PRINCIPAL IDEAL JORDAN ALGEBRAS

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ROBERT MELVIN ANDERSON

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This is to certify that the

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Charles Tsai
Major professor

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ABSTRACT

PRINCIPAL IDEAL JORDAN ALGEBRAS

By

Robert Melvin Anderson

The purpose of this paper is to study the structure of certain classes of principal ideal Jordan algebras. A Jordan algebra J over a commutative ring with identity is said to have property A (property B), if J satisfies the polynomial identities $2(x,y,xz) + (z,y,x^2) = 0$ and $(w,x,yz) + (z,x,yw) + (y,x,wz) = 0$, J has an identity, and each ideal A of J contains an element x such that A is equal to the intersection of all the quadratic ideals which contain x (such that $A = JU_x$). The main results of this paper are:

Theorem. If J is a Jordan algebra with property A, then $J = \oplus \sum_{i=1}^{n} J_i$ where each J_i has property A and is either a u-prime algebra or a u-primary algebra containing a nonzero nilpotent ideal.

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Theorem. A Jordan algebra J has property B if and only if $J = \oplus \sum_{i=1}^{n} J_i$ where each summand is a simple Jordan algebra with identity.

Theorem. Let J be a Jordan algebra over a field. J has property A if and only if $J = \oplus \sum_{i=1}^{n} J_i$ where each summand has property A, all but at most one of the summands are simple Jordan algebras, and each summand is either a simple Jordan algebra or contains only one proper u-prime ideal, which is nilpotent.

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INTRODUCTION

Let R be a commutative ring with identity. J is a Jordan algebra over R if J is an R-module with a product defined which satisfies the distributive laws, the polynomial identities $xy - yx = 0$ and $(x^2,y,x) = (x^2y)x - x^2(yx) = 0$, and $r(xy) = (rx)y = x(ry)$ for all $x,y \in J$ and $r \in R$.

For any Jordan algebra J, the operator $U_{x,y}$ is defined to be $zU_{x,y} = \{x,z,y\} = x(zy) + (xz)y - z(xy)$ for $x,y,z \in J$. $U_{x,x}$ is denoted by U_x. U_x and $U_{x,y}$ are related by the equation $U_{x+y} = U_x + U_y + 2U_{x,y}$. If A and B are subsets of J, then AU_B is defined to be the set of all finite sums of elements of the form aU_b where $a \in A$ and $b \in B$.

In a Jordan algebra J, a subset A is called a quadratic ideal if A is an R-module and $JU_A \subseteq A$. For any $x \in J$, JU_x is a quadratic ideal denoted by $[x]$. $Rx + JU_x$ is a quadratic ideal equal to the intersection of all the quadratic ideals which contain x and is denoted by $\langle x\rangle$. For example, if one considers the polynomial ring F[X] as a Jordan algebra over a field F, then [X] is equal to the principal ideal generated by X^2, and $\langle X\rangle$ is equal to the principal ideal generated by X.

A special Jordan algebra is a subspace of an associative algebra over a field of characteristic not equal to two which is closed under the Jordan product $x \cdot y = \frac{1}{2}(xy + yx)$.

In order to use the polynomial identities which arise from the study of Jordan algebras over a field of characteristic not equal to two, one could study Jordan algebras which have no elements of order two or, less restrictively, those which satisfy the polynomial identities $h(x,y,z) = 2(x,y,xz) + (z,y,x^2) = 0$ and $d(w,x,y,z) = (w,x,zy) + (z,x,yw) + (y,x,wz) = 0$ where $(x,y,z) = (xy)z - x(yz)$ (see appendix). In this paper it shall be assumed that all Jordan algebras satisfy the polynomial identities $h(x,y,z) = 0$ and $d(w,x,y,z) = 0$. Thus identities such as $U_{bU_a} = U_a U_b U_a$ ([2], p.52), and identities proved by Macdonald's theorem for Jordan algebras over fields hold in any Jordan algebra ([2], pp.40-47).

Macdonald's theorem. Any identity in three variables which is of degree of at most one in one of these variables, which holds for all special Jordan algebras, holds for all Jordan algebras.

This paper deals with finding results for Jordan algebras similar to the well known result for commutative, associative rings with identity that a ring is a principal ideal ring if and only if it is a finite direct sum of

principal ideal domains and special principal ideal rings ([5], pp.242-247). This result was generalized to noncommutative principal ideal rings, where it is assumed that each right ideal is principal [1]. In Jordan algebras, quadratic ideals seem to correspond with the one-sided ideals in the associative case; thus it would seem that a similar result for Jordan algebras could be obtained if it were assumed that each quadratic ideal were principal, but actually one only need assume that each ideal is a principal quadratic ideal of the form $\langle x \rangle$.

CHAPTER I

JORDAN ALGEBRAS WITH PROPERTY A

In this chapter it will be understood that all Jordan algebras are over an arbitrary commutative ring R with identity.

Lemma 1-1. In a Jordan algebra $xU_{y,zU_y} = zU_{y,xU_y}$ for any $x,y,z \in J$.

Proof. If J is a special Jordan algebra then:

$$\begin{aligned} xU_{y,zU_y} &= \{\{y,z,y\},x,y\} \\ &= \tfrac{1}{2}(yzyxy + yxyzy) \\ &= \tfrac{1}{2}\left(yz(xU_y) + (xU_y)zy\right) \\ &= zU_{y,xU_y}. \end{aligned}$$

By Macdonald's theorem the identity holds in any Jordan algebra.

Lemma 1-2. If C is an ideal in a Jordan algebra J and $aU_b \in C$ for any $a,b \in J$, then $aU_{\langle b\rangle} \subseteq C$. If A is a quadratic ideal and $AU_b \subseteq C$, then $AU_{\langle b\rangle} \subseteq C$.

Proof. For J an algebra over R, let $rb + dU_b$ be an arbitrary element of $\langle b\rangle$, where $d \in J$ and $r \in R$.

$$\begin{aligned} aU_{rb+dU_b} &= aU_{rb} + aU_{dU_b} + 2aU_{rb,dU_b} \\ &= r^2aU_b + aU_bU_dU_b + 2aU_{rb,dU_b} \end{aligned}$$

$= r^2(aU_b) + (aU_b)U_dU_b + 2rdU_{b,aU_b}$ by lemma 1-1.

Thus $aU_{rb+dU_b} \in C$, and $aU_{<b>} \subseteq C$. The second statement follows from the first.

Lemma 1-3. In a Jordan algebra, $b^n(aU_b) = (ab^n)U_b$ for $n \in N$.

Proof. Let R_x be the operator which multiplies on the right by x. $b^n(aU_b) = a(2R_b{}^2 - R_{b^2})R_{b^n}$

$= aR_{b^n}(2R_b{}^2 - R_{b^2})$ since

$R_{x^j}R_{x^i} = R_{x^i}R_{x^j}$ for any $i,j \in N$ by ([3], p.92). Thus $b^n(aU_b) = (ab^n)U_b$.

Lemma 1-4. In a Jordan algebra with identity, $(aU_{b^n})(cU_{b^n}) = b^{2n}U_{a,c}U_{b^n}$ where $a,b,c \in J$ and $n \in N$.

Proof. If J is a special Jordan algebra then:

$$\begin{aligned} U_{aU_b,cU_b} &= \tfrac{1}{2}\left(U_{(a+b)U_b} - U_{aU_b} - U_{cU_b}\right) \\ &= \tfrac{1}{2}\left(U_bU_{a+c}U_b - U_bU_aU_b - U_bU_cU_b\right) \\ &= \tfrac{1}{2}\left(U_b(U_a + U_c + 2U_{a,c})U_b - U_bU_aU_b - U_bU_cU_b\right) \\ &= U_bU_{a,c}U_b. \end{aligned}$$

Thus in any Jordan algebra $U_{aU_b,cU_b} = U_bU_{a,c}U_b$ and $(aU_b)(cU_b) = 1U_{aU_b,cU_b} = 1U_bU_{a,c}U_b = b^2U_{a,c}U_b$. Assume the statement is true for all $n \leq k$.

$$\begin{aligned} (aU_{b^{k+1}})(cU_{b^{k+1}}) &= \left((aU_b)U_{b^k}\right)\left((cU_b)U_{b^k}\right) \\ &= b^{2k}U_{aU_b,aU_b}U_{b^k} \\ &= b^{2k}(U_bU_{a,c}U_b)U_{b^k} \\ &= b^{2(k+1)}U_{a,c}U_{b^{k+1}}. \end{aligned}$$

Thus the lemma holds by induction.

Lemma 1-5. If J is a Jordan algebra then $(aU_b)c = 2aU_{b,cb} - (ac)U_b$ for any $a,b,c \in J$.

Proof. $(aU_b)c - 2aU_{b,cb} + (ac)U_b$

$= 2((ab)b)c - (ab^2)c - 2(ba)(cb) - 2((cb)a)b + 2((cb)b)a$
$\quad + 2((ac)b)b - (ac)b^2$

$= 2(((ab)b)c + ((cb)b)a + ((ac)b)b) - (ab^2)c - 2(ba)(cb)$
$\quad - 2((cb)a)b - (ac)b^2$

$= 2(2(ab)(bc) + b^2(ac)) - 2(ba)(cb) - (ab^2)c - 2((cb)a)b$
$\quad - (ac)b^2$ by the polynomial identity $d(a,b,b,c) = 0$,

$= 2(ab)(bc) + b^2(ac) - (ab^2)c - 2((cb)a)b = 0$ by the identity $d(b,a,b,c) = 0$.

Corollary 1-6. If A and B are ideals of a Jordan algebra J, then AU_B is an ideal of J.

Proof. For any $a \in A$, $b \in B$, and $c \in J$, then $(aU_b)c = -(ac)U_b + 2aU_{b,cb}$ by lemma 1-5. $2aU_{b,cb} = aU_{b+cb} - aU_b - aU_{cb} \in AU_B$. Therefore $(aU_b)c \in AU_B$ and AU_B is an ideal of J.

Lemma 1-7. Let J be a Jordan algebra and A be an ideal of the form $A = \langle a \rangle$. If $b \in J$, then $ba^n \in A^n$ when n is an odd positive integer, and $ba^n \in A^{n-1}$ when n is an even positive integer.

Proof. If n is an odd integer then $n = 2m + 1$ for some integer m. $a^n = aU_{a^m} \in A(U_A)^m$ which is an ideal by corollary 1-6. Therefore $ba^n \in A(U_A)^m \subseteq A^n$. If n is an even integer then $n = 2r$ for some integer r. $a^{2r-1} \in$

$A(U_A)^{r-1}$ where $A(U_A)^{r-1}$ is an ideal by corollary 1-6. $ba^n = ((a^{2r-1})a)b \in ((A(U_A)^{r-1})a)b \subseteq A(U_A)^{r-1} \subseteq A^{n-1}$.

Lemma 1-8. Let J be a Jordan algebra and C be an ideal of J of the form $\langle c \rangle$ for some $c \in C$. If m and n are positive integers such that $n \geq 2$, then $(aU_{c^m})c^n \in C^n U_{c^m} - (aU_{c^{m+1}})c^{n-2}$.

Proof. Suppose J is a special Jordan algebra.

$$\begin{aligned}(aU_{c^m})c^n &= \tfrac{1}{2}(c^m ac^{m+n} + c^{m+n}ac^m)\\ &= c^m(c\cdot a)c^{m+n-1} + c^{m+n-1}(c\cdot a)c^m\\ &\quad - \tfrac{1}{2}(c^{m+1}ac^{m+n-1} + c^{m+n-1}ac^{m+1})\\ &= ((a\cdot c)U_{c^m})c^{n-1} + c^{n-1}((a\cdot c)U_{c^m})\\ &\quad - \tfrac{1}{2}((aU_{c^{m+1}})c^{n-2} + c^{n-2}(aU_{c^{m+1}}))\\ &= 2((a\cdot c)U_{c^m})\cdot c^{n-1} - (aU_{c^{m+1}})\cdot c^{n-2}.\end{aligned}$$

Thus $(aU_{c^m})c^n = 2((ac)U_{c^m})c^{n-1} - (aU_{c^{m+1}})c^{n-2}$ in any Jordan algebra by Macdonald's theorem. Therefore

$$\begin{aligned}(aU_{c^m})c^n &= 2((ac)c^{n-1})U_{c^m} - (aU_{c^{m+1}})c^{n-2} \text{ by lemma 1-3,}\\ &\in C^n U_{c^m} - (aU_{c^{m+1}})c^{n-2}.\end{aligned}$$

Let $\delta : N \to N$ be a function defined as $\delta(n)$ equal to the greatest integer function of $\frac{1}{2}(n + 2)$. For example $\delta(1) = 1$; $\delta(2) = 2$: $\delta(3) = 2$; $\delta(4) = 3$, and $\delta(5) = 3$.

Lemma 1-9. If J is a Jordan algebra and C is an ideal of J of the form $\langle c \rangle$ for some $c \in C$, then $(aU_{c^{\delta(n)}})c^m \in C^m U_{c^{\delta(n)}} + [c^{\delta(m+n)}]$ for any positive integers n and m.

Proof. If m is an odd integer then:

$$(aU_c\delta(n))c^m = (ac^m)U_c\delta(n) \text{ by lemma 1-3,}$$
$$\in C^mU_c\delta(n) \text{ by lemma 1-7,}$$
$$\in C^mU_c\delta(n) + [c^{\delta(m+n)}].$$

If m is an even integer where m = 2t then:

$$(aU_c\delta(n))c^m \in C^mU_c\delta(n) + C^{m-2}U_c\delta(n)+1 + \ldots + C^2U_c\delta(n)+t-1$$
$$+ [c^{\delta(n)+t}] \text{ by lemma 1-8,}$$
$$\in (C^m + C^{m-2}U_c + \ldots + C^2U_ct-1)U_c\delta(n) + [c^{\delta(n)+t}]$$
$$\in C^mU_c\delta(n) + [c^{\delta(m+n)}].$$

Lemma 1-10. Let J be a Jordan algebra with identity and C be an ideal of J of the form <c>. For any positive odd integer n, if $C^n \subseteq [c^{\delta(n)}] + \sum_{i=1}^{\infty} Rc^{n+i-1}$ then C^n is an ideal.

Proof. If $x \in C^n$ then $x = aU_c\delta(n) + \sum_{i=1}^{\infty} r_i c^{n+i-1}$.

For any $b \in J$, $xb = (aU_c\delta(n))b + \sum_{i=1}^{\infty} r_i bc^{n+i-1} \in$

$(aU_c\delta(n))b + C^n$ by lemma 1-7, since n is an odd integer. $aU_c\delta(n) \in J(U_C)^{\delta(n)}$ which is an ideal by corollary 1-6, and thus $(aU_c\delta(n))b \in J(U_C)^{\delta(n)}$

$$\in JU_C(U_C)^{\delta(n)-1}$$
$$\in C^{2(\delta(n)-1)+1}$$
$$\in C^{2\delta(n)-1}$$
$$\in C^n.$$

Therefore $xb \in C^n$, and C^n is an ideal.

Lemma 1-11. Let J be a Jordan algebra with identity and $\mathcal{C}$ be an ideal of the form $<c>$ for some $c \in \mathcal{C}$. Let i and m be positive integers such that $1 \le i \le m$. If

$$\mathcal{C}^n \subseteq [c^{\delta(n)}] + \sum_{i=1}^{\infty} Rc^{n+i-1} \text{ for all } n \le m, \text{ then}$$

$$c^{2\delta(i)} U_{a,bU_c\delta(m)-\delta(i)} \in \mathcal{C}^m.$$

Proof. $c^{2\delta(i)} U_{a,bU_c\delta(m)-\delta(i)}$

$$= (c^{2\delta(i)}a)(bU_c\delta(m)-\delta(i)) + (c^{2\delta(i)}(bU_c\delta(m)-\delta(i)))a$$
$$- c^{2\delta(i)}(a(bU_c\delta(m)-\delta(i)))$$
$$= - \left(((bU_c\delta(m)-\delta(i))c)a\right)c^{2\delta(i)-1}$$
$$- \left(((bU_c\delta(m)-\delta(i))c^{2\delta(i)-1})a\right)c$$
$$+ c^{2\delta(i)}\left(a(bU_c\delta(m)-\delta(i))\right) + (ac)\left((bU_c\delta(m)-\delta(i))c^{2\delta(i)-1}\right)$$
$$+ (ac^{2\delta(i)-1})\left((bU_c\delta(m)-\delta(i))c\right) + \left(c^{2\delta(i)}(bU_c\delta(m)-\delta(i))\right)a$$
$$- c^{2\delta(i)}\left(a(bU_c\delta(m)-\delta(i))\right) \quad \text{by the use of the identity}$$

$$d(bU_c\delta(m)-\delta(i), a, c, c^{2\delta(i)-1}) = 0,$$

$$= - \left(((bc)U_c\delta(m)-\delta(i))a\right)c^{2\delta(i)-1}$$
$$- \left(((bc^{2\delta(i)-1})U_c\delta(m)-\delta(i))a\right)c$$
$$+ (ac)\left((bc^{2\delta(i)-1})U_c\delta(m)-\delta(i)\right)$$
$$+ (ac^{2\delta(i)-1})\left((bc)U_c\delta(m)-\delta(i)\right) + \left((bc^{2\delta(i)})U_c\delta(m)-\delta(i)\right)a$$

by lemma 1-3,

$\in \mathcal{C}^{2\delta(m)-1}$ by lemma 1-7 and lemma 1-10,

$\in \mathcal{C}^m$.

Theorem 1-12. If J is a Jordan algebra with identity and $\mathcal{C}$ is an ideal of J of the form $<c>$ for some $c \in \mathcal{C}$, then

$$\mathcal{C}^n \subseteq [c^{\delta(n)}] + \sum_{i=1}^{2\delta(n)-n} Rc^{n+i-1}.$$

Proof. The statement is true for $n = 1$ since $C = \langle c\rangle = [c] + Rc$. Assume that the statement is true for all $n \leq k$. Let i and m be positive integers such that $1 \leq i \leq m$ and $i + m = k + 1$. Let $w \in C^i$ and $z \in C^m$.

By the induction hypothesis, $w = aU_{c^{\delta(i)}} + \sum_{n=1}^{2\delta(i)-i} r_n c^{n+i-1}$ and $z = bU_{c^{\delta(m)}} + \sum_{n=1}^{2\delta(m)-m} s_n c^{n+m-1}$.

$$wz = (aU_{c^{\delta(i)}} + \sum_{n=1}^{2\delta(i)-i} r_n c^{n+i-1})(bU_{c^{\delta(m)}} + \sum_{n=1}^{2\delta(m)-m} s_n c^{n+m-1})$$

$$\in (aU_{c^{\delta(i)}})(bU_{c^{\delta(m)}}) + (\sum_{n=1}^{2\delta(i)-i} r_n c^{n+i-1})(bU_{c^{\delta(m)}})$$

$$+ (aU_{c^{\delta(i)}})(\sum_{n=1}^{2\delta(m)-m} s_n c^{n+m-1}) + \sum_{n=1}^{\infty} Rc^{k+n}$$

$$\in c^{2\delta(i)}U_{a,bU_{c^{\delta(m)-\delta(i)}}}U_{c^{\delta(i)}} + \sum_{n=1}^{2\delta(i)-i} r_n((bU_{c^{\delta(m)}})c^{n+i-1})$$

$$+ \sum_{n=1}^{2\delta(m)-m} s_n((aU_{c^{\delta(i)}})c^{n+m-1}) + \sum_{n=1}^{\infty} Rc^{k+n}$$

by lemma 1-4, where $U_{c^{\delta(m)-\delta(i)}}$ is the identity operator when $\delta(m) = \delta(i)$,

$$\in C^m U_{c^{\delta(i)}} + C^i U_{c^{\delta(m)}} + [c^{\delta(m+i)}] + \sum_{n=1}^{\infty} Rc^{k+n}$$

by lemma 1-9 and lemma 1-11,

$$\in \left([c^{\delta(m)}] + \sum_{n=1}^{2\delta(m)-m} Rc^{m+n-1}\right)U_{c^{\delta(i)}}$$

$$+ \left([c^{\delta(i)}] + \sum_{n=1}^{2\delta(i)-i} Rc^{i+n-1}\right)U_{c^{\delta(m)}}$$

$$+ [c^{\delta(m+i)}] + \sum_{n=1}^{\infty} Rc^{k+n}$$

by the induction hypothesis,

$$\in [c^{\delta(m)+\delta(i)}] + \sum_{n=1}^{\infty} Rc^{m+2\delta(i)+n-1} + \sum_{n=1}^{\infty} Rc^{i+2\delta(m)+n-1}$$

$$+ [c^{\delta(m+i)}] + \sum_{n=1}^{\infty} Rc^{k+n}$$

$$\in [c^{\delta(m+i)}] + \sum_{n=1}^{\infty} Rc^{n+m+i-1} + \sum_{n=1}^{\infty} Rc^{k+n}$$

$$\in [c^{\delta(m+i)}] + \sum_{n=1}^{\infty} Rc^{k+n}$$

$$\in [c^{\delta(k+1)}] + \sum_{n=1}^{2\delta(k+1)-(k+1)} Rc^{(k+1)+n-1}.$$

Thus $C^iC^m \subseteq [c^{\delta(k+1)}] + \sum_{n=1}^{2\delta(k+1)-(k+1)} Rc^{(k+1)+n-1}$.

$$C^{k+1} = \sum_{i+m=k+1} C^iC^m \subseteq [c^{\delta(k+1)}] + \sum_{n=1}^{2\delta(k+1)-(k+1)} Rc^{(k+1)+n-1}.$$

Therefore the theorem holds by induction.

Corollary 1-13. If J is a Jordan algebra with identity; A and C are ideals of J; C is of the form <c> for some $c \in C$, and $c^n \in A$, then $C^{2n} \subseteq A$.

Lemma 1-14. If J is a Jordan algebra with identity and A is an ideal of J of the form <a>, then A^n is an ideal of J for any odd positive integer n.

Proof. The lemma follows immediately from lemma 1-10 and theorem 1-12.

Lemma 1-15. If B is an ideal of a Jordan algebra J, and B is of the form <b> for some $b \in B$, then $2aU_{b,cb} \in$

$aU_{\mathcal{B}}$ for any $a,c \in J$.

Proof. $b + cb \in \mathcal{B}$ since $b, cb \in \mathcal{B}$. $2aU_{b,cb} = aU_{b+cb} - aU_b - aU_{cb} \in aU_{\mathcal{B}}$.

Lemma 1-16. Let C and $\mathcal{B}_i$ be ideals of the Jordan algebra J for $i = 1,2,\ldots,n$, where $\mathcal{B}_i$ is of the form $\langle b_i \rangle$ for some $b_i \in \mathcal{B}_i$. If $aU_{b_1}U_{b_2}\ldots U_{b_n} \in C$, for some $a \in J$ then $AU_{\mathcal{B}_1}U_{\mathcal{B}_2}\ldots U_{\mathcal{B}_n} \subseteq C$, where A is the principal ideal generated by a. In particular if $\mathcal{B}$ is an ideal of the form $\langle b \rangle$ and $aU_{b^n} \in C$ then $AU_{b^n} \subseteq C$.

Proof. For $n = 1$, $aU_{b_1} \in C$ implies $(aU_{b_1})d = 2aU_{b_1,db_1} - (ad)U_{b_1} \in C$ for any $d \in J$ by lemma 1-5. $2aU_{b_1,db_1} \in aU_{\mathcal{B}_1} \subseteq C$ by lemma 1-15 and lemma 1-2. Therefore $(ad)U_{b_1} \in C$ and $AU_{b_1} \subseteq C$. Thus by lemma 1-2, $AU_{\mathcal{B}_1} \subseteq C$. Assume the lemma holds for all $n \leq k$ and $aU_{b_1}U_{b_2}\ldots U_{b_{k+1}} \in C$. $(aU_{b_1}U_{b_2}\ldots U_{b_k})U_{b_{k+1}} \in C$ and by the case $n = 1$, $A_kU_{\mathcal{B}_{k+1}} \subseteq C$ where A_k is the principal ideal generated by $aU_{b_1}U_{b_2}\ldots U_{b_k}$. $aU_{b_1}U_{b_2}\ldots U_{b_k} \in A_k$ and by the induction hypothesis $AU_{\mathcal{B}_1}U_{\mathcal{B}_2}\ldots U_{\mathcal{B}_k} \subseteq A_k$. Therefore $AU_{\mathcal{B}_1}U_{\mathcal{B}_2}\ldots U_{\mathcal{B}_{k+1}} \subseteq (AU_{\mathcal{B}_1}U_{\mathcal{B}_2}\ldots U_{\mathcal{B}_k})U_{\mathcal{B}_{k+1}} \subseteq A_kU_{\mathcal{B}_{k+1}} \subseteq C$. Thus the lemma holds by induction.

Lemma 1-17. If J is a Jordan algebra and A is an ideal of the form $A = \langle a \rangle$, then $[a]$ is an ideal of J.

Proof. [a] is closed under addition by definition and under multiplication by lemma 1-4. If $x \in [a]$, then $x = dU_a \in JU_A$. For any $b \in J$, $xb \in JU_A$ since JU_A is an ideal by corollary 1-6. Thus:

$$xb = \sum_{i=1}^{n} \left(d_i U_{(r_i a + c_i U_a)}\right)$$

$$= \sum_{i=1}^{n} \left(d_i U_{r_i a} + d_i U_{c_i U_a} + 2 d_i U_{r_i a, c_i U_a}\right)$$

$$= \sum_{i=1}^{n} (r_i^2 d_i U_a + d_i U_a U_{c_i} U_a + 2 r_i d_i U_{a, c_i U_a})$$

$\in [a]$ since $A^2 \subseteq [a]$ by theorem 1-12. Therefore $xb \in [a]$, and [a] is an ideal of J.

An ideal of a Jordan algebra is said to be irreducible if it is not equal to a finite intersection of ideals strictly containing it.

Theorem 1-18. In a Jordan algebra with the ascending chain condition on ideals, every ideal is a finite intersection of irreducible ideals.

Proof. Let C be the family of ideals which are not the finite intersection of irreducilbe ideals. Suppose $C \neq \emptyset$, then there is a maximal element A in C. Since A cannot be irreducible, $A = \bigcap_{i=1}^{n} A_i$ where the A_i are ideals of J strictly containing A. By the maximality of A in C, $A_i = \bigcap_{j=1}^{n_i} A_{ij}$ where the A_{ij} are irreducible ideals. $A =$

$\bigcap_{i=1}^{n} \bigcap_{j=1}^{n_i} A_{ij}$ contradicting A belonging to C. Therefore $C = \emptyset$ and every ideal of J is a finite intersection of irreducible ideals.

Lemma 1-19. If J is a principal ideal Jordan algebra with identity, then J has the ascending chain condition on ideals.

Proof. If $A_1 \subseteq A_2 \subseteq \ldots$ is any ascending chain of ideals, then $A = \bigcup_{i=1}^{\infty} A_i$ is an ideal. Therefore A is generated by some element a. $a \in \bigcup_{i=1}^{\infty} A_i$ which implies that $a \in A_n$ for some n. Thus $A_n = A_k$ for $k \geq n$.

An ideal C in a Jordan algebra J is said to be u-primary if for any ideals A and B such that $AU_B \subseteq C$ then $A \subseteq C$ or $B^n \subseteq C$ for some positive integer n. A Jordan algebra is said to be u-primary if the ideal (0) is u-primary. A Jordan algebra is said to have property A if the algebra has identity and every ideal is of the form <x> for some element x in the ideal.

Theorem 1-20. If a Jordan algebra J has property A, then any irreducible ideal of J is u-primary.

Proof. Let C be any irreducible ideal and A and B be any ideals of J such that $AU_B \subseteq C$. $A = \langle a \rangle$ and $B = \langle b \rangle$. Let $M_i = \{ y \in J : YU_{b^i} \subseteq C$ where Y is the principal ideal generated by y $\}$. M_i is an ideal of J and $M_i \subseteq M_{i+1}$.

Therefore there exists a positive integer k such that $M_k = M_n$ for $n \geq k$ since J has the ascending chain condition on ideals by lemma 1-19.

$$\begin{aligned} \mathcal{B}^{2k+1} &\subseteq [b^{k+1}] + Rb^{2k+1} \quad \text{by theorem 1-12,} \\ &\subseteq [b^{k+1}] + RbU_{b^k} \\ &\subseteq [b^k]. \end{aligned}$$

Let $z \in (\mathcal{B}^{2k+1} + C) \cap (A + C)$. Therefore $z = dU_{b^k} + c = a' + c'$ where $c, c' \in C$, $a' \in A$, and $d \in J$. $zU_b = dU_{b^{k+1}} + cU_b = a'U_b + c'U_b$, but $(a'U_b + c'U_b)$, $cU_b \in C$. Therefore $dU_{b^{k+1}} \in C$. By lemma 1-16, $\mathcal{D}U_{b^{k+1}} \subseteq C$ where $\mathcal{D}$ is the principal ideal generated by d. Thus $d \in M_{k+1} = M_k$, and $dU_{b^k} \in C$. Therefore $C = (\mathcal{B}^{2k+1} + C) \cap (A + C)$, but $\mathcal{B}^{2k+1}$ is an ideal by lemma 1-14. Since C is irreducible, $\mathcal{B}^{2k+1} \subseteq C$ or $A \subseteq C$, and C is u-primary.

Lemma 1-21. If J is a Jordan algebra, then $U_{b^2}(U_aU_{b^2})^nU_a = (U_{b^2}U_a)^{2n+1}$.

Proof. For $n = 0$, $U_{b^2}U_a = U_{b^2}U_a$. Assume the statement holds for $n = k$, that is $U_{b^2}(U_aU_{b^2})^kU_a = (U_{b^2}U_a)^{2k+1}$.

$$\begin{aligned} U_{b^2}(U_aU_{b^2})^{k+1}U_a &= U_{b^2}(U_aU_{b^2})^kU_aU_{b^2}U_a \\ &= U_{b^2}U_{b^2}(U_aU_{b^2})^kU_aU_{b^2}U_a \\ &= U_{b^2}U_aU_{b^2}(U_aU_{b^2})^kU_aU_{b^2}U_a \\ &= U_{b^2}U_a(U_{b^2}U_a)^{2k+1}U_{b^2}U_a \\ &= (U_{b^2}U_a)^{2(k+1)+1}. \end{aligned}$$

Therefore the lemma holds by induction.

Lemma 1-22. If J is a Jordan algebra then $(aU_b)^{2^n} = b^2(U_aU_{b^2})^{r_n}U_aU_b$ where $r_n = 2^{n-1} - 1$.

Proof. For n = 1, $(aU_b)^2 = 1U_{aU_b} = 1U_bU_aU_b = b^2U_aU_b = b^2(U_aU_{b^2})^0U_aU_b$. Assume the statement holds for $n = k$, that is $(aU_b)^{2^k} = b^2(U_aU_{b^2})^{r_k}U_aU_b$.

$$
\begin{aligned}
(aU_b)^{2^{k+1}} &= \left((aU_b)^{2^k}\right)^2 \\
&= \left(b^2(U_aU_{b^2})^{r_k}U_aU_b\right)^2 \\
&= b^2U_{b^2}(U_aU_{b^2})^{r_k}U_aU_b \\
&= b^2U_aU_{b^2}(U_aU_{b^2})^{r_k}U_aU_b \\
&= b^2U_a(U_{b^2}U_a)^{2r_k+1}U_b \quad \text{by lemma 1-21,} \\
&= b^2(U_aU_{b^2})^{2r_k+1}U_aU_b \\
&= b^2(U_aU_{b^2})^{r_{k+1}}U_aU_b \quad \text{since } 2r_k + 1 = r_{k+1}.
\end{aligned}
$$

Therefore the lemma holds by induction.

An ideal $\mathcal{C}$ of a Jordan algebra J is said to be a u-prime ideal if for any ideals $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}U_{\mathcal{B}} \subseteq \mathcal{C}$ then either $\mathcal{A} \subseteq \mathcal{C}$ or $\mathcal{B} \subseteq \mathcal{C}$. A Jordan algebra is said to be u-prime if (0) is a u-prime ideal. For any ideal $\mathcal{A}$ of J, the u-prime radical $P(\mathcal{A})$ of $\mathcal{A}$ is defined to be the intersection of all the u-prime ideals in J which contain $\mathcal{A}$.

Lemma 1-23. Let $\mathcal{A}$ and $\mathcal{C}$ be ideals of a Jordan algebra with identity, and let $\mathcal{A}$ be of the form <a>. If $a^n \in \mathcal{C}$ for some positive integer n then $\mathcal{A} \subseteq P(\mathcal{C})$.

Proof. By corollary 1-13, $A^{2n} \subseteq C$. $\bar{A}^{2n} = (\bar{0})$ in $\bar{J} = J/C$. Since it is proved in [4] that $P((\bar{0}))$ contains all nilpotent ideals, $\bar{A} \subseteq P((\bar{0}))$. Therefore $A \subseteq P(C)$.

Theorem 1-24. If J is a Jordan algebra with property A and C is a u-primary ideal, then P(C) is a u-prime ideal. In particular, the u-prime radical of an irreducible ideal is u-prime.

Proof. Suppose P(C) is not a u-prime ideal, therefore there exists ideals A and B such that $AU_B \subseteq P(C)$ while neither A nor B is contained in P(C). Let $A = \langle a \rangle$ and $B = \langle b \rangle$. $aU_b \in P(C)$. $(aU_b)^{2^n} \in C$ for some integer 2^n by theorem 7 in [4].

$$b^2(U_aU_{b^2})^{r_n}U_aU_b \in C \text{ by lemma 1-22,}$$
$$1U_b(U_aU_{b^2})^{r_n}U_aU_b \in C$$
$$1U_b(U_aU_bU_b)^{r_n}U_aU_b \in C$$
$$JU_B(U_AU_BU_B)^{r_n}U_AU_B \subseteq C \text{ by lemma 1-16.}$$

Now $JU_B(U_AU_BU_B)^{r_n}U_A$ is an ideal of J by corollary 1-6. Therefore $JU_B(U_AU_BU_B)^{r_n}U_A \subseteq C$ or $B^m \subseteq C$ for some positive integer m since C is a u-primary ideal. $B^m \nsubseteq C$ for if $B^m \subseteq C$ then $B \subseteq P(C)$ by lemma 1-23, contradicting the choice of B. Therefore $JU_B(U_AU_BU_B)^{r_n}U_A \subseteq C$. By repeating this argument and by using the fact that no power of A or B can be contained in C, one obtains $J \subseteq C$. This contradicts the fact that P(C) is not a u-prime ideal. Therefore P(C) is a u-prime ideal.

Lemma 1-25. Let J be a Jordan algebra; $\mathcal{P}$ be a u-prime ideal, and $\mathcal{B}$ be an ideal of the form $\langle b\rangle$. If $a \in J$ such that $aU_b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Proof. If $aU_b \in \mathcal{P}$ then $\mathcal{A}U_b \subseteq \mathcal{P}$ where $\mathcal{A}$ is the principal ideal generated by the element a, by lemma 1-16. $\mathcal{A}U_{\mathcal{B}} \subseteq \mathcal{P}$ by lemma 1-2. Therefore $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$ and $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Lemma 1-26. If $\{ A_i \}_{i=1}^{n}$ is a set of ideals of a Jordan algebra J such that $\bigcap_{i=1}^{n} A_i = (0)$ and $A_i + J_i = J$ where $J_i = \bigcap_{k\neq i} A_k$, then $J = \oplus \sum_{i=1}^{n} J_i$.

Proof. For any $x \in J$, $x = x_i + y_i$ where $x_i \in J_i$, $y_i \in A_i$. $x - \sum_{k=1}^{n} x_k = (x - x_i) - \sum_{k\neq i} x_k = y_i - \sum_{k\neq i} x_k \in A_i$, since $x_k \in J_k \subseteq A_i$ for $i \neq k$. Thus $x - \sum_{k=1}^{n} x_k \in \bigcap_{i=1}^{n} A_i = (0)$, and $x = \sum_{k=1}^{n} x_k$. Therefore $J = \sum_{i=1}^{n} J_i$. The sum is direct since $J_i \cap \sum_{k\neq i} J_k \subseteq J_i \cap A_i = (0)$.

Lemma 1-27. Let J be a Jordan algebra with identity, $\mathcal{B}$ and $\{ A_i \}_{i=1}^{n}$ be ideals of J. If $\mathcal{B}$ is comaximal with each A_i then $\mathcal{B}$ is comaximal with $\bigcap_{i=1}^{n} A_i$.

Proof. $J = J^n = \prod_{i=1}^{n} (\mathcal{B} + A_i) \subseteq \mathcal{B} + \prod_{i=1}^{n} A_i \subseteq \mathcal{B} + \bigcap_{i=1}^{n} A_i \subseteq J$. Therefore $\mathcal{B}$ is comaximal with $\bigcap_{i=1}^{n} A_i$.

Lemma 1-28. Let J be a Jordan algebra with identity and $\{ A_i \}_{i=1}^{n}$ be a set of pairwise comaximal ideals. If $\bigcap_{i=1}^{n} A_i = (0)$, then $J = \oplus \sum_{i=1}^{n} J_i$ where $J_i \simeq J/A_i$.

Proof. By lemma 1-26 and lemma 1-27, $A_i + J_i = J$ and $J = \oplus \sum_{i=1}^{n} J_i$ where $J_i = \bigcap_{k\neq i} A_k$. Let $1 = \sum_{i=1}^{n} e_i$ where $e_i \in J_i$. Let $\gamma_i(a) = ae_i$, which is a homomorphism of J onto J_i. The kernel of γ_i, $K_i = \{ x \in J : xe_i = 0 \} = \sum_{k\neq i} J_k \subseteq A_i$, but $A_i \cap J_i = (0)$ and $\gamma_i(A_i) = (0)$. Therefore $K_i = A_i$, and $J_i \simeq J/A_i$.

Lemma 1-29. Let J be a Jordan algebra with identity and $\{ A_i \}_{i=1}^{n}$ be a set of ideals of J. The A_i are pairwise comaximal if and only if the $P(A_i)$ are pairwise comaximal.

Proof. If $A_i + A_k = J$, then $P(A_i) + P(A_k) = J$ since an ideal is contained in its u-prime radical. If $P(A_i) + P(A_k) = J$, then $1 = a_i + a_k$ where $a_i \in P(A_i)$ and $a_k \in P(A_k)$. $a_i^{\,n} \in A_i$ for some positive integer n. $1^n = (a_i + a_k)^n = a_i^{\,n} + a$ for some $a \in P(A_k)$. There exists a positive integer m such that $a^m \in A_k$. $1 = 1^{nm} = (a_i^{\,n} + a)^m \in (A_i + a)^m \subseteq A_i + a^m \subseteq A_i + A_k$. Therefore $J = A_i + A_k$.

Lemma 1-30. Let J be a Jordan algebra with property A. If P is a u-prime ideal of J, and $P \subsetneq A$, where A is an ideal of the form $A = \langle a \rangle$, then $P \subseteq \bigcap_{i=1}^{\infty} [a^i]$.

Proof. Let $P = \langle p \rangle$ and $p = ra + dU_a$.

$ap = ra^2 + (dU_a)a = rU_a + (da)U_a$ by lemma 1-3,

$= (r + da)U_a \in P$.

$a \notin P$, for if $a \in P$ then $A \subseteq P$ contradicting $P \subsetneq A$. Therefore by lemma 1-25, $r + da \in P \subset A$, and $r \in A$. Let $r = sa + bU_a$.

Thus $p = (sa + bU_a)a + dU_a$

$= sa^2 + (bU_a)a + dU_a$

$= sU_a + (ba)U_a + dU_a$ by lemma 1-3,

$= (s + ba + d)U_a$

$\in [a]$.

Therefore $P \subseteq [a]$. Assume $P \subseteq [a^k]$, then $p = cU_{a^k}$ for some $c \in J$. $cU_{a^k} = c(U_a)^k \in P$, therefore by lemma 1-16, $C(U_A)^k \subseteq P$ where C is the principal ideal generated by c. Since P is a u-prime ideal and $A \not\subseteq P$ then $C \subseteq P$. $c \in P \subseteq [a]$. Thus $c = eU_a$. $p = cU_{a^k} = eU_aU_{a^k} = eU_{a^{k+1}} \in [a^{k+1}]$.

Thus $P \subseteq [a^{k+1}]$ and by induction $P \subseteq \bigcap_{i=1}^{\infty} [a^i]$.

Lemma 1-31. Let J be a Jordan algebra with property A, and let P_1, P_2 be u-prime ideals of J. If $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$ then $P_1 + P_2 = J$.

Proof. Let $P_1 + P_2 = P = \langle p \rangle$. $P_1 \subsetneq P$ and $P_2 \subsetneq P$ since $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. By lemma 1-30, $P_1 \subseteq [p]$ and $P_2 \subseteq [p]$. Thus $P_1 + P_2 \subseteq [p] \subseteq \langle p \rangle = P_1 + P_2$, and $[p] = \langle p \rangle$. $p = aU_p$ for some $a \in J$. $1U_p = 1U_{aU_p} = 1U_pU_aU_p = p^2U_aU_p$. Thus $(1 - p^2U_a)U_p = 0 \in P_1$. $p \notin P_1$, for if $p \in P_1$ then $P \subseteq P_1$ contradicting $P_2 \not\subseteq P_1$. Therefore by lemma 1-25, $1 - p^2U_a \in P_1 \subseteq P$, and $1 \in P = P_1 + P_2$. Therefore $P_1 +$

$P_2 = J$.

Lemma 1-32. Let J be a Jordan algebra with property A. If $\{ C_i \}_{i=1}^n$ is a set of u-primary ideals of J such that $P = P(C_i)$ for $i = 1,2,\ldots,n$, then $C = \bigcap_{i=1}^{n} C_i$ is a u-primary ideal and $P = P(C)$.

Proof. Let $P = \langle p \rangle$, thus $p^{n_i} \in C_i$ for some positive integer n_i where $i = 1,2,\ldots,n$. Let $m = \max\{ n_i \}_{i=1}^n$. Thus $p^m \in C$ and $P \subseteq P(C)$ by lemma 1-23. Therefore $P = P(C)$. Let A and B be any ideals of J such that $AU_B \subseteq C$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. If $A \subseteq C_i$ for $i = 1,2,\ldots,n$, then $A \subseteq C$. If $A \not\subseteq C_j$ for some $j \in \{ 1,2,\ldots,n \}$, then $B^k \subseteq C_j \subseteq P$, implying that $B \subseteq P$, since C_j is a u-primary ideal and P is a u-prime ideal. Since $p^m \in C$, $P^{2m} \subseteq C$ by corollary 1-13. Thus $b^{k(2m)} \in P^{2m} \subseteq C$, and $B^{4km} \subseteq C$ by corollary 1-13. Therefore C is a u-primary ideal.

Lemma 1-33. If J is a Jordan algebra with property A, and C_1, C are u-primary ideals such that $P(C_1) \subsetneqq P(C)$, then $C_1 \subsetneqq C$.

Proof. Let $P(C_1) = P_1$ and $P(C) = P = \langle p \rangle$. P_1 and P are u-prime ideals by theorem 1-24. There exists and integer $m \geq 2$ such that $p^m \in C$. By theorem 1-12, $P^{2m} \subseteq [p^{m+1}] + Rp^{2m} + Rp^{2m+1} \subseteq C$. But $C_1 \subseteq P_1 \subseteq \bigcap_{i=1}^{\infty} [p^i] \subseteq [p^{2m}] \subseteq P^{2m} \subseteq C$ by lemma 1-30. Suppose $C_1 = C$, then

$p^m \in P_1$. By lemma 1-23, $P \subseteq P_1$ contradicting $P_1 \subsetneqq P$. Therefore $C_1 \subsetneqq C$.

Lemma 1-34. Let J be a Jordan algebra with property A. If A is an ideal of J, then there exists a finite set of pairwise comaximal, u-primary ideals $\{ C_i \}_{i=1}^{n}$ such that $A = \bigcap_{i=1}^{n} C_i$.

Proof. By theorem 1-18 and theorem 1-20, $A = \bigcap_{i=1}^{m} A_i$ where the A_i are u-primary ideals. Let $M = \{ 1,2,\ldots,m \}$, and $I = \{ i \in M : P(A_i) = P(A_1) \}$. Let $D_1 = \bigcap_{i \in I} A_i$. D_1 is a u-primary ideal with $P(A_1) = P(D_1)$ by lemma 1-32. Let $r \in M - I$, and $H = \{ i \in M : P(A_i) = P(A_r) \}$. Let $D_2 = \bigcap_{i \in H} A_i$. D_2 is a u-primary ideal with $P(D_2) = P(A_r) \neq P(D_1)$ by lemma 1-32. In this manner, one can construct from $\{ A_i \}_{i=1}^{m}$ the set $\{ D_i \}_{i=1}^{k}$ such that $A = \bigcap_{i=1}^{k} D_i$ and the D_i are u-primary ideals with the $P(D_i)$ distinct. Let $\{ C_i \}_{i=1}^{n}$ be the set of all minimal ideals of the set $\{ D_i \}_{i=1}^{k}$. Therefore $A = \bigcap_{i=1}^{n} C_i$. Suppose $P(C_i) + P(C_j) \neq J$ for some $i \neq j$, then without lose of generality one may assume by lemma 1-31 that $P(C_i) \subsetneqq P(C_j)$. By lemma 1-33, $C_i \subsetneqq C_j$ contradicting the choice of the C_j being minimal in the set $\{ D_i \}_{i=1}^{k}$. Therefore $P(C_i) + P(C_j) = J$ for $i \neq j$. By lemma 1-29, $\{ C_i \}_{i=1}^{n}$ is a set

of pairwise comaximal ideals.

Theorem 1-35. If J is a Jordan algebra with property A, then $J = \oplus \sum_{i=1}^{n} J_i$ where each J_i has property A and is either a u-prime algebra or a u-primary algebra containing a nonzero nilpotent ideal.

Proof. By lemma 1-34, $(0) = \bigcap_{i=1}^{n} C_i$ where the C_i are pairwise comaximal, u-primary ideals. Thus $J = \oplus \sum_{i=1}^{n} J_i$ where $J_i \simeq J/C_i$ by lemma 1-28. Each J_i has property A since it is a homomorphic image of J. If $C_i = P(C_i)$, then J_i is a u-prime algebra, if not then J_i is a u-primary algebra with a nilpotent ideal $P(C_i)/C_i$.

Lemma 1-36. Let J be a u-prime Jordan algebra with property A. If the u-prime ideals are pairwise comaximal, then J is simple.

Proof. Suppose J is not simple, then J contains a nontrival ideal A. $A = \bigcap_{i=1}^{n} C_i$ where the C_i are u-primary ideals by lemma 1-34. Thus J contains a nontrival u-primary ideal C. P(C) is a u-prime ideal by theorem 1-24, but J has (0) and J as its only u-prime ideals. If $P(C) = (0)$, then $C = (0)$ contradicting C being nontrivial. If $P(C) = J$, then $1 = 1^n \in C$ for some positive integer n: thus $C = J$ contradicting C being nontrivial.

Therefore J is simple.

A Jordan algebra J is said to have property B, if J has an identity and every ideal is of the form [x] for some x in the ideal. Property B is a special case of property A.

Lemma 1-37. If J is a Jordan algebra with property B, then the u-prime ideals are pairwise comaximal.

Proof. Suppose that P_1, P_2 are distinct u-prime ideals. Suppose $P_1 + P_2 \neq J$. By lemma 1-31, one may assume without loss of generality that $P_1 \not\subseteq P_2$. $P_1 = [p]$ for some $p \in P_1$. $p = aU_p$ for some $a \in J$. $1U_p = 1U_{aU_p} = 1U_pU_aU_p = p^2U_aU_p$. Thus $(1 - p^2U_a)U_p = 0 \in P_2$. By lemma 1-25, $p \in P_2$ or $(1 - p^2U_a) \in P_2$. $p \notin P_2$, for if $p \in P_2$ then $P_1 \subseteq P_2$ which is a contradiction. Thus $(1 - p^2U_a) \in P_2$ and $P_1 + P_2 = J$. Therefore the u-prime ideals of J are pairwise comaximal.

Lemma 1-38. If J is a Jordan algebra with property B, then J contains no nonzero nilpotent ideals.

Proof. Suppose A is a nilpotent ideal of J. $A = [a]$ for some $a \in A$. Thus $a = bU_a$ and $a^n = 0$ for some positive integer n. $a = bU_a = bU_{bU_a} = bU_aU_bU_a = bU_aU_bU_{bU_a} = b(U_aU_b)^2U_a = \dots = b(U_aU_b)^{2n}U_a \in A^{2n} = (0)$ by theorem 1-12. Therefore $A = (0)$.

Theorem 1-39. A Jordan algebra J has property B if and only if $J = \oplus \sum_{i=1}^{n} J_i$ where each summand is a simple Jordan algebra with identity.

Proof. If J has property B, then $J = \oplus \sum_{i=1}^{n} J_i$ where each summand is either a u-prime algebra or contains a nonzero nilpotent ideal by theorem 1-35. Since each summand has property B, it must be a u-prime algebra by lemma 1-38. Thus by lemma 1-37 and lemma 1-36, each summand is a simple Jordan algebra with identity. If $J = \oplus \sum_{i=1}^{n} J_i$ where each summand is a simple Jordan algebra with identity, then J has identity. Let A be any ideal, then $A = \oplus \sum_{i=1}^{n} A_i$. Let $e = \oplus \sum_{i=1}^{n} e_i$ where $e_i = 1_i$ if $A_i = J_i$ and $e_i = 0$ if $A_i = (0)$. Thus $e \in A$. $[e] = A$ since if $a \in A$ then $a = \oplus \sum_{i=1}^{n} b_i U_{e_i}$ where $b_i \in J_i$, and $a = (\sum_{i=1}^{n} b_i) U_e \in [e]$.

An element z of a Jordan algebra J is said to be an absolute zero divisor if $JU_z = (0)$.

Lemma 1-40. If J is a Jordan algebra with property A and J contains a nonzero nil ideal, then J contains a nonzero absolute zero divisor.

Proof. If $A = \langle a \rangle$ is a nonzero nil ideal then $a^n = 0$ for some positive integer n, and $A^{2n} = (0)$ by corollary 1-13. By lemma 1-14, A^m is an ideal for any positive odd integer m. Thus J contains a nilpotent ideal of index 2 or 3, since either A^{k-1} or A^{k-2} is an ideal of J where k is the nilpotent index of A. Let $B = \langle b \rangle$ be any nilpotent ideal of J of nilpotent index 2 or 3. If $b^2 = 0$ and $c \in J$ then:

$$\begin{aligned}
cU_b &= 2(cb)b - cb^2 \\
&= 2(cb)b \quad \text{since } b^2 = 0, \\
&= 2(rb + dU_b)b \quad \text{since } cb \in B = \langle b \rangle, \\
&= 2(db)U_b + 2rb^2 \text{ by lemma 1-3}, \\
&= 2(db)U_b \quad \text{since } b^2 = 0, \\
&= 2(sb + eU_b)U_b \text{ since } db \in B = \langle b \rangle, \\
&= 2sb^3 + 2eU_bU_b \\
&= 2sb^3 + 2eU_{b^2} \\
&= 0 \quad \text{since } b^2 = 0.
\end{aligned}$$

Therefore b is an absolute zero divisor. If $b^3 = 0$, and $b^2 \neq 0$ then for $c \in J$:

$$\begin{aligned}
cU_{b^2} &= 2(cb^2)b^2 - cb^4 \\
&\in B^3 = (0).
\end{aligned}$$

Therefore b^2 is an absolute zero divisor. Thus J contains a nonzero absolute zero divisor.

Theorem 1-41. Let J be a Jordan algebra with property A. If J contains no nonzero absolute zero divisors, then J is a direct sum of u-prime algebras.

Proof. By theorem 1-35, $J = \oplus \sum_{i=1}^{n} J_i$ where each J_i is either a u-prime algebra or a u-primary algebra containing a nonzero nilpotent ideal. By lemma 1-40, J contains a nonzero absolute zero divisor if some J_i is not a u-prime algebra. Therefore if J contains no nonzero absolute zero divisors, then J is a direct sum of u-prime algebras.

Any associative, commutative principal ideal ring with identity is a Jordan algebra with property A when it is considered as an algebra over itself, and thus it could appear as a summand in theorem 1-35.

Another example of a possible summand can be constructed from the vector space J over a field F, of characteristic not two, with a basis $\{ e_{11}, e_{12}, e_{21}, e_{22}, z \}$. Under the multiplication given below, which is obtained by putting the Jordan product on the noncommutative

	e_{11}	e_{12}	e_{21}	e_{22}	z
e_{11}	e_{11}	$\frac{1}{2}e_{12}$	$\frac{1}{2}e_{21}$	0	$\frac{1}{2}z$
e_{12}	$\frac{1}{2}e_{12}$	0	$\frac{1}{2}(e_{11}+e_{22})$	$\frac{1}{2}e_{12}$	0
e_{21}	$\frac{1}{2}e_{21}$	$\frac{1}{2}(e_{11}+e_{22})$	0	$\frac{1}{2}e_{21}$	0
e_{22}	0	$\frac{1}{2}e_{12}$	$\frac{1}{2}e_{21}$	e_{22}	$\frac{1}{2}z$
z	$\frac{1}{2}z$	0	0	$\frac{1}{2}z$	0

Jordan algebra described in ([3], p.147), J becomes a Jordan algebra over F. The ideals of J are: $(0) \subseteq \langle z \rangle \subseteq \langle e_{11}+e_{22} \rangle = J$, and thus J is a u-primary Jordan algebra with property A. Also the subalgebra generated by the set $\{ e_{11}, e_{22}, e_{12} + e_{21}, z \}$ has property A, and its ideals are: $(0) \subseteq \langle z \rangle \subseteq \langle e_{11} + e_{22} \rangle$.

The subalgebra T of the preceeding example, generated by the set $\{ 1, e, z \}$, where $1 = e_{11} + e_{22}$ and $e = e_{12} + e_{21}$, is a principal ideal Jordan algebra without property A. Its ideals are (0), N, C, $\mathcal{D}$, and T generated by the elements 0, z, 1+e, 1-e, and 1 respectively (the ideals C and $\mathcal{D}$ are not of the right form for T to have property A). (0) is an irreducible ideal of T, and $P((0)) = N$, which is not u-prime since $CU_{\mathcal{D}} \subseteq N$ and no power of either C or $\mathcal{D}$ is contained in N. In view of this example, theorem 1-24 cannot be generalized to include all principal ideal Jordan algebras.

Consider the Jordan algebra obtained by putting the Jordan product on the upper triangular 2x2 matrices over a field F of characteristic not two. This algebra is a principal ideal algebra, but does not have property A since the ideals $A = Fe_{11} + Fe_{12}$ and $B = Fe_{22} + Fe_{12}$ are not of the right form. (0) is not a finite intersection of ideals properly containing it, and thus it is irreducible. $AU_B = BU_A = (0)$, but neither A nor B is nilpotent. Thus (0) is irreducible but not u-primary. In view of these last two examples, the

methods of this paper cannot be generalized to include all principal ideal Jordan algebras.

CHAPTER II

JORDAN ALGEBRAS WITH PROPERTY A OVER A FIELD

In this chapter it will be understood that all Jordan algebras are over an arbitrary field F of characteristic not two. Let J be a Jordan algebra and A an ideal of the form $\langle a \rangle$. It was proved in Chapter I that $A^2 \subseteq [a]$, and a question which arises is whether or not $[a] \subseteq A^2$. If $[a] \subseteq A^2$ then all the powers of A are ideals which would simplify many of the proofs. When J is an algebra over a field, this question is partially answered in the following lemma.

Lemma 2-1. If J is a Jordan algebra with property A, and A is an ideal of J of the form $A = \langle a \rangle$, then either $a^2 \in [a^2]$ or $A^2 = [a]$ which is an ideal of J.

Proof. By lemma 1-17, [a] is an ideal of J; therefore $[a] = \langle cU_a \rangle$. Let $\overline{J} = J/A^3$. Thus $\overline{1}U_{\overline{a}} = \overline{a}^2 = f\overline{c}U_{\overline{a}} + \overline{b}U_{\overline{c}U_{\overline{a}}} = f\overline{c}U_{\overline{a}} + \overline{b}U_{\overline{a}}U_{\overline{c}}U_{\overline{a}} = f\overline{c}U_{\overline{a}}$ since $bU_aU_cU_a \in A^3$. If $f \neq 0$, then $\overline{c}U_{\overline{a}} = f^{-1}\overline{a}^2$, and $cU_a \in f^{-1}a^2 + A^3 \subseteq A^2$. Thus $[a] \subseteq A^2$. $A^2 \subseteq [a]$ by theorem 1-12. Therefore $A^2 = [a]$ which is an ideal of J. If $f = 0$, then $a^2 \in A^3$.

$a^2 = dU_{a^2} + ra^3$ by theorem 1-12,

$= dU_{a^2} + ra(a^2)$

$= dU_{a^2} + ra(dU_{a^2} + ra^3)$

$= dU_{a^2} + r(ad)U_{a^2} + r^2U_{a^2}$ by lemma 1-3,

$\in [a^2]$.

Lemma 2-2. Let J be a Jordan algebra with an ideal A of the form $A = \langle a \rangle$. If $a^2 \in [a^2]$, then $a^2 \in A^n$ for any $n \geq 1$. In particular, if A is a nilpotent ideal, then $a^2 = 0$.

Proof. Let $a^2 = bU_{a^2}$. Thus $a^2 = bU_{bU_{a^2}} = b(U_{a^2}U_b)^1U_{a^2} = b(U_{a^2}U_b)^1U_{bU_{a^2}} = b(U_{a^2}U_b)^2U_{a^2} = \ldots = b(U_{a^2}U_b)^nU_{a^2} \in A^n$.

Lemma 2-3 Let J be a Jordan algebra with property A. If A is a nilpotent ideal of J then $A^4 = (0)$.

Proof. Let $A = \langle a \rangle$. If $a^2 \in [a^2]$, then $a^2 = 0$ by lemma 2-2, and $A^3 = (0)$ by theorem 1-12. In view of lemma 2-1, it suffices to show that the lemma holds in the case when $[a] = A^2$ is an ideal. Let n be the smallest positive integer such that $a^n = 0$. Suppose $n > 4$. Without loss of generality it may be assumed that $n = 5$, since one could pass to the quotient algebra $\bar{J} = J/A^5$, since A^5 is an ideal by lemma 1-14.

$A^3 \subseteq Fa^3 + [a^2]$ by theorem 1-12,

$\subseteq Fa^3 + A^4$ since $[a^2] \subseteq A^4$ when $[a] = A^2$ is an ideal,

$\subseteq Fa^3 + Fa^4$ by theorem 1-12 and by the fact that

$a^5 = 0$. $A^3 = \langle f_1a^3 + f_2a^4\rangle$ since J has property A. $a^4 \in A^3$ thus $a^4 = f_3(f_1a^3 + f_2a^4) + bU_{f_1a^3+f_2a^4} = f_3f_1a^3 + f_3f_2a^4$ since $bU_{f_1a^3+f_2a^4} \in A^5 = (0)$. $0 = a^5 = f_3f_1a^4 + f_3f_2a^5 = f_3f_1a^4$, thus $f_3f_1 = 0$ and $f_1 = 0$ since $a^4 \neq 0$. Therefore $A^3 = \langle f_2a^4\rangle$. $a^3 \in A^3$ and $a^3 = f_4f_2a^4 + cU_{f_2a^4} = f_4f_2a^4$ since $cU_{f_2a^4} \in A^5 = (0)$. $a^4 = (a^3)a = (f_4f_2a^4)a = f_4f_2a^5 = 0$, contradicting the choice of n. Therefore $n \leq 4$. By theorem 1-12, $A^4 = [a^3] + Fa^4 + Fa^5$, but $[a^3] \subseteq A^5 = (0)$, when $[a] = A^2$. $A^4 = Fa^4 + Fa^5 = (0)$ since $a^4 = 0$.

Corollary 2-4. Let J be a Jordan algebra with property A. If A is an ideal of J, then $A^4 = A^n$ for $n \geq 4$.

Proof. Let $\overline{J} = J/A^m$ where m is an odd integer greater then n. $\overline{A}$ is a nil ideal of $\overline{J}$, and by lemma 2-3 $\overline{A}^4 = (\overline{0})$. Therefore $A^4 \subseteq A^m \subseteq A^n \subseteq A^4$, and $A^4 = A^n$.

Lemma 2-5. Let J be a Jordan algebra with property A, and let A be an ideal of J. If $a^n = 0$, then $A^n = (0)$ where $A = \langle a\rangle$.

Proof. If $n = 1$, $A = \langle 0\rangle = (0)$. If $n = 2$,

$A^2 \subseteq Fa^2 + A^3$ by theorem 1-12,

$\subseteq Fa^2 + Fa^3 + [a^2] = (0)$ by theorem 1-12.

In view of lemma 2-1 and lemma 2-2, it may be assumed that $A^2 = [a]$ is an ideal of J. If $n = 3$,

$A^3 \subseteq Fa^3 + [a^2]$ by theorem 1-12,

$\subseteq Fa^3 + A^4$ since $[a^2] \subseteq A^4$ when $A^2 = [a]$ is an ideal,

$\subseteq Fa^3 + Fa^4 + Fa^5 + [a^3] = (0)$ by theorem 1-12.

If $n \geq 4$, then $A^n = (0)$ by lemma 2-3.

Theorem 2-6. If J is a Jordan algebra with property A, then all the u-prime ideals are pairwise comaximal.

Proof. Let P_1 and P_2 be any distinct u-prime ideals of J. Without loss of generality one may assume that $P_1 \nsubseteq P_2$. Let $P_1 = \langle p_1 \rangle$. $P_1{}^4 = P_1{}^9 \subseteq [p_1{}^5] + Fp_1{}^9 \subseteq [p_1{}^4] \subseteq P_1{}^4$ by theorem 1-12 and corollary 2-4. Thus $[p] = P_1{}^4$ and $p = aU_p$ where $p = p_1{}^4$. $1U_p = 1U_{aU_p} = p^2U_aU_p$. $(1 - p^2U_a)U_p = 0 \in P_2$. By lemma 1-25, $p \in P_2$ or $1 - p^2U_a \in P_2$. If $p = p_1{}^4 \in P_2$, then $P_1 \subseteq P(P_2) = P_2$ by lemma 1-23, which contradicts $P_1 \nsubseteq P_2$. Thus $1 - p^2U_a \in P_2$, and $P_1 + P_2 = J$.

Theorem 2-7. Let J be a Jordan algebra. J has property A if and only if $J = \oplus \sum_{i=1}^{n} J_i$ where each summand has property A, all but at most one of the summands are simple Jordan algebras, and each summand is either a simple algebra or contains only one proper u-prime ideal, which is nilpotent.

Proof. If J has property A, then by theorem 1-35, $J = \oplus \sum_{i=1}^{n} J_i$ where each summand is either u-prime or contains a proper u-prime ideal which is nilpotent. Suppose that there are two summands J_i, J_k which do not have property B. There exist ideals A of J_i and B of J_k

which are not of the form [x] for some x in the ideal. $\mathcal{A} \oplus \mathcal{B}$ is an ideal of J, thus $\mathcal{A} \oplus \mathcal{B} = \langle a \oplus b \rangle$ for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. $a \in \mathcal{A} \oplus \mathcal{B}$, therefore $a = r(a \oplus b) + dU_{(a\oplus b)}$. $(1 - r)a = dU_a$ and $-rb = dU_b$ since the sum is direct. $1 - r \neq 0$ or $r \neq 0$, without loss of generality assume $r \neq 0$. Thus $b = r^{-1}dU_b$ and $\mathcal{B} = [b]$ contradicting the choice of $\mathcal{B}$. Therefore all but at most one of the summands have property B, and by theorem 1-39, they are a direct sum of simple Jordan algebras. If the remaining summand is u-prime, then it is simple by lemma 1-36 and theorem 2-6, otherwise it has only one proper u-prime ideal, which is nilpotent. If $J = \oplus \sum_{i=1}^{n} J_i$ where the summands satisfy the conditions given in the statement of the theorem, then J has identity since each summand does. Let J_i for $i = 2,3,\ldots,n$ have property B. If $\mathcal{C}$ is any ideal then $\mathcal{C} = \oplus \sum_{i=1}^{n} \mathcal{C}_i$ where $\mathcal{C}_i$ is an ideal of J_i. Thus $\mathcal{C}_1 = \langle c_1 \rangle$, $\mathcal{C}_k = [c_k]$ for $k = 2,3,\ldots,n$ and $c_i \in \mathcal{C}_i$. Let $c = \sum_{i=1}^{n} c_i$. There exist elements $d_k \in J_k$ such that $d_k U_{c_k} = c_k$ for $k = 2,3,\ldots,n$. Thus $d_k U_c = c_k \in \langle c \rangle$ for $k = 2,3,\ldots,n$, and $c_1 = c - \sum_{i=2}^{n} c_i \in \langle c \rangle$. Therefore $\langle c \rangle = \mathcal{C}$ and J has property A.

Theorem 2-8. Let J be a Jordan algebra with property A. If $Z(J) = \{ x \in J : JU_x = 0 \}$, then $Z(J)$ is an ideal of J of the form Fx for some $x \in J$.

Proof. Let $A = \langle a \rangle$ be the nil radical of J. Kevin McCrimmon has shown that $Z(J) \subseteq A$. Let n be the smallest positive integer such that $a^n = 0$. If $n = 1$, $A = \langle 0 \rangle = (0) = Z(J)$. If $n = 2$, $A^2 \subseteq [a] = JU_a = (0)$ as in lemma 1-40, and $Z(J) = A = Fa$. For $n > 2$, one may assume $[a] = A^2$ in view of lemma 2-1 and lemma 2-2. If $n = 3$, $A^3 = (0)$ by lemma 2-5. For $x \in Z(J)$, $x = ra + bU_a$ and

$$
\begin{aligned}
0 = 1U_x &= 1U_{ra+bU_a} \\
&= r^2U_a + 1U_{bU_a} + 2U_{ra,bU_a} \\
&= r^2U_a + a^2U_bU_a + 2r(ba)U_a \text{ by lemma 1-3,} \\
&= r^2a^2 \text{ since } a^2U_bU_a + 2r(ba)U_a \in A^3 = (0).
\end{aligned}
$$

Since $a^2 \neq 0$, $r = 0$ and $Z(J) \subseteq [a] = A^2$. Therefore $Z(J) = A^2 = \langle a' \rangle = Fa'$ for some $a' \in [a]$.

If $n = 4$, $A^4 = (0)$ by lemma 2-5. For $x \in Z(J)$, $0 = 1U_x = r^2a^2 + a^2U_bU_a + 2r(ba)U_a$ as in the case for $n = 3$. Thus $r^2a^2 \in A^3$. If $r \neq 0$ then $a^2 \in A^3$ and $a^3 \in A^4 = (0)$, which contradicts the choice of n. Therefore $r = 0$ and $Z(J) \subseteq [a] = A^2$. For $d \in [a]$, $d = cU_a$, then $eU_d = eU_{cU_a} = eU_aU_cU_a \in A^4 = (0)$. Therefore $Z(J) = [a] = A^2 = \langle a' \rangle = Fa'$ for some $a' \in [a]$.

APPENDIX

APPENDIX

POLYNOMIAL IDENTITIES IN JORDAN ALGEBRAS OVER A COMMUTATIVE RING WITH IDENTITY

For any set X, let N(X)' be the free monad generated by X ([2], pp.23-25), then the free module FR(X) over Z generated by N(X)' is the free nonassociative ring generated by X. Let $T(X)$ be the ideal generated by all the elements of the form $f(x,y) = (x,y,x^2) = (xy)x^2 - x(yx^2)$ and $g(x,y) = xy - yx$ where $x,y \in FR(X)$, then $FJR(X) = FR(X)/T(X)$ is the free Jordan ring generated by X.

Define $h(x,y,z) = (x,y,xz + zx) + (z,y,x^2)$ and $d(w,x,y,z) = (w,x,zy) + (z,x,yw) + (y,x,wz)$. Let $A(X) = \{ h(x,y,z) : x,y,z \in N(X)' \} \cup \{ d(w,x,y,z) : w,x,y,z \in N(X)' \}$, and $W(X) = A(X) \cup \{ g(x,y) : x,y \in N(X)' \}$. Let $M(X)$ be the ideal of FR(X) generated by W(X).

Lemma A-1. $FR(X) = M(X) \oplus P$ as modules, where P is a free module over Z.

Proof. Let $L(X) = \{ R_y : y \in N(X)' \} \cup \{ L_y : y \in N(X)' \} \cup \{$ identity operator $\}$. where R_y anf L_y are right and left multiplication by y respectively. Let $B(X) = \{ wT_1T_2...T_n : w \in W(X), T_i \in L(X), \text{ and } n \in N \}$.

$B(X)$ generates $M(X)$ as a Z-module. Let $Q(X)'$ be the free nonassociative algebra over the rational numbers Q (the vector space over Q with $N(X)'$ as a basis). $FR(X) \subsetneq Q(X)'$. Let $M(X)'$ be the subspace of $Q(X)'$ generated by $B(X)$. Let $H(X)$ be a basis of $M(X)'$ such that $H(X) \subsetneq B(X)$. Extend $H(X)$ to a basis $V(X)$ of $Q(X)'$ by elements of $N(X)'$. $H(X)$ is a basis over Z for $FR(X)$. Thus $M(X)$ is spanned by $H(X)$, and if P is the free Z-module generated by $V(X) - H(X)$, then $FR(X) = M(X) \oplus P$ as modules.

Lemma A-2. If $w,x,y,z \in FR(X)$, then $d(w,x,y,z)$, $g(x,y) \in M(X)$.

Proof. The lemma holds by multilinearity of $d(w,x,y,z)$ and $g(x,y)$ and by the definition of $M(X)$.

Lemma A-3. If $y \in FR(X)$ and $z \in N(X)'$, then $f(y,z) \in M(X)$.

Proof. Let $y = \sum_{i=1}^{n} n_i y_i$ where $n_i \in Z$ and $y_i \in N(X)'$. For $n = 1$, $f(n_1 y_1, z) = n_1{}^3 f(y_1, z) \in M(X)$. Assume that the statement holds for $n \leq k$. Then for $n = k + 1$:

$f(y,z) = (x + n_{k+1}y_{k+1}, z, (x + n_{k+1}y_{k+1})^2)$ where $x = \sum_{i=1}^{k} n_i y_i$,

$$\begin{aligned} &= (x,z,x^2) + n_{k+1}(x,z,xy_{k+1} + y_{k+1}x) \\ &\quad + n_{k+1}{}^2(x,z,y_{k+1}{}^2) + n_{k+1}(y_{k+1},z,x^2) \\ &\quad + n_{k+1}{}^2(y_{k+1},z,xy_{k+1} + y_{k+1}x) + n_{k+1}{}^3(y_{k+1},z,y_{k+1}{}^2) \end{aligned}$$

$$= f(x,z) + n_{k+1}{}^3 f(y_{k+1},z) + n_{k+1}{}^2 h(y_{k+1},z,x)$$
$$+ n_{k+1} h(x,z,y_{k+1})$$
$$= f(x,z) + n_{k+1}{}^3 f(y_{k+1},z) + n_{k+1}{}^2 h(y_{k+1},z,x)$$
$$+ n_{k+1} n_k d(y_k,z,y_{k+1},w) + n_{k+1} n_k{}^2 h(y_k,z,y_{k+1})$$
$$+ n_{k+1} n_k d(y_k,z,w,y_{k+1}) + n_{k+1} h(w,z,y_{k+1})$$

where $w = \sum_{i=1}^{k-1} n_i y_i$,

$$= f(y,z) + n_{k+1}{}^3 f(y_{k+1},z) + \sum_{i=1}^{k} n_{k+1}{}^2 n_i h(y_{k+1},z,y_i)$$
$$+ n_{k+1} n_k d(y_k,z,y_{k+1},w) + n_{k+1} n_k{}^2 h(y_k,z,y_{k+1})$$
$$+ n_{k+1} n_k d(y_k,z,w,y_{k+1}) + f(n_{k+1} y_{k+1} + w,z)$$
$$- f(w,z) - n_{k+1}{}^3 f(y_{k+1},z) - \sum_{i=1}^{k-1} n_{k+1}{}^2 n_i h(y_{k+1},z,y_i)$$

$\in M(X)$ by lemma A-2, the induction hypothesis, and by the definition of $M(X)$. Therefore the lemma holds by induction.

Lemma A-4. If $x,y \in FR(X)$ then $f(x,y) \in M(X)$.

Proof. Let $y = \sum_{i=1}^{n} n_i y_i$ where $y_i \in N(X)'$ and $n_i \in Z$.

$$f(x,y) = f(x, \sum_{i=1}^{n} n_i y_i) = \sum_{i=1}^{n} n_i f(x,y_i) \in M(X) \text{ by lemma A-3.}$$

Lemma A-5. Let J be any nonassociative ring with a generating set X. If the polynomial identities $g(x,y) = 0$, $f(x,y) = 0$, $h(x,y,z) = 0$, and $d(w,x,y,z) = 0$ are satisfied on X, then J is a Jordan ring. In particular if J is an algebra over a field of characterstic not

two or three, with the polynomial identities $g(x,y) = 0$ and $d(w,x,y,z) = 0$ being satisfied on X, then J is a Jordan algebra.

Proof. $FR(X)/M(X)$ is a Jordan ring by lemma A-2 and lemma A-4. Since $g(x,y) = 0$, $f(x,y) = 0$, $h(x,y,z) = 0$, and $d(w,x,y,z) = 0$ are satisfied on the generating set X of J, then J is a homomorphic image of $FR(X)/M(X)$ and is a Jordan ring. If J is an algebra over a field of characteristic not two or three then: $f(x,y) = \frac{1}{3}d(x,y,x,x) = 0$ and $h(x,y,z) = d(x,y,z,x) = 0$ for $x,y,z \in X$. Thus by the first part of the lemma, J is a Jordan algebra.

Lemma A-6. If $w,x,y,z \in FR(X)$, then $2h(x,y,z)$, $2d(w,x,y,z) \in T(X)$.

Proof. Let $h'(x,y,z) = h(x,y,z) + h(z,y,x)$. For $a,b,c \in FR(X)$, $h'(a,b,c) = f(a + c,b) - f(a,b) - f(c,b) \in T(X)$. Therefore $2d(w,x,y,z) = h'(w + y,x,z) - h'(w,x,z) - h'(y,x,z) + g(y,x)L_{zw} - g(y,x)L_wL_z + g(x,z)L_{yw} - g(x,z)L_wL_y + g(z,y)L_{xw} - g(z,y)L_wL_x \in T(X)$. For $n \in N$, $nh(a,b,c) + n^2h(b,c,a) = f(a + nb,c) - f(a,c) - f(nb,c) \in T(X)$. Therefore $2h(z,y,x) + 4h(x,y,z)$, $h(z,y,x) + h(x,y,z) \in T(X)$, and $2h(x,y,z) = 4h(x,y,z) + 2h(z,y,x) - 2(h(z,y,x) + h(x,y,z)) \in T(X)$.

Lemma A-7. Let S be any Jordan ring, if $t(x_1,x_2,\ldots,x_n) = 0$ is a polynomial identity for $FR(S)/M(S)$, then $2t(x_1,x_2,\ldots,x_n) = 0$ is a polynomial identity for S.

If in addition $h(x,y,z) = 0$ and $d(w,x,y,z) = 0$ are polynomial identities for S, then $t(x_1,x_2,\ldots,x_n) = 0$ is also a polynomial identity for S.

Proof. By lemma A-2 and lemma A-4, $FR(S)/M(S)$ is a Jordan ring, therefore there is a homomorphism $\gamma: FJR(S) \to FR(S)/M(S)$ such that $\gamma(s) = s$ for $s \in S$. By lemma A-6 and by the definition of $T(S)$, it follows that $f(x,y)$, $g(x,y)$, $2d(w,x,y,z)$, and $2h(x,y,z) \in T(S)$ for $w,x,y,z \in FR(S)$. Thus the kernel K of γ is the Z-module generated by $\{ aT_1T_2\ldots T_n : a \in A(S)$ and $T_i \in L(S) \}$ for $L(S)$ defined in lemma A-1. In $FJR(S)$, $t(x_1,x_2,\ldots x_n) \in K$ and $2t(x_1,x_2,\ldots x_n) = 0$. Since S is a homomorphic image of $FJR(S)$, then $2t(x_1,x_2,\ldots,x_n) = 0$ is a polynomial identity for S. If $h(x,y,z) = 0$ and $d(w,x,y,z) = 0$ are polynomial identities for S, then there is a homomorphism $\alpha: FJR(S) \to S$ such that $\alpha(s) = s$, and K is contained in the kernel of α. Therefore there is a homomorphism $\beta: FR(S)/M(S) \to S$ such that $\beta\gamma = \alpha$. Since S is a homomorphic image of $FR(S)/M(S)$, $t(x_1,x_2,\ldots,x_n) = 0$ is a polynomial identity for S.

Lemma A-8. If $t(x_1,x_2,\ldots,x_n) = 0$ is a polynomial identity, with integral coefficients, for every Jordan algebra over the rational numbers Q, then it is a polynomial identity for $FR(X)/M(X)$ for any set X.

Proof. Since $h(x,y,z) = 0$ and $d(w,x,y,z) = 0$ are polynomial identities for Jordan algebras over fields

of characteristic not two ([3], p.91), there is a homomorphism $\beta : FR(X)/M(X) \to FJ(X)'$ where $\beta(x) = x$ for $x \in X$ (where $FJ(X)'$ is the free Jordan algebra over Ω generated by X). Let $y \in FR(X)$ such that $\beta(\bar{y}) = 0$. $FR(X)$ is contained in $Q(X)'$, and $FJ(X)' \simeq Q(X)'/M(X)'$ where $M(X)'$ is the subspace generated by $M(X)$. Since $\beta(\bar{y}) = 0$, $y \in M(X)'$ and $y = \sum_{i=1}^{n} r_i z_i$ where $r_i \in \Omega$ and $z_i \in M(X)$. Let m be a common denominator of the r_i's, thus $r_i = n_i/m$ for $m, n_i \in Z$. Therefore $my = \sum_{i=1}^{n} n_i z_i \in M(X)$, and $m\bar{y} = 0$. By lemma A-1, $FR(X)/M(X)$ is isomorphic to a free module over Z, but this implies $\bar{y} = 0$. Thus β is a monomorphism, and any polynomial identity with integral coefficients for $FJ(X)'$ holds in $FR(X)/M(X)$.

Theorem A-9. Let S be any Jordan ring or Jordan algebra over a commutative ring with identity. If $t(x_1, x_2, \ldots x_n) = 0$ is a polynomial identity, with integral coefficients, for every Jordan algebra over Ω, then $2t(x_1, x_2, \ldots, x_n) = 0$ is a polynomial identity for S. If in addition $h(x,y,z) = 0$ and $d(w,x,y,z) = 0$ are polynomial identities for S, then $t(x_1, x_2, \ldots, x_n) = 0$ is a polynomial identity for S.

Proof. Since any Jordan algebra over a commutative ring with identity is a Jordan ring, the theorem follows immediately from lemma A-7 and lemma A-8.

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