\Gamma_1 \cup \Gamma_2$. #

Sufficient conditions in problem 5.1 continue

Remark 5.23. By using the necessary conditions derived so far and stated in the Remark 5.20, we could think of process after Corollary 5.6 as part of the establishing the sufficient conditions for having a local solution $v = \frac{1}{y} u$ in the following sense. We could establish a local solution near $a(0)$, Γ_1 as explained in Remark 5.7, and we could continue it along a strip neighborhood of $[-a(0),a(0)]$, where L^{-1} is analytic and imaginary valued and the local solution is bigger than φ except at $\pm a(0)$. We could then continue the

construction of the local solution to agree with the local solution near $-a(0)$, Γ_2 without any need to Lemmas 5.15, 5.16, 5.17, and Theorem 5.18 because we have now the formula for Γ_2 which is represented by m, the analytic continuation of L.

We can then apply Lemma 5.21 and Theorem 5.22 to show that L^{-1} is a conformal mapping in the subset of $\mathcal{O}$ bounded between Γ_1 and Γ_2. Thus the formula for the local solution can be extended to the whole disk $\mathcal{O}$. In order to show $\frac{1}{y} u = v$ is a local solution in the sense stated in Problem 5.1, it is enough to show

$\frac{1}{y} u - \varphi > 0$ in the domain bounded between Γ_1, Γ_2, or

$u - \psi > 0$ away from $(-a(0), a(0))$ there. #

Theorem 5.24. $\frac{1}{y} u - \varphi > 0$ in the domain $L(T_0) \cup [-a(0), a(0)] \cup \overline{L(T_0)}$, where by bar here we mean the complex conjugation.

Proof: It is enough to show $u - \psi > 0$ away from $(-a(0), a(0))$ in $L(T_0)$, because we can do similarly to show $u - \psi < 0$ away from this segment in $\overline{L(T_0)}$ by symmetry of L, L^{-1}, and also because $\frac{1}{y} u - \varphi > 0$ in a uniform strip around $(-a(0), a(0))$ bounded by Γ_1, Γ_2. Let us first investigate the sign of $u - \psi$ along the image of the arc segment $t = y_0 e^{i\theta}$, $0 \leq \theta \leq \theta_0$,

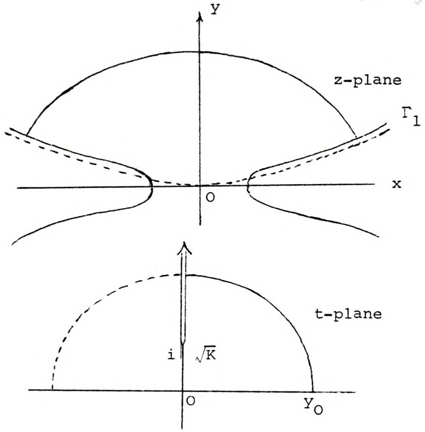

Figure 5.18

under the mapping L. Put $\theta = \frac{s}{y_0}$. Let $\lambda(s) = L(t(s)) = L(y_0 e^{i\frac{s}{y_0}})$ and $F(s) = u(\lambda(s)) - \psi(\lambda(s))$. Then $F(0) = 0$, $F'(0) = \nabla(u - \psi) \cdot \lambda'(s)\big|_{s=0} = 0$, and $F''(0) = \Delta(u - \psi)\|\lambda'(s)\|^2\big|_{s=0} = -\Delta\psi\|\lambda'(0)\|^2$ which is near $4y_0\|\lambda_u'(0)\|^2$ where the index u indicates unperturbed case. Thus $u - \psi > 0$ along $\lambda(s)$ uniformly with respect to perturbation parameters. Similarly we can argue and

show $u - \psi > 0$ along the image of the arc segment $t = y_0 e^{i\theta}$, $\pi - \theta_0 \leq \theta \leq \pi$ under the mapping L.

To show $u - \psi > 0$ elsewhere on $L(t)$, ie.., for $t = y_0 e^{i\theta}$, $\theta_0 \leq \theta \leq \pi - \theta_0$, let $z_p = a(y_0) + iy_0$. Then for any z on this arc we have

for unperturbed solution:

$$u_u(z) = \psi(z_0) - \frac{1}{2} \operatorname{Im} \int_{z_0}^{z} [\zeta^2 + 4(L_u^{-1}(\zeta))^2] d\zeta$$

for perturbed candidate:

$$u(z) = \psi(z_p) - \frac{1}{2} \operatorname{Im} \int_{z_p}^{z} [\zeta^2 + 4(L^{-1}(\zeta))^2] d\zeta \ .$$

Thus

$$\begin{aligned} u_u(z) - u(z) &= \psi(z_0) - \psi(z_p) - \frac{1}{2} \operatorname{Im} \int_{z_0}^{z_p} \zeta^2 d\zeta \\ &\quad - 2 \operatorname{Im} \int_{z_0}^{z} (L_u^{-1}(\zeta))^2 d\zeta \\ &\quad + 2 \operatorname{Im} \int_{z_p}^{z} (L^{-1}(\zeta))^2 d\zeta \\ &= \psi(z_0) - \psi(z_p) - \frac{1}{2} \operatorname{Im} \int_{z_0}^{z_p} \zeta^2 d\zeta \\ &\quad - 2 \operatorname{Im} \int_{z_p}^{z} [(L_u^{-1}(\zeta))^2 - (L^{-1}(\zeta))^2] d\zeta \\ &\quad - 2 \operatorname{Im} \int_{z_0}^{z_p} (L_u^{-1}(\zeta))^2 d\zeta \ . \end{aligned}$$

Since z_p is near z_0, thus $\psi(z_0) - \psi(z_p)$, $-\frac{1}{2}\,\mathrm{Im}\int_{z_0}^{z_p} \zeta^2 d\zeta$, and $-2\,\mathrm{Im}\int_{z_0}^{z_p}(L_u^{-1}(\zeta))^2 d\zeta$ are small. Putting $L_p^{-1}(\zeta) = t$ we get $(L_u^{-1}(\zeta))^2 - (L^{-1}(\zeta))^2 = [L_u^{-1}(L(t))]^2 - t^2$. Since $L(t)$ is uniformly (with respect to t) close to $L_u(t)$, thus $L_u^{-1}(L(t)) - t$ is near zero uniformly with respect to t. Hence $u_u(z) - u(z)$ is near zero for all such z. This implies $(u_u - \psi) - (u - \psi)$ is near zero for all z on the boundary. Since $u_u - \psi > 0$ on the noncontact set of the unperturbed problem, thus away from $\Gamma_1 \cup \Gamma_2$ on the arc we have $u - \psi > 0$. Hence $u - \psi \geq 0$ on the image of the semicircle under the map L vanishing only at z_p and $-\bar{z}_p$.

Now we want to show $u - \psi > 0$ everywhere in the domain bounded by Γ_1, Γ_2, the segment $[-a(0), a(0)]$, and the image of the semicircle under L. Suppose not. Then $u - \psi$ takes its minimum at some point in the interior of the domain. At such a point $\nabla(u - \psi) = 0$. Since

$$u(z) - \psi(z) = \frac{2}{3} y^3 - 2\,\mathrm{Im}\int_{a(0)}^{z} (L^{-1}(\zeta))^2 d\zeta$$

thus $\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x} = -2\,\mathrm{Im}[L^{-1}(z)]^2 = 0$, $\frac{\partial u}{\partial y} - \frac{\partial \psi}{\partial y} = 2y^2 - 2\,\mathrm{Re}[L^{-1}(z)]^2 = 0$. This shows $[L^{-1}(z)]^2 = y^2$, which means z on $\Gamma_1 \cup \Gamma_2$. This contradiction completes the proof.

Conclusions

5.25. Summary: We want to put the answer we got for the Problem 5.1 in a summary. Given ϕ near ϕ_0 in Z, and parameters K, $A = -\alpha^2$, $B = A^2$ fixed, as stated in the Remark 5.20, there is only one constant of integration to be chosen for the equation $a''(y) = f(y)$ because the other one is determined by symmetry. More specifically we observed that $a'(0) = 0$ and $a(0) > 0$ was to be chosen. We need also to see this point by considering the initial data at y_0 (as we did already) in the following way. We shall see that choosing $a'(y_0)$ near $\frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\Big|_{y=y_0}$, then (ii) and (iii) in the Remark 5.20 will determine α^2, $\phi(0)$, with $\frac{\phi(0) - 1}{(-1)^r \phi^{(r+1)}(0)} > 0$, in terms of other parameters, where $\phi^{(r+1)}$ is the first derivative of ϕ nonzero at 0 and (iv) will give $a(0)$. Then $a(y_0)$ will be completely determined by (v), that is, we have only one constant of integration, $a'(y_0)$, to be chosen. We stress that we choose $a'(y_0) - \frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\Big|_{y=y_0}$ as close to zero as we wish. We will show that $a(y_0)$ will then be close to x_0 and the answer to the Problem 5.1 will be unique.

Lemma 5.26. From (ii) in the Remark 5.20 we can find α^2 as a function of K, ϕ (or as a function of

$K, \alpha_0, \alpha_1, \alpha_2, \ldots,$ where $\phi(t^2) = \sum_{j=0}^{\infty} \alpha_j t^{2j}$). Moreover we have asymptotically $\frac{\alpha^2}{K} = (\log \frac{y_0}{\sqrt{K}}) O^+(1)$, where O^+ is the usual O with positive leading term.

Proof: By (ii) we have $a'(y_0) + \int_0^{y_0} \frac{\zeta^2 \phi(\zeta^2)}{(\zeta^2+K)^{3/2}} d\zeta = \alpha^2 \int_0^{y_0} \frac{\phi(\zeta^2)}{(\zeta^2+K)^{3/2}} d\zeta$. Put $\zeta = \xi\sqrt{K}$. Then

$$a'(y_0) + \int_0^{\frac{y_0}{\sqrt{K}}} \frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi = \frac{\alpha^2}{K} \int_0^{\frac{y_0}{\sqrt{K}}} \frac{\phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi > 0 , \tag{*}$$

where the integral on the right hand side stays away from zero because ϕ is near ϕ_0 in Z. This shows α^2 as a function of K, ϕ (or parameters $K, \alpha_0, \alpha_1, \alpha_2, \ldots$). To get the asymptotic representation, observe that for the integral on the left hand side we have

$$\int_0^{\frac{y_0}{\sqrt{K}}} \frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi = \int_0^1 \frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi + \int_1^{\frac{y_0}{\sqrt{K}}} \frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi$$

$$= \int_0^1 \frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} d\xi + \phi(0) \int_1^{\frac{y_0}{\sqrt{K}}} \frac{d\xi}{\xi}$$

$$+ \int_1^{\frac{y_0}{\sqrt{K}}} \left[\frac{\xi^2 \phi(K\xi^2)}{(\xi^2+1)^{3/2}} - \frac{\alpha_0}{\xi} \right] d\xi$$

$$= \int_0^1 \frac{\xi^2\phi(K\xi^2)}{(\xi^2+1)^{3/2}}\, d\xi + \phi(0) \log \frac{y_0}{\sqrt{K}}$$

$$+ \int_1^{\frac{y_0}{\sqrt{K}}} \left[\frac{\xi^2\phi(K\xi^2)}{(\xi^2+1)^{3/2}} - \frac{\alpha_0}{\xi}\right] d\xi \ ,$$

where the first integral on the right is positive and bounded (away from zero) for small K, and the integrand in the second one is:

$$\frac{\xi^3[\phi(K\xi^2) - \phi(0)] + \phi(0)[\xi^3 - (\xi^2+1)^{3/2}]}{\xi(\xi^2 + 1)^{3/2}}$$

But $(\xi^2 + 1)^{\frac{3}{2}} - \xi^3 \quad \dfrac{3\xi^4 + 3\xi^2 + 1}{(\xi^2+1)^{3/2} + \xi^3}$ and therefore

$$\int_1^{\frac{y_0}{\sqrt{K}}} \left[\frac{\xi^2\phi(K\xi^2)}{(1+\xi^2)^{3/2}} - \frac{\alpha_0}{\xi}\right] d\xi$$

$$= \int_1^{\frac{y_0}{\sqrt{K}}} \frac{\xi^2[\phi(K\xi^2) - \phi(0)]}{(\xi^2 + 1)^{3/2}}\, d\xi$$

$$+ \int_1^{\frac{y_0}{\sqrt{K}}} \frac{\phi(0)(3\xi^4 + 3\xi^2 + 1)d\xi}{\xi(\xi^2+1)^{3/2}[\xi^3 + (\xi^2+1)^{3/2}]} \ ,$$

where the second integral on the right hand side is positive and bounded for small K. For the first integral on the

right hand side, by going back to the ζ variable we have

$$\int_{\sqrt{K}}^{y_0} \frac{\zeta^2[\phi(\zeta^2) - \phi(0)]}{(\zeta^2 + K)^{3/2}}\, d\zeta,$$ which is bounded for small k

(easy to check). Substituting these results back in (*) yields

$$\phi(0) \log \frac{y_0}{\sqrt{K}} + O(1) = \frac{\alpha^2}{K} \int_0^{\frac{y_0}{\sqrt{K}}} \frac{\phi(K\xi^2)}{(1+\xi^2)^{3/2}}\, d\xi \qquad (**)$$

Hence asymptotically (as K approaches zero) we have:
$\frac{\alpha^2}{K} = (\log \frac{y_0}{\sqrt{K}}) O^+(1)$, where O^+ is the usual O with the leading term of $O^+(1)$ positive (as we see above).

<u>Remark.</u> If we put $\phi(t^2) = \phi(0)\theta(t^2)$, then (**) will have the form

$$\frac{\alpha^2}{K} \int_0^{\frac{y_0}{\sqrt{K}}} \frac{\theta(K\xi^2)}{(1+\xi^2)^{3/2}}\, d\xi = \log \frac{y_0}{\sqrt{K}} + \frac{1}{\alpha_0} O(1) ,$$

$$\theta(t^2) = \sum_{j=0}^{\infty} \beta_j t^{2j} , \qquad \beta_j = \frac{\alpha_j}{\alpha_0} .$$

<u>Lemma 5.27</u>. From (iii) in the Remark 5.20 we can find α_0 as a function of K, β_j, for all integers $j \geq 1$, with $\frac{\mu - 1}{\beta_{r+1}(-1)^{r+1}} \geq 0$, where $\mu = \frac{1}{\alpha_0}$ and β_{r+1} is the first nonzero among $\beta_1, \beta_2, \beta_3, \ldots$. Moreover

$\alpha_0 = 1$ if and only if $\beta_j = 0 \; \forall \; j \geq 1$ or $K = 0 = \alpha$.

Asymptotically we have $1 - \frac{1}{\alpha_0} = \beta_1 K[(\log \frac{y_0}{\sqrt{K}})o^+(1) + \frac{3}{4} + O(K \log \frac{y_0}{\sqrt{K}})]$ if $\beta_1 \neq 0$, and $(1 - \frac{1}{\alpha_0})(-1)^r = \beta_{r+1} K^{r+1}[(\log \frac{y_0}{\sqrt{K}})o^+(1) + o^+(1) + O(K \log \frac{y_0}{\sqrt{K}})]$ if $\beta_1 = \beta_2 = \cdots = \beta_r = 0$ and $\beta_{r+1} \neq 0$, $r \geq 1$.

<u>Proof</u>: Since $\frac{2}{\pi} \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \frac{(\alpha^2 - \zeta^2)}{(\zeta^2+K)^{3/2}} \phi(\zeta^2) d\zeta =$
$\frac{2}{\pi} \int_{\mathcal{C}_1} \frac{(\alpha^2-\zeta^2)\phi(\zeta^2)}{(\zeta^2 + K)^{3/2}} d\zeta + \frac{2}{\pi} \int_{\mathcal{C}_1} \frac{(\alpha^2-\zeta^2)\phi(\zeta^2)}{-(\zeta^2 + K)^{3/2}} (-d\zeta)$, where ζ is changed to $-\zeta$ in the last integral, thus

$\int_{\mathcal{C}_1 \cup \mathcal{C}_2} f(\zeta) d\zeta = \frac{2}{\pi} \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \frac{(\alpha^2 - \zeta^2)}{(\zeta^2+K)^{3/2}} \phi(\zeta^2) d\zeta = -4i$ if and only if (iii) in the Remark 5.20 holds. If we now add the infinity point to each Riemann sheet in Figure 5.19-a, then the double Riemann sheet will topologically be equivalent to a sphere, as in the Figure 5.19-b, with

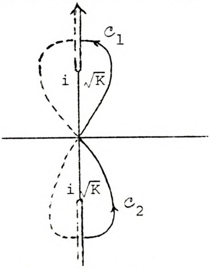

Figure 5.19-a

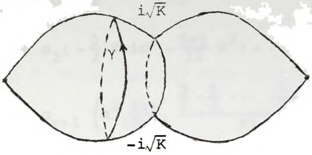

Figure 5.19-b

two (spherical) poles $\pm i\sqrt{K}$ on it. Then, we can integrate the above integral along the path γ, rather than $\mathcal{C}_1 \cup \mathcal{C}_2$, and the integrand has its singularity at infinity which is reduced to a singularity at zero if we make the change $\zeta = \frac{1}{\eta}$. Thus

$$\int_{\mathcal{C}_1 \cup \mathcal{C}_2} f(\zeta)d\zeta = \frac{2}{\pi} \int_\gamma \frac{(\alpha^2-\zeta^2)\phi(\zeta^2)}{(K+\zeta^2)^{3/2}}\, d\zeta$$

$$= \frac{2}{\pi} \sum_{j=0}^{\infty} \alpha_j \int_\gamma \frac{(\alpha^2-\zeta^2)}{(K-\zeta^2)^{3/2}}\, \zeta^{2j} d\zeta$$

$$= \frac{2}{\pi} \sum_{j=0}^{\infty} \alpha_j \left[\alpha^2 \int_{\partial B(0)} \frac{\eta^{1-2j}}{(1+K\eta^2)^{3/2}}\, d\eta - \int_{\partial B(0)} \frac{1}{(1+K\eta^2)^{3/2}} \frac{d\eta}{\eta^{2j+1}} \right],$$

where $\partial B(0)$ is the boundary of a small enough disk about 0 traversed in the positive direction Note $(1+K\eta^2)^{-\frac{3}{2}} = 1 - \frac{3}{2} K\eta^2 + \frac{\frac{3}{2}\cdot\frac{5}{2}}{2!} K^2\eta^4 - \frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!} + \cdots$. Hence

$$-4i = \frac{2}{\pi}\Bigg[-2\pi i\alpha_0 + \alpha_1(2\pi i\alpha^2 + \frac{3}{2}\pi iK) +$$

$$+ \alpha_2(-\frac{3}{2}K\pi i\alpha^2 - \frac{5\pi i}{32}K^2) + \cdots +$$

$$\alpha_{r+1}\Bigg(\alpha^2 \frac{2\pi i}{(2r)!} \frac{\frac{3}{2}\cdot\frac{5}{2}\cdots(\frac{3}{2}+r-1)}{r!}(-1)^r K^r$$

$$+ \frac{2\pi i}{(2r+2)!}\frac{\frac{3}{2}\cdot\frac{5}{2}\cdots\cdot(\frac{3}{2}+r)}{(r+1)!}(-1)^r K^{r+1} + \cdots\Bigg],$$

i.e., $\frac{1}{\alpha_0} - 1 = -\beta_1(\alpha^2 + \frac{3}{4}K) + \beta_2(\frac{3}{4}K\alpha^2 + \frac{5}{64}K^2) + \cdots +$

$$\beta_{r+1}\frac{\frac{3}{2}\cdot\frac{5}{2}\cdots(\frac{3}{2}+r-1)}{r!(2r)!}(-K)^{r+1}\left(\frac{\alpha^2}{K} + \frac{\frac{1}{2}(\frac{3}{2}+r)}{(r+1)^2(2r+1)}\right) + \cdots .$$

Thinking of this equation as a relation among $\mu = \frac{1}{\alpha_0}$, K, β_j, for all $j \geq 1$, and recalling from the last Remark, by (*) α^2 has only one term $K\frac{a'(y_0)}{\alpha_0} = \mu a'(y_0)K$ depending on μ. This term has small coefficient $Ka'(y_0)$, and therefore by applying the implicit function theorem in $\mu - 1 = -\beta_1(\alpha^2 + \frac{3}{4}K) + \beta_2(\frac{3}{4}K\alpha^2 + \frac{5}{64}r^2) + \cdots$ to find μ as a function of $K, \beta_1, \beta_2, \beta_3, \ldots$. We see from the equation

$$\mu - 1 = -\beta_1 K(\frac{\alpha^2}{K} + \frac{3}{4}) + \beta_2 K^2(\frac{3}{4}\frac{\alpha^2}{K} + \frac{5}{64}) + \cdots + \qquad (1)$$

$$\beta_{r+1}(-K)^{r+1}\frac{\frac{3}{2}\cdot\frac{5}{2}\cdots(\frac{3}{2}+r-1)}{r!(2r)!}$$

$$(\frac{\alpha^2}{K} + \frac{\frac{3}{4}+\frac{r}{2}}{(r+1)^2(2r+1)}) + \cdots$$

that if $\beta_{r+1} = \theta^{(r+1)}(0)$ is the first nonzero coefficient among $\beta_1, \beta_2, \beta_3, \ldots,$ then we must have $\frac{\mu - 1}{\beta_{r+1}(-1)^{r+1}} = \frac{1 - \alpha_0}{\alpha_{r+1}(-1)^{r+1}} \geq 0$ with equality case only when $K = 0$.

More precisely $\mu = 1$ if and only if all $\beta_j = 0$ or $K = 0$, i.e., if and only if $\phi \equiv 1$ or $K = 0 = \alpha = A = B$. Moreover asymptotically we have (by (1) above)

$$\frac{\mu-1}{-\beta_1} = \frac{\phi(0)-1}{\phi'(0)}$$

$$= K[(\log \frac{y_0}{\sqrt{K}})o^+(1) + \frac{3}{4} + O(K \log \frac{y_0}{\sqrt{K}})],$$

$$\text{if } \beta_1 \neq 0, \qquad (2)$$

and

$$\frac{(\mu-1)}{\beta_{r+1}}(-1)^{r+1} = \frac{\frac{3}{2}\cdot\frac{5}{2}\cdots(\frac{3}{2}+r-1)}{r!(2r)!} K^{r+1}[(\log \frac{y_0}{\sqrt{K}})o^+(1) + \frac{\frac{3}{4}+\frac{r}{2}}{(r+1)^2(2r+1)} + O(K \log \frac{y_0}{\sqrt{K}})]$$

if β_{r+1} is the first nonzero among $\beta_2, \beta_3, \beta_4, \ldots$. If all β_j, $j \geq 1$, are zero, then $\mu = 1$, $\alpha_0 = 1 = \phi(0)$ (no matter what K is - of course small). If $\beta_1 \neq 0$, then by (2) we have asymptotically the graph 5.20-a.

If $\beta_1 = 0 = \beta_2 = \cdots = \beta_r$ but $\beta_{r+1} \neq 0$ for some $r \geq 1$, then by (2) we have asymptotically the graph 5.20-b.

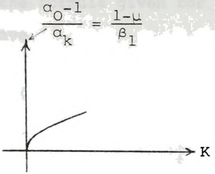

Figure 5.20-a

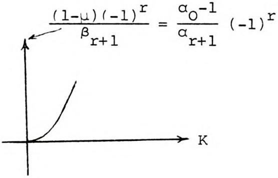

Figure 5.20-b

<u>Corollary 5.28</u>. Given K and θ as in the last Remark, Lemma 5.27 shows that μ (i.e., $\alpha_0 = \phi(0)$) can be found uniquely in terms of K, $\beta_j = \theta^{(j)}(0)$, $j \geq 1$. By Lemma 5.26 then α^2 can be found in terms of these parameters. #

<u>Lemma 5.29</u>. $a(0) = \sqrt{K}(\log \frac{y_0}{\sqrt{K}})0^+(1)$. Moreover $a(y_0)$ is really close to x_0.

Proof: By the formula given for $a(0)$ in the Remark 5.20 we have

$$a(0) = \sqrt{K} - \int_0^{i\sqrt{K}} (i\sqrt{K} - \zeta) f(\zeta) d\zeta$$

$$= \sqrt{K} + \frac{2}{\pi} \int_0^{\sqrt{K}} (\sqrt{K} - \sigma) \frac{(\alpha^2 + \sigma^2)\phi(-\sigma^2)}{(K - \sigma^2)^{3/2}} d\sigma$$

$$= \sqrt{K} + \frac{2}{\pi} \int_0^{\sqrt{K}} \frac{(\alpha^2 + \sigma^2)\phi(-\sigma^2)}{(\sqrt{K} - \sigma)^{1/2}(K + \sigma)^{3/2}} d\sigma$$

$$= \sqrt{K} + \frac{2}{\pi} \sqrt{K} \int_0^1 \frac{(\frac{\alpha^2}{K} + \xi^2)\phi(-K\xi^2) d\xi}{(1-\xi)^{1/2}(1+\xi)^{3/2}}$$

$$= \sqrt{K} + \frac{2}{\pi} \frac{\alpha^2}{\sqrt{K}} \int_0^1 \frac{\phi(-K\xi^2) d\xi}{(1-\xi)^{1/2}(1+\xi)^{3/2}} + \sqrt{K} \int_0^1 \frac{\xi^2 \phi(-K\xi^2) d\xi}{\sqrt{1-\xi}\sqrt{(1+\xi)^3}}$$

Recall that the leading term of $\phi(\zeta^2)$ is $\alpha_0 = \frac{1}{\mu} = \phi(0)$. Thus by substituting the value of α^2 from Lemma 5.26 we get $\sqrt{K}(\log \frac{y_0}{\sqrt{K}}) 0^+(1)$ as the leading term for $a(0)$:

$$a(0) = \sqrt{K} + \sqrt{K}(\log \frac{y_0}{\sqrt{K}}) 0^+(1) + \sqrt{K} \int_0^1 \frac{\xi^2 \phi(-K\xi^2)}{\sqrt{1-\xi}\sqrt{(1+\xi)^3}} d\xi$$

(negligible)

This shows the first part of the Lemma. For the second part observe that the Remark following Definition 2.8.1 indicates for the unperturbed problem $a_u(y) = -\frac{2}{\pi} y \log y + (c + \frac{2}{\pi})y - \frac{2}{\pi} \psi_1(y) \Rightarrow a_u(y_0) = y_0(c - \frac{2}{\pi} \log y_0) + \frac{2}{\pi} y_0 - \frac{2}{\pi} \psi_1(y_0)$, where $\psi_1(y) = 0(y^3)$ is analytic at 0.

For the perturbed problem

$$a(y) = a(0) + \int_0^y (y - \zeta)f(\zeta)d\zeta \Rightarrow$$

$$a(y_0) = a(0) + y_0 a'(y_0) - \int_0^{y_0} \zeta f(\zeta)d\zeta$$

By assumption of the original problem $a'(y_0)$ is close to $c - \frac{2}{\pi}\log y_0 = a_u'(y_0)$. Since $a(0)$ is negligible (by the above estimate), thus in order to show $a_u(y)$ is close to $a(y)$ we only need to compare $-\int_0^{y_0}\zeta f(\zeta)d\zeta$ with $\frac{2}{\pi} y_0 - \frac{2}{\pi}\psi_1(y_0)$. We proceed as follows

$$- \int_0^{y_0} \zeta f(\zeta)d\zeta = -\frac{2}{\pi}\int_0^{y_0} \zeta \frac{\alpha^2 - \zeta^2}{(\zeta^2+K)^{3/2}} \phi(\zeta^2)d\zeta$$

$$= -\frac{2}{\pi}\int_0^{y_0} \frac{\alpha^2\zeta}{(K+\zeta^2)^{3/2}} \phi(\zeta^2)d\zeta$$

$$+ \frac{2}{\pi}\int_0^{y_0} \frac{\zeta^3}{(K+\zeta^2)^{3/2}} \phi(\zeta^2)d\zeta$$

$$= -\frac{2}{\pi}\int_0^{y_0} \frac{\alpha^2\zeta}{(K+\zeta^2)^{3/2}} \phi(\zeta^2)d\zeta$$

$$+ \frac{2}{\pi}\int_0^{y_0} \left[\frac{\zeta^3}{(K+\zeta^2)^{3/2}} - 1\right]\phi(\zeta^2)d\zeta$$

$$+ \frac{2}{\pi}\int_0^{y_0}(\phi(\zeta^2) - 1 + 1)d\zeta$$

$$= -\frac{2}{\pi}\frac{\alpha^2}{\sqrt{K}}\int_0^{\frac{y_0}{\sqrt{K}}} \frac{\xi\phi(K\xi^2)}{(1+\xi^2)^{3/2}}\, d\xi$$

$$- \frac{2}{\pi}\sqrt{K}\int_0^{\frac{y_0}{\sqrt{K}}} \frac{(1+3\xi^2+3\xi^4)\phi(K\xi^2)}{(1+\xi^2)^{3/2}[\xi^3+(1+\xi^2)^{3/2}]}\,d\xi$$

$$+ \frac{2}{\pi}\int_0^{y_0}(\phi(\zeta^2)-1)d\zeta + \frac{2}{\pi}\,y_0 \ .$$

The last term hereis $\frac{2}{\pi}\,y_0$, therefore we only need to compare the other terms with $-\frac{2}{\pi}\,\psi(y_0)$. The first and the second terms on the right hand side above are negligible because of coefficients and the bounded integrals for small K. For the last integral $\frac{2}{\pi}\int_0^{y_0}(\phi(\zeta^2)-\phi(0))d\zeta +$ $\frac{2}{\pi}\int_0^{y_0}(\phi(0)-1)d\zeta$, where $\phi(0) = \alpha_0$ by our last power series expansion of $\phi(\zeta^2)$. The first term is close enough to $-\frac{2}{\pi}\,\psi_1(y_0)$ (both with leading qubic term, at least, with respect to y_0), and the second term $\frac{2}{\pi}(\alpha_0 - 1)y_0$ is negligible by the results of the Lemma 5.2.7. Here is the proof complete. #

<u>Remark</u>. As the above proof shows, the difference of $a(y_0)$, $x_0 = a_u(y_0)$ is $O(K^r) + O(y_0^3)$ for any $0 < r < \frac{1}{2}$. #

What we have proved so far can be summarized into the following (main result) theorem:

<u>Theorem 5.30</u>. Fix $z_0 = x_0 + iy_0$, $x_0 > 0$, $y_0 > 0$, on the unperturbed free boundary Γ_1 (in Example 2.5).

Consider A, B, K near 0, $K > 0$, and ϕ near ϕ_0 in Z, and f(y) with the branch of f for which $f(y_0) < 0$, and $a(y_0) + iy_0$ near $z_0 = x_0 + iy_0$, $a'(y_0)$ near $\frac{d}{dy}(H(y) - \frac{2}{\pi} y \log y)\Big|_{y=y_0}$. Then a necessary and sufficient condition that there exists a symmetric local solution in $\mathcal{O}$, an open fixed disk (independent of parameters K,A,B) about the origin contained in the domain of the local solution in Example 2.5, with part of the free boundary

$$\Gamma_1 : \begin{cases} x = a(y), & a''(y) = f(y) \\ a'(y)\Big|_{y_0} = a'(y_0), & a(y)\Big|_{y_0} = a(y_0) \end{cases}$$

near y_0 is

(i) $f(t) = \frac{2}{\pi} \frac{\alpha^2 - t^2}{(t^2+K)^{3/2}} \phi(t^2)$, where $A = -\alpha^2$, $B = \alpha^4$, $\alpha > 0$

(ii) $a'(y_0) = \int_0^{y_0} f(\zeta)d\zeta$

(iii) $\int_{\mathcal{C}} f(\zeta)d\zeta = -2i$

(iv) $a(y_0) = a(0) + y_0 a'(y_0) - \int_0^{y_0} \zeta f(\zeta)d\zeta$ near x_0, where

$$a(0) = \frac{1}{2} \int_{\mathcal{C}} \zeta f(\zeta)d\zeta = \sqrt{K} - \int_0^{i\sqrt{K}} (i\sqrt{K} - \zeta)f(\zeta)d\zeta .$$

Moreover these four conditions are consistent and given K and $\vartheta = \frac{1}{\phi(0)} \phi$ fixed, $\alpha_0 = \phi(0)$, $\alpha^2 = -A$, $B = A^2$

can be found uniquely as a function of the given parameters $(K, \theta'(0), \theta''(0), \ldots, \theta^{(r)}(0), \ldots)$. In particular the set of admissible parameters for the above necessary and sufficient conditions is not vacuous. Furthermore given the above fixed parameters the local solution is unique with only one constant of integration: $a'(y_0)$ to be chosen near $a_u'(y_0)$. #

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