INVERTING AND MONOTONE PROPERTIES OF COMPLEXES

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ABSTRACT

INVERTING AND MONOTONE PROPERTIES OF COMPLEXES

by Shashichand Fatehchand Kapoor

Doyle and Hocking introduced the concept of invertibility for topological spaces and then applied this idea to finite geometric simplicial complexes. Characterizations of 1- and 2-complexes with a single invert point were given by Doyle and Klassen respectively. Hocking proved that if K is a complex with $0 \leq \dim I(K) \leq \dim K$, then K is a multiple suspension.

In Chapter II we show that if a complex K has $\dim \{I(K)\} \geq 1$, then $CI(K) = I(K)$. For a complex K with $I(K) \neq \emptyset$ or S^0 and $p \in I(K)$ we show that there exists an inverting homeomorphism which fixes p and that $K - p$ is an open monotone union $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ where each $\mathcal{U}_i \overset{T}{=} \mathrm{Lk}\, p \times E^1$. For products of complexes it is shown that if K_1 and K_2 are non-degenerate connected complexes and $I(K_1 \times K_2) \neq \emptyset$ or S^0, then K_1, K_2 and $K_1 \times K_2$ are contractible provided $K_1 \times K_2$ is not a $(\dim K_1 \times K_2)$-sphere.

In Chapter III we discuss complexes with a single invert point by restricting the number of orbits under isotopy and by imposing the Brouwer Property on the complex. An interesting characterization of a 3-sphere is obtained in Theorem 3.6 when we show that a 3-complex with

Brouwer Property and $\dim\{I(K)\} \geq 1$ is necessarily a 3-sphere. A 3-complex with Brouwer Property and a single point invert set turns out to be a suspension of a closed 2-manifold with the suspension points identified.

In Chapter IV we discuss invert sets of suspensions of complexes with a single invert point. It is conjectured that if K is any complex with $I(K) = \{p\}$, then $I(\mathcal{S}(K)) = S^0$. For a 1-complex, we prove that this is true, and for 2- and 3-complexes we get the invert set as a 0-sphere if the complex has two orbits under isotopy. The uniqueness of the open cone neighborhood is used to show that local homology groups are invariant under triangulations of any complex. For any complex K with $I(K) = \{p\}$, we prove that if $I(\mathcal{S}(K)) \neq S^0$, then $\dim\{I(\mathcal{S}(K))\} \geq 2$. A result of Doyle on suspension rings in a double suspension is generalized to show that for any complex K, $I(\mathcal{S}^k(K)) \supseteq S^{k-1}$ for $k = 1,2,3, \cdots$.

In the last chapter we introduce the concept of an expanding n-star graph $E(n)$ as a monotone union of star graphs and show that all such graphs can be embedded in a plane. This concept suggests a possible generalization of the self-avoiding walks discussed by Kesten and generalizes a result of Doyle on complexes which are monotone unions of 1-cells.

INVERTING AND MONOTONE PROPERTIES OF COMPLEXES

By

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To

Bibiji and Bowji

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CHAPTER I

INTRODUCTION

The concept of invertible spaces was introduced by Doyle and Hocking in [3]. This lead to the investigation of such concepts as continuous invertibility, dimensional invertibility and local invertibility in [4], [5] and [6]. These papers discussed the above concepts with reference to general topological spaces. For manifolds they gave rise to some very interesting and useful results. We give below the relevant basic definitions and results.

Let X be a topological space. The symbol $\mathcal{H}(X)$ denotes the group of all homeomorphisms of X onto itself and $\mathcal{G}(X)$ denotes the subgroup of $\mathcal{H}(X)$ consisting of all maps in $\mathcal{H}(X)$ which are isotopic to the identity map on X.

<u>Definition 1.1</u> Let $p \in X$. Then p is an <u>invert point</u> of X if and only if for each open neighborhood U of p there exists $h \in \mathcal{H}(X)$ such that $h(X - U) \subset U$.

Here h is an <u>inverting map</u> for U. The collection of all invert points of X is called the <u>invert set</u> of X and is denoted by $I(X)$. X is called <u>invertible</u> if and only if $I(X) = X$.

Definition 1.2 Let $p \in X$. Then p is a continuous invert point of X if and only if for each open neighborhood U of p there exists $g \in \mathcal{G}(X)$ such that $g(X - U) \subset U$.

The set of all continuous invert points of X is called the continuous invert set of X and is denoted by $CI(X)$. Clearly, $CI(X) \subset I(X)$. X is said to be continuously invertible if and only if $CI(X) = X$.

In [1] Doyle discussed the invert set in a finite geometric simplicial complex. He proved that if K is a complex, then $I(K)$ carries subcomplexes of each triangulation of K. In other words, $I(K)$ is invariant under triangulations of K in this sense. He also showed that if K is a complex, then $I(K)$ is null, a point, or a finite simplicial sphere. The next two theorems give characterizations of 1- and 2-complexes with a single invert point.

Theorem 1.1 (Doyle) Let K be a 1-complex. Then $I(K) = \{p\}$ if and only if K is a set of r (≥ 2) simple closed curves meeting in p but otherwise disjoint in pairs (an r-leafed rose).

Theorem 1.2 (Klassen) Let K be a 2-complex. Then $I(K) = \{p\}$ if and only if

$$K = \left(\bigcup_{i=1}^{m} C_i^2 \right) \cup \left(\bigcup_{j=1}^{n} S_j^1 \right) ,$$

where (i) C_i^2 is a 2-cell, a 2-sphere, a pinched annulus or a pinched torus for $1 \leq i \leq m$ such that $C_s^2 \cap C_t^2 = \{p\}$ or a union of 1-spheres containing p for all $s \neq t$ and $1 \leq s \leq m$, $1 \leq t \leq m$; and (ii) S_j^1 is a 1-sphere which is disjoint from

$$[\left(\bigcup_{i=1}^{m} C_i^2\right) \cup \left(\bigcup_{\substack{r=1 \\ r \neq j}}^{n} S_r^1\right)] - p$$

for $1 \leq j \leq n$.

In [12] Klassen gave the characterization of a 1-complex with a 0-sphere as its invert set. For the purpose of simplicity, S^n will denote an n-sphere for $n = 0,1,2, \cdots$ and $\mathcal{S}^k(K)$ will denote the k-fold suspension of K with $\mathcal{S}^1(K)$ written as $\mathcal{S}(K)$. The cardinality of a set A will be written as $|A|$.

<u>Theorem 1.3 (Klassen)</u> Let K be a connected 1-complex. Then $I(K) = S^o$ if and only if $K = \mathcal{S}(F)$ where F is a set of finite number of points with $|F| \neq 2$.

In [9] Hocking generalized a result of Klassen and proved the following:

<u>Theorem 1.4 (Hocking)</u> A complex K is a suspension if and only if $I(K)$ contains a 0-sphere.

The next result discusses complexes K with $\dim I(K) = \dim K$.

<u>Theorem 1.5</u> Let K be an n-complex where $n \geq 1$. Then $I(K) = S^n$ if and only if $K \stackrel{T}{=} S^n$.

<u>Proof</u>. Klassen proved the result for $n = 1$ in [12]. Moreover, if $K = S^n$, then $I(K) = S^n$ (See [2] and [3]). So let K be an n-complex with $I(K) = S^n$. Then S^n is a subcomplex of K. Let $p \in S^n$ such that $p \in \text{Int } \sigma^n$ where σ^n is a principal n-simplex in S^n. Let U be an open neighborhood of p in σ^n and h the corresponding inverting map such that $h(K - U) \subset U$. Now $U \stackrel{T}{=} E^n$ and it can be so arranged that $K - U \stackrel{T}{=} E^n$. Then K is an n-manifold. Using the characterizations in [2] and [3], we get $K \stackrel{T}{=} S^n$.

In a recent unpublished work, Hocking proved the following result which shows that all complexes K with $0 \leq \dim I(K) \leq \dim K$ are multiple suspensions. First we state a lemma whose proof is omitted.

<u>Lemma 1.6</u> Let A^k be a k-simplex in a complex K and let A^k_1 be a k-simplex in the barycentric subdivision of A^k. Then $Lk(A^k_1, K') \stackrel{T}{=} Lk(A^k, K)$.

<u>Theorem 1.7 (Hocking)</u> If the n-complex K^n has $I(K) = S^k$, $0 \leq k \leq n$, then $K^n \stackrel{T}{=} \mathcal{S}^{k+1}(L)$.

<u>Proof</u>. Let A^k be a principal k-simplex in $I(K^n)$. Choose $p \in \text{Int } A^k$ such that p lies interior to a k-simplex in each barycentric subdivision of A^k. This is pos-

sible since a barycentric subdivision introduces finitely many points of the subdivision leading to a countable set of points by successive barycentric subdivisions.

Let U_o be a closed neighborhood of p in $\mathrm{Int}\, A^k$ such that the boundary of U_o relative to A^k is $\mathrm{Bd}\, U_o = S^{k-1}$ and $U_o = p \circ \mathrm{Bd}\, U_o = p \circ S^{k-1}$. Now $p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)$ is a closed neighborhood of p in K^n. Choose $q \in I(K^n) - A^k$. Then there exists $h_o \in \mathcal{H}(K^n)$ such that $h_o(q) = p$ and

$$h_o\Big(K^n - \mathrm{Int}\,\big(p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)\big)\Big)$$
$$\subset \quad \mathrm{Int}\,\big(p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)\big) \quad .$$

This may be rewritten as $h_o^{-1}(p) = q$ and

$$K^n - \mathrm{Int}\,\big(p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)\big)$$
$$\subset \quad h_o^{-1}\Big(\mathrm{Int}\,\big(p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)\big)\Big)$$

Passing to the barycentric subdivision, let $p \in A_1^k$ where A_1^k is a k-simplex in K'. Keeping p and $\mathrm{Bd}\, A^k$ pointwise fixed, shrink S^{k-1} to lie in $\mathrm{Int}\, A_1^k$ and then $U_1 = p \circ S_1^{k-1}$ is a closed neighborhood of p in A_1^k and $p \circ S_1^{k-1} \circ \mathrm{Lk}(A_1^k, K')$ is a closed neighborhood of p in K^n. By Lemma 1.6, $p \circ S_1^{k-1} \circ \mathrm{Lk}(A_1^k, K') \overset{T}{=} p \circ S^{k-1} \circ \mathrm{Lk}(A^k, K^n)$. Also, there exists $h_1 \in \mathcal{H}(K^n)$ such that $h_1(q) = p$ and

$$h_1\Big(K^n - \mathrm{Int}\,\big(p \circ S_1^{k-1} \circ \mathrm{Lk}(A_1^k, K')\big)\Big)$$
$$\subset \quad \mathrm{Int}\,\big(p \circ S_1^{k-1} \circ \mathrm{Lk}(A_1^k, K')\big) \quad ,$$

or $h_1^{-1}(p) = q$ and

$$K^n - \mathrm{Int}\left(p \circ S_1^{k-1} \circ Lk(A_1^k, K')\right)$$

$$\subset \quad h_1^{-1}\left(\mathrm{Int}\left(p \circ S_1^{k-1} \circ Lk(A_1^k, K')\right)\right) \quad .$$

Let $g_1 \in \mathcal{H}(K^n)$ such that

$$g_1\left(p \circ S^{k-1} \circ Lk(A^k, K^n)\right)$$

$$= \quad p \circ S_1^{k-1} \circ Lk(A_1^k, K').$$

Consider $h_1^{-1}\, g_1\left(\mathrm{Int}\left(p \circ S^{k-1} \circ Lk(A^k, K^n)\right)\right)$ as an open cone neighborhood of q. By construction, we get

$$K^n - p = \bigcup_{i=0}^{\infty} h_i^{-1}\, g_i\left(\mathrm{Int}\left(p \circ S^{k-1} \circ Lk(A^k, K^n)\right)\right) \quad .$$

By uniqueness of the open cone neighborhood (see [13]) we get

$$K^n - p \overset{T}{=} \mathrm{Int}\left(p \circ S^{k-1} \circ Lk(A^k, K^n)\right) \quad .$$

Then K^n is the 1-point compactification of $p \circ S^{k-1} \circ Lk(A^k, K^n)$, or K^n is homeomorphic to $S^o \circ S^{k-1} \circ Lk(A^k, K^n)$. The induction on k is now obvious, and we get

$$K \overset{T}{=} S^k \circ Lk(A^k, K^n) = \mathcal{S}^{k+1}(L),$$

where $L = Lk(A^k, K^n)$ is an (n-k-1)-complex. Also, Theorem 1.4 and Theorem 1.5 correspond to $k = 0$ and n respectively. This completes the proof.

The next theorem is due to Klassen (Theorem 4.1 of [12]). We present a simplified version of the proof.

<u>Theorem 1.8 (Klassen)</u> Let K be an n-complex with $I(K) = \{p\}$. Then $p \in CI(K)$ and consequently $CI(K) = I(K)$.

Proof. Let $V = \mathring{St}\, p$ be the open star of p. Let $h \in \mathcal{H}(K)$ be an inverting map for V. Then $h(K - V) \subset V$ and $h(p) = p$ since $I(K) = \{p\}$. Also, $K - V \subset h(V)$ implies that $h^{-1}(K - V) \subset V$. By uniform continuity and $h(p) = p$, there exists a neighborhood U of p such that $U \subset V$ and $h(U) \subset V$.

Let $r \in \mathcal{H}(K)$ such that r is the identity outside $\overline{V}$ and $rh^{-1}(K - V) \subset U$. Then $hrh^{-1}(K - V) \subset h(U) \subset V$. Now r can be accomplished by an isotopy r_t, $0 \le t \le 1$, such that $r_1 = r$. Then $g_t = hr_t h^{-1}$ is an inverting map for V with $g_o = id$ and $g_1(K - V) \subset V$. Thus $g_1 \in \mathcal{G}(K)$ and $p \in CI(K)$. Since $CI(K) \subset I(K)$, we get $CI(K) = I(K)$.

Remark. Let K be any complex with $I(K) = \{p,q\}$. We assert that $CI(K) = \emptyset$. If not, let $p \in CI(K)$ and U be any open neighborhood of p which excludes q. There exists an inverting map $g \in \mathcal{G}(K)$ such that $g(q) \in U$. But every point of the arc $g_t(q)$, $0 \le t \le 1$, is an invert point of K. This is a contradiction. However, Hocking proved the following theorem in [9].

Theorem 1.9 (Hocking) Let K be a complex such that $\dim I(K) \ge 1$ and $CI(K) \ne \emptyset$. Then $CI(K) = I(K)$.

Hocking conjectured in [9] that $\dim I(K) \ge 1$ is enough to imply $CI(K) = I(K)$. We prove this in the next chapter. The next theorem is also due to Hocking.

<u>Theorem 1.10 (Hocking)</u> For complexes P and Q, let $P = \mathcal{S}(Q)$. If $\dim I(P) \geq 1$, then (i) $I(Q) \subset Q \cap I(P)$ and (ii) $CI(Q) \subset Q \cap CI(P)$.

It was mentioned in [9] that if equality could be proved in Theorem 1.10, other well known results may then be used with this to prove the Poincaré Conjecture in dimension four. In other words, the concept of an invert set and some current problems in combinatorial topology are related.

Unless otherwise specified, we will follow the standard terminology of [10].

CHAPTER II

GENERAL RESULTS

<u>Theorem 2.1</u> Let K be a complex with $\dim I(K) \geq 1$. Then $CI(K) = I(K)$.

<u>Proof</u>. By Theorem 1.4 we can write $K = \mathcal{S}(L)$ with p and q as the vertices of suspension. Let $s \in L \cap I(K)$ and $h \in \mathcal{H}(K)$ such that $h(p) = s$. Let g_t $(0 \leq t \leq 1)$ be an isotopy such that $g_t h(p)$ moves away from s. This is possible since there is a product neighborhood of s in K. Let $f_t = h^{-1} g_t h$. We use this to move p. Since $f_o = h^{-1}(id)h = id$ and f_t is a homeomorphism, f_t is isotopic to the identity map. Also $h^{-1} g_t h(p) \neq p$.

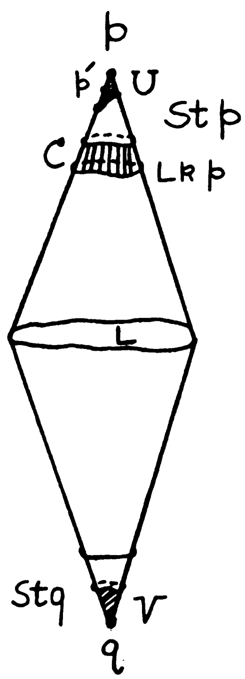

Figure 2.1

Let U be an open set in K containing p. For our purpose it will be enough to take $U \subset \mathring{S}t\, p$. Also, $Lk\, p$ is bicollared in K. Choose a collar C of $Lk\, p$ in $\mathring{S}t\, p$ such that $U \cap C = \emptyset$. Let $f_t \in \mathcal{G}(K)$ such that it is fixed outside $\mathring{S}t\, p - C$ and moves p to $f_t(p) = p'$ with $p' \in U$.

Since K is a suspension, there exists $\alpha_t \in \mathcal{G}(K)$ such that $\alpha_1(K - \mathring{S}t\, p) \subset V$ and $\alpha_t | \mathring{S}t\, p = id$, where V is a sufficiently small open neighborhood of q such that $V \subset \mathring{S}t\, q$. Let $\beta_t \in \mathcal{G}(K)$ such that $\beta_t | (K - \mathring{S}t\, q) = id$ and is such that it slides V away from q. This is done by arguments similar to that used above to construct f_t. Let $\gamma_t \in \mathcal{G}(K)$ such that $\gamma_t | (\mathring{S}t\, p - C) = id$ and slides $\beta_1(V)$ inside C. Finally, let $\delta_t \in \mathcal{G}(K)$ be such that $\delta_t | (K - \mathring{S}t\, p) = id$ and $\delta_1(p') = p$.

Now define $\emptyset_t = \delta_t \gamma_t \beta_t \alpha_t f_t$, $0 \leq t \leq 1$. Then $\emptyset_o = id$ and

$$\begin{aligned} \emptyset_1(K - \mathring{S}t\, p) &= \delta_1 \gamma_1 \beta_1 \alpha_1 f_1 (K - \mathring{S}t\, p) \\ &\subset \delta_1 \gamma_1 \beta_1 \alpha_1 (K - \mathring{S}t\, p) \\ &\subset \delta_1 \gamma_1 \beta_1 (V) \\ &\subset \delta_1 (C) \\ &\subset \mathring{S}t\, p. \end{aligned}$$

This shows that $\emptyset_1$ is the required inverting map for $\mathring{S}t\, p$ and is isotopic to the identity map on K. Hence $p \in CI(K)$. By Theorem 1.9, $CI(K) = I(K)$. This completes the proof.

<u>Theorem 2.2</u> Let K be a complex with $I(K) \neq \emptyset$ or $\dot{S}^o$. If $p \in I(K)$, then there is an inverting homeomorphism $f \in \mathcal{H}(K)$ such that $f(p) = p$.

<u>Proof</u>. If $I(K) = \{p\}$, then every inverting homeomorphism fixes p. So let $\dim I(K) = k \geq 1$. Without any loss of generality we may assume that $p \in \operatorname{Int} \sigma^k$, where σ^k is a principal k-simplex in $I(K)$. Let U be any open set containing p such that $U \cap \sigma^k = V \subset \operatorname{Int} \sigma^k$. Choose $q \in V$ and $p \neq q$. Let $W \subset V$, $p \notin W$, $q \in W$ such that that an open set A in K containing q has $A \cap \sigma^k = W$ and $A \subset U$.

Since $q \in I(K)$, there exists $h \in \mathcal{H}(K)$ such that $h(K - A) \subset A$ and $q \neq h(p) \in W$. Choose $g \in \mathcal{H}(K)$ such that (i) $gh(p) = p$ and (ii) $g(A) \subset U$. Define $f = g \circ h$. Then $f \in \mathcal{H}(K)$. Also,

$$f(K - U) = gh(K - U) \subset gh(K - A) \subset g(A) \subset U$$

$$\text{and} \quad f(p) = gh(p) = p.$$

<u>Corollary 2.3</u> Let K be a complex with $I(K) \neq \emptyset$ or S^o. Let $p \in I(K)$ and U be an open set containing p. Then there exists an open set $V \subset U$ and $p \in V$ such that some inverting homeomorphism fixes V pointwise.

<u>Proof</u>. As in the proof of Theorem 2.2, assume that $p \in \operatorname{Int} \sigma^k$. Since there is an inverting homeomorphism f which fixes p, by uniform continuity there exists a symmetric ball V in U, with p as center and such that $f|V = \mathrm{id}$.

Theorem 2.4 Let K be a complex with $\dim I(K) \geq 2$. Then $I(K)$ is continuously ω-homogeneously embedded in K.

Proof. By Theorem 2.1, $CI(K) = I(K)$. Let

$$A_n = \{a_1, a_2, \cdots, a_n\}$$

and $$B_n = \{b_1, b_2, \cdots, b_n\}$$

be any two sets of distinct points in $I(K)$. We can choose a triangulation T of K fine enough to ensure that there is a principal simplex σ in $I(K)$ such that $(A_n \cup B_n) \cap \mathrm{St}\,\sigma = \emptyset$. Let $p \in \mathrm{Int}\,\sigma \subset \mathring{\mathrm{St}}\,\sigma$. Since $p \in I(K)$, there exists $f \in \mathcal{H}(K)$ such that $f(K - \mathring{\mathrm{St}}\,\sigma) \subset \mathring{\mathrm{St}}\,\sigma$. Thus $f(A_n)$ and $f(B_n)$ are contained in $\mathrm{Int}\,\sigma$.

If $n = 1$, we use Lemma 0 of [1] to obtain $g \in \mathcal{G}(K)$ such that $g|(K - \mathrm{Int}\,\sigma) = \mathrm{id}$ and $gh(a_1) = f(b_1)$. Define $h = f^{-1}gf$. Then $h \in \mathcal{G}(K)$ and $h(A_1) = B_1$.

As induction hypothesis, assume that for all i such that $1 < i < n$, there exists $h \in \mathcal{G}(K)$ such that $h(A_i) = B_i$. Let $A_n = A_{n-1} \cup a_n$ and $B_n = B_{n-1} \cup b_n$. Then there exists $h \in \mathcal{G}(K)$ such that $h(A_{n-1}) = B_{n-1}$. If $h(a_n) = b_n$, we are done. Otherwise, let D be a closed set containing B_{n-1} and if needed, attach a collar C to D. Now there exists an isotopy θ_t which moves a_n to b_n in $\sigma - (C \cup D)$ leaving D fixed. Then $\alpha_t = \theta_t \cdot h$ is the required isotopy.

Remark. When $\dim I(K) = 1$, a similar result may be proved if we disregard the order of points in A_n and B_n.

Theorem 2.5 Let K be a complex with $I(K) \neq \emptyset$ or S^0 and $p \in I(K)$. Then $K - p$ is an open monotone union $\overset{\infty}{\underset{i=1}{(M)}}\mathcal{U}_i$, where $\mathcal{U}_i \overset{T}{=} \operatorname{Lk} p \times E^1$.

Proof. Let U be an open cone neighborhood of p and C be any compact set in $K - p$. Then there exists an inverting homeomorphism $h \in \mathcal{H}(K)$ such that $h(C) \subset U$. Consider $U = \operatorname{Lk} p \times [0,1)$ with $\operatorname{Lk} p \times 0$ identified with p. Then every compact set in $K - p$ is contained in a product space $\operatorname{Lk} p \times E^1$. Thus $K - p$ is the monotone union $\overset{\infty}{\underset{i=1}{(M)}}\mathcal{U}_i$, where $\mathcal{U}_i \overset{T}{=} \operatorname{Lk} p \times E^1$.

We observe that if K is a 1-complex with $I(K) \neq \emptyset$ or S^0 and $p \in I(K)$, then $K - p \overset{T}{=} F \times E^1$, where F is a finite set of points such that $|F| = 1$ for $I(K) = S^1$ and $|F| \geq 2$ for $I(K) = \{p\}$. If K is a 2-complex with $I(K) \neq \emptyset$ or S^0 and $p \in I(K)$, then

$$K - p \overset{T}{=} \begin{cases} E^1 \times E^1 & \text{if } I(K) = S^2 \\ B \times E^1 & \text{if } I(K) = S^1 \end{cases}$$

where B is a one point union of b (≥ 3) semi-open intervals. If $I(K) = \{p\}$, the cases are more complicating in view of Theorem 1.2. It may be possible to show that $K - p \overset{T}{=} G \times E^1$ where G is a graph.

Let G be a graph and consider $U = \underset{i=1}{\overset{\infty}{(\mathrm{M})}}\,\mathcal{U}_i$ where $\mathcal{U}_i \overset{T}{=} G \times E^1$. The 1-point compactification of U gives only one space K which is invertible at a point p. Thus monotone union property gives rise to a unique space in this sense and the failure of this property may not yield uniqueness. The following example is illustrative of the first part and serves as a counter example for many intuitive conjectures for complexes with a single point of invertibility.

<u>Example</u>. Let G be a graph which is a one point union of a 1-sphere and an open interval. Then the one point compactification of $\underset{i=1}{\overset{\infty}{(\mathrm{M})}}\,\mathcal{U}_i$, where $\mathcal{U}_i \overset{T}{=} L \times E^1$ and L is a union of two 1-spheres joined by an arc, is a pinched torus with a spanning disk. If we call this complex K then $I(K) = \{p\}$. We note that $K - p \overset{T}{=} G \times E^1$ and K is not a pinched suspension.

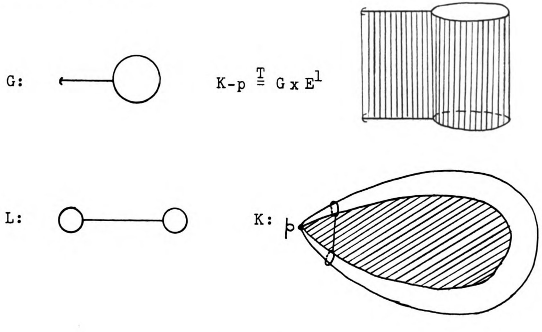

Figure 2.2

If G is a connected graph, consider $\mathcal{U}_i \stackrel{T}{=} G \times E^1$. We note that G can be embedded in E^3. Let v be a vertex in G of maximum degree d. We claim that $\mathcal{U}_i$ can be embedded in a d-book if G is a tree. In order to see this, note that vertices in G of degree 2 present no problem. The same is true for the open end of a 1-simplex. Since G is connected, let a vertex x of degree α be joined to a vertex y of degree β. Construct α- and β-books at x and y respectively. Let $\beta \leq \alpha$. Since these books have one page in common, they can be embedded in an α-book. A repeated application of the same argument yields the result. Since $G \times E^1$ contains a copy of $\underset{i=1}{\overset{\infty}{\textcircled{M}}}\mathcal{U}_i$, we conclude that the monotone union $\underset{i=1}{\overset{\infty}{\textcircled{M}}}\mathcal{U}_i$ can be embedded in a d-book. If G is not a tree, the number of pages in the book may have to be increased.

In [12] Klassen proved that if K is a 2-complex with $I(K) = \{p\}$, then $K \stackrel{T}{=} L \times [0,1]$ where $L \times 0 \cup L \times 1 \cup M \times [0,1]$ is identified with p and M is a finite set of points in a 1-complex L. This leads to the following result.

<u>Proposition 2.6</u> Let K be a 1-complex with $I(K) \neq \emptyset$. Let F be a finite set of points with $|F| = f$. Then

(a) $I(K) = \{p\}$ if and only if $K \stackrel{T}{=} F \times [0,1]$ where $f \geq 2$ and $F \times 0 \cup F \times 1$ is identified with p,

(b) $I(K) = \dot{S}^0$ if and only if $K \stackrel{T}{=} F \times [0,1]$ where $f \neq 2$ and $F \times 0$ and $F \times 1$ are identified with

p and q respectively

and (c) $I(K) = S^1$ if and only if $K \overset{T}{=} F \times [0,1]$ where $f = 1$ and $F \times 0 \cup F \times 1$ is identified with p or $f = 2$ and $F \times 0$ and $F \times 1$ are identified with p and q respectively.

<u>Proof</u>. Obvious from Theorems 1.1, 1.3 and 1.5.

Similarly it is possible to write down the corresponding result for a 2-complex in view of the earlier theorems. So far it has not been possible to factor higher dimensional complexes with a non-empty invert set in this fashion. The aim of the last proposition is to exhibit a factorization with $[0,1]$ as one of its factors, as compared to Theorem 2.5 in which $K - p$ can be written as a monotone union $\overset{\infty}{\underset{i=1}{\textcircled{\scriptsize M}}} \mathcal{U}_i$ where $\mathcal{U}_i$ has a factor $(0,1)$.

<u>Proposition 2.7</u> Let K be a 1-complex such that $I(K) = S^0$. If $I(K/I(K)) \neq \{p\}$, then $K = D^1$ (1-cell).

<u>Proof</u>. $I(K) = S^0$ implies that $K = \mathcal{S}(F)$ where F is a finite set of points with $|F| \neq 2$. Then $K/I(K)$ is a rose with $|F|$ leaves. $|F| \neq 1$ implies that $I\left(K/I(K)\right) = \{p\}$. Hence $|F| = 1$ and $K = D^1$.

<u>Proposition 2.8</u> Let K be a 2-complex such that $I(K) = S^1$. If $I\left(K/I(K)\right) \neq \{p\}$, then $K = D^2$ (2-cell).

<u>Proof</u>. We have $K = \mathcal{S}^2(F)$ where F is a finite set of points with $|F| \neq 2$. Clearly, $|F| \geq 3$ implies that $K/I(K)$ is a one point union of $|F|$ 2-spheres.

Thus $I\left(K/I(K)\right) = \{p\}$. For $|F| = 1$, $K = \mathcal{S}^2(F) = D^2$ and $I\left(K/I(K)\right) = I(S^2) = S^2$.

Next we quote from [1] a general theorem in this context.

Theorem 2.9 (Doyle) Let K be an n-complex which is not a point and let $I(K) \neq \emptyset$. If $K/I(K)$ is invertible, then K is a sphere or a cell if K is a manifold or has a free $(n-1)$-face.

In [6] it was proved that if $I(S \times T) \neq \emptyset$, then $I(S) \times I(T) \subseteq I(S \times T)$, and that the product of two invertible spaces is either invertible or has empty invert set. This, when applied to complexes gives some interesting results. For instance, if K_1 and K_2 are complexes such that $I(K_1) \neq \emptyset$, $I(K_2) \neq \emptyset$ and $I(K_1 \times K_2) \neq \emptyset$, then $\emptyset \neq I(K_1) \times I(K_2) \subseteq I(K_1 \times K_2)$. If $I(K_1 \times K_2) = (p,q)$, $I(K_1) \neq \emptyset$, and $I(K_2) \neq \emptyset$, then $I(K_1) = \{p\}$ and $I(K_2) = \{q\}$. We note that this may be vaccuously satisfied. Moreover, if $I(K_1) \supseteq S^o$ and $I(K_2) \supseteq S^o$, then $I(K_1 \times K_2) \neq \emptyset$ implies that $\dim I(K_1 \times K_2) \geq 1$.

Let $K_1 = S^{k_1}$, $K_2 = S^{k_2}$ where $k_1 \geq 1$ and $k_2 \geq 1$. Assume that $K_1 \times K_2 \neq S^{k_1+k_2}$. Then we assert that $I(K_1 \times K_2) = \emptyset$. Note that $I(K_1) = K_1$ and $I(K_2) = K_2$. If $I(K_1 \times K_2) \neq \emptyset$, then $K_1 \times K_2 \subseteq I(K_1 \times K_2)$. But $I(K_1 \times K_2) \subseteq K_1 \times K_2$ implies that $I(K_1 \times K_2) = K_1 \times K_2$. By Theorem 1.5 this implies that $K_1 \times K_2 = S^{k_1+k_2}$. This is a contradiction.

<u>Theorem 2.10 (Doyle)</u> Let K be an n-complex with $\dim I(K) = k \geq 0$ and $n > k$. Then $H_i(K)$ and $\pi_i(K)$ are trivial for $0 < i \leq k$.

<u>Proof</u>. For the groups under consideration it is enough to consider continuous cycles or singular spheres that lie in the k-skeleton of K. First note that their homotopy classes are all represented by maps having $I(K)$ as carrier.

Let σ^k be a k-simplex of $I(K)$. Since $n > k$, σ^k lies on the face of a (k+1)-simplex σ^{k+1}. If σ^{k+1} has more than one k-simplex in $I(K)$ then use its barycenter to ensure that $\sigma^{k+1} \cap I(K) = \sigma^k$. This means that each map with $I(K)$ as carrier can be homotoped away from $I(K)$ leaving Int σ^k uncovered. This completes the proof.

<u>Proposition 2.11</u> Let K_1 and K_2 be k_1- and k_2-complexes respectively with $\dim I(K_1 \times K_2) = k \geq 0$. If $k < k_1 + k_2$, then $\pi_i(K)$ is trivial for $0 < i \leq k$, where K is K_1, K_2 or $K_1 \times K_2$. Moreover, $H_i(K_1 \times K_2) = 0$ for $0 < i \leq k$.

<u>Proof</u>. Note that $\pi_i(K_1 \times K_2) = \pi_i(K_1) \oplus \pi_i(K_2)$ for $0 < i \leq k_1 + k_2$. Now apply the last theorem.

<u>Theorem 2.12</u> Let $K = K_1 \times K_2$ where K_1 and K_2 are non-degenerate connected complexes with $\dim K = k$. If $I(K) \neq \emptyset$ or S^0 and $K \neq S^k$, then K, K_1 and K_2 are contractible.

<u>Proof</u>. Select $p_2 \in K_2$ such that $K_1 \times p_2 \not\subset I(K)$

and then choose $q \in I(K)$ such that $q \notin K_1 \times p_2$. Let U be a neighborhood of q in K such that $U \cap K_1 \times p_2 = \emptyset$. By Theorem 2.1 there exists an isotopy $h_t \in \mathcal{G}(K)$ with $0 \le t \le \frac{1}{2}$ and such that $h_{\frac{1}{2}}(K_1 \times p_2) \subset U$.

By selecting U to be an open cone neighborhood and extending the isotopy h_t to a homotopy $\tilde{h}_t$ we get $\tilde{h}_t : K \times I \to K$ such that $0 \le t \le 1$ and $\tilde{h}_1(K_1 \times p_2) = q$. This shows that K_1 and K_2 are contractible. Since there is a retraction g of K onto $K_1 \times p_2$, $\tilde{h}_t \circ g$ gives the required contraction of K into a point.

For example, let K_1 = fake 3-cell and K_2 = 2-disk. Then $K_1 \times K_2$ is a 5-cell with $I(K_1 \times K_2)$ = 4-sphere. By the last theorem, K_1 is contractible. The result of the theorem is more effective in a negative sense. Thus, if P is a non-contractible complex such that it is a factor of another complex K. Then $I(K) = \emptyset$ or S^0.

Remark. Let K^n be an n-complex with $\dim I(K^n) = k$. If $k = n$, then $K^n \overset{T}{=} \mathcal{S}^n(F)$ with $|F| = 2$; and if $k = n-1$, then $K^n \overset{T}{=} \mathcal{S}^n(F)$ with $|F| \neq 2$, where F is a finite set of points. This follows from Theorems 1.5 and 1.7. When $k = 0$, we can prove the following.

Proposition 2.13 Let K be an n-complex and $I(K) = S^0$. Then $K = \mathcal{S}(L)$ where $I(L) = \emptyset$ or $\{p\}$.

Proof. Under the hypothesis, $K = \mathcal{S}(L)$ by Theorem 1.4. Assume that $I(L) \supsetneq S^0$. Again by Theorem 1.4, we get $L = \mathcal{S}(M)$. But then $K = \mathcal{S}^2(M)$ and $I(K)$ contains at least a 1-sphere, namely the suspension ring (Theorem 4 of [1]). This is a contradiction.

The next two results complete the discussion of 2-complexes in view of Theorems 1.2 and 1.5. A part of the result is obtainable from Theorem 1.7; but we present an intuitive and independent proof.

Proposition 2.14 Let K be a 2-complex. Then $I(K) = S^1$ if and only if $K = \mathcal{S}^2(F)$, where F is a finite set of points with $|F| \neq 2$.

Proof. If $|F| = 2$, then $\mathcal{S}^2(F) = S^2$ and we get an immediate contradiction by using Theorem 1.5. So let F be a finite set of points with $|F| \neq 2$. Since $K = \mathcal{S}^2(F)$ is a double suspension, by Theorem 4 of [1], $S^1 \subseteq I(K)$ where S^1 is the suspension ring. Clearly $I(K) \neq S^2$, otherwise $K = S^2$ and we can write $K = \mathcal{S}^2(F)$ where $|F| = 2$. Hence $I(K) = S^1$.

Let K be a 2-complex with $I(K) = S^1$. By Theorem 1.4, $K = \mathcal{S}(L)$ where L is a 1-complex and the vertices of suspension belong to $I(K)$. Since $I(K)$ is a 1-sphere, there exist x and y in L such that $\{x,y\} \subset I(K)$. A small product neighborhood of x in K is an n-book for some n. Since all points on the back of this n-book are in $I(K)$, it follows that K is locally euclidean except at points of $I(K)$ and hence L is locally euclidean except at x and y. This shows that $\{x,y\} \subseteq I(L)$. We cannot have $I(L)$ as a 1-sphere otherwise L becomes a 1-sphere by Theorem 1.5 and consequently K is a 2-sphere. Then $I(L) = \{x,y\}$ implies that $L = \mathcal{S}(F)$, where F is a finite set of points with $|F| \neq 2$ (Theorem 1.3). This completes the proof.

<u>Proposition 2.15</u> Let K be a connected 2-complex. Then $I(K) = S^0$ if and only if $K = \mathcal{S}(L)$, where L is a 1-comples which is not a suspension of a finite number of points.

<u>Proof</u>. If $I(K) = S^0$ we can write $K = \mathcal{S}(L)$ where L is a 1-complex. Assume that $L = \mathcal{S}(F)$ where F is a finite set of points. If $|F| = 2$, then $L = S^1$ and $K = \mathcal{S}(L) = S^2$, whence $I(K) = S^2$. If $|F| \neq 2$, by Theorem 2.14 we get $I(K) = S^1$. This proves the necessary part of the theorem.

For the sufficiency we note that if $K = \mathcal{S}(L)$ where $L \neq \mathcal{S}(F)$, then $I(K) \neq S^1$ or S^2. But $K = \mathcal{S}(L)$ implies that $S^0 \subseteq I(K)$. Hence $I(K) = S^0$.

We may also interpret the last result as follows. If K is a 2-complex with $I(K) = S^0$, then $K = \mathcal{S}(L)$ where L is a 1-complex. Now $I(L) \neq S^0$ or S^1 as in the proof. Hence $I(L) = \emptyset$ or $\{p\}$. The case $I(L) = \{p\}$ is covered by Theorem 1.1 and for $I(L) = \emptyset$ we note that L is a graph which is not homeomorphic to a two point union of arcs.

Finally, we note that as a consequence of Theorem

1.7, the nature of an n-complex K with $0 \leq \dim I(K) \leq n$ is determined as a multiple suspension. The general problem of investigating suspensions of complexes with an empty invert set gets involved with the Generalized Poincaré Conjecture. In this connection we refer to a paper by Edwards on Pseudo Cells. Let K be a five dimensional pseudo cell. Then $K \times I = I^6$ and $\mathcal{S}(K) = I^6$. This means that $I(K) = \emptyset$, since K is a manifold with boundary, and $I\left(\mathcal{S}(K)\right) = S^5$.

So we turn our attention to complexes with a single point invert set. Theorems 1.1 and 1.2 give characterizations of 1- and 2-complexes of this variety. The general problem appears to be quite complex. In the next two chapters we discuss some of the properties of such complexes with a single invert point.

CHAPTER III

ORBITS AND BROUWER PROPERTY

Definition 3.1 Let $x \in X$. Then the orbit of x under homeomorphisms is the set of all images of x under elements of $\mathcal{H}(X)$, and this is denoted by $O_H(X)$. The number of orbits of X under the action of $\mathcal{H}(X)$ is denoted by $NO_H(X)$.

Definition 3.2 For $x \in X$, the orbit of x under isotopies is the set of all images of x under elements of $\mathcal{G}(X)$ and is denoted by $O_I(X)$. The number of orbits of X under the action of $\mathcal{G}(X)$ is denoted by $NO_I(X)$.

The following proposition is obvious.

Proposition 3.1 Let K be an n-complex with $p \in I(K)$. Then (i) $NO_I(K) = 1$ if and only if $K = S^n$ for $n \geq 1$ or $K = \{p\}$.

(ii) $NO_I(K) = 2$ and $I(K) = \{p\}$ imply that $K - p$ is locally euclidean of dimension n.

(iii) $I(K) = S^o$ implies that $NO_I(K) = 0$.

Proposition 3.2 Let K be a connected n-complex with $p \notin I(K)$. If $\dim\{I(K)\} = k$, then $\dim\{O_I(p)\} > k$.

Proof. The proof is by induction on k. When $k = -1$, $I(K) = \emptyset$ and $\dim\{O_I(p)\} \geq 0$. For $k = 0$, $I(K)$ is a point or a 0-sphere. But $p \notin I(K)$ implies that p

is not a singularity of K and $\dim \{O_I(p)\} \geq 1$.

Assume that the result is true for all $k < m$. Let K be a connected n-complex with $\dim \{I(K)\} = m$, where $m \geq 1$. Let $p \notin I(K)$ and $\dim \{O_I(p)\} \leq m$. Under some triangulation T of K, let $O_I(p)$ be written as a union of open simplices. Then $L = \overline{O_I(p)}$ is a subcomplex of K under T and $\dim L \leq m$. Now $I(K) = S^m$ and each simplex of $I(K)$ is principal. Also, $S^m \cap L \neq \emptyset$. Let $M = S^m \cup L$ be a subcomplex of K under T. Then $\dim M = m$ and $S^m \subset I(M)$. This implies that $M = S^m$ and $L = \emptyset$. This is a contradiction. Hence $\dim \{O_I(p)\} > m$ and the proof is complete.

<u>Proposition 3.3</u> Let K be a 1-complex. Then

(i) $I(K) = \{p\}$ implies that $NO_H(K) = NO_I(K) = r+1$, where $r \geq 2$ is the number of leaves in K.

(ii) $I(K) = S^0$ implies that $NO_I(K) = 0$ and $NO_H(K) = f+2$, where f is the number of points over which K is a suspension and $f \neq 2$.

(iii) $I(K) = S^1$ implies that $NO_I(K) = NO_H(K) = 1$.

<u>Proof</u>. Follows from the definitions of orbits and earlier theorems.

<u>Proposition 3.4</u> Let K be a 2-complex with $I(K) = \{p\}$. Then (i) $NO_I(K) = 2$ implies that K is a pinched torus and (ii) $NO_I(K) = 3$ implies that K is one of the following: one point union of two pinched tori,

two 2-spheres, a pinched torus and a 2-sphere, a 2-sphere and a 1-sphere or a pinched torus and a 1-sphere.

Proof. For (i) we note that one orbit is necessary for p since $I(K) = \{p\}$. This implies that we cannot have free 1-simplices in K. So $K = \bigcup_{i=1}^{m} C_i^2$ as in Theorem 1.2. Also, $C_i^2 \cap C_j^2 = \{p\}$ for $i \neq j$ otherwise it is a 1-sphere and needs one orbit. Then clearly $m < 2$, otherwise $NO_I(K) \geq 3$. This shows that $m = 1$ and a pinched torus is the only possibility for K.

When $NO_I(K) = 3$, we write $K = \left(\bigcup_{i=1}^{m} C_i^2\right) \cup \left(\bigcup_{j=1}^{n} S_j^1\right)$ as in Theorem 1.2. If C_i^2 is a 2-cell, we get $K = C_i^2$ and $I(K) \neq \{p\}$. If C_i^2 is a pinched annulus, then $NO_I(K) \geq 4$. This leaves C_i^2 as a 2-sphere or a pinched torus. Clearly we must have $m = 1$ or 2, $n = 0$ or 1 and $C_i^2 \cap C_j^2 = \{p\}$ for $i \neq j$ otherwise the number of orbits exceed 3. With $m = 2$ we get the first three possibilities. If $m = 1$, we must have $r = 1$ otherwise $I(K) \neq \{p\}$. This yields the remaining possibilities.

For higher dimensional complexes with a single invert point, the restriction on the number of orbits under isotopy does not simplify the problem to any significant degree. It is useful to impose some extra restriction on the complex. For a 3-complex K with $I(K) = \{p\}$, the imposition of Brouwer Property and the restriction of $NO_I(K)$ leads to the following results. First we need the definition of Brouwer Property.

Definition 3.3 A topological space X has Brouwer property if and only if homeomorphic images in X of open subsets of X are also open subsets of X.

This definition follows G.T. Whyburn in [14]. The following results by Duda appear in [7]. By Brouwer's Theorem on the Invariance of Domain, Euclidean Spaces and manifolds have the Brouwer Property, whereas manifolds with non-empty boundary do not. If K is an n-complex with Brouwer Property, then every r-simplex σ^r, $r \leq n-1$, is the face of an n-simplex, every σ^{n-1} is the face of exactly two n-simplices, and if σ^s belongs to $St(\sigma^r)$ then $St(\sigma^r) - \sigma^s$ cannot contain the homeomorphic image of an open n-cell intersecting σ^r. If K is an n-complex with $n < 3$, then K has Brouwer Property if and only if K is an n-manifold. Also, there exist non-manifolds with Brouwer Property in all dimensions greater than 2.

Lemma 3.5 (a) Let K be an n-complex with $I(K) = \{p\}$. If K has Brouwer Property then so does $\mathrm{Lk}\, p$.

(b) Let $K = \mathcal{S}(L)$ have Brouwer Property. Then L has Brouwer Property.

Proof. If not, consider $\mathrm{Lk}\, p \times E^1$.

Remark. Let K be an n-complex with $I(K) = \{p\}$. Let $\mathcal{L}(K)$ be the subcomplex of K determined by the closed (n-1)-simplices which are faces of none, one, three or more n-simplices of K. Klassen proved that if $\mathcal{L}(K) = \emptyset$, then $p \in I(\mathcal{L}(K))$ (Theorem 4.11 of [12]). Moreover, $\mathcal{L}(K)$ was used to effect a separation of K useful for characterizing a 2-complex with a single invert point. Clearly $\mathcal{L}(K) = \emptyset$ if and only if $I\left(\mathcal{L}(K)\right) = \emptyset$. Also, $\mathcal{L}(K) = \emptyset$ is a necessary condition for a complex K with $I(K) = \{p\}$ to have Brouwer Property.

<u>Theorem 3.6</u> Let K be a 3-complex with Brouwer Property and $\dim\{I(K)\} \geq 1$. Then $K \overset{T}{=} S^3$.

<u>Proof</u>. Since $\dim\{I(K)\} \geq 1$, we can write $K = \mathcal{S}(L)$ where L is a 2-complex which has Brouwer Property since K has the same. Then L is a 2-manifold. Also, there exist x and y in L such that $\{x,y\} \subset L \cap I(K)$. Since L is a manifold, $L \subseteq I(K)$. Thus $K = \mathcal{S}(L) \subseteq I(K)$. Consequently $K = I(K)$ and by Theorem 1.5 we get $K \overset{T}{=} S^3$.

<u>Remark</u>. Let K be a 3-complex with $I(K) = S^0$ and having Brouwer Property. Then $K = \mathcal{S}(L)$ where L is a 2-complex with Brouwer Property by Lemma 3.5 and hence it is a 2-manifold M^2. It is possible that M^2 may be a disjoint union of m (≥ 1) 2-manifolds. From such a complex it is easy to obtain another with a single point invert set by identifying the two suspension points of $\mathcal{S}(L)$ as is the case in the next result.

<u>Theorem 3.7</u> Let K be a 3-complex with

Brouwer Property and $I(K) = \{p\}$. If $NO_I(K) = 2$, then K is a suspension of a closed 2-manifold M^2 with the suspension points identified at p.

Proof. We note that $Lk\,p$ has Brouwer Property by Lemma 3.5. Since $\dim\{Lk\,p\} = 2$, $Lk\,p$ is a closed 2-manifold. Out of the two orbits under isotopy, one orbit is necessary for p. This shows that K does not contain any simplex of dimension less than or equal to $(i-1)$ which is not a face of an i-simplex in K for $0 \leq i \leq 3$. Moreover $Lk\,p$ must have precisely two components, for if it has one, then $I(K) \supsetneq \{p\}$. Then K is a suspension over one of the components of $Lk\,p$ with the suspension points identified at p.

Corollary 3.8 Let K be a 3-complex with Brouwer Property and $I(K) = \{p\}$. Suppose that $NO_I(K) = 3$. Then

(i) $K = K_1 \cup K_2$ where $K_1 \cap K_2 = \{p\}$ and for $i = 1, 2$ K_i is a suspension of a 2-manifold with the suspension points identified at p or a cone over a 2-manifold from p,

or (ii) K is a suspension over a 2-manifold with the suspension points identified at p.

Proof. The proof proceeds as in the last theorem. However, since we have $NO_I(K) = 3$, it is possible to have two 3-complexes K_1 and K_2 with $K_1 \cap K_2 = \{p\}$ and each K_i behaving as in Theorem 3.7. This gives the first part of (i). But it is possible that $Lk\,p \cap K_i$ may be connected, in which case we get a cone over a 2-manifold from p. This completes case (i). The proof of (ii) is similar to that of the last theorem.

Let K be an n-complex with $I(K) = \{p\}$. Let $x \in K - p$ such that $\dim \{O_I(x)\} = k$ is minimal. If $O = O_I(x)$, then $\overline{O} = O \cup p$ and $p \in CI(\overline{O})$. Also, $\overline{O} - p$ is a k-manifold M^k with $\mathrm{Bd}\, M^k = \emptyset$. By an earlier remark, M^k has Brouwer Property and consequently $Lk(p,\overline{O})$ has Brouwer Property.

If $k = 1$, then $\overline{O} \overset{T}{=} S^1$. If $k = 2$, then $Lk(p,\overline{O})$ has dimension one and Brouwer Property. Thus it is a 1-manifold without boundary and so it is a collection of disjoint 1-spheres. If $Lk(p,\overline{O})$ is a 1-sphere then $\overline{O} \overset{T}{=} S^2$. If $Lk(p,\overline{O})$ is a collection of two disjoint 1-spheres, then $\overline{O} \overset{T}{=}$ a pinched torus. If $k = 3$, then $Lk(p,\overline{O})$ has dimension two and Brouwer Property, and is a 2-manifold without boundary. All this leads to the next result.

<u>Proposition 3.9</u> Let K be an n-complex with $I(K) = \{p\}$. Let $x \in K - p$ such that $\dim \{O_I(x)\} = k$ is minimal.

Then (i) $k = 1$ implies that $\overline{O_I(x)} \overset{T}{=} S^1$.

(ii) $k = 2$ implies that $Lk(p,\overline{O_I(x)})$ is a collection of disjoint 1-spheres

and (iii) $k = 3$ implies that $Lk(p,\overline{O_I(x)})$ is a 2-manifold without boundary.

In particular, the last proposition is useful for a 3-complex where the possible values of k are 1, 2 and 3, from which the nature of $Lk(p,\overline{O_I(x)})$ is available.

CHAPTER IV

SUSPENSIONS

<u>Theorem 4.1</u> Let K be a complex with $I(K) = \{p\}$. Then $p \in I(\mathcal{S}(K))$ if and only if $\dim\{I(\mathcal{S}(K))\} \geq 1$.

<u>Proof</u>. Let u and v be suspension points for $\mathcal{S}(K)$. Then $\{u,v\} \subseteq I(\mathcal{S}(K))$. If $p \in I(\mathcal{S}(K))$, then $|I(\mathcal{S}(K))| \geq 3$. This means that $\dim\{I(\mathcal{S}(K))\} \geq 1$. On the other hand, if $\dim\{I(\mathcal{S}(K))\} \geq 1$, then $I(K) \subseteq K \cap I(\mathcal{S}(K))$ (by Theorem 1.10). This shows that $p \in I(\mathcal{S}(K))$.

<u>Corollary 4.2</u> Let K be a complex with $I(K) = \{p\}$. Then $p \in I(\mathcal{S}(K))$ if and only if $CI(\mathcal{S}(K)) = I(\mathcal{S}(K))$.

<u>Proof.</u> If $p \in I(\mathcal{S}(K))$, then $\dim\{I(\mathcal{S}(K))\} \geq 1$ by the last result. Using Theorem 2.1 we get $CI(\mathcal{S}(K)) = I(\mathcal{S}(K))$. If $CI(\mathcal{S}(K)) = I(\mathcal{S}(K))$, then $|I(\mathcal{S}(K))| \geq 2$ since $\mathcal{S}(K)$ is a suspension. But $|I(\mathcal{S}(K))| = 2$ implies that $CI(\mathcal{S}(K)) = \emptyset$. This gives $\dim\{I(\mathcal{S}(K))\} \geq 1$. Now use Theorem 4.1.

<u>Proposition 4.3</u> Let K be an n-complex with $\dim\{I(K)\} = k \geq 1$. Then $\dim\{I(L)\} \leq k-1$ where $K = \mathcal{S}(L)$.

<u>Proof</u>. We use Theorems 1.4 and 1.10 to write $K = \mathcal{S}(L)$ with $I(L) \subseteq L \cap I(K)$. This shows that

$I(L) \subsetneq I(K)$. It is evident that equality is not possible as the vertices of suspension lie in $I(K)$ but not in $I(L)$. The result is now obvious.

Theorem 4.4 Let K be an n-complex with $I(K) = \{p\}$. If $I(\mathcal{S}(K)) \neq S^0$, then $\dim\{I(\mathcal{S}(K))\} \geq 2$.

Proof. Let $I(\mathcal{S}(K)) \supsetneq S^0$ and assume that $\dim\{I(\mathcal{S}(K))\} = 1$. Let u and v be the vertices of suspension used in obtaining $\mathcal{S}(K)$ from K. By Theorem 4.1 we note that $p \in I(\mathcal{S}(K))$. Also, there exists $q \in K$ such that $q \in I(\mathcal{S}(K))$ and $p \neq q$. Let U be an open neighborhood of p in k. Then there exists $h \in \mathcal{H}(K)$ such that $h(K - U) \subseteq U$. In particular, $h(q) \in U$. Now we can construct a sequence $\{h_i(q)\}$ converging to p in K and $h_i(q) \in U$ for $i = 1, 2, \cdots$. By suspending each h_i, we can show that $h_i(q) \in I(\mathcal{S}(K))$. By compactness and uniform continuity, this cannot happen unless $\dim\{I(\mathcal{S}(K))\} \geq 2$.

Corollary 4.5 Let K be an n-complex. If $\dim\{I(K)\} = 1$, then $K = \mathcal{S}(L)$ where $I(L)$ is empty or a 0-sphere.

Proof. We use Theorem 1.4 to write $K = \mathcal{S}(L)$. By Proposition 4.3, $\dim I(L) \leq 0$. Again, $I(L) = \{p\}$ is not possible by the last theorem.

Remark. We may compare the last result with that of Proposition 2.13.

From the standard results in topology, we note that if a complex K is a suspension of a complex L, then K is simply connected if and only if L is connected. Now let K be a complex such that $\dim\{I(K)\} \geq 2$. Then we assert that $\pi_1(K) = 1$. This is obvious in view of Theorem 1.7. For if $\dim\{I(K)\} = k$, then $K = \mathcal{S}^{k+1}(L)$ where $k \geq 2$. Hence K is the suspension of a connected complex. In other words, if K is an n-complex such that $\pi_1(K) \neq 1$, then $\dim\{I(K)\} \leq 1$. Moreover, if K is an n-complex with $\dim\{I(K)\} \geq 2$, then $\pi_1\big(K/I(K)\big) = 1$.

Let K be an n-complex with $I(K) = S^1$ and $\pi_1(K) \neq 1$. Then $K = \mathcal{S}(L)$ and L is not connected. Let $S^1 = \mathcal{S}(x \cup y)$ where $x, y \in L$. Then L has just two components. If u is a suspension vertex for K, u is a local cut point of K. Since I(K) consists of local cut points of continuous invertibility (see Theorem 2.1), we must have $\dim K = 1$ and $K = S^1$. This result can also be stated as follows. If K is an n-complex such that $I(K) = S^1$ and $K \neq S^1$, then $\pi_1(K) = 1$. Alternately, if K is an n-complex such that $K \neq S^1$ and $\pi_1(K) \neq 1$, then I(K) is empty, a single point, or a 0-sphere. We collect these results in the following proposition.

<u>Proposition 4.6</u> (a) $K = \mathcal{S}(L)$ is simply connected if and only if L is connected.

(b) $\dim\{I(K)\} \geq 2$ implies that $\pi_1(K) = 1$ and $\pi_1(K/I(K)) = 1$.

(c) $I(K) = S^1$ and $\pi_1(K) \neq 1$ implies that $K = S^1$, or $I(K) = S^1$ and $K \neq S^1$ implies that $\pi_1(K) = 1$, or $K \neq S^1$ and $\pi_1(K) \neq 1$ implies that $I(K) = \emptyset$, $\{p\}$ or S^0.

Next we discuss a few results on double suspensions. Let K be an n-complex with $I(K) = \{p\}$. Let a_1, b_1 be the vertices of suspension for $\mathcal{S}(K)$ and let a_2, b_2 be the vertices of suspension for $\mathcal{S}^2(K)$. We will write the double suspension of K as $D(K)$. The suspension ring is $a_1a_2 \cup a_2b_1 \cup b_1b_2 \cup b_2a_1$ and is written as $R = \langle a_1a_2b_1b_2 \rangle$. Doyle proved in [1] that $R \subseteq CI\big(D(K)\big)$.

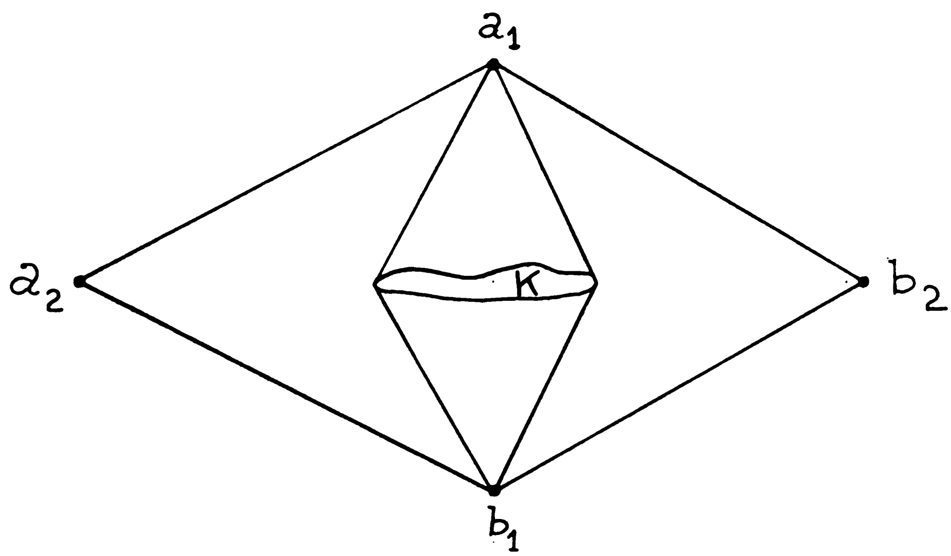

Figure 4.1

<u>Proposition 4.7</u> (i) $\dim\{I(D(K))\} \geq 1$

(ii) $I(D(K)) = CI(D(K))$

(iii) $I(\mathcal{S}(K)) \subseteq I(D(K))$ and

$CI(\mathcal{S}(K)) \subseteq CI(D(K))$.

<u>Proof</u>. Since $R \subseteq CI(D(K)) \subseteq I(D(K))$, we get (i) and (ii) by Theorem 2.1. For (iii) use Theorem 1.10.

Doyle proved in [1] that if $R \subsetneq I(D(K))$, then $\dim\{I(D(K))\} \geq 2$. We can now note the following results in view of the earlier theorems of this chapter. If $p \in I(\mathcal{S}(K))$ and $R \subsetneq I(D(K))$, then $CI(\mathcal{S}(K)) = I(\mathcal{S}(K))$ and has dimension 1 or more, and $CI(D(K)) = I(D(K))$ with dimension 2 or more. If $p \notin I(\mathcal{S}(K))$ and $R \subsetneq I(D(K))$, then $\emptyset = CI(\mathcal{S}(K)) \subsetneq I(\mathcal{S}(K)) = \{a_1, b_1\}$ and $CI(D(K)) = I(D(K))$ with dimension 2 or more. If $p \notin I(\mathcal{S}(K))$ and $R = I(D(K))$, then $\emptyset = CI(\mathcal{S}(K)) \subsetneq I(\mathcal{S}(K)) = \{a_1, b_1\} \subsetneq R = CI(D(K)) = I(D(K))$. We assert that it is impossible to have $p \in I(\mathcal{S}(K))$ and $R = I(D(K))$. Assume to the contrary. Then $CI(\mathcal{S}(K)) = I(\mathcal{S}(K))$ with dimension 1 or more by Theorems 2.1 and 4.1. Using Proposition 4.7 we get $R = CI(\mathcal{S}(K)) = I(\mathcal{S}(K)) = CI(D(K)) = I(D(K))$. But the disk spanned by $R \cup \text{arc}\,(a_1\, p\, b_1) \cup \text{arc}\,(a_2\, p\, b_2)$ must be contained in $I(D(K))$. This is a contradiction. This leads to the next result.

<u>Proposition 4.8</u> Let K be an n-complex with $I(K) = \{p\}$. Then (i) $p \in I(\mathcal{S}(K))$ implies that $R \subsetneq I(D(K))$ and

then

$$\dim\{I(\mathcal{S}(K))\} \geq 1 \quad \text{and} \quad \dim\{I(D(K))\} \geq 2,$$

(ii) $p \notin I(\mathcal{S}(K))$ and $R \subsetneq I(D(K))$ implies that

$$\emptyset = CI(\mathcal{S}(K)) \subsetneq I(\mathcal{S}(K)) = S^0 \subsetneq CI(D(K)) = I(D(K)) = S^{k \geq 2},$$

and (iii) $p \notin I(\mathcal{S}(K))$ and $R = I(D(K))$ implies that

$$\emptyset = CI(\mathcal{S}(K)) \subsetneq I(\mathcal{S}(K)) = S^0 \subsetneq CI(D(K)) = I(D(K)) = S^1.$$

In connection with double suspensions, we quote a theorem due to Doyle which provides a scheme for constructing complexes with precisely one invert point. For example, if K is a non-simply connected compact n-manifold, then $D(K)/R$ is precisely this type of complex.

<u>Theorem (Doyle)</u> Let K be a triangulated compact n-manifold. Then $I(D(K)) = R$, unless $D(K)$ is a sphere. Further, if $I(D(K)) = R$, then $D(K)/I(D(K))$ is locally an (n+2)-manifold except at one point.

<u>Theorem 4.9</u> Let K be an n-complex with $I(K) = \{p\}$ and $NO_I(K) = 2$. If $\dim\{I(\mathcal{S}(K))\} \geq 1$, then $\mathcal{S}(K) \overset{T}{=} S^{n+1}$.

<u>Proof</u>. Since $NO_I(K) = 2$, $K - p$ is locally euclidean of dimension n. Also, $\dim\{I(\mathcal{S}(K))\} \geq 1$ implies that there exists $x \in K - p$ such that $x \in I(\mathcal{S}(K))$. By homogeneity, $K - p \subset I(\mathcal{S}(K))$ and $K \subset I(\mathcal{S}(K))$ since

$p \in I(\mathcal{S}(K))$ in view of Theorem 4.1. Since $\dim K = n$, we get $\dim \{I(\mathcal{S}(K))\} = \dim \{\mathcal{S}(K)\} = n+1$. Now by Theorem 1.5 we get $\mathcal{S}(K) = S^{n+1}$.

<u>Corollary 4.10</u> Let K be an n-complex with $I(K) = \{p\}$ and $NO_I(K) = 2$. If $p \in I(\mathcal{S}(K))$, then $Lk(p,K)$ has Brouwer Property.

<u>Proof</u>. By Theorem 4.1, $\dim \{I(\mathcal{S}(K))\} \geq 1$. The last theorem gives $\mathcal{S}(K) = S^{n+1}$ and this has Brouwer Property since it is a manifold without boundary. By Lemma 3.5, both K and $Lk(p,K)$ have Brouwer Property.

Let K be an n-complex with $I(K) \neq \emptyset$. Assume that $p \in I(K)$ and $St\,p$ embeds in E^n. Now let $\mathcal{S}(K)$ have Brouwer Property and $\dim \{I(\mathcal{S}(K))\} \geq 1$. As noted earlier, K has Brouwer Property. Then $K \overset{T}{=} S^n$ and $\mathcal{S}(K) \overset{T}{=} S^{n+1}$. Consider the case when $I(K) = \{p\}$. If $\mathcal{S}(K)$ has Brouwer Property, then we must have $\dim \{I(\mathcal{S}(K))\} < 1$ or $I(\mathcal{S}(K)) = S^o$. We have the following:

<u>Proposition 4.11</u> Let K be an n-complex such that $n \geq 1$, $p \in I(K)$ and $St\,p$ embeds in E^n. Then

(i) $\mathcal{S}(K)$ has Brouwer Property and $\dim \{I(\mathcal{S}(K))\} \geq 1$ imply that $K \overset{T}{=} S^n$,

(ii) $\mathcal{S}(K)$ has Brouwer Property and $I(K) = \emptyset$, $\{p\}$ or S^o imply that $I(\mathcal{S}(K)) = S^o$,

and (iii) "$I(K) = S^o$" and "$\mathcal{S}(K)$ has Brouwer Property" are mutually exclusive.

Proof. We need show only (iii). Assume that $I(K) = S^0$ and $\mathcal{S}(K)$ has Brouwer Property. Then $K = \mathcal{S}(L)$ by Theorem 1.4, and $\mathcal{S}(K) = \mathcal{S}^2(L)$. By Doyle's theorem, we get $\dim \{I(\mathcal{S}(K))\} \geq 1$. Using (i) we get $K \overset{T}{=} S^n$ and this contradicts $I(K) = S^0$.

Proposition 4.12 Let K be a 2-complex with $I(K) = \{p\}$ and $NO_I(K) = 2$. Then $I(\mathcal{S}(K)) = S^0$.

Proof. Assume that $\dim \{I(\mathcal{S}(K))\} \geq 1$. By Theorem 4.9, $\mathcal{S}(K) \overset{T}{=} S^3$. Now we use Theorem 1.2 to get the result.

It may be useful to remark that $NO_I(K) = 2$ does not imply that $NO_I(\mathcal{S}(K)) = 2$. We can only say that $1 \leq NO_I(\mathcal{S}(K)) \leq 4$.

Let $n > 1$ and identify two antipodal points of S^n in a nice way to obtain an n-complex K. This may be called a generalized pinched torus. It is evident that $I(K) = \{p\}$ and $NO_I(K) = 2$. Moreover, $I(\mathcal{S}(K)) = S^0$ since $\mathcal{S}(K) \neq S^{n+1}$. This suggests the next set of results.

Proposition 4.13 Let K be an n-complex with $I(K) = \{p\}$ and $NO_I(K) = 2$. If K is not a homotopy n-sphere, then $I(\mathcal{S}(K)) = S^0$.

Proof. Let $\dim \{I(\mathcal{S}(K))\} \geq 1$. Then $\mathcal{S}(K) = S^{n+1}$ and K is a homotopy n-sphere. This proves the result.

Let K be an n-complex $(n \geq 2)$ such that $\mathcal{S}(K) = S^{n+1}$. Then K is a homotopy n-sphere. Let v be any vertex of K in the given triangulation. Then K and $Lk(v,K)$ have the Brouwer Property. Since $Lk(v,\mathcal{S}(K)) = \mathcal{S}\big(Lk(v,K)\big)$ and $\mathcal{S}(K) = S^{n+1}$, $\mathcal{S}\big(Lk(v,K)\big)$ has the integral homology groups of an n-sphere. Moreover,

$$\Phi : H_{i-1}\big(Lk(v,k)\big) \longrightarrow H_i\big(\mathcal{S}\, Lk(v,K)\big)$$

is an onto isomorphism for $2 \leq i \leq n$ with $H_o\big(Lk(v,K)\big) = Z$. Thus

$$H_k\big(Lk(v,K)\big) = \begin{cases} 0 & \text{for } 1 \leq k \leq n-2 \\ Z & \text{for } k = o,\ n-1 \end{cases}$$

<u>Remark</u>. The fact that the local homology groups are invariant under all triangulations of K can be justified by using the uniqueness of the open cone neighborhood (see [13]). Let v be any vertex of K under any triangulation. Consider $\overset{\circ}{St}\, v - v$. There exists a deformation of this onto $Lk\, v$. Now $\overset{\circ}{St}\, v$ is an open cone neighborhood of v. By Kwun's theorem, we get the result that the links of v are homeomorphic under all triangulations of K. This proves the desired result.

<u>Proposition 4.14</u> Let K be an n-complex with $n \geq 2$, $I(K) = \{p\}$ and $NO_I(K) = 2$. Let v be any vertex of K

under the given triangulation such that either (i) $H_k\big(Lk(v,K)\big) \neq 0$ for some k such that $1 \leq k \leq n-2$ or (ii) $H_k\big(Lk(v,K)\big) \neq Z$ for $k = 0$ or $n-1$. Then $I\big(\mathcal{S}(K)\big) = S^0$.

<u>Proof</u>. If we deny the assertion, then $\mathcal{S}(K) = S^{n+1}$ by Theorem 4.9. This contradicts preceeding remarks.

<u>Theorem 4.15</u> Let K be a 3-complex with $I(K) = \{p\}$ and $NO_I(K) = 2$. Then $I\big(\mathcal{S}(K)\big) = S^0$.

<u>Proof</u>. Assume that $\dim \{I\big(\mathcal{S}(K)\big)\} \geq 1$. Then by Theorem 4.9, $\mathcal{S}(K) \overset{T}{=} S^4$. Also from earlier remarks, we get

$$H_i\big(Lk(p,K)\big) = \begin{cases} 0 & \text{for } i = 1 \\ Z & \text{for } i = 0,2 \end{cases}$$

Moreover, $Lk(p,K)$ has Brouwer Property by Corollary 4.10. Then it is a 2-manifold without boundary with the prescribed homology groups. Thus $Lk(p,K) = S^2$ and $K = p\cdot Lk(p,K)$ is a 3-cell with a 2-sphere of invert points. This contradicts $I(K) = \{p\}$. If $Lk(p,K)$ is connected and simply connected, we get an immediate contradiction.

<u>Theorem 4.16</u> Let K be a 1-complex with $I(K) = \{p\}$. Then $I\big(\mathcal{S}(K)\big) = S^0$.

<u>Proof</u>. By Theorem 1.1, K is an r-leafed rose with $r \geq 2$. Assume that $I\big(\mathcal{S}(K)\big) \neq S^0$. Then by Theorem 4.4, $\dim \{I\big(\mathcal{S}(K)\big)\} = k \geq 2$. So there exists at least a 1-sphere of invert points of $\mathcal{S}(K)$ in K. Since K is a 1-complex, it can contain only a 1-sphere in it. Thus $k = 2$. Let

$I(\mathcal{S}(K)) = \mathcal{S}(S^1)$ where $S^1 \subset K$. Now K is an r-leafed rose. Hence S^1 must be one of the petals of this rose. Then $p \in I(\mathcal{S}(K))$ and the symmetry of the figure shows that every point of K is an invert point of $\mathcal{S}(K)$. This is easily seen to be impossible.

<u>Remark</u>. A more direct proof of the last theorem can also be given. Let $q \in K - p$. Let U be an open neighborhood of q in $\mathcal{S}(K)$. We can take U to be an open 2-cell. Clearly there does not exist any $h \in \mathcal{H}(\mathcal{S}(K))$ such that $h(\mathcal{S}(K) - U) \subset U$. In particular, we cannot have $h(p) \in U$. Hence $q \in K - p$ implies that $q \notin I(\mathcal{S}(K))$. This means that $|K \cap I(\mathcal{S}(K))| \leq 1$. But $\dim\{I(\mathcal{S}(K))\} \geq 2$ implies that $|K \cap I(\mathcal{S}(K))| \geq 2$. This shows that $K \cap I(\mathcal{S}(K)) = \emptyset$. By Theorem 4.1 we get $0 \leq \dim\{I(\mathcal{S}(K))\} < 1$, and the result is now obvious.

Let R be an r-leafed rose, $r \geq 2$. We now investigate $I(\mathcal{S}^k(R))$ for $k \geq 1$. The result for $k = 1$ is given in the last theorem. For $k = 2$, we note that $\mathcal{S}(R)$ is topologically the union of r 2-spheres with an arc common to all of them. Then $\mathcal{S}^2(R)$ is the union of r 3-spheres with a common 2-disk D^2. It is also easy to see that $I(\mathcal{S}^2(R)) = \mathrm{Bd}\, D^2 = S^1$. $\mathcal{S}^3(R)$ may be considered as a double suspension of $\mathcal{S}(R)$. Let $\{a_i, b_i\}$ denote the set of vertices of suspension for obtaining $\mathcal{S}^i(R)$ from $\mathcal{S}^{i-1}(R)$ with the obvious restrictions on i. Then $I(\mathcal{S}^3(R))$ contains the suspension ring $\langle a_2\, a_3\, b_2\, b_3 \rangle$.

Also, $I(\mathcal{S}^2(R)) = \langle a_1a_2b_1b_2\rangle$. Moreover, $\mathcal{S}^3(R)$ is a union of r 4-spheres with a common 3-disk D^3. We claim that $I(\mathcal{S}^3(R)) = \operatorname{Bd} D^3 = S^2$, the suspension of $\langle a_1a_2b_1b_2\rangle$. In order to see this, we observe that $CI(\mathcal{S}^2(R)) = \langle a_1a_2b_1b_2\rangle$ and $\langle a_2a_3b_2b_3\rangle \subseteq CI(\mathcal{S}^3(R))$. Then points of $\langle a_1a_2b_1b_2\rangle$ are equivalently embedded in $\mathcal{S}^3(R)$ and we are finished. The induction on k is now obvious. We have the following:

<u>Proposition 4.17</u> Let R be an r-leafed rose $(r \geq 2)$ and let $K = \mathcal{S}^k(R)$. Then $I(K) = S^{k-1}$ for $k \geq 1$.

The last result suggests a generalization of a result of Doyle in [1].

<u>Proposition 4.18</u> Let K be an n-complex. Then

$$I\left(\mathcal{S}^k(K)\right) \supseteq S^{k-1} \quad \text{for } k = 1,2,3,\cdots.$$

<u>Proof</u>. The result is true for $k = 1$ and 2 by Theorem 1.4 and Theorem 4 of [1] respectively. So assume that the result is true for $k \geq 3$. Let a_{k+1}, b_{k+1} be the vertices of suspension for getting $\mathcal{S}^{k+1}(K)$ from $\mathcal{S}^k(K)$. By induction hypothesis, $S^{k-1} \subseteq I(\mathcal{S}^k(K))$. Clearly, $\dim\{I(\mathcal{S}^{k+1}(K))\} \geq 1$ and by Theorem 1.10 we get $I(\mathcal{S}^k(K)) \subseteq I(\mathcal{S}^{k+1}(K))$. By homogeneity, the suspension of S^{k-1} from vertices a_{k+1} and b_{k+1} lies in $I(\mathcal{S}^{k+1}(K))$. Then $S^k \subseteq I(\mathcal{S}^{k+1}(K))$, and the proof is complete.

Remark. In view of the earlier theorems, we note that if $k \geq 2$ and $K = \mathcal{S}^k(L)$, then $CI(K) = I(K)$. This result, together with the last proposition, gives a simplified proof of a similar result in [9]. It is also clear that by taking successive suspensions, the dimension of the invert set is raised by at least one. By Proposition 4.17, we note that the dimension is raised precisely by one when we take the successive suspensions of a rose R, and in this sense the result is the best possible. Let K be an n-complex. Then

$$k-1 \leq \dim I\big(\mathcal{S}^k(K)\big) \leq n+k,$$

where $k \geq 1$; and $K = R$, $K = S^n$ respectively give the equality at the extremes.

Let F be a finite set of points. Then $\dim \{\mathcal{S}^k(F)\} = k$ for $k \geq 1$. By Proposition 4.18, $S^{k-1} \subseteq I\big(\mathcal{S}^k(F)\big)$. Now either (i) $S^{k-1} \subsetneq I\big(\mathcal{S}^k(F)\big)$ or (ii) $S^{k-1} = I\big(\mathcal{S}^k(F)\big)$. In case of (i) we get $I\big(\mathcal{S}^k(F)\big) = S^k$ since $\mathcal{S}^k(F)$ has dimension k. By Theorem 1.5 this means that $\mathcal{S}^k(F) \overset{T}{=} S^k$ and this is impossible unless $|F| = 2$. The following remark is now obvious and appears to be converse of the remark preceding Proposition 2.13.

Remark. Let F be a finite set of points and $k \geq 1$. Then (i) $|F| = 2$ implies that $I\big(\mathcal{S}^k(F)\big) = \mathcal{S}^k(F) = S^k$ and (ii) $|F| \neq 2$ implies that $I\big(\mathcal{S}^k(F)\big) = S^{k-1}$.

Let K be a complex such that $\dim\{I(K)\} \geq 1$. Then write $K = \mathcal{S}(L)$ with p and q as the vertices of suspension. Clearly there exists $v \in L \cap I(K)$. Now $\mathcal{S}(\mathring{S}t(v,L))$ gives a proper suspension neighborhood N of v. Without any loss of generality we may assume that $I(K) \not\subset N$, otherwise a smaller open set may be chosen in $\mathring{S}t(v,L)$ to get the desired result.

Choose $w \in (K - N) \cap I(K)$. Let U_1 be an open set containing w. There exists an inverting map h_1 such that $h_1(K - U_1) \subset U_1$. In particular, choose diameter of $U_1 < 1$. We get $h_1(N) \subset U_1$. Since $I(K)$ is continuously homogeneous, we get $h_1(v) = w$. By uniform continuity, $w \in \text{Int } h_1(N)$. Let U_2 be an open set containing w such that $U_2 \subset \text{Int } h_1(N)$ and diameter of $U_2 < \frac{1}{2}$. Then there exists $h_2 \in \mathcal{H}(K)$ such that $h_2(K - U_2) \subset U_2$. Again, we get $h_2(N) \subset U_2$ and $h_2(v) = w \in \text{Int } h_2(N)$. Proceeding inductively we get a sequence of inverting maps $\{h_i\}_{i=1}^{\infty}$ with the property that $w = \bigcap_{i=1}^{\infty} h_i(N)$ and diameter of $h_i(N) < \frac{1}{i}$. Then $w \in I(K)$ and has arbitrarily small suspension neighborhoods, and this shows that every point in $I(K)$ has this property. In particular, if C is an open cone neighborhood of v in L, then $\mathcal{S}(C)$ embeds in $C \times I$.

We can obtain the same result by the following argument. Let $w \in (K - N) \cap I(K)$. Let N_1 be an open cone neighborhood of w such that $N_1 \cap N = \emptyset$ and diameter of $N_1 < 1$. Then there exists $h_1 \in \mathcal{H}(K)$ such that $h_1(N) \subset \operatorname{Int} N_1$. Now $h_1(v) \in I(K)$. In $\operatorname{Int} h_1(N)$, select an open cone neighborhood N_2 of $h_1(v)$ with diameter $< \frac{1}{2}$. Inversion about $h_1(v)$ yields an inverting map h_2 such that $h_2(N) \subset \operatorname{Int} N_2$. Proceeding inductively we get a sequence $\{h_i\}_{i=1}^{\infty}$ of inverting maps such that $z = \bigcap_{i=1}^{\infty} h_i(N) \in I(K)$ and has arbitrarily small suspension neighborhoods. We now state the next result.

<u>Theorem 4.19</u> Let K be an n-complex with $\dim \{I(K)\} \geq 1$. Let $p \in I(K)$. Then p has arbitrarily small suspension neighborhoods.

<u>Remark</u>. Following the arguments leading to the last theorem and using Theorem 1.7, it is evident that if $\dim \{I(K)\} = k \geq 1$, then every invert point has arbitrarily small k-fold suspension neighborhoods.

Let K be an n-complex with $I(K) = \{p\}$. It was remarked earlier that if equality could be established in Theorem 1.10, the Poincaré Conjecture could be proved in dimension 4. Thus, the equality $I(K) = K \cap I(\mathcal{S}(K))$ for $\dim \{ I(\mathcal{S}(K)) \} \geq 1$ is stronger than the Poincaré Conjecture. If we use this for the complex K with a single invert point, the following result is obtained. Let $\dim \{I(\mathcal{S}(K))\} \geq 1$, Then

$$\{p\} = I(K) = K \cap I(\mathcal{S}(K)) .$$

But $\dim\{I(\mathcal{S}(K))\} \geq 1$ implies that $|K \cap I(\mathcal{S}(K))| \geq 2$. This is a contradiction. So we must have $I(\mathcal{S}(K)) = S^0$. Moreover, Theorem 4.16 for $n = 1$, Proposition 4.12 for $n = 2$ and Theorem 4.15 for $n = 3$, the last two with the additional hypothesis of two orbits under isotopy, indicate that the following result may be true. We end this chapter with this conjecture.

Conjecture. Let K be an n-complex with $I(K) = \{p\}$. Then $I(\mathcal{S}(K)) = S^0$.

CHAPTER V

AN APPLICATION TO GRAPHS

In a recent study, Doyle extended to a class of spaces called monotonic complexes, the result that every open connected set in E^n is a monotone union of closed n-cells. The relevant definition and the statement of his unpublished result are as follows.

<u>Definition 5.1</u> A simplicial complex K^n is <u>monotonic</u> if and only if $K^n = \bigcup_{i=1}^{p} K_i$ where each K_i is a subcomplex of K^n, K_1 is an n-simplex, and for $1 \leq i \leq p-1$, K_{i+1} is obtained from K_i by adding just one n-simplex L_i to K_i such that L_i and K_i have an (n-1)-simplex in common.

<u>Theorem (Doyle)</u> If K^n is a monotonic complex of dimension $n\ (\geq 2)$, then $K^n \overset{T}{=} \bigcup_{i=1}^{\infty} C_i$ where C_i is a closed n-cell and $C_i \subset C_{i+1}$ for $i = 1,2,3, \cdots$.

For $n \leq 1$, it was mentioned that a monotonic 0-complex is a point and that every connected 1-complex is monotonic.

<u>Remark</u>. Doyle proved that if K is a monotone union of 1-cells, then K is homeomorphic to one of the six figures

listed below.

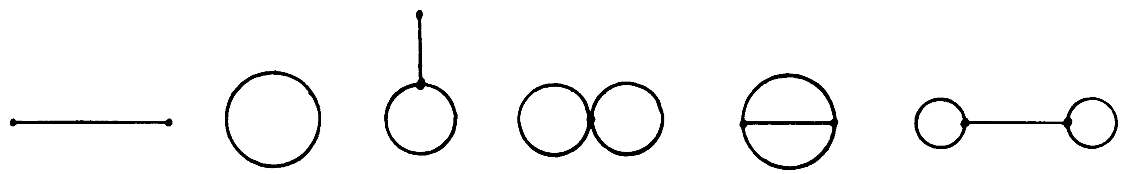

Figure 5.1

It was also remarked that five of these configurations represent the termination of a self-avoiding walk discussed by Kesten in [11]. These considerations lead to the following definition.

<u>Definition 5.2</u> If $E(n)$ is a graph such that it can be written as $\bigcup_{i=1}^{\infty} S_i(n)$ where $S_i(n)$ is a closed star graph of order n (≥ 2) and $S_i(n) \subset S_{i+1}(n)$ for $i = 1,2,3, \cdots$ then $E(n)$ is said to be an <u>expanding n-star graph</u>.

Then a monotone union of 1-cells may be written as $E(2)$ and has the non-homeomorphic forms given in Figure 5.1. Also, the generalization of self-avoiding walks of Kesten is immediate. By a direct counting process, it was possible to obtain the 30 configurations of $E(3)$ as given in Figure 5.2.

Remark. The collection of all expanding n-star graphs for $n \geq 2$ contains examples of all 1-complexes with a nonempty invert set. Moreover, we also obtain many examples of complexes with an empty invert set.

Definition 5.3 Let G be any graph. Then $D(k,G)$ will represent the number of vertices of G whose degree is greater than or equal to k.

Theorem 5.1 Let $E(n)$ be an expanding n-star graph. Then

$$D(k,E(n)) \leq 1 + [\frac{n}{k-2}]$$

for $3 \leq k \leq 2n$ and $n \geq 3$.

Proof. Let $E(n) = \bigcup_{i=1}^{\infty} S_i(n)$ where each $S_i(n)$ is a star graph with a vertex p such that $n \leq \rho(p) \leq 2n$. The maximum number of vertices in the graph with degree ≥ 3 is obtained if every end point of an arc meets the interior of that arc. Thus

$$D(3,E(n)) \leq 1 + [\frac{n}{1}].$$

For obtaining the maximum number of vertices of degree ≥ 4, the n ends of the arcs from p can be paired in such a way that every pair meets on the interior of an arc to produce a vertex of degree 4. Then

$D(4,E(n)) \leq 1 + [\frac{n}{2}]$. In general, $k-2$ ends have to meet on the interior of an arc to produce a vertex of degree k. But the n ends can be paired to produce at most $[\frac{n}{k-2}]$ vertices with degree k. This completes the proof.

Corollary 5.2 Let $E(n)$ be an expanding n-star graph and $n \geq 3$.

(a) If $n+3 \leq k \leq 2n$, then $D(k,E(n)) \leq 1$.

(b) If $k = n+1$ or $n+2$, then $D(k,E(n)) \leq 2$.

(c) If $n \geq 5$ and $k = n$, then $D(k,E(n)) \leq 2$.

Proof. Parts (a) and (b) follow from the last theorem. For part (c) we note that $n \geq 5$ implies $[\frac{n}{n-2}] = 1$.

Remarks. The preceding results show that we cannot have too many vertices of high degree in an expanding n-star graph. In fact, an expanding n-star graph is locally euclidean everywhere except at $(n+1)$ points at most. Moreover, if $E(n)$ is an expanding n-star graph and x is any vertex of $E(n)$ then $1 \leq \rho(x) \leq 2n$, and if $\rho(x) > n+2$ then x must be the center of $S_1(n)$ where $E(n) = \bigcup_{i=1}^{\infty} S_i(n)$.

Theorem 5.3 Let $E(n)$ be an expanding n-star graph and $n \geq 3$. Let $\rho = \max_{x \in E(n)} \{\rho(x)\}$. Then

$$(k-2)(D(k,E(n))-1) \leq n \leq \rho \quad \text{for} \quad 3 \leq k \leq 2n.$$

Proof. By Theorem 5.1 we get

$$D(k,E(n)) \leq 1 + [\frac{n}{k-2}] \leq 1 + \frac{n}{k-2} .$$

This gives $(k-2)(D(k,E(n))-1) \leq n$. Obviously $\rho \geq n$.

Using the standard terminology of graph theory, let K_n denote the complete graph on n vertices and let $K_{m,n}$

denote the complete bipartite graph on m and n vertices. From previous remarks and results we note that K_1, K_2, K_3, K_4, $K_{1,n}$ and $K_{2,n}$ are expanding r-star graphs for a suitable r. This naturally leads to the investigation of K_5 and $K_{3,3}$.

<u>Theorem 5.4</u> K_5 and $K_{3,3}$ are not expanding n-star graphs.

<u>Proof</u>. For K_5 we note that $\rho = 4$. If $k = 4$ then $D(4,K_5) = 5$. These values give $(k-2)\left(D(k,K_5)-1\right) = 8$ and there is no n which can satisfy the inequality in Theorem 5.3. Then K_5 cannot be an expanding n-star graph.

For $K_{3,3}$ we have $\rho = 3$. Taking $k = 3$, we get $D(3,K_{3,3}) = 6$. Now $(k-2)\left(D(k,K_{3,3})-1\right) = 5$ and again we conclude as before that $K_{3,3}$ is not an expanding n-star graph.

For K_4, it is evident that the inequality of Theorem 5.3 is satisfied for $\rho = 3$ and $3 \le k \le 2n$. The case with $k = 3$ is particularly interesting since it gives $n = 3$ and we note once again that K_4 occurs in $E(3)$. The preceeding theorem shows that Kuratowski's primitive skew curves are not expanding n-star graphs. In the next result we show that they cannot occur even as subgraphs of expanding n-star graphs.

<u>Theorem 5.5</u> Every expanding n-star graph is planar.

<u>Proof</u>. Assume that an expanding n-star graph can be

skew. Select the least n for which skewness occurs and write this graph as $E(n) = \bigcup_{i=1}^{\infty} S_i(n)$ where each $S_i(n)$ is a star graph of order n.

Let $f : S_1(n) \longrightarrow E(n)$ be a mapping which is one to one except on the end points of $S_1(n)$. Let $\widetilde{S}_1(n)$ be a star graph of order $(n - 1)$ obtained from $S_1(n)$ by deleting a semi-open branch of $S_1(n)$. Then $\widetilde{E}(n - 1) = f(\widetilde{S}_1(n))$ is an expanding $(n-1)$-star graph. By the minimality of n, $\widetilde{E}(n - 1)$ is planar.

This shows that every proper subset of $E(n)$ is planar. By Kuratowski's theorem, a graph is planar if and only if it has no subgraph homeomorphic with K_5 or $K_{3,3}$. Consequently, $E(n)$ must be K_5 or $K_{3,3}$. But this is impossible in view of Theorem 5.4 and the proof is complete.

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