MODIFICATIONS OF THE COBB-DOUGLAS FUNCTION TO DESTROY CONSTANT ELASTICITY AND SYMMETRY

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MODIFICATIONS OF THE COBB-DOUGLAS FUNCTION

To Destroy Constant Elasticity

and Symmetry

By

HAROLD O. CARTER

AN ABSTRACT

Submitted to the College of Agriculture of Michigan
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ABSTRACT

The objective of this study was to develop and appraise methods of modifying the Cobb-Douglas function to destroy characteristics of constant elasticity and symmetry. In past studies these characteristics were suspected of forcing certain undesirable restrictions on the fitted function.

The procedure was to make a transformation in the N-dimensional input space by replacing certain independent variables with dummy variables. Conceptually, these dummy variables introduced "ridge lines" on the production surface which imposed limits on the substitutability of X_i for X_j. The dummy variables were formulated mathematically into what is referred to as modification I and II as follows:

$$Z_j = PX_i \ (1-R^{X_j}) \qquad (1)$$

$$Z_j = PX_i \ (1-R^{\frac{X_j}{X_i}}) \qquad (2)$$

The "P" represents the ridge line proportion of X_j to X_i. The "R" is a ratio of a decreasing geometric series, the terms of which are the respective increments in the dummy variable Z_j due to sucessive unit increases in the independent variable X_j.

The prominent characteristics of both modification I and II were ascertained and their economic implications determined.

Modification I was fitted twice statistically using different estimates of the parameter "P" with the method of least squares.

Modification II was also fit by least squares to the same data that had been previously fitted by Vance Wagley, using the unmodified Cobb-Douglas function. Comparisons were made using various statistical measures.

The functions investigated did not appear to give a better fit statistically than the standard Cobb-Douglas function for these particular data. However, the modified function did indicate certain economic advantages over the unmodified function. These advantages permit the fitted function to show non-constant elasticity, conditions of symmetry more in agreement with empirical findings, and more realistic marginal value productivity estimates.

In addition, modification II in the labor input dimension gives strong indication of developing, with further work, into a usable production function that will show three stages of production simultaneously. Such preliminary findings suggest that research in this area should be extended.

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CHAPTER I

INTRODUCTION

Many of the problems in agricultural production economic research center around selecting and using appropriate equations to describe basic input-output relationships. It is difficult to devise equations that will express the very complicated "true relationships" found in the physical and social spheres of the agricultural sciences. Much of the difficulty is created by the various uncontrolled biological, climatic, and sociological variables which tend to obscure the "true relationships." The difficulty is aggravated still further by the inability of finite human minds to understand fully the nature and causality of certain relationships. Oftentimes, the best that can be done is to recognize that a functional relation does exist between certain variables. Experience and insight along with observation of data at hand help the researcher determine what type of equation is needed to express the relationship.

In productivity studies in agriculture one of the more widely used equations is a power function of the general form:

$$Y = aX_1^{b_1}, \ldots, X_i^{b_i}, \ldots, X_n^{b_n} \qquad (1.1)$$

This equation, as it is used in agricultural economics, is referred to as a Cobb-Douglas function.

Empirical research using this particular equation indicates that it has certain major advantages over other functions. Not least among these advantages is the fact that the function can be transformed into logarithmic form, and the parameters estimated by the very simple and expedient method of least squares. However, as in most situations, simplicity and expediency have a "price." In this case the "price" is certain shortcomings inherent within the mathematical form of the Cobb-Douglas function.

It will be the purpose of this study to investigate methodically certain modifications or alterations of the Cobb-Douglas equation capable of alleviating or lessening these shortcomings. Among the more prominent of these shortcomings are: (1) constant elasticity not only with respect to the specific X_i's but also in respect to all the variables $X_1, \ldots, X_n$ collectively. (2) intersection of Y and the $YX_1, \ldots, YX_n$ planes at $Y = 0$. (3) inability of the function to describe, simultaneously, any two relationships such as increasing positive, decreasing positive or negative marginal returns.

Organization of Thesis

The historical and theoretical background, to be presented in Chapter II, will place the problem in its proper context. It consists of a brief discussion of the origin and use of the Cobb-Douglas function with special reference to its use in agriculture. The discussion is followed by an analysis of the function, both mathematical and economical, with particular attention to the relevant characteristics.

Chapter III will deal mainly with a methodical investigation of two possible modifications in the Cobb-Douglas function. The characteristics of each modification are ascertained and their economic implications determined.

Chapter IV will present comparative applications and evaluations of the two modifications with the unmodified Cobb-Douglas function after each has been fitted to the same empirical data.

The summary and conclusions of the study will be presented in Chapter V.

CHAPTER II

THEORETICAL BACKGROUND

History and Origin of Cobb-Douglas Function

In 1928, Professor Paul Douglas,[1] while at the University of Chicago, computed indexes of labor and capital for American manufacturing industries.[2] With the help of a mathematician from Amherst College, Charles W. Cobb, he developed a formula which would measure the relative effects of labor and capital upon productivity during this period.[3]

The following equation is what is now referred to as the original Cobb-Douglas function.

$$P = bL^{k}C^{1-k} \qquad (2.1)$$

The dependent variable P represents the value of the total production of the industry. The C stands for the total fixed capital available for production, L represents labor used in production, and b is a constant. The exponents k and 1-k are coefficients of elasticity for P in respect to the independent variables labor and capital. The sum of the exponents were made equal to one which implies the assumption of constant returns to scale. It may be possible that Cobb's familiarity

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1 Now U. S. Senator Douglas from Illinois.

2 Paul H. Douglas, Theory of Wages, (New York: The Macmillan Co., 1934).

3 Paul H. Douglas and Charles W. Cobb, "A Theory of Production," American Economic Review, XVIII, Supplement, (March, 1928), pp. 139-165.

with Euler's theorem[4] affected his decision to force the exponents equal to one. This function is linear in logarithmic form, and the values of the b and k can be estimated by a modified[5] method of least squares. This equation was later modified on the recommendation of David Durand,[6] so that the sum of the exponents need not equal one. The resulting equation is what Professor Douglas used in his many manufacturing studies.

$$P = bL^{k}C^{j} \qquad (2.2)$$

The exponents k and j are the co-efficients of elasticity of P with respect to labor (L) and capital (C) while b is a constant. The function is linear in logarithmic form and the values of b, k and j can be estimated by the method of least squares.

Application of the Cobb-Douglas Function in Agriculture

Some of the first applications of this type of function in agriculture were made at Iowa State College by Tintner, Brownlee, and Heady. Tintner used farm business records from 609 Iowa farms for the year 1942 to derive productivity estimates of various input categories.[7]

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[4] Cf., R. G. D. Allen, Mathematical Analysis for Economists, (London: Macmillan and Co., Ltd., St. Martin's St., 1947), XII, p. 317. Briefly Euler's theorem states that if each input is attributed its marginal product, the total product, under specified conditions, will be exhausted.

[5] Gerhart Tintner, "A Note on the Derivation of the Production Functions from Farm Records," Econometrica, XII, No. 1, (January, 1944), pp. 31.

[6] David Durand, "Some Thoughts on Marginal Productivity with Special Reference to Professor Douglas' Analysis," Journal Political Economics, XLV, (December, 1937), pp. 740-758.

[7] Tintner, op. cit., pp. 26-34.

Tintner and Brownlee made similar estimates for 468 Iowa farms for the year 1939.[8] Heady studied a random sample of 738 Iowa farms.[9] Fienup, at Montana State College, also used a random sampling procedure to study productivity on Montana dry land crop farms.[10] Johnson applied the Cobb-Douglas analysis in studies of farms in the Purchase Area and western Kentucky.[11] He used what he refers to as a "purposive" sample in all of his studies. This means selecting sample farms that are not in scale line adjustment; thus reducing the intercorrelation among input categories and thereby increasing the reliability of the estimated regression coefficients. A similar study was made by Toon at Kentucky, also using a "purposive" sample.[12] Similar to Tintner and Brownlee, Drake at Michigan State College used farm account records to gain estimates of the marginal productivity of inputs, as well as study some of

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[8] Tintner and O. H. Brownlee, "Production Functions Derived from Farm Records," Journal of Farm Economics, XXVI, (August, 1944), pp. 566-571.

[9] Earl O. Heady, "Production Functions from a Random Sample of Farms," Journal of Farm Economics, XXVIII, No. 4, (November, 1946), pp. 989-1004.

[10] Darrell F. Fienup, Resource Productivity on Montana Dry Land Crop Farms, Mimeographed Circular 66, (Bozeman: Montana State College Agricultural Experiment Station, 1952).

[11] Glenn L. Johnson, Sources of Income on Upland Marshall County Farms, Progress Report No. 1, and Sources of Income on Upland McCracken County Farms, Progress Report No. 2, (Lexington: Kentucky Agricultural Experiment Station, 1953).

[12] Thomas G. Toon, The Earning Power of Inputs; Investment and Expenditures on Upland Grayson County Farms During 1951, Progress Report No. 7, (Lexington: Kentucky Agricultural Experiment Station, 1953).

the problems encountered in this approach.[13] Wagley, at Michigan State College, used a "purposive" sample in deriving the earning power of selected input categories on thirty-three Ingham County dairy farms.[14] This study, as well as Toon's, presents an excellent "cook-book" description of the computations involved in applying the Cobb-Douglas function to empirical data. Trant, also at Michigan State College, derived a method of adjusting marginal value productivity estimates for changing prices of both inputs and outputs.[15]

Other applications of Cobb-Douglas productivity functions have determined earning powers of certain inputs for individual enterprises. Heady used this approach in fitting power functions with pork dependent on both corn and protein.[16] P. R. Johnson, at North Carolina State College, fitted the Cobb-Douglas and other algebraic functions to fertilizer-yield data.[17]

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13 Louis Schneider Drake, "Problems and Results in the Use of Farm Account Records to Derive Cobb-Douglas Value Productivity Functions," Unpublished Ph.D. Dissertation, Department of Agricultural Economics, Michigan State College, 1952.

14 Robert Vance Wagley, "Marginal Productivity of Investments and Expenditures, Selected Ingham County Farms, 1952," Unpublished M. S. Thesis, Department of Agricultural Economics, Michigan State College, 1953.

15 Gerald Ion Trant, "A Technique of Adjusting Marginal Value Productivity Estimates for Changing Prices." Unpublished M. S. Thesis, Department of Agricultural Economics, Michigan State College, 1954.

16 Earl O. Heady, Roger C. Woodworth, Damon Catron, and Gordon C. Ashton, "An Experiment to Derive Productivity and Substitution Coefficients in Pork Production," Journal of Farm Economics, XXXV, (August, 1953), pp. 341-354.

17 Paul R. Johnson, "Alternative Functions for Analyzing A Fertilizer-Yield Relationship," Journal of Farm Economics, XXXV, (November, 1953), pp. 519-529.

More recently C. Beringer at Michigan State University developed concepts and methods of utilizing the Cobb-Douglas analysis to estimate the marginal value productivities of input categories in separate enterprises of multiple enterprise farms.

Relevant Characteristics of the Function

Returns to Scale

Consider the general equation in the following form:

$$Y = AX_1^{b_1},\ldots,X_i^{b_i} \quad X_j^{b_j},\ldots,X_n^{b_n} \qquad (2.3)$$

Y is the output or dependent variable, the A is a constant and the X_i's are independent variables. The b_i's are constant coefficients of elasticity for the Y in respect to the X_i's.[18] The sum of the regression coefficients ($\sum_{i=1}^{n} b_i$) indicates the nature of the returns to scale. Since the coefficients of elasticity are constant over the entire range of the

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[18] The equation for elasticity is:

$$E = \frac{\partial Y}{\partial X_i} \cdot \frac{X_i}{Y} \qquad (2.4)$$

Taking the partial derivative of Y with respect to X_i of (2.3), the resulting equation is:

$$\frac{\partial Y}{\partial X_i} = \frac{b_i \;E\;(Y)}{X_i} \qquad (2.5)$$

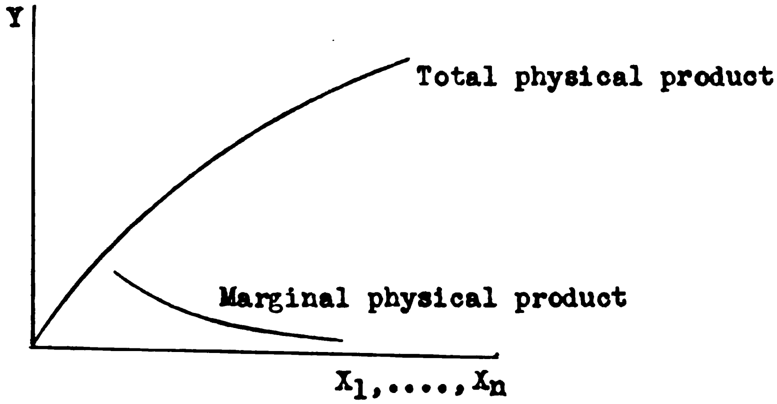
Solving (2.5) for the regression coefficient gives;

$$b_i = \frac{\partial Y}{\partial X_i} \cdot \frac{X_i}{Y} \qquad (2.6)$$

which is identically equal to the elasticity equation (2.4).

function, the elasticities of the dependent variable in respect to the independent variables are necessarily constant. That is, the function can show decreasing positive, increasing positive, constant, and negative marginal returns, singularly but not simultaneously.

Decreasing returns.--If the sum of the elasticity coefficients is greater than zero and less than one ($0 < \sum_{i=1}^{n} b_i < 1$), the function indicates total returns which increase at a decreasing rate. This case is illustrated in Figure 1.



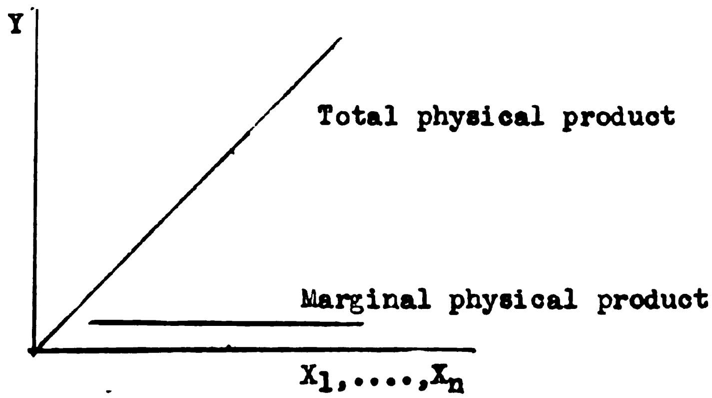
(used in any constant set of proportions)

Figure 1
An Example Showing Increasing
Returns at a Decreasing Rate

While the total product always increases at a decreasing rate, the marginal product decreases becoming asymptotic to the horizontal axis. This shortcoming cannot be considered too serious from the standpoint of economic analysis since it is irrational for an entrepreneur to operate within the range of increasing returns or negative marginal returns. Thus, it is not difficult to believe that most farm units are operating in the area of decreasing marginal returns to individual inputs.

Constant returns to scale.--If the sum of the elasticity coefficients is one ($\sum_{i=1}^{n} b_i = 1$), there exists constant returns to scale.

Such an equation is linearly homogeneous in the first degree,[19] and is illustrated in Figure 2. The total product in this case will go to



(used in any constant set of proportions)

Figure 2
An Example Showing Constant
Returns to Scale

infinity and the marginal product will be constant at a certain level. This is to say that, if the use of X_i is increased by a given percentage, output increases by the same percentage. This, in view of the law of diminishing returns, implies control over all measurable variables and that unmeasurable variables such as management and weather are randomly and normally distributed.

<u>Increasing returns to scale</u>.--If the sum of the elasticity coefficients is greater than one ($\sum_{i=1}^{n} b_i > 1$), the function indicates total returns which increase at an increasing rate. This case is illustrated in Figure 3. As the use of the X_i is expanded, the total physical product will go to infinity and the marginal physical product will always

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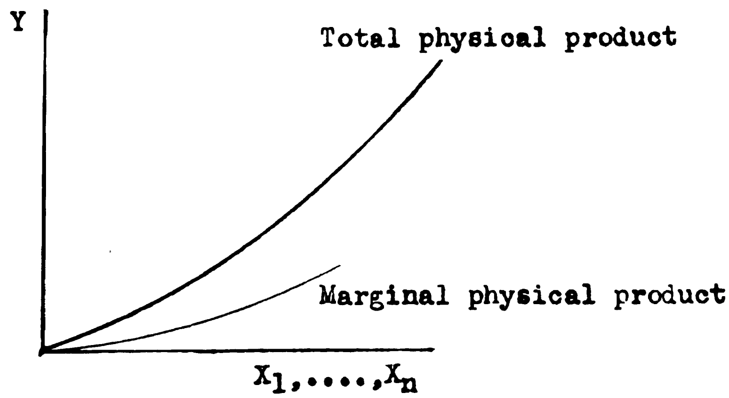
[19] Cf., R. G. D. Allen, op. cit., p. 315. Briefly $Z=f(x,y)$ is a linear homogeneous function if $f(\lambda x, \lambda y) = \lambda f(x, y,)$ for any points $(x, y,)$ and for any value of λ whatever.

increase.



(used in any constant set of proportions)

Figure 3
An Example of Increasing
Returns to Scale

Symmetry

The summetry of the function is illustrated in Figure 4, which shows that contour lines[20] in an X_iX_j plane become asymptotic to both the vertical and horizontal axes. This implies that there is an

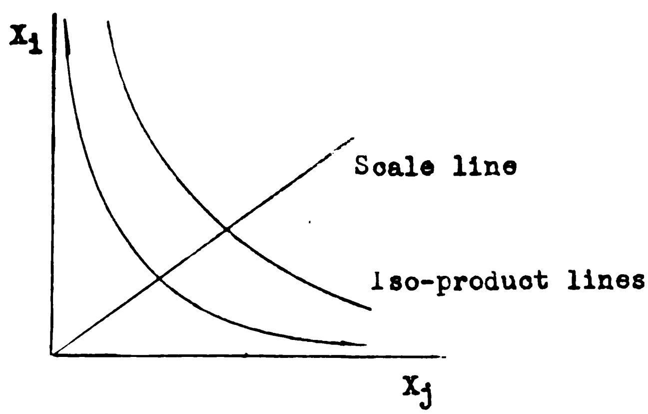


Figure 4
An Example Showing Contour Lines with
Characteristics of Symmetry and a
Scale Line with a Constant Slope

[20] Contour lines or iso-product lines show all combinations of two inputs which will product a given output.

unlimited range in which the proportions of any two inputs could be varied to produce a given level of output.[21]

The symmetry characteristic does not always correspond to reality. For example, output often can be produced with one input, i.e. a dairy cow may produce milk at a low level, utilizing only forage but no grain with other inputs constant.

The other illustration in Figure 4 shows the contour lines to have equal slope when X_i and X_j are used in the same proportion at successive levels of output. This results in straight scale lines in all subspaces of the N dimensional input space, but not in spaces involving output unless $\sum b_i$'s in the subspace equals one.[22]

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[21] Referring back to equation (2.3) let the dependent variable Y = C (constant level of output) as follows:

$$C = AX_1^{b1},\dots,X_i^{b_i}\ X_j^{b_j},\dots,X_n^{b_n} \qquad (2.7)$$

Solving (2.7) for X_i in terms of X_j, holding other inputs constant, yields:

$$X_i = \sqrt[b_i]{\frac{C}{AX_1^{b1}\ X_j^{b_j}\dots X_n^{b_n}}} \qquad (2.8)$$

If $X_i = 0$, then $X_i = \infty$ or if $X_i = 0$, then $X_j = \infty$.

[22] The proof of this in two dimensional subspaces is shown by first considering the general equation of the scale line:

$$\frac{MPP_{X_i}(Y)}{MPP_{X_j}(Y)} = \frac{P_{X_i}}{P_{X_i}} \qquad (2.9)$$

This relationship in terms of the Cobb-Douglas equation is secured by deriving the MPP of X_i and X_j using equation (2.5) and setting the results in a proportion equal to the price ratio PX_i/PX_j as follows:

Origin Always at the $Y = 0$, $X_1 = 0, \ldots, X_n = 0$ Point

The Cobb-Douglas possesses the characteristic of always originating at $Y = X_i = 0$. In addition, if any $X_i = 0$, then $Y = 0$. These characteristics are closely related to symmetry which shows output being produced with a combination of inputs but never in the absence of a single input.[23] Shortcomings of this nature limit the applicability of the function since empirical data often reflects input-output relationships which: (1) originate somewhere on the Y axis, (2) indicate output even in the absence of certain inputs. An example of (1) is applying fertilizer to a crop; the output even at the zero application may be high due to the natural fertility present in the soil, as shown in Figure 5. An example of (2) exists on farms that produce sizeable incomes without livestock investments, since their labor can be marketed in the form of cash crops.

$$\frac{\dfrac{b_i \; E(Y)}{X_i}}{\dfrac{b_j \; E(Y)}{X_j}} = \frac{PX_i}{PX_j} \tag{2.10}$$

Solving (2.10) for X_j in terms of X_i, the equation for the scale line in the X_iX_j plane is secured:

$$X_j = \frac{b_j \; PX_i}{b_i \; PX_j} \cdot X_i \tag{2.11}$$

As $\frac{b_j \; PX_i}{b_i \; PX_j} = K$ (a constant), $X_j = KX_i$ (2.12)

Thus, the scale line is straight in the two dimensional input space and by analogous reasoning the scale line could be shown to be straight in the N-dimensional input space.

[23] Refer back to Figure 4 and equation (2.8).

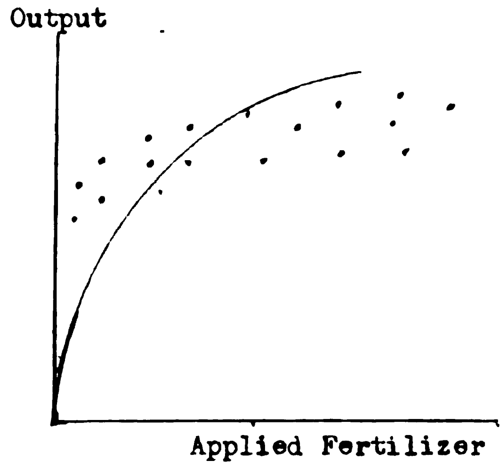

Figure 5
An Example Showing the Discrepancy That May Exist by Fitting a Cobb-Douglas Function to Fertilizer Input-output Data

Need for Modification

The review of the relevant characteristics of the unmodified Cobb-Douglas equation demonstrates, among other things, that its application is limited to data which reflect certain kinds of relationships. In order to broaden the use of such a convenient productivity function, alterations of the function to overcome these limitations are needed.

Mathematically there are several methods by which this power function could be altered to permit it to reflect certain desirable characteristics. However, any modification must be considered on the basis of (1) whether certain mathematical and economic specifications are met and (2) whether the function retains the important properties of expediency and simplicity in estimating the parameters. The following chapter will consider some possible modifications.

CHAPTER III

POSSIBLE MODIFICATIONS OF THE COBB-DOUGLAS FUNCTION

Though the need for this study has long been apparent, impetus for it was given in the form of criticism by Professor L. H. Brown[1] of certain value productivity estimates in the Wagley Cobb-Douglas study.[2] The estimates under criticism had the following characteristics:

1. The iso-product lines in the labor-livestock plane possessed the usual Cobb-Douglas characteristics of symmetry where there is an unlimited range in which the absolute amounts of labor and forage-livestock investment can be varied to produce a given level of output.

2. The iso-product lines in the labor-machinery plane possessed the same conditions of symmetry.

These characteristics indicate that a farmer with a fixed amount of labor available can increase the use of capital in the form of machinery and livestock indefinitely and, according to the function, continue to increase gross income. However, as Brown points out, the physical capacity of a man limits the amount of machinery and/or the volume of livestock he can handle and that after this capacity is reached it is illogical to assume the marginal value product of capital investments

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[1] Professor L. H. Brown is an extension specialist in agricultural economics at Michigan State University.

[2] R. V. Wagley, op. cit.

to be anything but zero or negative. Thus, it was thought the Wagley estimates might over-estimate the MVP's of machinery and forage-livestock investments and underestimate the MVP of relatively small amounts of labor.

These criticisms appeared justifiable as the Cobb-Douglas function does assume symmetry and constant elasticity which might force these undesirable restrictions on the fitted function in Wagley's study.

The current study shows that the shortcomings of constant elasticity and symmetry can be reduced by introducing a "ridge line" in factor-factor dimensions to correspond more closely to the actual production response. The ridge line can be visualized as a "crease" on the production surface which imposes limits on the substitutability between two inputs at a given level of output. The ridge line also can be considered as the loci of points where additional amounts of machinery and/or livestock used in conjunction with various fixed levels of labor result in zero marginal returns as a result of ever-present physical limitations.

Visualization of a new concept is an important step in research. However, conversion of a concept into mathematical or quantitative form is an equally important step for empirical application.

First, consider the general form of the standard Cobb-Douglas equation:

$$Y = aX_1^{b_1}, \ldots, X_i^{b_i}, \ldots, X_j^{b_j}, \ldots X_n^{b_n} \qquad (3.1)$$

Y is the dependent variable, the X_i's the independent variables, the b_i's the coefficients of elasticity of Y in respect to the X_i's, and the "a" is constant. To place the modification in proper perspective, the assumption is made that all input-output relationships possess the usual Cobb-Douglas form, except X_i and X_j for which empirical evidence shows there exists a limited range within which the proportions of X_i and X_j can be varied in producing a given output. The extreme limit of the range is the "ridge line" where additional X_j results in little change in output.

The mathematical procedure followed is to replace X_j by a dummy variable, Z_j, which is a function of both X_i and X_j. In algebraic terms $Z_j = f(X_i, X_j)$.

Modification I

The first modification defines the dummy variable as:

$$Z_j = PX_i(1-R^{X_j}) \qquad (3.2)$$

The P represents the ridge line proportion of X_j to X_i. R is the ratio of a decreasing geometric series, the terms of which are the respective increments in the dummy variable Z_j due to sucessive unit increases in X_j. X_i and X_j are the independent variables whose iso-product lines are under modification. For computational purposes, it is convenient to measure X_i and X_j in units such that $P = 1$. The dummy variable Z_j approaches PX_i, the total amount of X_j that can be associated with a given amount of X_i and still yield $MPP_{X_j}(Y) > 0$ as the input of the variable factor X_j becomes infinitely large.

Estimating the Parameters

Estimation of the "P".--It is difficult to estimate the "ridge line" proportion of X_j to X_i because other inputs vary which influence the substitutability of X_i and X_j, thus creating the problem of isolating the studied variables in order to determine estimates of "P." The estimate "P" is also affected considerably by the organizational set-up of the farms involved. A farm, for instance, with a well-arranged milking parlor can obviously handle a greater capital investment in livestock with a given supply of labor than a poorly organized farm. These disturbances must be taken into consideration by the researcher in evaluating past input-output studies to gain estimates of "P." In addition, consultation with farm management men provides reliable sources of such estimates.

Estimation of the "R."--The estimate of the constant "R" is limited to the area between zero and one so as to realize diminishing returns to the dummy variable Z_j. Between these limits, the value of the "R" depends on (1) the size of the units in which the X_j is measured, and (2) the nature of the input-output relationship between X_j and Y. A suggested method of determining the appropriate "R" is to make several reasonable estimates of "R" and then plot the Z_j function varying only X_j. The appropriate "R" will be determined by the curve which yields the closest approximation to the relationship shown in the data.

Some Relevant Characteristics of Modification I

Elasticity.--After modification I, the elasticity of the Y with respect to the input X_j is no longer constant, which is shown algebraically as follows:

The equation for elasticity is:

$$E = \frac{\partial Y}{\partial X_j} \cdot \frac{X_j}{Y} \qquad (3.3)$$

The modification I equation is written as:

$$Y = AX_1^{b1},\ldots,X_i^{bi}\left[PX_i(1-R^{X_j})\right]^{b_j},\ldots,X_n^{b_n} \qquad (3.4)$$

The partial derivative of Y with respect to X_j in (3.4) is:

$$\frac{\partial Y}{\partial X_j} = \frac{-b_j R^{X_j} \ln R \cdot E(Y)}{1-R^{X_j}} \qquad (3.5)$$

Substituting (3.5) in (3.3) the elasticity of Y with respect to X_j becomes:

$$E = \frac{-b_j \ln R \cdot R^{X_j} X_j}{1-R^{X_j}} \qquad (3.6)$$

As $K = -b_j \ln R$ (a constant), equation (3.6) simplifies to:

$$E = \frac{K \cdot R^{X_j} X_j}{1-R^{X_j}} \qquad (3.7)$$

Equation (3.6) shows that as X_j increases, the elasticity of Y with respect to X_j is no longer constant but increases in a constant ratio.

<u>Scale line of modification I.</u>--The scale line in the X_i and X_j plane of the modified function I is no longer a straight line. This is demonstrated algebraically by the following derivation of the scale line. The equation of a scale line in the X_iX_j dimension is:

$$\frac{MPP_{X_i}(Y)}{MPP_{X_j}(Y)} = \frac{PX_i}{PX_j} \qquad (3.8)$$

The marginal physical product (MPP_{X_i}) of X_i for equation (3.4) is:

$$\frac{\partial Y}{\partial X_i} = \frac{(b_i + b_j)\ E(Y)}{X_i} \qquad (3.9)$$

For the marginal physical product of X_j in equation (3.4), refer to (3.5). The scale line of the modified I equation (3.4) is derived by dividing equation (3.9) by (3.5) and setting the result equal to the price ratio as follows:

$$\frac{\dfrac{(b_i+b_j)\ E(Y)}{X_i}}{\dfrac{-b_j\ \ln R \cdot R^{X_j}\ E(Y)}{1-R^{X_j}}} = \frac{PX_i}{PX_j} \tag{3.10}$$

Solving (3.10) for X_i gives:

$$X_i = \frac{(b_i+b_j)\ PX_j \cdot (1-R^{X_j})}{PX_i\ (-b_j\ \ln R)\ R^{X_j}} \tag{3.11}$$

As $\dfrac{(b_i+b_j)\ PX_j}{(-b_j \cdot \ln R)PX_i} = K$, a constant, the final simplified scale line equation for (3.3) in the X_iX_j dimension becomes:

$$X_i = K \left[\frac{1-R^{X_j}}{R^{X_j}}\right] \tag{3.12}$$

Equation (3.12) shows that the slope of the scale line in the X_iX_j plane does not remain constant, but increases with each added increment of X_j.

<u>Symmetry</u>.--In Chapter II symmetry was defined as the phenomenon of the contour lines becoming asymptotic to the X_i and X_j axes. In modified form I, each contour line becomes asymptotic to the X_i axes and to X_i = a constant in the X_j dimension. This can be considered mathematically with the derivation of the contour line in the X_iX_j plane holding other inputs constant. Referring back to the modified form I (3.4), let Y = C (level of output) and P = 1. The result is:

$$C = AX_1^{b_1},\ldots,X_i^{b_i}\left[PX_i(1-R^{X_j})\right]^{b_j},\ldots,X_n^{b_n} \tag{3.13}$$

Solving (3.13) for X_i in terms of X_j yields:

$$X_i = \left[\frac{C}{aX_1^{b_1},\ldots,(1-R^{X_j})^{b_j},\ldots,X_n^{b_n}}\right]^{\frac{1}{b_i+b_j}} \tag{3.14}$$

In (3.14), as X_j increases, the denominator approaches a constant and X_i approaches a limit, which is shown in Figure 6. The equation for the line representing the limiting amount of X_i is:

$$X_i = \left[\frac{C}{aX_1^{b_1},\ldots,X_n^{b_n}}\right]^{\frac{1}{b_i+b_j}} \tag{3.14a}$$

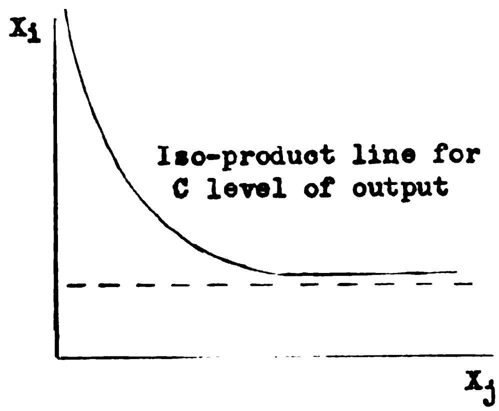

Figure 6
An Iso-product Line from Modification I
Equation in the X_iX_j Dimension

Disadvantages of Modification I

Difficulty in estimating parameters "P" and "R".--Estimating the "ridge line" proportion "P" is difficult, as stated, because organizations of farms vary so greatly within a given sample and this greatly affects the proportions that X_i to X_j can be varied. Estimating "R" is also difficult in spite of the fact that the range is narrowed to $0 < R < 1$ in order to realize diminishing marginal returns for the dummy variable.

Increasingly complex.--The complexities of deriving marginal value product estimates, iso-product lines, elasticities, and scale line

relationships, are increased as shown by equations (3.5), (3.8), (3.11) and (3.13).

Advantages of Modification I

Elasticity.--The elasticity of the X_j is no longer constant.

Scale lines.--The marginal rates of substitution between the X_i and X_j are not constant for successively higher output levels using the same proportions of X_i and X_j, and thus the scale lines are no longer straight.

Symmetry.--The characteristics of symmetry no longer exist to such a degree in the X_j plane which means that after the ridge line proportion is reached in varying X_j relative to fixed amount of X_i, $\frac{\partial Y}{\partial X_j}$ is no longer greater than zero.

Modification II

A second modification was investigated which in many respects is similar to modification I. It expresses mathematically the "ridge line" condition in the X_iX_j dimension as:

$$Z_j = PX_i \left(1 - R^{\frac{X_j}{X_i}}\right) \tag{3.15}$$

The Z_j, as in the first modification, represents a dummy variable which replaces X_j in equation (3.1). The "P" is the "ridge line" proportion of X_j to X_i. The R is the ratio of a decreasing geometric series, the terms of which are the respective increments in the dummy variable Z_j due to successive unit increases in X_j. The X_j and X_i for computational advantages are measured in units such that $P = 1$. The essential difference between this second modification and the first is that R, as shown in equation (3.15), is raised to a power which is the ratio of the relative amounts of X_j and X_i used. This results in the magnitude of the Z_j being more dependent on the variable X_i than in modification I. Thus, the elasticity of Y with respect to X_j increases as X_j is expanded but at a smaller constant ratio than modification I.

Estimating the Parameter for Modification II

Estimation of the "P".--The problems of estimating the parameter "P" are the same as discussed in conjunction with modification I.

Estimation of the "R".--As stated in modification I, the value of the "R" depends on (1) the size of the units in which X_j is measured, (2) the nature of the input-output relation between X_j and Y. In modification II the variable factor is measured as a ratio of the relative amounts of X_j and X_i, resulting in smaller units and thus a smaller value of "R".

Relevant Characteristics of Modification II

Elasticity.-- After modification II, the elasticity of the product with respect to the input X_j is no longer constant. This is shown mathematically as follows: The equation for elasticity is:

$$E = \frac{\partial Y}{\partial X_j} \cdot \frac{X_j}{Y} \qquad (3.16)$$

The modification II equation is:

$$Y = AX_1^{b_1}, \ldots, X_i^{b_i}, \ldots \left[PX_i(1-R^{\frac{X_j}{X_i}})\right]^{b_j}, \ldots X_n^{b_n} \qquad (3.17)$$

Taking the partial derivative of Y with respect to X_j in (3.17) yields:

$$\frac{\partial Y}{\partial X_j} = \frac{-b_j \cdot \ln R \cdot R^{\frac{X_j}{X_i}} E(Y)}{X_i \left(1-R^{\frac{X_j}{X_i}}\right)} \qquad (3.18)$$

Substituting (3.18) in (3.16), the elasticity of Y with respect to X_j becomes:

$$E = \frac{-b_j \cdot \ln R \cdot R^{\frac{X_j}{X_i}}}{1-R^{\frac{X_j}{X_i}}} \qquad (3.19)$$

As $K = b_j \ln R$ (constant), equation (3.19) simplifies to:

$$E = \frac{K \cdot R^{\frac{X_j}{X_i}}}{1-R^{\frac{X_j}{X_i}}} \qquad (3.20)$$

As the X_j increases in (3.20), holding X_i constant, the numerator tends to zero, the denominator tends to one, and the elasticity increases to infinity. Comparing the elasticity equations of modification I (3.7) and (3.20) for any given level of X_j, holding X_i and K the same, the elasticity of Y with respect to X_j will be greater in (3.7) than (3.20).

Symmetry.--In modification form II each contour line becomes asymptotic to the X_i axes and to X_i = a constant in the X_j dimension. Derivation of the contour lines in the X_iX_j space follows: Referring back to (3.17), let Y = C (constant level of output).

$$C = AX_1^{b_1}, \ldots, X_i^{b_i} \left[PX_i\left(1-R^{\frac{X_j}{X_i}}\right)\right]^{b_j}, \ldots, X_n^{b_n} \qquad (3.21)$$

If P = 1, the equation simplifies to:

$$1 - \left[\frac{C}{AX_1^{b_1},..X_i^{b_i+b_j}..,X_n^{b_n}}\right]^{\frac{1}{b_j}} = R^{\frac{X_j}{X_i}} \qquad (3.22)$$

Then, putting (3.22) in logarithmic form and solving for X_j yields:

$$\frac{\log\left[1-\left(\frac{C}{AX_1^{b_1},...,X_i^{b_i+b_j}..,X_n^{b_n}}\right)^{\frac{1}{b_j}}\right] X_i}{\log R} = X_j \qquad (3.23)$$

Equation (3.23) shows that $X_j \rightarrow 0$, when $X_i \rightarrow \infty$ and the X_i approaches a constant when $X_j \rightarrow \infty$, given all other inputs at a constant level other than zero.

Scale line.--The scale line for modification II in the X_iX_j space no longer has a constant slope as shown by the unmodified Cobb-Douglas. The scale line equation is represented by (3.8). Derivation of (3.8) in terms of the modified function II follows: The marginal product of X_i for equation (3.17) is:

$$\frac{\partial Y}{\partial X_i} = \frac{(b_i+b_j)\ E(Y)}{X_i} + \frac{b_j \ln R \cdot R^{\frac{X_j}{X_i}} X_j\ E(Y)}{X_i^2(1-R^{\frac{X_j}{X_i}})} \qquad (3.24)$$

For the marginal product of X_j refer back to equation (3.18). Dividing (3.24) by (3.18) and setting the results equal to the price ratio, PX_i/PX_j, defines the scale line in the X_iX_j space.

$$\frac{(b_i+b_j)(1-R^{\frac{X_j}{X_i}})}{b_j \ln R \cdot R^{\frac{X_j}{X_i}}} - \frac{1}{X_i} = \frac{PX_i}{PX_j} \qquad (3.25)$$

As $K = PX_j b_j \ln R$, $C = PX_i b_j \ln R$ and $A = PX_j(b_j+b_i)$, all constants, (3.25) simplifies to:

$$X_i = \frac{K\ R^{\frac{X_j}{X_i}}}{A(1-R^{\frac{X_j}{X_i}})+C\cdot R^{\frac{X_j}{X_i}}} \qquad (3.26)$$

This means that the marginal product of X_i decreases even more as the "ridge line" proportion of X_j to X_i is approached, and that the use of X_i is expanded proportionally more than X_j.

Advantages of Modification II

The advantages of modification II are much the same as with modification I. These are:

Elasticity.--The elasticity of the product with respect to X_j is no longer constant. However, as X_j increases, the elasticity of the modification I increases at a faster rate than the modification II equation. This may, or may not be an advantage, depending on the input-output relationship of the data under study.

Scale lines.--The marginal rates of substitution between X_j and X_i are no longer constant for successively higher output levels using the same proportions of X_i and X_j. This results in scale lines which do not have a constant slope. However, the slope of the scale line for the modified form II is slightly less at most points than for the modified form I because the elasticity of Y with respect to X_j is less.

Symmetry.--The contour lines in the X_iX_j space become asymptotic to the X_i axes and asymptotic to $X_i = C$, a constant, in the X_j space.

This partially eliminates the characteristic of symmetry present in the unmodified function.

Disadvantages of Modification II

Difficulty in estimating parameter "P" and "R".--The same difficulty exists in estimating the parameters as was discussed in modification I.

Complexity.--The complexity of estimating marginal value productivities, iso-product lines, and scale lines is increased over and above that for modification I as shown by equations (3.18), (3.20), and (3.26).

CHAPTER IV

STATISTICAL EVALUATION AND COMPARISON

It was shown in Chapter III that characteristics of symmetry and constant elasticity, possessed by the Cobb-Douglas equation, could have forced certain undesirable restrictions on Wagley's input-output estimates. A conceptual modification was introduced into the Cobb-Douglas function which can be visualized as a "ridge line" in the factor-factor dimension. This conceptual modification was formulated mathematically into what is referred to as modification I and II. In this chapter, the problem must be faced of which of the alternative functions, the original Cobb-Douglas, the modified Cobb-Douglas I or II best describes the data.

The procedure followed is to fit, by least squares using a logarithmic transformation,[1] the (1) unmodified Cobb-Douglas function, (2) modification I and (3) modification II, to a common set of data and evaluate the "goodness of fit" by various statistical measures.

Direct statistical tests are available to determine whether a significant reduction in variance is obtained when using the modified function compared to the original Cobb-Douglas, e.g., F-test on the sum

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[1] Cf. Gerhart Tintner, "A Note on the Derivation of Production Functions from Farm Records," op. cit., p. 27. Briefly, Tintner states that if the errors in the data are small and normally distributed, a logarithmic transformation of the variables will preserve the normality to a substantial degree. But even if the errors are not normally distributed and not independent, Tintner further states that we shall still get the best linear estimation by the application of the method of least squares. However, tests of significance are no longer valid.

of the squares of the residual quantities. Other methods to determine the best "fit" are in terms of such statistics as standard error of estimate, coefficient of determination, and standard error of regression coefficients. The standard errors of estimate indicate the closeness with which new estimated values may be expected to approximate the true but unknown values. The coefficient of determination measures the percentage of the variance in Y which is associated with $X_1, \ldots, X_n$. The standard error of the regression coefficients measures the accuracy of the estimated regression coefficients. These statistical measures have a precise and definite meaning when the assumptions of (1) normality and (2) randomness or at least independence with respect to the X_i's are met. The conditions to be met are:

$$E(u|X_i) = E(u) = 0 \qquad (4.1)$$

This states that the unexplained residual (u) given any value of any independent variable (X_i) is equal to the expected value of the unexplained residual (u) which is in turn equal to zero. The additional condition to insure normality and randomness or independence is:

$$\sigma_u \big| X_i = 1 = \sigma_u \qquad (4.2)$$

The σ_u is computed from a normal probability distribution. Equation (4.2) states that the standard error of the residuals given any value of any independent variable is equal to one if the residuals are randomly and normally distributed about the regression line. However, there is an inconsistency which exists when testing several functions fitted to the same data. These assumptions of normality and randomness or independence are made for each regression line. That is, the assumptions of

$E(u|X_i) = 0$ and $\sigma_u|X_i = 1$ cannot hold for an unmodified Cobb-Douglas fitted as well as for a modified Cobb-Douglas line fitted to the same data. Thus, if these assumptions are not met for both regression lines, and obviously they are not, statistical tests of comparison based on these assumptions become less meaningful and require more careful interpretation.

Evaluation of the Unmodified Cobb-Douglas Fit

The input-output data used in the evaluation were taken from a study of thirty-three purposively sampled Ingham County dairy farms located mainly on Miami, Hillsdale and Conover soils for the year 1952.[2] Purposive sampling is selecting farms that are not in scale line adjustment so as to (1) reduce the inter-correlation among input categories, (2) increase the variance of individual inputs categories and thereby decrease the standard error of the regression coefficients which increases their reliability.

In Wagley's study inputs were grouped into five categories with gross income (X_1) as follows:

- X_2, land, (acres)
- X_3, labor, (months)
- X_4, productive expenses, (dollars)
- X_5, livestock-forage investment, (dollars)
- X_6, machinery investment, (dollars)

The data were fitted by the Doolittle method of least squares[3]

Vance R. Wagley, op. cit., pp. 19-27.

[3] Mordecai Ezekiel, Methods of Correlation Analysis, 2nd Ed., (New York: John Wiley and Sons, Inc., 1949), pp. 455-485.

using the Cobb-Douglas equation which is linear in logarithmic form as follows: (4.3)

$$\log X_1 = \log a + b_2 \log X_2 + b_3 \log X_3 + b_4 \log X_4 + b_5 \log X_5 + b_6 \log X_6$$

The regression coefficients with their standard errors and the marginal value product at the geometric mean quantities are shown in Table I.

TABLE I

ESTIMATED REGRESSION COEFFICIENTS, STANDARD ERRORS, AND MARGINAL VALUE PRODUCTS AT THE GEOMETRIC MEANS, USING THE UNMODIFIED COBB-DOUGLAS FUNCTION, FOR SELECTED INGHAM COUNTY DAIRY FARMS, 1952

Input Category	Geometric[a] Mean Quantity	Regression Coefficients and Standard Error	Marginal Value Products
X_2, Land	130 Acres	.21107 ∓ .09868	$16.56
X_3, Labor	14 Months	.04166 ∓ .13083[b]	30.19
X_4, Expenses	$3,348	.25001 ∓ .11432	.76
X_5, Livestock-Forage Investment	$7,126	.44821 ∓ .08394	.64
X_6, Machinery Investment	$6,803	.12556 ∓ .10929[b]	.18

[a] Cf. F. E. Croxton and D. J. Cowden, Applied General Statistics, (New York: Prentice-Hall, Inc., 70 Fifth Avenue), p. 221. The geometric mean is defined as the Nth root of the product of N items which is written symbolically as:

$$\sqrt[N]{X_{i1} \cdot X_{i2} \cdot X_{i3}, \ldots, X_{iN}} \qquad (4.4)$$

The computation is usually carried out by means of logarithmic thus,

$$\log G = \frac{\log X_{i1} + \log X_{i2} + \log X_{i3} + , \ldots, + \log X_{iN}}{N} \qquad (4.5)$$

[b] Not significantly different from zero at 95 percent confidence by "t" test.

The coefficient of determination was found to be .92, indicating that ninety-two percent of the variance in the dependent variable gross income was associated with variations in the independent variables. The other eight percent may be associated with such unmeasurable factors as management and weather.

The standard error of estimate of the dependent variable ($\bar{S}$) was found to be .09028 which indicates that for sixty-seven out of one hundred farms randomly sampled from the same population, given 1952 conditions, gross income would be expected to fall between the fiducial limits of 8,287 and 12,570 dollars.

Modification I

The first modification was adapted to Wagley's data by the following procedure:

1. The input categories of land (X_2) and expenses (X_4) were used in the usual Cobb-Douglas form.

2. The input categories of livestock-forage (X_5) and machinery (X_6) were considered as capital investments whose earning power is limited, as pointed out in Chapter III, by the physical capacity of the labor (X_3). Therefore, in modification I, introduction of the "ridge line" on the production surface which imposes limits on the substitutability of X_5 for X_3, and X_6 for X_3, was accomplished by replacing X_5 and X_6 by the dummy variables Z_5 and Z_6. The dummy variables are defined as:

$$Z_5 = P_1 X_3 \left(1 - R^{X_5}\right) \qquad (4.6)$$

$$Z_6 = P_2X_3\ (1-R^{X_6}) \qquad (4.7)$$

In estimating the "P's", the problem was to determine:

1. The "ridge line" proportion of dollars invested in livestock-forage to labor.

2. The "ridge line" proportion of dollars invested in machinery to labor.

The following P's were secured in consultation with Professor L. H. Brown[4] who based his estimates on the observed operating ratios of capital to labor for numerous Michigan farms.

1. Thirteen hundred dollars of livestock-forage investment per month of labor.

2. One thousand dollars of machinery investment per month of labor.

In order to make the Z_5 and Z_6 easily computable, the livestock-forage investment was measured in thirteen hundred dollar units and the machinery investment in one thousand dollar units. This made $P = 1$ in Z_5 and Z_6.

The constant R was made equal to nine-tenths for both the Z_5 and Z_6 after considering the graphical relationship between X_5 and Z_5 and X_6 and Z_6 using different values of R. No statistical tests were applied.

The modified equation I is linear in logarithmic form and was fitted by the Doolittle method of least squares.[5]

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[4] Professor L. H. Brown is an extension specialist in Agricultural Economics, Michigan State University.

[5] Ezekiel, op. cit., pp. 455-485.

$$\log X_1 = \log a + b_2 \log X_2 + b_3 \log X_3 + b_4 \log X_4 + b_5 \log P_1X_3(1-R^{X_5}) + b_6 \log P_2X_3(1-R^{X_6}) \quad (4.8)$$

In modification I, the regression coefficients and the standard errors were computed. These are compared with the regression coefficients of the standard Cobb-Douglas fit in Table II.

A comparison of the elasticities for the input categories of land (X_2), labor (X_3)[6] and expenses (X_4) for the modified equation I with the standard Cobb-Douglas shows no significant difference at the 95 percent level. However, a comparison of the regression coefficients b_5 and b_6 would be meaningless because in the modified function the b_i's represent the elasticities of the dummy variables Z_5 and Z_6. A more meaningful comparison between the "fits" would be to compare the estimated marginal value products as shown in Table III.

Table III shows no significant difference between the MVP of land for the modification I and the unmodified function. However, the marginal value product of expenses for the modified form is greater than for the standard Cobb-Douglas estimate. A normal return on expenses is a dollar for a dollar.

The estimated MVP of labor for all quantities of labor is greater for the modified form I than the standard Cobb-Douglas function as shown in Figure 7 (Cf. modification I-b and II, discussions on labor).

The marginal value product of livestock-forage, at the geometric mean amount, for the modified function is greater than the unmodified

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6 See footnote b, Table II.

TABLE II

ESTIMATED REGRESSION COEFFICIENTS AND STANDARD ERRORS USING THE MODIFICATION I AND THE STANDARD COBB-DOUGLAS FIT FOR SELECTED INGHAM COUNTY DAIRY FARMS, 1952

Input	Geometric Mean Quantities	Regression[a] Coefficients for Modification I	Regression Coefficients for Standard Cobb-Douglas
Land X_2	130 Acres	.20321 ∓ .11416	.21105 ∓ .09868
Labor X_3	14 Months	-.52093 ∓ .19956[b]	.04166 ∓ .13083
Expenses X_4	\$3,348	.36461 ∓ .12425	.25001 ∓ .11432
Livestock-Forage X_5	\$7,126	.66225 ∓ .10855	.44821 ∓ .08394
Machinery X_6	\$6,803	-.08649 ∓ .07545[c]	.12556 ∓ .10929

[a] The regression coefficient for the livestock-forage and machinery indicate the elasticities of X_1 with respect to the dummy variables Z_5 and Z_6.

[b] The negative regression coefficient b_3 does not indicate a negative elasticity as shown by the following proof. According to definition the equation for the elasticity of X_1 with respect to X_3 is:

$$\frac{\partial X_1}{\partial X_3} \cdot \frac{X_3}{X_1} \qquad (4.9)$$

The partial derivative of the modified equation with respect to X_3 is:

$$\frac{\partial X_1}{\partial X_3} = \frac{(b_3+b_5+b_6)\ E(X_1)}{X_3} \qquad (4.10)$$

Substituting (4.10) back into (4.9) and simplifying yields:

$$E_{X_3} = b_3+b_5+b_6 \qquad (4.11)$$

Then, substituting the values computed into the equation reveals that the elasticity of X_1 with respect to X_3 is:

$$E_{X_3} = -.52093+.66225+(-.08649) = .05483 \qquad (4.12)$$

[c] The "t" test indicated that b_6 was not significantly different from zero at the 95 percent level.

TABLE III

ESTIMATED MARGINAL VALUE PRODUCTS COMPUTED FROM STANDARD COBB-DOUGLAS AND MODIFICATION I AT THE GEOMETRIC MEAN FOR SELECTED INGHAM COUNTY DAIRY FARMS, 1952

Input Categories	Standard Cobb-Douglas[a]	Modification I[b]
Land, (X_2)	16.56	15.38
Labor, (X_3)	30.19	39.26
Expenses, (X_4)	.76	1.10
Livestock-Forage, (X_5)	.64	.79
Machinery, (X_6)	.18	-.10[c]

[a] The marginal value products for the standard Cobb-Douglas is computed from the following form:

$$MVP_{X_i} = \frac{b_i \; E(X_1)}{X_i} \qquad (4.13)$$

[b] When using the modified equation (4.8), the problem of computing the MVP's becomes slightly more complex. The general procedure is to take a partial derivative of the dependent variable in respect to the independent variable.

$$\frac{\partial X_1}{\partial X_3} = \frac{(b_3+b_5+b_6) \; E(X_1)}{X_3} \qquad (4.14)$$

$$\frac{\partial X_1}{\partial X_5} = \frac{-b_5 R^{X_5} \ln R \cdot E(X_1)}{(1-R^{X_5})} \qquad (4.15)$$

$$\frac{\partial X_1}{\partial X_6} = \frac{-b_6 R^{X_6} \ln R \cdot E(X_1)}{(1-R^{X_6})} \qquad (5.16)$$

The method of deriving the MVP's for the X_2 and X_4 is the same as for standard Cobb-Douglas and is shown in footnote above.

[c] The "t" test indicated no significant difference from zero at 95 percent level.

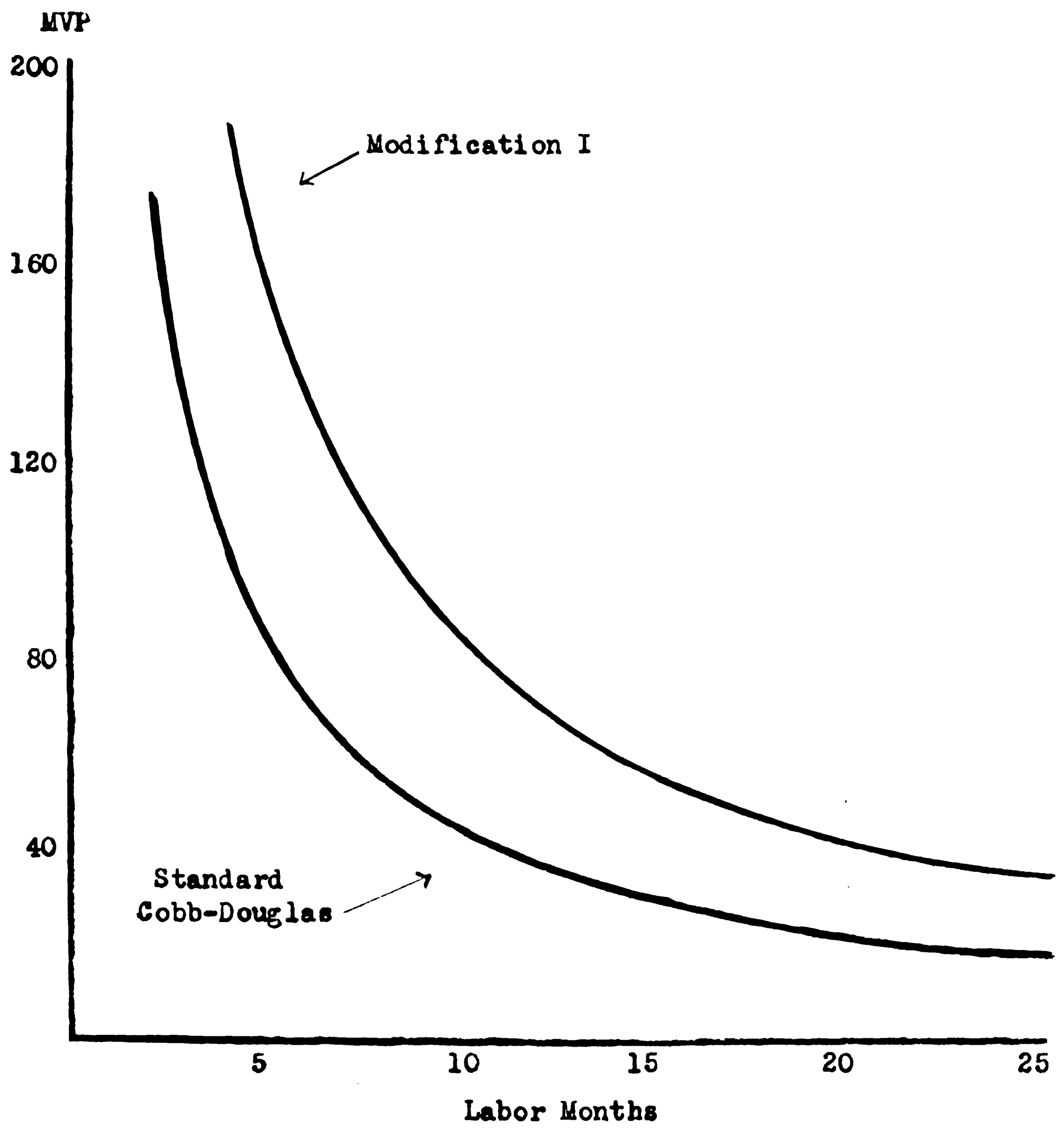

Figure 7
The Marginal Value Productivity of Labor, Modification I
Compared with the Standard Cobb-Douglas,
on Selected Ingham County Farms, 1952

Cobb-Douglas. However, this situation does not hold true when the amount of livestock-forage investment exceeds the "ridge line proportion." That is at the "ridge line" proportion of eighteen thousand dollars, (labor fixed at fourteen months and thirteen hundred dollars the estimated maximum amount of livestock-forage investment per month of labor) the MVP of livestock-forage in the unmodified Cobb-Douglas exceeds the MVP for the modified form I as shown in Figure 8.

The negative marginal value product of machinery for the modified function does not appear too meaningful in view of the author's knowledge of the sample farms.

Assuming the normality of the residuals, the standard error of estimate ($\bar{S}$) for the modification I was computed and found to be .09137. This indicates that of sixty-seven out of one hundred farms randomly sampled, from the same population conditions, gross income would be expected to fall between 8,267 and 12,590 dollars. The ($\bar{S}$) of the unrevised Cobb-Douglas fit is .09028 indicating that sixty-seven percent of farms randomly sampled from the same population, given 1952 conditions, would expect to have a gross income between 8,287 and 12,560 dollars. It is apparent that there is no significant difference between the standard error of estimates of the modified form I and the standard Cobb-Douglas, at the ninety-five percent level.

The coefficient of determination ($\bar{R}^2$) was computed to be .88 for the modified equation I which indicates that eighty-eight percent of the variance in gross income (X_1) is associated with variations in the independent variables. The coefficient of determination for the standard

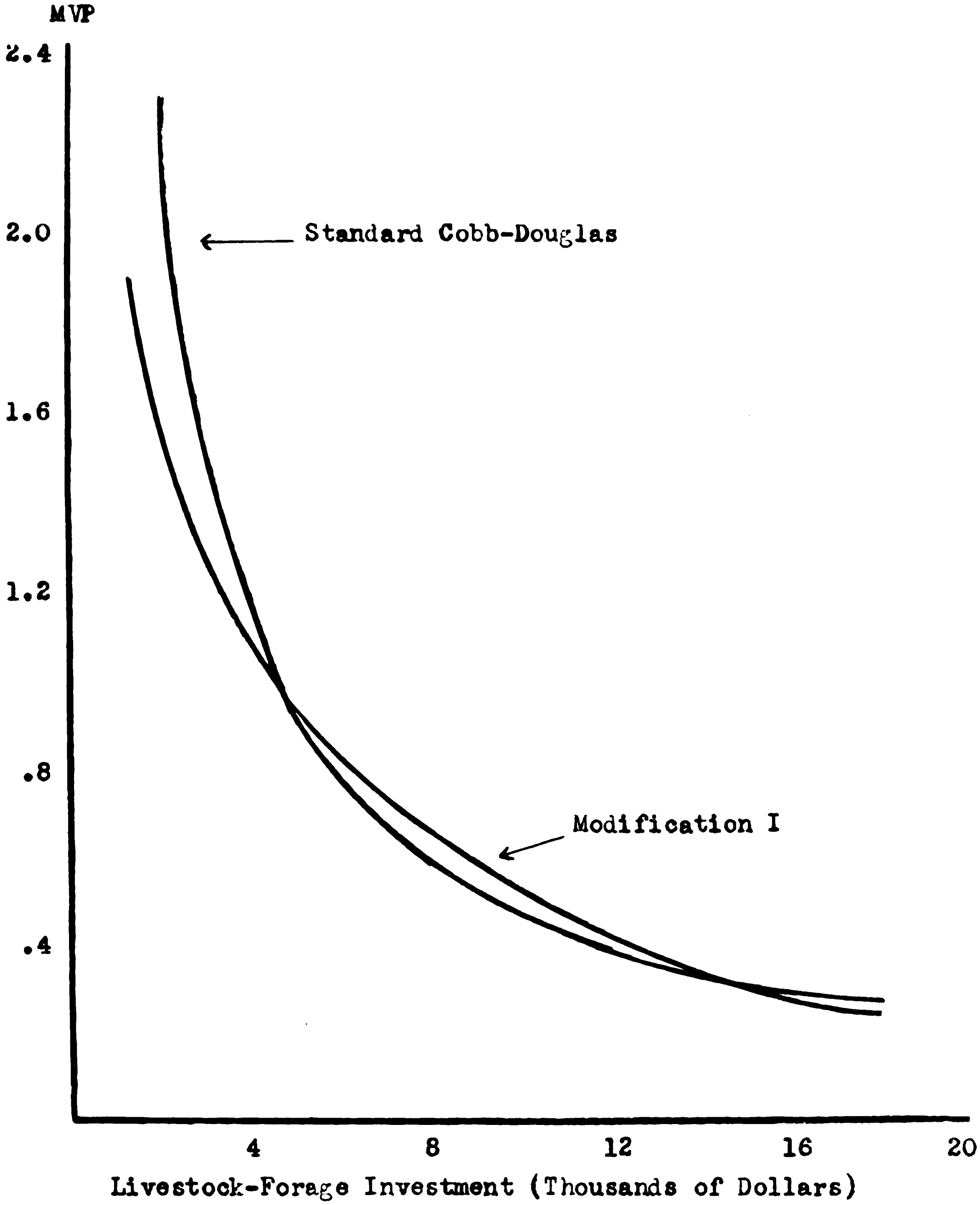

Figure 8
The Marginal Value Productivity of Livestock-Forage, Modification I Compared with the Standard Cobb-Douglas, on Selected Ingham County Farms, 1952

Cobb-Douglas was computed to be .92 and there appears to be no significant difference between these estimates.

Another method used to evaluate "goodness of fit" among the functions was to apply the F-test[7] on the variances of the computed residuals, i.e. the difference between the estimated gross income and the actual gross income. The standard error of the residual quantities for the modification I was found to be 1,777 dollars compared to 2,490 dollars for the standard Cobb-Douglas estimate. The reduction in the variance for the modification I proved to be significant at the ninety-five percent level.

The final conclusions that the "fit" of the modified function I is not superior to the standard Cobb-Douglas, in spite of the fact that there is a significant reduction in variance of the residuals, was drawn because:

1. It is meaningless to assume a negative or zero marginal value product for machinery in view of the farms studied.

2. The standard error of estimate and coefficient of determination between modification I and the unmodified function were not significantly different, at the ninety-five percent level.

Evaluation of Modification 1-b

An iteration of the first modification was made, which is referred to as modification 1-b, because only a few of the surveyed farms

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[7] George W. Snedecor, Statistical Methods, 4th Ed., (Ames: Iowa State College Press, 1946), pp.380.

had more than thirteen hundred dollars invested in livestock-forage per month of labor and/or one thousand dollars invested in machinery per month of labor.

For modification I-b a new ridge line proportion "P" was estimated of (1) seven hundred dollars investment in livestock-forage per month of labor, and (2) seven hundred dollars investment in machinery per month of labor. These new proportions included a larger share of the sample farms whose capital-labor ratio exceeded the estimated "ridge line" proportion as shown in Figures 9 and 10. Thus, it was expected that the estimated MVP of livestock-forage and machinery would decrease and the MVP of labor increase.

For computational simplicity the machinery and livestock investments were measured in seven hundred dollar units. Thus the P in both dummy variables, was made equal to one.

$$Z_5 = P_1 X_3 \,(1 - R^{X_5}) \qquad (4.17)$$

$$Z_6 = P_2 X_3 \,(1 - R^{X_6}) \qquad (4.18)$$

The value of the constant R was left at nine-tenths. With these changes the modification I-b was fitted in logarithmic form by the Doolittle method of least squares.[8] The regression coefficients were computed along with the standard errors and found to be:

$$b_2 = .12602 \mp .11335$$

$$b_3 = -.82817 \mp .26248$$

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[8] Ezekiel, op. cit., pp. 455-485

[9] The negative b_3 for the modification I-b does not indicate a negative elasticity with respect to labor (X_3) as shown in footnote b, Table II. The elasticity of gross income with respects to X_3 would be equal to the sum of b_3, b_5, and b_6 which is .12047.

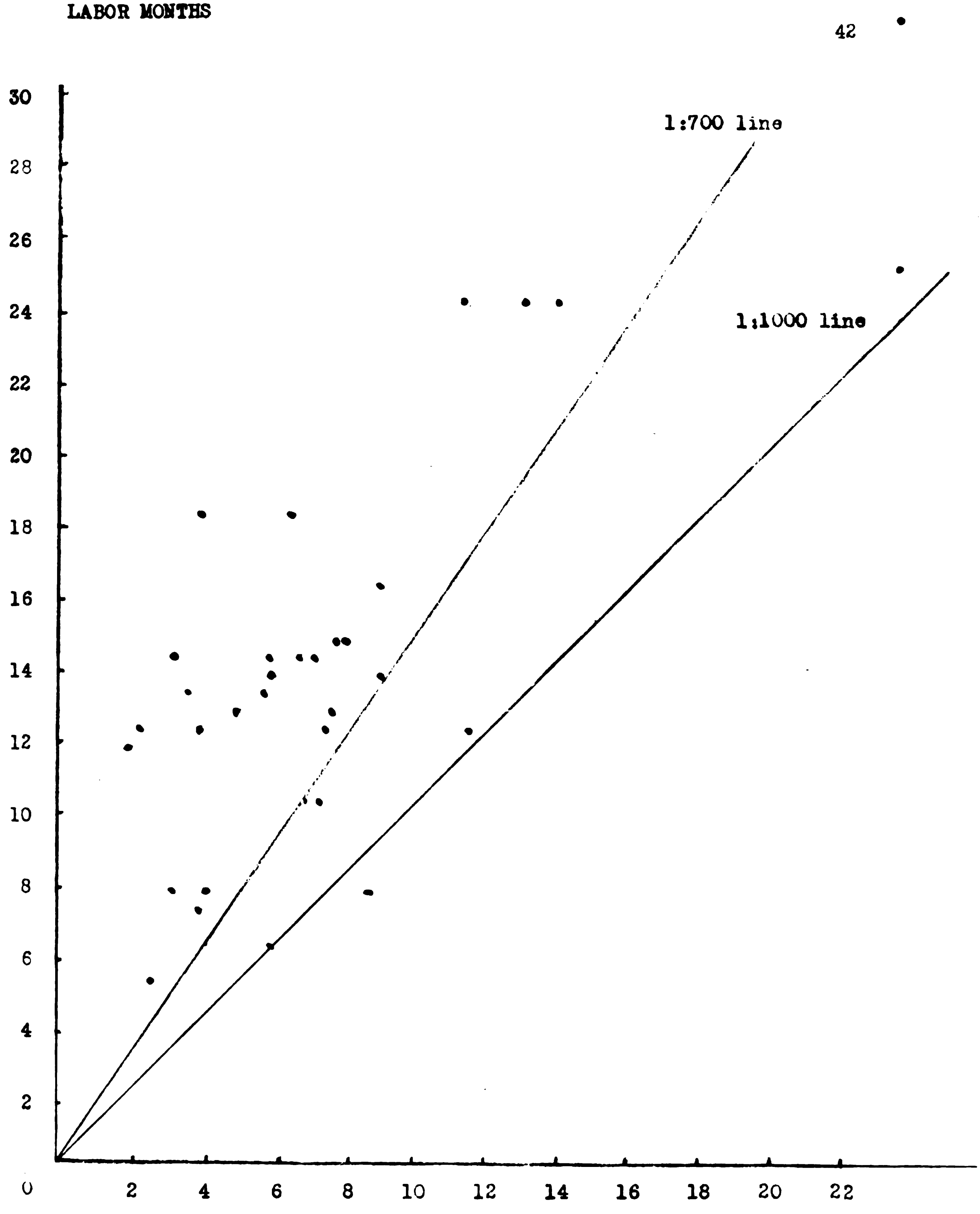

Figure 9
Graph Showing Range of Data for
Labor and Machinery

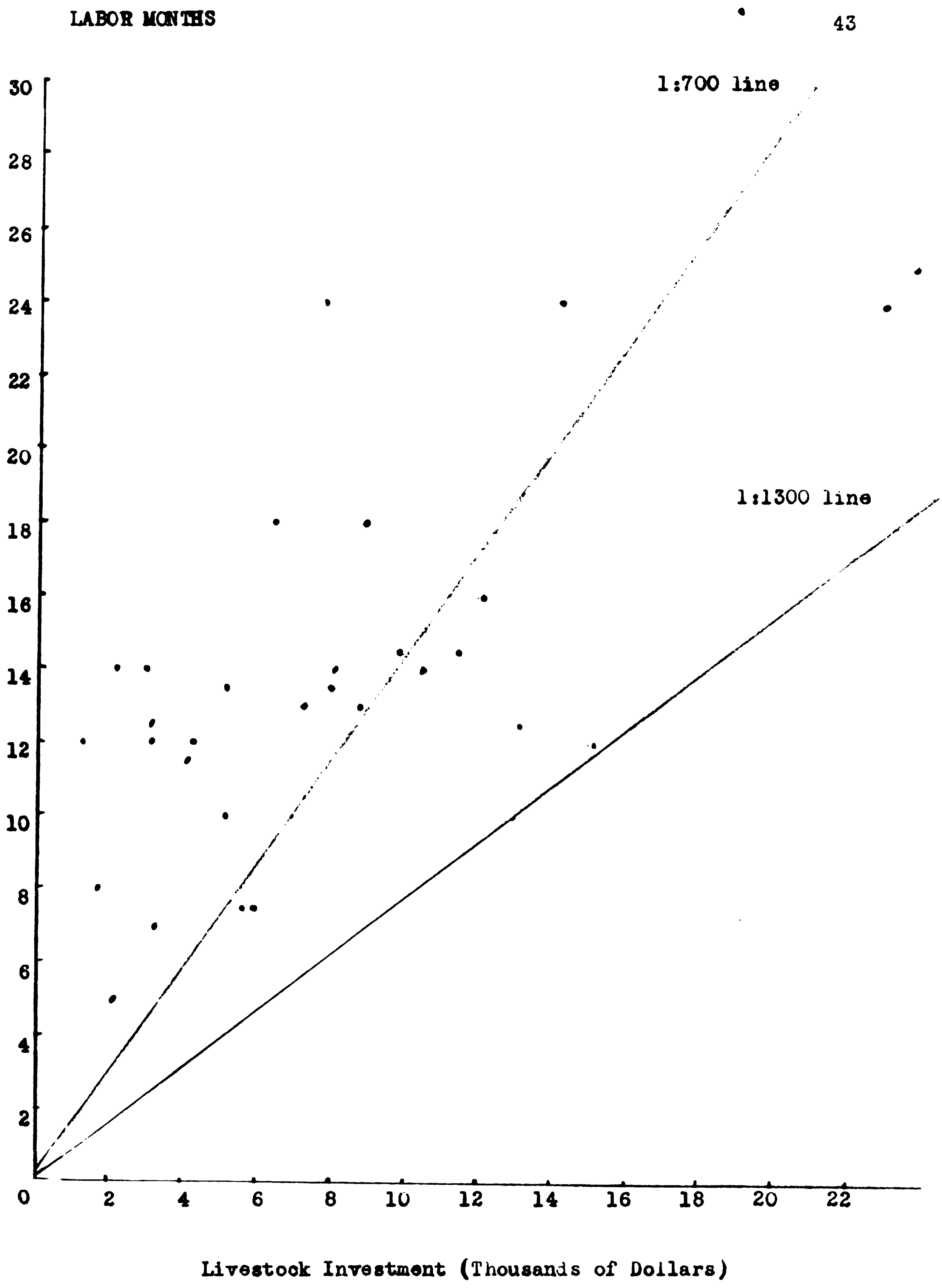

Figure 10
Graph Showing Range of Data for
Labor and Livestock

$$b_4 = .38438 \mp .12531$$

$$b_5 = .64612 \mp .13286$$

$$b_6 = .30252 \mp .17344$$

The constant "a" was computed to be 2.45972.

With these coefficients, the marginal value products were estimated and compared with the Cobb-Douglas marginal value products in Table IV.

The estimated marginal value products of livestock-forage for the modified equation I-b and the unrevised Cobb-Douglas are shown in Figure 11. Figure 11 indicates that at very low rates of investment of livestock-forage, the modified function I-b yields a higher marginal value product to this investment than the unmodified Cobb-Douglas. However, at higher rates of investment, which exceed the "ridge line" proportion, the marginal value product of the modified function I-b is less than the unmodified Cobb-Douglas estimates.

In Figure 12, the same type of comparison is made between the marginal value products of machinery investment, holding other inputs constant at the geometric mean. Figure 12 indicates that at low rates of investment of machinery, the modified function I-b yields a higher MVP than the standard Cobb-Douglas estimates. However as the investment of machinery exceeds the "ridge line" proportion, the MVP of the modified functions decreases at a faster rate than the modified function.

The estimated MVP of labor for the modified function I-b and the unrevised Cobb-Douglas are shown in Figure 13. Figure 13 indicates

TABLE IV

ESTIMATED MARGINAL VALUE PRODUCTS USING THE MODIFICATION I-b AND THE STANDARD COBB-DOUGLAS AT THE GEOMETRIC MEAN QUANTITIES FOR SELECTED INGHAM COUNTY FARMS, 1952

Input Category and Unit	Geometric Mean Quantities	Marginal Value[a] Product--Standard Cobb-Douglas	Marginal Value[b] Product--Modification I-b
Land, (X_2)	130 Acres	16.56	9.54
Labor, (X_3)	14 Months	30.19	86.27
Expenses, (X_4)	3,348	.76	1.16
Livestock-Forage, (X_5)	7,126	.64	.50
Machinery, (X_6)	6,803	.18	.23

[a] For method of derivation refer back to footnote a, Table III.

[b] For method of derivation refer back to footnote b, Table III.

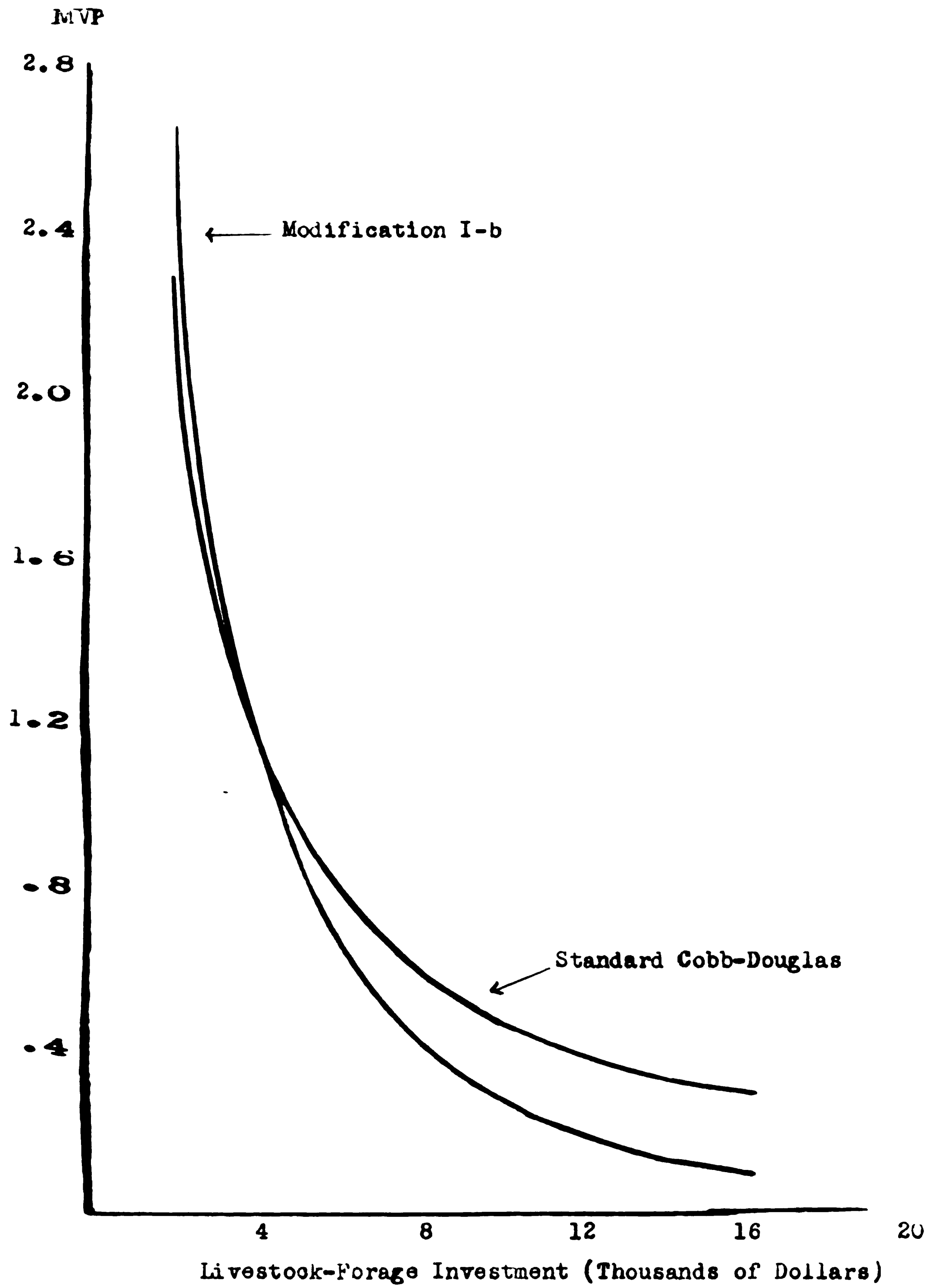

Figure 11
The Marginal Value Productivity of Livestock-Forage, Modification I-b Compared with Standard Cobb-Douglas, on Selected Ingham County Farms, 1952

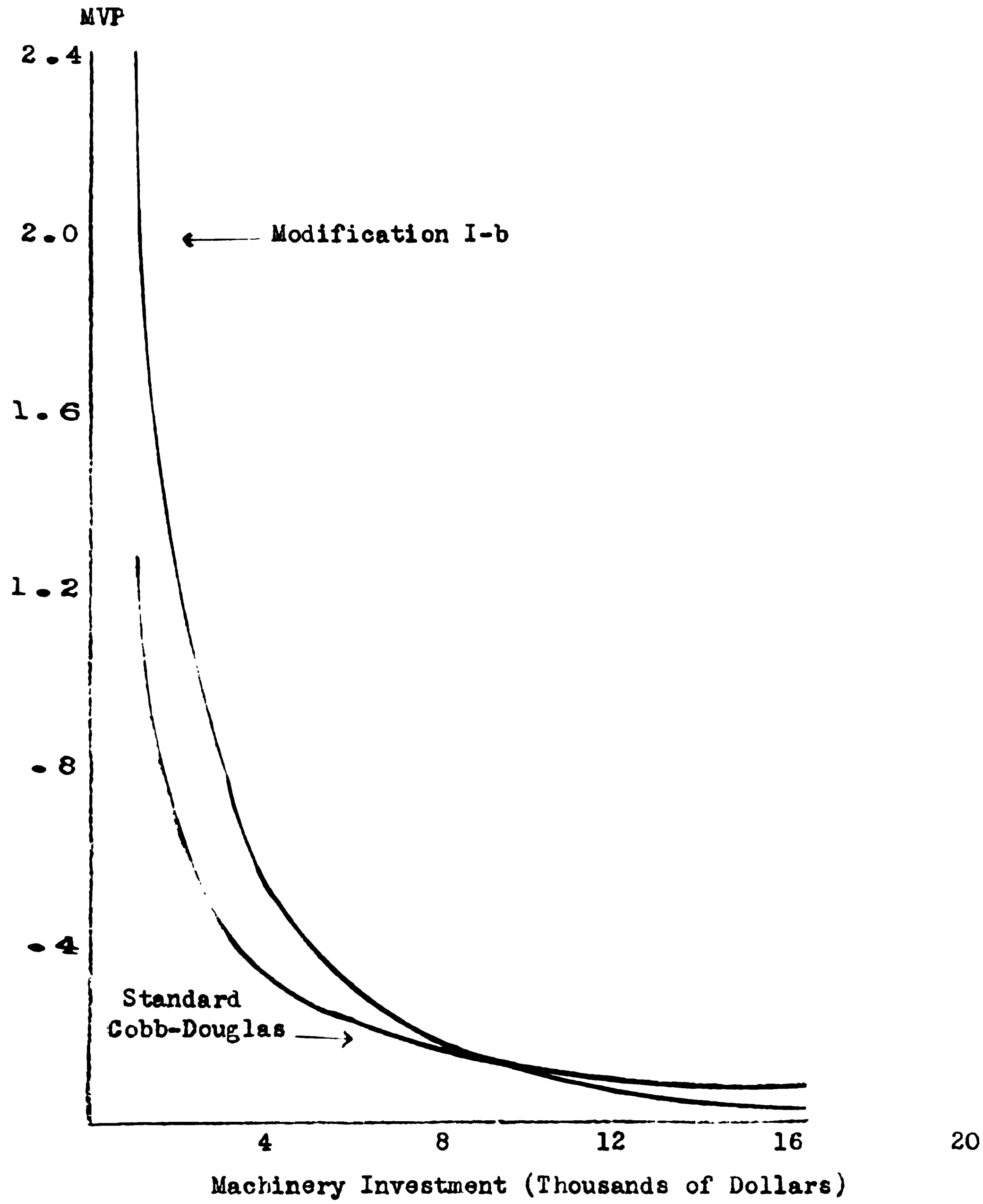

Figure 12
The Marginal Value Productivity of Machinery, Modification I-b Compared with the Standard Cobb-Douglas, on Selected Ingham County Farms, 1952

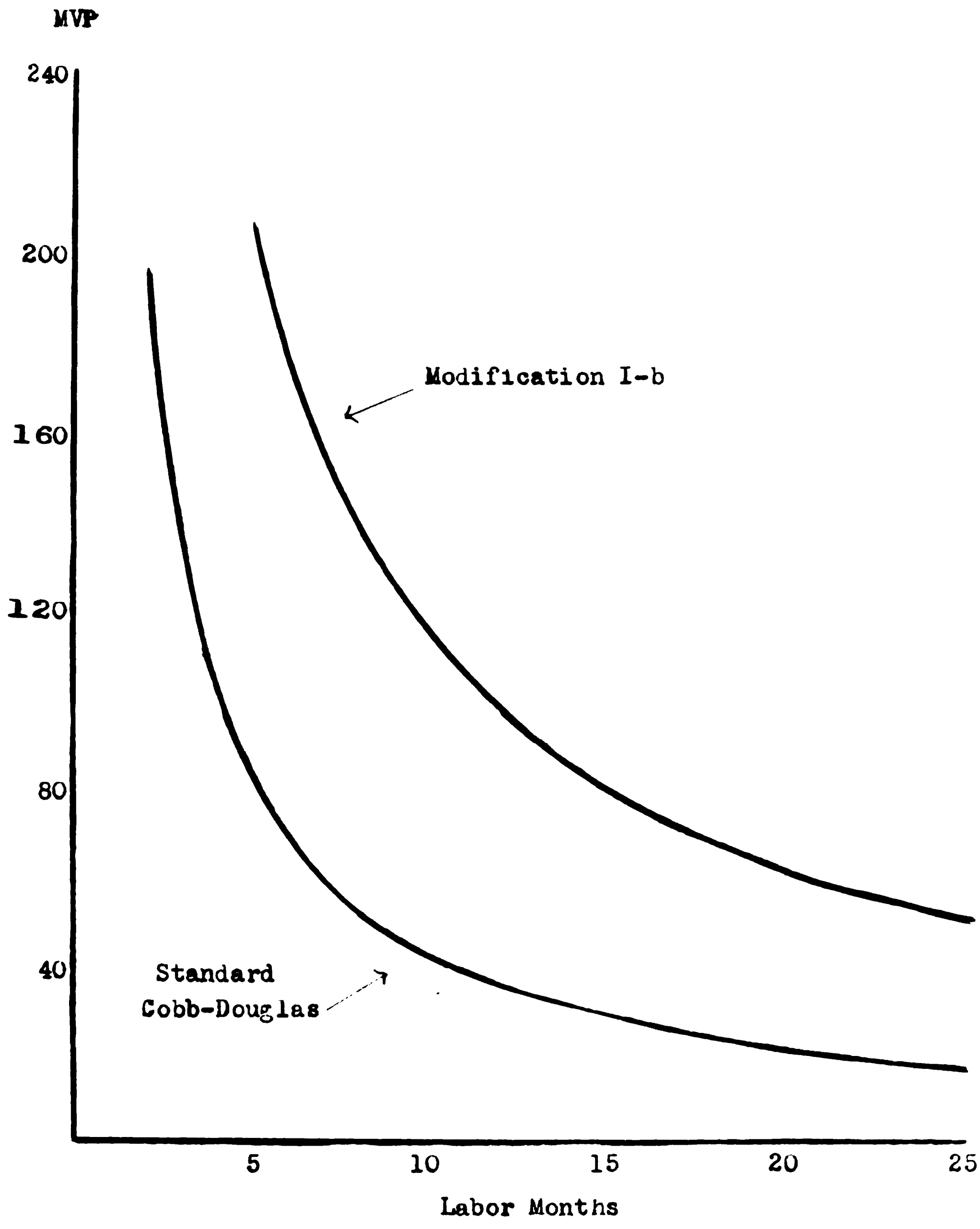

Figure 13
The Marginal Value Productivity of Labor, Modification I-b
Compared with the Standard Cobb-Douglas,
on Selected Ingham County Farms,
1952

that the MVP of labor for the modified function I-b is considerably higher, for all quantities of labor, than the unmodified Cobb-Douglas estimate.

These results, which show high relative MVP's for labor and low relative MVP's for livestock-forage and machinery for the modified function I-b, can be related back to Brown's original criticism of certain estimates in Wagley's Cobb-Douglas study. These criticisms were concerned with estimates that showed relatively low earning power for labor and relatively high earning power for livestock-forage and machinery when in fact the capital-labor operating ratio may have exceeded the physical capabilities of the operator. These estimates of modification I-b indicate that the Cobb-Douglas function can be altered to reflect more closely these types of relationships. However, the earning powers reflected by the modified function I-b do not give a significantly better statistical fit than the unmodified Cobb-Douglas for these particular data.

Using the F-test on the sum of the squares of the residuals to evaluate the "goodness of fit," the reduction in the sum of the squares for the modification I-b compared to the unmodified function did not prove to be significant at the ninety-five percent level.[10] The standard error of the computed residuals for the modification I was found to be 1,981 dollars compared to 2,490 dollars for the standard Cobb-Douglas, and 1,777 dollars for the modification I estimate.

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[10] Snedecor, op. cit., p. 380.

When applying the F-test on the residual variance quantities of the farms which had more than seven hundred dollars investment in machinery per month of labor and/or more than seven hundred dollars of livestock-forage per month of labor,[11] the reduction in the sum of the squares for the modification I-b as compared to the unmodified function did not prove to be significant at the ninety-five percent level.

The standard error of estimate ($\bar{S}$) about the regression line for the modified equation I-b was found to be .09468 in logarithms as compared to .09028 for the standard Cobb-Douglas, and there was no significant difference between these estimates at the ninety-five percent level. The ($\bar{S}$) for modification I was found to be .09137 which was not significantly different from ($\bar{S}$) of I-b, at the ninety-five percent level.

The coefficient of determination ($\bar{R}^2$) was .89 for the modified function I-b compared to .92 for standard Cobb-Douglas. This was not significantly different at the ninety-five percent level. The coefficient of determination for modification I was .88 which was not significantly different from the modification I-b estimate.

The conclusion that the statistical fit of modification I-b is not superior to the standard Cobb-Douglas was drawn because:

(1) the standard error of estimates ($\bar{S}$) were not significantly different, (2) the reduction in the variance of the residuals from all the farms using the modified equation I-b was slight, but not significant, and (3) the reduction in variances, of the farms whose capital

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[11] Refer back to Figures 9 and 10.

investments in livestock-forage and/or machinery exceeded the estimated "ridge line" proportion, was not significant.

Conclusions were drawn concerning the comparative fits of modification I-b and modification I as follows:

1. The standard error of estimate at the geometric mean showed no significant difference between the two fits.

2. The variance of the computed residual quantities was smaller, but not significantly, for modification I as compared to modification I-b.

3. The computed coefficients of determination were not significantly different between the fits.

Evaluation of Modification II

The same input-output data were fitted, by the Doolittle method of least squares,[12] using modification II which is linear in logarithmic form, as follows:

$$\log X_1 = \log a + b_2 \log X_2 + b_3 \log X_3 + b_4 \log X_4 + b_5 \log P_1 X_3 (1 - R^{\frac{X_5}{X_3}}) + b_6 \log P_2 X_3 (1 - R^{\frac{X_5}{X_3}}) \quad (4.16)$$

The estimates of the "ridge line" proportion of machinery investment to labor months, and livestock-forage investment to labor months, were identical to those in modification I-b. These estimates of "P" were seven hundred dollars machinery investment per month of labor and seven hundred dollars livestock-forage investment per month of labor.

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[12] Ezekiel, op. cit., pp. 405-485.

These "ridge line" proportions were used in preference to the modification I "P's" because they appeared more reasonable in view of the operating ratio of capital to labor for farms in the study. For computational purposes, the machinery and livestock-forage investments are measured in seven hundred dollar units making the "P's" equal to one. The constant R was made equal to three-tenths. This is smaller than previous estimates of R because the variable factor, capital, is measured in units relative to labor months and not in absolute terms (Cf. 4.16).

The marginal value products for the modified equation II and the unmodified Cobb-Douglas were computed at the geometric mean organization and are shown in Table V.

The marginal value products at the geometric mean for modification II and the unmodified function do not appear to be significantly different for land, machinery, and livestock-forage respectively. A comparison between the MVPs, of the modified equation II and the standard Cobb-Douglas, with varying amounts of (1) livestock-forage, (2) machinery investment, and (3) labor months shows relationships as represented in Figures 14, 15, and 16.

Figure 14 shows (1) the marginal value product for machinery investment with the modified equation II to be slightly greater than the Cobb-Douglas when using amounts of machinery investment less than the "ridge line" proportion and (2) that the marginal value product tends to decrease more than the unmodified Cobb-Douglas estimate when using amounts greater than the ridge line proportion.

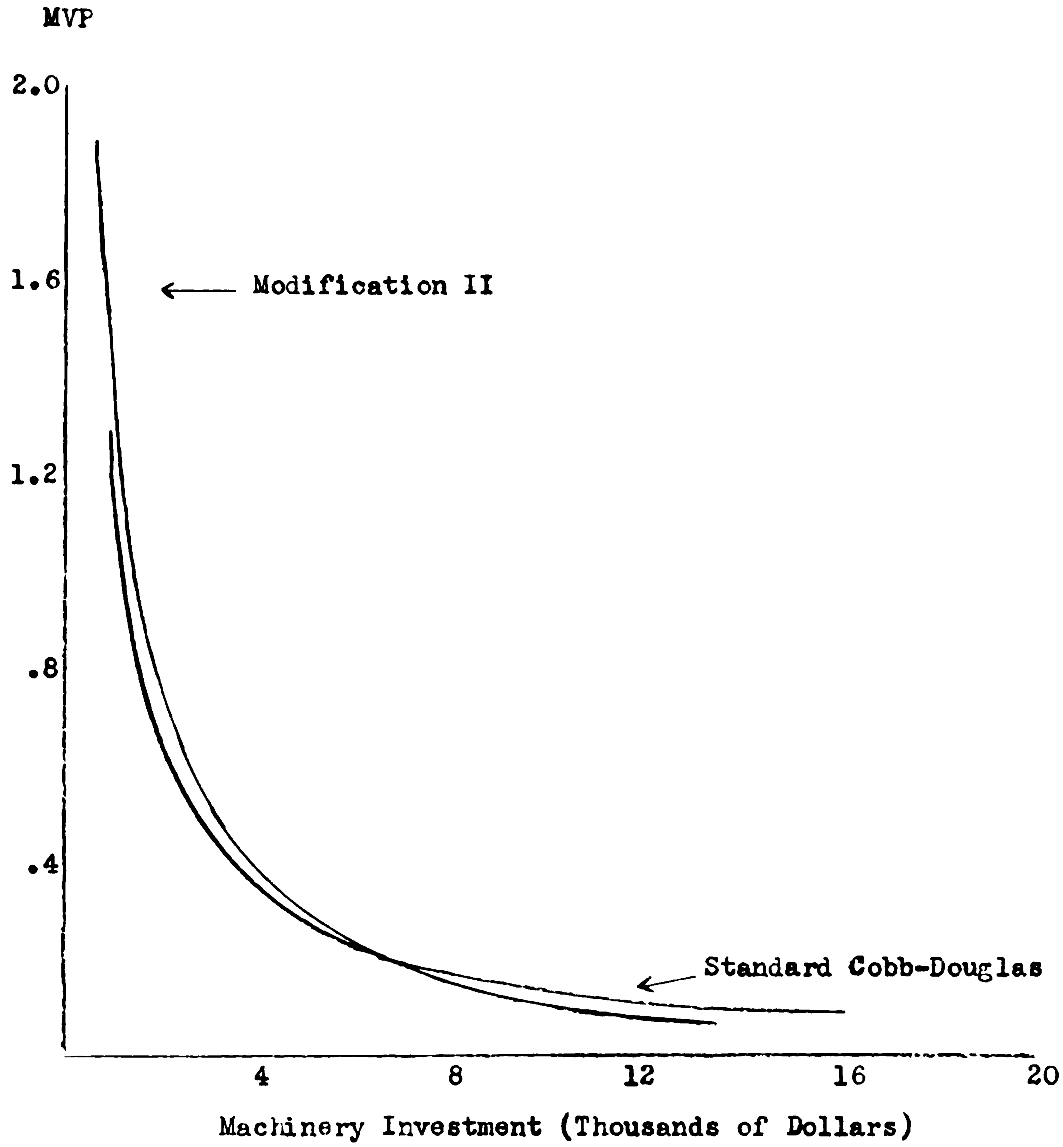

Figure 14
The Marginal Value Productivity of Machinery, Modification II Compared with the Standard Cobb-Douglas, on Selected Ingham County Farms, 1952

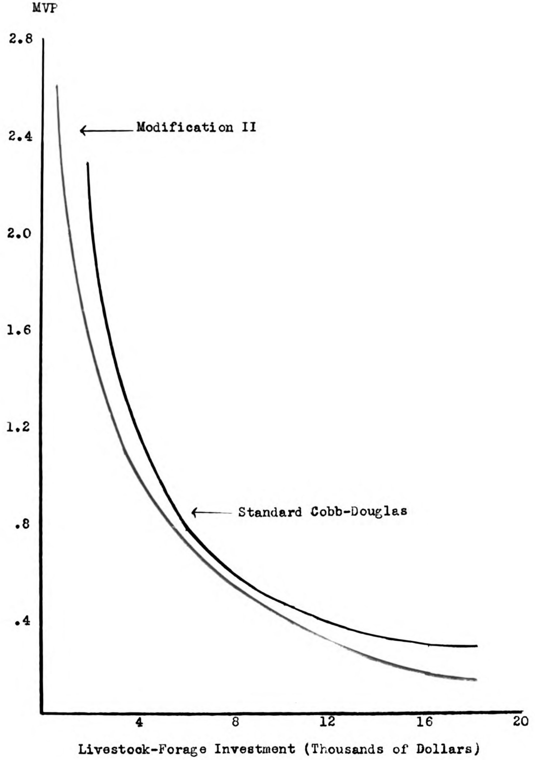

Figure 15
The Marginal Value Productivity of Livestock-Forage, Modification II Compared with the Standard Cobb-Douglas on Selected Ingham County Farms, 1952

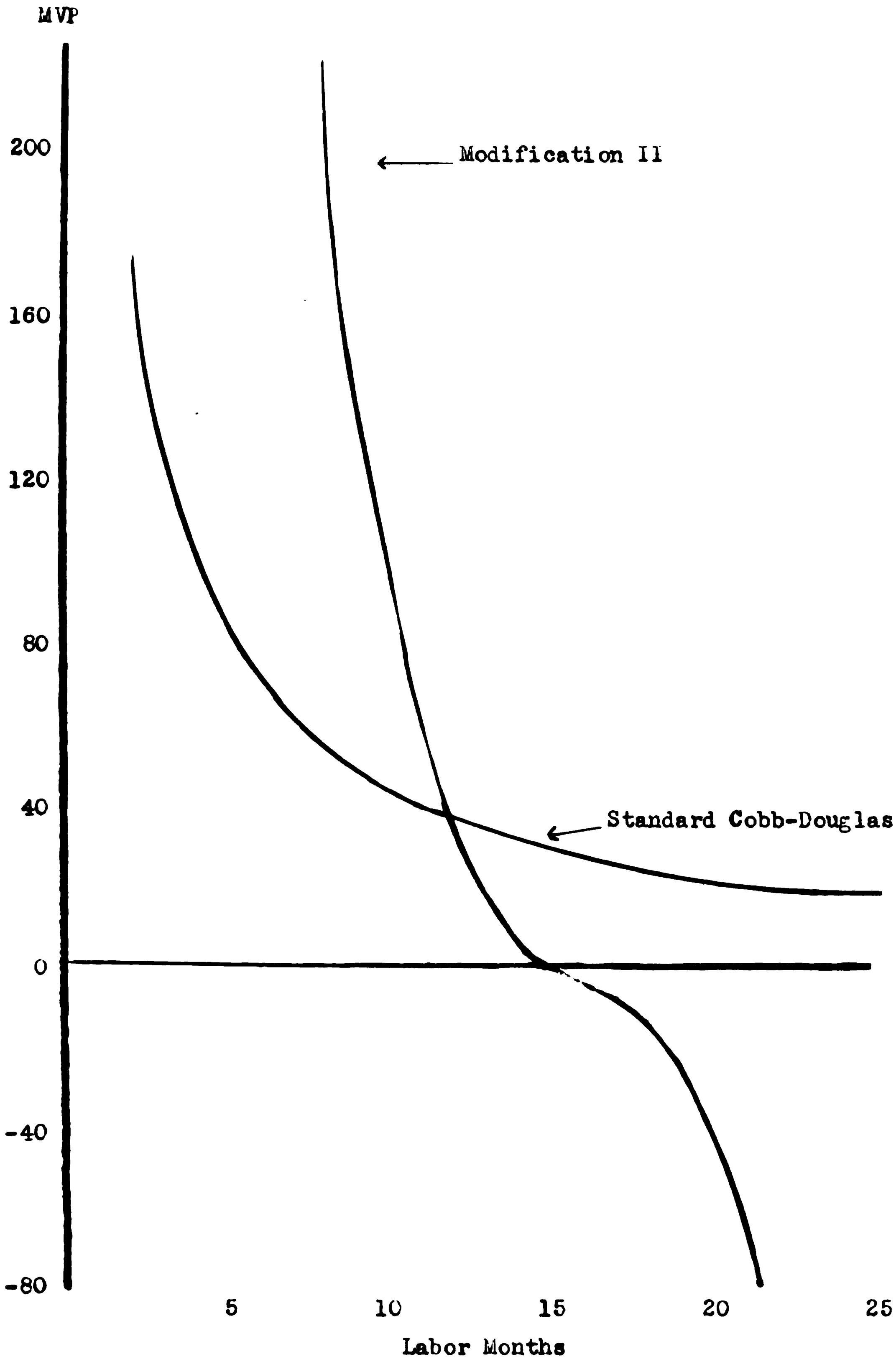

Figure 16
The Marginal Value Productivity of Labor, Modification II Compared with the Standard Cobb-Douglas, on Selected Ingham County Farms, 1952

TABLE V

ESTIMATED MARGINAL VALUE PRODUCTS USING THE MODIFICATION II AND THE STANDARD COBB-DOUGLAS AT THE GEOMETRIC MEAN QUANTITIES FOR SELECTED INGHAM COUNTY FARMS, 1952

Input Category and Unit	Geometric Mean Quantities	Marginal Value[a] Product--Standard Cobb-Douglas	Marginal Value[b] Product--Modification II
Land, (X_2)	130 Acres	16.56	13.67
Labor, (X_3)	14 Months	30.19	23.78
Expenses, (X_4)	3,348	.76	.93
Livestock-Forage, (X_5)	7,126	.64	.68
Machinery, (X_6)	6,803	.18	.22

[a] The MVP's were computed from the equation:

$$MVP_{X_i} = \frac{b_i \; E(X_1)}{X_i} \qquad (4.19)$$

[b] The MVP's are computed by taking a partial derivative of the dependent variable with respect to the individual inputs or independent variables.

$$X_1 = AX_2^{b_2}X_3^{b_3}X_4^{b_4}\left[P_1X_3(1-R^{\frac{X_5}{X_3}})\right]^{b_5}\left[P_2X_3(1-R^{\frac{X_6}{X_3}})\right]^{b_3} \qquad (4.20)$$

The MVP or partial derivation for the X_2 and X_4 variable is computed form (4.19). The MVP's for inputs X_3, X_5 and X_6 are more complex as follows:

$$\frac{\partial X_1}{\partial X_3} = \frac{(b_3+b_5+b_6)E(X_1)}{X_3} + \frac{b_5 \cdot R^{\frac{X_5}{X_3}} \ln R \cdot X_5 E(X_1)}{X_3^2(1-R^{\frac{X_5}{X_3}})} + \frac{b_6 R^{\frac{X_6}{X_3}} \ln R \cdot X_6 E(X_1)}{X_3^2(1-R^{\frac{X_6}{X_3}})} \qquad (4.21)$$

$$\frac{\partial X_1}{\partial X_5} = \frac{-b_5 R^{\frac{X_5}{X_3}} \ln R \cdot E(X_1)}{X_3(1-R^{\frac{X_5}{X_3}})} \qquad (4.22)$$

$$\frac{\partial X_1}{\partial X_6} = \frac{-b_6 R^{\frac{X_6}{X_3}} \ln R \cdot E(X_1)}{X_3(1-R^{\frac{X_6}{X_3}})} \qquad (4.23)$$

Figure 15 shows that the marginal value product of livestock-forage with modified equation II to be smaller at all points compared to the standard Cobb-Douglas. Furthermore, the divergence increases with the larger livestock-forage investment.

Figure 16 shows labor, using the modification II equation, to have an extremely high earning power of 504 dollars at five months and negative 94 dollars at the other extreme of eighteen months. The MVP of 504 dollars at five months is not so strange considering the capital-labor ratio at this point exceeds the estimated "ridge line" proportion,[13] and according to the nature of the modified function II the MVP of labor would be relatively high. No statistical significance was attached to the negative MVP because there were no farms available with fifteen or more months of labor and geometric mean proportions of other inputs, and thus extrapolating beyond the range of the data is meaningless. These results suggest that it is possible to reflect more than one stage of production by a transformation of the independent variables.

The standard error of estimate of the modification II was found to be .06382 compared to .09028 for the unmodified Cobb-Douglas estimate. This indicates that for sixty-seven out of one hundred randomly sampled farms from the same population, given 1952 conditions, the estimated gross income would fall within the fiducial limits of 8,808 dollars and 11,817 dollars using the modified II regression equation.

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[13] The livestock-forage and machinery investments are held constant at the geometric mean amount of 7,126 dollars and 6,803 dollars, respectively, while the estimated "ridge line" proportion of capital for five months of labor is 3,500 dollars.

This is compared to the fiducial limits of 8,227 dollars and 12,560 dollars using the standard Cobb-Douglas estimating equation. The difference in the ($\bar{S}$) was not significant at the ninety-five percent level.

The coefficient of determination ($\bar{R}^2$) for the modification II was found to be .89 compared with .92 for the standard Cobb-Douglas. The difference was not significant at the ninety-five percent level.

The F-test was applied to the variance of the computed residuals. The standard error of the residuals for the modification II was found to be 1,730 dollars compared to 2,490 dollars for the standard Cobb-Douglas. The reduction in the sum of the squares for modification II proved to be significant at the ninety-five percent level.

The same F-test was applied on the variance of the residuals for the specific farms having more than seven hundred dollars worth of machinery investment per month of labor and/or seven hundred dollars worth of livestock-forage investment per month of labor. The reduction in the variance was not significant at the ninety-five percent level.

Conclusions were drawn concerning the comparative statistical fits of modification II and the unmodified function as follows:

1. The standard error of estimate ($\bar{S}$) was not found to be significantly different for the modification II as compared to the unmodified function.

2. The variance of the computed residual quantities were significantly smaller with the modification II as compared with the unmodified function.

3. The computed coefficients of determination were not significantly different between the fits.

4. Estimated earning power of machinery and livestock-forage for modification II decreased relatively more than estimates for the unmodified function when using amounts of investments greater than "ridge line" proportions.

5. The MVP of labor for the modification II showed decreasing positive, and decreasing negative marginal returns. The estimates were not statistically significant but the transformation suggests greater possibilities with more applicable data.

Conclusions were drawn concerning the comparative statistical fits of modification II and modification I-b as follows:

1. The standard error of estimate ($\bar{S}$) was not significantly different for the modification II as compared to the modified function I-b.

2. The variance of the computed residual quantities was slightly smaller, but not significantly so, for the modification II compared with modified function I-b.

3. The computed coefficients of determination were not significantly different between the fits.

4. The variance of the computed residual quantities, for the farms whose capital-labor ratio exceeded the estimated "P," was slightly smaller but not significantly so, for the modification II as compared to I-b.

Conclusions were drawn concerning the comparative statistical fits of modification II and modification I as follows:

1. The standard error of estimate ($\bar{S}$) at the geometric mean

was not significantly different for the modification II compared to modification I.

2. The variance of the computed residual quantities were not significantly different for the two fits.

3. The computed coefficients of determination were not considered significantly different between the two fits.

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary

This study has considered a method of destroying the characteristics of symmetry and constant elasticity, inherent within the Cobb-Douglas function, which may force certain undesirable restrictions on the fitted function. A conceptual modification was introduced in the form of a "ridge line" in the factor-factor dimension to correspond more realistically to the actual production responses. The modifications were formulated mathematically by replacing the X_j with a dummy variable Z_j in the unmodified Cobb-Douglas equation. The $Z_j = f(X_i, X_j)$.

Modification I

The first modification defined the dummy variable as:

$$Z_j = PX_i(1-R^{X_j}) \tag{5.1}$$

The P represents the ridge line proportion of X_j to X_i. The R is the ratio of a decreasing geometric series, the terms of which are the respective increments in the dummy variable Z due to successive unit increases in the independent variables X_j.

The prominent characteristics of modification I are:

1. The elasticity of Y with respect to X_j is no longer constant.

2. The slope of the scale line in the X_iX_j plane varies with the amount of X_j present.

3. The contour in X_iX_j space becomes asymptotic to the X_i axes and to X_i = constant in the X_j dimension.

Modification II

The second modification defined the dummy variables as:

$$Z_j = PX_i(1-R^{\frac{X_j}{X_i}}) \tag{5.2}$$

The parameter P is the ridge line proportion of X_j to X_i, the R is a constant which is a ratio of a decreasing geometric series, the terms of which are the respective increments in the dummy variable Z_j, due to successive unit increases in the independent variable X_j. The difference between this second modification and the first is that R is raised to a power which is the ratio of the relative amounts of X_j and X_i used. This results in the magnitude of the Z_j being more dependent on the variable X_i than in modification I which causes the elasticity of Y with respect to X_j to increase more slowly than modification 1.

The characteristics of modification II are:

1. The elasticity of the Y with respect to the X_j varies with the amount of X_j present. Using modification I-b estimates of the parameters "P" and "R", the elasticity of X_j will be less than modification I estimates. This may or may not be an advantage depending on the nature of input-output data.

2. The scale lines no longer have constant slope which indicates variable rates of substitution between X_i and X_j when expanding the use of resources in the same proportion.

3. The contour lines as in modification 1, become asymptotic to the X_i axes and to X_i = constant in the X_j space.

4. The X_i variable indicates both positive decreasing and negative marginal returns.

Evaluation and Comparison

The problem was to determine which of the alternative functions, the unmodified Cobb-Douglas, modification I, or modification II best describe the data. The procedure was to fit by least squares the three functions and evaluate the "goodness of fit" by various statistical measures.

Input-output data used in the evaluation were taken from a study by Vance Wagley of thirty-three purposively sampled Ingham County farms, using six input categories meaningful with the dependent variable gross income.

<u>Evaluation of modification 1.</u>--The "ridge line" proportion "P" was estimated at (1) thirteen hundred dollars livestock-forage investment per month of labor, and (2) one thousand dollars machinery investment per month of labor.

The equation was fitted by least squares and the results indicated:

1. A negative MVP for machinery which is meaningless in view of the farms studied.

2. A significant reduction in the variance of the residual quantities compared to the unmodified function.

3. The standard error of estimate was not significantly

different from the unmodified Cobb-Douglas.

4. The coefficient of determination was not significantly different from the unmodified Cobb-Douglas.

5. The MVP of labor was relatively higher when using amounts less than the estimated "ridge line" proportion for modification I as compared to the unmodified function.

6. The MVP of livestock-forage was relatively less for amounts greater than the estimated "ridge line" proportion when using the modification I as compared to the unmodified function.

The conclusions were that the modification I equation did not yield a superior statistical fit to the unmodified Cobb-Douglas for these particular data. However, the function did permit a higher MVP for labor and a relatively low MVP for livestock-forage when using amounts greater than the ridge line proportion.

Evaluation of modification I-b.--An iteration of the first modification was made using a new "ridge line" proportion of (1) seven hundred dollars investment in livestock-forage per month of labor, and (2) seven hundred dollars investment in machinery per month of labor. The equation was fitted by least squares and the results indicated:

1. The reduction of the residual variance quantities was slight but not significant when using the modification I-b compared to the unmodified Cobb-Douglas.

2. The reduction in variance of the residual quantities of the farms whose capital investment in livestock-forage and/or machinery exceeded the estimated "ridge line" proportion was not significant when

using the modification I-b as compared to the unmodified function.

3. The standard error of estimate was not significantly different from the unmodified Cobb-Douglas estimate of ($\bar{S}$).

4. The coefficient of determination was not significantly different for the modification I-b and the unmodified function.

5. The MVP of labor for modification I-b indicated the same relationship as did modification I.

6. The MVP of livestock-forage for modification I-b indicated the same relationship as did modification I.

7. The MVP of machinery was relatively lower for amounts greater than the estimated "ridge line" proportion when using modification I-b as compared to the standard Cobb-Douglas function.

The conclusions were that modification I-b did not demonstrate a superior statistical fit for these particular data but it did show certain economic advantages. These advantages were in the form of non-constant elasticity for the dependent variable in respect to the modified independent variables, and further it permitted the fitted function to show more economically realistic iso-product relationships in the X_iX_j dimension.

Evaluation of modification II.--The same estimates of the "ridge line" proportion of machinery investment to labor months, and livestock-forage investment to labor months, were used as in modification I-b. The equation was fitted by least squares and the results indicated:

1. The standard error of estimate ($\bar{S}$) was not significantly different from the unmodified Cobb-Douglas estimate.

2. The reduction in the residual variance quantities was significant when using the modification II as compared to the unmodified Cobb-Douglas estimate.

3. The reduction in the variance of the residuals of the farms whose capital investments in livestock-forage and/or machinery exceeded the estimated ridge line proportion was not significant when using the modification II function as compared to the unmodified Cobb-Douglas equation.

4. The MVP of livestock-forage and machinery, for modification II, showed essentially the same relationships as did modification I and I-b.

5. The MVP of labor for the modification II was decreasing positive, and negative.

Conclusions

This study demonstrated that alterations on the Cobb-Douglas function can be realized by introducing various transformations on the independent variables. These transformations take the form of dummy variables in the N-dimensional input space.

The functions investigated did not appear to give a better fit statistically than the standard Cobb-Douglas function for these particular data. However, the modified functions did appear to have certain economic advantages over the unmodified function. These modifications permit the fitted function to show non-constant elasticity, conditions of symmetry more in agreement with empirical findings, and more economically realistic marginal value productivity estimates.

More specifically, the results of modification I indicate that the MVP estimates for labor at the geometric mean are not significantly different from the estimate derived by using the standard Cobb-Douglas. However, the apparent economic advantage of modification I is based on the high MVP estimates for labor when using small amounts in relation to the amount of investment in livestock-forage and machinery. It seems more reasonable economically for labor to reflect high earning power when there is a small supply being used concurrently with an abundance of livestock-forage and machinery. This is in opposition to the relatively low earning power of labor shown by past Cobb-Douglas studies. In addition, modification I showed MVP estimates for livestock-forage which were relatively low when using a large amount of livestock-forage investment with a small supply of labor. Estimates which reflect low earning powers of livestock-forage when there is an abundance of it in relation to small amounts of labor also seem more reasonable economically. The MVP of machinery for modification I was negative which is meaningless for the farms studied.

Modification I-b demonstrated that the modified function I can be made to show different relationships by adjusting the parameters in the dummy variables. The results indicated that even higher MVP estimates of labor can be realized, if the data warrants it, by lowering the "P" or ridge line proportion of combining labor to capital.

Modification II shows the most promise for additional research. The preliminary findings strongly indicate the possibility of developing, with further work, a usable production function that will reflect three stages of production, simultaneously.

Such preliminary results as these functions have shown suggest that more intensified research in this area should be forthcoming.

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