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ABSTRACT

SECOND ORDER SEQUENTIAL ESTIMATION OF THE MEAN EXPONENTIAL SURVIVAL TIME UNDER RANDOM CENSORING

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We study in this work a sequential estimator of the mean θ of an exponential distribution when the data is randomly right censored. The loss is measured by the sum of squared error loss of estimation and a linear cost function of the number of observations. Without any further conditions, second order expansions are provided for the expectation of the stopping time and for the risk. Also the asymptotic normality of the stopping time is demonstrated. Sequential interval estimation of θ is also considered.

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CHAPTER 1

INTRODUCTION

In several survival studies pertaining to clinical trials, lifetesting, reliability and epidemiological investigations the estimation of the mean survival time θ is of fundamental importance. This is usually based on the data gathered from a sample of $n(\geq 1)$ units as in a reliability study or lifetest. An analysis of the estimator $\hat{\theta}_n$ constructed would now be necessary before its practical application in a given situation. However, it is often the case that $\hat{\theta}_n$ (with n held fixed) is very hard to analyse, but its salient features become more apparent "in the limit as n tends to infinity". The consideration of large sample sizes is often inappropriate in many longitudinal studies where ethical reasons, high per unit costs and monitoring costs preclude implementation of statistical procedures which require genuinely large sample sizes for their proper utilization. This leads us to consider some sequential or quasi-sequential schemes that may effectively reduce the sample sizes required for efficient estimation of θ. Generally this would engender substantial savings in costs and on-test time with a reduction in the loss of experimental units and without serious loss of sensitivity or efficacy of the statistical investigations.

A common feature of several survival studies is that the lifetimes (or failure times) of the units under

observation may not be completely observable due to the presence of censorship. This is typically the case in a clinical trial in which patients under treatment may be lost to follow up due to withdrawals from study. In some situations competing risks, other than that under study, curtails observation of the duration variable of interest.

Suppose that the true survival time X of a specimen may be detered from complete observation by the action of a censoring variable Y, so that the only datum available to the investigator is (Z,δ), where $Z = \min(X,Y)$ and δ is 1 or 0 according as $X \le Y$ or $X > Y$. The random censorship model assumes that X and Y are independent variables. Suppose X has exponential survival function F, $F(t) = \exp(-t\theta^{-1})$, $t > 0$ and Y has censoring distribution $G(G(\cdot) = P\{Y > \cdot\})$. Both θ and G are unknown and we wish to estimate θ. If $c(>0)$ is the per unit cost of observation we place n items on test and record the data $\{(Z_i,\delta_i): 1 \le i \le n\}$. For an estimator $\hat{\theta}_n$ of θ we measure overall loss incurred by

$$L_n(c) = (\hat{\theta}_n - \theta)^2 + cn$$

and the preliminary objective is to minimize the expected loss, called the risk $R_n(c) = EL_n(c)$, by optimal choice of n. We exhibit this by obtaining the expansion

$$R_n(c) = \theta^2(E\delta)^{-1}\, n^{-1} + cn + O(n^{-2})$$

from which the optimal sample size N_c may be taken as the integer closest to $\theta(E\delta)^{-1/2}c^{-1/2}$. The corresponding optimal minimum risk is then R_{N_c}. Since both θ and G are unknown, N_c and R_{N_c} are not completely specified and therefore we are led naturally to consider an alternative sequential procedure for estimating θ. We propose such a scheme with a stopping rule $T(=T_c)$ and consequent estimation of θ by $\hat{\theta}_T$. The performance of the sequential procedure is described by comparing its risk $R_T = E(L_T(c))$, with that of the optimal "fixed sample scheme" risk R_{N_c}.

We say that the procedure $(T, \hat{\theta}_T)$ is <u>asymptotically risk efficient</u> if $R_T/R_{N_c} \to 1$ as $c \to 0$. Since $R_{N_c} = 2cN_c + O(c^{-1/2})$, we will have that $R_T - R_{N_c} = o(c^{-1/2})$.

The central thesis of this research is a careful analysis of the regret function $R(\theta, G, c) = R_T - R_{N_c}$, which may be viewed as the additional risk incurred in using the sequential scheme given by T_c over the fixed sample scheme N_c. We shall obtain the expansion

$$R(\theta, G, c) = Bc + o(c)$$

where the constant B will be explicitly computed. This also shows that the procedure $(T, \hat{\theta}_T)$ has <u>bounded regret</u> i.e. $R(\theta, G, c) = O(c)$. Additionally we obtain an expanion for the expected sample size ET_c and the asymptotic distribution of an appropriately normalized version of T_c.

Gardiner and Susarla (1984) were first to consider the above problem. They demonstrated the asymptotic risk efficiency. Hence the present work is a second order extension of their work.

The study of sequential point estimation of the exponential mean in the absence of censoring is taken up in Starr and Woodroofe (1972) and Woodroofe (1977). The stopping time $T_c = \inf\{n \geq m: n > \bar{X}_n c^{-1/2}\}$ is considered. Woodroofe (1977) obtains second order expansions for ET_c and the regret under the condition that $m \geq 3$.

To place our results in proper perspective, we present brief review of literature on sequential point estimation. Sequential procedures analogous to the one outlined here have been considered in the absence of censorship by several researchers beginning with the pioneering work of Robbins (1959) for the estimation of the mean of the normal population.

Let X_n, $n \geq 1$ be independent, identically distributed normal random variables with mean μ and standard deviation σ, both unknown. Consider the loss function $L_n(c) = (\bar{X}_n - \mu)^2 + cn$, for estimation of μ. The risk $R_n(c) = EL_n(c) = \sigma^2 n^{-1} + cn$. The integer nearest to $\sigma c^{-1/2}$ say N_c, minimizes the above risk. Since σ is unknown N_c is also unknown. Robbins suggested $T_c = \inf\ \{n \geq m: n > c^{-1/2}\hat{\sigma}_n\}$ as an alternative for N_c and conjectured that $R(\mu, c) = R_{N_c} - R_{T_c}$ is $O(c)$ where

$$R_{T_c} = E(\bar{X}_{T_c} - \mu)^2 + cET_c \quad \text{and} \quad R_{N_c} = \sigma^2 N_c^{-1} + cN_c.$$

Starr (1966) proved that the above procedure is asymptotically risk efficient if and only if $m \geq 3$. Later Starr and Woodroofe (1969) showed that $R(\mu, c)$ is $O(c)$ under the same condition. Woodroofe (1977) gave the second order expansions for $R(\mu, c)$ and ET_c. He showed that $R(\mu, c) = (1/2)c + O(c)$ if $m \geq 4$. This paper is a landmark in the theory of sequential estimation in the sense that it developed and applied entirely new techniques--those of nonlinear renewal theory to obtain the necessary second order expansions. A formulation and a proof of a general nonlinear renewal theorem was given by Lai and Siegmund (1977, 1979).

The above discussion strongly indicates the good performance of the sequential procedure for the normal case. But is this procedure good in general? Let $P(X_1 = 1) = \mu = 1 - P(X_1 = 0)$, $0 < \mu < 1$. Then for $m \geq 2$

$$\begin{aligned} R_{T_c}(c) &= E(\bar{X}_{T_c} - \mu)^2 + cET_c \\ &\geq \int_{\{X_1 = 1, \ldots, X_m = 1\}} (\bar{X}_m - \mu)^2 \, dP \\ &= (1 - \mu)^2 \mu^m > 0, \end{aligned}$$

and $R_{N_c} \approx 2(c\mu(1-\mu))^{1/2} \to 0$ as $c \to 0$. Hence $\lim_{c \to 0} \{R_{N_c}(c)/R_{T_c}(c)\} = 0$ and T is not asymptotically risk efficient. To remedy this situation, Chow and Robbins (1965) suggested that the initial sample size should go to infinity at an appropriate rate as $c \to 0$. Ghosh and

Mukhopadhyay (1979) exploited this fact and proved the asymptotic risk efficiency of T_c in the estimation of the mean (modified in view of the above fact) without the normality assumption, in the general nonparametric context, under the condition that the eighth moment is finite. Sen and Ghosh (1981) consider sequential point estimation of estimable parameters based on U-statistics under the condition that $E|g|^{2+\delta} < \infty$ for some $\delta > 0$, where g is the symmetric kernel corresponding to the parameter of interest. Estimation of the mean is a particular case with g as the identity function. Note the drastic reduction in the moment condition from 8 to $2 + \delta$, $\delta > 0$. Chow and Yu (1981) proved asymptotic risk efficiency for the mean problem independently of the above two references under the condition that $EX_1^{2+\delta} < \infty$ for some $\delta > 0$. Their result is a special case of the result of Sen and Ghosh (1981). Sequential point estimation of locaton based on some R-, L-, and M-estimators is discussed in Sen (1980). Sen's book (1981) has an excellent survey of the above mentioned article.

None of these results in the nonparametric context go beyond asymptotic risk efficiency. Chow and Martinsek (1982) were first to show that $R(\mu,c)$ is $O(c)$ for the mean problem under the assumption that $E\ X_1^{6+\delta} < \infty$ for some $\delta > 0$. Martinsek (1983) obtains second order expansions for $R(\mu,c)$ in the nonlattice case and bounds in the lattice case, under the condition that $E X_1^{8+\delta} < \infty$ for some $\delta > 0$. In the nonlattice case,

$$R(\mu,c) = (2 - (3/4)E(Z_1^2 - 1)^2 + 2E^2 Z_1^3)\ c + o(c),$$

where $Z_1 = (X_1 - \mu)\sigma^{-1}$. Thus if X_1 is symmetric $R(\mu,c) \leq 2\ c + O(c)$. That is, in the limit, one loses at most the cost of two observations when using the stopping rule T_c instead of N_c.

By way of contrast, it also follows that the regret can take arbitrarily large negative values as the distribution of the X_i's varies, even among symmetric distributions. To illustrate this, let $X_1, X_2 \ldots$ be i.i.d. with probability density function f,

$$f(x) = 2|x|^{-5}\ [|x|\geq 1]$$

where [A] denotes the indicator of set A. For $M > 1$ define

$$X_{iM} = X_i[|X_i| \leq M].$$

Then for each M, $X_{1M}, X_{2M} \ldots$ are i.i.d. and their common distribution is symmetric around zero. Thus $R(\mu,c) = (2 - (3/4)\log(M)/(1-M^{-2})^2 + 3/4)\ c + o(c)$. Clearly, as M tends to ∞, the coefficient of c in the above expression approaches $-\infty$. The above example is due to Martinsek (1983) and it provides an answer to the question raised by Starr and Woodroofe (1972) and discussed further by Woodroofe (1977), as to whether the coefficient in the regret expansion can ever take negative values. Although Woodroofe (1977) got positive values in the gamma and normal cases, in general it need not be positive, and in fact for distributions with large fourth moments (as in the example above) arbitrarily large negative values can be achieved.

In light of Martinsek (1983) there is a renewed hope that second order efficiency could possibly be established in other nonparametric problems reviewed above. The present work is one such example.

In Chapter 2, we develop the necessary prerequisites of nonlinear renewal theory and moments of randomly stopped sums. Most of the results are taken from Chapter 4 of Woodroofe's monograph (1982) and Chow, Robbins and Teicher (1965). Hence proofs have been omitted.

Chapter 3 is divided in to many sections. First three develop our model. In section 5 the main theorems are stated. Theorem 1 gives the second order expansion for ET_c. Theorem 2 asserts the asymptotic normality of T_c and Theorem 3 gives the second order expansion for the R_T. Proofs of these theorems are based on several lemmas. Some of them, which are of independent interest are stated and proved in section 4. Section 5 gives the proofs of the main theorems.

In Chapter 4, a related but a different problem of asymptotic fixed width sequential interval estimation for θ, is developed. Second order expansion for the stopping time involved in achieved as a bonus from techniques developed in Chapter 3.

CHAPTER 2

PRELIMINARIES

Most of the results in this chapter are taken from Chapter 4 of Woodroofe's monograph (1982). Hence proofs have been omitted.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{F}_n$, $n \geq 1$ be an increasing sequence of sub-sigma-algebrae of $\mathcal{F}$.

Definition 2.1. A random variable t is said to be a proper stopping time (with respect to $\mathcal{F}_n$, $n \geq 1$) if and only if t is positive integer valued and $\{t=n\} \in \mathcal{F}_n$ for all $n \geq 1$.

Definition 2.2. The random variables X_n, $n \geq 1$ are said to be independently adapted to $\mathcal{F}_n$, $n \geq 1$ if and only if X_n is $\mathcal{F}_n$ measurable and $\mathcal{F}_n$ is independent of the sequence X_k $k > n$, for every $n \geq 1$.

Theorem 2.1. (Wald's lemma) Let X_n, $n \geq 1$ be i.i.d. random variables which are independently adapted to increasing sigma-algebras $\mathcal{F}_n$, $n \geq 1$, let $S_n = X_1 + X_2 + \dots X_n$, $n \geq 1$ and let t be a proper stopping time for which $E\, t < \infty$. If X_1 has a finite mean μ, then

$$E\, S_t = \mu\, E\, t;$$

and furthermore

$$E\, (S_t - t\mu)^2 = \delta^2\, E\, t,$$

if X_1 has a finite variance δ^2.

Definition 2.3. (u.c.i.p.) A sequence Y_n, $n \geq 1$, of random variables is said to be uniform continuous in probability if and only if for every $\epsilon > 0$ there is a $\delta > 0$ for which

$$P\{\max_{0 \leq k \leq n\delta} |Y_{n+k} - Y_n| \geq \epsilon\} < \epsilon \quad \text{for all} \quad n \geq 1.$$

Remark 2.1. If Y_n, $n \geq 1$ converges to a finite limit with probability 1 as $n \to \infty$, then it is u.c.i.p..

Definition 2.4. A sequence Y_n, $n \geq 1$ of random variables are said to be stochastically bounded if and only if for every $\epsilon > 0$ there is a $c > 0$ for which

$$P\{|Y_n| > c\} < \epsilon \quad \text{for all} \quad n \geq 1.$$

In particular, if Y_n converges in distribution, then Y_n, $n \geq 1$, are stochastically bounded.

Example 2.1. Normalized partial sums. If $X_1, X_2, \ldots$ are i.i.d. with finite mean μ and finite positive variance δ^2, then $Y_n = \delta^{-1} n^{1/2}(S_n - n\mu)$ $n \geq 1$, is u.c.i.p..

Lemma 2.1. If Y_n, $n \geq 1$ and Z_n, $n \geq 1$ are u.c.i.p., then so is $Y_n + Z_n$, $n \geq 1$. If in addition Y_n, $n \geq 1$, and Z_n, $n \geq 1$, are stochastically bounded, and if ϕ is any continuous function on R^2, then $\phi(Y_n, Z_n)$, $n \geq 1$, is u.c.i.p..

Theorem 2.2. (Anscombe's theorem). Suppose that Y_1, $Y_2, \ldots$ are u.c.i.p.; let t_a, $a > 0$, be integer valued random variables for which $a^{-1} t_a$ converges to a finite positive constant c in probability and let $N_a = [ac]$, $a > 0$. Then

$$Y_{t_a} - Y_{N_a} \to 0 \text{ in probability as } a \to \infty .$$

If in addition, Y_n converges in distribution to a random variable Y, then Y_{t_a} converges in distribution to Y as $a \to \infty$.

We need vonBahr's (1965) extension of the central limit which asserts:

<u>Theorem 2.3</u>. Let X_i, $i \geq n$ be i.i.d. with finite mean μ, positive variance δ^2, and $E|X_1|^\alpha < \infty$ where $\alpha \geq 2$, then

$$E|\delta^{-1} n^{-1/2}(S_n - n\mu)|^\alpha \to \frac{2^{\alpha/2}}{\sqrt{\Pi}} \Gamma\ (1/2 + \alpha/2) .$$

The convergence of moments in Anscombe's theorem is examined next. The most general theorem available is by Chow and Yu (1981) which is as follows.

<u>Theorem 2.4</u>. Let $Y_1, Y_2, \ldots$ be independent random variables with $E\,Y_n = 0$ for all $n \geq 1$. Assume that for some $p \geq 2$, $\{|Y_n|^p, n \geq 1\}$ is uniformly integrable. Let $\mathcal{F}_n$ be a σ-algebra generated by $\{Y_1, Y_2, \ldots Y_n\}$ for each $n \geq 1$, $\mathcal{F}_0 = \{\phi, \Omega\}$, and let $\{M(b), b \epsilon B\}$ be proper $\mathcal{F}_n$-stopping times with $B \subset (0,\infty)$ such that $\{(b^{-1}M(b))^{p/2}, b \epsilon B\}$ is uniformly integrable. Let $W_n = \sum_{i=1}^{n} Y_i$. Then

$$\{|b^{-1/2} W_{M(b)}|^p, b \epsilon B\}$$

is uniformly integrable.

Following result is a part of Theorem 7 of Chow, Robbins, and Teicher (1965).

<u>Theorem 2.5</u>. If $X_1, X_2, \ldots$ are independent with $E X_n = 0$, $E X_n^4 < \infty$ and t is a proper $\mathcal{F}_n$-stopping time with $E t^2 < \infty$, then $E S_t^4 < \infty$ where $S_n = X_1 + X_2, \ldots X_n$, $n \geq 1$ and $\mathcal{F}_n = \sigma$-algebra generated by $\{X_1, X_2, \ldots, X_n\}$. The rest of the chapter is a review of linear and nonlinear renewal theory.

Let $S_n = X_1 + X_2, \ldots, X_n$, $n \geq 1$, be a random walk and for $a \geq 0$, let

$$\tau_a = \inf\{n \geq 1: S_n > a\}$$

be the time at which the random walk first reaches the height a, or ∞ if no such time exists. Next, define R_a on $\{\tau_a < \infty\}$ by $R_a = S_{\tau_a} - a$. Thus R_a is the excess of the random walk over the boundary a at the time which it first crosses a.

If $\mu = E(X_1) > 0$, then $S_n \to \infty$ with probability 1 by the strong law of large numbers so $\tau_a < \infty$ for all $a \geq 0$ with probability 1. It can be shown that $\tau_a < \infty$ for all $a \geq 0$ with prob. 1 if $\mu = 0$, too. It can be verified that τ_a is a proper stopping time if $\mu \geq 0$ and $\mathcal{F}_n = \sigma$-algebra generated by $\{X_1, X_2, \ldots, X_n\}$.

The following is a corollary of the classical renewal theorem.

<u>Theorem 2.6</u>. Suppose that $0 < \mu < \infty$. If F is nonarithmetic, then R_a has a limiting distribution H as $a \to \infty$, where

$$H(dr) = \frac{1}{E(S_\tau)} P(S_\tau > r)\, dr \qquad r \geq 0,$$

and $\tau = \inf\{n: S_n > 0\}$.

Theorem 2.7. If X_1 has a finite variance δ^2, then the mean of H is

$$\rho = \frac{\mu^2+\delta^2}{2\mu} - \sum_{k=1}^{\infty} k^{-1} E(S_k^-)$$

where $^-$ denotes the negative part.

We have a following important corollary of Theorem 2.6 and 2.7.

Corollary 2.1. Suppose $\mu > 0$, that $E\{\mathrm{Max}(0,X_1)^2\}<\infty$, and that F is nonarithmetic. Then

$$E(R_a) \to \rho$$

and

$$E(\tau_a) = \mu^{-1}(a+\rho) + o(1) \qquad \text{as} \qquad a \to \infty .$$

To study the counterparts of Theorem 2.6, 2.7 and Corollary 2.1 in the nonlinear case, we have the following set up. Let $X_1, X_2, \ldots$ denote i.i.d. random variables on $(\Omega,\mathcal{F},P)$ with finite, positive mean μ; and $S_n = X_1 + X_2,\ldots,X_n$, $n \geq 1$. In addition ξ_n, $n \geq 1$, denote random variables for which $(X_1,\xi_1),\ldots,(X_n,\xi_n)$ are independent of X_k, $k > n$, for every $n \geq 1$. The objective is to extend aspects of renewal theory to

$$Z_n = S_n + \xi_n, \qquad n \geq 1, \text{ under smoothness}$$

conditions on ξ_n, $n \geq 1$. Define $Z_0 = 0$, $\mathcal{F}_0 = \{\phi,\Omega\}$ and $\mathcal{F}_n = \sigma\{(X_k,\xi_k): k \leq n\}$, $n \geq 1$. Thus X_n, $n \geq 1$, are independently adapted to $\mathcal{F}_n$, $n \geq 1$. Next, let

$$\tau_a = \inf\{n \geq 1: S_n > a\},$$

$$t_a = \inf \{n \geq 1: Z_n > a\},$$

and

$$R_a = Z_{t_a} - a \qquad a \geq 0.$$

These notations and assumptions are used throughout the chapter.

Definition 2.5. The process ξ_n, $n \geq 1$, is said to be slowly changing if and only if:

(i) $\frac{1}{n} \text{Max} \{|\xi_1|, |\xi_2| \ldots |\xi_n|\} \to 0$ in probability as $n \to \infty$ and

(ii) ξ_n, $n \geq 1$, is u.c.i.p.

Remark 2.2. Observe that (i) holds if $\xi_n/n \to 0$ with probability 1 as $n \to \infty$.

Remark 2.3. If ξ_n^1, $n \geq 1$, and ξ_n^{11}, $n \geq 1$ are two slowly changing sequences, then $\xi_n = \xi_n^1 + \xi_n^{11}$ $n \geq 1$, defines another slowly changing sequence.

Example 2.2. Let $Y_1, Y_2 \ldots$ be i.i.d. with a finite mean υ and a finite, positive variance, then $\xi_n = n(\bar{Y}_n - \upsilon)^2$, $n \geq 1$, is slowly changing.

Lemma 2.2. If (i) holds and $N = N_a$ = the greatest integer in $a\mu^{-1}$, $a \geq 0$, then $t_a < \infty$ for all $a \geq 0$ with probability 1 and $t_a N_a^{-1} \to 1$ in probability as $a \to \infty$. In particular, $\xi_n n^{-1} \to 0$ with probability one, implies $t_a N_a^{-1} \to 1$ with probability 1 as $a \to \infty$.

Theorem 2.8 and Theorem 2.9 are generalizations of Theorem 2.7 and Corollary 2.1 in the nonlinear context.

<u>Theorem 2.8</u>. Suppose that X_1 is nonarithmetic and that ξ_n, $n \geq 1$ are slowly changing. Then $R_a = Z_{t_a} - a$ has a limiting distribution H, as $a \to \infty$, where

$$H(dr) = \frac{1}{E(S_\tau)} P\{S_\tau > r\}\, dr, \qquad r > 0,$$

and $\tau = \inf\{n: S_n > 0\}$. That is R_a has the same limiting distribution as $S_{\tau_a} - a$.

<u>Theorem 2.9</u>. Let A_n, $n \geq 1$ be F_n-measurable sets, and V_n, $n \geq 1$ be F_n-measurable random variables for which following conditions hold.

(1) $\sum_{n=1}^{\infty} P\left(\bigcup_{k=n}^{\infty} A_k'\right) < \infty$,

(2) $\xi_n = V_n$ on A_n, $n \geq 1$,

(3) $\{\max_{0 \leq k \leq n} |V_{n+k}|,\ n \geq 1\}$ are uniformly integrable,

(4) $\sum_{n=1}^{\infty} P\{V_n \leq n\epsilon\} < \infty$ for some ϵ, $0 < \epsilon < \mu$,

(5) $E(V_n) \to E(V)$ where V is some random variable,

(6) $P\{t_a \leq \epsilon N_a\} = o(N_a^{-1})$ as $a \to \infty$, $\epsilon > 0$,

where N_a = largest integer in $a\mu^{-1}$.

In addition, suppose X_1 has finite, positive variance δ^2, and that V_n, $n \geq 1$ are slowly changing and F is nonarthimetic, then

$$E(t_a) = \mu^{-1}(a+\rho-E(V)) + o(1) \quad \text{as } a \to \infty, \text{ where}$$

$$\rho = E(S_\tau^2)/2E(S_\tau) = \frac{\mu^2+\delta^2}{2\mu} - \sum_{k=1}^{\infty} k^{-1} E(S_k^-).$$

Theorem 2.8 and a variant of Theorem 2.9 were first proved by Lai and Siegmund (1977, 1979). Hagwood and Woodroofe (1982) simplified the second theorem. Theorem 2.9

as stated here is a slight modification of Theorem 4.5, Woodroofe (1982), the proof being essentially the same.

CHAPTER 3

SEQUENTIAL POINT ESTIMATION

1. The Model

Let X and Y be nonnegative independent random variables with survival functions F and G respectively i.e. $F(t) = P(X>t)$ and $G(t) = P(Y>t)$ for all $t \geq 0$. We assume that X is exponential with mean θ and $G(0) \neq 0$. Consider $Z = \min(X,Y)$ and $\delta = 1$ whenever $X \leq Y$ and 0 otherwise.

Suppose $\{(Z_i,\delta_i): 1 \leq i \leq n\}$ is a random sample of size n. We wish to estimate θ in presence of the nuisance parameter G. Consider the sequence of estimators $\hat{\theta}_n$, $n \geq 1$, of θ given by

$$\hat{\theta}_n = \bar{Z}_n \bar{\delta}_n^{-1} [\bar{\delta}_n \neq 0] \tag{3.1}$$

where the overscore denotes the corresponding sample mean and [A] denotes the indicator of Set A.

The loss incurred in estimation of θ by $\hat{\theta}_n$ is

$$L_n(c) = (\hat{\theta}_n - \theta)^2 + cn, \tag{3.2}$$

where c is the cost per observation.

2. Some Preliminary Formulae and Results

$$E\,\delta = P(X \leq Y) = \theta^{-1} \int_0^\infty e^{-x/\theta} G(x)dx = b.$$

$$E\,(Z) = \int_0^\infty P\,(Z>z)\,dz = \int_0^\infty F\,(z))G(z)dz$$

$$= \int_0^\infty e^{-x/\theta} G(x)dx\ .$$

Thus, $E(Z-\theta\delta) = 0$.

$\text{Var}\,(Z - \theta\delta) = E\,(Z - \theta\delta)^2 = \theta^2\, E\,\delta$.

The covariance matrix Σ for the vector (Z, δ) works out to be

$$\Sigma = \begin{bmatrix} 2\int_0^\infty x\, e^{-x/\theta}\, G(x)dx - \theta^2 b^2 & \theta^{-1}\int_0^\infty x e^{-x\theta} G(x)dx - \theta b^2 \\ \theta^{-1}\int_0^\infty x\, e^{-x/\theta}\, G(x)dx - \theta b^2 & b(1-b) \end{bmatrix}$$

Denote by (e_1, e_2) be a normal vector with mean 0 and covariance matrix Σ.

Observe that by the strong law of large numbers $\hat{\theta}_n$, $n \geq 1$ is a strongly consistent estimator of θ and by the central limit theorem we have,

$\sqrt{n}\,(\hat{\theta}_n - \theta)$ converges in distribution to normal random variable with mean 0 and variance $\sigma^2 = \theta^2\, b^{-1}$.

<u>Remark 3.1</u>. Since $P(\bar{\delta}_n = 0) = b^n \to 0$ at an exponential rate as $n \to \infty$ and all our scale factors will be algebric powers of n, we shall suppress terms involving $[\bar{\delta}_n = 0]$.

<u>Lemma 3.1.</u> For any $k \geq 1$ $E \sup_n \bar{\delta}_n^{-k} [\bar{\delta}_n \neq 0]$

$$\leq (2b^{-1})^k + \sum_{n=1}^{\infty} n^k\, P(\bar{\delta}_n \leq b/2) < \infty .$$

<u>Proof</u>. $E \sup_n \bar{\delta}_n^{-k} [\bar{\delta}_n \neq 0] = E \sup_n \bar{\delta}_n^{-k} [\bar{\delta}_n > b/2]$

$$+ E \sup_n \bar{\delta}_n^{-k} [n^{-1} \leq \bar{\delta}_n \leq b/2]$$

$$\leq (2b^{-1})^k + \sum_{n=1}^{\infty} n^k\, P(\bar{\delta}_n \leq b/2) < \infty .$$

<u>Lemma 3.2.</u> $E\,(\hat{\theta}_n - \theta)^2 = \sigma^2 n^{-1}$

$$+ n^{-2}\,\{-2\, b^{-3}\; E(\delta_1 - b)(Z_1 - \theta\delta_1)^2$$

$$+ 3b^{-4}\, Ee_2^2\, (e_1 - \theta e_2)^2\} + o(n^{-2}) .$$

Proof. By Taylor's theorem in two variables, we have

$$\hat{\theta}_n = \theta + b^{-1}(\bar{Z}_n - \theta\,\bar{\delta}_n) - b^{-2}(\bar{\delta}_n - b)(\bar{Z}_n - \theta b)$$
$$+ \theta b^{-2}(\bar{\delta}_n - b)^2 + \{\lambda_2^{-3}(\bar{\delta}_n - b)^2(\bar{Z}_n - \theta b)$$
$$- \lambda_1\,\lambda_2^{-4}(\bar{\delta}_n - b)^3\}$$

where λ_1 lies between $\bar{Z}_n$ and θb and λ_2 lies between $\bar{\delta}_n$ and b.

Thus $E\,(\hat{\theta}_n - \theta)^2$

$$= E\,b^{-2}(\bar{Z}_n - \theta\bar{\delta}_n)^2 + E\,\{b^{-2}(\bar{\delta}_n - b)(\bar{Z}_n - \theta b)$$
$$- \theta b^{-2}(\bar{\delta}_n - b)^2 - \lambda_2^{-3}(\bar{\delta}_n - b)^2(\bar{Z}_n - \theta b)$$
$$- \lambda_1\,\lambda_2^{-4}(\bar{\delta}_n - b)^3\}^2$$
$$- 2E\{b^{-3}(\bar{\delta}_n - b)(\bar{Z}_n - \theta\bar{\delta}_n)(\bar{Z}_n - \theta b)$$
$$- \theta b^{-3}(\bar{Z}_n - \theta\bar{\delta}_n)(\bar{\delta}_n - b)^2\}$$
$$+ 2\,Eb^{-1}(\bar{Z}_n - \theta\,\bar{\delta}_n)\,\{\lambda_2^{-3}(\bar{\delta}_n - b)^2(\bar{Z}_n - \theta b)$$
$$- \lambda_1\lambda_2^{-4}(\bar{\delta}_n - b)^3\}$$
$$= I_n + II_n + III_n + IV_n.$$

It can be easily checked that $I_n = \sigma^2 n^{-1}$.

$$III_n = -2\,b^{-3}\,E\,(\bar{\delta}_n - b)\,(\bar{Z}_n - \theta\bar{\delta}_n)^2$$
$$= -2b^{-3}\,n^{-3}\sum_{i=1}^{n} E\,(\delta_i - b)\,(Z_i - \theta\delta_i)^2$$
$$= -2b^{-3}\,n^{-2}\,E\,(\delta_1 - b)\,(Z_1 - \theta\delta_1)^2\,.$$

Now we shall consider IV_n.

Let $f(x,y) = 2b^{-4}y^2(x-\theta y)^2$ for any x,y real numbers. Let P_n denote the random variable $f(n^{1/2}(\bar{Z}_n-\theta b),\, n^{1/2}(\bar{\delta}_n-b))$

and $Q_n = 2n^2 b^{-1} (\bar{Z}_n - \theta\, \bar{\delta}_n)\{\lambda_2^{-3} (\bar{\delta}_n - b)^2 (\bar{Z}_n - \theta b)$
$- \lambda_1 \lambda_2^{-4} (\bar{\delta}_n - b)^3\} - P_n$.

Thus $n^2 IV_n = E\, P_n + E\, Q_n$.

By central limit theorem, P_n converges in distribution to $f(e_1, e_2)$ and Q_n converges to zero almost surely. Thus $2n^2 b^{-1} (\bar{Z}_n - \theta\bar{\delta}_n)\ \{\lambda_2^{-3} (\bar{\delta}_n - b)^2 (\bar{Z}_n - \theta b) - \lambda_1 \lambda_2^{-4} (\bar{\delta}_n - b)^3\}$ converges in distribution to $f(e_1, e_2)$.

Now to conclude that $n^2 IV$ converges to $Ef(e_1, e_2)$, we need to verify uniform integrability of $\{P_n + Q_n,\ n \geq 1\}$, which follows from the following facts. Since $0 < \lambda_2^{-p} < b^{-p} + \bar{\delta}_n^{-p}\ [\bar{\delta}_n \neq 0]$ by previous lemma, λ_2^{-p} is uniformly integrable for every $p > o$. Similarly λ_1^p is uniformly integrable for every $p > o$. Also $(n^{1/2} (\bar{Z}_n - \theta b))^p$ and $(n^{1/2} (\bar{\delta}_n - b))^p$ are uniformly integrable for every $p > o$. Similar computations for II_n gives the lemma.

3. Sequential Procedure.

Using (3.2) and lemma (3.1) we have,

$$R_n(c) = E\,(L_n(c)) = \sigma^2 n^{-1} + cn + O(n^{-2})\ .$$

For large n, N_c = nearest integer to $(c^{-1/2}\,\sigma)$ which minimizes the risk. Since σ is unknown, N_c, the optimal sample size is unknown and thus one is naturally led to explore a sequential scheme to estimate θ. Define a stopping rule

$$T_c = \min \{n \geq n_{1c}: \quad n > c^{-1/2} \hat{\sigma}_n\} \tag{3.3}$$

where $n_{1c} = c^{\frac{1}{-2(1+\alpha)}}$, $\alpha > o$.

$$\hat{\sigma}_n = \bar{Z}_n \bar{\delta}_n^{-3/2} [\bar{\delta}_n \neq 0] + [\bar{\delta}_n = 0] .$$

Note that $\hat{\sigma}_n$, $n \geq 1$, is strongly consistent for σ .

4. Lemmas

Let $0 < \epsilon < 1$ be fixed. Let n_{2c} and n_{3c} be the integer parts of $N_c(1-\epsilon)$ and $N_c(1+\epsilon)$ respectively. We may write T_c and N_c without the subscript c in the sequel. Also we shall freely write $c^{-1/2} \sigma$ for N. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{(Z_1,\delta_1),$ $(Z_2,\delta_2),\ldots,(Z_n,\delta_n)\}$.

Remark 3.2. The terms involving the random variables $[\bar{\delta}_N = 0]$, $[\bar{\delta}_T = 0]$ are left out without any further indication since $P(\bar{\delta}_N = 0)$ and $P(\bar{\delta}_T = 0)$ go to zero at an exponential rate as $c \to 0$ and all our scale factors will be algebric powers of c. $(P(\bar{\delta}_T = 0) = \sum_{n=m_{1c}}^{\infty} P(\bar{\delta}_n = 0,$ $T = n) \leq \sum_{n=n_{1c}}^{\infty} P(\bar{\delta}_n = 0) = b^{n_{1c}}(1-b)^{-1}.)$

Lemma 3.3. $\|\sup_{n\geq m} (\hat{\sigma}_n - \sigma)\|_p = O(m^{-1/2})$ for all $p > 0$ and $m \geq 1$.

Proof. $|\hat{\sigma}_n - \sigma|$

$$\leq \bar{\delta}_n^{-3/2}([\bar{\delta}_n \neq 0] \ (|\bar{Z}_n - \theta b| + \theta b^{-1/2}|\bar{\delta}_n^{3/2} - b^{3/2}|)$$

$$= \bar{\delta}_n^{-3/2}\, [\bar{\delta}_n \neq 0]\, |\bar{Z}_n - \theta b|$$
$$+\ \theta b^{-1/2}\, \bar{\delta}_n^{-3/2}\, [\bar{\delta}_n \neq 0]\, (\bar{\delta}_n^{1/2} + b^{1/2})^{-1}\, (\bar{\delta}_n$$
$$+\ \bar{\delta}_n^{1/2}\, b^{1/2} + b)\ |\bar{\delta}_n - b|$$
$$\leq \bar{\delta}_n^{-3/2}\, [\bar{\delta}_n \neq 0]\, |\bar{Z}_n - \theta b|$$
$$+\ (3\theta/2)(\bar{\delta}_n \wedge b)^{-5/2}\, [\bar{\delta}_n \neq 0]\ |\bar{\delta}_n - b| \qquad (3.3)$$
$$\leq \bar{\delta}_n^{-3/2}\, [\bar{\delta}_n \neq 0]\, |\bar{Z}_n - \theta b|$$
$$+\ (3\theta/2)\ (\bar{\delta}_n^{-5/2} + b^{-5/2})\ [\bar{\delta}_n \neq 0]\, |\bar{\delta}_n - b| \qquad (3.4)$$

The Schwarz inequality, lemma 3.1, and the maximal inequality for reverse martingales give the lemma.

Lemma 3.4. $P(T \leq n_{2c}) = O(c^p)$ for all $p > 0$

and

$$P(T \geq n_{2c}) = O(c^p) \quad \text{for all} \quad p > 0\ .$$

Proof. $P(T \leq n_{2c}) \leq P(\max_{n_{1c} \leq n \leq n_{2c}} |\hat{\sigma}_n - \sigma| > \epsilon\, \sigma)$

$$\leq P(\max_{n_{1c} \leq n} |\bar{Z}_n - \theta b| > \eta_1) + P(\max_{n_{1c} \leq n} |\bar{\delta}_n - b| > \eta_2) \qquad (3.5)$$

for some $\eta_1, \eta_2 > 0$. The above inequality is obtained by using (3.3) and a truncation argument similar to the proof of lemma 3.1. With the reverse martingale inequality, $n_{1c} = c^{\frac{1}{2(1+\alpha)}}$ and (3.5) imply the lemma.

Corollary 3.1. For all $p > 0$, $\{T_c N_c^{-1}\}^{-p}$: $0 < c < 1\}$ is uniformly integrable.

Proof. Let $k = (1-\epsilon)^{-p}$

$$\int [(T/N)^{-p} > k]\ (TN^{-1})^{-p} dP$$

$$\leq N^p \int [T/N > 1-\epsilon]\, dP = N^p P(T < N(1-\epsilon))$$

$$\leq o(1) \quad \text{as} \quad c \to o, \quad \text{by Lemma 3.4.}$$

Lemma 3.5. $\{(TN^{-1})^p : 0 < c < 1\}$ is uniformly integrable for all $p > 0$.

Proof. $TN^{-1} \leq 1 + \sigma^{-1} (\hat{\sigma}_{T-1} - \sigma) + c^{1/2} \sigma +$

$[T = n_{1c}]\, n_{1c}$

Lemma 3.1 and 3.3 imply the desired uniform integrability.

Lemma 3.6. For all $p > 0$, $\{(N^{-1/2}(T - N))^p$, $0 < c < 1\}$ is uniformly integrable.

Proof. By definition of T, we have

$$c^{-1/2}\, \hat{\sigma}_T < T \leq c^{-1/2}\, \hat{\sigma}_{T-1}\, P(T > n_{1c}) + n_{1c}\, P(T = n_{1c}) + 1.$$

Hence

$$|N^{-1/2}(T - N)|^p$$
$$\leq \operatorname{Max}\{|c^{-1/4}\sigma^{1/2}(\hat{\sigma}_T - \sigma)|^p, |c^{-1/4}\sigma^{1/2}(\hat{\sigma}_{T-1} - \sigma)|^p$$
$$+ \{c^{1/4} n_{1c}\, P(T = n_{1c})\}^p + 1\}. \tag{3.6}$$

Also

$$(c^{-1/4}|\hat{\sigma}_T - \sigma|)^p \leq k_p(c^{-p/4} a_T^p |\bar{Z}_T - \theta b|^p + c^{-p/4} b_T^p |\bar{\delta}_T - b|^p)$$

where $a_n = [\bar{\delta}_n \neq 0]\, \bar{\delta}^{-3/2}$, and

$$b_n = \theta(\bar{\delta}_n + \bar{\delta}_n^{1/2}\, b^{1/2} + b)(\bar{\delta}_n^{1/2} + b^{1/2})\, \bar{\delta}_n^{-3/2}\, b^{1/2}\, [\bar{\delta}_n \neq 0].$$

Thus by the Schwarz inequality,

$$E(c^{-1/4}|\hat{\sigma}_T - \sigma|)^p \leq k_p\, \{E^{1/2} a_T^{2p}\, E^{1/2}(c^{-1/4}|\bar{Z}_T - \theta b|)^{2p}$$
$$+ E^{1/2} b_T^{2p}\, E^{1/2}\, (\bar{\delta}_T - b)c^{-1/4})^{2p}\}.$$

$E(c^{-1/4}|\bar{Z}_T - \theta b|)^{2p} = O(1)$ by Lemmas 3.5 and 2.4 and Corollary 3.1

$E\, a_T^{2p} = O(1)$ by Lemma 3.1.

The other term is treated similarly to obtain uniform integrability of $\{(c^{-1/4}|\hat{\sigma}_T - \sigma|)^p : \quad 0 < c < 1 \}$. (3.7)
Furthermore by Lemma 3.4. $(c^{1/4} \, n_{1c}P(T = n_{1c}))^p = o(1)$. (3.8)
Hence by [3.6], [3.7], [3.8] to prove the lemma, we only need to show

$$E|c^{-1/4}\sigma^{1/2}(\sigma_{T-1}-\sigma)|^p = O(1) \quad \text{for all} \quad p > 0 \qquad (3.9)$$

Observe

$$(c^{-1/4}|\hat{\sigma}_{T-1} - \sigma|)^p \le \tilde{k}_p(c^{-p/4} \, a^p_{T-1}|\bar{Z}_{T-1} - \theta b|^p$$
$$+ \, c^{-p/4}b^p_{T-1}|\bar{\delta}_{T-1}-b|^p) \text{ and}$$
$$E \, c^{-p/4}|\bar{Z}_{T-1}-\theta b|^p \le \hat{k}_p(c^{-p/4} \, E|\bar{Z}_T - \theta b|^p +$$
$$E \, T^{-p}(Z_T - \theta b)^p \, c^{-p/4}) \qquad (3.10)$$

By Theorem 2.4 the first term on the right side off (3.10) is bounded.

$$E|Z_T - \theta b|^p = \sum_{n=n_{1c}}^{\infty} E|Z_n - \theta b|^p[T=n]$$
$$\le \sum_{n=n_{1c}}^{\infty} E|Z_n - \theta b|^p[T \ge n]$$
$$\le \sum_{n=1}^{\infty} E\,[T \ge n] \, E(|Z_n - \theta b|^p | F_{n-1})$$
$$= E|Z_1 - \theta b|^p E\,T. \qquad (3.11)$$

Using (3.11), The Schwarz inequality and Corollary 3.1, we have that the second term on the right hand side of (3.10) is bounded. For all $p > 0$, $E \, a^p_{T-1} = O(1)$ by the Lemma 3.1, and similarly $E \, b^p_{T-1} = O(1)$. Hence the lemma.

5. The Main Theorems.

Let $W_i = b^{1/2}\theta^{-1}+(3/2)(\theta b^{1/2})^{-1}(\delta_i-b)-(\theta^2 b^{1/2})^{-1}(Z_i-\theta b)$ and $S_n = \sum_{i=1}^{n} W_i$. Let S_n^- denote the negative part of S_n.

Let A be a 2×2 matrix defined as follows:

$$A = \begin{bmatrix} (\theta^3 b^{3/2})^{-1} & -(3/4)(\theta^2 b^{3/2})^{-1} \\ -(3/4)(\theta^2 b^{3/2})^{-1} & (3/8)(\theta b^{3/2})^{-1} \end{bmatrix}$$

Define $V = (e_1, e_2)A(e_1, e_2)'$. In the sequel no distinction in made between N and $c^{-1/2}\sigma$ and Remark 3.2 applies.

<u>Theorem 3.1</u>. $E\ T_c = N + \sigma\ (\rho - EV) + o(1)$ as $c \to 0$, where

$$\rho = (1/2)\{(3/4)\ \sigma^{-1} + (9/4)\sigma^{-1}b^{-1} - \sigma^{-3}b^{-3} \int xe^{-x/\theta}G(x)dx\} - \sum_{K=1}^{\infty} K^{-1}\ E\ S_K^-\ .$$

<u>Theorem 3.2</u>. $T_c^* = N^{1/2}\ (T-N)$ is asymptotically normal with mean zero and variance

$$(9/4)\theta^{-2} - 4^{-1}\theta^{-2}b + \theta^{-4}b^{-1}\int xe^{-x/\theta}\ G(x)dx.$$

<u>Theorem 3.3</u>. $R(\theta, G, c) = Bc + o(c)$ where

$$\begin{aligned} B = &-2\theta^{-3}b^{-2}E\ \{(Z_1 - \theta b) - 3/2\theta(\delta_1 - b)\}(Z_1 - \theta\delta_1)^2 \\ &-4\theta^{-4}b^{-3}\ \{E(e_1 - 3/2\theta e_2)(e_1 - \theta e_2)\}^2 \\ &-2\theta^{-2}b^{-2}\ E(e_1 - (3/2)\theta e_2)^2 \\ &+5\theta^{-4}b^{-3}\ E(e_1 - \theta e_2)^2\ (e_1 - (3/2)\theta e_2)^2 \\ &+3\theta^{-3}b^{-3}\ E(e_1 - \theta e_2)^2\ e_1 e_2 \\ &-(15/4)\theta^{-1}\sigma^{-1}\ E(e_1 - \theta e_2)^2\ e_2^2 - 2\sigma\ EV \\ &+ 6\ \theta^{-3}b^{-3}\ Ee_2\ (e_1 - \theta e_2)^2\ \{e_1 - (3/2)\theta e_2\} \\ &- 4b^{-3}\theta^{-3}E\ e_2(e_1 - \theta e_2)\ E(e_1 - \theta e_2)(e_1 - 3/2\theta e_2) \\ &- 2b^{-2}\theta^{-1}Ee_2(e_1 - 3/2\theta e_2). \end{aligned}$$

<u>Remark 3.3.</u> We note that in the absense of censoring,

the above results reduce to those given by Woodroofe (1977). The constant B turns out to be 3.

6. Proofs.

Let $D_c = T(\hat{\sigma}_T)^{-1} - c^{-1/2}$ and $\tau_c = \inf\{n \geq n_{1c}: S_n > c^{-1/2}\}$, with S_n, V as defined in the previous section.

Lemma 3.7. As $c \to 0$, D_c has a limiting distribution H, where $H(dr) = (ES_\tau)^{-1} P(S_\tau > r)\, dr$, $r > 0$, and τ denotes the first ladder epoch of S_n, $n \geq 1$. Thus D_c has the same limiting distribution as $S_{\tau_c} - c^{-(1/2)}$.

Proof. Using Taylor's theorem for two variables, we have

$$\begin{aligned} n(\hat{\sigma}_n)^{-1} = n\{\sigma^{-1} &+ (3/2)(\theta b^{1/2})^{-1}(\bar{\delta}_n - b) \\ &- (\theta^2 b^{1/2})^{-1}(\bar{Z}_n - \theta b) \\ &+ (3/8)\, \lambda_1^{-1/2}\, \lambda_2^{-1}\, (\bar{\delta}_n - b)^2 \\ &- (3/2)\lambda_1^{1/2}\lambda_2^{-2}\, (\bar{\delta}_n - b)(\bar{Z}_n - \theta b) \\ &+ \lambda_1^{3/2}\, \lambda_2^{-3}\, (\bar{Z}_n - \theta b)^2\}, \end{aligned}$$

where λ_1 and λ_2 lie between b and $\bar{\delta}_n$, θb and $\bar{Z}_n$ respectively. Thus $n(\hat{\sigma}_n)^{-1} = S_n + \S_n$ where

$$\begin{aligned} \S_n = n\{&(3/8)\, \lambda_1^{-1/2}\, \lambda_2^{-1}\, (\bar{\delta}_n - b)^2 \\ &- (3/2)\, \lambda_2^{1/2}\, \lambda_2^{-2}\, (\bar{\delta}_n - b)(\bar{Z}_n - \theta b) \\ &+ \lambda_1^{3/2}\lambda_2^{-3}\, (\bar{Z}_n - \theta b)^2\}. \end{aligned}$$

The W_i's are independent, identically distributed, and non-arithmetic. Also $E\, W_i = \sigma^{-1} > 0$ and $\{\S_n, n \geq 1\}$ is a slowly changing sequence. These follow from Example 2.1, Remark 2.1, Lemma 2.1, and Remark 2.3. Hence by Theorem

2.8, we have the result.

Remark 3.4. Though $\{D_c: \ 0 < c < 1\}$ may not be uniformly integrable, $\{(D_c\ \hat{\sigma}_T)^K: \ 0 < c < 1\}$ is, for all $K > 0$. Observe that

$$\begin{aligned} D_c\hat{\sigma}_T &= T - \hat{\sigma}_T c^{-1/2} < T - (T-1)[T>n_{1c}] \\ &\quad - \hat{\sigma}_{n_{1c}}\ c^{-1/2}[T=n_{1c}] \\ &= 1 + (T-1 - \hat{\sigma}_{n_{1c}}\ c^{-1/2})[T=n_{1c}]. \end{aligned}$$

Schwarz inequality and Lemmas 3.2, 3.4, 3.5 give us the necessary uniform integrability.

Lemma 3.8. (i) $\xi_n\ n^{-1/2} \to 0$ in probability as $n \to \infty$ and (ii) ξ_n converges in distribution to V.

Proof. (ii) implies (i). Application of the bivariate central limit theorem gives (ii).

Lemma 3.9. Let $A_n = \{\bar{Z}_n > 2^{-1}\theta b$ and $\bar{\delta}_n > 2^{-1}b\}$ and $V_n = \xi_n[A_n]$.

Then (i) $\sum_{n=1}^{\infty} P(\bigcup_{k\geq n} A_k^{'}) < \infty$.

(ii) V_n converges in distribution to V.

(iii) $\{\underset{0\leq k\leq n}{\mathrm{Max}} |V_{n+k}|,\ n\geq 1\}$ are uniformly integrable.

(iv) $\sum_{n=1}^{\infty} P(V_n \leq -n\beta) < \infty$ for some β, $0 < \beta < \sigma^{-1}$.

Proof. Note that $\bigcup_{k\geq n} A_k^{'} = \{\bigcup_{k\geq n} (\bar{Z}_k \geq 2^{-1}\theta b)\} \cup \{\bigcup_{k\geq n} (\bar{\delta}_k \leq 2^{-1}b)\}$.

Thus
$$\begin{aligned} P(\bigcup_{k\geq n} A_k^{'}) &\leq P(\underset{k\geq n}{\mathrm{Max}} |\bar{Z}_k - \theta b|^4 \geq 16^{-1}\theta^4 b^4) \\ &\quad + P(\underset{k\geq n}{\mathrm{Max}} |\bar{\delta}_k - b|^4 > 16^{-1}b)^4). \end{aligned}$$

By the reverse submartingale inequality,

$$P(\bigcup_{k\geq n} A_k') \leq \eta(E|\bar{Z}_n - \theta b|^4 + E|\bar{\delta}_n - b|^4) = O(n^{-2})$$

where η is a constant. Hence (i) obtains.

Since $[A_n] \to 1$ almost surely, by Lemma 3.8, V_n converges in distribution to V as $n \to \infty$. Hence (ii) obtains.

Let $\alpha > 1$.

$$E \underset{0\leq k\leq n}{\text{Max}} (n+k)^{\alpha}[\bar{\delta}_{n+k}\neq 0]\ \bar{\delta}_{n+k}^{-\alpha/2}\ \bar{Z}_{n+k}^{-\alpha}|\bar{\delta}_{n+k}-b|^{2\alpha}[A_{n+k}]$$

$$\leq (2^{-1}\theta b)^{-\alpha}(2^{-1}b)^{-\alpha/2}E \underset{0\leq k\leq n}{\text{Max}} (n+k)^{\alpha}|\bar{\delta}_{n+k}-b|^{2\alpha}$$

$$\leq (2^{-1}\theta b)^{-\alpha}(2^{-1}b)^{-\alpha/2}\ n^{-\alpha}\ E \underset{0\leq k\leq n}{\text{Max}} (n+k)^{2\alpha}|\bar{\delta}_{n+k}-b|^{2\alpha}$$

$$= (2^{-1}\theta b)^{-\alpha}(2^{-1}b)^{-\alpha/2}n^{-\alpha}\ E \underset{0\leq k\leq n}{\text{Max}} |\sum_1^k(\delta_i-b)|^{2\alpha}.$$

By the martingale inequality, the right hand side of the above inequality is bounded above by

$$(2^{-1}\theta b)^{-\alpha}(2^{-1}b)^{-\alpha/2}(2\alpha)^{-2\alpha}(2\alpha-1)^{-2\alpha}n^{-\alpha}E|\sum_1^{2n}(\delta_i-b)|^{2\alpha}.$$

By vonBahr's (1965) extension of the central limit theorem, (i.e. Theorem 3.2), $E\{(2n)^{1/2}|\bar{\delta}_n-b|\}^{2\alpha} \to 2^{\alpha}\ \pi^{-1/2}\Gamma(2^{-1}+\alpha)$.

Hence

$$\sup_n E \underset{0\leq k\leq n}{\text{Max}} (n+k)^{\alpha}[\bar{\delta}_{n+k}\neq 0]\bar{\delta}_{n+k}^{-\alpha/2}\ \bar{Z}_{n+k}^{-\alpha}|\bar{\delta}_{n+k}-b|^{2\alpha}[A_{n+k}] < \infty.$$

The above inequality implies that

$$\{\underset{0\leq k\leq n}{\text{Max}} (n+k)[\bar{\delta}_{n+k}\neq 0]\bar{\delta}_{n+k}^{-1/2}\ \bar{Z}_{n+k}^{-1}|\bar{\delta}_{n+k}-b|^{2}[A_{n+k}]:\ n\geq 1\}$$

is uniformly integrable.

Dealing similarly with the other terms in V_n, (iii) can be obtained. Finally,

$$P(V_n\leq n\beta) \leq P(-(3/2)\lambda_1^{1/2}\lambda_2^{-2}(\bar{\delta}_n-b)(\bar{Z}_n-\theta b)[A_n] < -\beta)$$

$$= P((3/2)[A_n]\lambda_1^{1/2}\lambda_2^{-2}(\bar{\delta}_n-b)(\bar{Z}_n-\theta b) > \beta)$$

$$\leq P((3/2)[A_n]\{(\theta b)^{-2} + \bar{Z}_n^{-2}\}|\bar{\delta}_n-b||\bar{Z}_n-\theta b| > \beta)$$

$$\leq P((3/2)(\theta b)^{-2}|(\bar{\delta}_n-b||Z_n-\theta b| > 2^{-1}\beta)$$

$$+ P\{(3/2)\bar{Z}_n^{-2}[A_n]|\bar{\delta}_n-b||\bar{Z}_n-\theta b| > 2^{-1}\beta\}$$

$$\leq P\{(3/2)(\theta b)^{-2}|(\bar{\delta}_n-b)(\bar{Z}_n-\theta b)| > 2^{-1}\beta)$$

$$+ P(3/2)(\theta b/2)^{-2}|\bar{\delta}_n-b||\bar{Z}_n-\theta b| > 2^{-1}\beta\}$$

By application of the Chebysev and Schwarz inequalities, we have

$$P(V_n \leq n\beta) \leq K\ E^{1/2}\ (\bar{\delta}_n-b)^4\ E^{1|2}(\bar{Z}_n-\theta b)^4 = O(n^{-2})$$

Hence (iv) obtains.

<u>Proof of Theorem 3.1</u>. Lemmas 3.9, 3.4 and Theorem 2.9 imply the theorem.

<u>Proof of Theorem 3.2</u>. Since $S_T + \xi_T - D_c = c^{-1/2}$, we have $-(S_T-\sigma^{-1}T)\ N^{-1/2}\sigma-\sigma\ N^{-1/2}(\xi_T-D_c) = N^{-1/2}(T-c^{-1/2}\sigma)$. Since $(S_n-\sigma^{-1}n)n^{-1/2}\sigma$ converges in distribution to normal random variable with mean zero and variance $(9/4)\theta^{-2} - 4^{-1}\theta^{-2}b + \theta^{-4}b^{-1}\int xe^{-x/\theta}G(x)dx$, the fact that $N^{-1}T \to 1$ almost surely, and Anscombe's theorem imply $(S_T-\sigma^{-1}T)N^{1/2}\sigma$ converges in distribution to above mentioned normal random variable. Similarly Lemma 3.8 and Anscombe's theorem imply that $\xi_T N^{-1/2}$ converges to zero in probability. By Lemma 3.7, $D_c N^{-1/2}$ converges to zero in probability. Thus $N^{1/2}(T-N)$ converges to normal random variable as stated in Theorem 3.2.

<u>Remark 3.5.</u> For the proof of Theorem 3.3, we shall use the following notation. Let $P_i = \theta^{-1}b^{-1/2}(Z_i-\theta\delta_i)$, $\tilde{S}_n = \sum_{i=1}^{n} P_i$ and $U_i = (\theta b)^{-1}\{Z_i-(3/2)\theta(\delta_i-b)\} - 1$. Also let $\hat{S}_n = \sum_{i=1}^{n}(\delta_i-b)$. It can be easily checked that the variance of $P_i = 1$.

<u>Lemma 3.10</u>. $E(\bar{\delta}_T-b)(\bar{Z}_T-\theta\bar{\delta}_T)^2$

$$= c\sigma^{-2}\{E(\delta_1-b)(Z-\theta\delta_1)^2 - 3b^{-1}\theta^{-1}Ee_2(e_1-\theta e_2)^2(e_1-(3/2)\theta e_2)$$
$$+ 2\theta^{-1}b^{-1}\ E(\delta_1-b)(Z_1-\theta\delta_1)\ E(e_1-\theta e_2)(e_1-3/2\theta e_2)$$
$$+ \theta\ Ee_2(e_1-3/2\theta e_2)\} + o(c).$$

<u>Proof</u>. In view of Remark 3.5,

$$\begin{aligned}
E(\bar{\delta}_T-b)(\bar{Z}_T-\theta\bar{\delta}_T)^2 &= \theta^2 b\ ET^{-1}\hat{S}_T\ \tilde{S}_T{}^2 \\
&= \theta^2 b\ E(T^{-3}-N^{-3})\ \hat{S}_T\tilde{S}_T^2 + \theta^2 b\ N^{-3}\ E\hat{S}_T\ \tilde{S}_T^2 \\
&= \theta^2 b\ N^{-3}\ ET^{-3}(N-T)(N^2+NT+T^2)\ \hat{S}_T\tilde{S}_T^2 + \theta^2 bN^{-3}E\hat{S}_T\tilde{S}_T^2 \\
&= \theta^2 bN^{-2}E(T^{-1}N)N^{-1/2}(N-T)(T^{-2}N^2+T^{-1}N+1)(\hat{S}_T N^{-1/2})(\tilde{S}_T^2\ N^{-1}) \\
&\quad + \theta^2 bN^{-3}E\ \hat{S}_T(\tilde{S}_T^2 - T) + \theta^2 bN^{-3}E\ \hat{S}_T T \\
&= I + II + III. \qquad (3.12)
\end{aligned}$$

Using the facts that $T\sigma_T^{-1} - c^{1/2} = D_c$ and $\hat{\sigma}_T\,\sigma^{-1} - 1$

$$= \theta^{-1}b^{-1}\ \{\bar{Z}_T - \theta b - (3/2)\theta(\bar{\delta}_T-b) - (3/2)\theta^{-1}b^{-2}(Z_T-\theta b)(\bar{\delta}_T-b)$$
$$+ (15/8)\sigma^{-1}\bar{Z}_T(\bar{\delta}_T-b)^2\beta^{-7/2}\}$$

for some β between b and $\bar{\delta}_T$, we have

$$\begin{aligned}
N^{-1/2}(N-T) &= -N^{1/2}(\hat{\sigma}_T\sigma^{-1}-1) \\
&= -N^{1/2}\theta^{-1}b^{-1}\{\bar{Z}_T-\theta b-(3/2)\ \theta(\bar{\delta}_T-b) \\
&\quad - (3/2)\theta^{-1}b^{-2}(\bar{Z}_T-\theta b)(\bar{\delta}_T-b)
\end{aligned}$$

$$+ (15/8)\sigma^{-1}\bar{Z}_T(\bar{\delta}_T-b)^2\beta^{-7/2}\}. \tag{3.13}$$

Theorem 2.4, Remark 3.4, Anscombe's theorem, $\beta^{-7/2} < b^{-7/2} + \bar{\delta}_T^{-7/2}$, Corollary 3.1 and (3.13) imply that

$$I = -3c\sigma^{-2}\{\theta^{-1}b^{-1}E(e_1-3/2\theta e_2)\ e_2(e_1-\theta e_2)^2\} + o(c). \tag{3.14}$$

Computation of the term II is briefly scatched below since it is similar to the proof of the lemma in Chow and Martinsek (1982).

Note that $(\hat{S}_n,\mathcal{F}_n)$ and $(\tilde{S}_n^2-n,\mathcal{F}_n)$ are martingales. Also

$$2E\hat{S}_T(\tilde{S}_T^2-T) = -\{E(\hat{S}_T-\tilde{S}_T^2 + T)^2 - E\hat{S}_T^2 - E(\tilde{S}_T^2-T)^2\}. \tag{3.15}$$

By Wald's lemma, $E\hat{S}_T^2 = E(\delta_1-b)^2\ ET$ (3.16)

Theorem 1 and lemmas 6,8 of Chow, Robbins and Teicher (1965) imply

$$\begin{aligned} E(\hat{S}_T-\tilde{S}_T^2 + T)^2 &= E(\delta_1-b)^2\ ET + (P_1^2-1)^2\ ET + 4\sum_{j=1}^{T} \tilde{S}_{j-1}^2 \\ &\quad + 4EP_1^3\ ET\tilde{S}_T - 2E(\delta_1-b)(P_1^2-1)\ ET \\ &\quad - 4E(\delta_1-b)P_1ET\tilde{S}_T \end{aligned}$$

and

$$E(\tilde{S}_T^2-T)^2=E(P_1^2 - 1)ET + 4E\sum_{j=1}^{T} \tilde{S}_{j-1}^2 + 4EP_1^3\ ET\tilde{S}_T. \tag{3.17}$$

(3.15), (3.16), (3.17) and Wald's lemma imply that

$$\begin{aligned} E\hat{S}_T(\tilde{S}_T^2-T) &= E(\delta_1-b)(P_1^2-1)ET + 2E(\delta_1-b)P_1ET\tilde{S}_T \\ &= E(\delta_1-b)(P_1^2-1)ET + 2E(\delta_1-b)P_1E(T-N)\tilde{S}_T. \end{aligned} \tag{3.18}$$

Now (3.18), lemma 3.5, (3.13), Remark 3.4 and Anscombe's theorem, $\beta^{-7/2} < b^{7/2} + \bar{\delta}_T^{-7/2}$ and Theorem 2.4 imply that

$$\begin{aligned} II = c\sigma^{-2}\theta^2b\{&E(\delta_1-b)(Z_1-\theta\delta_1)^2 \\ &+ 2\theta^{-1}b^{-1}E(\delta_1-b)(Z_1-\theta\delta_1)E(e_1-3/2\theta e_2)(e_1-\theta e_2)\} + o(c) \end{aligned}$$

$$= c\sigma^{-2}\{E(\delta_1-b)(Z_1-\theta\delta_1)^2$$

$$+ 2\theta^{-1}b^{-1}Ee_1(e_1-\theta e_2)E(e_1-3/2\theta e_2)(e_1-\theta e_2)\} + o(c) \quad (3.19)$$

By Wald's lemma,

$$III = \theta^2 b\ N^{-3}\ E\hat{S}_T(T-N).$$

Again, use of (3.13) and arguments simiIiar to (3.19) give us

$$III = c\sigma^{-2}\ \theta\ Ee_2\ (e_1-3/2\ \theta e_2) + o(c). \quad (3.20)$$

The lemma is proved by (3.12), (3.14), (3.19) and (3.20).

The next lemma asserts that $T^{-1}\tilde{S}_T^2$ and D_c are asymptotically independent.

<u>Lemma 3.11</u>. $P(T^{-1}\tilde{S}_T^2 \leq x,\ D_c \leq y) \to L(x)H(y)$ as $c \to 0$ for every $x,\ y \geq 0$, where H is as described in Lemma 3.7 and L is the distribution function of a chisquare random variable with one degree of freedom.

<u>Proof</u>. Proof is similar to the lemma in Martinsek (1983).

<u>Proof of Theorem 3</u>.

$$c^{-1}R(\theta,G,c)=(c^{-1}\sigma^2E\ \bar{P}_T^2-ET)+ 2(ET-N) - 2c^{-1}b^{-1}\sigma^2E(\bar{\delta}_T-b)\bar{P}_T^2$$

$$+ 2c^{-1}\sigma E\bar{P}_T\ \{\lambda_2^{-3}(\bar{\delta}_T-b)^2(\bar{Z}_T-\theta b) - \lambda_1\lambda_2^{-4}(\bar{\delta}_T-b)^3\}$$

$$+ c^{-1}\{b^{-2}(\bar{\delta}_T-b)(\bar{Z}_T-\theta b)-\theta b^{-2}(\bar{\delta}_T-b)^2-\lambda_2^{-3}(\bar{\delta}_T-b)^2(\bar{Z}_T-\theta b)$$

$$+ \lambda_1\lambda_2^{-4}(\bar{\delta}_T-b)^3\}^2$$

$$+ \{2\theta^{-2}b^{-2}E(\delta_1-b)(Z_1-\theta\delta_1)^2\ -3\theta^{-2}b^{-3}Ee_2^2(e_1-\theta e_2)^2\}+o(1)$$

$$= I + II + III + IV + V + VI + o(1).$$

II and III are given by theorem 1 and lemma 3.10 respectively.

$$IV = 2\ \theta^{-2}b^{-3}\ E(e_1-\theta e_2)^2 e_2^2 + o(1) \qquad (3.21)$$

by lemma 3.5, the central limit theorem, Anscombe's theorem and by Theorem 2.4. Similarly

$$V = \theta^{-2}b^{-3}E(e_1-\theta e_2)^2 e_2^2 + o(1). \qquad (3.22)$$

Now $I = E\ \tilde{S}_T^{\ 2}\{(N^{-1}T)^{-2}-1\}$. By Taylor's theorem,

$$I = -2\ E\ \tilde{S}_T^2(N^{-1}T-1) + 3\ E\ \tilde{S}_T^2\lambda^{-4}(N^{-1}T-1)^2, \qquad (3.23)$$

where λ lies between 1 and $N^{-1}T$.

$$E\ \tilde{S}_T^2\ \lambda^{-4}(N^{-1}T-1)^2$$

$$= ETN^{-1}\ \{T(\bar{Z}_T-\theta\bar{\delta}_T)^2\theta^{-2}b^{-1}\}\ \lambda^{-4}\{N^{-1/2}(T-N)\}^2$$

$$= ETN^{-1}\{T(\bar{Z}_T-\theta\bar{\delta}_T)^2\theta^{-2}b^{-1}\}\lambda^{-4}\{N^{1/2}(\hat{\sigma}_T\sigma^{-1}-1)$$
$$+ N^{-1/2}D_c\hat{\sigma}_T\}^2.$$

Using (3.13), Remark 3.4 and reasoning similar to (3.21) gives us the convergence of the above term to

$$E\ \theta^{-2}b^{-1}(e_1-\theta e_2)^2(e_1-(3/2)\theta e_2)^2\sigma^{-2}b^{-3}. \qquad (3.24)$$

Now consider the first term in the right hand side of (3.23).

$$E\ \tilde{S}_T^2(N^{-1}T-1) = E\ \tilde{S}_T^2(\hat{\sigma}_T\sigma^{-1}-1) + E\ \tilde{S}_T^2\ N^{-1}\ D_T\hat{\sigma}_T. \qquad (3.25)$$

Using (3.13) we have,

$$E\ \tilde{S}_T^2(\hat{\sigma}_T\sigma^{-1}-1)$$

$$= \theta^{-1}b^{-1}E\tilde{S}_T^2(\bar{Z}_T-3/2\theta(\bar{\delta}_T-b)-\theta b\}-\ 3/2\theta^{-1}b^{-2}E\tilde{S}_T^2(Z_T-\theta b)(\bar{\delta}_T-b)$$
$$+ (15/8)\ E\ \tilde{S}_T^2\sigma^{-1}\bar{Z}_T(\bar{\delta}_T-b)^2\beta^{-7/2}$$
$$= (i) + (ii) + (iii)$$

By Wald's Lemma,

$$(i) = E\ \tilde{S}_T^2(\sum_{i=1}^{T}\ u_i)T^{-1}$$
$$= E(\tilde{S}_T^2-T)(\Sigma\ u_i)N^{-1} + E(\tilde{S}_T^2-T)(\Sigma u_i)(T^{-1}-N^{-1})$$

$$= \tilde{I} + \tilde{\tilde{II}}.$$

As in the lemma of Chow and Martinsek (1982), $\tilde{I}$ can be evaluated as

$$\tilde{I} = 2^{-1}N^{-1}ET\{Eu_1^2-E(P_1^2-u_1-1)^2 + E(P_1^2-1)^2\}$$
$$+ 2E\{(u_1+1)P_1\}E(T-N)\tilde{S}_T N^{-1}.$$

Lemmas 3.4, 3.13 and the fact that $N^{1/2}(T-N) = N^{1/2}(\sigma_T\sigma^{-1}-1) + N^{-1/2}D_c\hat{\sigma}_T$, we have the convergence of $\tilde{I}$ to

$$1/2\ \{E\ u_1^2-E(P_1^2-u_1-1)^2 + E(P_1^2-1)^2\}$$
$$+ \theta^{-2}b^{-3|2}\ E(u_1+1)P_1E(e_1-3/2\theta e_2)(e_1-\theta e_2).$$

$$\tilde{\tilde{II}} = E\ \tilde{S}_T^2(N-T)(\sum_{i=1}^{T} u_i)\ N^{-1}T^{-1} - N^{-1}E(N-T)(\sum_1^T u_i)$$

converges to

$$E(e_1-(3/2)\theta e_2)^2\theta^{-2}b^{-2}-E(e_1-\theta e_2)^2(e_1-(3/2)\theta e_2)^2\theta^{-4}b^{-3}.$$

Thus

$$(i) = 1/2\{E\ u_1^2-E(e_1^2-u_1-1)^2 + E(P_1^2-1)^2\}$$
$$+ 2E(u_1+1)P_1(e_1-(3/2)\theta e_2)(e_1-\theta e_2)\theta^{-4}b^{-3}$$
$$+ E(e_1-(3/2)\theta e_2)^2\theta^{-2}b^{-2}-E(e_1-\theta e_2)^2(e_1-(3/2)\theta e_2)^2\theta^{-4}b^{-3}$$
$$+ o(1).$$

$(ii) = -(3/2)\theta^{-3}b^{-3}(e_1-\theta e_2)^2e_1e_2 + o(1)$ and

$(iii) = (15/8)\theta^{-1}\sigma^{-1}E(e_1-\theta e_2)^2e_2^2 + o(1).$

Remark 3.4 and lemma 3.11 imply that

$$E\tilde{S}_T^2\ N^{-1}\ D_T\hat{\sigma}_T = \sigma\rho + o(1).$$

Putting together all the above terms we have theorem 3.

CHAPTER 4

SEQUENTIAL INTERVAL ESTIMATION

1. Sequential Procedure.

It can be easily checked that

$$n^{1/2}(\hat{\theta}_n-\theta) \xrightarrow{L} N(0,\sigma^2(\theta)) \tag{4.1}$$

where $\sigma^2(\theta) = \theta^2 b^{-1}$. For a given $d > 0$ and $\alpha \epsilon (0,1)$, in view of (4.1) let us take $I_n = (\hat{\theta}_n - d, \hat{\theta}_n + d)$ with $n(d)$ defined by

$$n(d) = \inf\{k \geq 1: \ k \geq Z^2_{\alpha/2} \sigma^2 d^{-2}\} \tag{4.2}$$

where Z_α is the upper 100α percentage point of the standard normal distribution. Since σ is unknown, the specification of the 'optimal' sample size (4.2) can not be made. We therefore led naturally to construct a sequential procedure in which the sample size is a positive integer valued random variable $N = N(d)$ and the desired confidence interval for θ is $I_N = [\hat{\theta}_N - d, \hat{\theta}_N + d]$. The sequential procedure (N, I_N) is said to be <u>asymptotically consistent</u> if for every θ positive,

$$\lim_{d \downarrow 0} P(\theta \in I_N) \geq 1 - \alpha, \tag{4.3}$$

and is said to be <u>asymptotically efficient</u> if for all θ positive

$$\lim_{d \downarrow 0} E_\theta\{N(d)|n(d)\} \leq 1. \tag{4.4}$$

Following stopping time N is a slight modification of the stopping time defined by Gardiner, Susarla and VanRyzin (1985).

$$N(d) = \inf\{k > k_{1d} : k \geq (Z_{\alpha/2} d^{-1})^2 \hat{\sigma}_k^2\}$$

where $k_{1d} = \text{int } (Z_{\alpha/2}/d)^{1/2(1+\lambda)}$ for $\lambda > 0$. They prove the following theorem.

Theorem 4.1. (N, I_N) is both asymptotically consistent and efficient. In fact

$$P\{\theta \epsilon I_N\} \to 1-\alpha \tag{4.5}$$

and

$$E\ \{N(d) | n(d)\} \to 1 \quad \text{as} \quad d \to 0. \tag{4.6}$$

In the spirit of Theorem 3.1 and 3.3 one expects to achieve second order expansions for $E\ N$ and $P(\theta \epsilon I_N)$. Theorem (4.2) gives the second order expansion for $E\ N$. The expansion of $P(\theta \epsilon I_N)$ remains an open problem.

In their paper, Chow and Robbins (1965) illustrate a general methodology for the construction of the fixed width sequential confidence intervals for the mean of the population. Sen (1981) contains several refinements of these methods that have been successfully applied to obtain parallel results for the other functionals of the underlying distribution. Woodroofe (1977) gives the second order expansions for $E\ N$ and $P\{\theta \epsilon I_N\}$ for the normal mean problem. So far this is the only second order computation available in the literature for the fixed width sequential problem.

2. The Theorem.

Let

$$B = \begin{bmatrix} 3\theta^{-4}b^{-1} & -3\theta^{-3}b^{-1} \\ -3\theta^{-3}b^{-1} & 3\theta^{-2}b^{-1} \end{bmatrix}$$

and (e_1, e_2) as defined in Chapter 3 section 2. Let $W = (e_1, e_2)\ B\ (e_1, e_2)'$.

Theorem 4.2. $E\ N = \sigma^2\{d^{-2}(Z_{\alpha/2})^2 + \sigma - E\ W\} + o(1)$ as $d \to 0$, where

$$\rho = 2^{-1}(3\theta^{-2}b + 9\theta^{-2} - 4\theta^{-4}b^{-1} \int_0^\infty x\ e^{x/\theta}G(x)dx)$$
$$- \sum_{k=1}^{\infty} k^{-1}\ E(S_k^-)$$

Proof. Using Taylor's Theorem for two variables, we have

$$\begin{aligned} k\hat{\sigma}_k^{-2} &= k\{\sigma^{-2} - 2\theta^{-3}(\bar{Z}_k - \theta b) + 3\theta^{-2}(\bar{\delta}_k - b) \\ &\quad + 3\lambda_1^{-4}\lambda_2^3(\bar{Z}_k - \theta b)^2 \\ &\quad - 6\lambda_1^{-3}\lambda_2^2\ (\bar{Z}_k - \theta b)(\bar{\delta}_k - b) \\ &\quad + 3\lambda_1^{-2}\lambda_2\ (\bar{\delta}_k - b)^2\} \\ &= \hat{S}_k + \hat{\xi}_k. \end{aligned}$$

where

$$S_k = \sum_{i=1}^{k} \hat{X}_i,\quad \hat{X}_i = \sigma^{-2} - 2\theta^{-3}(Z_i - \theta b) + 3\theta^{-2}(\delta_i - b)$$

and

$$\hat{\xi}_k = n\{3\lambda_1^{-4}\lambda_2^3(\bar{Z}_k - \theta b)^2 - 6\lambda_1^{-3}\lambda_2^2(\bar{Z}_k - \theta b)(\bar{\delta}_k - b)$$
$$+ 3\lambda_1^{-2}\lambda_2\ (\bar{\delta}_k - b)^2\}.$$

It can be easily checked that $\hat{\xi}_k$ is slowly changing. Note that $\hat{X}_i$ is nonarithmetic and $E\,\hat{X}_i = \sigma^{-2} > 0$, further $\hat{X}_i$'s are independent identically distributed random variables. Hence by the Lai and Siegmund theorem (Theorem 2.8), we have the following result. $U_d = N\hat{\sigma}_N^{-2} - (Z_{\alpha/2})^2\, d^{-2}$ has limiting distribution M, as $d \to 0$. Where $M(dr) = (E\,\hat{S}_\tau)^{-1}\, P(\hat{S}_\tau > r)dr$, $r>0$ and τ is the first ladder epoch of $\hat{S}_k$, $k \geq 1$. Lemmas similar to 3.4, 3.8, 3.9 can be proved by exactly the same methods. Theorem 2.9 then implies the theorem.

3. Concluding Remarks.

It would be desirable to remove exponentiality assumption from the above problem and in the point estimation problem discussed in the previous chapter. Gardiner and Susarla (1983) is an attempt in this direction. They allow X to have any survival function F (not necessarily the exponential) and under fairly general moment conditions on F and G, they show the asymptotic risk efficiency of their procedure. The problem in this set up is very hard to work with since the estimator of the mean is an integral with respect to product limit estimator of F.

Instead of a purely nonparametric approach, it may be easier to study robust procedures with respect to contamination of the exponential distribution.

Repeated significance testing in the context of the censored model discussed in this dissertation will also be of interest, particularly from the point of view of clinical trials and other applications in Medicine.

It is hoped that the techniques developed in Chapter 3 would be useful to show 'bounded regret' in other sequential nonparametric problems such as procedures based on U-statistics and also on L-, M-, R- estimators of location.

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