AUTOMORPHISM GROUPS OF GRAPHS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY

Julian Kateley, Jr.

1963

THESIS

This is to certify that the

thesis entitled

AUTOMORPHISM GROUPS of GRAPHS

presented by

Julian Kateley, Jr.

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in E.E.

G. P. Weeg
Major professor W.C.K.

Date August 6, 1963

O-169

AUTOMORPHISM GROUPS OF GRAPHS

By

Julian Kateley, Jr.

AN ABSTRACT OF A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1963

ABSTRACT

AUTOMORPHISM GROUPS OF GRAPHS

by Julian Kateley, Jr.

A graph is defined to be a system (S, R), where S in a finite, non-null set and R is a subset of S x S, or simply a relation on R. No other restrictions are put on R so that the graphs considered are oriented or directed and may have loops or slings.

After carefully defining the preliminary mathematical concepts used, a study is made of the problem of finding the automorphism group of a graph. Certain well known techniques are presented in a form applicable to oriented graphs. A special technique is formulated which considerably simplifies the otherwise difficult task of calculating the automorphism groups of a graph.

Some consideration is given to certain classes of special graphs. These are (1) interchange graphs, (2) graph products, (3) self-complementary graphs, and (4) graphs having specified groups.

AUTOMORPHISM GROUPS OF GRAPHS

By

Julian Kateley, Jr.

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1963

ACKNOWLEDGEMENT

No acknowledgement can possibly convey the credit due the many individuals who have contributed to this thesis. The continued assistance, guidance and patience of Dr. G. P. Weeg; the support and counsel of Dr. L. W. Von Tersch; and the cooperation and encouragement of the author's family are but a few of the factors that have made this thesis possible.

TABLE OF CONTENTS

Page

I. INTRODUCTION 1

II. BASIC CONCEPTS AND THE DEFINITION OF A GRAPH 5

III. THE AUTOMORPHISM GROUP OF A GRAPH . . . 10

IV. PROCEDURES FOR CALCULATION OF THE AUTOMORPHISM GROUP OF A GRAPH 16

V. SPECIAL GRAPHS 32

VI. CONCLUSION 50

BIBLIOGRAPHY 52

LIST OF FIGURES

		Page
3.1	The graph of Example 3.1	15
4.1	The graphs of Example 4.1	18
4.2	The graph of Example 4.2	19

LIST OF TABLES

Page

4.1 An Intersection Table for Example 4.4 28

4.2 Analysis of the cycle (1) for Example 4.4 29

4.3 Analysis of the cycle starting with 13 for Example 4.4 29

4.4 Analysis of the cycle starting with 15 for Example 4.4 30

I. INTRODUCTION

Graphs have been the object of study for at least the last one hundred years, both as a mathematical discipline in themselves and also as an important tool in various other fields. Thus graphs are used in the study of electrical networks, switching circuits, and communication networks to mention but a few of the areas related to Electrical Engineering. Graphs are used also in branches of Chemistry, Physics, Biology, Psychology, Philosophy and Sociology.

Much of the work to date has concentrated on what may be characterized as the topological properties of graphs. Thus the connectivity properties of graphs, the path, circuit and tree properties, and such other properties as the chromatic number of a graph are all in the general category of topological properties of graphs.

The objective of this thesis is to set forth certain algebraic properties of graphs. D. König in his book, "Theorie der endlichen and unendlichen Graphen," Leipzig (1936) poses questions of an algebraic nature about graphs. For example, König asks, "When can a given abstract group be set up as the group of a graph, and if possible how can the graph be constructed?" In spite of the early origin of this question in König's classic book on graphs, literature in the area of algebraic studies of graphs is meager and of fairly recent origin. Ore [O1] in his book on graphs devotes only one short chapter to the groups of graphs. C. Berge in his book, "The Theory of Graphs," London (1962) makes no reference per se to algebraic properties of graphs. Several of the pertinent works are cited in the text of this thesis, but as evidence of the limited publications on this subject, these references are also cited here. For example, Frucht [F1] has briefly examined the groups of isomorphic graphs. Frucht [F2] and Sabidussi [S1], [S2] have studied the problem posed by König of finding graphs with given groups. Kagno [K1], [K2] has investigated certain types of graphs and presented their groups. Sabidussi [S3] has studied graph products, and finally, Sabidussi [S4] has studied interchange graphs or graph derivatives. Cre [O1] apparently includes all other papers in his bibliography pertinent to the subject of algebraic properties of graphs.

Unavoidably, use is made of topological properties of graphs in this thesis, not only because there is no clear dividing line between topological and algebraic properties, but also because certain topological properties result in interesting and useful algebraic properties.

Nevertheless, the definition of a graph is based on algebraic concepts as presented in Section II. Algebraic concepts leading to the definition of the group of a graph are presented in Section III. The intent is that these two sections establish a rigorous basis for the following two sections, which contain the main results of this thesis.

Of central interest, in this and other algebraic studies of graphs, is the group of the graph. For graphs of even reasonable size, the direct calculation of these groups is extremely difficult. The results of Section IV make feasible the hand calculation of groups of graphs of moderate size even when computer calculations were previously too lengthy.

Certain results of Section IV also provide a basis for an attack on the problem posed by König, as discussed in Section V. In particular, though several authors have examined his question, a completely satisfactory answer has not yet been given. As will be made evident in this thesis, the answer so far given for König's question is that every finite group is isomorphic to the automorphism group of a graph.

The graph constructed having the desired property, however, contains many more vertices than the group does symbols. This rather avoids the real intent of the problem, namely, what groups are groups of graphs where the number of vertices and the number of group symbols are equal? It is known that for a cyclic group generated by a single cycle that there is no graph having that group when the graph is non-oriented (see Kagno [K1]). In this thesis, it is shown that this restriction does not exist if oriented graphs are considered, indeed, it is shown that for a broad class of groups the corresponding oriented graphs do exist.

Section V also includes consideration of certain other algebraic properties of special kinds of graphs including the problem of finding a graph of a given group, if there is such a graph. The special kinds of graphs considered also develop insight into this problem.

For some reason not fully understood by this student, all of the papers cited investigate only non-oriented graphs. As a consequence of the results presented in this thesis, such a restriction is not only unnecessary but undesirable for certain problems.

As a final introductory comment, it should be noted that the symbol Δ is used throughout this thesis to indicate the completion of a proof.

II. BASIC CONCEPTS AND THE DEFINITION OF A GRAPH

So as to define properly the concepts peculiar to this thesis and to provide continuity to the text, certain standard mathematical concepts and definitions will be given here. Those definitions, which though they might not be original with this thesis, but for which there may be some controversy in the literature, will be assigned a numbered definition. It is thereby hoped that the definitions and theorems of this and succeeding sections follow in a natural and comprehensible way.

Every mathematical system ultimately rests upon certain undefinable concepts. In set theory, the undefinables are commonly taken to be "set", "element", and "belongs to". Thus, no attempt is made to define set. However, a set is said to be composed of elements. Moreover a set is said to be well defined if, given any object, it is possible to decide whether or not this object belongs to the set.

The set of those elements having a specified property P is denoted by $\{x \mid x \text{ has property } P\}$. The set with no elements is called the null set, and is denoted by ϕ. Given any two sets A and B, A is a subset of B, denoted by $A \subseteq B$, if each element of A is an element of B. A is a proper subset of B, denoted by $A \subset B$, if A is a subset of B, A is not the null set, and there is at least one element of B which is not an element of A. Two sets A and B are equal, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$.

The Cartesian product of a set of sets $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples

$[a_1, a_2, \cdots, a_n]$, where a_i is in A_i for $i = 1, 2, \cdots, n$. Two elements $[b_1, b_2, \cdots, b_n]$ and $[c_1, c_2, \cdots, c_n]$ of $A_1 \times A_2 \times \cdots \times A_n$ are equal if and only if $b_i = c_i$ for all b_i and c_i in A_i. This assumes some appropriate definition of equality of elements in each A_i and will either be apparent from the context in which the sets are used or else will be defined.

This equality is a special case of the general concept called a relation on a set. The idea of a relation is frequently encountered in set theory, but its definition tends to vary from author to author. So as to avoid confusion, the following definition is used in this thesis.

Definition 2.1 A <u>binary relation</u> R from a set A into a set B is a subset of R of A x B. If [a, b] is in R, it is common to say that a is related to b and to write aRb. The <u>domain</u> of R is the set of all elements of A which are related by R to at least one element of B, thus dom R = $\{a \text{ in } A \mid aRy \text{ for some } y \text{ in } B\}$. The <u>range</u> of R is the set of all elements of B to which at least one element of A is related by R, thus range R = $\{b \text{ in } B \mid xRb \text{ for some } x \text{ in } A\}$. A binary relation from A into A is called a <u>relation in</u> A.

Definition 2.2 A <u>graph</u> A is an ordered pair A = (S, R) where S is a finite, non-null set and R is a binary relation in S.

The elements of S are commonly called the <u>vertices</u> of the graph and the elements of R are commonly called the <u>arcs</u> of the graph. As defined above, a graph may have "loops". A <u>loop</u> is an arc [s, s] for some s in S. Here the arcs of a graph are oriented. The arcs being oriented results from the elements of R being ordered pairs of vertices.

A relation R is a <u>symmetric relation</u> if [a, b] in R implies [b, a] in R for all [a, b] in R. There is of course no implication that [a, b] = [b, a]. A relation R in a set S is a <u>reflexive</u> relation if [a, a] is in R for all a in S. A relation R in S is an anti-reflexive relation if [a, a] is not in R for any a in S. If, for a graph A = (S, R), R is required to be reflexive, then for every vertex of S, there is an arc which is a loop. If R is required to be anti-reflexive, then A has no loops.

The definition of a connected graph is not standard in the literature. So as to give a precise meaning to that idea, a series of definitions must be given.

Definition 2.3 Given a graph A = (S, R), a subgraph B of A is an ordered pair B = (T, Q) such that $T \subseteq S$, $Q \subseteq R$ and $Q \subseteq T \times T$. As defined, a subgraph is a graph. B is a <u>proper subgraph</u> if $T \subset S$.

Definition 2.4 Given a graph A = (S, R) where S is a set of n or more elements, a <u>path</u> of A is a subgraph B = (T, Q) of A such that $T = \{t_i \mid 1 \leq i \leq n, \text{ all } t_i \text{ distinct}\}$, and such that $Q = \{t_i, t_{i+1} \mid 1 \leq i \leq n - 1\}$ for $n > 1$. Thus there is a path <u>from</u> t_1 <u>to</u> t_n.

If there is a path from t_1 to t_n in a graph, there need not in general be a path from t_n to t_1. However, given a graph A = (S, R) with R a symmetric relation, if there is a path from s in S to t in S, then there is a path from t to s.

Definition 2.5 Given a graph $A = (S, R)$ and a vertex s in S, let $\underline{J_s}$ be the set of vertices of S such that $J_s = \{t \text{ in } S \mid \text{there exists a path from } s \text{ to } t\}$.

Definition 2.6 A set of vertices $T \subseteq S$ of a graph $A = (S, R)$ is <u>open</u> if, given any s in T, $J_s \subseteq T$. Also, the subgraph (T, Q) is said to be an <u>open subgraph</u> of A.

Definition 2.7 A graph $A = (S, R)$ is <u>strongly connected</u> if, given any s and t in S, there exists a path from s to t.

Set union and intersection are now defined since these are required in the definition of a connected graph. The <u>union</u> of a set $\{A_i \mid i = 1, 2, \cdots\}$ of sets, denoted by $\bigcup_i A_i$, is the set of all elements which are in at least one of the A_i. The <u>intersection</u> of a set $\{A_i \mid i = 1, 2, \cdots\}$ of sets, denoted by $\bigcap_i A_i$, is the set of all elements which are in all of the A_i. If the number of sets is finite, then union may be denoted by $\bigcup_i A_i = A_1 \cup A_2 \cup \cdots \cup A_n$, $i = 1, 2, \cdots, n$, and intersection may be denoted by $\bigcap_i A_i = A_1 \cap A_2 \cap \cdots \cap A_n$, $i = 1, 2, \cdots, n$.

The following definition is based on a similar definition from topology.

Definition 2.8 A graph $A = (S, R)$ is <u>not connected</u> if there exists non-null open sets $U \subseteq S$ and $V \subseteq S$ such that $U \cup V = S$ and $U \cap V = \phi$; otherwise A is <u>connected</u>.

Definition 2.9 The <u>complement</u> A' of a graph $A = (S, R)$ is the ordered pair

$$A' = (S, [S \times S] - R)$$

Thus the complement of a graph is a graph.

A well known fact concerning the complement of a graph is presented in Theorem 2.1. Since its proof is straightforward it is included here. The theorem itself will find application in the latter part of this thesis.

Theorem 2.1 If a graph $A = (S, R)$ is not connected, then the complement A' of A is strongly connected.

Proof: Since A is not connected, there are at least two non-null open sets $U \subseteq S$ and $V \subseteq S$ such that $U \cup V = S$ and $U \cap V = \Phi$. Then $[u_i, v_j]$ and $[v_j, u_i]$ are both in $S \times S - R$ for all u_i in U and all v_j in V. Moreover, for any u_k and u_ℓ in U and for any v_j in V, $[u_k, v_j]$ and $[v_j, u_\ell]$ are in $S \times S - R$. Thus $A' = (S, S \times S - R)$ is strongly connected. Δ

III. THE AUTOMORPHISM GROUP OF A GRAPH

The automorphism group of a graph could be defined directly. However, since the concepts leading up to the definitions of an automorphism and of a group are also otherwise useful in this thesis, a less direct approach is used.

A <u>function</u> F from a set A to a set B is a binary relation from A into B which satisfies the additional properties:

(1) dom $F \neq \emptyset$

(2) if aFb_1 and aFb_2, then $b_1 = b_2$.

The notation $b = F(a)$ is commonly used and means aFb. Two functions F and G are equal if dom F = dom G and $F(a) = G(a)$ for all a in dom F. The domain of F is extended to include functions of subsets of dom F so that the notation F(C) is used where $F(C) = \{b \text{ in range } F \mid b = F(c) \text{ for all } c \text{ in } C \subseteq \text{dom } F\}$.

A function F from A to B is a function from A <u>onto</u> B if range F = B. A function F from A to B is said to be <u>one-to-one</u> if $F(a) = F(x)$ implies $a = x$ for all a and x in dom F.

A <u>binary operation</u> o on a set A is a function F from A x A into A. A binary operation o on A is closed if dom F = A x A. The notation aob = c is commonly used and means $c = F([a, b])$ where [a, b] is in A x A and c is in A.

An <u>abstract system</u> (S, R, O) is a non-null set S, a set R of binary relations in S, and a set O of closed operations on S. Either R or O may be null, but not both.

Let (S, R, O) and (T, Q, P) be two abstract systems. A function H from S into T is a <u>homomorphism</u> from (S, R, O) into (T, Q, P) provided that there is a relation q in Q corresponding to every relation r in R, and an operation p in P corresponding to every operation o in O such that

(1) if [a, b] is in r, then [H(a), H(b)] is in q for all a and b in S, and for every r in R.

(2) H(aob) = H(a) p H(b) for all a and b in S, and for every o in O.

If T = range H, then H is a homomorphism from (S, R, O) <u>onto</u> (T, Q, P). A function H from S onto T is an <u>isomorphism</u> from (S, R, O) onto (T, Q, P) if and only if H is a one-to-one homomorphism from (S, R, O) onto (T, Q, P). An isomorphism from (S, R, O) onto (S, R, O) is an <u>automorphism</u> on (S, R, O).

Since a graph A = (S, R) is an abstract system consisting of a non-null set S and a single binary relation R in S, an automorphism f on A is a one-to-one function from S onto S such that aRb if and only if f(a)Rf(b) for all a and b in S. The set G of all such automorphisms on a graph A = (S, R) is a group, as is well known.

A <u>group</u> is a system (G, o) consisting of a set G together with a binary operation o on G such that

(1) the binary operation is closed; i.e., aob is in G for all a and b in G;

(2) the binary operation o is an associative operation; i.e., (aob)oc = ao(boc) for all a, b and c in G;

(3) there is a left identity element e in G such that eoa = a for all a in G;

(4) for each a in G, there is a left inverse element d in G such that doa = e.

Given a group[1] (G, o), it can be shown that

(1) there exists a right inverse for each element in G; that is, there is an element a^{-1} in G such that aoa^{-1} = e for each a in G;

(2) there exists a right identity element e in G; that is, aoe = a for all a in G;

(3) there is only one identity element in G;

(4) the left and right inverse elements are unique.

The inverse of an element a is denoted as usual by a^{-1}, as in (1) above.

Now it is possible to show that the set F = $\{f\}$ of all automorphisms f on a graph A = (S, R) is a group (G, $*$). To do this, it is first necessary to define the operation $*$. Let f_1 and f_2 be automorphisms on the graph A = (S, R). Then the operation $*$ is defined by $f_1 * f_2(a) = f_1[f_2(a)]$ for all a in S, and is commonly called the composition of f_1 and f_2.

---

[1] As is commonly done, both the set G and the system (G,o) will be referred to as a group, with the actual meaning apparent from the content.

It can be shown in general, that the set of all automorphisms on a system together with the composition of these automorphisms is a group. Nevertheless, since it is informative, the proof is presented here for the case when the system is a graph.

Theorem 3.1 Let $A = (S, R)$ be a graph. The set of all automorphisms on A together with the composition of the automorphisms is a group, G(A).

Proof: Let f_i, f_j and f_k be any automorphisms on A, and let o be the operation of composition.

(1) $f_i o f_j$ is an automorphism. To show this, it is sufficient to show that $f_i o f_j$ is a one-to-one homomorphism from S onto S.

First, $f_i o f_j$ is a one-to-one function because if $f_i o f_j(a) = f_i o f_j(b)$, then $f_i[f_j(a)] = f_i[f_j(b)]$, hence $f_j(a) = f_j(b)$, hence $a = b$.

Next, to show that $f_i o f_j$ is an onto function, it is sufficient to show that there exists an a in S such that $f_i o f_j(a) = b$ for any b in S. To show this, let $a = f^{-1}_j o f^{-1}_i(b) = f^{-1}_i[f_i^{-1}(b)]$. Then

$$\begin{aligned} f_i o f_j(a) &= f_i[f_j(a)] \\ &= f_i \{ f_j[f_j^{-1}(f_i^{-1}(b))]\} \\ &= f_i[f_i^{-1}(b)] \\ &= b. \end{aligned}$$

Finally $f_i o f_j$ is a homomorphism because if aRb, then $f_j(a) R f_j(b)$.

(2) $f_i o(f_j o f_k) = (f_i o f_j) o f_k$ because

$$[(f_i o f_j) o f_k](a) = [f_i o f_j]\, [f_k(a)]$$
$$= f_i(f_j[f_k(a)]$$
$$= f_i\ [f_j o f_k(a)]$$
$$= f_i o[f_j o f_k(a)]$$
$$= [f_i o(f_j o f_k)]\ (a)$$

for all a in S.

(3) The function e such that e(a) = a for all a in S is an automorphism. Also eof = f for any automorphism f in F since

$$eof(a) = ef(a) = f(a).$$

(4) There exists a g such that gof = e, because if f(a) = b for any a and b in S, then define g(b) = a. Now gof(a) = g[f(a)] = g(b) = a = e(a) therefore gof = e. △

Example 3.1. Let S = {1, 2, 3} and let R = {[1, 2], [2, 3], [3, 1]}. Then G(S, R) = {I, (123), (321)}. Standard cycle permutation notation is used here in identifying the group elements. A <u>permutation</u> on a set S is a one-to-one function from S onto S. Given a permutation f on S, a <u>cycle</u> of f is a permutation g on $C \subseteq S$ such that g(C) = C and such that for all subsets $D \subset C$, g(D) is not a subset of D. Thus a permutation is the composition of cycles. The permutation f_1 = (123) is a function such that $f_1(1) = 2$, $f_1(2) = 3$, and $f_1(3) = 1$; similarly for f_2 = (132). The entry I = (1) (2) (3) and is such that I(1) = 1, I(2) = 2, and I(3) = 3. Thus I is the identity element of G(S, R). Further, the composition $f_1 o f_2$ is (123)(132) and equals the identity element as does $f_2 o f_1$. Thus $f_1 = f_2^{-1}$ and $f_2 = f_1^{-1}$. Figure 2.1 shows this graph as commonly drawn.

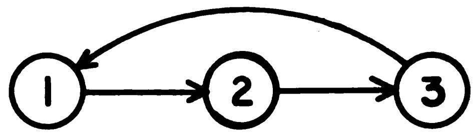


Figure 3.1. The graph of Example 3.1.

This is a simple example of a graph so its group is not difficult to find. In general, however, the automorphism group of a graph is very difficult to find. This is one of the main problems considered in this thesis. Several theorems to follow will considerably simplify the task of finding these groups.

IV. PROCEDURES FOR CALCULATION OF THE AUTOMORPHISM GROUP OF A GRAPH

While in general, it is difficult to find the automorphism group of a graph, Ore [O1, pp. 239-240], has noted several graph-group properties which simplify this task. Ore's discussion is somewhat restrictive and not very detailed. In this section, a more general and detailed discussion is given of some of Ore's work and in addition, a technique is presented which considerably simplifies the task of finding group elements of even relatively large graphs.

The development to follow requires the introduction of several numbers associated with groups and graphs. Thus, the number of elements in a set S, denoted by $\#(S)$, is called the order of the set S. The order of a graph $A = (T,R)$ is $\#(T)$ and is equal to the number of vertices in the graph. The order of a group $(G, *)$ is $\#(G)$. The degree of a permutation group which is the set of all automorphisms on the system (S; R; O) is $\#(S)$. For example, the group

$$F = \{(1)(2)(3)(4)(5)(6),\ (12)(34)(56)\}$$

is of order 2 and of degree 6.

A graph $A = (S, R)$ such that $R = S \times S$ is called a complete graph. The group G of all automorphisms on (S; R; O) such that $\#(G) = [\#(S)]!$ is called the symmetric group on $\#(S)$ symbols and is commonly denoted by Σ_n, where $n = \#(S)$. It is easy to show that $G(S, S \times S) = \Sigma_n$, where $n = \#(S)$.

Kagno [K2] has shown that the complement A' of a graph A = (S, R) has G(A') = G(A). However, since Kagno considers only connected symmetric graphs without loops, the proof of this useful result is given in a more general form.

Theorem 4.1 If A' is the complement of the graph A = (S, R), then G(A') = G(A).

Proof: It is sufficient to show that if f is any permutation in G(A), then f is also in G(A') and if g is any permutation in G(A'), then g is also in G(A).

First, suppose f is in G(A). It is sufficient to show that [a, b] is in S x S - R if and only if [f(a), f(b)] is in S x S - R for all a and b in S. Assume then that [a, b] is in S x S - R. Then [a, b] is not in R, so [f(a), f(b)] is not in R. Consequently, [f(a), f(b)] is in S x S - R. Thus if f is in G(A), then f is in G(A').

The second part of the proof, namely, that if g is in G(A'), then g is in G(A), is identical in nature to this first part and thus is omitted. Δ

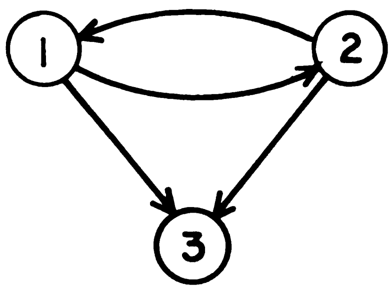
However, if the groups of two graphs are identical, the graphs need not be complements nor in any other way related, as is shown by the following example.

Example 4.1 Let (S, R) and (T, Q) be two graphs, with S = T = {1, 2, 3}, and

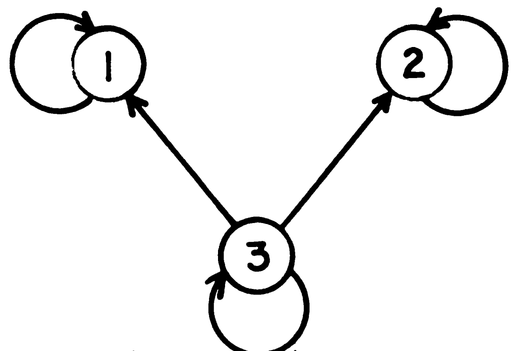
$$R = \{[1, 2], [1, 3], [2, 1], [2, 3]\},$$

$$Q = \{[1, 1], [1, 3], [2, 2], [2, 3], [3, 1], [3, 2], [3, 3]\}.$$

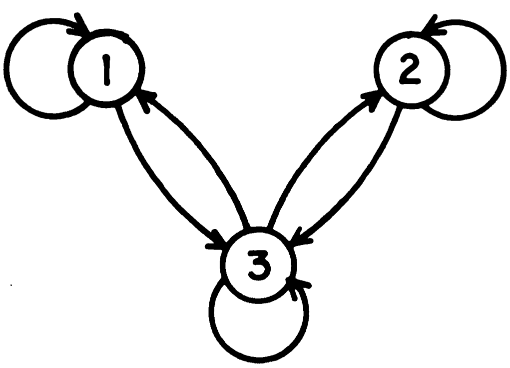
Then G(S, R) = G(T, Q) and (T, Q) is not the complement of (S, R). The graphs and their complements are shown in Figure 4.1. All four graphs have the same group. Note that (S, R) and (S, R)' are connected but not strongly connected, (T, Q) is strongly connected and (T, Q)' is not connected.



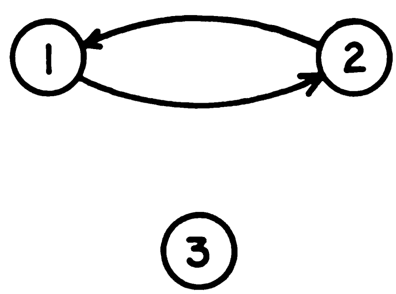
(a) Graph (S, R)



(b) Graph (S, R)'



(c) Graph (T, Q)



(d) Graph (T, Q)'

Figure 4.1. The graphs of Example 4.1.

Ore [Ol, p. 239], notes that the determination of the group of a graph which is not connected can be reduced to the problem of finding the groups of the connected subgraphs of the graph. This requires the notion of the direct product of groups, here defined.

Definition 4.1 The direct product of the groups $(G_1\ o_1)$, $(G_2,\ o_2)$, $\cdots$ $(G_n,\ o_n)$ is the system $(G,\ o)$ consisting of the Cartesian product $G = G_1 \times G_2 \times \cdots \times G_n$ and the operation o defined by

$$[f_1,\ f_2,\ \cdots,\ f_n]\ o\ [g_1,\ g_2,\ \cdots,\ g_n] = [f_1 o_1 g_1,\ f_2 o_2 g_2, \cdots,\ f_n\ o_n g_n]$$

for all $[f_1,\ f_2,\ \cdots,\ f_n]$ and $[g_1,\ g_2,\ \cdots,\ g_n]$ in G. The direct product G of these groups is denoted by $G_1 \times G_2 \times \cdots \times G_n$. It is known that the direct product of groups is a group.

The determination of the group of a graph in terms of its connected subgraph is detailed in the next theorem.

So as to make this next theorem more comprehensible, it is preceded by the following example.

Example 4.2 Find the automorphism group of the graph $A = (S, R)$, where $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and $R = \{[2, 1], [2, 3], [4, 5], [5, 4], [6, 7], [7, 6], [8, 9], [9, 8], [9, 10], [9, 11], [10, 11], [11, 10]\}$. This graph is shown in Figure 4.2.

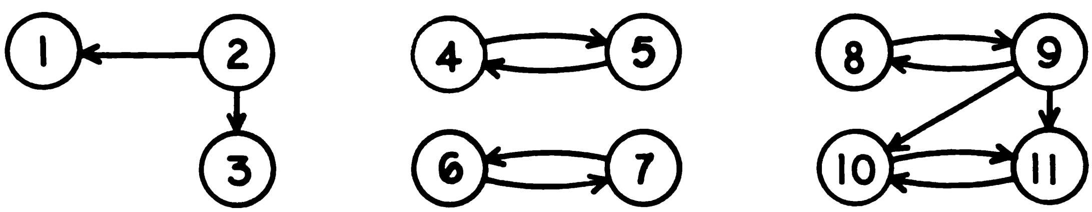


Figure 4.2 The graph of Example 4.2.

The graph A can be decomposed into the graphs $A_1 = (S_1, R_1)$, $A_2 = (S_2, R_2)$, and $A_3 = (S_3, R_3)$ where $S_1 = \{1, 2, 3\}$, $R_1 = \{[2, 1], [2, 3]\}$, $S_2 = \{4, 5, 6, 7\}$, $R_2 = \{[4, 5], [5, 4], [6, 7], [7, 6]\}$, $S_3 = \{8, 9, 10, 11\}$, and $R_3 = \{[8, 9], [9, 8], [9, 10], [9, 11], [10, 11], [11, 10]\}$.

Note that $S = \bigcup_i S_i$ and $R = \bigcup_i R_i$, $i = 1, 2, 3$, and $S_i \cap S_j = \emptyset$ for $i \neq j$ and $i, j = 1, 2, 3$.

The graph $A_2 = (S_2, R_2)$ can be further decomposed into isomorphic graphs, $A_{21} = (S_{21}, R_{21})$ and $A_{22} = (S_{22}, R_{22})$ where $S_{21} = \{4, 5\}$, $R_{21} = \{[4, 5], [5, 4]\}$, $S_{22} = \{6, 7\}$, and $R_{22} = \{[6, 7], [7, 6]\}$.

An automorphism of A is simply a re-arrangement of the vertices of A which "leaves the figure unchanged." But such changes can be determined piecemeal for each A_1, A_2 and A_3. Thus

$$G(A_1) = \{(1)(2)(3), (13)\}$$

$$G(A_2) = \{(4)(5)(6)(7), (45), (67), (45)(67), (46)(57)\ (47)(56)\}$$

$$G(A_3) = \{(8)(9)(10)(11), (10, 11)\}.$$

Moreover, any composition of permutations, one from each of $G(A_1)$, $G(A_2)$, and $G(A_3)$ is still an automorphism of A, and in fact there are no other automorphisms of A. Thus $G(A) = G(A_1) \times G(A_2) \times G(A_3)$.

These intuitively obvious results are now stated formally in the next theorem. A similar statement is made by Ore [O1, p. 239], but once again, for symmetric graphs. The proof of the theorem is for arbitrary R, and since the proof is long, it will be omitted.

Theorem 4.2 Let the graph A = (S, R) be composed of the subgraphs

$$A_1 = (S_1, R_1),\ A_2 = (S_2, R_2),\ \cdots,\ A_n = (S_n, R_n),$$

such that

$$A = (\bigcup_i S_i, \bigcup_i R_i),\ i = 1, 2, \cdots, n,$$

and

$$S_i \cap S_k = \emptyset\ ,\ \text{for } i \neq k \text{ and } i, k = 1, 2, \cdots, n.$$

Further, let the subgraph $A_i = (S_i, R_i)$, $i = 1, 2, \cdots, n$ be composed of the connected subgraphs

$$A_{ij} = (S_{ij}, R_{ij}),\ j = 1, 2, \cdots, m_i,$$

such that the graphs A_{ij} for $j = 1, 2, \cdots, m_i$ are isomorphic to each other but not to other subgraphs of A, and such that

$$A_i = (\bigcup_j S_{ij}, \bigcup_j R_{ij}),\ j = 1, 2, \cdots, m_i,$$

and

$$S_{ix} \cap S_{iy} = \emptyset \ \text{ for } x \neq y \text{ and } x, y = 1, 2, \cdots, m_i.$$

Then

$$G(A) = G(A_1) \times G(A_2) \times \cdots \times G(A_n).$$

This leaves the problem of finding the group of a graph composed only of isomorphic subgraphs. Frucht [F1] has stated a theorem which can be used to find these groups. For completeness, that theorem is presented here in a restated and more general form. The theorem uses the concept of simply isomorphic permutation groups. Two isomorphic permutation groups are <u>simply isomorphic</u> if they are of equal degree. Since the proof of this theorem is lengthy, but otherwise not complicated, and since such proofs are commonly omitted, this proof is omitted here.

Theorem 4.3 Let $A = (S, R)$ be a graph composed of the connected subgraphs

$$A_1 = (S_1, R_1),\ A_2 = (S_2, R_2),\ \cdots,\ A_m = (S_m, R_m),$$

such that the graphs A_i, $i = 1, 2, \cdots, m$ are all isomorphic and such that

$$A = (\bigcup_i S_i, \bigcup_i R_i),\quad i = 1, 2, \cdots, m,$$

and

$$S_i \cap S_j = \emptyset \quad \text{for } i \neq j \text{ and } i, j = 1, 2, \cdots, m.$$

Then the groups $G_i = G(A_i)$ are all of the same degree, of the same order and so they are simply isomorphic one to another. If $h = \#(G_i)$ and $n = \#(S_i)$, then $G(A)$ is of order $m!\, h^m$ and of degree mn. The elements of $G(A)$ can be described in the following way. Let $S_i = \{s_{i1}, s_{i2}, \cdots, s_{in}\}$, for $i = 1, 2, \cdots, m$. Form the matrix

$$M = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{bmatrix}$$

An element of $G(A)$ is any permutation of the rows of M followed by any permutation in $G(S_i)$.

Relying on these theorems, a technique is now developed which simplifies the task of finding the group elements of a graph.

Definition 4.2 Given a graph $A = (S, R)$, let

$$F_s = \{u \text{ in } S \mid (s, u) \text{ is in } R\},$$

$$T_t = \{v \text{ in } S \mid (v, t) \text{ is in } R\}.$$

F_s is the set of vertices u in S with arcs from s to u and T_t is the set of vertices v in S with arcs from v to t. F_s and T_t are not independent for a given graph because $T_t = \{s \mid t \text{ in } F_s\}$ and $F_s = \{t \mid s \text{ in } T_t\}$.

Definition 4.3 Given a graph $A = (S, R)$ with s in S, let

$$o(s) = \#(F_s),$$

$$i(s) = \#(T_s).$$

If R is symmetric, then $F_s = T_s$ for all s in S and $o(s) = i(s)$. These two numbers are commonly called the degree of the vertex s when R is symmetric. For R in general, o(s) is the number of arcs from s to vertices in the graph and i(s) is the number of arcs from vertices to s.

Definition 4.4 Given a graph, let

$$O_k = \{s \mid o(s) = k\}, \quad k = 0, 1, 2, \ldots$$

$$I_j = \{t \mid i(t) = j\}, \quad j = 0, 1, 2, \ldots$$

Thus O_k is the set of vertices in S having "out degree" k and I_j is the set of vertices in S having "in degree" j. Note that $\bigcup_k O_k = S$, $O_k \cap O_\ell = \emptyset$ for $k \neq \ell$, $\bigcup_j I_j = S$ and $I_j \cap I_m = \emptyset$ for $j \neq m$.

If a graph $A = (S, R)$ has a subset $X \subset S$ such that $F_x = \emptyset$ for all x in X, then $o(x) = 0$ and $X \subseteq O_0$. If a graph $A = (S, R)$ has a subset $Y \subset S$ such that $T_y = \emptyset$ for all y in Y, then $i(x) = 0$ and $Y \subseteq I_0$.

The automorphisms of the graph $A = (S, R)$ are functions from F_s onto F_s, from T_s onto T_s, from O_k onto O_k and from I_j onto I_j as is shown by the following theorems.

Theorem 4.4 Let $A = (S, R)$ be a graph. If f is in G(A), then

$$f(O_k) = O_k,$$

$$f(I_j) = I_j.$$

for all O_k and I_j defined by (S, R).

Proof: Suppose $i(a) = j$ for some a in S. Then there are vertices $y_1, y_2, \cdots, y_j$ in S such that $[y_i, a]$ is in R, for $i = 1, 2, \cdots, j$. Since f is in G(A), $[f(y_i), f(a)]$ is in R. Thus $i[f(a)] \geq j$. If there is a b in S such that $b \neq f(y_i)$ for $i = 1, 2, \cdots, j$ and $[b, f(a)]$ is in R, then $[f^{-1}(b), a]$ is in R, hence $f^{-1}(b) = y_i$ for some $i = 1, 2, \cdots, j$. But this implies $b = f(y_i)$, a contradiction. Hence $i[f(a)] = j$ and f(a) is in I_j. Since this is true for any a in S such that $i(a) = j$, then $f(I_j) = I_j$. The proof for O_k is identical except for obvious changes. Δ

According to well known set theory theorems,

$$f(\bigcup_i A_i) = f(A_1) \cup f(A_2) \cup \cdots \cup f(A_n),$$

$$g(\bigcap_i B_m) = g(B_1) \cap g(B_2) \cap \cdots \cap g(B_m),$$

where f is a function defined on A_i, $i = 1, 2, \cdots, n$, and g is a function defined on B_i, $i = 1, 2, \cdots, m$. These relations are implicit in the proof of the next theorem and are also used in the proof of theorems in the latter part of this section.

Theorem 4.5 Let $A = (S, R)$ be a graph. If f is in $G(A)$, then

$$f(F_s) = F_{f(s)}$$

$$f(T_t) = T_{f(t)}$$

for all s and t in S such that $F_s \neq \emptyset$ and $T_t \neq \emptyset$.

Proof: Suppose b is in F_a. Then $[a, b]$ is in R, so $[f(b), f(a)]$ is in R. This implies that $f(b)$ is in $F_{f(a)}$, so $f(F_a) \subseteq F_{f(a)}$.

Suppose c is in $F_{f(a)}$. Then $[f(a), c]$ is in R and $[a, f^{-1}(c)]$ is in R because f is in $G(A)$. This implies that $f^{-1}(c)$ is in F_a, so $f^{-1}(F_{f(a)}) \subseteq F_a$, or $F_{f(a)} \subseteq f(F_a)$. But then $f(F_a) = F_{f(a)}$.

The proof for T_a is similar. Δ

The requirements $F_s \neq \emptyset$ and $T_t \neq \emptyset$ in Theorem 4.5 are not restrictive since $o(s) = 0$ or $i(s) = 0$ and the group functions of these vertices are considered in Theorem 4.4.

The following theorems consider the converse situation from that of the previous two. It is assumed that a permutation on the vertices of the graph is given, and sufficient conditions for this permutation to be an automorphism of the graph are given.

Theorem 4.6 Let $A = (S, R)$ be a graph and let f be a permutation on S. If

$$f(T_a) = T_{f(a)}$$

for all a in S such that $T_a \neq \emptyset$, then f is in $G(A)$.

Proof: To show that f is in $G(A)$, it is sufficient to show that if $[b, a]$ is in R, then $[f(b), f(a)]$ is in R and if $[c, a]$ is not in R, then $[f(c), f(a)]$ is not in R for all a, b and c in S. Suppose $[b, a]$ is in

R. Then b is in T_a, so that f(b) is in $f(T_a) = T_{f(a)}$. Hence [f(b), f(a)] is in R therefore, [b, a] in R implies [f(b), f(a)] is in R.

Suppose [c, a] is not in R. Then c is not in T_a. Since f is a permutation on S, c not in T_a implies f(c) is not in $f(T_a)$. But f(c) not in $f(T_a) = T_{f(a)}$ implies [f(c), f(a)] is not in R. Thus [c, a] not in R implies [f(c), f(a)] is not in R. Therefore, f is in G(A). Δ

Theorem 4.7. Let A = (S, R) be a graph and let f be a permutation on S. If

$$f(F_a) = F_{f(a)}$$

for all a in S such that $F_a \neq \emptyset$, then f is in G(A).

Proof: The proof of this theorem is identical to that of Theorem 4.6, except for obvious changes. Δ

Theorems 4.6 and 4.7 constitute the converse of the theorems in Theorem 4.5. However, the converse of Theorem 4.4 is not true as is shown by the following example.

Example 4.3 Let A = (S, R) be the graph $S = \{1, 2, 3\}$, $R = \{[1, 1], [1, 2], [2, 1], [2, 3], [3, 2]\}$. For this graph, $O_1 = \{3\}$, $O_2 = \{1, 2\}$, $I_1 = \{3\}$ and $I_2 = \{1, 2\}$. For the permutation f = (12), $f(O_1) = O_1$, $f(O_2) = O_2$, $f(I_1) = I_1$ and $f(I_2) = I_2$. Yet G(A) = I and f is not in G(A).

A simple combination of Theorems 4.4 and 4.5 yields the following theorem.

Theorem 4.8. Let A = (S, R) be a graph. If f is in G(A), then

$$f(F_a \cap I_j) = F_{f(a)} \cap I_j,$$

$$f(T_b \cap I_j) = T_{f(b)} \cap I_j,$$

$$f(F_a \cap O_k) = F_{f(a)} \cap O_k,$$

$$f(T_b \cap O_k) = T_{f(b)} \cap O_k,$$

for all I_j and O_k defined by A and for all a and b in S such that $F_a \neq \emptyset$ and $T_b \neq \emptyset$, and assuming $f(\emptyset) = \emptyset$.

Proof: Since all four statements have essentially the same kind of proof, the proof of only one is given.

$$f(T_b \cap O_k) = f(T_b) \cap (O_k) = T_{f(b)} \cap O_k. \quad \triangle$$

Of all the theorems in this section, the following theorem yields the most powerful device for calculation of the group elements of a graph.

Theorem 4.9 Let A = (S,R) be a graph and let f be a permutation on S. If $f(F_a \cap I_j) = F_{f(a)} \cap I_j$ for all I_j defined by A and for all a in S with the special provision that $f(\emptyset) = \emptyset$, then f is in G(A).

Proof: Use is made of the set-theoretic properties of functions of the union and intersection of sets as given on page 24.

$$\begin{aligned} \text{If } f(F_a \cap I_j) &= F_{f(a)} \cap I_j \text{ as stated, then} \\ \bigcup_j f(F_a \cap I_j) &= f[\bigcup_j (F_a \cap I_j)] \\ &= f[F_a \cap (\bigcup_j I_j)] \\ &= f(F_a \cap S) \\ &= f(F_a). \end{aligned}$$

While

$$\begin{aligned} \bigcup_j (F_{f(a)} \cap I_j) &= F_{f(a)} \cap (\bigcup_j I_j) \\ &= F_{f(a)} \cap S = F_{f(a)}. \end{aligned}$$

Hence $f(F_a) = F_{f(a)}$ and by Theorem 4.7, f is in G(A). Δ

Theorem 4.9 is true for the other three intersection equations, that is for

$$f(T_b \cap I_j) = T_{f(b)} \cap I_j,$$
$$f(F_a \cap O_k) = F_{f(a)} \cap O_k,$$
$$f(T_b \cap O_k) = T_{f(b)} \cap O_k.$$

The use of Theorem 4.9 in the determination of group elements of a graph is illustrated by the following example.

Example 4.4 Let A = (S, R) with S = $\{1, 2, 3, 4, 5, 6\}$ and R = $\{[1, 2], [1, 3], [2, 2], [2, 4], [3, 4], [3, 5], [4, 4], [4, 6], [5, 1], [5, 6], [6, 2], [6, 6]\}$. Table 4.1 shows a table of all $F_a \cap I_j$ and $F_a \cap O_k$.

	$I_1 = \{1, 3, 5\}$	$I_3 = \{2, 4, 6\}$	$O_2 = S$
$F_1 = \{2, 3\}$	3	2	2, 3
$F_2 = \{2, 4\}$	⋮	2, 4	2, 4
$F_3 = \{4, 5\}$	5	4	4, 5
$F_4 = \{4, 6\}$	⋮	4, 6	4, 6
$F_5 = \{1, 6\}$	1	6	1, 6
$F_6 = \{2, 6\}$	⋮	2, 6	2, 6

Table 4.1. An Intersection Table for Example 4.4

It is apparent from the I_1 column or the I_3 column of the table that only cycles starting with 13, 15, 35, 53, 51, 31, 24, 26, 46, 64, 62, and 42 need be considered for permutations in G(A). This is not apparent from the O_2 column, and while all permutations in G(A) could be determined from the O_2 column, it is much easier to use the I_1 and I_3 columns. For example, suppose a possible permutation is to have a cycle (1). Then (1) implies (2) and (3), (2) implies (24) or (2) and (4). This and the rest of the analysis for (1) is carried out in Table 4.2, with : meaning "implies", I meaning "impossible", and C indicating the order in which the closures are determined.

(1) C_r : (2) C_g, (3)C_q	(6) C_h : (2)C_f, (6)C_b
(3) C_p : (5) C_o, (4)C_k	(2) : 24I
(4) : 46I	(2) C_ℓ : (2) C_c, (4)C_d
(4) C_j : (4) C_a, (6) C_i	(5) C_n : (1) C_ℓ, (6)C_m
(6) : 26I	

Table 4.2. Analysis of cycle (1).

Thus the identity is the only element of G(A) with the cycle (1). The above analysis can be done for any cycle consisting of just one element of S and always yields the identity permutation. Thus the identity element is the only element in G(A) and so any other element in G(A) permutes every vertex of A.

Consider the analysis for the cycle starting with 13 as carried out in Table 4.3, with closure order omitted.

13C : 35C 24C	46C : 46C 62C
24I : 26I	62C : 62C 24C
24C : 24C 46C	35C : 51C 46C
46I : 42 66I	51C : 13C 62C

Table 4.3. Analysis for the cycle starting with 15.

From Table 4.3, it is apparent that $f = (135)(246)$ is in $G(A)$.

Consider next the cycle starting with 15, as carried out in Table 4.4.

15C :	31C	36C	64C :	64C	26C
26C :	26C	42C	31C :	53C	42C
42C :	42C	64C	53C :	15C	64C

Table 4.4. Analysis for the cycle starting with 15.

From Table 4.4, $f = (153)(264)$ is in $G(A)$. Since this exhausts all possible cycles to be considered, $G(A) = \{I, (135)(246), (153)(264)\}$.

A computer program was written[1] which will calculate the group of a given graph. Since it is interesting to compare the method of this program with the technique of Theorem 4.9, the method used in this program is described using the following definitions.

Definition 4.5 Given a graph $A = (S, R)$ of order n, the <u>connection matrix</u> $C(A)$ is the n x n matrix $C(A) = (a_{ij})$, where

$$a_{ij} = 1 \quad \text{if } [s_i, s_j] \text{ is in } R,$$
$$a_{ij} = 0 \quad \text{if } [s_i, s_j] \text{ is not in } R$$

for s_i and s_j in S.

Definition 4.6 Given a set S of order n and a permutation f on S, the <u>permutation matrix</u> $P(f)$ is the n x n matrix $P(f) = (b_{ij})$ where $b_{ij} = 1$ if and only if, $f(s_i) = s_j$ for s_i and s_j in S, and $b_{ij} = 0$ otherwise.

---
[1] This program was written by Mrs. Elizabeth Phillips of the Computer Laboratory at Michigan State University.

Note that $P(f) = P'(f^{-1})$ where C' is the transpose of C. A permutation f on S is a group element of $G(A)$ of the graph $A = (S, R)$ if and only if $P(f) \cdot C(A) \cdot P'(f) = C(A)$. The computer program presently in use is given $C(A)$, generates all possible permutations on S, and forms this matrix product. For the graph of Example 4.4, there are $\#(S)! = 6! = 720$ permutations to be tested. For a graph of order 6, the present computer program requires 18.75 minutes of computer time on a Control Data 160-A digital computer.

The same test can be used to check the permutations determined by the method of Theorem 4.9. Since not all possible permutations need be considered, the task of finding group elements of a graph is considerably simplified. In fact, use of Theorem 4.9 allows hand calculation of group elements where the order of the graph is too large to allow calculation by a direct method with a computer.

V. SPECIAL GRAPHS

Having considered the automorphism group of a graph, it would seem reasonable to ask about the effect of the group permutations on the arcs of the graph. The following definition establishes the basis for several ideas about functions of arcs of graphs.

Definition 5.1 Given a graph $A = (S, R)$ and its automorphism group $G(A)$, let $F(A)$ be the set of permutations on R such that, for each g in $G(A)$, f_g is in $F(A)$ if and only if

$$f_g[a, b] = [g(a), g(b)]$$

for all $[a, b]$ in R.

Theorem 5.1 There is a homomorphism from $G(A)$ onto $F(A)$.

Proof: It is sufficient to show that there exists a correspondence π associating elements of $F(A)$ with elements of $G(A)$ such that

(1) π is a function from $G(A)$ into $F(A)$.

(2) π is an operation preserving function, where in each system the single binary operation is taken to be composition, so that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all g_1 and g_2 in $G(A)$.

(3) π is an onto function, i.e., for every f in $F(A)$ there is a g in $G(A)$ such that $\pi(g) = f$.

Let π be defined by $\pi(g) = f_g$ if

$$f_g[a, b] = [g(a), g(b)]$$

for all $[a, b]$ in R.

That π is function from $G(A)$ onto $F(A)$ follows directly from the way in which π is defined.

To show that π is operation preserving, let $\pi(g_i) = f_i$ for $i = 1, 2, 3$ and let $g_1g_2 = g_3$. Then $\pi(g_1g_2) = \pi(g_3) = f_3$, and

$$f_3[a, b] = [g_3(a), g_3(b)] = [g_1g_2(a), g_1g_2(b)]$$

for all $[a, b]$ in R. Let $g_2(a) = c$, $g_2(b) = d$. $[c, d]$ is in R for all $[a, b]$ in R. So

$$\begin{aligned} f_3[a, b] &= [g_1(c), g_1(d)] = f_1[c, d] \\ &= f_1[g_2(a), g_2(b)] = f_1f_2[a, b]. \end{aligned}$$

Thus $\pi(g_3) = \pi(g_1)\,\pi(g_2)$ and so $\pi(g_1g_2) = \pi(g_1g_2) = \pi(g_1)\pi(g_2)$.$\Delta$

It follows from Theorem 5.1 that F(A) is a group and it might seem at first that F(A) would be isomorphic to G(A). This is not so in general, as is shown by the following example.

Example 5.1 Let $S = \{1, 2, 3, 4\}$ and let $R = [1, 2], [2, 1]$. Then $G(A) = \{I, (12), (34), (12)(34)\}$ and $\pi(I) = \pi(34) = ([1, 2])([2, 1])$, and $\pi(12) = \pi(12)(34) = ([1, 2]\ [2, 1])$. Thus π is not a one-to-one function. This is not a short-coming of π, but rather of A. In this example, the graph has two perfectly isolated vertices. A <u>perfectly isolated vertex</u> s is one for which $o(s) = 0$ and $i(s) = 0$.

The condition for π to be a one-to-one function and therefore, an isomorphism from G(A) to F(A) is given in the following theorem.

Theorem 5.2. Let $A = (S, R)$ be a graph and let π be the function as defined in Theorem 5.1. If A has less than two perfectly isolated vertices, then π is an isomorphism from G(A) onto F(A).

Proof: It has been shown in Theorem 5.1 that π is a homomorphism from G(A) onto F(A). Thus it is sufficient to show only that π is a one-to-one function; i.e., if $\pi(g_1) = \pi(g_2)$, then $g_1 = g_2$ for all g_1 and g_2 in G(A).

If $\pi(g_1) = \pi(g_2)$, then $[g_1(a), g_1(b)] = [g_2(a), g_2(b)]$ for all $[a, b]$ in R. Thus $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$.

Suppose A contains no perfectly isolated vertices. Then every vertex in S appears in some element of R, and $g_1(x) = g_2(x)$ for all x in S, either because x = a or because x = b for all [a, b] in R. Therefore, $g_1 = g_2$.

Suppose A contains exactly one perfectly isolated vertex s in S. Then $g_1(s) = g_2(s) = s$ for all g_1 and g_2 in G(A). Again, either $g_1(a) = g_2(a)$ for all a in $S - \{s\}$ or $g_1(b) = g_2(b)$ for all b in $S - \{s\}$. Thus, $g_1 = g_2$. $\triangle$

From Theorem 4.1 a graph A and its complement A' both have the same group, that is G(A) = G(A'), a fact pointed out by Kagno [K2] for non-oriented graphs. From Theorem 2.1, either A or A' has no perfectly isolated vertices. Assuming A has no perfectly isolated vertices, A' may or may not have isolated vertices. Thus, G(A') is at least homomorphic to F(A') and $\#[F(A')] \mid \#[F(A)]$.

Suppose A = (S, R) is a graph with k perfectly isolated vertices. Let K be the set of perfectly isolated vertices. Let $B = (S, R \cup (K \times K))$. Then three cases are of interest. (1) If $2k \neq \#(S)$, then G(A) = G(B). (2) If $2k = \#(S)$, and $R \neq (S - K) \times (S - K)$, then G(B) = G(A). (3) If 2k = S and $R = (S - K) \times (S - K)$, then $G(A) \subset G(B)$ and G(B) is the group as given by Theorem 4.3.

Since an automorphism group $F(A)$ does exist for the vertices of $A = (S, R)$, it would seem that some relation or operation on R is preserved by the elements of $F(A)$. This is indeed true as shown by the following definition and theorem.

Definition 5.2 Let $A = (S, R)$ be a graph, let x, y, z be in S and let a, b be in R. Then

$$a r_1 b \quad \text{if} \quad a = [x, y] \text{ and } b = [z, y]$$

$$a r_2 b \quad \text{if} \quad a = [y, x] \text{ and } b = [y, z]$$

$$a r_3 b \quad \text{if} \quad a = [x, y] \text{ and } b = [y, z]$$

The relations thus defined have many interesting properties in themselves. For example, r_1 and r_2 are symmetric relations. A relation r is said to be <u>transitive</u> if arb and brc implies arc; r_1 and r_2 are transitive relations.

The fact that these relations are preserved by $F(A)$ is presented in the next theorem. This theorem is presented without proof since the form of the proof is the same as the form of the proof given for Theorem 3.1. Also the important part of the proof of the following theorem is supplied by Theorems 5.1 and 5.2.

Theorem 5.3 Let $A = (S, R)$ be a graph and let $B = (R, \{r_1, r_2, r_3\})$ be a system. Then the elements of $F(A)$ are automorphisms on B.

It is interesting to note that B may be thought of as graphs. If R is symmetric, then $r_1 = r_2 = r_3 = r$ and $B = (R, r)$ is what Ore [O1, p. 245] calls the interchange graph $B = I(A)$. Sabidussi [S4] calls B the graph derivative of A. Both Ore and Sabidussi present several results for these non-oriented graphs.

The n-th interchange, denoted by $I^n(A)$ of the graph A is the graph A_n where $A_1 = I(A)$, $A_2 = I(A_1) = I[I(A)] = I^2(A)$, etc. Under certain specified conditions on A not developed here, $I^n(A)$ exists, and $F[I^n(A)]$ is homomorphic to A for all n. Thus, an entire family of homomorphic groups is generated by $I^n(A)$.

Another special type of graph that has received some attention in the literature is the graph which is the product of two or more graphs. For example, Sabisussi [S3] has presented a detailed study of the product of non-oriented graphs without loops. Much of the work of Sabidussi can be extended to the more general oriented graph with loops, thus graph product definitions are given which extend those of Sabidussi to this more general case. In the following definitions, it is convenient to use the concept of the projection p_i from the Cartesian product of sets onto its i-th coordinate, so that for any $s = [s_1, s_2, \cdots, s_i, \cdots, s_n]$ in $S_1 \times S_2 \times \cdots \times S_i \times \cdots \times S_n$, $p_i(s) = s_i$ in S_i for $i = 1, 2, \cdots, n$. In the following definitions $I = \{1, 2, \cdots, n\}$ is assumed to be the index set.

Definition 5.3 Let $\{A_i = (S_i, R_i) \mid i \text{ in } I\}$ be a set of graphs. The <u>Cartesian product</u> A_c of these graphs is the system (S, R_c) where

(1) $S = S_1 \times S_2 \times \cdots \times S_n$;

(2) for a, b in S, [a, b] is in R_c if and only if there exists a k in I such that $[p_k(a), p_k(b)]$ is in R_k, and $p_j(a) = p_j(b)$ for all j in $I - \{k\}$.

Definition 5.4 Let $\{A_i = (S_i, R_i) \mid i \text{ in } I\}$ be a set of graphs. The <u>direct product</u> A_d of these graphs is the system (S, R_d) where

(1) $S = S_1 \times S_2 \times \cdots \times S_n$.

(2) for a, b in S, $[a, b]$ is in R_d if and only if there exists a non-null subset $K \subseteq I$ such that $[p_k(a), p_k(b)]$ is in R_k for k in K, and $p_j(a) = p_j(b)$ for all j in I - K.

Thus these products of a set of graphs are graphs possibly with loops and not necessarily non-oriented. This suggests the following problem. Given a graph $A = (S, R)$, is it possible to factor A into two (or more) graphs $A_1 = (S_1, R_1)$, $A_2 = (S_2, R_2)$ such that A is the product of A_1 and A_2? At least a partial answer can be given to this question. To emphasize the problem, the following example is presented.

Example 5.2 Let $A_1 = (S_1, R_1)$ and $A_2 = (S_2, R_2)$ be two graphs with $S_1 = \{1, 2\}$, $R_1 = \{[1, 2]\}$, $S_2 = \{3, 4, 5\}$ and $R_2 = \{[3, 4], [4, 3], [4, 4], [5, 4], [5, 5]\}$. Then the Cartesian product is $A_c = (S, R_c)$, and the direct product is $A_d = (S, R_d)$ where $S = \{[1, 3], [1, 4], [1, 5], [2, 3], [2, 4], [2, 5]\}$, and letting $u = [1, 3]$, $v = [1, 4]$, $w = [1, 5]$, $x = [2, 3]$, $y = [2, 4]$, $z = [2, 5]$, then

$$R_c = \{[u, v], [u, x], [v, u], [v, v], [v, y], [w, v], [w, w], [w, z], [x, y], [y, x], [y, y], [z, y], [z, z]\}$$

$$R_d = R_c \cup \{[w, y], [v, x], [u, y]\}.$$

Actually, the two graph products defined are not independent, for given one it is always possible to determine the other directly. Also, the existence of certain arcs in the graph product is determined by the existence of certain other arcs in the product. The following theorem exhibits these details.

Theorem 5.4 Let $A_c = (S, R_c)$ be the Cartesian product and let $A_d = (S, R_d)$ be the direct product of the graphs $A_1 = (S_1, R_1)$ and $A_2 = (S_2, R_2)$. Let u and v be in S_1 and w and x be in S_2.

(1) If $[u, w] R_c [u, x]$ then $[v, w] R_c [v, x]$ and $[v, w] R_d [v, x]$

(2) If $[u, w] R_c [v, w]$ then $[u, x] R_c [v, x]$ and $[u, x] R_d [v, x]$

(3) If $[u, w] R_c [u, x]$ and $[u, x] R_c [v, x]$, then $[u, w] R_d [v, x]$

Proof: The first two statements follow directly from the definitions of Cartesian and direct products. Statement (3) is true because from $[u, w] R_c [u, x]$ it follows that $w R_2 x$; and from $[u, x] R_c [v, x]$ it follows that $u R_1 v$. Therefore $[u, w] R_d [v, x]$. Δ

Theorem 5.4 can be generalized to any two dimensional cross section of any product of more than two graphs. Furthermore, it is much easier to draw the product graph when the results of Theorem 5.4 are used.

The problem of factoring a graph $A = (S, R)$ will be considered assuming the factors $A_1 = (S_1, R_1)$ and $A_2 = (S_2, R_2)$ are both without loops. This assumption simplifies the enumeration of arcs in A, as given in the following theorem.

Theorem 5.5 Let $A_1 = (S_1, R_1)$ and $A_2 = (S_2, R_2)$ be two graphs, both without loops. Then

$$\# (R) = \#(S_1)\, \#(R_2) + \#(S_2)\, \#(R_1),$$

$$\# (S) = \#(S_1)\, \#(S_2)$$

where $A = (S, R)$ is the Cartesian product of A_1 and A_2.

Proof: Clearly $\#(S) = \#(S_1)\,\#(S_2)$. Next $[i, x]R[i, y]$ for all $i = 1, 2, \cdots, \#(S_1)$ and for all $[x, y]$ in R_2, for a total of $\#(S_1)\#(R_2)$ such elements in R. Similarly there are $\#(S_2)\#(R_1)$ elements $(z, i)R(w, i)$ in R. Δ

Theorem 5.5 provides a basis for considering the problem of factoring a graph. If the graph $A = (S, R)$ is of prime order, then no factoring is possible. So assume $\#(S) = mn$. Then a necessary condition for (S,R) to be the Cartesian product of (S_1, R_1) and (S_2, R_2) is that $\#(S_1) = m$, $\#(S_2) = n$, and

$$mx + ny = \#(R)$$

where $x = \#(R_2)$ and $y = \#(R_1)$. Thus it is necessary to find integral solutions to this equation with the additional restrictions that $0 \leq y \leq m(m - 1)$ and $0 \leq x \leq n(n - 1)$. This is a Diophantine problem and has solutions if and only if the greatest common divisior of m and n divides $\#(R)$. Let $d = (m, n)$ denote the greatest common divisor of m and n. Then if d divides $\#(R)$ all solutions are given by

$$x = x^* + \frac{n}{d}\, t$$
$$y = y^* - \frac{m}{d}\, t$$

where $x = x^*$ and $y = y^*$ is any particular solution and t is any integer, positive, negative, or zero.

Several methods exist for finding x^* and y^*. The requirements $0 \leq y \leq m(m - 1)$ and $0 \leq x \leq n(n-1)$ assure a finite number of solutions.

Although only the product of pairs of graphs is considered from Theorem 5.4 on, these results can be extended to products of more than two graphs. However, the Diophantine problem in more than two variables is complicated. Also, in determining the factors of a graph, all possible factors of the order of the graph may be considered and even if a set of factors is found that gives a solution to the Diophantine equation, it is still necessary to determine the individual vertex sets and their arcs for each graph in the product. Primarily, the Diophantine solutions will indicate what factors are not suitable for the factoring of a graph.

A third type of graph of special interest is the self-complementary graph. The self-complementary graph is defined as follows.

Definition 5.5 A graph $A = (S, R)$ is self-complementary if there is a permutation h on S such that $[a, b]$ is in R if and only if $[h(a), h(b)]$ is not in R. The permutation h is said to leave the graph self-complementary. The set of all such permutations is denoted by $H(S,R)$.

The connection matrix $C(A)$ exhibits several useful properties of self-complementary graphs. One of the more useful properties is this. For a self-complementary graph $A = (S,R)$, a permutation h on S is in $H(A)$ if and only if $c_{ab} = \bar{c}_{h(a)h(b)}$ for all c_{ab} in $C(A)$. Here, if $c = 0$, then $\bar{c} = 1$, and if $c = 1$, then $\bar{c} = 0$. That this is true is shown as follows. First assume h is in $H(A)$. Suppose $c_{ab} = 1$ so that aRb. Then $[h(a), h(b)]$ is not in R so that $c_{h(a)h(b)} = 0$. Then suppose

$c_{ab} = 0$ so that [a, b] is not in R. Then [h(a), h(b)] is in R so that $c_{h(a)h(b)} = 1$. Thus, for h in H(a), $c_{ab} = \bar{c}_{h(a)h(b)}$ for all c_{ab} in C(A).

Next, assume $c_{ab} = \bar{c}_{h(a)h(b)}$ for h a permutation on S and for all c_{ab} in C(A). Thus, if [a, b] is in R, then [h(a), h(b)] is not R or if [a, b] is not in R, then [h(a), h(b)] is in R. Since this is true for all a and b in S, h is in H(A).

Several other features of self-complementary graphs are obvious in light of the above discussion, namely:

(1) Every permutation h in H(S, R) leaves no element s in S fixed. This means that h(s) ≠ s. Suppose it were true that h(s) = s for some s in S. The $c_{ss} = c_{h(s)h(s)}$ for c_{ss} in C(A). But this is impossible, as shown by the above discussion.

(2) A self-complementary graph A has an equal number of ones and zeros on the main diagonal of C(A). This is true because $c_{aa} = \bar{c}_{h(a)h(a)}$ for c_{aa} in C(A) and all a in S.

(3) A self-complementary graph A = (S, R) is of even order. This follows directly from statement (2), above.

(4) A self-complementary graph A = (S, R) has an equal number of ones and zeros off the main diagonal of C(A). This follows from the above discussion.

(5) Every cycle of a self-complementary permutation h is of even length. Suppose some cycle of h were of odd length, say $(s_1, s_2, \cdots, s_{2n+1})$. Then for $c_{s_1 s_2}$ in C(A), $c_{s_1 s_2} = c_{h^m(s_1)h^m(s_2)}$ for m even and $c_{s_1 s_2} = \bar{c}_{h^m(s_1)h^m(s_2)}$ for m odd.

But $h^{2n+1}(s_1) = s_1$ and $h^{2n+1}(s_2) = s_2$. So

$$c_{h^{2n+1}(s_1)h^{2n+1}(s_2)} = c_{s_1 s_2} = \overline{c}_{s_1 s_2}$$

which is impossible. Thus, all cycles of h are even in length.

These properties of self-complementary graphs are used in the theorems presented. The following theorem is unusual in that it proves the existence of a self-complementary graph for any even positive integer.

Theorem 5.6 If n is an even positive integer, then there exists a self-complementary graph of order n.

Proof: The proof consists of constructing the self-complementary graph (S, R) for which $h = (12)(34), \cdots, (n-1, n)$ is in H(S, R). This is done by filling in the entries of C(S, R) by letting $c_{m-1,j} = 1$ and $c_{m,j} = 0$ for $m = 2, 4, 6, \cdots, n$ and $j = 1, 2, \cdots, n$. Then consider c_{ij} and $c_{h(i)h(j)}$. Since the latter entry is in the row above or below c_{ij}, then $c_{ij} = \overline{c}_{h(i)h(j)}$. Hence (S, R) is self-complementary. Δ

The next theorem presented depends on two relationships between the elements of H(S, R) and G(S, R), namely:

(1) If h_1 and h_2 are in H(S, R) for the self-complementary graph (S, R), then $h_1 h_2$ is in G(S, R). This is true because $c_{s_i, s_j} = \overline{c}_{h_i(s_i),\ h_1(s_j)} = c_{h_1 h_2(s_i),\ h_1 h_2(s_j)}$ for all $c_{s_i s_j}$ in C(S, R).

(2) If h is in H(S, R) and g is in G(S, R) for the self-complementary graph (S, R), then hg is in H(S, R). This is true because

$$c_{s_i, s_j} = c_{g(s_i), g(s_j)} = \bar{c}_{hg(s_i), hg(s_j)}$$

for all $c_{s_i s_j}$ in $C(S, R)$.

These two properties show that $[H(S,R)]^2 \subseteq G(S,R)$ and $H(S,R)G(S,R) \subseteq H(S,R)$. It is necessary to define normal subgroup to complete these results.

A subgroup A of a group B is said to be a <u>normal subgroup</u> if $x^{-1}A\,x = A$ for all x in B.

Theorem 5.7 If (S, R) is a self-complementary graph, then $G(S,R) \cup H(S,R)$ is a subgroup of the symmetric group on S; the order of $G(S,R)$ is equal to the order of $H(S,R)$ and $G(S,R)$ is a normal subgroup of $G(S,R) \cup H(S,R)$.

Proof: It is first sufficient to show that $H^2(S,R) = G(S,R)$ and $H(S,R)G(S,R) = G(S,R)H(S,R) = H(S,R)$. To show that $H^2(S,R) = G(S,R)$, observe that if h_1, h_2, and h_3 are in $H(S,R)$, then $h_1h_2 = h_1h_3$ implies $h_2 = h_3$; thus the order of $H^2(S,R)$ is greater than or equal to that of $H(S,R)$; hence the order of $G(S,R)$ is greater than or equal to that of $H(S,R)$. But since $H(S,R)G(S,R) \subseteq H(S,R)$ then the order of $H(S,R)$ is greater than or equal to that of $G(S,R)$ by a similar argument. But then the order of $H(S,R)$ is that of $G(S,R)$, so that

$$\#[H^2(S,R)] = \#[G(S,R)] = \#[H(S,R)].$$

Hence $H^2(S,R) = G(S,R)$ and $G(S,R)H(S,R) = H(S,R)G(S,R) = H(S,R)$.

Next, that $G(S,R) \cup H(S,R)$ is a group follows by a simple verification of the group postulates.

Finally, since $gG = G = Gg$ for any g in G and $hG = H = Gh$ for any h in H, then G is a normal subgroup of $G \cup H$. Δ

The following example is intended to illustrate some of these ideas.

Example 5.3 Let $S = \{1, 2, 3, 4\}$ and let

$R_1 = \{[1, 2], [1, 3], [1, 4], [2, 2], [3, 1], [3, 2], [3, 3], [3, 4]\}$,

$R_2 = \{[1, 2], [1, 3], [1, 4], [2, 2], [3, 3], [4, 1], [4, 2], [4, 3]\}$,

$R_3 = \{[1, 2], [1, 3], [1, 4], [2, 1], [2, 2], [2, 3], [2, 4], [3, 3]\}$.

Then $A_1 = (S, R_1)$, $A_2 = (S, R_2)$ and $A_3 = (S, R_3)$ are self-complementary graphs with

$H(A_1) = \{(12)(34)\}$

$G(A_1) = \{I\}$

$H(A_2) = \{(12)(34), (13)(24), (1234), (1342)\}$,

$G(A_2) = \{I, (23), (14), (23)(14)\}$,

$H(A_3) = \{(13)(24)\}$

$G(A_3) = \{I\}$.

As a closing topic for this section, the problem of producing graphs with a given group is considered. Specifically, the problem is this. Given a permutation group F of degree n, find a graph (S, R) of order n such that the automorphism group $G(S, R) = F$.

Kagno [K2] has shown that there is no non-oriented graph whose group is the cyclic group generated by a single cycle. If oriented graphs are used to realize the graph having a given group, then this cyclic group generated by a single n-cycle always has a graph as shown by the following theorem.

Theorem 5.8 If F is the cyclic group generated by the permutation $(1, 2, \cdots, n)$, then there exists a graph $A = (S, R)$ such that $G(A) = F$.

Proof: The proof consists of specifying $C(A) = (a_{ij})$ for $A = (S, R)$ so that $G(A) = F$. Let

$$a_{i,\, i+1} = 1 \text{ for } i = 1, 2, \cdots, n-1$$

$$a_{n,\, 1} = 1,$$

$$a_{11} = a_{22} = \cdots = a_{nn},$$

and let $a_{ij} = 0$ otherwise.

First, $(1, 2, \cdots, n)$ is an automorphism of (S, R). This is true because

$$a_{ij} = a_{i+1,\, j+1} \qquad \text{for } i, j < n$$

while

$$a_{nj} = a_{1,\, j+1} \qquad \text{for } j < n$$

and

$$a_{i,n} = a_{i+1,\, 1} \qquad \text{for } i < n.$$

Hence $F \subseteq G(S, R)$.

Next, let π be in $G(S, R)$. Then

$$a_{i,j} = a_{\pi(i),\pi(j)} \qquad \text{for } i, j = 1, 2, \cdots, n.$$

Suppose $\pi(i) = i + k$ (where if $i + k > n$, then $\pi(i) = i + k - n$). Then $\pi(i+1) = i + k + 1$, since $a_{i,\, i+1} = a_{i+k,\, \pi(i+1)}$ and then the only 1 in row $i + k$ is at column $i + k + 1$. Therefore, $\pi(1) = 1 + k$, $\pi(2) = 2 + k$, etc. But then $\pi = (1, 2, \cdots, n)^k$ is in F. Δ

Actually, it is possible to specify a graph for a more general type of group, namely for a special cyclic group, where a cyclic group on n symbols is the group generated by all powers of a single permutation on n symbols. The particular cyclic group and a graph for this group are given by the following theorem.

Theorem 5.9 Let F be the group of degree n generated by the permutation $\pi = p_1 p_2 \cdots p_q$ where the p_i, $i = 1, 2, \cdots, q$ are the cycles composing π. Let the length of p_i be a_i for $i = 1, 2, \cdots, q$. If $(a_i, a_j) = 1$ for $i \neq j$ and $i, j = 1, 2, \cdots, q$, there is an oriented graph A of order n such that $G(A) = F$.

Proof: First, G can be written as the product of the subgroups $G_1, G_2, \cdots, G_q$ where G_i is the cyclic group generated by p_i, $i = 1, 2, \cdots, q$. This is proved by observing that p_i is in G, since $\pi^{k_i} = p_i$ for some integer k_i and for $i = 1, 2, \cdots, q$. Next, the order of G_i is a_i, hence the order of the product $G_1 G_2 \cdots G_q$ is $a_1 a_2 \cdots a_q$; moreover, $G_1 G_2 \cdots G_q$ is a group since $G_i G_j = G_j G_i$ for $i, j = 1, 2, \cdots, q$. Finally, the order of G is the same as that of $G_1 G_2, \cdots, G_q$.

Let $p_i = (s_{o, p_i}, s_{1, p_i}, \cdots s_{r_{p_i} - 1, p_i})$ for $i = 1, 2, \cdots, q$. Then the graph $A_i = (S_i, R_i)$ given by $[s_{j, p_i}, s_{j+1(r_{p_i}), p_i}]$ in R for $j = 0, 1, \cdots, r_{p_i} - 1$ has $G_i = G(A_i)$. Moreover, the graph

$$A = (\bigcup_i S_i, \bigcup_i R_i), \quad i = 1, 2, \cdots, q$$

has for its group $G(A) = G(A_1) \times \cdots \times G(A_q) = G$. Δ

On the basis of the previous two theorems, it would seem reasonable to attempt to prove that a graph always exists for any cyclic group. While no such proof is given in this thesis, the following example tends to support the conjecture that a graph exists for any cyclic group. In this example, the given group is not of the type covered by the previous theorem.

Example 5.4 Let $S = \{1, 2, 3, 4, 5, 6\}$ and let

$$F = \{(1234)(56),\ (13)(24),\ (1432)(56),\ I\}$$

be the given group. Then for

$$C(A) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

it can easily be shown that $G(A) = F$.

An approach to the general problem of finding a graph $A = (S, R)$ given a permutation group F on S such that $G(S, R) = F$, is to symbolically fill in $C(A)$ for some f in F using the property $c_{ij} = c_{f(i)f(j)}$ for all c_{ij} in $C(A)$. This process is illustrated by the following example.

Example 5.5 Let $f = (12)(34)(56)$ be in the given group F on $S = \{1, 2, 3, 4, 5, 6\}$. Then $c_{11} = c_{22} = a$, $c_{12} = c_{21} = b$, $c_{13} = c_{24} = c$, and so on for all c_{ij} in $C(A)$, so that for

$$C_f = \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & c & f & e \\ g & h & i & j & k & m \\ h & g & j & i & m & k \\ n & p & q & r & s & t \\ p & n & r & q & t & s \end{bmatrix}$$

f is in G(S, R) where (S, R) is determined by any assignment of ones or zeros to C_f entries so that $C_f = C(S, R)$.

Thus, if it is desired to determine a graph A = (S, R) of order n for which G(A) = F for some predetermined group F of degree n, it would appear that the following would be sufficient:

(1) Construct a C_f for each f in F.

(2) Assign the value 0 or 1 to each entry of each C_f avoiding contradictions if possible.

(3) Determine all subgroups D of Σ_n which contains F as a subgroup. For each d in D-F, select an i and j and set $C_{ij} \neq C_{d(i)d(j)}$, if possible.

The result would be a connection matrix of a graph whose group is possibly F.

If it is impossible to complete step (2) or step (3), then F is not the group of any graph.

The above procedure involves extensive calculations. Possibly, the intersection table technique as developed in Section IV could be used to simplify these calculations.

One final observation is of interest. Most large groups are not specified by giving the elements of the group, but rather by specifying some defining characteristic of the group. Thus, the group to graph problem should be considerably simplified for groups so specified as is the situation with the groups considered in Theorem 5.8 and 5.9.

VI. CONCLUSION

The original interest in graphs which prompted this study resulted from a problem in switching circuits. Because of the possible application of groups of graphs to such problems, the subject of algebraic properties of graphs was considered. Without regard to possible applications, this study in itself, presented some difficult and interesting problems.

For this reason, this thesis completely ignores the possible applications of this work. Two such applications deserve mention. First, the original switching circuit problem involved state merging and state reduction in sequential machines. Since the merger diagram of a flow table is actually a graph of the (S, R) type, the possibility of an algebraic attack on the merger problem should be considered. Secondly, the problem of counting the total number of non-isomorphic trees in a network has received considerable attention in the circuit theory literature. Since trees of a given network that are isomorphic are related by the automorphism group of one such tree, it would seem reasonable to attempt to classify and to count trees by use of algebraic techniques of the sort used in this thesis.

Any application of this work requires a re-evaluation of the type of graph which is studied. It is easy to specialize the graph (S, R) to the non-oriented or to the loopless case. But, some applications of graphs require multiple arcs from one vertex to another. This is not possible with the graph (S, R), and is one shortcoming of this type of graph. Conversely, many of the results in this thesis can be extended to include a more general type of system having sets of relations rather than just a single relation. Thus, a study of the algebraic properties of systems, in general, would seem to be an area for possible further work.

There are several other areas of possible further work using just the graph (S, R) as the basic system. For example, the problem of finding a graph of a given group requires much more study. It should be possible to express the group of a graph product in terms of the groups of the graphs. Self-complementary graphs, oriented and non-oriented, offer many problems that could be studied. These are but a few of the many areas involving algebraic concepts as presented in this thesis.

BIBLIOGRAPHY

[F1] Robert Frucht, "On the groups of repeated graphs," Bull. Amer. Math. Society, v. 55 (1949), pp. 418-420.

[F2] ——————, "Graphs of degree three with a given abstract group," Canad. J. of Math., v. 1 (1949), pp. 365-378.

[F3] ——————, "A one-regular graph of degree three," Canad. J. of Math., v. 4 (1952), pp. 240-247.

[K1] I. N. Kagno, "Linear graphs of degree ≤ 6 and their groups," Amer. J. of Math., v. 68 (1946), pp. 505-520.

——————, Correction to [K1], v. 69 (1947), p. 872

——————, Correction to [K1], v. 77 (1955), p. 392

[K2] ——————, "Desargues' and Pappus' graphs and their groups," Amer. J. of Math., v. 69 (1947), pp. 859-862.

[O1] Oystein Ore, Theory of Graphs. Rhode Island: American Mathematical Society, 1962.

[S1] Gert Sabidussi,"Graphs with given group and given graph-theoretical properties," Canad. J. Math., v. 9(1957), pp. 515-525.

[S2] ——————, "On the minimum order of graphs with given automorphism group," Monatsh. Math., v. 63 (1959), pp. 124-127.

[S3] ——————, "Graph Multiplication," Math. Zeitschr., v. 72 (1960), pp. 446-457.

[S4] ——————, "Graph Derivatives," Math. Zeitschr., v. 76 (1961), pp. 385-401.

ROOM USE ONLY