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INVARIANT CURVES FOR NUMERICAL METHODS
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HAI THANH DOAN

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SHUI N. CHOW
Major professor

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INVARIANT CURVES FOR NUMERICAL METHODS AND THE HOPF BIFURCATION

By

Hai Thanh Doan

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ABSTRACT

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METHODS AND THE HOPF BIFURCATION

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Hai Thanh Doan

We consider the problem of tracking a family of periodic orbits using numerical methods.

First it is shown that if an one parameter family $\dot{x} = f(\alpha,x)$ has a family of periodic orbits bifurcating from a stationary solution then when $\dot{x} = f(\alpha,x)$ is approximated by a convergent single-step method, the resulting difference equation possesses a family of invariant curves bifurcating from the same stationary solution. The result is then extended to convergent, strongly stable, linear multistep methods. The results also show that the rates of convergence toward the invariant curves are roughly the same for all these methods. Finally, we discuss the time delayed equations.

To

My Parents

and

anh Son, Long, Van, Lam

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INTRODUCTION

Many physical and biological systems are governed by differential equations depending on a parameter. The state variable x satisfies an equation of the form

$$(*) \qquad \dot{x}(t) = f(\alpha, x(t))$$

where $\alpha \in \mathbb{R}$, $x \in$ some Banach space, usually $\mathbb{R}^n$. Of particular interest to us is the case when $(*)$ possesses a family of periodic orbits bifurcating from some stationary solution. The global behavior of these orbits have been studied by several authors (see [1], [3], [12]). Generally speaking, since f is nonlinear it is impossible to compute such behavior explicit. Often one has to resort to some numerical procedure. Typically $(*)$ is then approximated by some difference equation of the form

$$(**) \qquad x_{m+1} = F(\alpha, x_{m+1-k}, \ldots, x_m)$$

It is reasonable to expect that under favorable circumstances the solutions of $(**)$ stay close to the periodic orbits of $(*)$.

In this present work we will show that under appropriate conditions, (**) possesses a family of invariant curves.

In chapter 1 we consider the case when (**) arises from some convergent, single-step method. First we show that if $x \in \mathbb{R}^2$ and (*) has a Hopf bifurcation of periodic orbits at α_0 then generically, (**) has a Hopf bifurcation of invariant curves at $\alpha_0 + O(h)$. The case $x \in \mathbb{R}^n$, $n \geq 3$ follows from the center manifold theorem. We then establish the existence and continuation of invariant curves for (**) away from the bifurcation point. This means that (**) has a family of invariant curves which is at least piecewise continuous in α. These curves may be thought of as the approximation to the family of periodic orbits of (*).

In chapter 2 we extend the above results to convergent, strongly stable, multistep methods.

Finally, in chapter 3 we concentrate on time-delayed equations. Some numerical implications will also be discussed.

1. SINGLE-STEP METHODS

1.1 Preliminaries

In this section we shall provide some basic definitions that will be needed throughout the chapter. Mainly we will be concerned with a system of O.D.E.

$$(1.1) \qquad \dot{x} = f(x); \quad x \in \mathbb{R}^n;$$

f is sufficiently smooth,

together with a finite approximation arising from some explicit, single-step method. This yields a difference equation of the form

$$(1.2) \qquad x_{m+1} = x_m + h\phi(h, x_m)$$

where h is the step size of the method employed.

Definition 1.1.1: (1.2) is said to be a convergent approximation of (1.1) if the solution of $\{x_n\}$ of (1.2) with starting point x_0 satisfies

$$\lim_{\substack{h \to 0 \\ nh \to t}} x_n = x(t)$$

where $x(t)$ is the theoretical solution of the initial value problem

$$\dot{x}(t) = f(x(t)); \quad x(0) = x_0$$

Definition 1.1.2: (1.2) is said to be a consistent approximation of (1.1) if $\lim_{h\to 0} \phi(h,x) = f(x)$.

The following theorem is standard and can be found, for example, in [9].

Theorem 1.1.3: (1.2) is convergent iff it is consistent.

In this chapter we are primarily interested in obtaining the invariant curves for (1.2). Thus we need the following.

Definition 1.1.4: A curve $\Gamma \subset \mathbb{R}^n$ is said to be invariant under (1.2) if $T(\Gamma) = \Gamma$ where T is the mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by (1.2).

Definition 1.1.5: An invariant curve Γ of T is said to be attracting under T if $T^k(x)$ spiral toward Γ as $k \to \infty$ for all x sufficiently close to Γ.

1.2 Hopf Bifurcation in $\mathbb{R}^n$

Consider an one-parameter family of O.D.E.

(2.1) $\quad \dot{x}(t) = f(\alpha,x(t)); \quad \alpha \in \mathbb{R}; \quad x \in \mathbb{R}^n; \quad n \geq 2$.

Suppose that $f(\alpha,0) = 0$ for all α near 0 and that $f(\alpha,x)$ is as smooth as needed in α and x. Let

$$A(\alpha) = \partial f/\partial x(\alpha,0) \quad \text{and} \quad g(\alpha,x) = f(\alpha,x) - A(\alpha)x \ .$$

Assume further that $A(\alpha)$ has a pair of complex conjugate eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$ satisfying the Hopf bifurcation conditions

$$(2.2) \qquad \mathrm{Re}\,\lambda'(0) > 0; \quad \mathrm{Re}\,\lambda(0) = 0; \quad \mathrm{Im}\,\lambda(0) \neq 0$$

while all other eigenvalues of $A(\alpha)$ have negative real parts and stay uniformly away from the imaginary axis for all α near 0. It was shown by Hopf ([10]) that there exists a family of periodic orbits emanating from the zero solution as α varies across 0.

Now suppose we approximate (2.1) by the Euler method. We then obtain a difference equation of the form

$$(2.3) \qquad \begin{aligned} x_{m+1} &= x_m + hf(\alpha, x_m) \\ &= x_m + hA(\alpha)x_m + hg(\alpha, x_m) \end{aligned}$$

Assume temporarily that $x \in \mathbb{R}^2$, then $A(\alpha)$ is a 2×2 matrix with eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$, then there exists a family of nonsingular matrices $P(\alpha)$ such that

$$P(\alpha)^{-1}A(\alpha)P(\alpha) = \begin{bmatrix} \mathrm{Re}\,\lambda(\alpha) & -\mathrm{Im}\,\lambda(\alpha) \\ \mathrm{Im}\,\lambda(\alpha) & \mathrm{Re}\,\lambda(\alpha) \end{bmatrix} \overset{\text{def}}{=} B(\alpha) .$$

As usual we may assume that $P(\alpha)$ is as smooth in α as needed.

Set $x = P(\alpha)y$ and $x_m = P(\alpha)y_m$, then (2.1) becomes

(2.4) $$\dot{y} = B(\alpha)y + g_1(\alpha,y)$$

and (2.3) becomes

(2.5) $$y_{m+1} = y_m + hB(\alpha)y_m + hg_1(\alpha,y_m)$$

where $g_1(\alpha,y) = P(\alpha)^{-1}g(\alpha,P(\alpha)y)$ is as smooth in α and y as needed. Also $g_1(\alpha,y) = O(|y|^2)$, i.e., $g_1(\alpha,y)$ has a second order zero in y at $y = 0$.

Following Lanford [11], we now identify $\mathbb{R}^2$ with the complex plane by writing $z = u_1 + iu_2$ to denote the vector $[u_1,u_2]^T$ in $\mathbb{R}^2$ (^{T} denotes transposition). Equation (2.4) then becomes

(2.4′) $$\dot{z} = \lambda(\alpha)z + g_2(\alpha,z)$$

where $g_2(\alpha,z) = O(|z|^2)$.

Similarly (2.5) becomes

(2.5′) $$z_{m+1} = (1 + h\lambda(\alpha))z_m + hg_2(\alpha,z) .$$

We now bring (2.4′) and (2.5′) to their normal form.

<u>Lemma 1.2.1</u>: Let ϕ_α be the family of mappings from $\mathbb{C}$ to $\mathbb{C}$ defined by (2.5′). Then for $h > 0$ sufficiently small there exists an α-dependent change of coordinates of the form

$$z = w + \gamma(h,\alpha,w) \quad \text{where} \quad \gamma(h,\alpha,w) = O(|w|^2)$$

such that in the new coordinates $\phi_\alpha(w) = w_1$ where

$$(2.6) \quad w_1 = (1 + h\lambda(\alpha))w - ha(h,\alpha)|w|^2 w + hg_3(h,\alpha,w)$$

where $g_3(h,\alpha,w) = O(|w|^5)$.

Similarly there exists a change of coordinates of the form

$$z = w + \tilde{\gamma}(\alpha,w) \quad \text{where} \quad \tilde{\gamma}(\alpha,w) = O(|w|^2)$$

so that (2.4′) becomes

$$(2.7) \qquad \dot{w} = \lambda(\alpha)w - \tilde{a}(\alpha)|w|^2 w + g_4(\alpha,w)$$

where $\tilde{a}(\alpha) = \lim_{h\to 0} a(h,\alpha)$ and $g_3(h,\alpha,w)$ converges uniformly to $g_4(\alpha,w)$ on a neighborhood of $(\alpha,w) = (0,0)$ as $h \to 0$.

Proof. We rewrite (2.4′) and (2.5′) as

$$(2.4'') \qquad \dot{z} = \lambda(\alpha)z + A_2(\alpha,z) + g_5(\alpha,z)$$

$$(2.5'') \qquad z_1 = [1 + h\lambda(\alpha)]z + hA_2(\alpha,z) + hg_5(\alpha,z)$$

where $A_2(\alpha,z)$ is homogeneous of degree two.

$$A_2(\alpha,z) = a_0(\alpha)z^2 + a_1(\alpha)z\bar{z} + a_2(\alpha)\bar{z}^2$$

and $g_5(\alpha,z) = O(|z|^3)$.

For (2.5″) we consider a change of coordinates of the form

$$z = w + \gamma(h,\alpha,w)$$

where $\gamma(h,\alpha,w) = \gamma_0(h,\alpha)w^2 + \gamma_1(h,\alpha)w\bar{w} + \gamma_2(h,\alpha)\bar{w}^2$. Substitute in (2.5″) we obtain

$$(2.8) \qquad w_1 + \gamma(w_1) = [1 + h\lambda(\alpha)][w + \gamma(w)] + hA_2[w + \gamma(w)] + hO(|w|^3)$$

where for convenience we have suppressed α and h.
Since $w_1 = [1 + h\lambda(\alpha)]w + hO(|w|^2)$ we have

$$\gamma(w_1) = \gamma[(1 + h\lambda(\alpha))w] + hO(|w|^2)$$

Also $A_2[w + \gamma(w)] = A_2(w) + O(|w|^3)$. Thus (2.5″) yields

$$(2.9) \qquad w_1 = [1 + h\lambda(\alpha)]w + [1 + h\lambda(\alpha)]\gamma(w) + hA_2(w) - \gamma[(1 + h\lambda(\alpha))w] + hO(|w|^3)$$

We now choose γ so that

$$(2.10) \qquad [1 + h\lambda(\alpha)]\gamma(w) + hA_2(w) - \gamma[(1 + h\lambda(\alpha))w] \equiv 0$$

This can be done by choosing $\gamma_0, \gamma_1, \gamma_2$ so that

$$(2.11\,a) \qquad [1 + h\lambda(\alpha)]\gamma_0(h,\alpha) + ha_0(h,\alpha) - [1 + h\lambda(\alpha)]^2\gamma_0(h,\alpha) = 0$$

(2.11 b) $$[1+h\lambda(\alpha)]\gamma_1(h,\alpha)+ha_1(h,\alpha) - [1+h\lambda(\alpha)][1+h\overline{\lambda(\alpha)}]\gamma_1(h,\alpha) = 0$$

(2.11 c) $$[1+h\lambda(\alpha)]\gamma_2(h,\alpha)+ha_2(h,\alpha) - [1+h\overline{\lambda(\alpha)}]^2\gamma_2(h,\alpha) = 0$$

or

(2.12 a) $$\gamma_0(h,\alpha) = a_0(h,\alpha)/[\lambda(\alpha)(1+h\lambda(\alpha))]$$

(2.12 b) $$\gamma_1(h,\alpha) = a_1(h,\alpha)/[\overline{\lambda(\alpha)}(1+h(\alpha))]$$

(2.12 c) $$\gamma_2(h,\alpha) = a_2(h,\alpha)/[2\overline{\lambda(\alpha)}+h\overline{\lambda(\alpha)}^2-\lambda(\alpha)]$$

We note that since $\operatorname{Im}\lambda(0) \neq 0$ the right hand side of (2.12) are well defined for all α near 0 and h small. (2.5″) then becomes

(2.13) $$w_1 = [1+h\lambda(\alpha)]w+hg_6(h,\alpha,w)$$

where $g_6(h,\alpha,w) = O(|w|^3)$.

From (2.12) we have

$$\lim_{h\to 0}\gamma_0(\alpha,h) = \tilde{a}_0(\alpha)/\lambda(x) \overset{\text{def}}{=} \tilde{\gamma}_0(\alpha)$$

$$\lim_{h\to 0}\gamma_1(\alpha,h) = \tilde{a}_1(\alpha)/\overline{\lambda(\alpha)} \overset{\text{def}}{=} \tilde{\gamma}_1(\alpha)$$

$$\lim_{h\to 0}\gamma_2(\alpha,h) = \tilde{a}_2(\alpha)/[2\overline{\lambda(\alpha)}-\lambda(\alpha)] \overset{\text{def}}{=} \tilde{\gamma}_2(\alpha)$$

Rewrite (2.5″) as

$$(z_1 - z)/h = \lambda(\alpha)z + g_2(\alpha,z)$$

and by taking the limits as $h \to 0$, it is easy to see that under the change of coordinates

$$z = w + \tilde{\gamma}_0(\alpha)w^2 + \tilde{\gamma}_1(\alpha)w\bar{w} + \tilde{\gamma}_2(\alpha)\bar{w}^2$$

(2.4″) becomes

$$(2.14) \qquad \dot{w} = \lambda(\alpha)w + g_7(\alpha,w)$$

where $g_6(h,\alpha,w)$ converges uniformly to $g_7(\alpha,w)$ on some neighborhood of $(\alpha,w) = (0,0)$ as $h \to 0$.

Assuming that we have made such changes and rewrite w by z so that (2.13) and (2.14) become

$$(2.15) \qquad z_1 = [1 + h\lambda(\alpha)]z + hg_6(h,\alpha,z)$$

$$(2.16) \qquad \dot{z} = \lambda(\alpha)z + g_7(\alpha,z)$$

We now try to eliminate all terms of degree 3 in (2.15) and (2.16). First we rewrite them as

$$(2.15') \qquad z_1 = [1 + h\lambda(\alpha)]z + hA_3(h,\alpha,z) + hg_8(h,\alpha,z)$$

$$(2.16') \qquad \dot{z} = \lambda(\alpha)z + \tilde{A}_3(\alpha,z) + g_9(\alpha,z)$$

where

$$A_3(h,\alpha,z) = a_0(h,\alpha)z^3 + a_1(h,\alpha)z^2\bar{z} + a_2(h,\alpha)z\bar{z}^2 + a_3(h,\alpha)\bar{z}^3$$

$$\tilde{A}(\alpha,z) = \tilde{a}_0(\alpha)z^3 + \tilde{a}_1(\alpha)z^2\bar{z} + \tilde{a}_2(\alpha)z\bar{z}^2 + \tilde{a}_3(\alpha)\bar{z}^3$$

$$\lim_{h\to 0} a_j(h,\alpha) = \tilde{a}_j(\alpha); \qquad j = 0,\ 1,\ 2,\ 3;$$

and $g_8(h,\alpha,z)$ converges uniformly to $g_9(\alpha,z)$ on some neighborhood of $(\alpha,z) = (0,0)$ as $h \to 0$.

As before we consider a change of coordinates of the form

$$z = w + \gamma(h,\alpha,w)$$

where now

$$\gamma(h,\alpha,w) = \gamma_0(h,\alpha)w^3 + \gamma_1(h,\alpha)w^2\bar{w} + \gamma_2(h,\alpha)w\bar{w}^2 + \gamma_3(h,\alpha)\bar{w}^3$$

(2.15′) then becomes

$$(2.15'') \qquad w_1 + \gamma(w_1) = [1 + h\lambda(\alpha)][w + \gamma(w)] + hA_3[w + \gamma(w)] + hO(|w|^4)$$

or

$$(2.15''') \qquad w_1 = [1 + h\lambda(\alpha)]w + [1 + h\lambda(\alpha)]\gamma(w) + hA_3(w) - \gamma[(1 + h\lambda(\alpha))w] + O(|w|^4)$$

where again we have suppressed α and h for convenience. Ideally we would like to choose γ so that

$$(2.17)\qquad [1+h\lambda(\alpha)]\gamma(w) + hA_3(w) - \gamma[(1+h\lambda(\alpha))w] \equiv 0$$

or

$$(2.18\,a)\qquad [1+h\lambda(\alpha)]\gamma_0(h,\alpha) + ha_0(h,\alpha) - [1+h\lambda(\alpha)]^3\gamma_0(h,\alpha) = 0$$

$$(2.18\,b)\qquad [1+h\lambda(\alpha)]\gamma_1(h,\alpha) + ha_1(h,\alpha) - [1+h\lambda(\alpha)]^2[1+h\overline{\lambda(\alpha)}]\gamma_1(h,\alpha) = 0$$

$$(2.18\,c)\qquad [1+h\lambda(\alpha)]\gamma_2(h,\alpha) + ha_2(h,\alpha) - [1+h\lambda(\alpha)][1+h\overline{\lambda(\alpha)}]^2\gamma_2(h,\alpha) = 0$$

$$(2.18\,d)\qquad [1+h\lambda(\alpha)]\gamma_3(h,\alpha) + ha_3(h,\alpha) - [1+h\overline{\lambda(\alpha)}]^3\gamma_3(h,\alpha) = 0$$

However the equation (2.18 b) can not be solved in general because

$$[1+h\lambda(\alpha)] - [1+h\lambda(\alpha)]^2[1+h\overline{\lambda(\alpha)}] = [1+h\lambda(\alpha)][1-|1+h\lambda(\alpha)|^2]$$

and for h small there exists an α near 0 such that $|1+h\lambda(\alpha)| = 1$.

Thus we choose

$$(2.19\,a) \quad \gamma_0(h,\alpha) = ha_0(h,\alpha)/[(1+h\lambda(\alpha))^3 - (1+h\lambda(\alpha))]$$
$$= a_0(h,\alpha)/[2\lambda(\alpha) + 3h\lambda(\alpha)^2 + h^2\lambda(\alpha)^3]$$

$$(2.19\,b) \quad \gamma_1(h,\alpha) = 0$$

$$(2.19\,c) \quad \gamma_2(h,\alpha)$$
$$= ha_2(h,\alpha)/[(1+h\overline{\lambda(\alpha)})^2(1+h\lambda(\alpha)) - (1+h\lambda(\alpha))]$$
$$= a_2(h,\alpha)/[2\overline{\lambda(\alpha)} + h\overline{\lambda(\alpha)}^2 + 2h|\lambda(\alpha)|^2 + h^2\lambda(\alpha)\overline{\lambda(\alpha)}^2]$$

$$(2.19\,d) \quad \gamma_3(h,\alpha)$$
$$= ha_3(h,\alpha)/[(1+h\overline{\lambda(\alpha)})^3 - (1+h\lambda(\alpha))]$$
$$= a_3(h,\alpha)/[3\overline{\lambda(\alpha)} + 3h\overline{\lambda(\alpha)}^2 + h^2\overline{\lambda(\alpha)}^3 - \lambda(\alpha)]$$

then (2.15′) becomes

$$(2.20) \quad w_1 = [1+h\lambda(\alpha)]w + ha_1(h,\alpha)|w|^2 w + hO(|w|^4)$$

Since

$$\lim_{h\to 0} \gamma_0(h,\alpha) = \tilde{a}_0(\alpha)/2\lambda(\alpha) \overset{\text{def}}{=} \tilde{\gamma}_0(\alpha)$$

$$\lim_{h\to 0} \gamma_2(h,\alpha) = \tilde{a}_2(\alpha)/2\overline{\lambda(\alpha)} \overset{\text{def}}{=} \tilde{\gamma}_2(\alpha)$$

$$\lim_{h\to 0} \gamma_3(h,\alpha) = \tilde{a}_3(\alpha)/[3\overline{\lambda(\alpha)} - \lambda(\alpha)] \overset{\text{def}}{=} \tilde{\gamma}_3(\alpha)$$

under the change of coordinates

$$z = w + \tilde{\gamma}_0(\alpha)w^3 + \tilde{\gamma}_2(\alpha)w\bar{w}^2 + \tilde{\gamma}_3(\alpha)\bar{w}^3$$

(2.16′) becomes

$$\dot{w} = \lambda(\alpha)w + \tilde{a}_1(\alpha)|w|^2 w + O(|w|^4) \tag{2.21}$$

where $\tilde{a}_1(\alpha) = \lim_{h\to 0} a_1(h,\alpha)$.

Repeating the above process we can eliminate all terms of degree 4 in (2.20) and (2.21). Set $a(h,\alpha) = -a_1(h,\alpha)$ and $\tilde{a}(\alpha) = -\tilde{a}_1(\alpha)$ we obtain the lemma.

We now introduce the polar coordinates by setting $r = |w|$ and $r_1 = |w_1|$. Since

$$\begin{aligned} &|1 + h\lambda(\alpha) - ha(h,\alpha)|w|^2| \\ &\quad = |1 + h\lambda(\alpha)| - hb(h,\alpha)|w|^2 + h^2 O(|w|^4) \end{aligned}$$

where $b(h,\alpha) = [1 + h \operatorname{Re} \lambda(\alpha)]a(h,\alpha)/|1 + h\lambda(\alpha)|$, (2.20) becomes

$$r_1 = |1 + h\lambda(\alpha)|r - hb(h,\alpha)r^3 + hf_1(h,\alpha,r,\theta) \tag{2.22 a}$$

where $f_1(h,\alpha,r,\theta) = O(r^5)$.

Similarly we set $\theta = \arg(w)$ and $\theta_1 = \operatorname{Arg}(w_1)$ then since

$$\operatorname{Arg}[1 + h\lambda(\alpha) - ha(h,\alpha)|w|^2] = \operatorname{Arg}[1 + h\lambda(\alpha)] + hO(|w|^2)$$

we have

(2.22 b) $$\theta_1 = \theta + h\varphi(h,\alpha) + hf_2(h,\alpha,r,\theta)$$

where $h\varphi(h,\alpha) \overset{\text{def}}{=} \operatorname{Arg}[1 + h\lambda(\alpha)]$ and $f_2(h,\alpha,r,\theta) = O(r^2)$. By taking the limits as $h \to 0$, (2.21) becomes

(2.23 a) $$\dot{r} = \operatorname{Re}\lambda(\alpha)r - \tilde{b}(\alpha)r^3 + \tilde{f}_1(\alpha,r)$$

(2.23 b) $$\dot{\theta} = \operatorname{Im}\lambda(\alpha) + \tilde{f}_2(\alpha,r)$$

Now suppose that $\tilde{b}(0) > 0$. Let $\tau > 0$ be fixed and let $p(t)$ and $\varphi(t)$ be the solutions of (2.23) satisfying the initial conditions

$$p(0) = r; \qquad \varphi(0) = \theta$$

Define a mapping $\int_\alpha : (r,\theta) \to (\bar{r},\bar{\theta})$ where

$$\bar{r} = p(\tau); \qquad \bar{\theta} = \varphi(\tau)$$

Note that

(2.24 a) $$\bar{r} = e^{\operatorname{Re}\lambda(\alpha)\tau}\, r - b_1(\alpha)r^3 + F_1(\alpha,r,\theta)$$

(2.24 b) $$\bar{\theta} = \theta + \operatorname{Im}\lambda(\alpha)\tau + G_1(\alpha,r,\theta)$$

where

$$b_1(\alpha) = \begin{cases} \tilde{b}(\alpha)e^{\mathrm{Re}\,\lambda(\alpha)\tau}\,(e^{2\,\mathrm{Re}\,\lambda(\alpha)\tau} - 1)/(2\,\mathrm{Re}\,\lambda(\alpha)) & \text{for } \alpha \neq 0 \\ \tilde{b}(0)\tau \quad \text{for} \quad \alpha = 0 & \end{cases}$$

$$F_1(\alpha,r,\theta) = O(r^5)$$

$$G_1(\alpha,r,\theta) = O(r^2)$$

Now set

$$(2.25\text{ a}) \qquad V(h,\alpha) = |1 + h\lambda(\alpha)|^2 - 1 = 2h\,\mathrm{Re}\,\lambda(\alpha) + h^2|\lambda(\alpha)|^2$$

$$(2.25\text{ b}) \qquad V_1(h,\alpha) = 2\,\mathrm{Re}\,\lambda(\alpha) + h|\lambda(\alpha)|^2$$

then $V_1(0,0) = 0$ and $\partial V/\partial\alpha(0,0) = 2\,\mathrm{Re}\,\lambda'(0) > 0$. By the Implicit Function Theorem there exists an α_h such that

$$V_1(h,\alpha_h) = V(h,\alpha_h) = 0$$

In fact a little calculation shows that

$$(2.26) \qquad \alpha_h = -h|\lambda(0)|^2/2\,\mathrm{Re}\,\lambda'(0) + O(h^2)$$

Let ϕ_α be the mapping as defined by (2.22). Define

$\pi_\alpha = \phi_\alpha^N$ where $N = [\tau/h]$ = the greatest integer $\leq \tau/h$.

Using the fact that ϕ_α is the Euler approximation of (2.23), it is easy to see that

$$\phi_\alpha(r,\theta) = (r_N, \theta_N)$$

where

(2.27 a) $$r_N = |1 + h\lambda(\alpha)|^N r - b_2(h,\alpha)r^3 + F_2(h,\alpha,r,\theta)$$

(2.27 b) $$\theta_N = \theta + \mathrm{Im}\,\lambda(\alpha)\tau + G_2(h,\alpha,r,\theta)$$

where $\lim_{h\to 0} b_2(h,\alpha) = b_1(\alpha)$ and $F_2(h,\alpha,r,\theta)$, $G_2(h,\alpha,r,\theta)$ converge uniformly to $F_1(\alpha,r,\theta)$, $G_1(\alpha,r,\theta)$ on some neighborhood of $(\alpha,r) = (0,0)$ as $h \to 0$.

We need the following

Theorem 1.2.2 (Ruelle-Taken): Let $\phi_\mu : (r,\theta) \to (r_1,\theta_1)$ be a family of sufficiently smooth mappings from $\mathbb{R}^2 \to \mathbb{R}^2$ where

(2.28 a) $$r_1 = (1+\mu)r - b_1(\mu)r^3 + O(r^5)$$

(2.28 b) $$\theta_1 = \theta + a(\mu) + b_2(\mu)r^2 + O(r^4)$$

If $b_1(0) > 0$ and $a(0) \neq 0$ then there exists $\delta > 0$ such that ϕ_μ has an attracting curve for each μ satisfying

$0 < \mu < \delta$. This curve is unique on a neighborhood of $r = 0$ and depends continuously on μ.

<u>Proof</u>: See [11] or [13].

Since $\text{Re}\,\lambda(0) = 0$; $\text{Re}\,\lambda'(0) > 0$ and $b_1(0) > 0$, we could apply Theorem 1.2.2 to obtain an attracting invariant curve for (2.24) for $0 < \alpha < \epsilon$. It is easy to see that this curve is invariant under the flow of (2.23) and hence is in fact a periodic orbit of (2.23).

Similarly as $\lim_{h\to 0} b_2(h,\alpha) = b_1(\alpha)$ and $b_1(\alpha)$ is continuous for α near 0 we have $b_2(h,\alpha_h) > 0$ for h sufficiently small. Theorem 1.2.2 then yields a family of attracting invariant curves for (2.27) for α satisfying

$$\alpha_h \le \alpha \le \epsilon_h$$

α_h is given in (2.26) while ϵ_h is such that

$$|1 + h\lambda(\epsilon_h)|^N \le \delta_h$$

where $\delta_h > 0$ is the constant provided by Theorem 1.2.2.

Since $F_2(h,\alpha,r,\theta)$, $G_2(h,\alpha,r,\theta)$ converge uniformly to $F_1(\alpha,r,\theta)$, $G_1(\alpha,r,\theta)$ on some neighborhood of $(\alpha,r) = (0,0)$. An easy application of Ascoli-Arzela theorem implies that the partial derivatives of F_2 and G_2 converge uniformly to the corresponding partial derivatives of F_1 and G_1 as $h \to 0$. This means that we could choose δ_h to be independent of h provided that h is sufficiently small.

We then choose ϵ_h to be independent of h. Thus π_α possesses an invariant curve for each α satisfying $\alpha_h < \alpha < \epsilon$ where α_h is given by (2.26) and $\epsilon > 0$ is some constant independent of h. This result is needed for the continuation of the invariant curves toward a Hopf bifurcation which will be discussed in the next section.

Now suppose that Γ_α is invariant under π_α then so is $\phi_\alpha(\Gamma_\alpha)$ for

$$\pi_\alpha(\phi_\alpha(\Gamma_\alpha)) = \phi_\alpha^{N+1}(\Gamma_\alpha) = \phi_\alpha(\pi_\alpha(\Gamma_\alpha)) = \phi_\alpha(\Gamma_\alpha).$$

Uniqueness implies that $\phi_\alpha(\Gamma_\alpha) = \Gamma_\alpha$ and hence Γ_α is invariant under ϕ_α. Thus ϕ_α has a family of invariant curves depending continuously on α for $\alpha_h \leq \alpha < \epsilon$.

In case $\tilde{b}(0) < 0$ we could apply Theorem 1.2.2 to $\int_\alpha^{-1}$ to obtain a family of periodic orbits for (2.1) which bifurcates subcritically from the zero solution, i.e., there exists a nontrivial periodic orbit for (2.1) for each α satisfying $-\epsilon < \alpha < 0$ for some $\epsilon > 0$. These orbits are repelling. Similarly, we could apply Theorem 1.2.2 to π_α^{-1} to obtain a family of invariant curves depending continuously on α for $-\epsilon < \alpha < \alpha_h$ where $\epsilon > 0$ can be choosen to be independent of h. These curves are invariant and repelling under ϕ_α.

We now turn our attention to higher dimension cases, i.e., $x \in \mathbb{R}^n$; $n \geq 3$. We thus have a system of O.D.E.

(2.29) $$\dot{x} = f(\alpha,x) = A(\alpha)x + g(\alpha,x); \quad x \in \mathbb{R}^n$$

and the approximation

(2.30) $$x_{m+1} = x_m + hf(\alpha,x_m)$$

By making an α dependent change of coordinates, we can decompose (2.29) into

(2.31a) $$\dot{y} = B(\alpha)y + g_1(\alpha,y,z)$$

(2.31b) $$\dot{z} = C(\alpha)z + g_2(\alpha,y,z)$$

and (2.30) into

(2.32a) $$y_{m+1} = y_m + hB(\alpha)y_m + hg_1(\alpha,y_m,z_m)$$

(2.32b) $$z_{m+1} = z_m + hC(\alpha)z_m + hg_2(\alpha,y_m,z_m)$$

where

$$y,\ y_m,\ y_{m+1} \in \mathbb{R}^2$$
$$z,\ z_m,\ z_{m+1} \in \mathbb{R}^{n-2}$$

$B(\alpha)$ = 2×2 matrix with eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$

$C(\alpha)$ = $(n-2) \times (n-2)$ matrix with spectrum in the left half plane for all α near 0.

$$g_1 : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$$

$$g_2 : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$$

Also g_1 and g_2 have second order zeroes in (y,z) at $(0,0)$.

We note that the eigenvalues of $Id + hB(\alpha)$ are $1 + h\lambda(\alpha)$ and $1 + h\overline{\lambda(\alpha)}$ which cross the unit circle when α crosses α_h (given in (2.26)). While the eigenvalues of $Id + hC(\alpha)$ stay inside the unit circle.

Again we let $\tau > 0$ be fixed and $N = [\tau/h]$.

Let $y(t)$, $z(t)$ be the solutions of (2.31) with initial conditions

$$y(0) = y \, ; \qquad z(0) = z \, .$$

We then define $\int_\alpha (y,z) = (\overline{y},\overline{z})$ where

$$\overline{y} = y(\tau) \, ; \qquad \overline{z} = z(\tau) \, .$$

Also let ϕ_α be the mapping given by (2.32) and define

$$\pi_\alpha = \phi_\alpha^N$$

As before we have

(2.33 a) $$\overline{y} = e^{B(\alpha)\tau} y + G_1(\alpha,y,z)$$

(2.33 b) $$\overline{z} = e^{C(\alpha)\tau} z + G_2(\alpha,y,z)$$

and since ϕ_α is the Euler approximation of (2.31) we have

$$\pi_\alpha(y,z) = (y_N, z_N)$$

where

(2.34 a) $$y_N = \tilde{B}(\alpha)y + G_3(h,\alpha,y,z)$$

(2.34 b) $$z_N = \tilde{C}(\alpha)z + G_5(h,\alpha,y,z)$$

where

$$\tilde{B}(\alpha) = [Id + hB(\alpha)]^N ,$$

$$\tilde{C}(\alpha) = [Id + hC(\alpha)]^N ,$$

$G_3(h,\alpha,y,z)$, $G_4(h,\alpha,y,z)$ have second order zeroes in (y,z) at $(0,0)$ and converge uniformly to $G_1(\alpha,y,z)$, $G_2(\alpha,y,z)$ on some neighborhood of $(\alpha,y,z) = (0,0,0)$ as $h \to 0$. Note also that

$$\lim_{h\to 0} \tilde{B}(\alpha) = e^{B(\alpha)\tau} , \qquad \lim_{h\to 0} \tilde{C}(\alpha) = e^{C(\alpha)\tau}$$

in some matrix norm.

We need the following

Theorem 1.2.3: Let ψ be a mapping of a neighborhood of 0 in a Banach space X into X. We assume that ψ is C^{k+1}, $k \geq 1$ and that $\psi(0) = 0$. We further assume that $D\psi(0)$ has spectral radius 1 and the spectrum of $D\psi(0)$ splits

into a part on the unit circle with generalized eigenspace Y of finite dimension and the remainder which is at a non-zero distance from the unit circle with corresponding generalized eigenspace Z. Then there exists $\varepsilon > 0$ and a C^k mapping u from $\{y \in Z : |y| < \varepsilon\}$ into Z with a second order zero at zero such that

a) The manifold $\Gamma_u = \{(y,z) \mid z = u(y);\ |y| < \varepsilon\} \subset Y \oplus Z$, i.e., the graph of u is invariant under ψ in the sense that if $|y| < \varepsilon$ and if $\psi(y,u(y)) = (y_1,z_1)$ with $|y_1| < \varepsilon$ then $z_1 = u(y_1)$.

b) The manifold Γ_u is locally attracting for ψ in the sense that if $|y| < \varepsilon$, $|z| < \varepsilon$ and if $(y_n,z_n) = \psi^n(y,z)$ are such that $|y_n| < \varepsilon$, $|z_n| < \varepsilon$ for all $n > 0$ then $\lim_{n\to\infty} |z_n - u(y_n)| = 0$.

<u>Proof</u>: See [11] or [13]. We also remark that ε depend on $|D\psi^j(x)|$, $j = 0,\dots,k+1$ for x in some neighborhood of 0 in $\mathbb{R}^n$.

Let S_0 be the mapping

$$S_0 : (\alpha,y,z) \to (\alpha,\bar{y},\bar{z})$$

where $\bar{y}$, $\bar{z}$ are given by (2.33). Assuming that $f(\alpha,x)$ is C^{k+1} for some $k \geq 1$ then S_0 is a C^{k+1} mapping and satisfies all the hypotheses in Theorem 1.2.3. Thus it has a C^k invariant manifold

$$\sum_0 = \{(\alpha,y,z) \mid z = u_0(\alpha,y);\ |y| < \varepsilon;\ |\alpha| < \varepsilon\}$$

for some $\varepsilon > 0$.

Similarly, let S_h be the mapping

$$S_h : (\alpha,y,z) \to (\alpha,\pi_{\alpha+\alpha_h}(y,z))$$

then S_h is also C^{k+1} and satisfies all the hypotheses in Theorem 1.2.3. Thus it also has a C^k invariant manifold

$$\sum_h = \{(\alpha,y,z) \mid z = u_h(\alpha,y);\ |y| < \varepsilon_h;\ |\alpha| < \varepsilon_h\}$$

for some $\varepsilon_h > 0$.

Since $\alpha_h = O(h)$, S_h converges uniformly to S_0 on some neighborhood of $|\alpha,y,z| = (0,0,0)$ as $h \to 0$. Using the Ascoli-Arzela theorem, we can show that all the derivatives of S_h converge uniformly to the corresponding derivatives of S_0 as $h \to 0$. Thus we could choose $\varepsilon_h = \varepsilon$ for sufficiently small h, i.e., $\sum_0$ and $\sum_h$ are well defined on a common neighborhood of 0.

Now note that u_0 and u_h satisfy

(2.35) $$u_0[\alpha,e^{B(\alpha)\tau}\, y + G_1(\alpha,y,u_0(\alpha,y))] = e^{C(\alpha)\tau}\, u_0(\alpha,y) + G_2(\alpha,y,u_0(\alpha,y))$$

(2.36) $$u_h[\alpha,\tilde{B}(\alpha+\alpha_h)y + G_3(h,\alpha+\alpha_h,y,u_h(\alpha,y))] = \tilde{C}(\alpha+\alpha_h)u_h(\alpha,y) + G_4(h,\alpha+\alpha_h,y,u_h(\alpha,y))$$

In general the center manifold Σ_0 is not unique. Hence we would not expect that u_h converges uniformly to u_0. On the other hand, Wan has shown in [14] that the k^{th} order Taylor expansions are the same for each manifold. This means that $D^j u_0(0,0)$ are unique for $j \leq k$. Similarly $D^j u_h(0,0)$ are also unique for $j \leq k$. Since u_h and its j^{th} derivatives for $j \leq k$ are uniformly bounded on a neighborhood of $(\alpha,y) = (0,0)$ for all h sufficiently small, using the Ascoli-Arzela theorem we can show that $D^j u_h(0,0)$ converge to $D^j u_0(0,0)$ for $j \leq k-1$.

From (2.33a) and (2.34a) we have

$$\bar{y} = e^{B(\alpha)\tau} y + G_1(\alpha,y,u_0(\alpha,y)) \tag{2.37}$$

$$y_N = \tilde{B}(\alpha+\alpha_h)y + G_3(h,\alpha+\alpha_h,y,u_h(\alpha,y) \tag{2.38}$$

Note that the change of coordinates for ϕ_α in Lemma 1.2.1 depend only on $D^j\phi_\alpha(0)$ for $0 \leq j \leq 4$ and that the $(k-1)^{th}$ order Taylor expansion of (2.38) at $(\alpha,y) = (0,0)$ converges to that of (2.37). Thus assuming that $k \geq 5$, we could transform (2.37) into

$$\bar{r} = e^{\mathrm{Re}\lambda(\alpha)\tau} r - b_1(\alpha)r^3 + F_1(\alpha,r,\theta) \tag{2.39a}$$

$$\bar{\theta} = \theta + \mathrm{Im}\lambda(\alpha)\tau + F_2(\alpha,r,\theta) \tag{2.39b}$$

and (2.38) into

$$r_N = |1+h\lambda(\alpha)|^N r - b_2(h,\alpha)r^3 + F_3(h,\alpha,r,\theta) \tag{2.40a}$$

(2.40b) $\theta_N = \theta + \mathrm{Im}\lambda(\alpha)\tau + F_4(h,\alpha,r,\theta)$

where $b_2(h,\alpha) \to b_1(\alpha)$; $F_1(\alpha,r,\theta) = O(r^5)$; $F_2(\alpha,r,\theta) = O(r^2)$; $F_3(h,\alpha,r,\theta) = O(r^5)$; $F_4(h,\alpha,r,\theta) = O(r^2)$. But F_3 and F_4 need not converge to F_1 and F_2.

Assuming that $b_1(0) > 0$ we then can proceed as before to show that (2.40) possesses a family of invariant curves for all α satisfying $\alpha_h \leq \alpha \leq \delta_h$ where $\alpha_h = O(h)$.

We now show that there exists a $\delta > 0$ independent of h such that $\delta_h \geq \delta$ for all h sufficiently small. Suppose the contrary then there exists a sequence $\{h_n\}$ where $h_n \to 0$ as $n \to \infty$ such that $\delta_{h_n} \to a \leq 0$. By the Ascoli-Arzela theorem, the sequence $\{u_{h_n}(\alpha,y)\}$ has a subsequence, which we again call $\{u_{h_n}(\alpha,y)\}$, converging uniformly to some function $u(\alpha,y)$ on $\{|\alpha| < \varepsilon, |y| < \varepsilon\}$. The function $u(\alpha,y)$ satisfies (2.35) and hence defines a center manifold for S_0. The equation (2.37) and (2.38) becomes

(2.41) $$\bar{y} = e^{B(\alpha)\tau}y + G_1(\alpha,y,u(\alpha,y))$$

(2.42) $$y_N = \tilde{B}(\alpha+\alpha_{h_n})y + G_3(h_n,\alpha+\alpha_{h_n},y,u_{h_n}(\alpha,y))$$

where now the right hand side of (2.42) converges uniformly to that of (2.41) on $\{|\alpha| < \varepsilon, |y| < \varepsilon\}$. This means that (2.42) possesses a family of invariant curves for α satisfying $\alpha_h \leq \alpha \leq \delta$ for some $\delta > 0$ independent of

h. This contradicts the assumption that $\delta_{h_n} \to 0$. Hence there must exist $\delta > 0$ so that $\delta_h \geq \delta$ for all h sufficiently small.

Now let Γ_α be an invariant curve for (2.33) as obtained above then it is easy to see that $\phi_\alpha(\Gamma_\alpha)$ is also invariant under (2.33). Moreover since Σ_h is locally attracting, Γ_α and $\phi_\alpha(\Gamma_\alpha)$ must lie on Σ_h and hence induce two invariant curves for (2.38). Uniqueness implies that these two curves are the same and hence $\Gamma_\alpha = \phi_\alpha(\Gamma_\alpha)$. This means that Γ_α is invariant under ϕ_α as expected.

Also note that the invariant curve for (2.38) induced by Γ_α is attracting under (2.38). This means that Γ_α is attracting under (2.33) on the manifold Σ_h. Using the fact that Σ_h is locally attracting it follows easily that Γ_α is attracting globally.

The case $b_1(0) < 0$ can be treated similarly. Thus we have

Theorem 1.2.4: If the term $b_1(\alpha)$ in (2.39) satisfies $b_1(0) > 0$ $(b_1(0) < 0)$ then the difference equation (2.3) possess a family of invariant curves depending continuously on α for $\alpha_h < \alpha < \epsilon$ $(-\epsilon < \alpha < \alpha_h)$ where $\epsilon > 0$ is independent of h provided that h is sufficiently small and $\alpha_h = O(h)$. Also in the case $b_1(0) > 0$, the invariant curves are attracting under (2.3).

Remark 1.2.5: Thus far we have assumed that (2.3) arises from the Euler method. However all the arguments given still hold true if some other explicit, convergent, single-step method is employed. Implicit single-step methods will be discussed in Remark 1.3.4.

Remark 1.2.6: Theorem 1.2.4 also holds if some of the eigenvalues of $A(\alpha)$ have positive real parts. What we need is a center Manifold theorem more general than the one in Theorem 1.2.3. We now have a mapping of the form

$$\psi : (x,y,z) \to (x_1,y_1,z_1)$$

$$x_1 = Ax + f_1(x,y,z)$$

$$y_1 = By + f_2(x,y,z)$$

$$z_1 = Cz + f_3(x,y,z)$$

where $\|A\| < 1$, $\|B^{-1}\| < 1$ in some matrix norm and the spectrum of C lies on the unit circle. Also f_1, f_2, f_3 are C^{k+1} in x,y,z and have second order zeros at 0.

First, we could construct a center-stable manifold for ψ of the form

$$\{y = u(x,z) \mid |x| < \varepsilon,\ |z| < \varepsilon\}$$

for some $\varepsilon > 0$. The existence of such manifolds can be proved in the same manner as in the proof of the Center Manifold Theorem given in Theorem 1.2.3 (see [13]). We then have

$$x_1 = Ax + f_1(x,u(x,z),z)$$

$$z_1 = Cz + f_3(x,u(x,z),z)$$

Now we could use Theorem 1.2.3 to obtain a center manifold for ψ.

1.3 Existence and Continuation

We now prove the existence of the invariant curves away from the Hopf bifurcation. First we temporarily fix α and write (2.1) as

$$(3.1) \qquad \dot{x} = f(x)\ ; \quad x \in \mathbb{R}^n\ ; \quad n \geq 2$$

Assume that (3.1) has a ω-periodic orbit Γ whose characteristic multipliers $\mu_1 = 1, \mu_2, \ldots, \mu_n$ satisfying $|\mu_j| < 1$ for $2 \leq j \leq k$ and $|\mu_j| > 1$ for $k+1 \leq j \leq n$. Suppose (3.1) is approximated by some convergent, explicit, single-step method. We then obtain a difference equation

(3.2) $$x_{m+1} = x_m + hf(x_m) + h^2 F(h, x_m)$$

We now show that (3.2) has an invariant curve near Γ for h sufficiently small. The case $k = n$ or $k = 1$ was proved in [2]. Thus we are primarily interested in the case $1 < k < n$.

From Hale [7], there exists a local coordinate system along Γ of the form (ρ, θ) so that (3.1) is equivalent to

(3.3 a) $$\dot{\rho} = A(\theta)\rho + f_1(\theta, \rho)$$

(3.3 b) $$\dot{\theta} = 1 + f_2(\theta, \rho)$$

where $f_1(\theta,\rho) = O(|\rho|^2)$; $f_2(\theta,\rho) = O(|\rho|)$. $A(\theta)$, $f_1(\theta,\rho)$, $f_2(\theta,\rho)$ are ω-periodic in θ.

The variational equation for (3.3) is

(3.4) $$d\rho \mid d\theta = A(\theta)\rho$$

which has characteristic multipliers $\mu_2, \ldots, \mu_n$. Let $X(\theta)$ be a fundamental matrix solution of (3.4) then so is $X(\theta + \omega)$. Thus there exists a nonsingular matrix D such that

$$X(\theta + \omega) = X(\theta)D .$$

Let B be a real matrix such that $D^2 = e^{2B\omega}$. Define

$$P(\theta) = X(\theta)e^{-B\theta}$$

then $P(\theta + 2\omega) = X(\theta + 2\omega)e^{-B(\theta+2\omega)} = P(\theta)$.

Also since $X(\theta)$ is a fundamental matrix solution of (3.4) we have

$$dP(\theta)/d\theta + P(\theta)B = A(\theta)P(\theta) \tag{3.5}$$

Set $\rho = P(\theta)r$ then

$$\dot{\rho} = d(P(\theta)r)/dt$$

$$= d(P(\theta))/d\theta\ \dot{\theta}\ r + P(\theta)\dot{r}$$

which gives using (3.5) and (3.3)

$$A(\theta)P(\theta)r = [A(\theta)P(\theta) - P(\theta)B]r + P(\theta)\dot{r} + O(|r|^2)$$

or

$$\dot{r} = Br + f_3(\theta,r)$$

Thus (3.1) is equivalent to

$$\dot{r} = Br + f_3(\theta,r) \tag{3.4 a}$$

$$\dot{\theta} = 1 + f_4(\theta,r) \tag{3.4 b}$$

where f_3, f_4 are 2ω-periodic in θ, $f_3(\theta,r) = O(|r|^2)$; $f_4(\theta,r) = O(|r|)$; B is a $(n-1) \times (n-1)$ matrix such that

$e^{2B\omega}$ has eigenvalues $\mu_2^2,\dots,\mu_n^2$.

Let $u : \mathbb{R}^n \to \mathbb{R}^n$, $v : \mathbb{R}^n \to \mathbb{R}^n$ be such that

$$r = u(x); \quad \theta = v(x) \quad \text{for all } x \text{ near } \Gamma .$$

As usual we can assume that u and v are as smooth as needed. We note that

$$\dot{r} = du(x)/dx \cdot \dot{x} = du(x)/dx\ f(x)$$

$$\dot{\theta} = dv(x)/dx \cdot \dot{x} = dv(x)/dx\ f(x)$$

Set $r_1 = u(x_1)$; $\theta_1 = u(x_1)$ where

$$x_1 = x + hf(x) + h^2F(h,x)$$

then

$$\begin{aligned}(3.5\text{ a})\qquad r_1 &= u(x + hf(x) + h^2F(h,x)) \\ &= u(x) + du(x)/dx[hf(x) + h^2F(h,x)] + O(h^2) \\ &= r + h\dot{r} + O(h^2) \\ &= r + hBr + hf_3(\theta,r) + h^2F_3(h,\theta,r)\end{aligned}$$

where in general $F_3(h,\theta,0) \not\equiv 0$.

Similarly

$$\begin{aligned}(3.5\text{ b})\qquad \theta_1 &= v(x + hf(x) + h^2F(h,x)) \\ &= \theta + h\dot{\theta} + O(h^2) \\ &= \theta + h + hf_4(\theta,r) + h^2F_4(h,\theta,r)\end{aligned}$$

where in general $F_4(h,\theta,0) \neq 0$.

Let $\lambda_2,\ldots,\lambda_n$ be the eigenvalues of B. By rearranging if necessary we can assume that

$$e^{2\lambda_j\omega} = \mu_j^2; \qquad j = 2,\ldots,n .$$

In particular this means that $\operatorname{Re}\lambda_j < 0$ for $j = 2,\ldots,k$ while $\operatorname{Re}\lambda_j > 0$ for $j = k+1,\ldots,n$.

Let Y = generalized eigenspace of B corresponding to the eigenvalues $\lambda_2,\ldots,\lambda_k$

$\mathbb{Z}$ = generalized eigenspace of B corresponding to the eigenvalues $\lambda_{k+1},\ldots,\lambda_n$.

By projecting to Y and Z we can write (3.5) as

(3.6 a) $$y_1 = (Id + hC)y + hf_5(\theta,y,z) + h^2F_5(h,\theta,y,z)$$

(3.6 b) $$z_1 = (Id + hD)z + hf_6(\theta,y,z) + h^2F_6(h,\theta,y,z)$$

(3.6 c) $$\theta_1 = \theta + h + hf_7(\theta,y,z) + h^2F_7(h,\theta,y,z)$$

where $f_5(\theta,y,z)$ and $f_6(\theta,y,z)$ have second order zeroes while $f_7(\theta,y,z)$ has first order zero in (y,z) at $(0,0)$. All the functions are as smooth as needed.

Also there exist $c, d > 0$ such that

$$\|Id + hC\| \leq 1 - hc$$

and

$$\|(Id + hD)^{-1}\| \leq 1 - hd$$

for h sufficiently small.

We have the following

Lemma 1.3.1: Let

$\mathcal{B}_Y$ = {continuous 2ω-periodic functions $y(\theta)$ with values in Y such that $|y(\theta)| \leq \delta$ and $|y(\theta_1) - y(\theta_2)| \leq \delta|\theta_1 - \theta_2|$}

$\mathcal{B}_Z$ = {continuous 2ω-periodic functions $z(\theta)$ with values in Z such that $|z(\theta)| \leq \delta$ and $|z(\theta_1) - z(\theta_2)| \leq \delta|\theta_1 - \theta_2|$}

where $\delta > 0$ is to be determined.

Then for every $z(\theta) \in \mathcal{B}_Z$ there exists a unique $y(\theta) \in \mathcal{B}_Y$ such that

$$(3.7)\quad y[\theta + h + hf_7(\theta,y(\theta),z(\theta)) + h^2F_7(h,\theta,y(\theta),z(\theta))] = [Id + hC]y(\theta) + hf_5(\theta,y(\theta),z(\theta)) + h^2F_5(h,\theta,y(\theta),z(\theta))$$

Similarly for every $y(\theta) \in \mathcal{B}_Y$ there exists a unique $z(\theta) \in \mathcal{B}_Z$ such that

$$(3.8)\quad z[\theta + h + hf_7(\theta,y(\theta),z(\theta)) + h^2F_7(h,\theta,y(\theta),z(\theta))] = [Id + hD]z(\theta) + hf_6(\theta,y(\theta),z(\theta)) + h^2F_6(h,\theta,y(\theta),z(\theta))$$

Proof: We first prove (3.7). So assume that $z(\theta) \in \mathcal{B}_Z$ is given. Define a mapping $T_z : \mathcal{B}_Y \to \mathcal{B}_Y$ as follows:

Let $g(\theta) \in \mathcal{B}_Y$. Fix θ and let $\bar{\theta}$ be such that

$$(3.9) \qquad \theta = \bar{\theta} + h + hf_7(\bar{\theta}, g(\bar{\theta}), z(\bar{\theta})) + h^2 F_7(h, \bar{\theta}, g(\bar{\theta}), z(\bar{\theta})) \pmod{2\omega}$$

such $\bar{\theta}$ exists and is unique (mod 2ω) since the right hand side of (3.9) is strictly increasing and covers an interval of length 2ω as $\bar{\theta}$ varies from 0 to 2ω.

$T_z\, g$ is defined to be the function satisfying

$$(3.10) \qquad T_z\, g(\theta) = [Id + hC]g(\bar{\theta}) + hf_5(\bar{\theta}, g(\bar{\theta}), z(\bar{\theta})) + h^2 F_5(h, \bar{\theta}, g(\bar{\theta}), z(\bar{\theta}))$$

Clearly $T_z\, g(\theta)$ is 2ω-periodic. Also

$$|T_z\, g(\theta)| \le \|Id + hC\|\, |g(\bar{\theta})| + h|f_5(\bar{\theta}, g(\bar{\theta}), z(\bar{\theta}))| + h^2 |F_5(h, \bar{\theta}, g(\bar{\theta}), z(\bar{\theta}))$$

Now $\|Id + hC\| \le 1 - hc$.

For convenience we let N denote a bound for the norms of f_5, f_6, f_7, F_5, F_6, F_7 and some of the first and second derivatives of these functions for $|y| \le \delta$; $|z| \le \delta$.

Since $f_5(\theta,y,z)$ has a second order zero in (y,z) at $(0,0)$, we have

$$|f_5(\theta,y,z)| \le N\delta^2 .$$

Also $|F_5(h,\theta,y,z)| \le N$.

Hence

$$|T_z\, g(\theta)| \le (1-hc)\delta + hN^2\delta + h^2N$$

$$= \delta\,[1 - hc + hN\delta + h^2N/\delta\,]$$

Thus if we choose $\delta = Kh$ where $K \ge 2N/c$ then for h sufficiently small

$$1 - hc + hN\delta + h^2N\,/\,\delta < 1$$

which gives $|T_z\, g(\theta)| \le \delta$.

To see that $T_z\, g(\theta)$ is Lipschitz continuous, let θ_1, θ_2 be given and let $\overline{\theta}_1$, $\overline{\theta}_2$ be such that

$$\theta_1 = \overline{\theta}_1 + h + hf_7(\overline{\theta}_1, g(\overline{\theta}_1), z(\overline{\theta}_1)) + h^2F_7(h,\overline{\theta}_1, g(\overline{\theta}_1), z(\overline{\theta}_1)) \tag{3.11 a}$$

$$\theta_2 = \overline{\theta}_2 + h + hf_7(\overline{\theta}_2, g(\overline{\theta}_2), z(\overline{\theta}_2)) + h^2F_7(h,\overline{\theta}_2, g(\overline{\theta}_2), z(\overline{\theta}_2)) \tag{3.11 b}$$

Now

$$|f_7(\bar{\theta}_1, g(\bar{\theta}_1), z(\bar{\theta}_1)) - f_7(\bar{\theta}_2, g(\bar{\theta}_2), z(\bar{\theta}_2))|$$

$$\leq N\delta |\bar{\theta}_1 - \bar{\theta}_2| + N|g(\bar{\theta}_1) - g(\bar{\theta}_2)| + N|z(\bar{\theta}_1) - z(\bar{\theta}_2)|$$

$$\leq 3N\delta \; |\bar{\theta}_1 - \bar{\theta}_2|$$

The first estimate follows from the fact that as $f_7(\theta, y, z)$ has a first order zero in (y, z) at $(0, 0)$ so does $\partial f_7(\theta, y, z)/\partial\theta$.

Similarly

$$|F_7(h, \bar{\theta}_1, g(\bar{\theta}_1), z(\bar{\theta}_1)) - F_7(h, \bar{\theta}_2, g(\bar{\theta}_2), z(\bar{\theta}_2))|$$

$$\leq N[|\bar{\theta}_1 - \bar{\theta}_2| + |g(\bar{\theta}_1) - g(\bar{\theta}_2)| + |z(\bar{\theta}_1) - z(\bar{\theta}_2)|]$$

$$\leq N(2\delta + 1) \; |\bar{\theta}_1 - \bar{\theta}_2|$$

Substract (3.11 b) from (3.11 a) and take absolute values, we obtain

$$|\theta_1 - \theta_2| \leq |\bar{\theta}_1 - \bar{\theta}_2| + [3hN\delta + h^2 N(2\delta + 1)] \; |\bar{\theta}_1 - \bar{\theta}_2|$$

Hence

$$|\bar{\theta}_1 - \bar{\theta}_2| \leq [1 - hN(3\delta + 2h\delta + h)]^{-1} \; |\theta_1 - \theta_2|$$

Therefore

$$
\begin{aligned}
&|T_z\, g(\theta_1) - T_z\, g(\theta_2)| \\
&\quad \le \|Id + hC\|\; |g(\overline{\theta}_1) - g(\overline{\theta}_2)| \\
&\quad + h|f_5(\overline{\theta}_1, g(\overline{\theta}_1), z(\overline{\theta}_1)) - f_5(\overline{\theta}_2, g(\overline{\theta}_2), z(\overline{\theta}_2))| \\
&\quad + h^2|F_5(h, \overline{\theta}_1, g(\overline{\theta}_1), z(\overline{\theta}_1)) - F_5(h, \overline{\theta}_2, g(\overline{\theta}_2), z(\overline{\theta}_2))| \\
&\quad \le [(1 - hc)\delta + 3hN\delta^2 + Nh^2(2\delta + 1)]\; |\overline{\theta}_1 - \overline{\theta}_2| \\
&\quad \le [(1 - hc)\delta + 3hN\delta^2 + Nh^2(2\delta + 1)] \\
&\qquad [1 - hN(3\delta + 2h\delta + h)]^{-1}\; |\theta_1 - \theta_2| \\
&\quad \le \delta|\theta_1 - \theta_2|
\end{aligned}
$$

if we choose $\delta = Kh$ where $K \ge 2N/c$ and if h is sufficiently small.

Hence $T_z\, g \in \mathcal{B}_Y$.

We now show that T_z is a contraction map from $\mathcal{B}_Y$ to $\mathcal{B}_Y$.

Let $g_1(\theta)$, $g_2(\theta) \in \mathcal{B}_Y$. Fix θ and let θ_1 and θ_2 be such that

$$
\begin{aligned}
(3.12\text{ a})\qquad \theta &= \theta_1 + h + hf_7(\theta_1, g_1(\theta_1), z(\theta_1)) \\
&\quad + h^2F_7(h, \theta_1, g_1(\theta_1), z(\theta_1))
\end{aligned}
$$

$$
\begin{aligned}
(3.12\text{ b})\qquad \theta &= \theta_2 + h + hf_7(\theta_2, g_2(\theta_2), z(\theta_2)) \\
&\quad + h^2F_7(h, \theta_2, g_2(\theta_2), z(\theta_2))
\end{aligned}
$$

Substract and solve for $|\theta_1 - \theta_2|$ we obtain

$$|\theta_1 - \theta_2| \le h|f_7(\theta_1, g_1(\theta_1), z(\theta_1)) - f_7(\theta_2, g_2(\theta_2), z(\theta_2))|$$
$$+ h^2|F_7(h, \theta_1, g_1(\theta_1), z(\theta_1)) - F_7(h, \theta_2, g_2(\theta_2), z(\theta_2))|$$

Note that

$$|f_7(\theta_1, g_1(\theta_1), z(\theta_1)) - f_7(\theta_2, g_2(\theta_2), z(\theta_2))|$$
$$\le N\delta[|\theta_1 - \theta_2| + |g_1(\theta_1) - g_2(\theta_2)| + |z(\theta_1) - z(\theta_2)|$$

and

$$|g_1(\theta_1) - g_2(\theta_2)| \le |g_1(\theta_1) - g_2(\theta_1)| + |g_2(\theta_1) - g_2(\theta_2)|$$
$$\le \|g_1 - g_2\|_{C^0} + \delta|\theta_1 - \theta_2|$$

where $\|g_1 - g_2\|_{C^0} = \underset{\theta}{\mathrm{Max}}\ |g_1(\theta) - g_2(\theta)|$.

Also

$$|F_5(h, \theta_1, g_1(\theta_1), z(\theta_1)) - F_5(h, \theta_2, g_2(\theta_2), z(\theta_2))|$$
$$\le N[|g_1(\theta_1) - g_2(\theta_2)| + |z(\theta_1) - z(\theta_2)| + |\theta_1 - \theta_2|]$$

Hence

$$|\theta_1 - \theta_2| \le |\theta_1 - \theta_2|[h(N\delta^2 + N\delta + N\delta^2) + h^2N(2\delta + 1)]$$
$$+ \|g_1 - g_2\|_{C^0}[hN\delta + h^2N]$$

Thus

$$|\theta_1 - \theta_2| \le N_1(h\delta + h^2)\|g_1 - g_2\|_{C^0}$$

for some constant $N_1 > 0$ independent of h and δ. Now

$$\text{(3.13 a)} \quad T_z\, g_1(\theta) = (Id + hC)g_1(\theta_1) + hf_5(\theta_1, g(\theta_1), z(\theta_1)) + h^2 F_5(h, \theta_1, g_1(\theta_1), z(\theta_1))$$

$$\text{(3.13 b)} \quad T_z\, g_2(\theta) = (Id + hC)g_2(\theta_2) + hf_5(\theta_2, g_2(\theta_2), z(\theta_2)) + h^2 F_5(h, \theta_2, g_2(\theta_2), z(\theta_2))$$

Therefore

$$\begin{aligned} |T_z\, g_1(\theta) - T_z\, g_2(\theta)| &\le \|Id + hC\|\ |g_1(\theta) - g_2(\theta)| \\ &+ h\ |f_5(\theta_1, g_1(\theta_1), z(\theta_1)) - f_5(\theta_2, g_2(\theta_2), z(\theta_2))| \\ &+ h^2 |F_5(h, \theta_1, g_1(\theta_1), z(\theta_1)) - F_5(h, \theta_2, g_2(\theta_2), z(\theta_2))| \end{aligned}$$

Note that

$$\begin{aligned} &|f_5(\theta_1, g_1(\theta_1), z(\theta_1)) - f_5(\theta_2, g_2(\theta_2), z(\theta_2))| \\ &\le N\delta^2 |\theta_1 - \theta_2| + N\delta\, [|g_1(\theta_1) - g_2(\theta_2)| + |z(\theta_1) - z(\theta_2)|] \\ &\le N\delta \|g_1 - g_2\|_{C^0} + 3N\delta^2 |\theta_1 - \theta_2| \end{aligned}$$

Similarly

$$\begin{aligned} &|F_5(h, \theta_1, g_1(\theta_1), z(\theta_1)) - F_5(h, \theta_2, g_2(\theta_2), z(\theta_2))| \\ &\le N[|\theta_1 - \theta_2| + |g_1(\theta_1) - g_2(\theta_2)| + |z(\theta_1) - z(\theta_2)|] \\ &\le N\|g_1 - g_2\|_{C^0} + (2N\delta + N)\, |\theta_1 - \theta_2| \end{aligned}$$

Hence

$$\begin{aligned}|T_z\, g_1(\theta) - T_z\, g_2(\theta)| &\le (1-hc)[\|g_1 - g_2\|_{C^0} + \delta|\theta_1 - \theta_2|] \\ &\quad + h[N\delta\|g_1 - g_2\|_{C^0} + 3N\delta^2|\theta_1 - \theta_2|] \\ &\quad + h^2[N\|g_1 - g_2\|_{C^0} + (2N\delta + N)|\theta_1 - \theta_2|] \\ &\le [1 - hc + N(h\delta + h^2)]\|g_1 - g_2\|_{C^0} \\ &\quad + [\delta(1-hc) + Nh(3\delta^2 + 2h\delta + h)]|\theta_1 - \theta_2| \\ &\le N_2\|g_1 - g_2\|_{C^0}\end{aligned}$$

where

$$\begin{aligned}N_2 &= 1 - hc + N(h\delta + h^2) \\ &\quad + [\delta(1-hc) + Nh(3\delta^2 + 2h\delta + h)]hN_1(\delta + h) < 1\end{aligned}$$

if $\delta = Kh$ and h is sufficiently small.

By the Contraction Mapping Principle, T_z has a fixed point which is the solution of (3.7).

The existence of a function $z(\theta)$ which solves (3.8) can be obtained in exactly the same fashion. First we assume that $y(\theta) \in \mathcal{B}_Y$ is given, we then construct a mapping $T_Y : \mathcal{B}_z \to \mathcal{B}_z$ given by

$$\begin{aligned}(3.14)\quad g(\bar{\theta}) &= [Id + hD]T_y\, g(\theta) + hf_6(\theta, y(\theta), T_y\, g(\theta)) \\ &\quad + h^2F_6(h, \theta, y(\theta), T_y\, g(\theta))\end{aligned}$$

where $\bar{\theta}$ is given by

(3.15) $\overline{\theta} = \theta + h + hf_7(\theta, y(\theta), g(\theta)) + h^2 F_7(h, \theta, y(\theta), g(\theta))$

We can then proceed to show that T_y is well defined and is a contraction map and hence has a fixed point which is the function $z(\theta)$ that solves (3.8). Q.E.D.

If all the nontrivial characteristic multipliers $\mu_2, \ldots, \mu_n$ satisfy $(\mu_m) < 1$ $(\mu_j > 1)$ then by letting $z(\theta) \equiv 0$ $(y(\theta) \equiv 0)$; Lemma 2.3.2 provides an attracting (repelling) invariant curve for (3.2).

In the more general case, i.e., when not all μ_j satisfy $|\mu_j| < 1$ (or $|\mu_j| > 1$) we actually need a stronger result than Lemma 2.1. More precisely, let $z_1(\theta)$, $z_2(\theta)$ be two functions in $\mathcal{B}_Z$, they then define two mappings T_1, T_2 from $\mathcal{B}_Y$ to $\mathcal{B}_Y$ with fixed points $y_1(\theta)$ and $y_2(\theta)$. We need to show that

$$\|y_2 - y_1\|_{C^0} \le K_1 \|z_1 - z_2\|_{C^0} \quad \text{for some } K_1 < 1.$$

We have

(3.16 a) $$y_1(\theta_1) = [Id + hC] y_1(\theta) + hf_5(\theta, y_1(\theta), z_1(\theta)) + h^2 F_5(h, \theta, y_1(\theta), z_1(\theta))$$

where θ_1 is given by

(3.16 b) $$\theta_1 = \theta + h + hf_7(\theta, y_1(\theta), z_1(\theta)) + h^2 F_7(h, \theta, y_1(\theta), z_1(\theta))$$

and

$$(3.17\,a) \qquad y_2(\theta_2) = [Id + hC]y_2(\theta) + hf_5(\theta, y_2(\theta), z_2(\theta)) + h^2F_5(h, \theta, y_2(\theta), z_2(\theta))$$

where

$$(3.17\,b) \qquad \theta_2 = \theta + h + hf_7(\theta, y_2(\theta), z_2(\theta)) + h^2F_7(h, \theta, y_2(\theta), z_2(\theta))$$

From (3.16 b) and (3.17 b) we obtain

$$|\theta_1 - \theta_2| \leq N_2(h^2 + \delta h)[\|y_1 - y_2\|_{C^0} + \|z_1 - z_2\|_{C^0}]$$

for some constant $N_2 > 0$ independent of h and δ.

From (3.16 a) and (3.17 a) we obtain

$$|y_1(\theta_1) - y_2(\theta_2)| \leq (1 - hc)\|y_1 - y_2\|_{C^0} + N(h\delta + \delta^2)[\|y_1 - y_2\|_{C^0} + \|z_1 - z_2\|_{C^0}]$$

Hence

$$|y_1(\theta_1) - y_2(\theta_1)| \leq |y_1(\theta_1) - y_2(\theta_2)| + |y_2(\theta_2) - y_2(\theta_1)| \leq |y_1(\theta_1) - y_2(\theta_2)| + \delta|\theta_1 - \theta_2|$$

This yields

$$\|y_1 - y_2\|_{C^0} \le \|y_1 - y_2\|_{C^0}[1 - hc + (N + \delta N_2)(h\delta + \delta^2)]$$
$$+ \|z_1 - z_2\|_{C^0}[(N + \delta N_2)(h\delta + \delta^2)]$$

or

$$\|y_1 - y_2\|_{C^0} \le K_1 \, \|z_1 - z_2\|_{C^0}$$

where

$$K_1 = (N + \delta N_2)(h\delta + \delta^2)[hc - (N + \delta N_2)(h\delta + \delta^2)]^{-1} < 1$$

if $\delta = Kh$ and h is sufficiently small.

Similarly given two functions $y_1(\theta)$, $y_2(\theta)$ in $\mathcal{B}_Y$, the solutions $z_1(\theta)$, $z_2(\theta)$ of (3.8) then satisfy

$$\|z_1 - z_2\|_{C^0} \le K_2\|y_1 - y_2\|_{C^0}$$

where $K_2 < 1$ for h small.

Now we can prove that the mapping defined by (2.6) possesses an invariant curve. First we set

$$y^{(0)}(\theta) = z^{(0)}(\theta) \equiv 0 .$$

Inductively given $z^{(k)}(\theta)$ in $\mathcal{B}_Z$, we let $y^{(k+1)}(\theta)$ be the function which is the fixed point of the mapping $T_{z}(k)$. We then let $z^{(k+1)}(\theta)$ be the fixed point of $T_{y^{(k+1)}}$. We then have

$$\|z^{(k+1)} - z^{(k)}\|_{C^0} \leq K_2\|y^{(k+1)} - y^{(k)}\|_{C^0}$$
$$\leq K_2K_1\|z^{(k)} - z^{(k-1)}\|_{C^0}$$

Similarly we also have

$$\|y^{(k+1)} - y^{(k)}\|_{C^0} \leq K_2K_1\|y^{(k)} - y^{(k-1)}\|_{C^0}$$

Hence the sequences $y^{(k)}(\theta)$, $z^{(k)}(\theta)$ converge uniformly to some functions $y(\theta)$, $z(\theta)$.

The curve $\overline{\Gamma} = \{(\theta, y(\theta), z(\theta)) \mid 0 \leq \theta \leq 2\omega\}$ is the desired invariant curve.

Note that $\overline{\Gamma}$ lies in a δ-neighborhood of Γ where $\delta = Kh$. Thus if $h \to 0$, we have $\overline{\Gamma} \to \Gamma$ in the C^0 norm.

Now suppose we allow (3.1) to depend on a real parameter α

(3.18) $$\dot{x} = f(\alpha, x)$$

Suppose (3.8) has a periodic orbit Γ_0 at $\alpha = \alpha_0$ whose nontrivial characteristic multipliers satisfy $|\mu_j| \neq 1$ for $j = 2,\ldots,n$. Using the fact that the Poincare map for (3.18) has a nonsingular Jacobian at $\alpha = \alpha_0$, we obtain a family of periodic orbits Γ_α depending continuously on α for all α near α_0.

Let

$$(3.19) \qquad x_{m+1} = x_m + hf(\alpha,x_m) + h^2F(h,\alpha,x_m)$$

be a difference approximation of (3.18) which arises from some explicit, convergent, single-step method. We now could repeat the arguments given earlier to construct the sequences $y^{(k)}(\alpha,\theta)$, $z^{(k)}(\alpha,\theta)$ depending continuously on α for α near α_0. As before these sequences converge uniformly to $y(\alpha,\theta)$, $z(\alpha,\theta)$ which give a family of invariant curves for (3.19) depending continuously on α for α near 0.

We have proved:

<u>Theorem 1.3.2</u>: Suppose that the system

$$(3.18) \qquad \dot{x} = f(\alpha,x)\ ; \qquad x \in \mathbb{R}^n\ ;$$

f is sufficiently smooth in α and x ,

has a periodic orbit Γ at $\alpha = \alpha_0$ whose characteristic multipliers $\mu_1 = 1, \mu_2,\ldots,\mu_n$ satisfy $|\mu_j| \neq 1$ for $j = 2,\ldots,n$. Then for h sufficiently small, the approximating difference equation

$$(3.19) \qquad x_{m+1} = x_m + hf(\alpha,x_m) + h^2F(h,\alpha,x_m),$$

which arises from some explicit, convergent, single-step method, has a fmily of invariant curves depending continuously on α for α near α_0. As $h \to 0$, these curves converge to the periodic orbits of (3.18). Also if $|\mu_j| < 1$ ($|\mu_j| > 1$) for $j = 2,\ldots,n$ then these curves are attracting (repelling) under (3.19).

Remark 1.3.3: Suppose that (3.18) has a stationary solution for all α near α_0 and that it satisfies the Hopf bifurcation conditions at $\alpha = \alpha_0$. Suppose also that the quantity $b_1(0)$ as in (2.24 a) is positive, then there exists an $\alpha_1 > \alpha_0$ such that (3.18) has a non-trivial attracting periodic orbit for each α satisfying $\alpha_0 < \alpha < \alpha_1$. From Theorem 1.3.2, (3.19) has a family of invariant curves depending on α for $\alpha_2 < \alpha < \alpha_3$ where $\alpha_2 - \alpha_0 = O(h)$ and $\alpha_3 - \alpha_1 = O(h)$. By choosing h small we can make α_2 close to α_0 and then using Theorem 1.2.4, we could trace along the invariant curves back to the stationary solution. Thus we also have continuation of invariant curves for (3.19) near the Hopf bifurcation.

It is well known that the periodic orbit of (3.18) could undergo some secondary bifurcation if one of the nontrivial characteristic multipliers crosses the unit circle (see [12]). We were unable to establish similar results for the invariant curves of (3.19) however.

Remark 1.3.4: Suppose that an implicit, single-step method is employed. We then have a difference equation of the form

$$(3.20) \qquad x_{m+1} = x_m + h\beta_1 f(x_m) + h\beta_0 f(x_{m+1}) + h^2 F(h, x_m, x_{m+1})$$

Fix x and consider the mapping

$$T : y \to x + h\beta_1 f(x) + h\beta_0 f(y) + h^2 F(h,x,y)$$

Let $|x| < M$ for some $M > 0$, by choosing h sufficiently small, T is a contraction map and hence has a fixed point x_1. We thus have

$$x_1 = x + h\beta_1 f(x) + h\beta_0 f(x_1) + h^2 F(h,x,x_1)$$

Also if f and F are sufficiently smooth then x_1 will be smooth in h and x. Since $x_1 = x + O(h)$ we have

$$x_1 = x + h\beta_1 f(x) + h\beta_0 f(x) + O(h^2) \tag{3.21}$$

Assuming that $\beta_0 + \beta_1 = 1$, (3.21) becomes

$$x_1 = x + hf(x) + h^2 F_1(h,x) \tag{3.22}$$

for some function $F_1(h,x)$. We now would apply Tneorem 1.2.4 and Theorem 1.3.2 to obtain invariant curves for (3.22) which are also invariant curves for (3.20).

2. LINEAR MULTISTEP METHODS

2.1 Preliminaries

In this chapter we will approximate

$$\dot{x} = f(x) \; ; \qquad x \in \mathbb{R}^n \tag{1.1}$$

by some linear multistep method. We thus obtain a difference equation of the form

$$x_{m+1} = \sum_{j=1}^{k} \alpha_j \, x_{m+1-j} + h \sum_{j=0}^{k} \beta_j \, f(x_{m+1-j}) \tag{1.2}$$

We will assume throughout the chapter that the method employed is convergent. This means that

$$\sum_{j=0}^{k} \alpha_j = 0 \qquad (\alpha_0 = -1)$$

and $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$.

We also assume that the quantity

$$\sum_{j=0}^{k} \beta_j \neq 0$$

Since (1.2) defines a mapping from $\mathbb{R}^{kn} \to \mathbb{R}^n$ we need a new definition for invariant curves.

Definition 2.1.1: **A curve** $\Gamma \subset \mathbb{R}^n$ is said to be invariant under (1.2) if for every point $x \in \Gamma$, there exist $k-1$ points on Γ $x_{-1},\ldots,x_{-(k-1)}$ such that the points generated by (1.2) with starting points, $x_{-(k-1)},\ldots,x_{-1},x$ all lie on Γ.

Definition 2.1.2: A curve $\Gamma \subset \mathbb{R}^n$ which is invariant under (1.2) is said to be attracting under (1.2) if given any point x sufficiently close to Γ, there exist $k-1$ points $x_{-1},\ldots,x_{-(k-1)}$ such that the points generated by (1.2) with starting points $x_{-(k-1)},\ldots,x_{-1},x$ spiral toward Γ.

2.2 The Hopf Bifurcation

As before we have a family of O.D.E.

$$(2.1) \qquad \dot{x} = f(\mu,x) = A(\mu)x + g(\mu,x); \qquad x \in \mathbb{R}^n$$

where $f(\mu,x)$ is as smooth as needed in μ and x, $g(\mu,x) = O(|x|^2)$ and $A(\mu)$ satisfies the Hopf bifurcation conditions at $\mu = 0$, i.e., $A(\mu)$ has a pair of complex conjugate eigenvalues, $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that

$$(2.2) \qquad \operatorname{Re} \lambda(0) = 0; \quad \operatorname{Re} \lambda'(0) > 0; \quad \operatorname{Im} \lambda(0) \neq 0 .$$

Assume further that all other eigenvalues of $A(\mu)$ stay uniformly away from the imaginary axis for all μ near 0.

Now suppose we approximate (2.1) using some explicit, convergent linear multistep method. We then obtain a difference equation of the form

$$x_{m+1} = \sum_{j=1}^{k} \alpha_j\, x_{m+1-j} + h \sum_{j=1}^{k} \beta_j\, f(\mu, x_{m+1-j}) \tag{2.3}$$

where $\sum_{j=1}^{k} \alpha_j = 1$ and $\sum_{j=1}^{k} j\,\alpha_j = \sum_{j=1}^{k} \beta_j \neq 0.$

Define $y_m = [x_{m+1-k}^T \cdots x_m^T]^T \in \mathbb{R}^{kn}$ (T denotes tranposition). The difference equation (2.3) is then equivalent to

$$y_{m+1} = B\, y_m + hG(\mu, y_m) \tag{2.4}$$

where

$$B = \begin{bmatrix} 0 & I & 0 & . & . & . & 0 \\ . & 0 & I & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 & I \\ \alpha_k I & . & . & . & . & . & \alpha_1 I \end{bmatrix}$$

and

$$G(\mu, y_m) = [0^T \cdots 0^T (\sum_{j=1}^{k} \beta_j f(\mu, x_{m+1-j}))^T]^T$$

where in B, O and I denote the $n \times n$ zero and identity matrices while in $G(\mu, y_m)$, O represents the n-dimensional zero vector.

Assuming that the method employed is strongly stable, i.e., the roots of the polynomial

$$p(\epsilon) = \sum_{j=0}^{k} \alpha_j \, \epsilon^{k-j}$$

are $1, \epsilon_1, \ldots, \epsilon_k$ where $|\epsilon_j| < 1$ for $j = 2, \ldots, k$.

Note that B then has 1 as an eigenvalue with multiplicity n while all other eigenvalues of B have absolute values less than 1. Let

$\mathbb{Z}$ = n-dimensional eigenspace of B corresponding to the eigenvalue 1.

W = $n(k-1)$ dimensional generalized eigenspace of B corresponding to the remaining spectrum of B.

We then can decompose $\mathbb{R}^{kn}$ as:

$$y = Py + (I - P)y \quad \text{for} \quad y \in \mathbb{R}^{kn}$$

where P is a projection matrix along $\mathbb{Z}$ such that $I - P$ is a projection matrix along W.

We now rewrite (2.4) as

$$z_{m+1} = z_m + hPG(\mu, z_m + w_m) \tag{2.5 a}$$

$$w_{m+1} = E\, w_m + h(I - P)G(\mu, z_m + w_m) \tag{2.5 b}$$

where $z_m = P\, y_m$, $z_{m+1} = P\, y_{m+1} \in \mathbb{Z}$

$$w_m = (I-P)y_m, \quad w_{m+1} = (I-P)y_{m+1} \in W$$

Note that the spectrum of E lies inside the unit circle.

Although we are interested in the case $h > 0$, the equation (2.5) however is well-defined for $|h| < \delta$ for some $\delta > 0$. Thus (2.5) defines a mapping

$$S : (h,\mu,z_m,w_m) \to (h,\mu,z_{m+1},w_{m+1})$$

on some neighborhood of $(0,0,0,0)$.

It is easy to see that S satisfies all the hypotheses in the Center Manifold Theorem (Theorem 1.2.3) and thus has an invariant manifold of the form

$$w = u(h,\mu,z); \quad |h| < \delta; \quad |\mu| < \delta; \quad |z| < \delta$$

for some $\delta > 0$.

Note that $u(0,\mu,z) \equiv 0$ and hence we can write

$$u(h,\mu,z) = h\, v(h,\mu,z)$$

where $v(h,\mu,z)$ is uniformly bounded on

$$\mathcal{D} = \{(h,\mu,z) \mid |h| < \delta; \quad |\mu| < \delta; \quad |z| < \delta\} .$$

From (2.5 a) we obtain

$$(2.6)\qquad z_{m+1} = z_m + hPG(\mu, z_m + hv(h,\mu,z_m)) = z_m + hPG(\mu,z_m) + h^2PF(h,\mu,z_m)$$

where $F(h,\mu,z)$ is uniformly bounded on $\mathcal{D}$. Now suppose that

$$z_m = [x^T \cdots x^T]^T$$

where $x \in \mathbb{R}^n$, then

$$G(\mu,z_m) = [0^T \cdots 0^T\ x_1^T]^T$$

where 0 is the n-dimensional zero vector and $x_1 \in \mathbb{R}^n$ is given by

$$x_1 = \sum_{j=1}^{k} \beta_j\ f(\mu,x)$$

Assuming temporarily that the roots ε_j of $p(\varepsilon) = \sum_{j=0}^{k} \alpha_j\ \varepsilon^{k-j}$ are all distinct. Note that W is spanned by the columns of

$$\begin{bmatrix} I & . & . & . & I \\ \varepsilon_2 I & . & . & . & \varepsilon_k I \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \varepsilon_2^{k-1} I & . & . & . & \varepsilon_k^{k-1} I \end{bmatrix}$$

where I is the $n \times n$ identity matrix.

Set

$$D = \begin{bmatrix} I & I & \cdot & \cdot & I \\ I & \epsilon_2 I & \cdot & \cdot & \epsilon_k I \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ I & \epsilon_2^{k-1} I & \cdot & \cdot & \epsilon_k^{k-1} I \end{bmatrix}$$

Since ϵ_j are distinct, D is invertible. Thus there exists $b \in \mathbb{R}^{kn}$ such that

$$G(\mu, z_m) = Db$$

Now if

$$b = [b_1{}^T \cdots b_k{}^T]^T$$

where $b_j \in \mathbb{R}^n$ then

$$P\ G(\mu, z_m) = [b_1{}^T \cdots b_1{}^T]^T$$

To find out what b_1 is we note that the $n \times n$ block in the upper right corner of D^{-1} is of the form dI where I is the $n \times n$ identity matrix and d is given by

$$d = (-1)^{k-1}\ d_1 \mid d_2$$

where

$$d_1 = \begin{vmatrix} 1 & . & . & . & 1 \\ \epsilon_2 & . & . & . & \epsilon_k \\ . & . & . & . & . \\ . & . & . & . & . \\ \epsilon_2^{k-2} & . & . & . & \epsilon_k^{k-2} \end{vmatrix} = \prod_{\substack{j > i \\ 2 \le i, j \le k}} (\epsilon_j - \epsilon_i)$$

and

$$d_2 = \begin{vmatrix} 1 & 1 & . & . & 1 \\ 1 & \epsilon_2 & . & . & \epsilon_k \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \epsilon_2^{k-1} & . & . & \epsilon_k^{k-1} \end{vmatrix} = \prod_{j=2}^{k} (\epsilon_j - 1) \prod_{\substack{j>i \\ 2 \le i, j \le k}} (\epsilon_j - \epsilon_i)$$

Hence

$$d = (-1)^{k-1} / \prod_{j=1}^{k} (\epsilon_j - 1)$$

$$= 1 / \prod_{j=1}^{k} (1 - \epsilon_j)$$

$$= 1 / p'(1)$$

$$= 1 / \sum_{j=0}^{k} \beta_j$$

Therefore

$$b_1 = d \sum_{j=0}^{k} \beta_j \; f(\mu, x) = f(\mu, x)$$

which gives

$$P\,G(\mu,z_m) = [f(\mu,x)^T \cdots f(\mu,x)^T]^T \tag{2.7}$$

In case ε_j are not all distinct, e.g., $\varepsilon_2 = \varepsilon_3$. If $\varepsilon_2 \neq 0$ we use

$$D = \begin{bmatrix} I & I & 0 & I & . & I \\ I & \varepsilon_2 I & \varepsilon_2 I & \varepsilon_4 I & . & \varepsilon_k I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ I & \varepsilon_2^{k-1} I & (k-1)\varepsilon_2^{k-1} I & \varepsilon_4^{k-1} I & . & \varepsilon_k^{k-1} I \end{bmatrix}$$

while if $\varepsilon_2 = 0$ we use

$$D = \begin{bmatrix} I & I & 0 & I & . & I \\ I & 0 & I & \varepsilon_4 I & . & \varepsilon_k I \\ . & . & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ I & 0 & 0 & \varepsilon_4^{k-1} I & . & \varepsilon_k^{k-1} I \end{bmatrix}$$

(2.7) is still valid in either case.

Thus (2.6) induces a mapping on $\mathbb{R}^n$ of the form

$$x_{m+1} = x_m + hf(\mu,x_m) + h^2 F_1(h,\mu,x_m) \tag{2.8}$$

where $F_1(h,\mu,x_m)$ is a component of $PF(h,\mu,z_m)$, here $z_m = [x_m^T \cdots x_m^T]^T$.

Proceed as in Chapter 1, we obtain a family of invariant curves for (2.8) which bifurcates from the zero solution. This implies a similar result for the difference equation (2.3).

Thus we have

Theorem 2.1.1: If the system (2.1) is approximated by a strongly stable, convergent, explicit, linear multistep method then we also obtain a family of invariant curves bifurcating from the zero solution as in Theorem 1.2.4.

Remark 2.1.2: As will be discussed in Remark 2.3.5, some of the requirements in Theorem 2.1.1 may be weaker.

2.3 Existence and Continuation

Suppose that the system

$$(3.1) \qquad \dot{x} = f(x) \; ; \qquad x \in \mathbb{R}^n$$

has a periodic orbit Γ with characteristic multipliers $\mu_1 = 1, \mu_2, \ldots, \mu_n$ satisfying $|\mu_j| < 1$ for $2 \le j \le \ell$ and $|\mu_j| > 1$ for $\ell + 1 \le j \le n$. If (3.1) is approximated by some convergent, explicit, multistep method, we would obtain a difference equation of the form

$$(3.2) \qquad x_{m+1} = \sum_{j=1}^{k} \alpha_j \, x_{m+1-j} + h \sum_{j=1}^{k} \beta_j \, f(x_{m+1-j})$$

Let $\Gamma = \{p(\theta) \mid 0 \leq \theta \leq \omega\}$ be the periodic orbit of (3.1), i.e., $p(\theta)$ is a w-periodic function in θ and satisfies

$$dp(\theta)/d\theta = f(p(\theta))$$

We now imbed Γ in $\mathbb{R}^{kn}$ by define

$$\tilde{\Gamma} = \{\tilde{P}(\theta) \mid 0 \leq \theta \leq \omega\}$$

where $\tilde{p}(\theta) = [p(\theta - (k-1)n)^T \cdots p(\theta)^T]^T$. Similarly given a sequence $x_{m-k+1},\ldots,x_m$ in $\mathbb{R}^n$ we define

$$z_m = [x_{m-k+1}{}^T \cdots x_m{}^T]^T$$

The difference equation (3.2) then defines a mapping $T : \mathbb{R}^{kn} \to \mathbb{R}^{kn}$ by

$$Tz_m = z_{m+1}$$

where $z_{m+1} = [x_{m-k+2}{}^T \cdots x_{m+1}{}^T]^T$.

Define

$$\int(\theta) = \partial T/\partial z\ (\tilde{p}(\theta)) \tag{3.3}$$

Then for

$$z = [y_1{}^T \cdots y_k{}^T]^T \text{ where } y_j \in \mathbb{R}^n$$

we have

$$\int(\theta)\, z = [y_2^T \cdots\cdot y_k^T\ y_{k+1}^T]^T$$

where

$$(3.4) \qquad y_{m+1} = \sum_{j=1}^{k} \alpha_i\, y_{m+1-j} + h \sum_{j=0}^{k} \beta_j\, \partial f/\partial x\, (P(\theta + (1-j)h) y_{m+1-j}$$

We now develop a local coordinate system around $\tilde{\Gamma}$. Assuming that μ_j are distinct then the variational equation

$$(3.5) \qquad dy(\theta)/d\theta = \partial f(\partial x(P(\theta))y\,; \qquad y \in \mathbb{R}^n$$

has n linearly independent solutions of the form

$$q_1(\theta) = dP(\theta)/d(\theta) \overset{\text{def}}{=} v_1(\theta)$$

$$q_j(\theta) = e^{\lambda_j \theta}\, v_j(\theta) \quad \text{for} \quad 2 \leq j \leq n$$

where $v_j(\theta)$ are 2ω-periodic in θ and λ_j are real numbers such that $e^{2\lambda_j \omega} = \mu_j^2$. We have choosen $v_j(\theta)$ to be 2ω-periodic in order to ensure that λ_j are real. Also as $\{q_j(\theta) \mid 1 \leq j \leq k\}$ are linearly independent at any θ, so are $\{v_j(\theta) \mid 1 \leq j \leq k\}$.

As $v_j(\theta)$ form a moving coordinate system around Γ, we would like to obtain a moving coordinate system around $\tilde{\Gamma}$ based on these functions.

Set $\lambda_1 = 0$ so that $q_j(\theta) = e^{\lambda_j \theta}\, v_j(\theta)$ for $1 \leq j \leq n$. Define

(3.6) $$z_j(\theta) = [u_{j1}(\theta)^T \cdots u_{jk}(\theta)^T]^T \in \mathbb{R}^{kn}$$

where $u_{j\ell}(\theta) = e^{-\lambda_j(k-\ell)h} v_j(\theta - (k-\ell)h) \in \mathbb{R}^n$.

Note that

$$z_j(\theta) = e^{-\lambda_j\theta} [q_j(\theta - (k-1)h)^T \cdots q_j(\theta)^T]^T$$

Hence using the facts that the local truncation error induced by a convergent method is of order $O(h^2)$ and that $q_j(\theta)$ is a solution of (3.5), we obtain

(3.7) $$\begin{aligned} &\int(\theta)\, z_j(\theta) \\ &= e^{-\lambda_j\theta}[q_j(\theta-(k-2)h)^T \cdots q_j(\theta)^T q_j(\theta+h)^T]^T + O(h^2) \\ &= e^{h\lambda_j} z_j(\theta+h) + O(h^2) \end{aligned}$$

Now assume that the method employed in (3.2) is strongly stable, i.e., the polynomial $p(\varepsilon) = \sum_{j=0}^{k} \alpha_j \varepsilon^{k-j}$ has k roots $1, \varepsilon_2, \ldots, \varepsilon_k$ where $|\varepsilon_j| < 1$ for $j = 2,\ldots,k$. Also for simplicity we will assume that ε_j are all distinct.

For $m = 2,\ldots,k$ we define

(3.8) $$e_{mj}(\theta) = [v_j(\theta)^T\ \varepsilon_m v_j(\theta)^T \cdots \varepsilon_m^{k-1} v_j(\theta)^T]^T$$

for $j = 1,\ldots,n$. We then have

(3.9) $$\begin{aligned} \int(\theta)\, e_{mj}(\theta) &= \varepsilon_m e_{mj}(\theta) + O(h) \\ &= \varepsilon_m e_{mj}(\theta+h) + O(h) \end{aligned}$$

assuming that $v_j(\theta)$ are at least C^1.

Now note that when $h = 0$, $z_j(\theta)$ becomes

$$z_j(\theta) = [v_j(\theta)^T \cdots v_j(\theta)^T]^T$$

Hence if

$$\sum_{j=1}^{n} a_{1j} z_j(\theta) + \sum_{m=2}^{k} \sum_{j=1}^{n} a_{mj} e_{mj}(\theta) = 0$$

then the linear independence of $v_j(\theta)$ implies that

$$a_{1j} + \sum_{m=2}^{k} \epsilon_m^i a_{mj} = 0$$

for $j = 1,2,\ldots,n$ and $i = 0,1,\ldots,k-1$. This in turn implies

$$\begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \epsilon_2 & \cdot & \cdot & \cdot & \epsilon_2^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \epsilon_k & \cdot & \cdot & \cdot & \epsilon_k^{k-1} \end{bmatrix} \begin{bmatrix} a_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a_{kj} \end{bmatrix} = 0$$

which yields $a_{mj} = 0$ for $1 \leq m \leq k$; $1 \leq j \leq n$. Let

$D(h,\theta)$ = determinant of the $(kn) \times (kn)$ matrix whose columns are $z_1(\theta),\ldots,z_n(\theta)$, $e_{21}(\theta),\ldots,e_{2n}(\theta),\ldots\ldots,e_{k1}(\theta),\ldots,e_{kn}(\theta)$

We have just shown that $D(0,\theta) \neq 0$ for all θ. Since $D(h,\theta)$ is continuous in h, θ and periodic in θ there exists a $\delta > 0$ such that

$$D(h,\theta) \neq 0 \quad \text{for} \quad 0 \le h < \delta \quad \text{for all} \quad \theta .$$

Hence the vectors $z_1(\theta),\ldots,z_n(\theta)$, $e_{21}(\theta),\ldots,e_{2n}(\theta)$, $\ldots$, $e_{k1}(\theta),\ldots,e_{kn}(\theta)$ are linearly independent at any θ if h is sufficiently small.

In the general case when not all μ_j are distinct, (3.5) has a fundamental matrix solution of the form $X(\theta) = V(\theta)e^{\Lambda\theta}$ where $V(\theta)$ is 2ω-periodic and Λ is a real $n \times n$ matrix whose eigenvalues are $\lambda_1,\ldots,\lambda_n$ where $e^{2\lambda_j w} = \mu_j^2$. We simply choose $v_j(\theta)$ to be the columns of $V(\theta)$ and $q_j(\theta)$ the columns of $X(\theta)$.

Similarly if ε_j are not all distinct, e.g., $\varepsilon_2 = \varepsilon_3$ we choose

$$e_{2j} = \text{as before}$$

$$e_{3j} = \begin{cases} [0 \;\; \varepsilon_2 v_j(\theta)^T \;\cdots\; (k-1)\varepsilon_2^{k-1}\, v_j(\theta)^T]^T & \text{if} \quad \varepsilon_2 \neq 0 \\ [0 \;\; v_j(\theta)^T \;\; 0 \;\cdots\; 0]^T & \text{if} \quad \varepsilon_2 = 0 \end{cases}$$

We then can proceed as before to show that the vectors $z_1(\theta),\ldots,z_n(\theta)$, $e_{21}(\theta),\ldots,e_{2n}(\theta),\ldots,e_{k1}(\theta),\ldots,e_{kn}(\theta)$ are linearly independent at any θ.

Thus we have the following

<u>Lemma 2.3.1</u>: There exists a neighborhood of $\tilde{\Gamma}$ in $\mathbb{R}^{kn}$ such that for any z in this neighborhood, there exists a unique (θ,r,w) such that

$$(3.9) \qquad z = \tilde{p}(\theta) + Z(\theta)r + E(\theta)w$$

where $Z(\theta)$ is the $(kn) \times (n-1)$ matrix whose columns are $z_2(\theta),\ldots,z_n(\theta)$, $E(\theta)$ is the $(kn) \times [(k-1)n]$ matrix whose columns are $e_{mj}(\theta)$; $j = 1,\ldots,n$; $m = 2,\ldots,k$. Also $r \in \mathbb{R}^{n-1}$ and $w \in \mathbb{R}^{(k-1)n}$.

<u>Proof</u>: Set $F(\theta,r,w,z) = z - [\tilde{p}(\theta) + Z(\theta)r + E(\theta)w]$ then

$$F(\theta,0,0,\tilde{p}(\theta)) = 0$$

and

$$\partial F/\partial\theta(\theta,0,0,\tilde{p}(\theta)) = z_1(\theta)$$

$$\partial F/\partial r(\theta,0,0,\tilde{p}(\theta)) = Z(\theta)$$

$$\partial F/\partial w(\theta,0,0,\tilde{p}(\theta)) = E(\theta)$$

Since $[z_1(\theta),Z(\theta),E(\theta)]$ is invertible at all θ, Lemma 3.1 follows from the Implicit Function Theorem and the fact that $\tilde{\Gamma}$ is compact. Q.E.D.

We note that since $Z(\theta)$ depends continuously on h and $E(\theta)$ is independent of h. The local coordinates (3.9) is valid for all $|r| < \delta$, $|w| < \delta$ where $\delta > 0$ can be chosen to be independent of h provided that $h > 0$

is sufficiently small. Also by assuming that the system (3.1) is sufficiently smooth, θ, r, w will depend as smoothly on z as needed. Let v_1, v_2, v_3 be appropriate functions such that

$$\theta = v_1(z); \quad r = v_2(\theta); \quad w = v_3(z)$$

for all z sufficiently close to $\tilde{\Gamma}$.

Let $z = \tilde{p}(\theta) + Z(\theta)r + E(\theta)w$ be sufficiently close to $\tilde{\Gamma}$, then

$$Tz = T(\tilde{p}(\theta)) + \int(\theta)[Z(\theta)r + E(\theta)w] + hg_1(h,\theta,r,w) \tag{3.10}$$

where $g_1(h,\theta,r,w)$ has second order zero in (r,w) at $(0,0)$.

We have

$$T(\tilde{p}(\theta)) = \tilde{p}(\theta + h) + O(h^2) \tag{3.11 a}$$

$$\int(\theta)Z(\theta) = Z(\theta + h)A + O(h^2) \tag{3.11 b}$$

$$\int(\theta)E(\theta) = E(\theta + h)D + O(h) \tag{3.11 c}$$

where

$$Z(\theta) = [z_1(\theta) \ldots z_n(\theta)]$$

$$E(\theta) = [e_{21}(\theta) \ldots e_{2n}(\theta) \ldots\ldots e_{k1}(\theta) \ldots e_{kn}(\theta)]$$

$$A = e^{\Lambda h}$$

and D is a $[(k-1)n] \times [(k-1)n)$ matrix satisfying $\|D\| \leq \epsilon < 1$.

Set $\overline{z} = \tilde{p}(\theta + h) + Z(\theta + h)Ar + E(\theta + h)Dw$ then

$$Tz - \overline{z} = hg_1(h,\theta,r,w) + hG_1(h,\theta,w) + h^2F_1(h,\theta,r,w)$$

the first term arises from (3.10), the second from (3.11 c) and the third from (3.11 a) and (3.11 b). The function $g_1(h,\theta,r,w)$ has second order zero in (r,w), $G_1(h,\theta,w)$ has first order zero in w while $F_1(h,\theta,0,0) \not\equiv 0$ in general.

Thus if $Tz = \tilde{p}(\theta_1) + Z(\theta_1)r_1 + E(\theta_1)w_1$ then

(3.12 a) $$\begin{aligned}\theta_1 &= v_1(Tz)\\ &= v_1(\overline{z}) + \partial v_1 \,|\, \partial z \;(\overline{z})\; [Tz - \overline{z}] + O(h^2)\\ &= \theta + h + hg_2(h,\theta,r,w) + hG_2(h,\theta,r,w) + h^2F_2(h,\theta,r,w)\end{aligned}$$

where $g_2(h,\theta,r,w)$ has second order zero in (z,w) at $(0,0)$ while $G_2(h,\theta,r,w)$ has first order zero in w.

Similarly we have

(3.12 b) $$r_1 = Ar + hg_3(h,\theta,r,w) + hG_3(h,\theta,r,w) + h^2F_3(h,\theta,r,w)$$

(3.12 c) $$w_1 = Dw + hg_4(h,\theta,r,w) + hG_4(h,\theta,r,w) + h^2F_4(h,\theta,r,w)$$

where g_3 and g_4 have second order zeroes in (r,w) while G_3 and G_4 have first order zeroes in w.

We now show that (3.12) possesses an invariant curve.

Rearrange the eigenvalues $1 + e^{h\lambda_j}$; $j = 2,\ldots,n$ of A so that $\operatorname{Re} \lambda_j < 0$ for $j = 2,\ldots,\ell$ and $\operatorname{Re} \lambda_j > 0$ for $j = \ell + 1,\ldots,n$. Let

Y = generalized eigenspace of A corresponding to the eigenvalues $1 + e^{h\lambda_j}$; $j = 2,\ldots,\ell$.

$\mathbb{Z}$ = generalized eigenspace of A corresponding to the eigenvalues $1 + e^{h\lambda_j}$; $j = \ell + 1,\ldots,n$.

By projecting to Y and $\mathbb{Z}$ we can write (3.12) as

$$\theta_1 = \theta + h + hg_5(h,\theta,y,z,w) + hG_5(h,\theta,y,z,w) + h^2F_5(h,\theta,y,z,w) \tag{3.13 a}$$

$$y_1 = By + hg_6(h,\theta,y,z,w) + hG_6(h,\theta,y,z,w) + h^2F_6(h,\theta,y,z,w) \tag{3.13 b}$$

$$z_1 = Cz + hg_7(h,\theta,y,z,w) + hG_7(h,\theta,y,z,w) + h^2F_7(h,\theta,y,z,w) \tag{3.13 c}$$

$$w_1 = Dw + hg_8(h,\theta,y,z,w) + hG_8(h,\theta,y,z,w) + h^2F_8(h,\theta,y,z,w) \tag{3.13 d}$$

where g_5, g_6, g_7, g_8 have second order zeroes in (y,z,w) at $(0,0,0)$; G_5, G_6, G_7, G_8 have first order zeroes in w.

For h sufficiently small we also have

$$\|B\| \leq 1 - hb$$

$$\|C^{-1}\| \leq 1 - hc$$

$$\|D\| \leq \varepsilon < 1$$

for some constant $b, c > 0$.

As in Chapter 1, we introduce the spaces

$\mathcal{B}_Y$ = {continuous 2ω-periodic functions $y(\theta)$ with values in Y such that $|y(\theta)| \leq \delta$ and $|y(\theta_1) - y(\theta_2)| \leq \delta|\theta_1 - \theta_2|$

$\mathcal{B}_{\mathbb{Z}}$ = {continuous 2ω-periodic functions $z(\theta)$ with values in $\mathbb{Z}$ such that $|z(\theta)| \leq \delta$ and $|z(\theta_1) - z(\theta_2)| \leq \delta|\theta_1 - \theta_2|$}

$\mathcal{B}_W$ = {continuous 2ω-periodic functions $w(\theta)$ with values in $\mathbb{R}^{(k-1)n}$ such that $|w(\theta)| \leq h\delta$ and $|w(\theta_1) - w(\theta_2)| \leq h\delta|\theta_1 - \theta_2|$}

Lemma 2.3.2: Given $z(\theta) \in \mathcal{B}_Z$, there exists a unique pair of functions $y(\theta)$, $w(\theta)$ in $\mathcal{B}_Y$ and $\mathcal{B}_W$ respectively such that

$$\begin{aligned}(3.14\text{ a})\qquad y(\theta_1) = {} & By(\theta) + hg_6(h,\theta,y(\theta),z(\theta),w(\theta)) \\ & + hG_6(h,\theta,y(\theta),z(\theta),w(\theta)) \\ & + h^2F_6(h,\theta,y(\theta),z(\theta),w(\theta))\end{aligned}$$

$$(3.14\,b) \qquad w(\theta_1) = Dw(\theta) + hg_8(h,\theta,y(\theta),z(\theta),w(\theta)) + hG_8(h,\theta,y(\theta),z(\theta),w(\theta)) + h^2F_8(h,\theta,y(\theta),z(\theta),w(\theta))$$

where θ_1 is given by

$$(3.15) \qquad \theta_1 = \theta + h + hg_5(h,\theta,y(\theta),z(\theta),w(\theta)) + hG_5(h,\theta,y(\theta),z(\theta),w(\theta)) + h^2F_5(h,\theta,y(\theta),z(\theta),w(\theta))$$

Similarly given a pair of functions $y(\theta), w(\theta)$ in $\mathcal{B}_Y$ and $\mathcal{B}_W$ respectively, there exists a unique function $z(\theta)$ in $\mathcal{B}_{\mathbb{Z}}$ such that

$$(3.16) \qquad z(\theta_1) = Cz(\theta) + hg_7(h,\theta,y(\theta),z(\theta),w(\theta)) + hG_7(h,\theta,y(\theta),z(\theta),w(\theta)) + h^2F_7(h,\theta,y(\theta),z(\theta),w(\theta))$$

where θ_1 is given by (3.15).

Proof: Let $z(\theta) \in \mathcal{B}_{\mathbb{Z}}$ be given. We then construct a mapping $T_z : \mathcal{B}_Y \times \mathcal{B}_W \to \mathcal{B}_Y \times \mathcal{B}_W$ as follows:

Let $u(\theta)$, $v(\theta)$ be a pair of functions in $\mathcal{B}_Y$ and $\mathcal{B}_W$ respectively. Fix θ and let $\bar{\theta}$ be such that

$$(3.17)\qquad \theta = \bar\theta + h + hg_5(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + hG_5(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + h^2F_5(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) \pmod{2\omega}$$

such $\bar\theta$ exists since the right hand side of (3.17) is strictly increasing and covers an interval of length 2ω as $\bar\theta$ varies from 0 to 2ω.

We then define $T_z(u,v) = (\tilde u,\tilde v)$ where

$$(3.18\text{a})\qquad \tilde u(\theta) = Bu(\bar\theta) + hg_6(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + hG_6(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + h^2F_6(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta))$$

$$(3.18\text{b})\qquad \tilde v(\theta) = Dv(\bar\theta) + hg_8(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + hG_8(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)) + h^2F_8(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta))$$

Let N denote a bound for the norms of g_6, g_7, g_8, G_6, G_7, G_8, F_6, F_7, F_8 and some of their first and second derivatives on the neighborhood $|y| \le \delta$; $|z| \le \delta$; $|w| \le h\delta$. Using the fact that $g_6(h,\theta,y,z,w)$ has a second order zero in (y,z,w) at $(0,0,0)$ we have

$$|g_6(h,\bar\theta,u(\bar\theta),z(\bar\theta),v(\bar\theta)| \le N[|u(\bar\theta)|^2 + |z(\bar\theta)|^2 + |v(\bar\theta)|^2] \le 2N\delta^2 + h^2N\delta^2$$

Similarly since $G_6(h,\theta,y,z,w) = O(|w|)$ we have

$$|G_6(h,\overline{\theta},u(\overline{\theta}),z(\overline{\theta}),v(\overline{\theta})| \leq N|v(\overline{\theta})|$$
$$\leq hN\delta$$

and of course

$$|F_6(h,\overline{\theta},u(\overline{\theta}),z(\overline{\theta}),v(\overline{\theta})| \leq N$$

Hence (3.11 a) gives

$$|\tilde{u}(\theta)| \leq (1-hb)|u(\overline{\theta})| + 2hN\delta^2 + h^3N\delta^2 + h^2N\delta + h^2N$$
$$\leq \delta[1 - hb + 2hN\delta + h^3N\delta + h^2N + h^2N/\delta]$$
$$\leq \delta$$

if $\delta = Kh$ where $K \geq 2N/b$ and h is sufficiently small.

Similarly we also have

$$|\tilde{v}(\theta)| \leq \varepsilon|v(\overline{\theta})| + 2hN\delta^2 + h^3N\delta^2 + h^2N\delta + h^2N$$
$$\leq h\delta[\varepsilon + 2N\delta + h^2N\delta + hN + hN/\delta]$$
$$\leq h\delta$$

if $\delta = Kh$ where $K > N/(1-\varepsilon)$ and h is sufficiently small.

To see that $\tilde{u}(\theta)$ and $\tilde{v}(\theta)$ are Lipschitz continuous, let θ_1, θ_2 be given and let $\overline{\theta}_1$, $\overline{\theta}_2$ be such that

$$(3.19a)\quad \theta_1 = \bar{\theta}_1 + h + hg_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1)) + hG_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1)) + h^2F_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1))$$

$$(3.19b)\quad \theta_2 = \bar{\theta}_2 + h + hg_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2)) + hG_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2)) + h^2F_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2))$$

We have

$$\begin{aligned}|g_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1)) - g_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2))| &\leq N\delta^2|\bar{\theta}_1 - \bar{\theta}_2| + N\delta|u(\bar{\theta}_1) - u(\bar{\theta}_2)| + N\delta|z(\bar{\theta}_1) - z(\bar{\theta}_2)| \\ &\quad + N\delta|v(\bar{\theta}_1) - v(\bar{\theta}_2)| \\ &\leq (3N\delta^2 + hN\delta^2)\ |\bar{\theta}_1 - \bar{\theta}_2|\end{aligned}$$

The first estimate follows from the fact that as $g_5(h,\theta,y,z,w)$ has second order zero in (y,z,w) at $(0,0,0)$ so does $\partial g_5(h,\theta,y,z,w)/\partial\theta$.

Similarly since $G_5(h,\theta,y,z,w) = O(|w|)$ we also have $\partial G_5(h,\theta,y,z,w)/\partial\theta = O(|w|)$; $\partial G_5(h,\theta,y,z,w)/\partial y = O(|w|)$ and $\partial G_5(h,\theta,y,z,w)/\partial z = O(|w|)$. Hence

$$|G_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1)) - G_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2))$$

$$\leq Nh\delta\,[|\bar{\theta}_1 - \bar{\theta}_2| + |u(\bar{\theta}_1) - u(\bar{\theta}_2)| + |z(\bar{\theta}_1) - z(\bar{\theta}_2)|]$$

$$+ N|v(\bar{\theta}_1) - v(\bar{\theta}_2)|$$

$$\leq Nh\delta\,(2 + 2\delta)\ |\bar{\theta}_1 - \bar{\theta}_2|$$

Also

$$|F_5(h,\bar{\theta}_1,u(\bar{\theta}_1),z(\bar{\theta}_1),v(\bar{\theta}_1)) - F_5(h,\bar{\theta}_2,u(\bar{\theta}_2),z(\bar{\theta}_2),v(\bar{\theta}_2))|$$

$$\leq N[|\bar{\theta}_1 - \bar{\theta}_2| + |u(\bar{\theta}_1) - u(\bar{\theta}_2)| + |z(\bar{\theta}_1) - z(\bar{\theta}_2)|$$

$$+ |v(\bar{\theta}_1) - v(\bar{\theta}_2)|]$$

$$\leq N[1 + 2\delta + h\delta]\ |\bar{\theta}_1 - \bar{\theta}_2|$$

Hence

$$|\bar{\theta}_1 - \bar{\theta}_2| \leq \gamma_1 |\theta_1 - \theta_2|$$

where

$$\gamma_1 = [1 - h(3N\delta^2 + hN\delta^2) - h^2 N\delta(2 + 2\delta) - h^2 N(1 + 2\delta + h\delta)]^{-1}$$

Note that the estimates for g_j, G_j, F_j are the same as those for g_5, G_5, F_5 when $j = 6,7$.

Therefore

$$
\begin{aligned}
|\tilde{u}(\theta_1) - \tilde{u}(\theta_2)| &\le \|B\| \; |u(\overline{\theta}_1) - u(\overline{\theta}_2)| \\
&\quad + h|g_6(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - g_6(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\quad + h|G_6(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - G_6(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\quad + h^2|F_6(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - F_6(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\le [(1-hb)\delta + h(3N\delta^2 + hN\delta^2) + h^2N\delta(2+2\delta) \\
&\qquad + h^2N(1+2\delta+h\delta)] \quad |\overline{\theta}_1 - \overline{\theta}_2| \\
&\le \delta \; \gamma_2\gamma_1 \; |\theta_1 - \theta_2|
\end{aligned}
$$

where

$$\gamma_2 = 1 - hb + h(3N\delta + hN\delta) + h^2N(2+2\delta) + h^2N(1+2\delta+h\delta)/\delta$$

Note that

$$\gamma_2\gamma_1 \le 1$$

if $\delta = Kh$ where $K \ge 2N/b$ and if h is sufficiently small.

Hence $\tilde{u} \in \mathcal{B}_Y$.

Similarly

$$
\begin{aligned}
|\tilde{v}(\theta_1) - \tilde{v}(\theta_2)| &\le \varepsilon |v(\overline{\theta}_1) - v(\overline{\theta}_2)| \\
&\quad + h|g_8(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - g_8(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\quad + h|G_8(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - G_8(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\quad + h^2|F_8(h,\overline{\theta}_1,u(\overline{\theta}_1),z(\overline{\theta}_1),v(\overline{\theta}_1)) \\
&\qquad - F_8(h,\overline{\theta}_2,u(\overline{\theta}_2),z(\overline{\theta}_2),v(\overline{\theta}_2))| \\
&\le [\varepsilon h\delta + h(3N\delta^2 + hN\delta^2) + h^2 N\delta(2+2\delta) \\
&\qquad + h^2 N(1 + 2\delta + h\delta)] \quad |\overline{\theta}_1 - \overline{\theta}_2| \\
&\le h\delta \;\; \gamma_3\gamma_1 \;\; |\theta_1 - \theta_2|
\end{aligned}
$$

where

$$\gamma_3 = \varepsilon + 3N\delta + hN\delta + hN(2+2\delta) + hN(1 + 2\delta + h\delta)/\delta$$

where $\gamma_3\gamma_1 \le 1$ if $\delta = Kh$ where $K > N/(1-\varepsilon)$ and if h is sufficiently small.

Thus $\tilde{v}(\theta) \in \mathcal{B}_W$.

We now show that T_z is a contraction map. To that end let (u_1,v_1) and $(u_2,v_2) \in \mathcal{B}_Y \times \mathcal{B}_W$. Fix θ and let θ_1 and θ_2 be such that

$$
\begin{aligned}
\text{(3.20 a)} \qquad \theta = \theta_1 &+ h + hg_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) \\
&+ hG_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) \\
&+ h^2 F_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1))
\end{aligned}
$$

$$(3.20\text{b}) \qquad \theta = \theta_2 + h + hg_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2)) + hG_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2)) + h^2F_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2))$$

Note that

$$|g_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) - g_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2))|$$
$$\leq N\delta^2|\theta_1 - \theta_2| + N\delta[|u_1(\theta_1) - u_2(\theta_2)| + |z(\theta_1) - z(\theta_2)| + |v_1(\theta_1) - v_1(\theta_2)|]$$
$$\leq |\theta_1 - \theta_2| \; [3N\delta^2 + hN\delta^2] + N\delta[\|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}]$$

Similarly

$$|G_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) - G_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2))|$$
$$\leq Nh\delta[|\theta_1 - \theta_2| + |u_1(\theta_1) - u_2(\theta_2)| + |z(\theta_1) - z(\theta_2))] + N|v_1(\theta_1) - v_2(\theta_2)|$$
$$\leq (Nh\delta + 3Nh\delta^2)|\theta_1 - \theta_2| + Nh\delta\|u_1 - u_2\|_{C^0} + N\|v_1 - v_2\|_{C^0}$$

$$|F_5(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) - F_5(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2))|$$

$$\leq N[|\theta_1 - \theta_2| + |u_1(\theta_1) - u_2(\theta_2)| + |z(\theta_1) - z(\theta_2| + |v_1(\theta_1) - v_2(\theta_2)|]$$

$$\leq N[1 + 2\delta + h\delta]|\theta_1 - \theta_2| + N[\|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}]$$

Hence

$$|\theta_1 - \theta_2| \leq N_1[(h\delta + h^2\delta + h^2)\|u_1 - u_2\|_{C^0} + (h\delta + h + h^2)\|v_1 - v_2\|_{C^0}]$$

where $N_1 > 0$ is some constant independent of δ and h.

From

(3.21 a) $$\tilde{u}_1(\theta) = Bu_1(\theta_1) + hg_6(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) + hG_6(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1)) + h^2F_6(h,\theta_1,u_1(\theta_1),z(\theta_1),v_1(\theta_1))$$

(3.21 b) $$\tilde{u}_2(\theta) = Bu_2(\theta_2) + hg_6(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2)) + hG_6(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2)) + h^2F_6(h,\theta_2,u_2(\theta_2),z(\theta_2),v_2(\theta_2))$$

We have

$$|\tilde{u}_1(\theta) - \tilde{u}_2(\theta)| \le (1 - hb)(\|u_1 - u_2\|_{C^0} + \delta|\theta_1 - \theta_2|)$$
$$+ hN[(3\delta^2 + h\delta^2)|\theta_1 - \theta_2| + \delta(\|u_1 - u_2\|_{C^0}$$
$$+ \|v_1 - v_2\|_{C^0})]$$
$$+ hN[(h\delta + 3h\delta^2)|\theta_1 - \theta_2|$$
$$+ h\delta\|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}]$$
$$+ h^2N[(1 + 2\delta + h\delta)|\theta_1 - \theta_2|$$
$$+ \|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}]$$
$$\le (1 - hb_1)\|u_1 - u_2\|_{C^0} + hK_1\|v_1 - v_2\|_{C^0}$$

for some constants $b_1, K_1 > 0$. Hence

$$\|\tilde{u}_1 - \tilde{u}_2\|_{C^0} \le (1 - hb_1)\|u_1 - u_2\|_{C^0} + hK_1\|v_1 - v_2\|_{C^0}$$

Similarly from

$$\text{(3.22 a)} \quad \tilde{v}_1(\theta) = Dv_1(\theta_1) + hg_8(h, \theta_1, u_1(\theta_1), z(\theta_1), v_1(\theta_1))$$
$$+ hG_8(h, \theta_1, u_1(\theta_1), z(\theta_1), v_1(\theta_1))$$
$$+ h^2F_8(h, \theta_1, u_1(\theta_1), z(\theta_1), v_1(\theta_1))$$

$$\text{(3.22 b)} \quad \tilde{v}_2(\theta) = Dv_2(\theta_2) + hg_8(h, \theta_2, u_2(\theta_2), z(\theta_2), v_2(\theta_2))$$
$$+ hG_8(h, \theta_2, u_2(\theta_2), z(\theta_2), v_2(\theta_2))$$
$$+ h^2F_8(h, \theta_2, u_2(\theta_2), z(\theta_2), v_2(\theta_2))$$

We have

$$
\begin{aligned}
|\tilde{v}_1(\theta) - \tilde{v}_2(\theta)| \le \; & \varepsilon(\|v_1 - v_2\|_{C^0} + h\delta|\theta_1 - \theta_2|) \\
& + hN[(3\delta^2 + h\delta^2)|\theta_1 - \theta_2| \\
& \quad + \delta(\|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0})] \\
& + hN[(h\delta + 3h\delta^2)|\theta_1 - \theta_2| \\
& \quad + h\delta\|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}] \\
& + h^2N[(1 + 2\delta + h\delta)|\theta_1 - \theta_2| \\
& \quad + \|u_1 - u_2\|_{C^0} + \|v_1 - v_2\|_{C^0}] \\
\le \; & \varepsilon_1\|v_1 - v_2\|_{C^0} + h^2K_2\|u_1 - u_2\|_{C^0}
\end{aligned}
$$

for some $0 < \varepsilon_1 < 1$, $K_2 > 0$, assuming that $\delta = Kh$ and h is sufficiently small.

Hence

$$\|\tilde{v}_1 - \tilde{v}_2\|_{C^0} \le \varepsilon_1\|v_1 - v_2\|_{C^0} + h^2K_2\|u_1 - u_2\|_{C^0}$$

Since the matrix $\begin{bmatrix} 1 - hb_1 & hK_2 \\ h^2K_2 & \varepsilon_1 \end{bmatrix}$ has spectrum inside the unit circle, T_z is a contraction map. The fixed point of T_z is the pair of functions $(y(\theta), w(\theta))$ in $\mathcal{B}_Y \times \mathcal{B}_W$ which satisfy (3.14).

The proof of the second part of Lemma 2.3.2 is similar and hence omitted. Q.E.D.

If the nontrivial characteristic multipliers $\mu_2, \ldots, \mu_n$ of (3.1) satisfy $|\mu_j| < 1$ for $j = 2, \ldots, n$,

then by letting $z(\theta) \equiv 0$, Lemma 2.3.2 provides an attracting invariant curve under T which then yields an attracting invariant curve for (3.2).

In the more general case, i.e., when not all μ_j satisfy $|\mu_j| < 1$, we actually need a stronger result than Lemma 2.3.2. More precisely, let $z_1(\theta)$ and $z_2(\theta)$ be two functions in $\mathcal{B}_Z$. Let (y_1, w_1) and (y_2, w_2) be the fixed points of T_{z_1} and T_{z_2} as provided by Lemma 2.3.2.

Let θ be fixed and θ_1, θ_2 be such that

$$\text{(3.23 a)} \quad \theta = \theta_1 + h + hg_5(h, \theta_1, y_1(\theta_1), z_1(\theta_1), w_1(\theta_1)) + hG_5(h, \theta_1, y_1(\theta_1), z_1(\theta_1), w_1(\theta_1)) + h^2 F_5(h, \theta_1, y_1(\theta_1), z_1(\theta_1), w_1(\theta_1))$$

$$\text{(3.23 b)} \quad \theta = \theta_2 + h + hg_5(h, \theta_2, y_2(\theta_2), z_2(\theta_2), w_2(\theta_2)) + hG_5(h, \theta_2, y_2(\theta_2), z_2(\theta_2), w_2(\theta_2)) + h^2 F_5(h, \theta_2, y_2(\theta_2), z_2(\theta_2), w_2(\theta_2))$$

Solving for $|\theta_1 - \theta_2|$ we obtain

$$|\theta_1 - \theta_2| \leq N_2[(h\delta + h^2\delta + h^2)\|y_1 - y_2\|_{C^0} + (h\delta + h^2\delta + h^2)\|z_1 - z_2\|_{C^0} + (h\delta + h + h^2)\|w_1 - w_2\|_{C^0}]$$

Hence from

$$(3.24\text{ a})\quad y_1(\theta) = By_1(\theta_1) + hg_6(h,\theta_1,y_1(\theta_1),z_1(\theta_1),w_1(\theta_1)) + hG_6(h,\theta_1,y_1(\theta_1),z_1(\theta_1),w_1(\theta_1)) + h^2F_6(h,\theta_1,y_1(\theta_1),z_1(\theta_1),w_1(\theta_1))$$

$$(3.24\text{ b})\quad y_2(\theta) = By_2(\theta_2) + hg_6(h,\theta_2,y_2(\theta_2),z_2(\theta_2),w_2(\theta_2)) + hG_6(h,\theta_2,y_2(\theta_2),z_2(\theta_2),w_2(\theta_2)) + h^2F_6(h,\theta_2,y_2(\theta_2),z_2(\theta_2),w_2(\theta_2))$$

We obtain

$$\|y_1 - y_2\|_{C^0} \leq (1 - hb_1)\|y_1 - y_2\|_{C^0} + hK_1\|w_1 - w_2\|_{C^0} + h^2K_3\|z_1 - z_2\|_{C^0}$$

for some constants $b_1,\ K_1,\ K_3 > 0$.

Similarly we also have

$$\|w_1 - w_2\|_{C^0} \leq \varepsilon_1\|w_1 - w_2\|_{C^0} + hK_2\|y_1 - y_2\|_{C^0} + h^2K_3\|z_1 - z_2\|_{C^0}$$

Conversely, suppose two pairs (y_1,w_1) and (y_2,w_2) in $\mathcal{B}_Y \times \mathcal{B}_W$ are given then Lemma 2.3.2 provides two functions $z_1,\ z_2$ in $\mathcal{B}_{\mathbb{Z}}$ which are the fixed points of mappings defined by (y_1,w_1) and (y_2,w_2) respectively. We then have

$$\|z_1 - z_2\|_{C^0} \leq (1 - hc_1)\|z_1 - z_2\|_{C^0} + hK_4\|w_1 - w_2\|_{C^0} + h^2K_5\|y_1 - y_2\|_{C^0}$$

As in Chapter 1, we can proceed to construct a sequence $(y^{(\kappa)}(\theta),\ z^{(\kappa)}(\theta), w^{(\kappa)}(\theta))$ which converges uniformly to $(y(\theta), z(\theta), w(\theta))$. These functions provide a curve invariant under (3.2). Thus we have

Theorem 2.3.3: The difference equation (3.2) possesses an invariant curve which lies inside an $O(h)$-neighborhood of the corresponding periodic orbit of (3.1).

Remark 2.3.4: As in Chapter 1, if (3.1) depends on a parameter α, we also obtain the continuation of the invariant curves for (3.2) away from the bifurcation points and near the Hopf bifurcation.

Remark 2.3.5: Thus far we have assumed that the method employed is strongly stable, convergent, explicit, linear multistep method. We now try to weaken some of the above conditions. First suppose some nonlinear method is used and yields a difference equation

(3.25) $$x_{m+1} = \sum_{j=1}^{\kappa} \alpha_j\, x_{m+1-j} + h \sum_{j=1}^{\kappa} \beta_j\, f(x_{m+1-j}) + h^2 G(x_{m+1-\kappa}, \ldots, x_m)$$

where the term $h^2\ G(x_{m+1-\kappa}, \ldots, x_m)$ corresponds to the nonlinear part of the method employed. Assuming that

$$\sum_{j=0}^{\kappa} \alpha_j = 0$$

$$\sum_{j=1}^{\kappa} \beta_j = \sum_{j=1}^{\kappa} j\alpha_j \neq 0$$

and the roots of the polynomial $p(\varepsilon) = \sum_{j=0}^{\kappa} \alpha_j \varepsilon^{\kappa-j}$ are $1, \varepsilon_2, \ldots, \varepsilon_\kappa$ where $|\varepsilon_j| < 1$ for $i = 2,\ldots,\kappa$. Then all the arguments given in this chapter still hold true since the term $h^2 G(x_{m+1-\kappa},\ldots,x_m)$ is very small when h is small. Thus we do have invariant curves for (3.25) provided that h is sufficiently small.

Now suppose an implicit method is used to yield

$$(3.26) \quad x_{m+1} = \sum_{j=1}^{\kappa} \alpha_j x_{m+1-j} + h \sum_{j=0}^{\kappa} \beta_j f(x_{m+1-j}) + h^2 G(x_{m+1-\kappa},\ldots,x_m,x_{m+1})$$

then by writing the above equation as a mapping from $\mathbb{R}^{\kappa n} \to \mathbb{R}^{\kappa n}$ and apply the argument given in Remark 1.3.4, we obtain an equivalent, explicit difference equation of the form

$$(3.27) \quad x_{m+1} = \sum_{j=1}^{\kappa} \alpha_j x_{m+1-j} + h \sum_{j=1}^{\kappa} (\beta_j - \beta_0 \alpha_j) f(x_{m+1-j}) + h^2 G_1(x_{m+1-\kappa},\ldots,x_m)$$

Note that

$$\sum_{j=1}^{\kappa} (\beta_j - \beta_0\alpha_j) = \sum_{j=1}^{\kappa} \beta_j - \beta_0 \sum_{j=1}^{\kappa} \alpha_j = \sum_{j=0}^{\kappa} \beta_j$$

thus we can proceed to obtain invariant curves for (3.27) which are also invariant under (3.26).

The requirement that the method employed is strongly stable is important, however. For example suppose we study the system

$$\dot{x} = ax; \quad x \in \mathbb{R}^n; \quad a < 0 \tag{3.28}$$

using the midpoint rule which yields the difference equation

$$x_{m+1} = x_{m-1} + 2h\, a\, x_m \tag{3.29}$$

Note that $x = 0$ is a solution which is attracting under the flow of (3.28). On the other hand since the characteristic equation for (3.29) is

$$\lambda^2 - 2h\, a\, \lambda + 1 = 0$$

which has two roots

$$\lambda_1 = ha + [1 + (ha)^2]^{1/2}$$

$$\lambda_2 = -1/\lambda_1$$

where $|\lambda_1| < 1$ and $|\lambda_2| > 1$. It is easy to see that the degenerate invariant curve $x = 0$ is not attracting under (3.29).

3. TIME DELAYED EQUATION

3.1 The Hopf Bifurcation

Consider a time delayed equation of the form

$$\text{(1.1)} \qquad \dot{x} = f(\alpha, x(t-1))$$

$$= a(\alpha)x(t-1) + g(\alpha, x(t-1))$$

where $x \in \mathbb{R}$, $g(\alpha,x) = O(|x|^2)$ and $f(\alpha,x)$ is as smooth in α and x as needed.

The linearized equation for (1.1) is

$$\text{(1.2)} \qquad y = a(\alpha)y(t-1)$$

with characteristic equation

$$\text{(1.3)} \qquad \lambda - a(\alpha)e^{-\lambda} = 0$$

Assume that (1.3) has a pair of complex conjugate solutions $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$ satisfying

$$\text{(1.4)} \qquad \operatorname{Re} \lambda(0) = 0; \ \operatorname{Re} \lambda'(0) > 0; \ \operatorname{Im} \lambda(0) \neq 0$$

while all other solutions have negative real part and stay uniformly away from the imaginary axis. It is then well known that (1.1) possesses a family of periodic orbits bifurcating from the zero solution as α crosses 0 (see [4] or [8]).

Now suppose we approximate (1.1) using the Euler method with step size $h = 1/n$ where n is a positive integer. We then obtain a difference equation

$$x_{\kappa+1} = x_\kappa + ha(\alpha)x_{\kappa-n} + hg(\alpha, x_{\kappa-n}) \tag{1.5}$$

Since (1.5) requires $n+1$ starting points, it is natural to think of it as a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. To that end we set

$$z_\kappa = [x_{\kappa-n} \ldots x_\kappa]^T$$

$$z_{\kappa+1} = [x_{\kappa-n+1} \ldots x_{\kappa+1}]^T$$

where T denotes tranposition. (1.5) then defines a mapping $U : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$Uz_\kappa = z_{\kappa+1} = A(\alpha)z_\kappa + hg(\alpha, z_\kappa) \tag{1.6}$$

where $A(\alpha)$ is an $(n+1) \times (n+1)$ matrix

$$A(\alpha) = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & . \\ . & . & . & . & . & 1 & 0 \\ 0 & 0 & . & . & . & 0 & 1 \\ ha(\alpha) & 0 & . & . & . & 0 & 0 \end{bmatrix}$$

and $G(\alpha, z_\kappa)$ is a $(n+1)$ vector.

$$G(\alpha, z_K) = [0 \ldots 0 \; g(\alpha, x_{K-n})]^T$$

The characteristic equation for $A(\alpha)$ is

$$\omega^{n+1} - \omega^n - ha(\alpha) = 0 \qquad (1.7)$$

Set $\omega = e^{h\beta}$ then (1.7) can be written as

$$(e^{h\beta} - 1)/h - a(\alpha)e^{-\beta} = 0 \qquad (1.8)$$

We have the following

Lemma 3.1.1: Let λ be a solution of (1.3) satisfying $\lambda \neq -1$. Then for $h > 0$ sufficiently small there exists a solution $\beta(h)$ of (1.8) such that $\lim_{n \to 0} \beta(h) = \lambda$

Proof: Set

$$v(h,\beta) = \begin{cases} (e^{h\beta} - 1)/h & \text{if} \quad h \neq 0 \\ \beta & \text{if} \quad h = 0 \end{cases}$$

then v is continuously differentiable in β and h. Define

$$F(h,\beta) = v(h,\beta) - a(\alpha)e^{-\beta}$$

then

$$F(0,\lambda) = \lambda - a(\alpha)e^{-\lambda} = 0$$

and

$$\partial F(0,\lambda)/\partial\lambda = 1 + a(\alpha)e^{-\lambda} = 1 + \lambda \neq 0$$

The lemma follows from the Implicit Function Theorem.

Generically we may assume that $a(0) \neq -e^{-1}$ so that -1 is not a root of $p(\lambda) = \lambda - a(\alpha)e^{-\lambda}$ for α near 0. Since $p'(\lambda) = 1 + a(\alpha)e^{-\lambda}$, the above assumption implies that all the roots of $p(\lambda)$ are distinct.

Now note that every solution λ of (1.3) satisfies

$$\text{Re}\ \lambda = a(\alpha)e^{-\text{Re}\ \lambda} \cos\ (\text{Im}\ \lambda)$$

which implies that $\text{Re}\ \lambda < M_1$ for some constant M_1. Also

$$\text{Im}\ \lambda = a(\alpha)e^{-\text{Re}\ \lambda} \sin\ (\text{Im}\ \lambda)$$

which yields $|\text{Im}\ \lambda| < M_2$ for some $M_2 > 0$.

Since (1.3) is entire in λ, it follows that given any real number b, there exist only a finite number of solutions of (1.3) say, $\lambda_1, \ldots, \lambda_K$ with real parts $> b$.

Similarly for h sufficiently small, (1.8) also has a finite numbers of solutions with real parts $> b$. These solutions, being bounded in a compact set, must converge to λ_j, $j = 1, \ldots, \kappa$ as $h \to 0$. Thus there are κ such solutions $\beta_1, \ldots, \beta_\kappa$ and

$$|\beta_j - \lambda_j| \le Mh \quad \text{for} \quad j = 1,\dots,\kappa$$

for some constant $M > 0$.

Let $\lambda_1(\alpha)$, $\lambda_2(\alpha)$ be the pair of solutions of (1.3) satisfying (1.4). Then (1.7) has a pair of complex conjugate solutions $\omega_1(\alpha)$, $\omega_2(\alpha)$ such that

$$\omega_j(\alpha) = 1 + h\lambda_j(\alpha) + O(h^2) \qquad j = 1,2 \tag{1.9}$$

These $\omega_j(\alpha)$ cross the unit circle as α crosses some value $\alpha_h = O(h)$.

From the argument above it also follows that all other solutions of (1.7) satisfy

$$|\omega| \le 1 - hb$$

for some $b > 0$. We now could proceed as in Chapter 1 to obtain a family of invariant curves for (1.5) which bifurcates from the zero solution as α crosses 0.

To see that these curves approximate the periodic orbits of (1.1) we introduce

$$\mathcal{B} = \{\text{Lipschitz continuous functions on } [-1,0]\}$$

then $\mathcal{B}$ is a Banach space with norm

$$|\varphi| = |\varphi(0)| + \sup_{\substack{-1\le\theta_1,\theta_2\le 0 \\ \theta_1\ne\theta_2}} [|\varphi(\theta_1) - \varphi(\theta_2))/|\theta_1 - \theta_2|]$$

For a vector $z = [z_1 \dots z_{n+1}]^T \in \mathbb{R}^{n+1}$ we define $I_n(z)$ to be a function in $\mathcal{B}$ with values at $-1, -1+h, \dots, 0$ equal to $z_1, \dots, z_{n+1}$ and is linear on each of the intervals $[-1+jh, -1+(j+1)h]$ for $j = 0, \dots, n-1$.

Conversely given a function $\varphi \in \mathcal{B}$, we define $J_n(\varphi)$ to be a vector in $\mathbb{R}^{n+1}$ whose components equal to the values of φ at $-1, -1+h, \dots, 0$.

Let $\varphi \in \mathcal{B}$ be given with $|\varphi| \leq \delta$ for some $\delta > 0$. Let $x(t)$ be the solution of

$$\begin{cases} \dot{x}(t) = f(x(t-1)) & \text{for} \quad t \geq 0 \\ x(t) = \varphi(t) & \text{for} \quad -1 \leq t \leq 0 \end{cases}$$

let $\pi : \mathcal{B} \to \mathcal{B}$ be the mapping defined by

$$\pi\varphi(0) = x(1+\theta) \quad \text{for} \quad -1 \leq \theta \leq 0$$

Similarly define $\pi_n : \mathcal{B} \to \mathcal{B}$ by $\pi_n = I_n U^n J_n$

then
$$\pi\varphi(\theta_2) - \pi\varphi(\theta_1) = \int_{\theta_1}^{\theta_2} f(\varphi(s))\,ds$$

while

$$\pi_n\varphi(\theta_2) - \pi_n\varphi(\theta_1) = \int_{\theta_1}^{\theta_2} E_n\, f(\varphi(s))\,ds$$

where $E_n\, f(\varphi(s))$ is the step function which equals to $f(\varphi(-1+jh))$ on $[-1+jh, -1+(j+1)h]$ for $j = 0, \dots, n-1$.

Since $\varphi(s)$ is Lipschitz continuous with Lipschitz constant δ, $f(\varphi(s))$ is also Lipschitz continuous with

Lipschitz constant $N\delta$ for some $N > 0$. This implies that

$$|f(\varphi(s)) - E_n f(\varphi(s))| \leq N\delta h$$

and hence

$$|\pi_n\varphi - \pi\varphi| \leq N\delta h$$

or $\pi_n\varphi$ converges uniformly to $\pi\varphi$ as $n \to \infty$ for $|\varphi| \leq \delta$.

Let $\varphi \in \mathcal{B}$ be such that $|\varphi| \leq \delta$ then for any $\varphi_1 \in \mathcal{B}$ we have

$$\pi(\varphi + t\varphi_1)(\theta) = \varphi(0) + t\varphi_1(0) + \int_{-1}^{\theta} f(\varphi(s) + t\varphi_1(s))ds$$

which gives for $-1 \leq \theta_1 \leq \theta_2 \leq 0$

$$D\pi(\varphi)\varphi_1(\theta_2) - D\pi(\varphi)\varphi_1(\theta_1) = \int_{\theta_1}^{\theta_2} f'(\varphi(s))\varphi_1(s)ds$$

Similarly we also have

$$D\pi_n(\varphi)\varphi_1(\theta_2) - D\pi_n(\varphi)\varphi_1(\theta_1) = \int_{\theta_1}^{\theta_2} E_n f'(\varphi(s))\varphi_1(s)ds$$

This implies

$$|D\pi_n(\varphi)\varphi_1 - D\pi(\varphi)\varphi_1| \leq N_1 h\delta|\varphi_1|$$

for some $N_1 > 0$. Hence

$$\|D\Pi_n(\varphi) - D\Pi(\varphi)\| \leq N_1 h\delta$$

Let $X(\alpha)$ = eigenspace of $D\Pi(0)$ corresponding to the eigenvalues $e^{\lambda(\alpha)}$ and $e^{\overline{\lambda(\alpha)}}$, then there exists a subspace $Y(\alpha)$ of $\mathcal{B}$ such that $\mathcal{B} = X(\alpha) \oplus Y(\alpha)$. By choosing some appropriate basis for $X(\alpha)$ and projecting to $X(\alpha)$ and $Y(\alpha)$ we can decompose Π as $\Pi : (x,y) \to (\bar{x},\bar{y})$ where

(1.10 a) $$\bar{x} = A(\alpha)x + F(\alpha,x,y)$$

(1.10 b) $$\bar{y} = B(\alpha)y + G(\alpha,x,y)$$

where $x,\bar{x} \in \mathbb{R}^2$; $y,\bar{y} \in Y(\alpha)$; $A(\alpha)$ is a 2×2 matrix with eigenvalues $\lambda(\alpha)$ and $\overline{\lambda(\alpha)}$; $B(\alpha)$ is a linear mapping from $Y(\alpha)$ to $Y(\alpha)$ with spectrum inside and away from the unit circle. $F : \mathbb{R} \times \mathbb{R}^2 \times Y(\alpha) \to \mathbb{R}^2$; $G : \mathbb{R} \times \mathbb{R}^2 \times Y(\alpha) \to Y(\alpha)$. Also F and G have second order zeroes in (x,y) at $(0,0)$.

Similarly we can decompose Π_n as $\Pi_n : (x,y) \to (x_n,y_n)$ where

(1.11 a) $$x_n = A_n(\alpha)x + hC_n(\alpha)y + F_n(\alpha,x,y)$$

(1.11 b) $$y_n = B_n(\alpha)y + hD_n(\alpha)x + G_n(\alpha,x,y)$$

The extra terms $hC_n(\alpha)y$ and $hD_n(\alpha)x$ appear due to the fact that $X(\alpha)$ and $Y(\alpha)$ are not the eigenspaces of Π_n.

Using the fact that $\pi_n(\varphi)$ and $D\pi_n(\varphi)$ converge uniformly to $\pi(\varphi)$ and $D\pi(\varphi)$ respectively for all φ satisfying $|\varphi| \leq \delta$, we have $A_n(\alpha) \to A(\alpha)$ as bounded linear operators from X to X, $B_n(\alpha) \to B(\alpha)$ as linear operators from Y to Y. Also $F_n(\alpha,x,y)$, $G_n(\alpha,x,y)$ and their derivatives converge uniformly to $F(\alpha,x,y)$, $G(\alpha,x,y)$ and the corresponding derivatives on a neighborhood of $(\alpha,x,y) = (0,0,0)$.

We now could proceed as in Chapter 1 to obtain a family of invariant curves for $\pi_n(\alpha)$ for $\alpha_n < \alpha < \varepsilon$ or $-\varepsilon < \alpha < \alpha_n$ in the generic case where $\alpha_n \to 0$ as $n \to 0$ and $\varepsilon > 0$ is independent of n.

<u>Theorem 3.1.2</u>: If (1.1) is approximated by the Euler method then the resulting difference equation (equation (1.5)), in the generic case, possesses a family of invariant curves depending continuously on α for either $\alpha_n < \alpha < \varepsilon$ or $-\varepsilon < \alpha < \alpha_n$ where $\alpha_n \to 0$ as $n \to 0$ while $\varepsilon > 0$ is independent of n.

<u>Remark 3.1.3</u>: As in Chapter 1, Theorem 3.1.2 still holds true if some convergent single step method other than the Euler method is employed or when some solutions of the characteristic equation (1.3) have positive real parts but stay uniformly away from the imaginary axis for all α near 0. It is also true if (1.1) has a more general form such as

$$\dot{x} = f(\alpha, x(t), x(t-1)); \quad x \in \mathbb{R}^n$$

provided that the corresponding characteristic equation still has a pair of complex conjugate solutions satisfying the Hopf bifurcation (1.4) while all other solutions stay uniformly away from the imaginary axis for α near 0.

3.2 Existence and Continuation

As in Chapter 1, we now show the existence of the invariant curves away from the bifurcation points. More precisely we now have a time delayed equation

$$(2.1) \qquad \dot{x} = f(x(t-1)); \quad x \in \mathbb{R}$$

which has an ω-periodic solution $p(t)$. This defines a periodic orbit Γ in $C = C([-1,0], \mathbb{R})$ by

$$\Gamma = \{P_s \mid 0 \leq s \leq \omega\},$$

where $P_s \in C$ is given by

$$P_s(\theta) = P(s+\theta) \quad \text{for} \quad -1 \leq \theta \leq 0 .$$

We assume that no nontrivial characteristic multipliers of Γ lie on the unit circle. Thus by rearrangement if necessary, we may assume that the multipliers $\mu_1, \mu_2, \ldots, \mu_\ell, \ldots$ of Γ satisfy

$$\mu_1 = 1; \quad |\mu_j| > 1 \quad \text{for} \quad 2 \leq j \leq \ell;$$

$$\text{and} \quad |\mu_j| < 1 \quad \text{for} \quad j \geq \ell + 1$$

Now suppose we approximate (2.1) using the Euler method with step size $h = 1/n$ for some positive integer n. This yields a difference equation

$$x_{\kappa+1} = x_\kappa + hf(x_{\kappa-n}) \tag{2.2}$$

which induces a mapping $U : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$U : z_\kappa \to z_{\kappa+1}$$

where

$$z_\kappa = [x_{\kappa-n} \ldots x_\kappa]^T$$

$$z_{\kappa+1} = [x_{\kappa-n+1} \ldots x_{\kappa+1}]^T$$

Set $N = [\omega \mid h]$ and define

$$S = U^N$$

Also let $\mathcal{B}$, I_n, J_n be as in section 3.1. The mapping S then yields a mapping from $\mathcal{B} \to \mathcal{B}$ given by

$$\pi_n = I_n S J_n \ .$$

Let $\pi : \mathcal{B} \to \mathcal{B}$ be the period map of (2.11), i.e., given a function $\varphi \in \mathcal{B}$, we solve the initial value problem

$$\dot{x}(t) = x(t-1); \ x(t) = \varphi(t) \quad \text{for} \quad -1 \leq t \leq 0$$

and define

$$\pi\varphi(\theta) = x(\omega + \theta) \quad \text{for} \quad -1 \leq \theta \leq 0$$

By repeating the argument given in section 3.1 several times if necessary we can show that

$$|\pi_n\varphi - \pi\varphi| \le N_1 h$$

$$\|D\pi_n(\varphi) - D\pi(\varphi)\| \le N_2 h$$

for all φ near Γ, i.e., $\varphi \in \mathcal{B}$ and $|\varphi - P_s|$ is small for some $s \in [0,\omega)$.

Note that $D\pi(P_s)$ has eigenvalues $\mu_1,\ldots,\mu_\ell,\ldots$ for all s. Thus we can decompose $\mathcal{B}$ as

$$\mathcal{B} = E(s) \oplus K(s)$$

where $E(s)$ is the one dimensional eigenspace of $D\pi(P_s)$ corresponding to the eigenvalue $\mu_1 = 1$. It is easy to see that $\dot{P}_s$ is a basis for $E(s)$.

Let V be a neighborhood of P_0 in $\mathcal{B}$. Define

$$H(s,\phi,z) = P_s + \phi - z$$

for $s \in \mathbb{R}$, $\phi \in K(s)$, $z \in V$. Then

$$H(0,0,P_0) = 0$$

and the derivative of H with respect to s, ϕ evaluated at $s = 0$, $\phi = 0$ and the pair (σ,ψ), $\sigma \in \mathbb{R}$, $\psi \in K(0)$ is $\dot{P}_0\sigma + \psi$. Since $\dot{P}_0$ is a basis for $E(0)$ and $E(0)$, $K(0)$ are linearly independent, the above derivative has a bounded inverse. The implicit function theorem implies

there is a $\delta > 0$ and unique $s(z)$ and $\phi(z)$ continuously differentiable with respect to z for $|z - P_0| < \delta$ so that $H(s(z),\phi(z),z) = 0$. Since Γ is compact we can apply the above argument a finite number of time to obtain

<u>Lemma 3.2.1</u>: There exists a neighborhood W of Γ in $\mathcal{B}$ such that for any $z \in W$ there exists unique $s \in [0,\omega)$ and $\phi \in K(s)$ such that $z = P_s + \phi$.

Let $z = P_s + \phi \in W$. Then

$$\pi_n z = \pi z + hF(h,z)$$

where F and its derivative with respect to z are uniformly bounded for $z \in W$ and

$$\pi z = \pi(P_s) + D\pi(P_s)\phi + g(s,\phi)$$

$$= P_s + D\pi(P_s)\phi + g(s,\phi)$$

where $g(s,\phi) = O(|\phi|^2)$. Note also that $D\pi(P_s)\phi \in K(s)$.

Suppose $\pi_n z = P_{\bar{s}} + \bar{\phi}$ where $\bar{\phi} \in K(\bar{s})$ then since $\bar{s}$ and $\bar{\phi}$ depend continuously differentiably on $\pi_n z$ and hence on z, we have

(2.3a) $$\bar{s} = s + g_1(s,\phi) + hF_1(h,s,\phi)$$

(2.3b) $$\bar{\phi} = D\pi(P_s)\phi + g_2(s,\phi) + hF_2(h,s,\phi)$$

where $g_j(s,\phi) = O(|\phi|^2)$ for $j = 1,2$ while $F_j(h,s,\phi)$ and their derivatives are uniformly bounded for $j = 1,2$.

Assuming that $|\mu_j| < 1$ for all $j \geq 2$ then $\|D\pi(P_s)\| \leq \rho < 1$ for all s. Define

$$\mathcal{U} = \{\text{continuous } \omega\text{-periodic functions } u(s) \text{ with values in } \beta \text{ such that } u(s) \in K(s),\ |u(s)| \leq \delta \text{ and } |u(s_1) - u(s_2)| \leq \delta |s_1 - s_2|\}$$

where $\delta > 0$ is to be determined.

As in Chapter 1, given $u \in \mathcal{U}$ we define $\mathcal{T}u$ to be the function satisfying

$$\text{(2.4a)} \qquad \mathcal{T}u(s) = D\pi(P_\tau)u(\tau) + g_2(\tau, u(\tau)) + hF_2(h, \tau, u(\tau))$$

where τ is such that

$$\text{(2.4b)} \qquad s = \tau + g_1(\tau, u(\tau)) + hF_1(h, \tau, u(\tau))$$

Since $\|D\pi(P_s)\| \leq \rho < 1$ we can proceed as in Chapter 1 to show that $\mathcal{T}u \in \mathcal{U}$ if $\delta = Mh$ for some constant $M > 0$ and h is sufficiently small.

Now given $u_1, u_2 \in \mathcal{U}$. We have

$$\text{(2.5a)} \qquad \mathcal{T}u_1(s) = D\pi(P_{\tau_1})u_1(\tau_1) + g_2(\tau_1, u_1(\tau_1)) + hF_2(h, \tau_1, u_1(\tau_1))$$

$$\text{(2.5b)} \qquad \mathcal{T}u_2(s) = D\pi(P_{\tau_2})u_2(\tau_2) + g_2(\tau_2, u_2(\tau_2)) + hF_2(h, \tau_2, u_2(\tau_2))$$

where τ_1 and τ_2 are given by

(2.6a) $$s = \tau_1 + g_1(\tau_1, u_1(\tau_1)) + hF_1(h, \tau_1, u_1(\tau_1))$$

(2.6b) $$s = \tau_2 + g_1(\tau_2, u_2(\tau_2)) + hF_2(h, \tau_2, u_2(\tau_2))$$

From (2.6a) and (2.6b) we have

$$|\tau_1 - \tau_2| \leq N_1(\delta + h) \; \|u_1 - u_2\|$$

where $\|u_1 - u_2\| = \sup_s |u_1(s) - u_2(s)|$ and $N_1 > 0$ is some constant independent of δ and h. Hence (2.5) gives

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\| \leq [\rho + N_2(h + \delta)] \; \|u_1 - u_2\|$$

for some constant $N_2 > 0$ if $\delta = Kh$ and h sufficiently small.

Hence for h sufficiently small, $\mathcal{T}$ is a contraction map which has a fixed point. This yields an invariant curve for (2.3).

In the general case when $|\mu_j| > 1$ for $2 \leq j \leq \ell$ and $|\mu_j| < 1$ for $j > \ell$ where $\ell \geq 2$ we decompose β as $\beta = K_1(s) \oplus K_2(s)$ where $K_1(s)$ is the generalized eigenspace of $D\pi(P_s)$ corresponding to $\mu_2, \ldots, \mu_\ell$. We then define

$\mathcal{U}_j$ = {continuous ω-periodic functions $u(s)$ with values in β such that $u(s) \in K_j(s)$, $|u(s)| \le \delta$ and $|u(s_1) - u(s_2)| \le \delta |s_1 - s_2|$} for $j = 1,2$.

Given a function $u_1 \in \mathcal{U}_1$, we define a mapping $\mathcal{T} : \mathcal{U}_2 \to \mathcal{U}_2$ by

$$(2.7a)\quad u_1(s) + \mathcal{T}u_2(s) = D\pi(P_\tau)(u_1(\tau) + u_2(\tau)) + g_2(\tau, u_1(\tau) + u_2(\tau)) + hF_2(h, \tau, u_1(\tau) + u_2(\tau))$$

for $u_2 \in \mathcal{U}_2$ where τ is given by

$$(2.7b)\quad s = \tau + g_1(\tau, u_1(\tau) + u_2(\tau)) + hF_1(h, \tau, u_1(\tau) + u_2(\tau))$$

As before we can show that $\mathcal{T}$ is a contraction map and hence has a fixed point $u_2 \in \mathcal{U}_2$ such that

$$(2.8a)\quad u_1(s) + u_2(s) = D\pi(P_\tau)(u_1(\tau) + u_2(\tau)) + g_2(\tau, u_1(\tau) + u_2(\tau)) + hF_2(h, \tau, u_1(\tau) + u_2(\tau))$$

where

$$(2.8b)\quad s = \tau + g_1(\tau, u_1(\tau) + u_2(\tau)) + hF_1(h, \tau, u_1(\tau) + u_2(\tau))$$

Conversely given $u_2 \in \mathcal{U}_2$ there exists $u_1 \in \mathcal{U}_1$ such that (2.8) is satisfied.

As in Chapter 1 we now construct sequences of functions $u_1^{(\kappa)}$ and $u_2^{(\kappa)}$ in $\mathcal{U}_1$ and $\mathcal{U}_2$ by defining

$$u_1^{(0)}(s) \equiv u_2^{(0)}(s) = 0$$

Inductively given $u_1^{(\kappa)}$ we construct a contraction map $\mathcal{J}_2 : \mathcal{U}_2 \to \mathcal{U}_2$ and let $u_2^{(\kappa+1)}$ be the fixed point of $\mathcal{J}_2$. Using $u_2^{(\kappa+1)}$ we then construct a contraction map $\mathcal{J}_1 : \mathcal{U}_1 \to \mathcal{U}_1$ and let $u_1^{(\kappa+1)}$ be its fixed point. As in Chapter 1 we can show that

$$\|u_1^{(\kappa+1)} - u_1^{(\kappa)}\| \leq K_1 \|u_2^{(\kappa+1)} - u_2^{(\kappa)}\|$$

$$\|u_2^{(\kappa+1)} - u_2^{(\kappa)}\| \leq K_2 \|u_1^{(\kappa)} - u_1^{(\kappa-1)}\|$$

where $K_1, K_2 < 1$ if $\delta = Mh$ and h is sufficiently small.

Thus $u_1^{(\kappa)}$ and $u_2^{(\kappa)}$ converge to some functions u_1 and u_2 as $\kappa \to \infty$. These functions define an invariant curve for (2.3).

Now let $\bar{\Gamma}$ be an invariant curve for π_n then so is $I_n U J_n(\bar{\Gamma})$. Uniqueness implies that $I_n U J_n(\bar{\Gamma}) = \bar{\Gamma}$ or $\bar{\Gamma}$ is invariant under $I_n U J_n$. This means that $J_n(\bar{\Gamma})$, as a curve in $\mathbb{R}^{n+1}$ is invariant under U.

Thus we have

<u>Theorem 3.2.2</u>: For $h = 1/n$ sufficiently small, (2.2) has an invariant curve in $\mathbb{R}^{n+1}$ which, when imbedded in β, converges to the periodic orbit of (2.1) as $h \to 0$.

<u>Remark 3.2.3</u>: As in Chapter 1, we also obtain the continuation of invariant curves away from the bifurcation points and near the Hopf bifurcation. Also Theorem 3.2.2 is still true when a more general system such as

$$\dot{x} = f(x(t), x(t-1)); \quad x \in \mathbb{R}^n$$

is approximated by some convergent single-step method.

<u>Remark 3.2.4</u>: In [5], Georg reported that in his numerical studies, the Euler method provided the best result in tracking the periodic orbits. Also decreasing the step size in many case did not improve the rate of convergence. These observations can be explained by noticing that under any convergent single-step method or linear convergent, strongly stable multistep method the rates of convergence to the invariant curves of the resulting difference equations are roughly the same as the rates of convergence to the periodic orbits. Thus for example it is erroneous to assume that the Runge-Kutta methods would yield the invariant curves more quickly than the Euler method.

In case all the nontrivial characteristic multipliers are inside the unit circle, the invariant curves are attracting if the step size h is sufficiently small. Thus if the round-off error is small compared to h, we could obtain approximations to those curves numerically using only some standard numerical method such as the Euler method.

If one or more of the multipliers cross the unit circle, the corresponding invariant curves are no longer attracting. Thus some special procedure is required. One such procedure involves transforming the problem of finding the periodic orbits into the problem of solving the equation $F(\alpha,x) = \pi(\alpha,x) - x = 0$ where $\pi(\alpha,x)$ is the period map (or Poincare map). $\pi(\alpha,x)$ and its Jacobian can be approximated using some standard numerical method. The Newton method or some of its modified form can be used to generate the approximations to the solutions of $F(\alpha,x) = 0$. Discussions and numerical results of the above procedure can be found in [5] and [6].

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