A NEW METHOD FOR CASCADE SYNTHESIS OF 1-PORT PASSIVE NETWORKS WITH RECIPROCAL AND NONRECIPROCAL LOSSLESS 2-PORTS DESCRIBED BY SCATTERING PARAMETERS

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CHIH-YU KAO

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This is to certify that the

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ABSTRACT

A NEW METHOD FOR CASCADE SYNTHESIS OF 1-PORT PASSIVE NETWORKS WITH RECIPROCAL AND NONRECIPROCAL LOSSLESS 2-PORTS DESCRIBED BY SCATTERING PARAMETERS

by Chih-yu Kao

The synthesis procedure presented in this thesis can be considered as an extension of the Darlington and the Talbot synthesis procedures. However, the synthesis procedure makes use of the scattering parameters and nonreciprocal elements are admitted in the realization. An existence theorem is stated and proved in the thesis, which serves as the basis for the synthesis procedure. This theorem states that given a reflection coefficient the corresponding 1-port network (RLCTΓ) can be realized in terms of a lossless 2-port network (LCTΓ), N_1, called the elementary section, terminated on another 1-port network (RLCTΓ), N_2. The proof of the theorem is such that it establishes the existence of this configuration and also describes a new synthesis procedure. The principal features of this synthesis method are:

(1) One simply deals with real polynomials rather than real rational functions.

(2) In each cycle of the procedure both the networks N_1 and N_2 are characterized simultaneously and it is not necessary to realize N_1 for the application of the synthesis procedure to the next cycle.

(3) All the computations require only the division of real polynomials which can be accomplished by means of the modified Routh's array described in the thesis. Due to the existence of such a simple algorithm it is feasible to carry out the actual computation by digital computers.

(4) All the elementary lossless 2-port networks are fully characterized and given in a table. By referring to this table, the parameters obtained for N_1 in each synthesis cycle yield an immediate realization of N_1.

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CHAPTER I

INTRODUCTION

The problem of cascade synthesis of passive electrical networks has been studied by several authors [DA 1, TA 1, HA 1, YO 1, RU 1, BE 4]. However, the formulation of the problem as well as the techniques developed by these authors differ from each other. In Darlington's synthesis method [DA 1], a driving-point immittance function is assumed to be given and this function is realized as a reciprocal lossless 2-port network terminated in a resistance. The reciprocal lossless 2-port network is then realized by cascade connected combinations of four different kind of sections, called type A, B, C and D, as shown in Fig. 1.1.

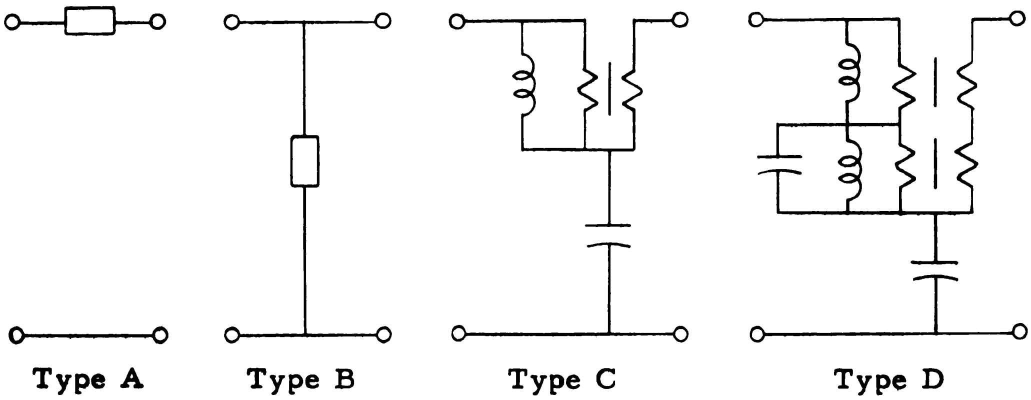


Fig. 1.1

The branches in the first two types consist of either a single inductance or capacitance, or a series-tuned circuit or a parallel-tuned circuit. Type C is called the Brune section which

realizes either pair of purely imaginary or purely real transmission zeros which are symmetrically located with respect to the origin. Finally, type D is called the Darlington section which realizes complex transmission zeros. If the numerator of the even part of the immittance function is not a perfect square, the numerator and the denominator of the immittance function are multiplied by a suitable Hurwitz polynomial, the so-called surplus factor. Such a factor, however, is always avoided if gyrators are admitted in the realization [HA 1]. Talbot [TA 1], on the other hand, used a chain matrix approach, as Piloty did, and presented a method for synthesizing reactance 2-port networks by factorizing the chain matrix into the product of two such matrices of desired degrees. Youla [YO 1] generalized Richard's theorem and defined a set of indexes and a polynomial chain matrix which are related to the real and imaginary parts of the given immittance functions. The element values of the various sections are obtained in closed form in terms of three or six indexes depending upon the complexity of the section. In this method, the gyrator is also included to take the nonreciprocal realization into consideration.

The scattering parameter formulation of passive n-port electrical networks and the properties of the scattering matrix have been treated in the literature by several authors [BE 1, BE 3, OO 2 and others]. The scattering parameters describe the performance of a network under any specified terminating conditions. The power transferred from a generator with a finite internal impedance to a resistive load is frequently best handled by scattering matrix [CA 2]. However, there are networks which are called degenerate or double degenerate which do not possess either an impedance or an admittance matrix or both [BE 1].

Rubin and Carlin [RU 1, RU 2] presented a cascade synthesis procedure for lossless reciprocal and nonreciprocal 2-port networks. "Nonreciprocal" transmission zeros are realized by four canonic nonreciprocal 2-port networks which are analogous to Darlington's A, B, C and D networks. Belevitch [BE 4] utilized the fact that the product of two passive scattering matrices is a passive scattering matrix. The corresponding n-port network is realized by interconnecting the component n-port networks by gyrators.

The synthesis method presented in this thesis can be considered as an extention of the Darlington and the Talbot synthesis procedures and utilizes the scattering parameters. In this procedure nonreciprocal sections are also allowed.

Each elementary section is analyzed on the basis of the necessary and sufficient conditions for a 2 x 2 matrix to be the scattering matrix of a lossless 2-port network and formulas are obtained for the determination of its element values. Since gyrators are allowed, the Darlington type D section can be considered as cascade connected two Brune sections each of which is series-series connected with a gyrator. This enables us to avoid surplus factors which may be needed in Darlington's procedure. Note that the degrees of the numerator and the denominator polynomials of the entries of the scattering matrices corresponding to these elementary sections do not exceed 2. This gives great simplification in actual computations.

The synthesis procedure is based on an existence theorem and the division algorithm described in the thesis. The existence theorem simply states that given a reflection coefficient, the parameters of a 2 x 2 scattering matrix corresponding to a lossless elementary section and the reflection coefficient for the remaining 1-port network do in fact exist. The division algorithm yields a simple computation for the determination of the said parameters. Once these parameters (polynomials) are established, the corresponding elementary section as well as the characterization of the remaining 1-port network are obtained simultaneously.

CHAPTER II

SCATTERING MATRICES OF PASSIVE 2-PORT NETWORKS

2.1. Introduction

The scattering parameter formulation of passive n-port electrical networks and the properties of the scattering matrix have been treated in the literature by several authors [BE 1, BE 3, OO 2 and others]. Procedures for realizing the scattering matrices with reciprocal and nonreciprocal passive n-port networks are also available. However, the realization procedures for reciprocal and nonreciprocal passive n-port networks were considered separately. Recently, all of these procedures have been integrated in a book by Newcomb [NE 1].

In this chapter, the well known results on the necessary and sufficient conditions for a given matrix of a passive lossless n-port network containing positive inductors (L), positive capacitors (C), ideal transformers (T) and gyrators (Γ) are summarized. Since our primary interest in this thesis is the cascade realization of 2-port LCTΓ networks, the relations among the entries of the corresponding 2 x 2 scattering matrix are emphasized and these basic relations are expressed in terms of real polynomials in a complex variable $\lambda = \sigma + j\omega$.

At the end of this chapter, a table of elementary 2-port sections together with their scattering matrices is given. Some of these sections are to be used as the basic sections in the cascade synthesis procedure presented in this thesis.

2.2 Scattering Matrix of a Passive Lossless N-Port Network

Consider the n-port network in Fig. 2.2.1. Let $v_1, v_2, \ldots, v_n$ and $i_1, i_2, \ldots, i_n$ be the port voltages and port currents, respectively. The scattering matrix S of the n-port network is defined by

$$S(V + I) = V - I \tag{2.2.1}$$

where

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} \tag{2.2.2}$$

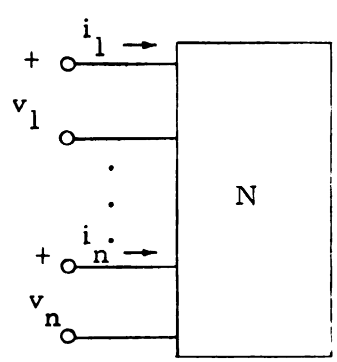


Fig. 2.2.1

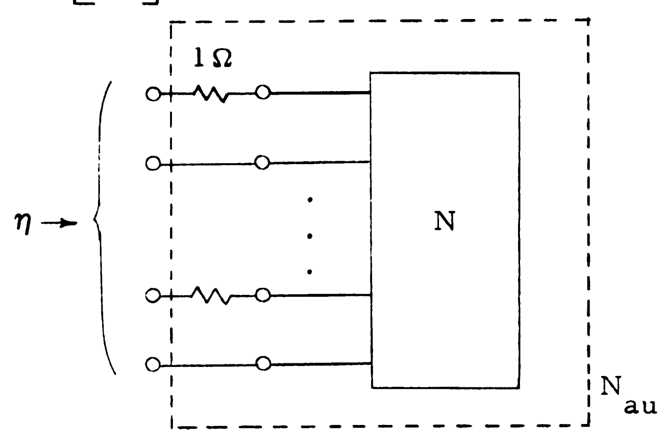


Fig. 2.2.2

The network N_{au} in Fig. 2.2.2, obtained from the n-port network N by augmenting each of its ports by a unit resistance, has a terminal admittance matrix η, which is related to S by

$$S = U_n - 2\eta \tag{2.2.3}$$

where U_n is a unit matrix of order n. This relation can be derived easily by a direct inspection of Fig. 2.2.2. If the original n-port has the terminal impedance matrix Z, then the following relations are immediate,

$$S = (Z - U_n)(Z + U_n)^{-1} \tag{2.2.4}$$

$$Z = \eta^{-1} - U_n = (U_n + S)(U_n - S)^{-1} \tag{2.2.5}$$

Similar relations hold for the terminal admittance matrix for the original n-port network.

The power input to the n-port network is given by

$$\mathrm{Re}(V^{T*}I) = \frac{1}{4}(V^{T*} + I^{T*})(U_n - S^{T*}S)(V + I) \tag{2.2.6}$$

where Re denotes "the real part of," the superscript asterisk and T denote the complex conjugation and the transposition, respectively. Also, as will be used later, the subscript asterisk is defined as

$$S_*(\lambda) = S(-\lambda).$$

The terminal admittance matrix η, of the augmented network is a positive real matrix, whose entries are necessarily finite on the imaginary axis, because of the unit terminal resistances. Therefore, by Eq. (2.2.3), the entries of S are analytic in the right half plane, including $j\omega$-axis, i.e., the denominators of these entries are strictly Hurwitz polynomials. Such matrices are called Hurwitzian [BE 1]. The power input to the passive network can not be negative, hence, as implied by Eq. (2.2.6) the Hermitian matrix

$U_n - S^{T*}S$ is non-negative definite. Since $S^*(\lambda) = S_*(\lambda)$ for $\lambda = j\omega$, then $U_n - S_*^T S$ is also Hermitian for $\lambda = j\omega$. In general, a matrix having this property is called para-Hermitian and is said to be non-negative definite if the associated Hermitian form is non-negative definite on $\lambda = j\omega$. Hence $U_n - S_*^T S$ is para-Hermitian and non-negative definite.

For a passive lossless network, since the power input is zero, Eq. (2.2.6) implies

$$U_n = S^{T*}S$$

i.e., S is a unitary matrix for $\lambda = j\omega$. However, $\lambda^* = -\lambda$ for $\lambda = j\omega$ and the relation $S_*^T S = U_n$ holds on the entire imaginary axis. Hence $S_*^T S = U_n$ holds everywhere. Therefore, S is also called para-unitary. The realizability conditions for the scattering matrix S by means of a passive lossless n-port network containing positive inductors, capacitors, ideal transformers and gyrators can now be stated as in the following theorem.

Theorem 2.2.1: [BE 3, OO 1, NE 1] The necessary and sufficient conditions for a matrix S to be the scattering matrix of an n-port network containing positive inductors, capacitors, ideal transformers and gyrators are:

1. S is Hurwitzian.
2. S is para-unitary

Only a sketch of the proof of this theorem will be given here. The well known theorems listed in Section 2.5 are needed for the proof. The necessity of the conditions has been shown in the preceding discussion, therefore, only the sufficiency of the conditions will be demonstrated.

Since S is Hurwitzian and unitary for $\lambda = j\omega$, i.e., $U_n = S^{T*}(j\omega)S(j\omega)$, then $U_n - S^{T*}(\lambda)S(\lambda) \geq 0$ for $\sigma > 0$, hence S is bounded-real. By Theorem 2.5.1 it can be seen easily that $(S + U_n)$ and $(U_n - S)$ are both positive-real.

If $(U_n - S)^{-1}$ exists, then $Z = (U_n + S)(U_n - S)^{-1}$ is positive-real (Theorem 2.5.2) and is realizable as the terminal impedance matrix of an n-port network. Since S is para-unitary, we have

$$Z = (U_n + S)(U_n - S)^{-1} = - (U_n + S_*^T)(U_n - S_*^T)^{-1} = - Z_*^T$$

which implies that Z is actually the impedance matrix of a lossless n-port network.

If $U_n - S$ is singular and has a normal rank r, r > 0, then there exists a real constant orthogonal matrix N such that

$$N^T SN = S' \dotplus U_{n-r}$$

with S' being bounded-real and $(U_r - S')^{-1}$ existing. For S', there exists Z' which is positive-real and has a network representation. By similar reasoning as before, we have

$$Z' = - Z'^T_*$$

which implies that the realized network is lossless. The transformation matrix N corresponds to the turns-ratio matrix of an ideal transformer network of 2n-ports.

As a result, S can be realized by a 2n-port ideal transformer with n-r of its ports being open circuited and the remaining r ports being terminated with an r-port network with a terminal impedance matrix Z'. This proves that the conditions 1 and 2 are sufficient.

2.3 Scattering Matrices of Passive Lossless 2-port Networks

It was shown in the foregoing section that the necessary and sufficient conditions for an n x n matrix S to be the scattering matrix of a passive lossless n-port network are:(1) S be Hurwitzian, and (2) S be para-unitary. Condition (1) merely states that the denominator of each entry of S is strictly Hurwitzian and the degree of the numerator can not exceed that of the denominator.

From condition (2), for the 2-port case, further relations on the entries of S follow. If

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad (2.3.1)$$

where S_{ij}'s are real rational functions of $\lambda(i, j = 1, 2)$, then

$$SS_*^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} S_{11*} & S_{21*} \\ S_{12*} & S_{22*} \end{bmatrix} = U_2 \qquad (2.3.2)$$

or

$$\begin{aligned} S_{11}S_{11*} + S_{12}S_{12*} &= 1 \\ S_{11}S_{21*} + S_{12}S_{22*} &= 0 \\ S_{21}S_{11*} + S_{22}S_{12*} &= 0 \\ S_{21}S_{21*} + S_{22}S_{22*} &= 1 \end{aligned} \qquad (2.3.3)$$

Let $S_{ij} = s_{ij}/s$, where s is the least common denominator, then we have

$$
\begin{aligned}
s_{11}s_{11*} + s_{12}s_{12*} &= ss_* \\
s_{11}s_{21*} + s_{12}s_{22*} &= 0 \\
s_{21}s_{11*} + s_{22}s_{12*} &= 0 \\
s_{21}s_{21*} + s_{22}s_{22*} &= ss_*
\end{aligned}
\qquad (2.3.4)
$$

Also by considering $S^T_* S = U_2$ the following relations can be obtained.

$$s_{11}s_{11*} + s_{12}s_{12*} = ss_* \qquad (2.3.5a)$$

$$s_{12}s_{11*} + s_{22}s_{21*} = 0 \qquad (2.3.5b)$$

$$s_{11}s_{11*} = s_{22}s_{22*} \qquad (2.3.5c)$$

$$s_{12}s_{12*} = s_{21}s_{21*} \qquad (2.3.5d)$$

From Eq. (2.3.5b) we have

$$s_{22} = -s_{12}s_{11*}/s_{21*} \qquad (2.3.6)$$

If $f_o = \text{GCD}(s_{12}, s_{21*})$, i.e., $s_{12} = f_o\theta_*$ and $s_{21*} = f_o\phi$ and GCD $(\theta_*, \phi) = 1$, then f_o cancels in Eq. (2.3.6) and the remaining factor of s_{21*} must divide s_{11*}. Therefore we have

$$
\begin{aligned}
s_{11*} &= h_{o*}\phi \\
s_{12} &= f_o\theta_* \\
s_{21} &= f_{o*}\phi_*
\end{aligned}
$$

Now, Eq. (2.3.5d) yields the following relation.

$$f_o\theta_* f_{o*}\theta = f_{o*}\phi_* f_o\phi$$

or

$$\theta_*\theta = \phi_*\phi \qquad (2.3.7)$$

Since GCD $(\theta_*, \phi) = 1$, it follows that

$$\phi = \pm\theta \qquad (2.3.8)$$

Therefore,

$$s_{12} = f_o\theta_* = \pm f_o\phi_* \tag{2.3.9}$$

$$s_{21} = \pm f_{o*}\theta_* = f_{o*}\phi_* \tag{2.3.10}$$

$$s_{11} = \pm h_o\theta_* = h_o\phi_* \tag{2.3.11}$$

$$s_{22} = -h_{o*}\theta_* = \mp h_{o*}\phi_* \tag{2.3.12}$$

As a conclusion, ϕ_* is a common factor of s_{11}, s_{12}, s_{21} and s_{22}, and by Eq. (2.3.5a) $\phi\phi_*$ must divide ss_*. As ϕ_* and s can not have common factors, (otherwise s would not be the least common denominator) ϕ_* must divide s_*. Therefore, ϕ divides s and is a strictly Hurwitz polynomial. Let $s = \phi g_o$, then

$$S = \frac{1}{\phi g_o}\begin{bmatrix} h_o\phi_* & \pm f_o\phi_* \\ f_{o*}\phi_* & \mp h_{o*}\phi_* \end{bmatrix}$$

$$= \frac{1}{\phi^2 g_o}\begin{bmatrix} h_o\phi\phi_* & \pm f_o\phi\phi_* \\ f_{o*}\phi\phi_* & \mp h_{o*}\phi\phi_* \end{bmatrix} \tag{2.3.13}$$

or

$$S = \frac{1}{Q}\begin{bmatrix} P & \pm R \\ R_* & \mp P_* \end{bmatrix} \tag{2.3.14}$$

In this expression, Q is a strictly Hurwitz polynomial. Note that, due to Eq. (2.3.5a), the polynomials P, Q and R satisfy the relation

$$PP_* + RR_* = QQ_* \tag{2.3.15}$$

2.4 Elementary Lossless 2-Port Networks

In this section, the elementary lossless 2-port networks which constitute the basic sections for a cascade realization are tabulated together with the corresponding scattering matrices. As can be seen, these matrices are of the form

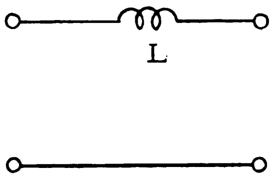
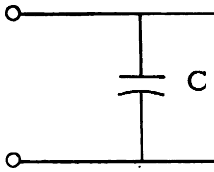
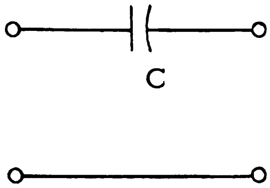
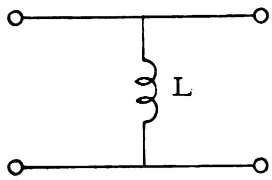
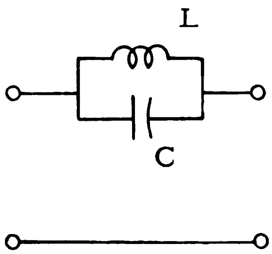
$$S = \frac{1}{Q} \begin{bmatrix} P & \pm R \\ R_* & \mp P_* \end{bmatrix}$$

with

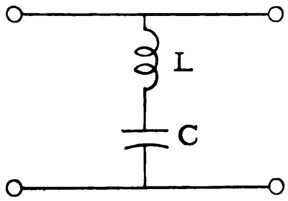
$$QQ_* - PP_* = RR_*$$

Note that any more complicated section can be obtained by cascading some of these elementary sections. Hence it is not necessary to include such complicated sections in the tabulation. For example, the Darlington-D section does not appear in the following list for it could be obtained by cascading two elementary sections NC3.

TABLE I

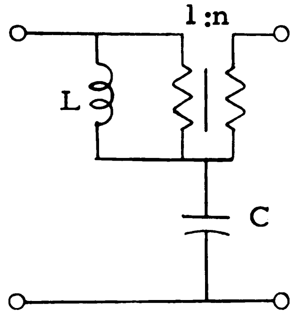
Type	Lossless 2-Port	Scattering Matrix	Trans. Zeros
A1		$\frac{1}{\frac{L}{2}\lambda + 1}\begin{bmatrix} \frac{L}{2}\lambda & 1 \\ 1 & \frac{L}{2}\lambda \end{bmatrix}$	$\lambda_o = \infty$
A2		$\frac{1}{\frac{C}{2}\lambda + 1}\begin{bmatrix} -\frac{C}{2}\lambda & 1 \\ 1 & -\frac{C}{2}\lambda \end{bmatrix}$	$\lambda_o = \infty$
B1		$\frac{1}{\lambda + \frac{1}{2C}}\begin{bmatrix} \frac{1}{2C} & \lambda \\ \lambda & \frac{1}{2C} \end{bmatrix}$	$\lambda_o = 0$
B2		$\frac{1}{\lambda + \frac{1}{2L}}\begin{bmatrix} -\frac{1}{2L} & \lambda \\ \lambda & -\frac{1}{2L} \end{bmatrix}$	$\lambda_o = 0$
C1		$\frac{1}{\lambda^2 + \frac{1}{2C}\lambda + \frac{1}{LC}}\begin{bmatrix} \frac{\lambda}{2C} & \lambda^2 + \frac{1}{LC} \\ \lambda^2 + \frac{1}{LC} & \frac{1}{2C} \end{bmatrix}$	$\lambda_o = \pm j\frac{1}{\sqrt{LC}}$

C2



$$\frac{1}{\lambda^2 + \frac{1}{2L}\lambda + \frac{1}{LC}} \begin{bmatrix} -\frac{\lambda}{2L} & \lambda^2 + \frac{1}{LC} \\ \lambda^2 + \frac{1}{LC} & -\frac{\lambda}{2L} \end{bmatrix} \qquad \lambda_o = \pm j \frac{1}{\sqrt{LC}}$$

C3

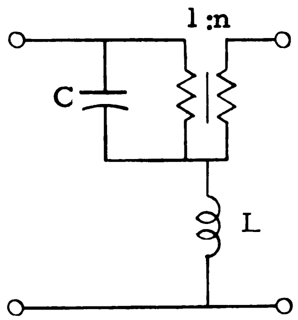


$$\frac{1}{\frac{n^2+1}{2n}\lambda^2 + \frac{L(n-1)^2 + C}{2nLC}\lambda + \frac{1}{nLC}} \times$$

$$\times \begin{bmatrix} \frac{1-n^2}{2n}\lambda^2 + \frac{L(n-1)^2 - C}{2nLC}\lambda & \lambda^2 + \frac{1}{nLC} \\ \lambda^2 + \frac{1}{nLC} & -\frac{1-n^2}{2n}\lambda^2 + \frac{L(n-1)^2 - C}{2nLC}\lambda \end{bmatrix}$$

$$\lambda_o = \pm \frac{1}{\sqrt{-nLC}}$$

C4

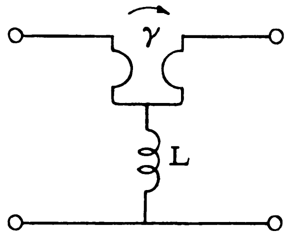


$$\frac{1}{\lambda^2 + \frac{(n-1)^2 L + C}{2LC}\lambda + \frac{n^2+1}{2LC}} \times$$

$$\times \begin{bmatrix} \frac{(n-1)^2 L - C}{2LC}\lambda + \frac{1-n^2}{2LC} & \lambda^2 + \frac{n}{LC} \\ \lambda^2 + \frac{n}{LC} & \frac{(n-1)^2 L - C}{2LC}\lambda - \frac{1-n^2}{2LC} \end{bmatrix}$$

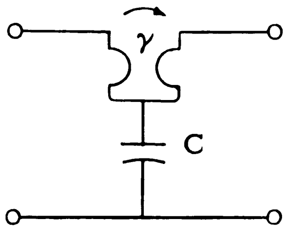
$$\lambda_o = \pm \sqrt{\frac{-n}{LC}}$$

NB1



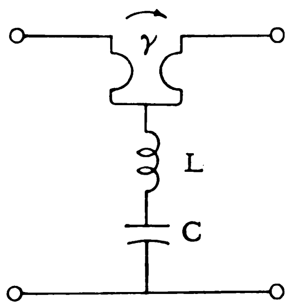
$$\frac{1}{\lambda + \frac{\gamma^2+1}{2L}} \begin{bmatrix} \frac{\gamma^2-1}{2L} & \lambda - \frac{\gamma}{L} \\ \lambda + \frac{\gamma}{L} & \frac{\gamma^2-1}{2L} \end{bmatrix} \qquad \lambda_o = \pm \frac{\gamma}{L}$$

NB2



$$\frac{1}{\frac{\gamma^2+1}{2\gamma}\lambda + \frac{1}{\gamma C}} \begin{bmatrix} \frac{\gamma^2-1}{2\gamma} & -\lambda + \frac{1}{\gamma C} \\ \lambda + \frac{1}{\gamma C} & \frac{\gamma^2-1}{2\gamma}\lambda \end{bmatrix} \qquad \lambda_o = \pm \frac{1}{\gamma C}$$

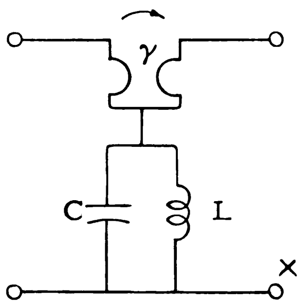
NC1



$$\frac{1}{\lambda^2 + \frac{\gamma^2+1}{2L}\lambda + \frac{1}{LC}} \begin{bmatrix} \frac{\gamma^2-1}{2L}\lambda & \lambda^2 - \frac{\gamma}{L}\lambda + \frac{1}{LC} \\ \lambda^2 + \frac{\gamma}{L}\lambda + \frac{1}{LC} & \frac{\gamma^2-1}{2L}\lambda \end{bmatrix}$$

$$\lambda_o = \frac{\pm\frac{\gamma}{L} \pm \sqrt{\frac{\gamma^2}{L^2} - \frac{4}{LC}}}{2}$$

NC2

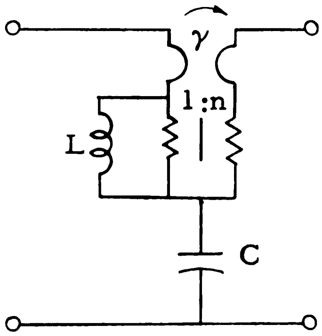


$$\frac{1}{\frac{\gamma^2+1}{2\gamma}\lambda^2 + \frac{1}{\gamma C}\lambda + \frac{\gamma^2+1}{2\gamma LC}} \times$$

$$\times \begin{bmatrix} \frac{\gamma^2-1}{2\gamma}\lambda^2 + \frac{\gamma^2-1}{2\gamma LC} & -\lambda^2 + \frac{1}{\gamma C}\lambda - \frac{1}{LC} \\ \lambda^2 + \frac{1}{\gamma C}\lambda + \frac{1}{LC} & \frac{\gamma^2-1}{2\gamma}\lambda^2 + \frac{\gamma^2-1}{2\gamma LC} \end{bmatrix}$$

$$\lambda_o = \frac{\pm\frac{1}{\gamma C} \pm \sqrt{\frac{1}{\gamma^2 C^2} - \frac{4}{LC}}}{2}$$

NC3

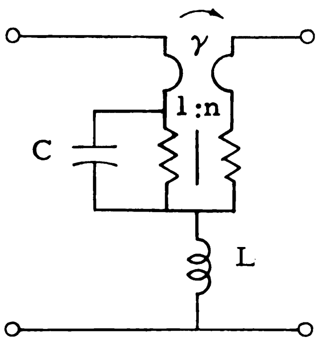


$$\lambda_o = \frac{\pm \frac{\gamma}{nL} \pm \sqrt{\frac{\gamma^2}{n^2 L^2} - \frac{4}{nLC}}}{2}$$

$$\frac{1}{\frac{n^2+1}{2n}\lambda^2 + \frac{(n-1)^2 L + (\gamma^2+1)C}{2nLC}\lambda + \frac{1}{nLC}} \times$$

$$\times \begin{bmatrix} \frac{1-n^2}{2n}\lambda^2 + \frac{(n-1)^2 L + (\gamma^2-1)C}{2nLC}\lambda & \lambda^2 - \frac{\gamma}{nL}\lambda + \frac{1}{nLC} \\ \lambda^2 + \frac{\gamma}{nL}\lambda + \frac{1}{nLC} & -\frac{1-n^2}{2n}\lambda^2 + \frac{(n-1)^2 L + (\gamma^2-1)C}{2nLC}\lambda \end{bmatrix}$$

NC4



$$\lambda_o = \frac{\pm \frac{\gamma}{C} \pm \sqrt{\frac{\gamma^2}{C^2} - \frac{4n}{LC}}}{2}$$

$$\frac{1}{\lambda^2 + \frac{(n-1)^2 L + (\gamma^2+1)C}{2LC}\lambda + \frac{n^2+1}{2LC}} \times$$

$$\times \begin{bmatrix} \frac{(n-1)^2 L + (\gamma^2-1)C}{2LC}\lambda + \frac{1-n^2}{2LC} & \lambda^2 - \frac{\gamma}{C}\lambda + \frac{n}{LC} \\ \lambda^2 + \frac{\gamma}{C}\lambda + \frac{n}{LC} & \frac{(n-1)^2 L + (\gamma^2-1)C}{2LC}\lambda - \frac{1-n^2}{2LC} \end{bmatrix}$$

2.5 Appendix to Chapter II

The definitions and theorems used in the foregoing sections are stated in this section. The proofs for these theorems and the statements of the definitions are found in [NE 1].

Definition 2.5.1. Bounded-real Matrix: (Def. 4.1 in [NE 1])

An n x n matrix $S(\lambda)$ is called bounded-real if it satisfies all the following conditions:

1. $S(\lambda)$ is holomorphic in $\sigma > 0$.
2. $S^*(\lambda) = S(\lambda^*)$ in $\sigma > 0$.
3. $U_n - S^{T*}(\lambda)S(\lambda) \geq 0$ in $\sigma > 0$.

Definition 2.5.2. Positive-real Matrix: (Def. 4.2 in [NE 1])

An n x n matrix $A(\lambda)$ is called positive-real if it satisfies all of the following conditions:

1. $A(\lambda)$ is holomorphic in $\sigma > 0$.
2. $A^*(\lambda) = A(\lambda^*)$ in $\sigma > 0$.
3. $A_H(\lambda) \geq 0$ in $\sigma > 0$.

where $A_H(\lambda)$ is the Hermitian part of $A(\lambda)$.

Theorem 2.5.1 (Theorem 5.12 in [NE 1])

If an n x n matrix S is bounded-real, then the two matrices B and C defined by

$$S = U_n - 2B = 2C - U_n$$

are both positive-real.

Theorem 2.5.2 (Theorem 5.14 in [NE 1])

If an n x n matrix S is bounded-real, with U_n-S of rank $r \neq 0$, then there exists a real, constant, orthogonal matrix N such that

$$N^T SN = S' \dotplus U_{n-r}$$

with S' bounded-real and U_r-S' non-singular. Further,

$$A = (U_r + s')(U_r - S')^{-1}$$

is positive-real.

CHAPTER III

REALIZATION OF TRANSMISSION ZEROS

3.1 Transmission Zeros

If the reflection coefficient of a 1-port RLCTΓ network is written in the form

$$S_1(\lambda) = \frac{P(\lambda)}{Q(\lambda)} \tag{3.1.1}$$

then the transmission zeros of this network are defined as the zeros of the real rational function $\frac{RR_*}{QQ}$, where $RR_* = QQ_* - PP_*$. In other words, the zeros of RR_*, after the cancellations of common factors with QQ, and $2(\delta Q-\delta R)$ zeros at infinity are called transmission zeros, where δ denotes "the degree of."

Consider the scattering matrix of a lossless 2-port LCTΓ network,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{P_1}{Q_1} & \pm \frac{R_{12}}{Q_1} \\ \frac{R_{12*}}{Q_1} & \mp \frac{P_{1*}}{Q_1} \end{bmatrix} \tag{3.1.2}$$

The transmission zeros of this lossless 2-port network are defined as the zeros of the real rational function $S_{12}S_{21}$. Thus, the transmission zeros consist of $2(\delta Q_1 - \delta R_{12})$ zeros at infinity plus the zeros of $R_{12}R_{12*}$, except for possible cancellation with Q_1^2. When this 2-port network is terminated in a 1-port RLCTΓ network with the reflection coefficient $S_2 = \frac{\zeta - 1}{\zeta + 1} = \frac{P_2}{Q_2}$, then the reflection coefficient of the resultant 1-port network is

$$S_1 = S_{11} + \frac{S_2 S_{12} S_{21}}{1 - S_2 S_{22}} \tag{3.1.3}$$

or

$$\frac{P}{Q} = \frac{P_1 Q_2 \pm Q_{1*} P_2}{Q_1 Q_2 \pm P_{1*} P_2} \tag{3.1.4}$$

Since

$$\frac{RR_*}{QQ} = \frac{QQ_* - PP_*}{QQ} = \frac{R_{12}R_{12*}(Q_2 Q_{2*} - P_2 P_{2*})}{(Q_1 Q_2 \pm P_{1*} P_2)^2} \tag{3.1.5}$$

the transmission zeros of the original lossless 2-port network, in general, are contained in those of this 1-port network.

The transmission zeros of a 1-port RLCTΓ network can also be defined by considering the even part of the given driving-point impedance (or admittance) function Z_1 (or Y_1). Indeed, since

$$Z_1 = \frac{1 + S_1}{1 - S_1} = \frac{Q + P}{Q - P} \tag{3.1.6}$$

and

$$\mathrm{Ev}\, Z_1 = \frac{1}{2}(Z_1 + Z_{1*})$$

$$= \frac{RR_*}{2(Q - P)(Q_* - P_*)} \tag{3.1.7}$$

it becomes evident that the transmission zeros of a 1-port RLCTΓ network are the zeros of $\mathrm{Ev}\, Z_1$ except those of RR_* which are also the zeros of Q^2.

For a lossless 2-port network, if $(U_2 + S)$ and $(U_2 - S)$ are non-singular, the corresponding terminal impedance matrix Z and admittance matrix Y exist and have the following forms.

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1+S_{11}-S_{22}-S_{11}S_{22}+S_{12}S_{21}}{1-S_{11}-S_{22}+S_{11}S_{22}-S_{12}S_{21}} & \dfrac{2\,S_{12}}{1-S_{11}-S_{22}+S_{11}S_{22}-S_{12}S_{21}} \\ \dfrac{2\,S_{21}}{1-S_{11}-S_{22}+S_{11}S_{22}-S_{12}S_{21}} & \dfrac{1-S_{11}+S_{22}-S_{11}S_{22}-S_{12}S_{21}}{1-S_{11}-S_{22}+S_{11}S_{22}-S_{12}S_{21}} \end{bmatrix} \quad (3.1.8)$$

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-S_{11}+S_{22}-S_{11}S_{22}+S_{12}S_{21}}{1+S_{11}+S_{22}+S_{11}S_{22}-S_{12}S_{21}} & \dfrac{-2\,S_{12}}{1+S_{11}+S_{22}+S_{11}S_{22}-S_{12}S_{21}} \\ \dfrac{-2\,S_{21}}{1+S_{11}+S_{22}+S_{11}S_{22}-S_{12}S_{21}} & \dfrac{1+S_{11}-S_{22}-S_{11}S_{22}+S_{12}S_{21}}{1+S_{11}+S_{22}+S_{11}S_{22}-S_{12}S_{21}} \end{bmatrix} \quad (3.1.9)$$

It is clear from the above equations that the zeros of Z_{12} (Y_{12}) and Z_{21} (Y_{21}) are contained in S_{12} and S_{21}, respectively.

In order to give a physical interpretation of transmission zeros, consider a lossless 2-port network, with the terminal impedance matrix Z, terminated in a 1-port RLCTΓ network ζ, called load, as shown in Fig. 3.1.1 The following relation is evident.

$$i_3 = \frac{Z_{21}}{\zeta + Z_{22}} i_1 \qquad (3.1.10)$$

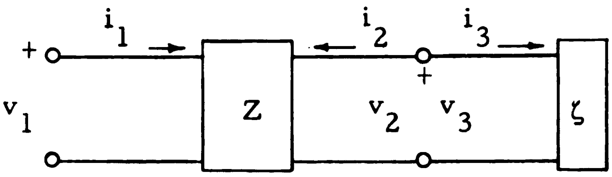


Fig. 3.1.1

If Z_{21} vanishes at a real frequency ω_o, i.e., $\lambda_o = j\omega_o$, then $i_3 = 0$ for a sinusoidal excitation with an angular frequency of ω_o applied at port 1. This indicates that the average power transmitted to the load is zero at frequency ω_o. Similarly, when a load is connected to port 1 and a sinusoidal excitation with an angular frequency of ω_o is applied at port 2, if $Z_{12}(j\omega_o) = 0$ no power is transmitted to the load. Same discussion can be applied to the terminal admittance matrix of the lossless 2-port network. Therefore, the physical meaning of the transmission zeros for real frequencies is that the power transmission from one port to the other is zero.

3.2 Two Useful Theorems and the Division Algorithm

In this section, two theorems and a division algorithm are presented which are important for the synthesis procedure discussed in the several later sections. These theorems deal with the existence of a second and a first degree polynomials passing through some fixed points given in the complex plane. On the other hand, the division algorithm provides computational facilities in the synthesis procedure. Although the proofs of these theorems as well as the proof of the procedure for division algorithm are elementary, because of their importance, this section is devoted to a fairly complete discussion of these proofs.

Theorem 3.2.1. Let $\frac{P}{Q}$ be a real rational function in which $\delta P \leq \delta Q$ and Q is strictly Hurwitz. If $\frac{P}{Q} \cdot \frac{P_*}{Q_*} = 1$ at $\lambda = \lambda_o = \sigma_o + j\omega_o$, where $\sigma_o \geq 0$ and $\omega_o > 0$ are finite, then there exist polynomials P_1 and Q_1 with $\delta P_1 \leq \delta Q_1 = 2$ such that

$$\frac{P_1(\lambda_o)}{Q_1(\lambda_o)} = \frac{P(\lambda_o)}{Q(\lambda_o)} \quad \text{and} \quad \frac{P_{1*}(\lambda_o)}{Q_{1*}(\lambda_o)} = \frac{P_*(\lambda_o)}{Q_*(\lambda_o)}$$

Proof: Part I, $\sigma_o > 0$.

Assume

$$P_1(\lambda) = a_2\lambda^2 + a_1\lambda + a_o \tag{3.2.1}$$

$$Q_1(\lambda) = b_2\lambda^2 + b_1\lambda + b_o \tag{3.2.2}$$

and

$$P(\lambda_o) = \alpha_1 + j\beta_1 \tag{3.2.3}$$

$$Q(\lambda_o) = \alpha_2 + j\beta_2 \tag{3.2.4}$$

where α_1, α_2, β_1, β_2, a_o, a_1, a_2, b_o, b_1, and b_2 are real.

Since

$$\frac{P_1(\lambda_o)}{Q_1(\lambda_o)} = \frac{P(\lambda_o)}{Q(\lambda_o)} \quad \text{and} \quad \frac{P_1(-\lambda_o)}{Q_1(-\lambda_o)} = \frac{P(-\lambda_o)}{Q(-\lambda_o)}$$

taking $\lambda_o = \sigma_o + j\omega_o$, from Eq. (3.2.1) and Eq. (3.2.2), we have

$$
\begin{aligned}
P_1(\lambda_o) &= (\sigma_o^{\,2} - \omega_o^{\,2})a_2 + \sigma_o a_1 + a_o + j(2\sigma_o\omega_o a_2 + \omega_o a_1) \\
&= (\alpha_1 + j\beta_1)(k_1 + jk_2) \\
&= (\alpha_1 k_1 - \beta_1 k_2) + j(\alpha_1 k_2 + \beta_1 k_1)
\end{aligned}
\qquad (3.2.5)
$$

$$
\begin{aligned}
Q_1(\lambda_o) &= (\sigma_o^{\,2} - \omega_o^{\,2})b_2 + \sigma_o b_1 + b_o + j(2\sigma_o\omega_o b_2 + \omega_o b_1) \\
&= (\alpha_2 + j\beta_2)(k_1 + jk_2) \\
&= (\alpha_2 k_1 - \beta_2 k_2) + j(\alpha_2 k_2 + \beta_2 k_1)
\end{aligned}
\qquad (3.2.6)
$$

$$
\begin{aligned}
P_1(-\lambda_o) &= (\sigma_o^{\,2} - \omega_o^{\,2})a_2 - \sigma_o a_1 + a_o + j(2\sigma_o\omega_o a_2 - \omega_o a_1) \\
&= (\alpha_2 + j\beta_2)(\ell_1 + j\ell_2) \\
&= (\alpha_2 \ell_1 - \beta_2 \ell_2) + j(\alpha_2 \ell_2 + \beta_2 \ell_1)
\end{aligned}
\qquad (3.2.7)
$$

$$
\begin{aligned}
Q_1(-\lambda_o) &= (\sigma_o^{\,2} - \omega_o^{\,2})b_2 - \sigma_o b_1 + b_o + j(2\sigma_o\omega_o b_2 - \omega_o b_1) \\
&= (\alpha_1 + j\beta_1)(\ell_1 + j\ell_2) \\
&= (\alpha_1 \ell_1 - \beta_1 \ell_2) + j(\alpha_1 \ell_2 + \beta_1 \ell_1)
\end{aligned}
\qquad (3.2.8)
$$

where k_1, k_2, ℓ_1 and ℓ_2 are real constants. Equating the real and the imaginary parts in each of the above equations, we have

$$
\begin{bmatrix}
(\sigma_o^2-\omega_o^2) & \sigma_o & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 & \beta_1 \\
2\sigma_o\omega_o & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 & -\alpha_1 \\
0 & 0 & 0 & (\sigma_o^2-\omega_o^2) & \sigma_o & 1 & 0 & 0 & -\alpha_2 & \beta_2 \\
0 & 0 & 0 & 2\sigma_o\omega_o & \omega_o & 0 & 0 & 0 & -\beta_2 & -\alpha_2 \\
(\sigma_o^2-\omega_o^2) & -\sigma_o & 1 & 0 & 0 & 0 & -\alpha_2 & \beta_2 & 0 & 0 \\
2\sigma_o\omega_o & -\omega_o & 0 & 0 & 0 & 0 & -\beta_2 & -\alpha_2 & 0 & 0 \\
0 & 0 & 0 & (\sigma_o^2-\omega_o^2) & -\sigma_o & 1 & -\alpha_1 & \beta_1 & 0 & 0 \\
0 & 0 & 0 & 2\sigma_o\omega_o & -\omega_o & 0 & -\beta_1 & -\alpha_1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_2 \\ a_1 \\ a_o \\ b_2 \\ b_1 \\ b_o \\ \ell_1 \\ \ell_2 \\ k_1 \\ k_2
\end{bmatrix}
\qquad (3.2.9)
$$

After elementary row operations, Eq. (3.2.9) can be reduced to

$$\begin{bmatrix} 4\sigma_o\omega_o & 0 & 0 & 0 & 0 & 0 & (c\beta_2-2\sigma_o\omega_o\alpha_2) & (c\alpha_2+2\sigma_o\omega_o\beta_2) & (c\beta_1-2\sigma_o\omega_o\alpha_1) & (c\alpha_1+2\sigma_o\omega_o\beta_1) \\ 0 & 4\sigma_o\omega_o & 0 & 0 & 0 & 0 & (c\beta_1-2\sigma_o\omega_o\alpha_1) & (c\alpha_1+2\sigma_o\omega_o\beta_1) & (c\beta_2-2\sigma_o\omega_o\alpha_2) & (c\alpha_2+2\sigma_o\omega_o\beta_2) \\ 0 & 0 & 4\sigma_o\omega_o & 0 & 0 & 0 & -\beta_2 & -\alpha_2 & -\beta_1 & -\alpha_1 \\ 0 & 0 & 0 & 4\sigma_o\omega_o & 0 & 0 & -\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2 \\ 0 & 0 & 0 & 0 & -2\sigma_o & 0 & -\alpha_2 & \beta_2 & \alpha_1 & -\beta_1 \\ 0 & 0 & 0 & 0 & 0 & -2\sigma_o & -\alpha_1 & \beta_1 & \alpha_2 & -\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\omega_o\alpha_2-\sigma_o\beta_2) & (-\omega_o\beta_2-\sigma_o\alpha_2) & (-\omega_o\alpha_1+\sigma_o\beta_1) & (\omega_o\beta_1+\sigma_o\alpha_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & (\omega_o\alpha_1-\sigma_o\beta_1) & (-\omega_o\beta_1-\sigma_o\alpha_1) & (-\omega_o\alpha_2+\sigma_o\beta_2) & (\omega_o\beta_2+\sigma_o\alpha_2) \end{bmatrix} \begin{bmatrix} a_o \\ b_o \\ a_2 \\ b_2 \\ a_1 \\ b_1 \\ \ell_1 \\ \ell_2 \\ k_1 \\ k_2 \end{bmatrix} = 0 \qquad (3.2.10)$$

where $c = \sigma_o^2 - \omega_o^2$.

The normal rank of the coefficient matrix in Eq. (3.2.10) is 8 and the number of unknowns is 10. It is clear from Eq. (3.2.10) that a_o, b_o, a_2, b_2, a_1, b_1 and two out of ℓ_1, ℓ_2, k_1 and k_2 can be expressed in terms of the remaining two variables.

Consider the following matrix formed by the last two rows and the last four columns of the coefficient matrix in Eq. (3.2.10)

$$\begin{bmatrix} (\omega_o a_2-\sigma_o\beta_2) & -(\omega_o\beta_2+\sigma_o a_2) & -(\omega_o a_1-\sigma_o\beta_1) & (\omega_o\beta_1+\sigma_o a_1) \\ (\omega_o a_1-\sigma_o\beta_1) & -(\omega_o\beta_1+\sigma_o a_1) & -(\omega_o a_2-\sigma_o\beta_2) & (\omega_o\beta_2+\sigma_o a_2) \end{bmatrix} \quad (3.2.11)$$

The determinants formed by any two columns of the above matrix are given as follows.

1) columns 1 and 2.

$$\begin{vmatrix} (\omega_o a_2-\sigma_o\beta_2) & -(\omega_o\beta_2+\sigma_o a_2) \\ (\omega_o a_1-\sigma_o\beta_1) & -(\omega_o\beta_1+\sigma_o a_1) \end{vmatrix} = (\sigma_o^2+\omega_o^2)(a_1\beta_2-\beta_1 a_2) \quad (3.2.12)$$

2) columns 3 and 4.

$$\begin{vmatrix} -(\omega_o a_1-\sigma_o\beta_1) & (\omega_o\beta_1+\sigma_o a_1) \\ -(\omega_o a_2-\sigma_o\beta_2) & (\omega_o\beta_2+\sigma_o a_2) \end{vmatrix} = -(\sigma_o^2+\omega_o^2)(a_1\beta_2-\beta_1 a_2) \quad (3.2.13)$$

3) columns 1 and 3.

$$\begin{vmatrix} (\omega_o a_2-\sigma_o\beta_2) & -(\omega_o a_1-\sigma_o\beta_1) \\ (\omega_o a_1-\sigma_o\beta_1) & -(\omega_o a_2-\sigma_o\beta_2) \end{vmatrix} = (\omega_o a_1-\sigma_o\beta_1)^2-(\omega_o a_2-\sigma_o\beta_2)^2 \quad (3.2.14)$$

4) columns 2 and 4.

$$\begin{vmatrix} -(\omega_o\beta_2+\sigma_o a_2) & (\omega_o\beta_1+\sigma_o a_1) \\ -(\omega_o\beta_1+\sigma_o a_1) & (\omega_o\beta_2+\sigma_o a_2) \end{vmatrix} = (\omega_o\beta_1+\sigma_o a_1)^2-(\omega_o\beta_2+\sigma_o a_2)^2 \quad (3.2.15)$$

As will be seen in the following discussion that the other two possible cases are actually not necessary for further considerations. However they are also listed for completeness.

5) columns 1 and 4.

$$\begin{vmatrix} (\omega_o a_2 - \sigma_o \beta_2) & (\omega_o \beta_1 + \sigma_o a_1) \\ (\omega_o a_1 - \sigma_o \beta_1) & (\omega_o \beta_2 + \sigma_o a_2) \end{vmatrix} = (\omega_o^2 - \sigma_o^2)(a_2\beta_2 - a_1\beta_1) + \sigma_o\omega_o(a_2^2 - a_1^2 + \beta_1^2 - \beta_2^2) \qquad (3.2.16)$$

6) columns 2 and 3.

$$\begin{vmatrix} -(\omega_o \beta_2 + \sigma_o a_2) & -(\omega_o a_1 - \sigma_o \beta_1) \\ -(\omega_o \beta_1 + \sigma_o a_1) & -(\omega_o a_2 - \sigma_o \beta_2) \end{vmatrix} = (\omega_o^2 - \sigma_o^2)(a_2\beta_2 + a_1\beta_1) + \sigma_o\omega_o(a_2^2 - \beta_2^2 - a_1^2 + \beta_1^2) \qquad (3.2.17)$$

In order to show that $\delta Q_1 = 2$, we must have $b_2 \not\equiv 0$ for a set of ℓ_1, ℓ_2, k_1 and k_2. Thus it has to be shown that

$$\beta_1\ell_1 + a_1\ell_2 + \beta_2 k_1 + a_2 k_2 \not\equiv 0$$

or

$$[\beta_1 \;\; a_1 \;\; \beta_2 \;\; a_2] \begin{bmatrix} \ell_1 \\ \ell_2 \\ k_1 \\ k_2 \end{bmatrix} \not\equiv 0 \qquad (3.2.18)$$

A) If

$$a_1\beta_2 - \beta_1 a_2 \neq 0$$

then both Eq. (3.2.12) and Eq. (3.2.13) are not equal to zero. ℓ_1 and ℓ_2 can be obtained in terms of k_1 and k_2 or vice versa. Further, left hand side of Eq. (3.2.18) becomes

$$\frac{1}{(\sigma_o^2+\omega_o^2)(a_1\beta_2-\beta_1 a_2)}\left[\begin{array}{c} 2\sigma_o(\sigma_o\beta_2-\omega_o a_2)(a_1\beta_2-\beta_1 a_2)+\omega_o(\sigma_o\beta_1-\omega_o a_1)\times \\ \times(a_1^2+\beta_1^2-a_2^2-\beta_2^2) \\ 2\sigma_o(\omega_o\beta_2+\sigma_o a_2)(a_1\beta_2-\beta_1 a_2)+\omega_o(\sigma_o a_1+\omega_o\beta_1)(a_1^2+\beta_1^2-a_2^2-\beta_2^2)\end{array}\right]\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{3.2.19}$$

Now, we have to show these two entries do not vanish simultaneously. If first entry vanishes, then

$$\frac{(\sigma_o\beta_2-\omega_o a_2)}{(\sigma_o\beta_1-\omega_o a_1)} = \frac{\omega_o(a_2^2+\beta_2^2-a_1^2-\beta_1^2)}{2\sigma_o(a_1\beta_2-\beta_1 a_2)} \tag{3.2.20}$$

Since $a_1\beta_2-\beta_1 a_2 \neq 0$ and $a_1^2+\beta_1^2-a_2^2-\beta_2^2 < 0$, similarly, if second entry is zero, we have

$$\frac{(\sigma_o a_2+\omega_o\beta_2)}{(\sigma_o a_1+\omega_o\beta_1)} = \frac{\omega_o(a_2^2+\beta_2^2-a_1^2-\beta_1^2)}{2\sigma_o(a_1\beta_2-\beta_1 a_2)} \tag{3.2.21}$$

From Eqs. (3.2.20) and (3.2.21) we now have

$$\frac{(\sigma_o\beta_2-\omega_o a_2)}{(\sigma_o\beta_1-\omega_o a_1)} = \frac{(\sigma_o a_2+\omega_o\beta_2)}{(\sigma_o a_1+\omega_o\beta_1)}$$

which is equivalent to

$$(\sigma_o^2+\omega_o^2)(a_1\beta_2-\beta_1 a_2) = 0$$

This result contradicts the assumption $a_1\beta_2-\beta_1 a_2 \neq 0$. Therefore, the entries in Eq. (3.2.19) can not vanish simultaneously and this proves that b_2 can be chosen to be nonzero.

B) If

$$\alpha_1\beta_2-\beta_1\alpha_2 = 0 \text{ and } \omega_o\alpha_1-\sigma_o\beta_1 \neq 0$$

then Eqs. (3.2.14) and (3.2.15) should be considered. If Eq. (3.2.14) is equal to zero, then we have

$$\omega_o\alpha_1-\sigma_o\beta_1 = \pm(\omega_o\alpha_2-\sigma_o\beta_2) \qquad (3.2.22)$$

Since $\alpha_1\beta_2-\beta_1\alpha_2 = 0$ or $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$ or $\frac{\alpha_1 \pm \alpha_2}{\alpha_2} = \frac{\beta_1 \pm \beta_2}{\beta_2}$,

Eq. (3.2.22) becomes

$$\omega_o(\alpha_1 \mp \alpha_2) = \sigma_o(\beta_1 \mp \beta_2)$$

or

$$\frac{\sigma_o}{\omega_o} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_1}{\beta_1} \qquad (3.2.23)$$

Similarly, for Eq. (3.2.15) we have

$$\frac{\omega_o}{\sigma_o} = -\frac{\alpha_2}{\beta_2} = -\frac{\alpha_1}{\beta_1} \qquad (3.2.24)$$

As a result, Eq. (3.2.14) does not vanish and ℓ_1 and k_1 can be expressed in terms of ℓ_2 and k_2. In this case, the left hand side of Eq. (3.2.18) becomes

$$\frac{\omega_o}{\omega_o\alpha_1-\sigma_o\beta_1}\left[(\alpha_1^2+\beta_1^2) \quad (\alpha_1\alpha_2+\beta_1\beta_2)\right]\begin{bmatrix}\ell_2\\ k_2\end{bmatrix}$$

It is evident that b_2 can be taken as non-zero, since $\alpha_1^2+\beta_1^2 \neq 0$.

If $\alpha_1\beta_2-\beta_1\alpha_2 = 0$ and $\omega_o\alpha_1-\sigma_o\beta_1 = 0$, which implies $\omega_o\alpha_2-\sigma_o\beta_2 = 0$, $\omega_o\beta_1+\sigma_o\alpha_1 \neq 0$ and $\omega_o\beta_2+\sigma_o\alpha_2 \neq 0$, then Eq. (3.2.15) does not vanish and ℓ_2 and k_2 can be expressed in terms of ℓ_1 and k_1. The left hand side of Eq. (3.2.18) becomes

$$\frac{-\omega_o}{\omega_o\beta_1+\sigma_o\alpha_1}\left[(\alpha_1^2+\beta_1^2) \quad (\alpha_1\alpha_2-\beta_1\beta_2)\right]\begin{bmatrix}\ell_1\\ k_1\end{bmatrix}$$

and it is evident that b_2 has non-zero solutions. Since the above cases are sufficient to have $b_2 \neq 0$, the vanishing determinants in Eqs. (3.2.16) and (3.2.17) need not be considered.

Part II, $\sigma_o=0$ and $\omega_o > 0$.

In this case, Eq. (3.2.9) becomes

$$\begin{bmatrix} -\omega_o^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 & \beta_1 \\ 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 & -\alpha_1 \\ 0 & 0 & 0 & -\omega_o^2 & 0 & 1 & 0 & 0 & -\alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & -\beta_2 & -\alpha_2 \\ -\omega_o^2 & 0 & 1 & 0 & 0 & 0 & -\alpha_2 & \beta_2 & 0 & 0 \\ 0 & -\omega_o & 0 & 0 & 0 & 0 & -\beta_2 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & -\omega_o^2 & 0 & 1 & -\alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_o & 0 & -\beta_1 & -\alpha_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_2\\ a_1\\ a_o\\ b_2\\ b_1\\ b_o\\ \ell_1\\ \ell_2\\ k_1\\ k_2\end{bmatrix} = 0 \qquad (3.2.25)$$

After the elementary row operations, we have

$$\begin{bmatrix} -\omega_o^2 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 & \beta_1 & 1 & 0 \\ 0 & \omega_o & 0 & 0 & 0 & 0 & -\beta_1 & -\alpha_1 & 0 & 0 \\ 0 & 0 & -\omega_o^2 & 0 & 0 & 0 & -\alpha_2 & \beta_2 & 0 & 1 \\ 0 & 0 & 0 & \omega_o & 0 & 0 & -\beta_2 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & \beta_2 & \alpha_1 & -\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 & -\alpha_2 & -\beta_1 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 & \beta_1 & \alpha_2 & -\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ b_2 \\ b_1 \\ \ell_1 \\ \ell_2 \\ k_1 \\ k_2 \\ a_o \\ b_o \end{bmatrix} = 0 \quad (3.2.26)$$

Similar to the case where $\sigma_o > 0$, the coefficient matrix of Eq. (3.2.26) has a normal rank 8. Therefore, a_2, a_1, b_2, b_1, ℓ_1, ℓ_2, k_1 and k_2 can be expressed in terms of a_o and b_o if

$$\begin{vmatrix} -\alpha_2 & \beta_2 & \alpha_1 & -\beta_1 \\ -\beta_2 & -\alpha_2 & -\beta_1 & -\alpha_1 \\ -\alpha_1 & \beta_1 & \alpha_2 & -\beta_2 \\ -\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2 \end{vmatrix} \neq 0 \quad (3.2.27)$$

Since $P_*(j\omega_o) = P^*(j\omega_o)$ which implies $\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2$, the above determinant equals to

$$\begin{vmatrix} -\alpha_2 & \beta_2 & \alpha_1 & -\beta_1 \\ -\beta_2 & -\alpha_2 & -\beta_1 & -\alpha_1 \\ -\alpha_1 & \beta_1 & \alpha_2 & -\beta_2 \\ -\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2 \end{vmatrix} = -[(\alpha_2^2+\beta_2^2) - (\alpha_1^2+\beta_1^2)]^2 = 0$$

Hence the rank of the coefficient matrix is at most equal to 7.

Now, let us assume $\alpha_2 \neq 0$ and perform some elementary row operations on the matrix formed by the last 4 rows of the coefficient matrix; the last 4 equations become

$$\begin{bmatrix} -\alpha_2 & \beta_2 & \alpha_1 & -\beta_1 \\ 0 & -(\alpha_2^2+\beta_2^2) & -(\alpha_1\beta_2+\beta_1\alpha_2) & \beta_1\beta_2-\alpha_1\alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \\ k_1 \\ k_2 \end{bmatrix} = 0 \qquad (3.2.28)$$

Since α_2 and β_2 can not both be equal to zero, if, e.g., $\alpha_2 = 0$, then $\beta_2 \neq 0$ and we have

$$\begin{bmatrix} 0 & \beta_2 & \alpha_1 & -\beta_1 \\ -\beta_2^2 & 0 & -\beta_1\beta_2 & -\alpha_1\beta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \\ k_1 \\ k_2 \end{bmatrix} = 0 \qquad (3.2.29)$$

From Eqs. (3.2.28) and (3.2.29) we can see that the rank of the coefficient matrix is 6 and a_2, a_1, b_2, b_1, ℓ_1 and ℓ_2 can be found in terms of k_1, k_2, a_o and b_o. The third equation in Eq. (3.2.26) gives

$$\omega_o^2 b_2 = -\alpha_2 k_1 + \beta_2 k_2 + b_o$$

and since α_2, β_2 and b_o are not simultaneously equal to zero, b_2 has a non-zero solution.

As a conclusion, the polynomials P_1 and Q_1 of the forms given by Eqs. (3.2.1) and (3.2.2), respectively, exist and the leading coefficient of Q_1 can always be made non-zero. Q.E.D.

Theorem 3.2.2 Let $\frac{P}{Q}$ be a real rational function in which $\delta P \leq \delta Q$ and Q is strictly Hurwitz. If $\frac{PP_*}{QQ_*} = 1$ at $\lambda = \lambda_o = \sigma_o$ where $\sigma_o \geq 0$ (ω_o=0) then there exist polynomials P_1 and Q_1 with $\delta P_1 \leq \delta Q_1 = 1$ such that

$$\frac{P_1(\lambda_o)}{Q_1(\lambda_o)} = \frac{P(\lambda_o)}{Q(\lambda_o)} \quad \text{and} \quad \frac{P_{1*}(\lambda_o)}{Q_{1*}(\lambda_o)} = \frac{P_*(\lambda_o)}{Q_*(\lambda_o)} .$$

Proof:

Assume

$$P_1(\lambda) = a_1\lambda + a_o \tag{3.2.30}$$

$$Q_1(\lambda) = b_1\lambda + b_o \tag{3.2.31}$$

and

$$P(\lambda_o) = \alpha_1 \tag{3.2.32}$$

$$Q(\lambda_o) = \alpha_2 \tag{3.2.33}$$

then

$$P(-\lambda_o) = m\alpha_2 \tag{3.2.34}$$

$$Q(-\lambda_o) = m\alpha_1 \tag{3.2.35}$$

where a_1, a_o, b_1, b_o, α_1, α_2 and m are real numbers. In Eqs. (3.2.30) and (3.2.31), if we let $\lambda = \pm\sigma_o$, then

$$P_1(\sigma_o) = a_1\sigma_o + a_o = \ell a_1 \qquad (3.2.36)$$

$$Q_1(\sigma_o) = b_1\sigma_o + b_o = \ell a_2 \qquad (3.2.37)$$

$$P_1(-\sigma_o) = -a_1\sigma_o + a_o = ka_2 \qquad (3.2.38)$$

$$Q_1(-\sigma_o) = -b_1\sigma_o + b_o = ka_1 \qquad (3.2.39)$$

where ℓ and k are real constants.

These can be written in a matrix form.

$$\begin{bmatrix} \sigma_o & 1 & 0 & 0 & -a_1 & 0 \\ 0 & 0 & \sigma_o & 1 & -a_2 & 0 \\ -\sigma_o & 1 & 0 & 0 & 0 & -a_2 \\ 0 & 0 & -\sigma_o & 1 & 0 & -a_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_o \\ b_1 \\ b_o \\ \ell \\ k \end{bmatrix} = 0 \qquad (3.2.40)$$

Equation (3.2.40) can be simplified as

$$\begin{bmatrix} \sigma_o & 0 & 1 & 0 & -a_1 & 0 \\ 0 & \sigma_o & 0 & 1 & -a_2 & 0 \\ 0 & 0 & 2 & 0 & -a_1 & -a_2 \\ 0 & 0 & 0 & 2 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_o \\ b_o \\ \ell \\ k \end{bmatrix} = 0$$

If $\sigma_o \neq 0$, then a_1, b_1, a_o and b_o can be solved in terms of ℓ and k.

$$\begin{bmatrix} a_1 \\ b_1 \\ a_o \\ b_o \end{bmatrix} = \frac{1}{2\sigma_o} \begin{bmatrix} a_1 & -a_2 \\ a_2 & -a_1 \\ \sigma_o a_1 & \sigma_o a_2 \\ \sigma_o a_2 & \sigma_o a_1 \end{bmatrix} \begin{bmatrix} \ell \\ k \end{bmatrix} \qquad (3.2.41)$$

Hence nonzero b_1 can be obtained.

If $\sigma_o = 0$, then $a_1 = \pm a_2$ and the only condition to be satisfied is

$$b_o = \pm a_o \neq 0 \qquad (3.2.42)$$

b_1 and a_1 can always be chosen arbitrarily.

If $\sigma_o = \infty$, then $a_1 = \pm a_2$ also and the only condition to be satisfied is

$$b_1 = \pm a_1 \neq 0 \qquad (3.2.43)$$

Therefore, it is always possible to find polynomials P_1 and Q_1 of the forms given in Eqs. (3.2.30) and (3.2.31) such that the leading coefficient of Q_1 is non-zero. Q.E.D.

<u>Division algorithm:</u>

The division algorithm described here is essentially the Euclidean algorithm which has identical steps as the Routh algorithm. More specifically, each cycle of division is exactly the Euclidean algorithm for polynomials [BI 1]. For each cycle, the coefficients of the quotient and the remainder polynomials are obtained by cross multiplication as in the Routh algorithm.

Let P and Q be real polynomials of degree m and n, respectively, with GCD(P, Q) $\equiv 1$ and $n \geq m > 0$, where GCD denotes "the greatest common divisor of." Let P and Q be written in the following forms.

$$P(\lambda) \equiv a_m \lambda^m + a_{m-1}\lambda^{m-1} + \ldots\ldots\ldots + a_1\lambda + a_o$$

$$Q(\lambda) \equiv b_n \lambda^n + b_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + b_1\lambda + b_o$$

where $a_m \neq 0$ and $b_n \neq 0$.

By the application of the Euclidean algorithm, from the polynomials $P(\lambda)$ and $Q(\lambda)$, we obtain a set of identities.

$$\begin{aligned} Q(\lambda) &\equiv q_o(\lambda)P(\lambda) + r_1(\lambda) \\ P(\lambda) &\equiv q_1(\lambda)r_1(\lambda) + r_2(\lambda) \\ r_1(\lambda) &\equiv q_2(\lambda)r_2(\lambda) + r_3(\lambda) \qquad (3.2.44) \\ &\vdots \\ r_{\rho-1}(\lambda) &\equiv q_\rho(\lambda)r_\rho(\lambda) + r_{\rho+1} \end{aligned}$$

where $\delta P > r_1(\lambda)$, $\delta r_i > \delta r_{i+1}$ for i = 1, 2, ..., ρ and $r_{\rho+1}$ is a nonzero constant.

The above identities can also be written as

$$r_1(\lambda) \equiv Q(\lambda) - q_o(\lambda)P(\lambda)$$

$$r_2(\lambda) \equiv -q_1(\lambda)Q(\lambda) + [1 + q_o(\lambda)q_1(\lambda)]\, P(\lambda) \qquad (3.2.45)$$

$$r_3(\lambda) \equiv [1 + q_1(\lambda)q_2(\lambda)]Q(\lambda) - [q_o(\lambda) + q_2(\lambda) + q_o(\lambda)q_1(\lambda)q_2(\lambda)]\, P(\lambda)$$

....

By using the bracket symbol notation introduced by Stieltjes [ST 1], which is defined by

$$[0] = 1$$

$$[q_o] = q_o$$

$$[q_o, q_1] = 1+q_oq_1$$

$$\vdots$$

and the recurrence formula

$$[q_o, q_1, \ldots\ldots, q_n] = [q_o, q_1, \ldots\ldots, q_{n-1}]q_n + [q_o, q_1, \ldots\ldots, q_{n-2}] \qquad (3.2.46)$$

the identities given in Eq. (3.2.45) take on the forms

$$r_1(\lambda) \equiv [0]Q(\lambda) - [q_o(\lambda)]\, P(\lambda)$$

$$r_2(\lambda) \equiv -[q_1(\lambda)]Q(\lambda) + [q_o(\lambda), q_1(\lambda)]P(\lambda) \qquad (3.2.47)$$

$$r_3(\lambda) \equiv [q_1(\lambda), q_2(\lambda)]Q(\lambda) - [q_o(\lambda), q_1(\lambda), q_2(\lambda)]P(\lambda)$$

$$\vdots$$

$$r_i(\lambda) \equiv (-1)^{i+1}\{ [q_1(\lambda), q_2(\lambda), \ldots, q_{i-1}(\lambda)]\, Q(\lambda) - [q_o(\lambda), q_1(\lambda), \ldots, q_{i-1}(\lambda)]\, P(\lambda)\}$$

To obtain r's and q's, a modified Routh's algorithm is used. This algorithm deals with two polynomials instead of one which is used in the original Routh's algorithm. Arrange the coefficient of $Q(\lambda)$ and $P(\lambda)$ so as to form the first two rows of the array,

$$b_n, b_{n-1}, b_{n-2}, \ldots\ldots\ldots\ldots, b_2, b_1, b_o$$

$$a_m, a_{m-1}, a_{m-2}, \ldots\ldots\ldots\ldots, a_2, a_1, a_o$$

The coefficients in third row of the array are obtained by cross multiplication exactly as in the Routh algorithm as follows.

$$C_{n-1} = \frac{a_m b_{n-1} - b_n a_{m-1}}{a_m}, \quad C_{n-2} = \frac{a_m b_{n-2} - b_n a_{m-2}}{a_m}, \ldots\ldots,$$

$$C_{n-i} = \frac{a_m b_{n-i} - b_n a_{m-i}}{a_m}, \ldots\ldots$$

If the degree of the polynomial corresponding to the third row is greater or equal to that corresponding to the second row, a new row is generated similarly. This is repeated until the degree of the remainder polynomial becomes less than that of the divisor which corresponds to the second row of the array. This cycle yields the pair $(r(\lambda), q(\lambda))$. Note that $r(\lambda)$ is formed by summing the coefficients of the last row each of which is multiplied by the respective degree of λ. Similarly, $q(\lambda)$ is obtained by first dividing each leading entry of the rows by a_m, then summing the leading coefficients of each row in the cycle, except those rows which are replica of the second row, which is multiplied by respective degree of λ.

The above cycle now is repeated, if necessary, several times for the last two rows of the array. Since at the end of each cycle, the inequalities $\delta r_1 > \delta r_2 > \delta r_3 \ldots$ hold, there will be a final cycle for which a zero remainder is obtained. Thus, r's and q's used in the Euclidean algorithm or Eq. (3.2.47) can be constructed easily by the help of the array.

3.3 Existence of Scattering Parameters Corresponding to a Selected Simple Set of Transmission Zeros

Darlington [DA 1] has shown that the driving-point impedance or admittance function, $F(\lambda)$, of an RLC 1-port network can be realized by a lossless 2-port network terminated in a unit resistance. However, in this realization procedure, it may be necessary to multiply the numerator and the denominator of $F(\lambda)$ by the same strictly Hurwitz polynomial, called the surplus factor. Augmentation of $F(\lambda)$ by such a polynomial will necessarily increase the number of reactive elements to be used in the realization of $F(\lambda)$. Hazony extended Darlington's synthesis procedure to non-reciprocal 1-port networks which eliminates the use of such surplus factors.

Consider a lossless 2-port network N terminated in an impedance ζ as shown in Fig. 3.3.1. Let

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{3.3.1}$$

with $S_{21} = \pm S_{12*}$, be the scattering matrix of N and let S_1 and S_2 be the reflection coefficients, respectively, of N at port 1 when port 2 is terminated in ζ and of ζ. Then

$$S_2 = \frac{\zeta - 1}{\zeta + 1} \qquad (3.3.2)$$

and

$$S_1 = S_{11} + \frac{S_2 S_{12} S_{21}}{1 - S_2 S_{22}} \qquad (3.3.3)$$

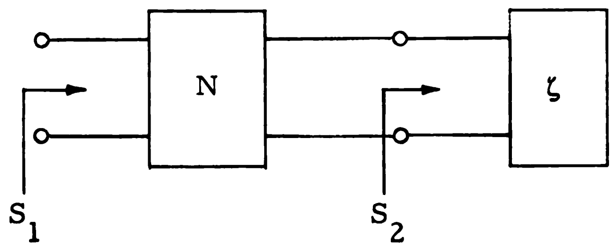


Fig. 3.3.1

If the terminating impedance $\zeta = 1$, i.e., $S_2 = 0$, then Eq. (3.3.3) becomes

$$S_1 = S_{11} \qquad (3.3.4)$$

which implies that the reflection coefficient corresponding to a driving-point impedance can be considered as the entry in 1x1 position of a 2x2 scattering matrix corresponding to the network obtained by the Darlington synthesis procedure for this driving-point impedance.

Let

$$S_1 = \frac{P}{Q} \qquad (3.3.5)$$

where P and Q are real polynomials in λ and Q is strictly Hurwitz. By Eq. (2.3.15), we have

$$RR_* = QQ_* - PP_* \tag{3.3.6}$$

which gives all the transmission zeros other than those at infinity. It is evident that the number of transmission zeros at infinity is $2(\delta Q - \delta R)$.

Consider Eq. (3.3.3) and let

$$S_{11} = \frac{P_1}{Q_1} \tag{3.3.7}$$

$$S_{12} = \frac{R_{12}}{Q_1} \tag{3.3.8}$$

$$S_{21} = \frac{R_{21}}{Q_1} \tag{3.3.9}$$

then from Eq. (2.3.14), we have

$$S_{22} = \mp \frac{P_{1*}}{Q_1} \tag{3.3.10}$$

where S_{22} assumes the negative sign if $R_{21} \equiv R_{12*}$ and the positive if $R_{21} \equiv -R_{12*}$. Further, let

$$S_2 = \frac{P_2}{Q_2} \tag{3.3.11}$$

Note that in Eqs. (3.3.7) through (3.3.11), P's, Q's and R's are real polynomials in λ.

Substituting Eqs. (3.3.7) through (3.3.11) into Eq. (3.3.3), the following relation can be obtained.

$$\frac{P}{Q} = \frac{P_1}{Q_1} + \frac{\dfrac{R_{12}R_{21}P_2}{Q_1\,Q_1\,Q_2}}{1 - \dfrac{P_2}{Q_2}\left(\mp\dfrac{P_{1*}}{Q_1}\right)}$$

$$= \frac{P_1Q_2 \pm Q_{1*}P_2}{Q_1Q_2 \pm P_{1*}P_2} \qquad (3.3.12)$$

where again, the upper or lower signs are used if $R_{21} \equiv R_{12*}$ or $R_{21} \equiv -R_{12*}$, respectively. From Eq. (3.3.12), we have

$$P \equiv P_1Q_2 \pm Q_{1*}P_2 \qquad (3.3.13)$$

$$Q \equiv Q_1Q_2 \pm P_{1*}P_2 \qquad (3.3.14)$$

Although for the most general decomposition of Eq. (3.3.12) the left hand side of Eqs. (3.3.13) and (3.3.14) should contain an arbitrary real polynomial, as will be seen in the proof of Theorem 3.3.1, without loss of generality, this polynomial can always be considered as unity. Substituting Eqs. (3.3.13) and (3.3.14) into Eq. (3.3.6), we have

$$RR_* \equiv QQ_* - PP_*$$

$$\equiv (Q_1Q_{1*} - P_1P_{1*})(Q_2Q_{2*} - P_2P_{2*}) \qquad (3.3.15)$$

On the other hand, for S_2, since

$$Q_2Q_{2*} - P_2P_{2*} = R_2R_{2*}$$

then the following relation can be obtained immediately.

$$RR_* = R_{12}R_{12*}R_2R_{2*} \qquad (3.3.16)$$

Equation (3.3.16) clearly shows that the transmission zeros of the original driving-point impedance can always be split into two parts; the first part, $R_{12}R_{12*}$, corresponds to a lossless 2-port network and the remaining part, R_2R_{2*}, corresponds to the terminating RLC network. In particular, the first part $R_{12}R_{12*}$ can be taken in a relatively simple form as to correspond to an elementary section discussed in Section2.4. Therefore, the synthesis procedure requires the proof of the fact that a simple set of transmission zeros can be realized by an elementary section and the information on the remaining transmission zeros are contained in a terminating impedance $\zeta(\lambda)$, or the corresponding reflection coefficient S_2. In other words, the cascade synthesis is justified if, after the selection of $R_{12}R_{12*}$, the existence of the real polynomials P_1, Q_1, P_2 and Q_2 is shown such that

$$\frac{1}{Q_1}\begin{bmatrix} P_1 & \pm R_{12} \\ R_{12*} & \mp P_{1*} \end{bmatrix}$$

is para-unitary while $\frac{P_2}{Q_2}$ is a bounded-real function. To this end, we shall now consider the following theorem.

<u>Theorem 3.3.1</u> Let $S_1 = \frac{P}{Q}$ be a real rational function in a complex variable $\lambda = \sigma + j\omega$ with the properties that $QQ_* - PP_* = RR_*$ and $GCD(P, Q) \equiv 1$. If Q is strictly Hurwitz and $\left|\frac{P}{Q}\right| \leq 1$ on $j\omega$-axis, then there exist polynomials P_1, Q_1, R_{12}, P_2, Q_2 and R_2 satisfying the relations

(1) $$\frac{P}{Q} = \frac{P_1Q_2 \pm Q_{1*}P_2}{Q_1Q_2 \pm P_{1*}P_2} \qquad (3.3.17)$$

(2) $$R_{12}R_{12*} \equiv Q_1Q_{1*} - P_1P_{1*} \qquad (3.3.18)$$

(3) $R_2 R_{2*} \equiv Q_2 Q_{2*} - P_2 P_{2*}$ (3.3.19)

(4) $R\,R_* \equiv R_{12} R_{12*} R_2 R_{2*}$ (3.3.20)

where in (1) only the upper or only the lower signs are to be used, such that

$$\delta Q_1 = 1 \quad \text{if} \quad \delta R_{12} \leq 1$$

$$\delta Q_1 = 2 \quad \text{if} \quad \delta R_{12} = 2$$

$$\left| \frac{P_1}{Q_1} \right| \leq 1 \quad \text{for} \quad \lambda = j\omega$$

$$\left| \frac{P_2}{Q_2} \right| \leq 1 \quad \text{for} \quad \lambda = j\omega$$

with Q_1 and Q_2 being strictly Hurwitz polynomials.

Proof: Since Q is strictly Hurwitz and $\left|\frac{P}{Q}\right| \leq 1$ for $\lambda = j\omega$, S_1 is a reflection coefficient for a driving-point impedance. By Eq. (3.2.6)

$$RR_* \equiv QQ_* - PP_* \tag{3.3.21}$$

is an even polynomial whose zeros are the transmission zeros which lie symmetrically about both the real and the imaginary axes as shown in Fig. 3.3.2. All the $j\omega$-axis zeros are necessarily of even multiplicity including those at infinity which will exist when $\delta Q > \delta R$.

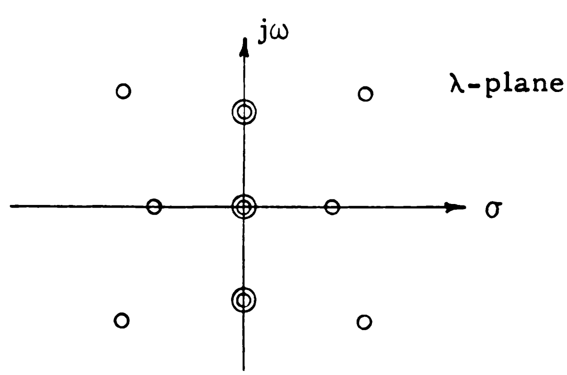


Fig. 3.3.2

Since RR_* is obtained directly from $QQ_* - PP_*$, once $R_{12}R_{12*}$ is selected, R_2R_{2*} follows immediately. Note that if $\delta Q > \delta R$, then $R_{12}R_{12*}$ can be selected as a constant (polynomial of zero degree). Due to the distribution of the zeros of RR_* in the complex λ-plane, RR_* can be factored as follows.

$$RR_* \equiv K(-\lambda^2)^{\ell} \prod_m (-\lambda^2 + a_m^2) \prod_n [(\lambda^2+b_n^2)^2 - c_n^2\lambda^2] \tag{3.3.22}$$

where ℓ , m and n are non-negative integers and K is a positive constant. Therefore, it is always possible to take R_{12} as of degree two, one or zero.

By Theorems 3.2.1 and 3.2.2, there exist polynomials P_1 and Q_1 with $\delta P_1 \leq \delta Q_1 \leq 2$ such that

$$\frac{P_1(\lambda_o)}{Q_1(\lambda_o)} = \frac{P(\lambda_o)}{Q(\lambda_o)} \tag{3.3.23}$$

and

$$\frac{P_{1*}(\lambda_o)}{Q_{1*}(\lambda_o)} = \frac{P_*(\lambda_o)}{Q_*(\lambda_o)} \tag{3.2.24}$$

where λ_o is a zero of $R_{12}R_{12*}$, which is also a zero of RR_*. Equation (3.3.23) implies that $PQ_1 - P_1Q$ is divisible by $R_{12}R_{12*}$. Let the quotient be P_2, i.e.,

$$R_{12}R_{12*}P_2 \equiv \pm(PQ_1 - P_1Q) \tag{3.3.25}$$

Since $Q_1Q_{1*} - P_1P_{1*} = 0$ at $\lambda = \lambda_o$, we have

$$\frac{P_1(\lambda_o)}{Q_1(\lambda_o)} = \frac{Q_{1*}(\lambda_o)}{P_{1*}(\lambda_o)} \tag{3.3.26}$$

By substituting Eq. (3.3.26) to Eq. (3.3.23), we obtain

$$\frac{Q_{1*}(\lambda_o)}{P_{1*}(\lambda_o)} = \frac{P(\lambda_o)}{Q(\lambda_o)} \tag{3.3.27}$$

This implies that $Q_{1*}Q - P_{1*}P$ is divisible by $R_{12}R_{12*}$. Calling the quotient Q_2, we have

$$R_{12}R_{12*}Q_2 \equiv \pm (Q_{1*}Q - P_{1*}P) \tag{3.3.28}$$

Equations (3.3.25) and (3.3.28) can be rewritten as follows.

$$P_2 = \pm \frac{PQ_1 - P_1Q}{R_{12}R_{12*}} \tag{3.2.29}$$

$$Q_2 = \pm \frac{Q_{1*}Q - P_{1*}P}{R_{12}R_{12*}} \tag{3.3.30}$$

From these expressions the following relation can be established.

$$\begin{aligned} Q_2Q_{2*} - P_2P_{2*} \equiv \frac{1}{R_{12}^2 R_{12*}^2} (&Q_1Q_{1*}QQ_* + P_1P_{1*}PP_* - P_1Q_{1*}P_*Q \\ &- P_{1*}Q_1PQ_* - PP_*Q_1Q_{1*} - P_1P_{1*}QQ_* \\ &+ P_1Q_{1*}P_*Q + P_{1*}Q_1PQ_*) \\ \equiv \frac{1}{R_{12}^2 R_{12*}^2} &(Q_1Q_{1*} - P_1P_{1*})(QQ_* - PP_*) \end{aligned} \tag{3.3.31}$$

By Theorems 3.2.1 and 3.2.2, Q_1 and P_1 can be obtained to satisfy

$$R_{12}R_{12*} \equiv Q_1Q_{1*} - P_1P_{1*}$$

Since

$$QQ_* - PP_* \equiv R_{12}R_{12*}R_2R_{2*},$$

Eq. (3.3.31) becomes

$$Q_2Q_{2*} - P_2P_{2*} \equiv R_2R_{2*}$$

which is essentially that in Eq. (3.3.19).

From Eqs. (3.3.18) and (3.3.19), we have

$$\left|\frac{P_1}{Q_1}\right| \leq 1 \quad \text{for} \quad \lambda = j\omega$$

$$\left|\frac{P_2}{Q_2}\right| \leq 1 \quad \text{for} \quad \lambda = j\omega$$

To show that Q_1 and Q_2 are strictly Hurwitz polynomials, we multiply Eq. (3.3.25) by P_{1*} and then add it to Eq. (3.3.28) multiplied by Q_1, i.e.,

$$\begin{aligned} \pm R_{12}R_{12*}(Q_1Q_2 \pm P_{1*}P_2) &\equiv (Q_1Q_{1*} - P_1P_{1*})Q \\ &\equiv R_{12}R_{12*}Q \end{aligned}$$

or

$$Q \equiv \pm (Q_1Q_2 \pm P_{1*}P_2) \qquad (3.3.32)$$

Similarly,

$$P \equiv \pm (P_1Q_2 \pm Q_{1*}P_2) \qquad (3.3.33)$$

It is to be noted that the signs appearing in front of the parentheses in Eqs. (3.3.32) and (3.3.33) are to be taken so that both are positive or negative. Similarly, the signs appearing in the parentheses must be taken to be both either positive or negative, or finally

$$\frac{P}{Q} = \frac{P_1Q_2 \pm Q_{1*}P_2}{Q_1Q_2 \pm P_{1*}P_2} .$$

In Eq. (3.3.32), since Q_1Q_2 is regular in the right half λ-plane and Q does not vanish on jω-axis, hence by Rouché's theorem Q_1Q_2 has the same number of zeros in the right half λ-plane as Q does. Therefore, Q_1 and Q_2 are strictly Hurwitz polynomials. Q.E.D.

3.4 Construction of the Polynomials P_1, Q_1, P_2 and Q_2

As is seen in the previous section that Theorem 3.3.1 establishes the existence of the polynomials P_1, Q_1, P_2 and Q_2. However, the computation of these polynomials require further considerations. For this reason we shall first consider the following pair of equations which are obtained in the proof of Theorem 3.3.1.

$$R_{12}R_{12*}P_2 \equiv PQ_1 - P_1Q \qquad (3.4.1)$$

$$R_{12}R_{12*}Q_2 \equiv Q_{1*}Q - P_{1*}P \qquad (3.4.2)$$

Since for a given pair of polynomials (P, Q) there exist unique polynomials $X(\lambda)$ and $Y(\lambda)$ with $\delta X < \delta Q$ and $\delta Y < \delta P$ [BO 1] such that

$$X(\lambda)P(\lambda) - Y(\lambda)Q(\lambda) \equiv 1 \qquad (3.4.3)$$

then Eq. (3.4.1) can be modified by using the identity in Eq. (3.4.3) as follows,

$$R_{12}R_{12*}P_2(XP - QY) \equiv PQ_1 - P_1Q$$

or

$$(R_{12}R_{12*}XP_2 - Q_1)P \equiv (R_{12}R_{12*}YP_2 - P_1)Q \qquad (3.4.4)$$

Since by assumption $GCD(P, Q) \equiv 1$, the polynomials $(R_{12}R_{12*}XP_2 - Q_1)$ and $(R_{12}R_{12*}YP_2 - P_1)$ must be divisible by Q and P, respectively, hence the quotients are equal to a polynomial, say J. Therefore,

$$R_{12}R_{12*}XP_2 - QJ \equiv Q_1 \qquad (3.4.5)$$

$$R_{12}R_{12*}YP_2 - PJ \equiv P_1 \ . \qquad (3.4.6)$$

When the polynomials $R_{12}R_{12*}X$ and Q ($R_{12}R_{12*}Y$ and P) are given, a pair of polynomials P_2 and J satisfying the identity in Eq. (3.4.5) [(3.4.6)] can be determined. To see this, consider the polynomials $R_{12}R_{12*}X$ and Q. Applying the division algorithm described in the previous section to this pair of polynomials, we have

$$\begin{aligned}
R_{12}R_{12*}X &\equiv q_oQ + r_1 \\
Q &\equiv q_1r_1 + r_2 \\
r_1 &\equiv q_2r_2 + r_3 \\
&\vdots \\
r_{i-2} &\equiv q_{i-1}r_{i-1} + r_i \\
r_{i-1} &\equiv q_ir_i + r_{i+1} \\
r_i &\equiv q_{i+1}r_{i+1} + r_{i+2} \\
&\vdots \\
r_{\rho-1} &\equiv q_\rho r_\rho + r_{\rho+1}
\end{aligned}$$

with $\delta Q > \delta r_1 > \delta r_2 > \ldots > \delta r_i > \delta r_{i+1} > \delta r_{i+2} > \ldots > \delta r_{\rho+1} = 0$

Assume that one of the remainder polynomials, say r_i, has the degree which is equal to that of Q_1, i.e.,

$$\delta r_i = \delta Q_1 \tag{3.4.8}$$

Also consider the following expressions for the remainder polynomials r_i, r_{i+1} and r_{i+2}.

$$r_i = (-1)^{i+1}\{[q_1, q_2, \ldots, q_{i-1}]R_{12}R_{12*}X - [q_o, q_1, \ldots, q_{i-1}]Q\} \tag{3.4.9}$$

$$r_{i+1} = (-1)^{i+2}\{[q_1, q_2, \ldots, q_i]R_{12}R_{12*}X - [q_o, q_1, \ldots, q_i]Q\} \tag{3.4.10}$$

$$r_{i+2} = (-1)^{i+3}\{[q_1, q_2, \ldots, q_{i+1}]R_{12}R_{12*}X - [q_o, q_1, \ldots, q_{i+1}]Q\} \tag{3.4.11}$$

If r_i is a strictly Hurwitz polynomial, we take

$$Q_1 = kr_i \tag{3.4.12}$$

where k is a real constant. From Eq. (3.4.9) we have

$$P_2 = (-1)^{i+1}k[q_1, q_2, \ldots, q_{i-1}] \tag{3.4.13}$$

Note that

$$Q = r_i[q_1, q_2, \ldots, q_{i-1}, \frac{r_{i-1}}{r_i}] \tag{3.4.14}$$

and therefore

$$\delta Q = \delta P_2 + \delta r_{i-1} \tag{3.4.15}$$

Since

$$\delta r_{i-1} = \delta q_i + \delta r_i$$

we have

$$\delta Q = \delta P_2 + \delta q_i + \delta r_i$$

$$= \delta P_2 + \delta q_i + \delta Q_1 \qquad (3.4.16)$$

On the other hand,

$$\delta Q_2 = \delta Q - \delta Q_1$$

and Eq. (3.4.16) implies

$$\delta Q_2 = \delta P_2 + \delta q_i$$

$$> \delta P_2 \qquad (3.4.17)$$

If r_i is not a strictly Hurwitz polynomial, then a linear combination of r_i with r_{i+1} is required to obtain Q_1. Note that, in general, several remainder polynomials will be necessary for the construction of Q_1. However, as stated in a theorem from algebra which is given in the following without proof, it will be sufficient to consider only $\delta Q_1 + 1$ remainder polynomials of different degrees.

Theorem 3.4.1 Consider the set of real polynomials $\{A_i(\lambda) | \delta A_i = i, \; i = 0, 1, \ldots, n\}$. Then there exist real numbers a_i such that every polynomial $B(\lambda)$ of degree $m \leq n$ can be expressed as a linear combination of the first m polynomials in the set, i.e.,

$$B(\lambda) \equiv a_o A_o(\lambda) + a_1 A_1(\lambda) + \ldots + a_m A_m(\lambda).$$

In particular, when $\delta Q_1 \leq 2$, in general, consideration of r_i, r_{i+1} and r_{i+2} will be required, where the degrees of r_i,

r_{i+1} and r_{i+2} are assumed to be 2, 1 and 0, respectively. However, it will be shown in the following that the strictly Hurwitz polynomial Q_1 can be constructed from the polynomials r_i and r_{i+1} alone. The linear combination of r_i and r_{i+1} implies that the degree of P_2 is given by

$$\begin{aligned} \delta P_2 &= \delta[q_1, q_2, \ldots, q_{i-1}, q_i] \\ &= \delta[q_1, q_2, \ldots, q_{i-1}] + \delta q_i \end{aligned} \tag{3.4.18}$$

Since

$$\begin{aligned} \delta Q &= \delta[q_1, q_2, \ldots, q_{i-1}] + \delta r_{i-1} \\ &= \delta[q_1, q_2, \ldots, q_{i-1}] + \delta q_i + \delta r_i \\ &= \delta P_2 + \delta Q_1 \end{aligned} \tag{3.4.19}$$

hence

$$\delta P_2 = \delta Q - \delta Q_1 = \delta Q_2 \tag{3.4.20}$$

On the other hand if we consider the linear combination of three remainders r_i, r_{i+1} and r_{i+2}, then

$$\delta P_2 = \delta[q_1, q_2, \ldots, q_i, q_{i+1}] \tag{3.4.21}$$

and consequently we have

$$\begin{aligned} \delta P_2 + \delta Q_1 &= \delta[q_1, q_2, \ldots, q_i, q_{i+1}] + \delta r_i \\ &= \delta[q_1, q_2, \ldots, q_i] + \delta q_{i+1} + \delta r_i \\ &= \delta Q + \delta q_{i+1} \end{aligned} \tag{3.4.22}$$

which implies

$$\delta P_2 > \delta Q_2 \tag{3.4.23}$$

This is a contradiction to the existence theorem stated in the previous section.

As a conclusion we state that, if $\delta r_i = \delta Q_1$, then the strictly Hurwitz polynomial Q_1 can always be obtained as a linear combination of the polynomials r_i and r_{i+1}.

In Eq. (3.4.7), if $\delta r_{i-1} > \delta Q_1$ and $\delta r_i < \delta Q_1$, then $\delta q_i \geq 2$ for $\delta Q_1 = 2$ or 1.

1) If $\delta Q_1 = 2$ and $\delta r_i = 1$, then Eq. (3.4.16) becomes

$$\begin{aligned} \delta Q &= \delta P_2 + \delta q_i + \delta r_i \\ &> \delta P_2 + \delta Q_1 \end{aligned} \tag{3.4.24}$$

By multiplying the remainder polynomial r_i by λ, the corresponding degree of P_2 is increased by 1, i.e.,

$$\begin{aligned} \delta P_2 &= \delta[q_1, q_2, \ldots, q_{i-1}] + 1 \\ &\leq \delta Q - \delta Q_1 \end{aligned} \tag{3.4.25}$$

Therefore, by considering the linear combination of r_i and λr_i, the strictly Hurwitz polynomial Q_1 can be constructed.

2) If $\delta Q_1 = 2$ and $\delta r_i = 0$ which implies $\delta q_i \geq 3$, then in order to construct the second degree polynomial q_1, r_i has to be multiplied by λ^2. In this case, however, the degree of the corresponding P_2 will exceed that of Q_2. This contradicts the existence theorem given in the previous section, hence the case under consideration cannot occur.

3) If $\delta Q_1 = 1$ and $\delta r_i = 0$, then r_i must be multiplied by

λ and the degree of the corresponding P_2 becomes

$$\delta P_2 = \delta[q_1, q_2, \ldots, q_{i-1}] + 1$$

Since $\delta q_i \geq 2$, then

$$\delta Q = \delta[q_1, q_2, \ldots, q_{i-1}] + \delta q_i + \delta r_i$$

$$\geq \delta P_2 + \delta Q_1 \qquad (3.4.26)$$

Thus the strictly Hurwitz polynomial Q_1 can be obtained from the linear combination of r_i and λr_i.

When the polynomials P_2 and J obtained from Eq. (3.4.5) are substituted into Eq. (3.4.6) we obtain

$$PQ_1 - R_{12}R_{12*}P_2 = QP_1 \qquad (3.4.27)$$

a) If $\delta(PQ_1) > \delta(R_{12}R_{12*}P_2)$, then $\delta(PQ_1) = \delta(QP_1)$ and since $\delta Q \geq \delta P$, we have

$$\delta Q_1 \geq \delta P_1 \qquad (3.4.28)$$

b) If $\delta(PQ_1) = \delta(R_{12}R_{12*}P_2)$, then $\delta(QP_1) \leq \delta(PQ_1)$ and consequently

$$\delta Q_1 \geq \delta P_1$$

c) If $\delta(PQ_1) < \delta(R_{12}R_{12*}P_2)$, then $\delta(QP_1) = \delta(R_{12}R_{12*}P_2)$ and since $\delta R_{12} \leq \delta Q_1$ and $\delta Q \geq \delta P_2 + \delta Q_1 \geq \delta(R_{12}P_2)$, we have

$$\delta P_1 \leq \delta R_{12} \leq \delta Q_1$$

Thus the polynomial P_1 obtained from Eq. (3.4.6) satisfies the degree condition for the elementary section to be realized.

It is then demonstrated that the polynomials P_1, Q_1, P_2 and Q_2 can be obtained from Eqs. (3.4.1) and (3.4.2) by the application of the division algorithm.

3.5 Further Discussion on "Construction of the Polynomials P_1, Q_1, P_2 and Q_2"

The polynomials P_1 and Q_1 obtained in the previous section are based on the conditions that Q_1 is strictly Hurwitz and $\delta Q_1 \geq \delta P_1$. This does not guarantee that P_1 and Q_1 will be of the forms as those corresponding to the elementary sections given in Section 2.4. However, it will become apparent from the following discussion that the polynomials P_1 and Q_1 of the desired forms can always be obtained provided that an additional condition is imposed to the linear combination of the remainder polynomials used to generate P_1 and Q_1. Note that this approach has a simple network interpretation and the following discussion is actually based on this interpretation.

Consider a lossless 2-port network N such that the degree of the least common denominator of the entries in the corresponding scattering matrix S' does not exceed 2. Let N be cascaded with an ideal transformer of turns ratio 1:n as shown in Fig. 3.5.1-(a).

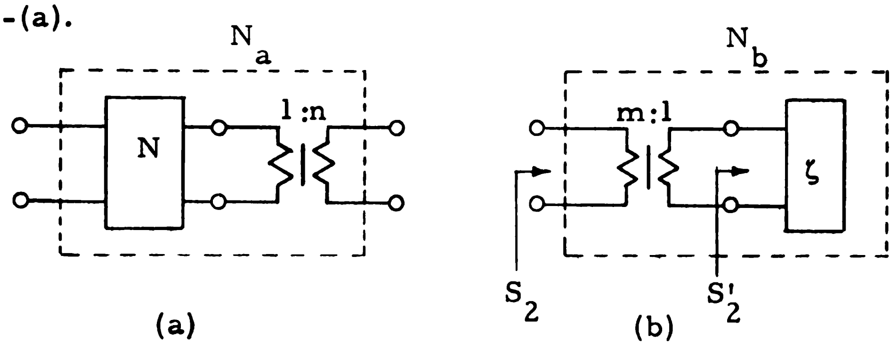


Fig. 3.5.1

The scattering matrix S' of N is of the form

$$S' = \begin{bmatrix} \dfrac{P'_1}{Q'_1} & \dfrac{R'_{12}}{Q'_1} \\[2ex] \dfrac{R'_{21}}{Q'_1} & \dfrac{\mp P'_{1*}}{Q'_1} \end{bmatrix} \tag{3.5.1}$$

A simple analysis yields that the scattering matrix S of the augmented network N_a is given by

$$S = \begin{bmatrix} \dfrac{P_1}{Q_1} & \dfrac{R_{12}}{Q_1} \\[2ex] \dfrac{R_{21}}{Q_1} & \dfrac{\mp P_{1*}}{Q_1} \end{bmatrix} \tag{3.5.2}$$

$$= \begin{bmatrix} \dfrac{(1+n^2)P'_1 - (1-n^2)(\mp Q'_{1*})}{(1+n^2)Q'_1 - (1-n^2)(\mp P'_{1*})} & \dfrac{2nR'_{12}}{(1+n^2)Q'_1 - (1-n^2)(\mp P'_{1*})} \\[3ex] \dfrac{2nR'_{21}}{(1+n^2)Q'_1 - (1-n^2)(\mp P'_{1*})} & \dfrac{(n^2-1)Q'_1 + (n^2+1)(\mp P'_{1*})}{(1+n^2)Q'_1 - (1-n^2)(\mp P'_{1*})} \end{bmatrix} \tag{3.5.3}$$

Thus,

$$Q_1 = (1+n^2)Q'_1 - (1-n^2)(\mp P'_{1*}) \tag{3.5.4}$$

$$P_1 = (1+n^2)P'_1 - (1-n^2)(\mp Q'_{1*}) \tag{3.5.5}$$

$$R_{12} = 2nR'_{12} \tag{3.5.6}$$

$$R_{21} = 2nR'_{21} \tag{3.5.7}$$

From Eq. (3.5.4), Q_1 can be rewritten as

$$Q_1 = (Q_1' \mp P_{1*}') + n^2(Q_1' \pm P_{1*}') \qquad (3.5.8)$$

The driving-point impedance at port 2 of N when port 1 is terminated in 1 ohm resistance is equal to the ratio of the polynomials $(Q_1' \mp P_{1*}')$ and $(Q_1' \pm P_{1*}')$. (These are, respectively, the numerator and the denominator polynomials of this impedance function.) These polynomials are Hurwitz. Furthermore, when one of them vanishes at a point $j\omega_o$ on $j\omega$-axis, the other does not vanish there. Therefore, the denominator polynomial given by Eq. (3.5.4) is strictly Hurwitz.

On the other hand, due to the bounded-real property of $\frac{P_1'}{Q_1'}$, the absolute values of the coefficients of P_1' do not exceed the corresponding coefficients, all positive, of Q_1'. Therefore, by proper selection of the parameter n, one of the coefficients of P_1 in Eq. (3.5.5) can be made zero which yields the forms appearing in the expressions for the elementary sections discussed in Section 2.4.

Consider now a 1-port network ζ augmented by an ideal transformer of turns ratio m: 1 as shown in Fig. 3.5.1-(b). If the reflection coefficient for ζ is denoted by $S_2' = \frac{P_2'}{Q_2'}$ then it follows that the reflection coefficient of the augmented network N_b is given by

$$S_2 = \frac{P_2}{Q_2} = \frac{(1+m^2)P_2' - (1-m^2)Q_2'}{(1+m^2)Q_2' - (1-m^2)P_2'} \qquad (3.5.9)$$

For the above reasoning S_2 is also a bounded-real function, as it should.

Now with the aid of foregoing discussions it becomes clear that in the realization of a given reflection coefficient $S_1 = \frac{P}{Q}$ we may first extract an elementary section which is included in Table I and the remaining 1-port network will now have a reflection coefficient S_2 and is still bounded-real. Indeed, when the procedure described in this thesis is applied, S_1 is first realized as in the form given in Fig. 3.5.2-(a). However, inserting two cascade connected ideal transformers of turns ratios 1:n and n:1 between the networks N and ζ, S_1 remains unaltered. Considering the above discussions, now n can be selected so that the network N_a in Fig. 3.5.2-(b) becomes identical to one of the

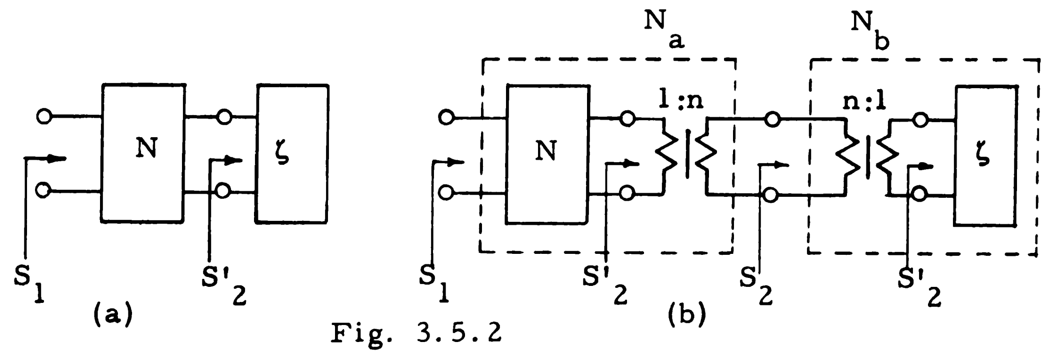


Fig. 3.5.2

elementary sections of Table I which is terminated on a new 1-port network N_b whose reflection coefficient is bounded-real and completely known.

In numerical computation of the polynomials P_1, Q_1, P_2 and Q_2 it is advantageous to consider both division arrays corresponding to the pairs of polynomials, $(R_1 R_{1*} X, Q)$ and $(R_1 R_{1*} Y, P)$. Since both arrays yield the same P_2 and J, they contain the same quotients up to a certain step. Note that, when only one division array is used, the step at which one should stop and determine the desired polynomials is actually the step where both division arrays deviate to having identical quotients. Therefore, simultaneous consideration of two division arrays yields the information as to where one should stop. Once this final step is determined, the polynomials P_1, Q_1, P_2 and Q_2 are constructed as described in the previous section.

CHAPTER IV

SYNTHESIS PROCEDURE AND EXAMPLES

4.1 Synthesis Procedure

The synthesis procedure described in this section is based on the result of Chapter III. As is indicated, one always has the liberty of ordering the transmission zeros. This synthesis procedure can be applied to a given reflection coefficient as well as to the driving-point immittance function of a 1-port RLCTΓ network. If an immittance function is given, it is first converted into the reflection coefficient and then the transmission zeros are determined by Eq. (3.3.6). Following is the step by step description of the synthesis procedure.

1. Obtain the reflection coefficient S_1: This step is omitted if S_1 is given. However, if 1-port RLCTΓ network is characterized by the immittance function, then the reflection coefficient of the network is

$$S_1 = \frac{Z_1 - 1}{Z_1 + 1} = \frac{1 - Y_1}{1 + Y_1} = \frac{P}{Q} \qquad (4.1.1)$$

where Z_1 and Y_1 are, respectively, the driving-point impedance and admittance functions of a 1-port RLCTΓ network. Since the numerator and the denominator polynomials of Z_1 or Y_1 are assumed to be relatively prime, the polynomials P and Q are also relatively prime.

2. Determine the transmission zeros: By using Eq. (3.3.6), which is repeated here for convenience

$$RR_* = QQ_* - PP_* \tag{4.1.2}$$

the finite transmission zeros are determined since these are the zeros of the even polynomial RR_*. The multiplicity of the transmission zero at infinity is given by $2(\delta Q - \delta R)$. Thus the locations and the multiplicities of all transmission zeros are determined. Further, RR_* can be factored such that each factor corresponds to an elementary lossless 2-port network described in Section 2.4. More specifically, we shall take each factor to be in one of the following forms:

$$1,\ -\lambda^2,\ (-\lambda^2 + a^2),\ (\lambda^2 + b^2)^2 - c^2\lambda^2,\ (\lambda^2 + d^2)^2, \tag{4.1.3}$$

where a, b, c and d are real and non-zero constants.

3. Obtain polynomials X and Y: From P and Q, by using the division algorithm, the polynomials X and Y with $\delta P > \delta Y$ and $\delta Q > \delta X$ are obtained uniquely which satisfy the identity,

$$XP - YQ \equiv 1.$$

4. Select the transmission zeros corresponding to an elementary section to be realized: Select R_1R_{1*} as one of the factors of RR_* given in Eq. (4.1.3).

5. Perform the division algorithm for $R_1R_{1*}X$ and Q; and also for $R_1R_{1*}Y$ and P: Two division algorithms are continued until different quotients show up.

6. Obtain the polynomials P_1, Q_1 and P_2: Take the linear combination of the remainders obtained in step 5 with their proper degrees. Then, together with the relation

$$R_1R_{1*} = Q_1Q_{1*} - P_1P_{1*}$$

the polynomials P_1, Q_1 and P_2 are obtained.

7. Obtain the polynomial Q_2: The polynomial Q_2 is obtained from the following identity,

$$R_1R_{1*}Q_2 \equiv Q_{1*}Q - P_{1*}P \tag{4.1.4}$$

since all other polynomials are already known.

This completes a cycle of realization of an elementary section. Repeating the above cycle for other selected transmission zeros, in the final cycle either both P_2 and Q_2 become constants or in the cycle before the last, P_2 and Q_2 are in the forms which correspond to an elementary section. For the latter case, the terminating resistance is 1 ohm.

4.2 Example I

Realize the driving-point impedance Z_1 given by

$$Z_1 = \frac{\lambda^4 + 2\lambda^3 + 6\lambda^2 + 8\lambda + 4}{\lambda^4 + 2\lambda^3 + 6\lambda^2 + 2\lambda + 4}$$

in cascaded 2-port LCTΓ networks terminated in a resistance.

Solution:

1. The corresponding reflection coefficient is

$$S_1 = \frac{P}{Q} = \frac{Z_1 - 1}{Z_1 + 1} = \frac{3\lambda}{\lambda^4 + 2\lambda^3 + 6\lambda^2 + 5\lambda + 4} \qquad (4.2.1)$$

2. Let

$$P = 3\lambda \qquad (4.2.2)$$

$$Q = \lambda^4 + 2\lambda^3 + 6\lambda^2 + 5\lambda + 4 \qquad (4.2.3)$$

then the transmission zeros are given by

$$RR_* = QQ_* - PP_* = (\lambda^2 + 2)^4 \qquad (4.2.4)$$

i.e., the transmission zeros are located on the imaginary axis at $\pm j\sqrt{2}$ with the multiplicities of 4.

3. To obtain polynomials X and Y for the given polynomials P and Q, we have the following division array.

Q	1	2	6	5	4
P	3	0			
	2	6	5	4	
	3	0			
	6	5	4		
	3	0			
	5	4			
	3	0			
	4				

From the above array we have,

$$q_o = \frac{1}{3}\lambda^3 + \frac{2}{3}\lambda^2 + 2\lambda + \frac{5}{3} \qquad (4.2.5)$$

$$r_1 = 4 \qquad (4.2.6)$$

Hence

$$X = (-1) \cdot \frac{1}{4} \cdot [q_o]$$

$$= -\frac{1}{12}(\lambda^3 + 2\lambda^2 + 6\lambda + 5) \qquad (4.2.7)$$

$$Y = (-1) \cdot \frac{1}{4} \cdot [0]$$

$$= -\frac{1}{4} \qquad (4.2.8)$$

4. For the first cycle of realization, an elementary section with $R_1R_{1*} = (\lambda^2 + 2)^2$ will be extracted.

5. Since R_1R_{1*} is selected, we consider

$$R_1R_{1*}X = -\frac{1}{12}(\lambda^7 + 2\lambda^6 + 10\lambda^5 + 13\lambda^4 + 28\lambda^3 + 28\lambda^2 + 24\lambda + 20) \qquad (4.2.9)$$

$$R_1R_{1*}Y = -\frac{1}{4}\lambda^4 - \lambda^2 - 1 \qquad (4.2.10)$$

The division arrays for $R_1R_{1*}X$ and Q; and for $R_1R_{1*}Y$ and P are given in the following.

$R_1R_{1*}X$	$\frac{-1}{12}$	$\frac{-2}{12}$	$\frac{-10}{12}$	$\frac{-13}{12}$	$\frac{-28}{12}$	$\frac{-28}{12}$	$\frac{-24}{12}$	$\frac{-20}{12}$
Q	1	2	6	5	4			
	0	$\frac{-4}{12}$	$\frac{-8}{12}$	$\frac{-24}{12}$	$\frac{-28}{12}$	$\frac{-24}{12}$	$\frac{-20}{12}$	
	1	2	6	5	4			
	0	0	$\frac{-8}{12}$	$\frac{-8}{12}$	$\frac{-20}{12}$			

which gives

$$q_o = -\frac{1}{12}\lambda^3 - \frac{1}{3}\lambda \tag{4.2.11}$$

$$r_1 = -\frac{2}{3}\lambda^2 - \frac{2}{3}\lambda - \frac{5}{3} \tag{4.2.12}$$

and

$R_1R_{1*}Y$	$-\frac{1}{4}$	0	-1	0	-1
P	3	0			
	0	-1	0	-1	
		3	0		
		0	-1		

which gives

$$p_o = \frac{1}{12}\lambda^3 - \frac{1}{3}\lambda$$

$$t_1 = -1 \tag{4.2.13}$$

6. In the above arrays, the quotients for the next steps are different. Since r_1 is strictly Hurwitz and of desired degree, then Q_1 and P_1 can be expressed as follows.

$$Q_1 = kr_1 \tag{4.2.14}$$

$$P_1 = kt_1 \tag{4.2.15}$$

Further, since

$$Q_1Q_{1*} - P_1P_{1*} = k^2 \cdot \frac{4}{9}(\lambda^2 + 2)^2 = R_1R_{1*} \tag{4.2.16}$$

then

$$k = \pm \frac{3}{2} \tag{4.2.17}$$

Taking $k = -\frac{3}{2}$, we have

$$Q_1 = \lambda^2 + \lambda + \frac{5}{2} \tag{4.2.18}$$

$$P_1 = \frac{3}{2} \tag{4.2.19}$$

Therefore,

$$\hat{P}_2 = -\frac{3}{2}[0] = -\frac{3}{2} \tag{4.2.20}$$

7. The polynomials $\hat{Q}_2$ is

$$\hat{Q}_2 = \frac{1}{R_1 R_{1*}} (Q_{1*}Q - P_{1*}P)$$

$$= \lambda^2 + \lambda + \frac{5}{2} \tag{4.2.21}$$

The elementary section described by P_1, Q_1 and R_1, and the remaining network described by $\hat{P}_2$ and $\hat{Q}_2$ are shown in Fig. 4.2.1.

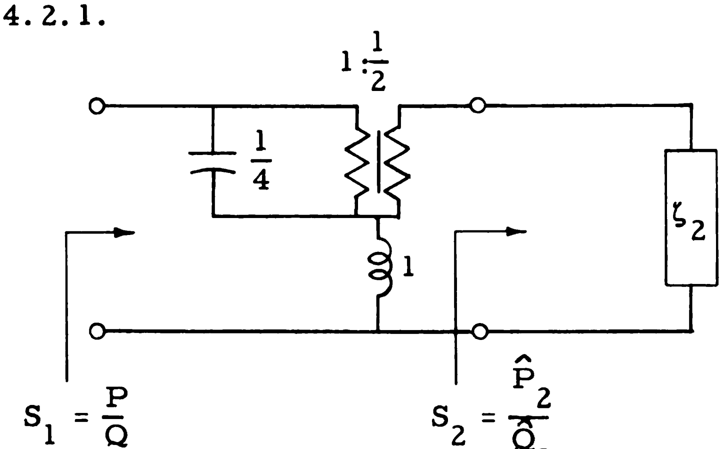


Fig. 4.2.1

For the remaining section, since $\hat{P}_2$ and $\hat{Q}_2$ correspond to an elementary section with

$$\hat{Q}_2\hat{Q}_{2*} - \hat{P}_2\hat{P}_{2*} = (\lambda^2 + 2)^2 = R_2 R_{2*}$$

the terminating resistance is 1 ohm.

Thus, the complete realization is now given in Fig. 4.2.2.

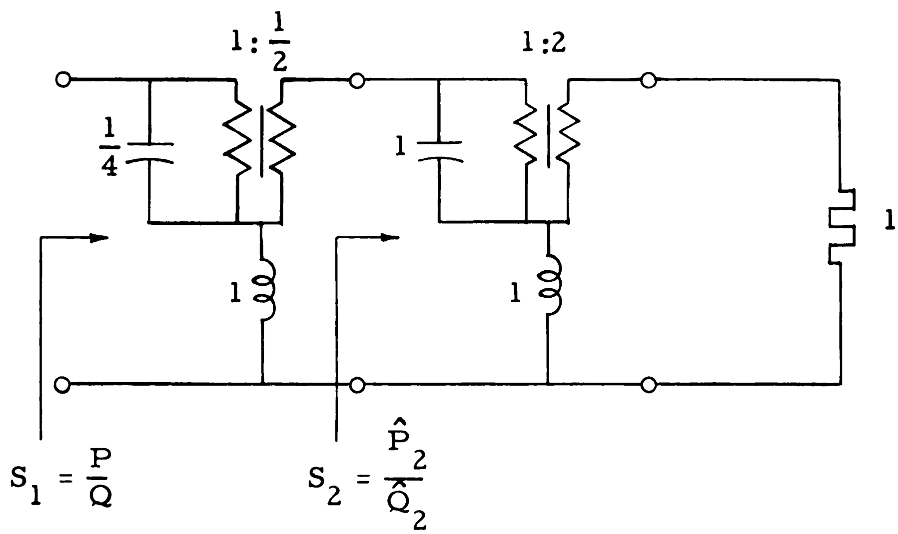


Fig. 4.2.2

4.3 Example II

Synthesize a cascade network whose reflection coefficient is given by

$$S_1 = \frac{P}{Q} = \frac{2\lambda^3 + 8\lambda^2 + 3\lambda - 1}{6\lambda^3 + 12\lambda^2 + 7\lambda + 1} \tag{4.3.1}$$

Solution:

Without indicating the steps of the synthesis procedure explicitely we first consider the transmission zeros. Since

$$P = 2\lambda^3 + 8\lambda^2 + 3\lambda - 1 \tag{4.3.2}$$

$$Q = 6\lambda^3 + 12\lambda^2 + 7\lambda + 1 \tag{4.3.3}$$

then the transmission zeros are determined from

$$RR_* = QQ_* - PP_* = 8\lambda^4(1 - 4\lambda^2) \tag{4.3.4}$$

It can be seen easily that the transmission zeros are located at the origin with multiplicity of 4 and on the real axis at $\pm\frac{1}{2}$ with multiplicities of 1. To obtain X and Y, we form the following division array.

Q	6	12	7	1
P	2	8	3	-1
	-12	-2	4	
	$\frac{23}{3}$	$\frac{11}{3}$	-1	
	-12	-2	4	
	$\frac{43}{18}$	$\frac{14}{9}$		
	$\frac{250}{43}$	4		
	$\frac{43}{18}$	$\frac{14}{9}$		
	$\frac{4 \cdot 99}{43^2}$			

which gives

$$q_o = 3$$

$$q_1 = -\frac{1}{6}\lambda - \frac{23}{36}$$

$$q_2 = -\frac{12 \cdot 18}{43}\lambda + \frac{18 \cdot 250}{43^2}$$

$$r_3 = \frac{4 \cdot 99}{43^2}$$

Therefore,

$$X = (-1)^3 \frac{1}{r_3} [q_o, q_1, q_2]$$

$$= -\frac{3 \cdot 43}{11}\lambda^2 - \frac{174}{11}\lambda - \frac{79}{22} \qquad (4.3.5)$$

$$Y = (-1)^3 \frac{1}{r_3} [q_1, q_2]$$

$$= -\frac{43}{11}\lambda^2 - \frac{144}{11}\lambda + \frac{57}{22} \qquad (4.3.6)$$

Take $R_1 R_{1*} = 1 - 4\lambda^2$. Since $(1 - 4\lambda^2)$ can be factored in two different ways as $(1 - 2\lambda)(1 + 2\lambda)$ or $(-1 - 2\lambda)(-1 + 2\lambda)$, there are two different elementary sections corresponding to these transmission zeros: one factorization corresponds to the relation $R_{21} = R_{12*}$ and the other to $R_{21} = - R_{12*}$. These two cases will be considered separately.

1) If $R_{21} = R_{12*}$, i.e., $R_{12} = 1 - 2\lambda$, then we have

$$R_1 R_{1*} X = \frac{12 \cdot 43}{11} \lambda^4 + \frac{4 \cdot 17}{11} \lambda^3 + \frac{29}{11} \lambda^2 - \frac{174}{11} \lambda - \frac{79}{22} \qquad (4.3.7)$$

$$R_1 R_{1*} Y = \frac{4 \cdot 43}{11} \lambda^4 + \frac{4 \cdot 144}{11} \lambda^3 + \frac{71}{11} \lambda^2 - \frac{144}{11} \lambda - \frac{57}{22} \qquad (4.3.8)$$

The division arrays for $R_1 R_{1*} X$ and Q, and for $R_1 R_{1*} Y$ and P are given as follows.

$R_1 R_{1*} X$	$\frac{12 \cdot 43}{11}$	$\frac{4 \cdot 174}{11}$	$\frac{29}{11}$	$\frac{-174}{11}$	$\frac{-79}{22}$
Q	6	12	7	1	
	$\frac{-8 \cdot 42}{11}$	$\frac{-573}{11}$	$\frac{-260}{11}$	$\frac{-79}{22}$	
	6	12	7	1	
	9	12	$\frac{3}{2}$		
	4	6	1		
	9	12	$\frac{3}{2}$		
	$\frac{2}{3}$	$\frac{1}{3}$			
	$\frac{15}{2}$	$\frac{3}{2}$			
	$\frac{2}{3}$	$\frac{1}{3}$			

which yields

$$q_o = \frac{2\cdot 43}{11}\lambda - \frac{56}{11}$$

$$r_1 = 9\lambda^2 + 12\lambda + \frac{3}{2}$$

$$q_1 = \frac{2}{3}\lambda + \frac{4}{9}$$

$$r_2 = \frac{2}{3}\lambda + \frac{1}{3}$$

$$q_2 = \frac{27}{2}\lambda$$

$$r_3 = \frac{15}{2}\lambda + \frac{3}{2}$$

$R_1R_{1*}Y$	$\frac{4\cdot 43}{11}$	$\frac{4\cdot 144}{11}$	$\frac{-157}{11}$	$\frac{-144}{11}$	$\frac{57}{22}$
P	2	8	3	-1	
	$\frac{-7\cdot 16}{11}$	$\frac{-415}{11}$	$\frac{-29\cdot 2}{11}$	$\frac{57}{22}$	
	2	8	3	-1	
	3	10	$\frac{-5}{2}$		
	$\frac{4}{3}$	$\frac{14}{3}$	-1		
	3	10	$\frac{-5}{2}$		
	$\frac{2}{9}$	$\frac{1}{9}$			
	$\frac{17}{2}$	$\frac{-5}{2}$			
	$\frac{2}{9}$	$\frac{1}{9}$			

which yields

$$p_o = \frac{2\cdot 43}{11}\lambda - \frac{56}{11}$$

$$t_1 = 3\lambda^2 + 10\lambda + \frac{5}{2}$$

$$p_1 = \frac{2}{3}\lambda + \frac{4}{9}$$

$$t_2 = \frac{2}{9}\lambda + \frac{1}{9}$$

$$p_2 = \frac{27}{2}\lambda$$

$$t_3 = \frac{17}{2}\lambda - \frac{5}{2}$$

Since the constant terms of q_2 and p_2 would be different if the divisions were carried one more step, both division arrays are stopped. Next, we consider the linear combinations of the remainder polynomials r_3, r_2, t_3 and t_2 to satisfy the strictly Hurwitz character of Q_1 and the relation $R_1R_{1*} = Q_1Q_{1*} - P_1P_{1*}$. Let

$$\begin{aligned} Q_1 &= n(r_3 - kr_2) \\ &= n[(\frac{15}{2} - \frac{2}{3}k)\lambda + (\frac{3}{2} - \frac{1}{3}k)] \end{aligned} \tag{4.3.9}$$

$$\begin{aligned} P_1 &= n(t_3 - kt_2) \\ &= n[(\frac{17}{2} - \frac{2}{9}k)\lambda - (\frac{5}{2} + \frac{1}{9}k)] \end{aligned} \tag{4.3.10}$$

Then

$$Q_1Q_{1*} - P_1P_{1*} = n^2(\frac{8}{81}k^2 - \frac{14}{9}k - 4)(1 - 4\lambda^2) \tag{4.3.11}$$

For Q_1 to be strictly Hurwitz, the bounds for k are

$$k < \frac{9}{2} \qquad \text{and} \qquad k > \frac{45}{4} \tag{4.3.12}$$

Referring to the corresponding elementary section and noting that P_1 has only λ term, we have $k = -\frac{45}{2}$ which is in agreement with the bounds. Thus from

$$n^2(\frac{8}{81}k^2 - \frac{14}{9}k - 4) = 1 \tag{4.3.13}$$

we have

$$n^2 = \frac{1}{81}$$

or

$$n = \pm \frac{1}{9} \tag{4.3.14}$$

If we let $n = \frac{1}{9}$, we have

$$P_1 = \frac{1}{9}(t_3 - kt_2) = \frac{3}{2}\lambda \tag{4.3.15}$$

$$Q_1 = \frac{1}{9}(r_3 - kr_2) = \frac{5}{2}\lambda + 1 \tag{4.3.16}$$

On the other hand, since

$$\begin{aligned} n(r_3 - kr_2) &= n\{R_1R_{1*}X[q_1, q_2] - Q[q_o, q_1, q_2] \\ &\quad + R_1R_{1*}X[q_1]k - Q[q_o, q_1]k\} \\ &= n\{R_1R_{1*}X[q_1, (q_2+k)] - Q[q_o, q_1, (q_2+k)]\} \end{aligned} \tag{4.3.17}$$

hence

$$\hat{P}_2 = n[q_1, (q_2 + k)] = \lambda^2 - \lambda - 1 \tag{4.3.18}$$

$$\begin{aligned} \hat{Q}_2 &= \frac{1}{R_1R_{1*}}[(1 - \frac{5}{2}\lambda)(6\lambda^3 + 12\lambda^2 + 7\lambda + 1) \\ &\quad + \frac{3}{2}\lambda(2\lambda^3 + 8\lambda^2 + 3\lambda - 1)] \\ &= 3\lambda^2 + 3\lambda + 1 \end{aligned} \tag{4.3.19}$$

Thus, the realization for the first cycle is shown in Fig. 4.3.1

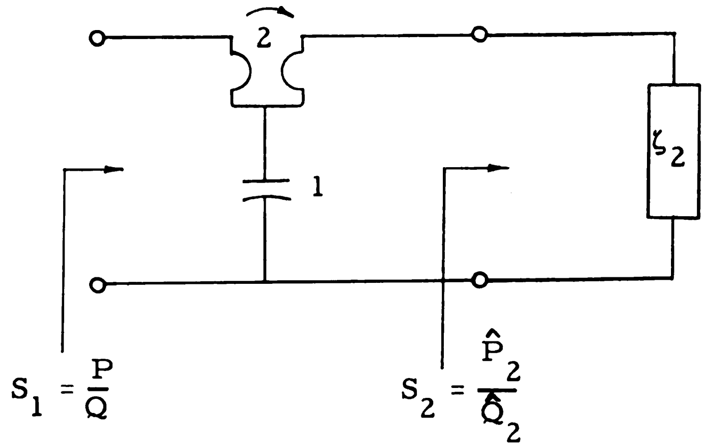


Fig. 4.3.1

In the following realization cycles, the notations X, Y, q_i's, p_i's, r_i's and t_i's are repeatedly used. To obtain X and Y from $\hat{P}_2$ and $\hat{Q}_2$, we again form the division array for $\hat{P}_2$ and $\hat{Q}_2$.

$\hat{Q}_2$	3	3	1
$\hat{P}_2$	1	-1	-1
	6	4	
	$-\frac{5}{3}$	-1	
	6	4	
	$\frac{1}{9}$		

which gives

$$q_o = 3$$
$$q_1 = \frac{1}{6}\lambda - \frac{5}{18}$$
$$r_2 = \frac{1}{9}$$

and

$$X = (-1)^2 \frac{1}{r_2} [q_o, q_1] = \frac{9}{2}\lambda + \frac{3}{2} \qquad (4.3.20)$$

$$Y = (-1)^2 \frac{1}{r_2}[q_1] = \frac{3}{2}\lambda - \frac{5}{2} \tag{4.3.21}$$

For $R_2 R_{2*} = -\lambda^2$, we have

$$R_2 R_{2*} X = -\frac{9}{2}\lambda^3 - 9\lambda^2 \tag{4.3.22}$$

$$R_2 R_{2*} Y = -\frac{3}{2}\lambda^3 + \frac{5}{2}\lambda^2 \tag{4.3.23}$$

Then form the division arrays for $R_2 R_{2*} X$ and $\hat{Q}_2$, and for $R_2 R_{2*} Y$ and $\hat{P}_2$.

$R_2 R_{2*} X$	$\frac{-9}{2}$	$\frac{-3}{2}$	0	0
$\hat{Q}_2$	3	3	1	
	3	$\frac{3}{2}$	0	
	3	3	1	
	$\frac{-3}{2}$	-1		
	1	1		
	$\frac{-3}{2}$	-1		

which gives

$$q_o = -\frac{3}{2}\lambda + 1$$

$$r_1 = -\frac{3}{2}\lambda - 1$$

$$q_1 = -2\lambda$$

$$r_2 = \lambda + 1$$

and

$R_2R_{2*}Y$	$\frac{-3}{2}$	$\frac{5}{2}$	0	0
$\hat{P}_2$	1	-1	-1	
	1	$\frac{-3}{2}$	0	
	1	-1	-1	
	$\frac{-1}{2}$	1		
	1	-1		
	$\frac{-1}{2}$	1		

which gives

$$p_o = -\frac{3}{2}\lambda + 1$$

$$t_1 = -\frac{1}{2}\lambda$$

$$p_1 = -2\lambda$$

$$t_2 = \lambda - 1$$

Let

$$Q_2 = n(-r_2 + kr_1)$$

$$= n[(-1 - \frac{3}{2}k)\lambda - (1 + k)] \quad (4.3.24)$$

$$P_2 = n(-t_2 + kt_1)$$

$$= n[(-1 - \frac{1}{2}k)\lambda + (1 + k)] \quad (4.3.25)$$

then

$$Q_2Q_{2*} - P_2P_{2*} = -n^2(2k + 2k^2)\lambda^2 \quad (4.3.26)$$

Referring to the corresponding elementary section in Table I, P_2 is either a constant or it has only the λ term. However, when P_2

has only λ term, then Q_2 would not be strictly Hurwitz due to the vanishing constant term. Therefore, P_2 can only be a constant. With this conclusion, we have

$$k = -2 \tag{4.3.27}$$

which yields a strictly Hurwitz polynomial Q_2. From Eq. (4.3.26), we have

$$n^2 = \frac{1}{4}$$

or

$$n = \pm \frac{1}{2}$$

If we take $n = \frac{1}{2}$, we have

$$P_2 = -\frac{1}{2} \tag{4.3.28}$$

$$Q_2 = \lambda + \frac{1}{2} \tag{4.3.29}$$

$$\hat{P}_3 = \lambda + 1 \tag{4.3.30}$$

$$\hat{Q}_3 = 3\lambda + 1 \tag{4.3.31}$$

The realization, after the second cycle, is shown in Fig. 4.3.2.

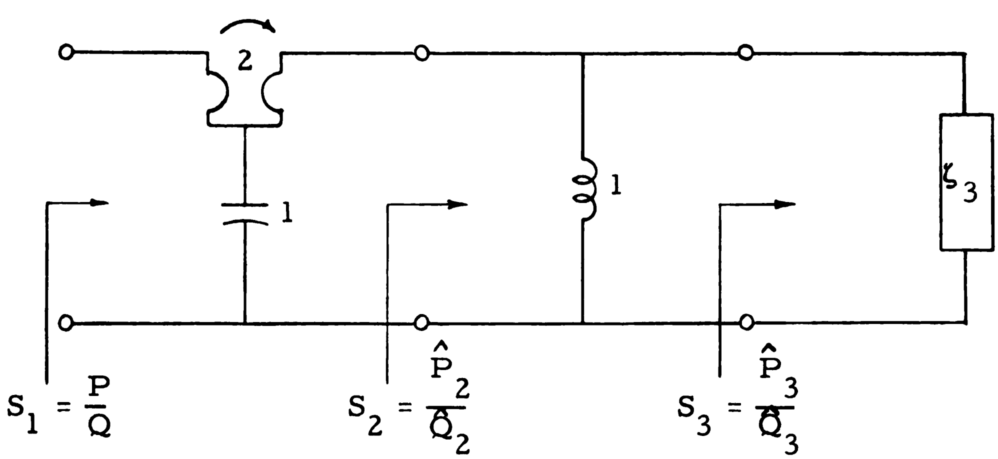


Fig. 4.3.2

To obtain X and Y for $\hat{P}_3$ and $\hat{Q}_3$, form the division array for $\hat{P}_3$ and $\hat{Q}_3$, and we have

$\hat{Q}_3$	3	1
$\hat{P}_3$	1	1
	-2	

which gives

$$q_o = 3$$

$$r_1 = -2$$

and

$$X = \frac{3}{2}$$

$$Y = \frac{1}{2}$$

For $R_3 R_{3*} = -\lambda^2$, we have

$$R_3 R_{3*} X = -\frac{3}{2}\lambda^2 \qquad (4.3.32)$$

$$R_3 R_{3*} Y = -\frac{1}{2}\lambda^2 \qquad (4.3.33)$$

Consider the division arrays for $R_3 R_{3*} X$ and $\hat{Q}_3$, and for $R_3 R_{3*} Y$ and $\hat{P}_3$.

$R_3 R_{3*} X$	$\frac{-3}{2}$	0	0
$\hat{Q}_3$	3	1	
	$\frac{1}{2}$	0	
	3	1	

which yields

$$q_o = -\frac{1}{2}\lambda$$

$$r_1 = \frac{1}{2}\lambda$$

and

$R_3R_{3*}Y$	$\frac{-1}{2}$	0	0
$\hat{P}_3$	1	1	
	$\frac{1}{2}$	0	
	1	1	

which yields

$$p_o = -\frac{1}{2}\lambda$$

$$t_1 = \frac{1}{2}\lambda$$

Let

$$Q_3 = n(r_1 - k\hat{Q}_3)$$

$$= n[(\frac{1}{2} - 3k)\lambda - k] \qquad (4.3.34)$$

$$P_3 = n(t_1 - k\hat{P}_3)$$

$$= n[(\frac{1}{2} - k)\lambda - k] \qquad (4.3.35)$$

The constant term of P_3 can not vanish, otherwise the constant term of Q_3 would vanish too. Hence it is necessary to take $k = \frac{1}{2}$, for which Q_3 is strictly Hurwitz. Since

$$Q_3Q_{3*} - P_3P_{3*} = n^2(2k - 8k^2)\lambda^2 = R_3R_{3*} \qquad (4.3.36)$$

we have

$$n = \pm 1.$$

Taking $n = -1$, we further have

$$P_3 = \frac{1}{2}$$

$$Q_3 = \lambda + \frac{1}{2}$$

$$\hat{P}_4 = 1$$

$$\hat{Q}_4 = 3$$

Thus the final realization is as shown in Fig. 4.3.3.

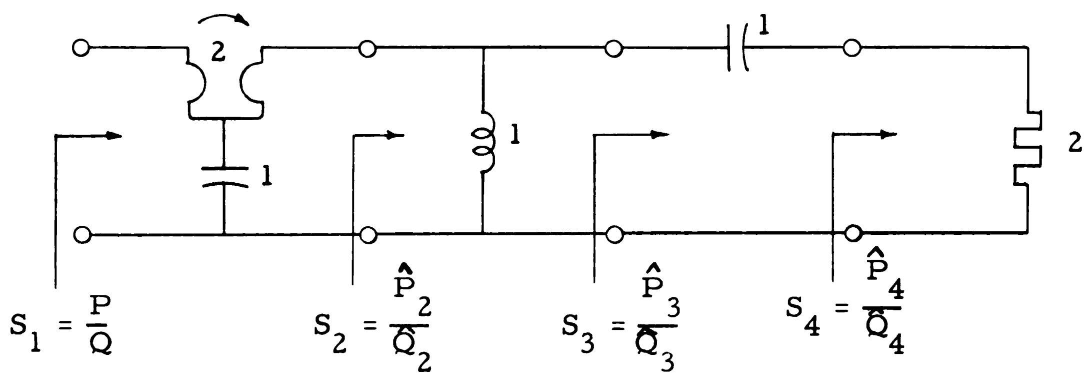


Fig. 4.3.3

2) If $R_{21} = -R_{12*}$, i.e., $R_{12} = -1 + 2\lambda$, then the division arrays for the first cycle are the same as those in case (1). In this case P_1 has a constant term only, hence we have

$$k = \frac{9 \cdot 17}{4} \quad \text{and} \quad n = \pm \frac{1}{9}$$

Taking $n = -\frac{1}{9}$ and from Eqs. (4.3.9) and (4.3.10), we have

$$P_1 = \frac{3}{4} \tag{4.3.37}$$

$$Q_1 = 2\lambda + \frac{5}{4} \tag{4.3.38}$$

$$\hat{P}_2 = \lambda^2 + \frac{7}{2}\lambda + 2 \tag{4.3.39}$$

$$\hat{Q}_2 = 3\lambda^2 + \frac{9}{2}\lambda + 2 \tag{4.3.40}$$

By repeating the steps in case (1), one will have the realization as shown in Fig. 4.3.4.

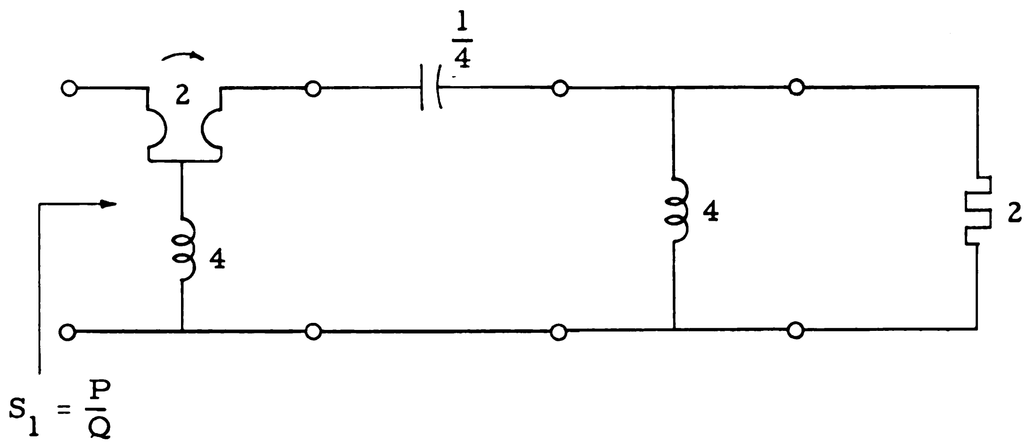


Fig. 4.3.4

CHAPTER V

CONCLUSION AND FURTHER PROBLEMS

A method for cascade synthesis of 1-port passive networks by means of successive extraction of 2-port elementary sections of Table I is fully discussed. Each of the elementary sections is characterized by scattering parameters. The use of nonreciprocal elements enables us to consider each of the elementary sections with not more than two reactive elements. For this reason, it is sufficient that the existence theorem stated in Section 3.3 (Theorem 3.3.1) is to be restricted for $\delta Q_1 \leq 2$.

The synthesis procedure is based on the step by step realization of the simple sets of transmission zeros of a given reflection coefficient. In each step, the realization consists of simple manipulation on polynomials, viz., the division algorithm and the linear combination of certain polynomials. At the end of each step, informations are obtained which are sufficient for the determination of elementary section to be extracted (whose element values can be determined later), and for generating the reflection coefficient for the remaining 1-port network.

The procedure described in this thesis is useful in the filter synthesis. In general, the filter synthesis is reduced to the realization of the reflection coefficient S_1 with specified transmission zeros. The computation of the key polynomials are accomplished by the use of the division array in a straightforward manner.

It is suggested as a further problem that one may consider complicated elementary sections. In this case, however, the existence theorem (Theorem 3.3.1) must be extended and such an extension should follow a different approach than that considered in this thesis.

Another area of investigation is the extension of the present method to the n-port cascade synthesis by essentially using the idea of Belevitch [BE 3] but carrying the computation by the method described in this thesis.

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