PTOLEMAIC METRIC SPACES AND THE CHARACTERIZATION OF GEODESICS BY VANISHING METRIC CURVATURE

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ABSTRACT

PTOLEMAIC METRIC SPACES AND THE CHARACTERIZATION
OF GEODESICS BY VANISHING METRIC CURVATURE

by David Clifford Kay

In a metric space, the characterization of a geodesic among all rectifiable arcs by the identical vanishing of its metric curvature--from now on referred to as the fundamental theorem--has curiously required the space to be locally ptolemaic. Such a metric space is one whose metric xy satisfies the inequality

$$pq \cdot rs + pr \cdot qs \geqslant ps \cdot qr$$

in some neighborhood of each point of the space. Haantjes did obtain the theorem for his concept of curvature in a class of metric spaces that includes, for example, spherical space, but his proof involves certain variations of the ptolemaic inequality. Counterexamples in L_1 spaces show that some condition is necessary. Part I of this thesis is devoted to discovering how restrictive the ptolemaic inequality is, with emphasis on those metric spaces introduced and studied previously by H. Busemann called G-spaces. On the other hand, it is possible that other inequalities may imply the fundamental theorem. This is explored in Part II, and finally, a new concept of curvature is proposed for which the fundamental theorem can be proved in a wide class of metric spaces which includes locally ptolemaic G-spaces, two-dimensional Riemannian spaces, and Banach spaces with strictly convex unit sphere.

More specifically, in Part I a concept of space curvature called median curvature is introduced, analogous to that studied by Busemann. This concept is compared with that of Busemann and the following theorems are proved:

A G-space having non-positive median space curvature is locally ptolemaic.

In a Riemannian space the conditions non-positive median curvature, non-positive Busemann curvature, non-positive classical curvature at every point, and being locally ptolemaic, are equivalent.

Other theorems of Part I are:

A straight G-space with convex differentiable spheres is euclidean iff it is ptolemaic and satisfies the parallel axiom.

A Finsler space is Riemannian with non-positive Busemann curvature iff it is locally ptolemaic.

A symmetric Hilbert geometry (where the absolute is a symmetric convex surface) is hyperbolic iff it is locally ptolemaic.

In Part II, two weaker forms of "non-positive median curvature" are introduced: If for each point p in a G-space there exists a neighborhood V_p in which the space has unique joins and a constant γ_p with $0 < \gamma_p < 1$ such that if a, b, and c be any three points of V_p and m the midpoint of the segment joining a and b, then the inequality

$$\gamma_p \cdot mc \leqslant \sqrt{\tfrac{1}{2} ac^2 + \tfrac{1}{2} bc^2 - \tfrac{1}{4} ab^2}$$

holds, the space is said to have weakly non-positive median space curvature, while if the inequality

$$mc \leqslant \tfrac{1}{2}\,(ac + bc)$$

holds, the space is said to have feebly non-positive median space curvature. (Non-positive median curvature is defined by the former inequality with $\gamma_p = 1$ for all p.) The theorem obtained is:

In any G-space which has both weakly and feebly non-positive median space curvature, the fundamental theorem is valid for both the Menger and the Haantjes definitions of the curvature of an arc.

The class of spaces for which this theorem is valid includes all Riemannian surfaces and certain (non-euclidean) Minkowski spaces. Also in Part II is introduced the following concept of curvature for arcs in G-spaces: If $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ are any three sequences of points on an arc such that for each i, $q_i < r_i < s_i$, $q_i r_i = r_i s_i$, and m_i is the unique midpoint of the segment joining q_i and s_i, then that arc is said to have transverse curvature $\kappa_T'(p)$ at p iff

$$\lim_{i \to \infty} \frac{8\, m_i r_i}{q_i s_i^{\,2}} = \kappa_T'(p).$$

This curvature is compared with classical concepts in euclidean space and a formula for it is derived in certain Minkowski spaces. Then the fundamental theorem is proved for this curvature in G-spaces having feebly non-positive median space curvature.

PTOLEMAIC METRIC SPACES AND THE CHARACTERIZATION OF GEODESICS BY VANISHING METRIC CURVATURE

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PREFACE

The author is deeply indebted to his major professor, Dr. L. M. Kelly, for his inspiration during the preparation of this thesis. Appreciation is due him especially for his patience during the actual writing and preparation of the final draft. The author also wishes to extend his appreciation to his parents, Mr. and Mrs. Clifford I. Kay for providing for the typing and printing of the manuscript.

To Helen

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INTRODUCTION

Various definitions of curvature for arcs in general metric spaces have been advanced. Menger [14] was perhaps the first to propose an acceptable definition, and he made immediate efforts to show that an arc in a metric space is a segment if and only if his curvature identically vanishes along the arc. He concocted a variety of examples showing that this is not the case, but eventually did show with the help of Alt and Beer that the property is characteristic of segments in E^n. Schoenberg [16] extended this theorem to arcs in metric ptolemaic spaces (see p. 5 for the definition) while Haantjes [11], using a definition of curvature suggested by Finsler, established a similar theorem in a class of metric spaces somewhat more general than ptolemaic spaces. Simple examples in L_1 spaces of dimension two show that the characterization theorem is not extendible to general linear spaces using any of the definitions thus far considered. It has been conjectured by L. Kelly and others that an arc with identically vanishing Menger or Haantjes-Finsler curvature in a strictly convex Banach space must be a segment, and one of the principal objectives of this thesis is the study of this conjecture.

Somewhat more generally we attempt here to extend the fundamental characterization theorem to as wide a class of metric spaces

as possible, while at the same time we seek to gain a deeper insight into the scope of the class of ptolemaic metric spaces. Our efforts to verify the conjecture are unsuccessful, but they lead to the introduction of a new curvature κ_T (defined in Section 9) which coincides with the variously studied curvatures in classical spaces and for which we can prove the following satisfying theorem:

Theorem: An arc A in a strictly convex Banach space is a segment if and only if for each point $p \in A$, $\kappa_T(p) = 0$.

Actually the class of spaces for which this theorem is valid is much wider than the strictly convex Banach spaces. It includes all spaces each two points of which can be joined by a unique segment (straight spaces in the language of Busemann [6]) and in which the length of the median of a triangle is less than or equal to one-half the sum of the adjacent sides. The theorem can be "localized" and in the local form such spaces include spherical, euclidean, hyperbolic, elliptic and in fact all Riemannian spaces. In Section 8 we obtain the precise relationship between κ_T in a Minkowski space and classical curvature in the associated euclidean space thus making contact with Busemann's work [5] in this direction.

Freese [9] in his Missouri dissertation proves that two-dimensional Riemannian space with everywhere non-positive Gauss curvature is ptolemaic, a fact which is actually implicit in the work of Wald on the metrization of Gauss curvature [20] . We introduce a new concept of space curvature somewhat analogous to that of Busemann and prove that any space with non-positive curvature in this sense is ptolemaic. This implies the Freese result in any

dimension.

We had hoped to establish that any G-space (defined in Section 5) having negative curvature in the sense of Busemann is ptolemaic. This proved to be rather difficult and in our efforts to construct examples we discovered the interesting fact that any symmetric ptolemaic Hilbert geometry (defined in Section 5) is hyperbolic. The concept of curvature we introduce also provides the analogue of the P. Kelly and Strauss theorem [13] to the effect that any Hilbert geometry with unisigned curvature is hyperbolic.

We should note in conclusion two other possibly significant failures. We have been unable to find an example of a "respectable" space (e.g., a G-space) in which the fundamental curvature characterization theorem for geodesics is not valid. We have also been unable to construct an example of a space of negative curvature in the sense of Busemann which is not Riemannian.

PART I

1. Preliminaries. If M is any set, a real-valued function d defined over $M \times M$ is a distance function on M if for any $p_i \in M$,

(a) $d(p_1,p_2) \geqslant 0$.

(b) $d(p_1,p_2) = 0$ iff (if and only if) $p_1 = p_2$.

(c) $d(p_1,p_2) = d(p_2,p_1)$.

(d) $d(p_1,p_3) \leqslant d(p_1,p_2) + d(p_2,p_3)$, referred to as the triangle inequality. A set M with a distance function is a metric space and $d(p_i,p_j)$ is the distance from p_i to p_j. The distance will be represented here more simply by the symbol p_ip_j and the number p_ip_j itself by a Greek character, the Latin alphabet being reserved primarily for points in M.

The topological conventions of Busemann [6] will be adopted. For the convenience of the reader a list of the more frequently used concepts appears below.

The sequence $\{p_i\}$, $i = 1, 2, 3, \cdots$, where $p_i \in M$, converges to $p \in M$ iff $\lim_{i\to\infty} p_ip = 0$.

A function $f: M_1 \longrightarrow M_2$ mapping M_1 into M_2 is continuous iff $\{f(p_i)\}$ converges to $f(p)$ whenever $\{p_i\}$, $i = 1, 2, 3, \cdots$, converges to p. A function $f: M_1 \longrightarrow M_2$ is an isometry (or a congruence) iff f is an onto mapping and $p_ip_j = f(p_i)f(p_j)$ for all $p_k \in M_1$. Such a mapping is clearly continuous and in fact a homeomorphism.

A metric space is compact iff every sequence contains a convergent subsequence or alternatively iff every infinite subset has an accumulation element. It is said to be finitely compact iff every bounded subset is compact.

A neighborhood about $p \in M$, to be denoted usually by V_p, is the open sphere $V(p,\delta)$ for some $\delta > 0$, that is, the sets of points $x \in M$ such that $px < \delta$.

A compact subset S of M has the property that every open covering by neighborhoods $\{V_p\}$, $p \in S$, can be refined to a finite subcovering, $V_{p_1}, V_{p_2}, \cdots, V_{p_n}$.

A definition of particular importance to this thesis is the following:

Definition 1.1: A quadruple of points p_1, p_2, p_3, and p_4 in a metric space is ptolemaic iff

$$p_ip_j \cdot p_kp_m + p_ip_k \cdot p_jp_m \geqslant p_ip_m \cdot p_jp_k$$

where (i, j, k, m) is any permutation of $(1, 2, 3, 4)$. A space M is said to be ptolemaic iff each of its quadruples are ptolemaic. It is locally ptolemaic iff each point $p \in M$ has a neighborhood V_p which is ptolemaic.

Euclidean space of dimension n, denoted by E^n, consists of all ordered n-tuples $(x_1,x_2,\cdots,x_n)$ of real numbers x_i metrized by defining

$$xy = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $x = (x_1,x_2,\cdots,x_n)$ and $y = (y_1,y_2,\cdots,y_n)$.

An isometric image in M of the euclidean segment is a metric segment in M. The images of the endpoints of the euclidean segment are the endpoints of the metric segment and the metric segment is said to be a segment joining those endpoints. $S(p,q)$ will denote the

segment joining p and q whenever it is unique in M. If $S(p,q)$ exists for each pair of points (p,q) in M it is said to have unique segments.

The point q is metrically between p and r iff $pq + qr = pr$ and $p \neq q \neq r$, and the symbol (pqr) will denote this situation. If p, q, and r are any three distinct points of a segment then one is always metrically between the other two.

A metric space is convex iff each pair of points has a point between them.

Theorem (Menger): Each two points of a complete convex metric space are joined by a metric segment. (The proof of this may be found in Busemann [6].)

A curve in a metric space M is a continuous map of E^1 into M; an arc is a homeomorphism of a euclidean segment. A geodesic is a curve which is locally a metric segment, that is, each point of the curve has a neighborhood in which each of its subarcs is a metric segment.

If (pmq) holds and $pm = mq$ then m is called a midpoint of the pair p, q. If $S(p,q)$ exists m must be in $S(p,q)$ and it is then unique and called the midpoint of the segment $S(p,q)$, denoted by m_{pq}.

If the points p, q, and r in M determine unique segments then the set union of those segments is the triangle with vertices p, q, and r, and is denoted by $T(p,q,r)$. The median of the triangle $T(p,q,r)$ from p is the segment $S(p,m_{qr})$, if it exists.

This section will be concluded with a definition which proves to be relevant to ptolemaic spaces.

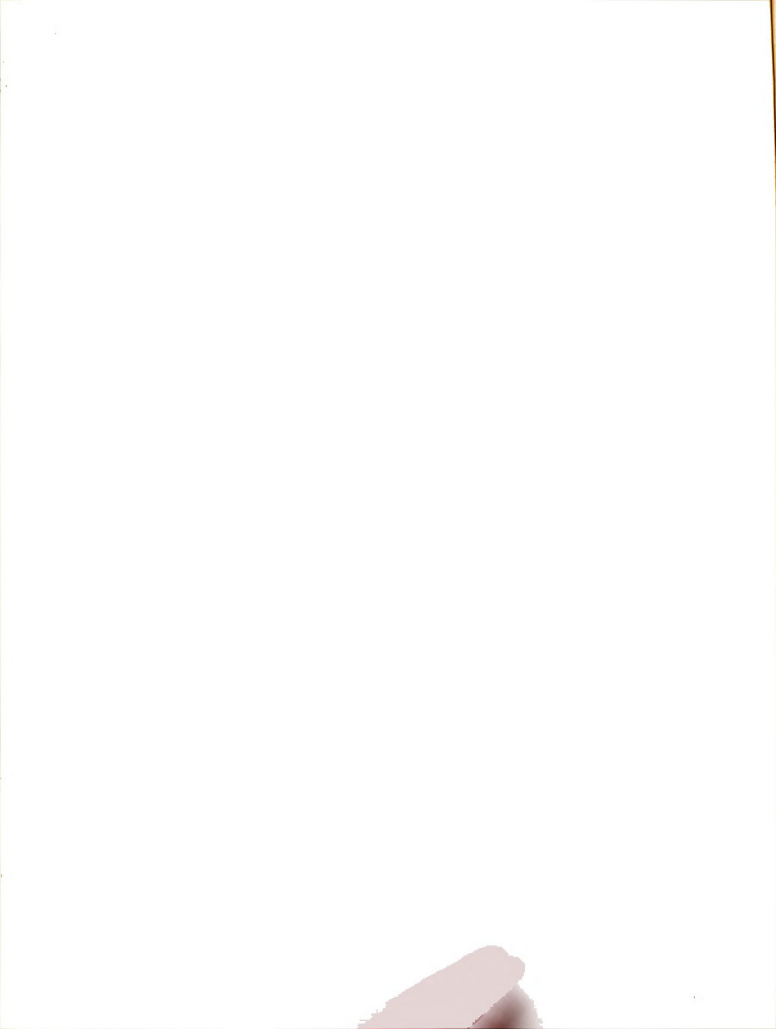
Definition 1.2: The median inequality for a complete convex metric space M is for any point triple (p,q,r) and any midpoint m of q and r the inequality

$$pm^2 \leqslant \tfrac{1}{2}\, pq^2 + \tfrac{1}{2}\, pr^2 - \tfrac{1}{4}\, qr^2.$$

Remark: The significance of this inequality may be seen from the fact that if $(p,q,r) \longrightarrow (p',r',q')$ is an isometry of M into E^2 with m' the midpoint of S(q',r'), and if the median inequality holds in M, then from the formula for the length of the median of a triangle in terms of its sides in E^2 it follows that $pm \leqslant p'm'$; this applies even if p', q', and r' are collinear. Moreover, if the median inequality holds in a complete convex metric space M, the space will have unique segments. For, suppose m is a midpoint of q and r, and p is any other midpoint. Then in the isometry $(p,q,r) \longrightarrow (p',q',r')$ into E^2, p' coincides with m' and the above observation produces $pm \leqslant 0$. That is, $pm = 0$, and $p = m$. Thus, midpoints are unique in M, and it easily follows that segments are unique also.

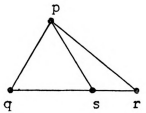
2. The Ptolemaic and Median Inequalities Related. A stronger form of the median inequality is obtained, from which it will be proved that in any complete convex metric space the median inequality implies the ptolemaic inequality.

Lemma 2.1: Let M be any complete convex metric space in which the median inequality holds, and therefore in which unique segments exist. Then if p, q, r, and s are any four points with $s \in S(q,r)$, the inequality



$$ps^2 \leqslant \frac{sr}{qr}\cdot pq^2 + \frac{qs}{qr}\cdot pr^2 - qs\cdot sr$$

holds. Moreover, if M is such that the strict median inequality holds for any three non-linear points (that is, not on a segment), then the above inequality is also strict for p,q,r non-linear points, q≠s≠r.



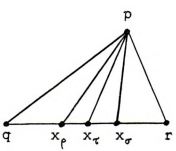
Proof: Since S(q,r) is isometric to the interval $0 \leqslant \xi \leqslant qr$ by the mapping $x \longrightarrow \xi$, $x \in S(q,r)$, put $\tau = qx/qr = \xi/qr$, which defines a linear map from S(q,r) to the unit interval $0 \leqslant \tau \leqslant 1$; then denote x by x_τ. The desired inequality for $x_\tau = s$ then reads

$$(2.1) \qquad px_\tau^2 \leqslant (1-\tau)\, pq^2 + \tau\cdot pr^2 - \tau(1-\tau)\, qr^2.$$

This will first be proved for τ a diadic rational $m/2^n$, by induction on n. Let $R(n) = \{1/2^n, 3/2^n, 5/2^n, \cdots, (2^n-1)/2^n\}$. For n = 1, if τ is in R(n) then $\tau = \frac{1}{2}$ and x_τ is the midpoint of S(q,r), so the proposition follows by the median inequality itself. Suppose it has been proved for all integers k less than n and $\tau \in R(k)$. Let τ be in R(n) and suppose $\tau = (2m-1)/2^n$ where $1 \leqslant m \leqslant 2^{n-1}$. Put $\rho = (m-1)/2^{n-1}$ and $\sigma = m/2^{n-1}$ so that ρ and σ are each in some R(k) for $k < n$. Then the induction hypothesis gives

(2.2)
$$px_\rho^2 \leqslant (1-\rho)pq^2 + \rho\cdot pr^2 - \rho(1-\rho)qr^2$$

(2.3)
$$px_\sigma^2 \leqslant (1-\sigma)pq^2 + \sigma\cdot pr^2 - \sigma(1-\sigma)qr^2.$$



Note that x_τ is the midpoint of $S(x_\rho, x_\sigma)$, since

$$x_\rho x_\tau = qx_\tau - qx_\rho = (\tau - \rho)qr$$

while

$$x_\tau x_\sigma = qx_\sigma - qx_\tau = (\sigma - \tau)qr$$

and $(\tau - \rho)qr = 2^{-n} \cdot qr = (\sigma - \tau)qr$. Observe also that $\tau = \frac{1}{2}(\rho + \sigma)$.

Applying the median inequality to $T(p, x_\rho, x_\sigma)$ it follows that

$$px_\tau^2 \leqslant \tfrac{1}{2}\, px_\rho^2 + \tfrac{1}{2}\, px_\sigma^2 - \tfrac{1}{4}\, x_\rho x_\sigma^2,$$

or, using $x_\rho x_\sigma = (\sigma - \rho)qr$,

$$px_\tau^2 \leqslant \tfrac{1}{2}\, px_\rho^2 + \tfrac{1}{2}\, px_\sigma^2 - \tfrac{1}{4}\, (\sigma - \rho)^2 qr^2. \tag{2.4}$$

From (2.2) and (2.3),

$$\begin{aligned} px_\tau^2 \;&\leqslant\; \tfrac{1}{2}\,(1 - \rho)pq^2 + \tfrac{1}{2}\,\rho \cdot pr^2 - \tfrac{1}{2}\,(\rho - \rho^2)qr^2 + \\ &\quad + \tfrac{1}{2}\,(1 - \sigma)pq^2 + \tfrac{1}{2}\,\sigma \cdot pr^2 - \tfrac{1}{2}\,(\sigma - \sigma^2)qr^2 - \tfrac{1}{4}(\sigma - \rho)^2 qr^2 \\ &= (1 - \tfrac{1}{2}\rho - \tfrac{1}{2}\sigma)pq^2 + \tfrac{1}{2}(\rho + \sigma)pr^2 - (\tfrac{1}{2}\rho + \tfrac{1}{2}\sigma - \tfrac{1}{4}\rho^2 - \tfrac{1}{2}\rho\sigma - \tfrac{1}{4}\sigma^2)qr^2 \\ &= (1 - \tau)pq^2 + \tau \cdot pr^2 - (\tau - \tau^2)qr^2 \end{aligned} \tag{2.5}$$

which is the desired inequality for τ in $R(n)$. Hence the inequality (2.1) holds by induction for all n and $\tau \in R(n)$, that is, if τ is a diadic rational. Since the diadic rationals are dense on the interval $(0,1)$ a sequence $\{\tau_n\}$ of diadic rationals can be found which converge to τ, given $s = x_\tau$, so that the corresponding sequence $\{x_{\tau_n}\}$ converges to x_τ and by the continuity of the metric, $\lim_{n \to \infty} px_{\tau_n} = px_\tau$. But since (2.1) holds for each τ_n we have for each n,

$$px_{\tau_n}^2 \leqslant (1 - \tau_n)pq^2 + \tau_n \cdot pr^2 - \tau_n(1 - \tau_n)qr^2. \tag{2.6}$$

Taking the limit as $n \longrightarrow \infty$, the desired result (2.1) is obtained.

Now suppose the strict median inequality holds for non-linear points in M, and let p, q, and r be non-linear, with $q \neq s \neq r$. Put $s = x_\tau$ and locate x_ρ and x_σ any two points on $S(q,r)$ such

that x_τ is the midpoint of $S(x_\rho, x_\sigma)$. Again it follows that $\tau = \frac{1}{2}(\rho+\sigma)$ except that here, ρ, σ, and τ are not necessarily diadic rationals. Nevertheless the inequalities (2.2) and (2.3) are now established for arbitrary ρ and σ, and the strict inequality holds in (2.4). It then follows that in the subsequent steps the inequalities are strict in (2.5) which gives the strict inequality in (2.1), completing the proof.

Remark: It follows from the lemma that if the median inequality holds in M and $(p,q,r) \longrightarrow (p',q',r')$ is an isometry into E^2 with s' that point on $S(q',r')$ such that $q's' = qs$, then $ps \leqslant p's'$. For it is clear that

$$\begin{aligned} ps &\leqslant \frac{sr}{qr}\cdot pq + \frac{qs}{qr}\cdot pr - qs\cdot sr \\ &= \frac{s'r'}{q'r'}\cdot p'q' + \frac{q's'}{q'r'}\cdot p'r' - q's'\cdot s'r' \\ &= p's', \end{aligned}$$

making use of an elementary formula valid in E^2.

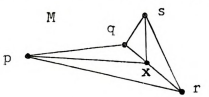
Theorem 2.1: If the median inequality holds in a complete convex metric space M, then M is ptolemaic.

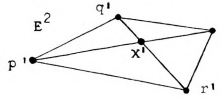
Proof: It is trivial that if four points are not distinct, or if they are linear, then they are ptolemaic. It may then be assumed that the four points are distinct and non-linear. Let the points be represented by p, q, r, and s, so ordered that among $pq\cdot rs$, $pr\cdot qs$, and $ps\cdot qr$, the value $ps\cdot qr$ is maximal. It then suffices to prove the single inequality

$$pq\cdot rs + pr\cdot qs \geqslant ps\cdot qr \tag{2.7}$$

for all cases. Suppose $(p,q,r) \longrightarrow (p',q',r')$ is an isometry

into E^2. Choose a second isometry $(q,r,s) \longrightarrow (q',r',s')$ into E^2 so that if both triples (p,q,r) and (q,r,s) are non-linear, p'





and s' fall on opposite sides of the lines through q' and r'. The points p' and s' are then distinct, for otherwise both triples (p,q,r) and (q,r,s) would be linear, and since M has unique segments from the median inequality this would imply that all four points were linear, contrary to assumption. Again, the line through p' and s' in E^2 will always have exactly one point x' in common with the line through q' and r' or else these lines would coincide implying that p, q, r, and s are linear. There are then two cases: (a) x' falls on $S(q',r')$ or, (b) x' is exterior to $S(q',r')$.

(a) $\underline{x' \text{ falls on } S(q',r')}$. Let x be that point on $S(q,r)$ such that $qx = q'x'$. By the lemma and the remark following it,

(2.8) $$px \leqslant p'x' \quad \text{and} \quad sx \leqslant s'x'$$

so that $px + xs \leqslant p'x' + x's' = p's'$ and the triangle inequality $ps \leqslant px + xs$ imply

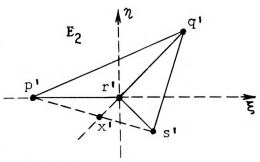
(2.9) $$ps \leqslant p's'.$$

From the fact that E^2 is ptolemaic, (2.9) implies

(2.10) $$pq \cdot rs + pr \cdot qs = p'q' \cdot r's' + p'r' \cdot q's' \geqslant p's' \cdot q'r' \geqslant ps \cdot qr.$$

(b) $\underline{x' \text{ is exterior to } S(q',r')}$. Consider the sum of the angles (p',q',r') and (r',q',s') in E^2 and the sum of angle (p',r',q') and angle (q',r',s'). One of these sums must be greater than a straight angle. Since the following argument may be applied

to either case it may be assumed that the latter sum is greater than a straight angle. If both angles in this sum are $\geqslant \frac{\pi}{2}$ then $p'q' > q'r'$ and $s'q' > q'r'$ so that $pq > qr$ and $qs > qr$ and it follows that



$$pq \cdot rs + pr \cdot qs > qr \cdot rs + pr \cdot qr = (rs + pr) \cdot qr \geqslant ps \cdot qr.$$

The remaining cases are when one of the angles of this sum is acute. Again it may be assumed without loss of generality that angle (q',r',s') is acute and angle (p',r',q') is obtuse or straight. Finally, it may be assumed that $ps > p's'$, for otherwise (2.9) holds and (2.10) follows as it did in (a). Introduce a coordinate system (ξ,η) in E^2 with the origin at r', p' on the negative ξ-axis, and s' in the fourth quadrant or on the negative η- axis. Since $\frac{\pi}{2} < \text{angle } (p',r',q') \leqslant \pi$, q' will either lie in the first quadrant or on the positive ξ-axis, hence its coordinates (α,β) will be such that $\alpha > 0$, $\beta \geqslant 0$. Let the coordinates of s' be (γ,δ) with $\gamma > 0$, $\delta < 0$. Consider the minor arc A of the circle with center at r' and radius ρ equal to $r's'$, whose endpoints are $s' = (\gamma,\delta)$ and $t' = (\rho,0)$. The real-valued function $f(z') = p'z'$ for $z' \in A$ is continuous, and $f(s') = p's' < ps$ while $f(t') = p't' = p'r' + r't' = pr + rs \geqslant ps$. Hence there exists a point $s'' \neq s'$ on A such that $f(s'') = ps$. If s'' has coordinates (γ',δ') then $\gamma' > \gamma$ and $\delta' > \delta$ and it follows that

$$\begin{aligned} q's'^2 - q's''^2 &= (\alpha - \gamma)^2 + (\beta - \delta)^2 - (\alpha - \gamma')^2 - (\beta - \delta')^2 \\ &= 2\alpha(\gamma' - \gamma) + 2\beta(\delta' - \delta) + (\gamma^2 + \delta^2) - (\gamma'^2 + \delta'^2) \\ &> \rho^2 - \rho^2 = 0 \end{aligned}$$

or $q's' > q's''$ and therefore

(2.11) $$qs > q's''.$$

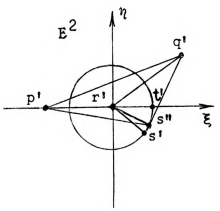
Applying the ptolemaic inequality to the quadruple p', q', r', and s'' in E^2,

$$p'q' \cdot r's'' + p'r' \cdot q's'' \geq p's'' \cdot q'r'$$

or, since $p's'' = ps$,

$$pq \cdot rs + pr \cdot q's'' \geq ps \cdot qr$$

which, together with (2.11), implies



$$pq \cdot rs + pr \cdot qs > pq \cdot rs + pr \cdot q's'' \geq ps \cdot qr,$$

completing the proof.

Remark: In E^2 the ptolemaic inequality is strict for distinct points in all but two cases, so the applications made of it in the above proof result in the strict inequality in all but those two cases. The two cases when equality holds for points in E^2 are when the points are linear or cyclic. An examination of the proof of Case (a) above reveals that since the points p', q', r', and s' cannot be linear, the only possible way for the ptolemaic equality to hold for those points is for them to be the vertices of a cyclic quadrilateral with diagonals $S(p', s')$ and $S(q', r')$. Then x' is an interior point of $S(q', r')$ which implies from the lemma that the strict inequality for (2.9), (2.10), and therefore for (2.7). In Case (b) the argument either reduced to Case (a) or the strict inequality was obtained for (2.7). This proves

Corollary 2.1: If the strict median inequality holds for non-linear triples in M and p, q, r, and s are any four points in M, the only occurrence of equality for (2.7) is the case when either the four points are not distinct or when they are linear.

3. The Ptolemaic Inequality and Riemannian Spaces. A brief description of a Riemannian n-manifold M will be given for the convenience of the reader. Many terms will be encountered which involve basic concepts of topology and are not defined here. The reader is referred to Synge and Schild [18] for details concerning tensor calculus.

(a) M is an n-manifold with a differentiable structure of class C^2 at least. That is, M is a connected separable topological space such that for each point p in M there is an open neighborhood V_p homeomorphic to an open neighborhood of E^n, n fixed for all $p \in M$, such that if this homeomorphism be denoted by h_p and its inverse by h_p^{-1} then whenver two neighborhoods V_p and V_q have non-void intersection the transformation $h_q h_p^{-1}$ from $E^n(x^1,x^2,\cdots,x^n)$ into $E^n(y^1,y^2,\cdots,y^n)$ is of the form

$$y^i = y^i(x^1,x^2,\cdots,x^n), \quad i = 1, 2, \cdots, n \tag{3.1}$$

where the y^i have continuous second order partial derivatives at least.

(b) There exists, with respect to the coordinate systems of M corresponding to these homeomorphisms into E^n a positive definite fundamental or metric covariant tensor $g_{ij}(x^1,x^2,\cdots,x^n)$ of the second order defined at each point for each coordinate system. That is, there exists an $n \times n$ positive definite matrix $[g_{ij}]$ which transforms by means of (3.1) from one coordinate system to another as follows (summation indicated by repeated indices):

$$g'_{ij} = g_{rs} \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^j}$$

where g'_{ij} corresponds to the components of the matrix in the

coordinate system $(y^1, y^2, \cdots, y^n)$. One usually makes the assumption that the g_{ij} are each of class C^3 at least.

(c) In terms of this tensor the arc length of curves of class C^1 at least, $x^i = x^i(\tau)$, $i = 1, 2, \cdots, n$, is defined as

$$\Lambda = \int_\alpha^\beta \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} \, d\tau$$

by means of which M is metrized by taking xy to be the number

$$\text{Inf} \int_\alpha^\beta \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} \, d\tau$$

taken over all curves $x^i = x^i(\tau)$ such that x corresponds to $\tau = \alpha$ and y to $\tau = \beta$.

Observe that in the case $n = 3$ and with

$$g_{ij}(x^1, x^2, x^3) \equiv 1 \quad \text{when } i = j,$$
$$\equiv 0 \quad \text{otherwise,}$$

the above formula reduces to the usual one for differentiable curves in E^3,

$$\Lambda = \int_\alpha^\beta \sqrt{\left(\frac{dx^1}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2 + \left(\frac{dx^3}{d\tau}\right)^2} \, d\tau$$

and thus M is E^3 in this case.

A few well-known results from Riemannian geometry will be needed. One of these is that every point has a neighborhood in which each pair of points may be joined by a unique metric segment, and that there exists a family of geodesics of class C^2, passing through each point, a unique one for each direction.

The angle between two geodesics may be defined as follows: Suppose two geodesics represented by $x_1^i = x_1^i(\tau)$ and $x_2^i = x_2^i(\tau)$, $i = 1, 2, \cdots, n$, emanate from a certain point $a \in M$ with the respective directions $(u_j^1, u_j^2, \cdots, u_j^n)$, $j = 1, 2$, where $u_1^i = dx_1^i/d\tau$, $u_2^i = dx_2^i/d\tau$,

$i = 1, 2, \cdots, n$. The measure A of the angle between those geodesics is

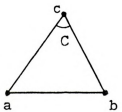
$$A = \operatorname{Cos}^{-1} \frac{g_{ij} u_1^i u_2^j}{\sqrt{g_{ij} u_1^i u_1^j}\ \sqrt{g_{ij} u_2^i u_2^j}} .$$

There is a satisfactory concept of curvature in M, referred to as Gauss curvature in the case $n = 2$ which is a generalization of Gauss curvature of surfaces in E^3. The curvature K at a point p in M for $n > 2$ is defined corresponding to each two-dimensional surface through p as the Gauss curvature of that surface. It is possible to characterize locally euclidean spaces (those for which each point has a neighborhood which is isometric to a neighborhood in E^n) among all Riemannian manifolds by the condition $K = 0$ at every point (see Synge and Schild [18]). The following results concerning curvature are from Cartan [7] , p. 261, for non-positive curvature, and from Alexandrov [1] for positive curvature.

The Cosine Inequality: If $K \leqslant 0$ in a Riemannian space M there exists a neighborhood V of each point in which the sides of every triangle T(a,b,c) with vertices in V satisfy

(3.2) $$ab^2 \geqslant ac^2 + bc^2 - 2ac \cdot bc \cos C$$

where C is the measure of the angle at c. If $K > 0$ at some point p in M, then there exists a surface which contains two geodesics through p such that on that surface there is a neighborhood V in which the sides of every triangle T(a,b,c) with vertices in V satisfy (writing $\overline{pq}$ as the metric of the surface)



(3.3) $$\overline{ab}^2 < \overline{ac}^2 + \overline{bc}^2 - 2\overline{ac} \cdot \overline{bc} \cos C.$$

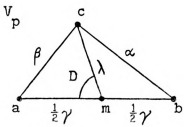
The following theorem was obtained for the 2-dimensional case by Freese [9] , using methods which cannot be extended to higher dimensions. The following theorem, therefore, is somewhat stronger, and its proof does not involve dimension in any way.

Theorem 3.1: If M is any Riemannian space of arbitrary dimension and with $K \leqslant 0$ at every point, then M is locally ptolemaic.

Proof: Since $K \leqslant 0$ at any point $p \in M$ there exists a neighborhood V_p which has unique segments and in which (3.2) holds for every triangle with vertices in V_p. It will be shown that V_p is a convex metric space in which the median inequality holds. V_p is already a convex metric space by assumption. Let a, b, and c be any three points of V_p and m the midpoint of a and b. Since $T(a,c,m)$ and $T(c,m,b)$ exist (3.2) may be applied. Let D be the measure of angle (a,m,c) and put $\alpha = bc$, $\beta = ac$, $\gamma = ab$, and $\lambda = cm$. Then

$$\beta^2 \geqslant \tfrac{1}{4}\gamma^2 + \lambda^2 - \gamma\lambda \cos D$$

$$\alpha^2 \geqslant \tfrac{1}{4}\gamma^2 + \lambda^2 + \gamma\lambda \cos D$$



which implies

$$\alpha^2 + \beta^2 \geqslant \tfrac{1}{2}\gamma^2 + 2\lambda^2.$$

That is, $\lambda^2 \leqslant \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - \frac{1}{4}\gamma^2$, which is the median inequality. By Theorem 1.2 V_p is ptolemaic which proves M is locally ptolemaic.

It might be conjectured that in certain cases the local condition of non-positive curvature could guarantee the ptolemaic inequality in the large. But that this is not the case even for K strictly negative everywhere may be seen in the example of the surface of revolution in $E^3(x,y,z)$ obtained by rotating the curve $y = x^2 + 1$,

$z = 0$ about the x-axis. This surface has negative curvature everywhere and by the above theorem is locally ptolemaic. But observe the points a, b, d, and c equally spaced on the circle generated by the point (0,1) on the generating curve. With xy denoting the metric on this surface, $ab = bd = cd = ac = \frac{\pi}{2}$ while $ad = bc = \pi$, but $ab \cdot cd + ac \cdot bd = \frac{\pi^2}{2}$ while $ad \cdot bc = \pi^2$ so that

$$ab \cdot cd + ac \cdot bd < ad \cdot bc$$

violating the ptolemaic inequality for arbitrary points.

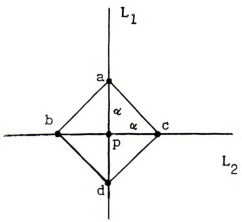
The converse of Theorem 3.1 also holds.

Theorem 3.2: If M is a Riemannian space and M is locally ptolemaic then $K \leqslant 0$ everywhere.

Proof: Suppose $K > 0$ for some point p of M. This means that there is a two-dimensional neighborhood about p such that every triangle T(a,b,c) in it satisfies (3.3) Let L_1 and L_2 be two geodesics in that neighborhood meeting at p orthogonally. Consider points a and d on L_1 on opposite sides of p with $\overline{ap} = \overline{dp} = \alpha$ and points b and c on L_2 on opposite sides of p so that $\overline{bp} = \overline{cp} = \alpha$. Then from (3.3), $\overline{ab}$, $\overline{bd}$, $\overline{cd}$, and $\overline{ac}$ are each less than $\sqrt{2}\alpha$ and

$$ab \cdot cd + ac \cdot bd \leqslant \overline{ab} \cdot \overline{cd} + \overline{ac} \cdot \overline{bd} < 2\alpha^2 + 2\alpha^2 = 2\alpha \cdot 2\alpha = \overline{ad} \cdot \overline{bc} = ad \cdot bc$$

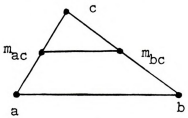
for all sufficiently small α, so regardless of the n-dimensional neighborhood taken about p there always exist a quadruple of points violating the ptolemaic inequality. But this contradicts the fact that M is locally ptolemaic, which then proves $K \leqslant 0$.



4. Metric Curvatures in Riemannian Space. In this section we study two metric concepts of curvature in the context of metric spaces with locally unique segments. In particular we investigate the relationships of these curvatures to the ptolemaic inequality in Riemannian spaces. After Busemann [6] we state

Definition 4.1: A metric space M which locally has unique segments is said to have non-positive Busemann curvature denoted by $K_B \leqslant 0$ iff each point $p \in M$ has a neighborhood V_p such that if a, b, and c are any three points in V_p then m_{ac} and m_{bc} exist and

(4.1) $$m_{ac}m_{bc} \leqslant \tfrac{1}{2} ab.$$



The space is said to have zero Busemann curvature denoted by $K_B = 0$ iff

(4.2) $$m_{ac}m_{bc} = \tfrac{1}{2} ab$$

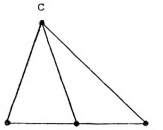
always holds, and non-negative Busemann curvature denoted by $K_B \geqslant 0$ iff

(4.3) $$m_{ac}m_{bc} \geqslant \tfrac{1}{2} ab$$

always holds. If the strict inequality holds in (4.1) for non-linear triples a, b, and c the space is said to have negative Busemann curvature denoted by $K_B < 0$, and if strict inequality holds in (4.3) for non-linear triples it is said to have positive Busemann curvature denoted by $K_B > 0$.

Definition 4.2: A metric space M which locally has unique segments is said to have non-positive median curvature denoted by $K_M \leqslant 0$ iff each point $p \in M$ has a neighborhood V_p such that if a, b, and c are any three points in V_p then m_{ab} exists and

(4.4) $m_{ab}c^2 \leqslant \frac{1}{2} ac^2 + \frac{1}{2} bc^2 - \frac{1}{4} ab^2$.



The space is said to have zero median curvature denoted by $K_M = 0$ iff

(4.5) $m_{ab}c^2 = \frac{1}{2} ac^2 + \frac{1}{2} bc^2 - \frac{1}{4} ab^2$

always holds, and non-negative median curvature denoted by $K_M \geqslant 0$ iff

(4.6) $m_{ab}c^2 \geqslant \frac{1}{2} ac^2 + \frac{1}{2} bc^2 - \frac{1}{4} ab^2$

always holds. If the strict inequalities (4.4) or (4.6) hold for non-linear triples, as in Definition 4.1 the space is said to have respectively negative and positive median curvature denoted in the obvious way.

It should be noticed that the inequality (1.2) referred to earlier as the median inequality is precisely the condition (4.4) affirming that the space has non-positive median curvature. It then follows by a previous theorem that a space which has non-positive median curvature is locally ptolemaic. It is clear that this is not the case with non-positive Busemann curvature, for this class of spaces includes Minkowskian spaces which are not in general ptolemaic.

It is important to observe that for both space curvatures K_B and K_M the condition "non-positive curvature" does not necessarily include the conditions "negative curvature" and "zero curvature" as mutually exclusive cases. That is, non-zero curvature for a space does not imply that the space has either positive or negative curvature.

Theorem 4.1: In any metric space which has unique segments locally, the conditions negative, non-positive, zero, non-negative, and positive median curvature respectively imply negative, non-positive, zero, non-negative, and positive Busemann curvature.

Proof: To handle the above cases at once, let the following notation be used. If a metric space S has $K_M \leqslant 0$ write $K_M(S) \leqslant 0$, and similarly for the other cases. Corresponding to each metric space S for which $K_M(S)$ is defined, define a binary relation $R_S(\alpha,\beta)$ on the real numbers as follows:

$$\begin{cases} R_S(\alpha,\beta) & \text{iff } \alpha<\beta \text{ if } K_M(S) < 0 \\ R_S(\alpha,\beta) & \text{iff } \alpha\leqslant\beta \text{ if } K_M(S) \leqslant 0 \\ R_S(\alpha,\beta) & \text{iff } \alpha=\beta \text{ if } K_M(S) = 0 \\ R_S(\alpha,\beta) & \text{iff } \alpha\geqslant\beta \text{ if } K_M(S) \geqslant 0 \\ R_S(\alpha,\beta) & \text{iff } \alpha>\beta \text{ if } K_M(S) > 0 \end{cases}$$

Let p be any point in the space. In each of the five cases there exists a neighborhood V_p in which for any three non-linear points x, y, and z in V_p and w the midpoint of x and z, $R_S(yw^2, \frac{1}{2}xy^2 + \frac{1}{2}yz^2 - \frac{1}{4}xz^2)$ holds. Let W_p be the neighborhood about p with radius one-half that of V_p and suppose a, b, and c are any three points of W_p. If ρ is the radius of V_p and m_{bc} is the midpoint of b and c, then

$$\begin{aligned} pm_{bc} &\leqslant pc + cm_{bc} \\ &= pc + \tfrac{1}{2}bc \\ &\leqslant pc + \tfrac{1}{2}(pb + pc) \\ &< \tfrac{1}{2}\rho + \tfrac{1}{2}(\tfrac{1}{2}\rho + \tfrac{1}{2}\rho) = \rho \end{aligned}$$

so that a, c, and m_{bc} lie in V_p. Let m_{ac} be the midpoint of a

and c and put $\alpha = bc$, $\beta = ac$, $\gamma = ab$, $\gamma' = m_{ac}m_{bc}$, and $\gamma'' = m_{bc}a$. If a, b, and c are linear, it is clear that $\gamma' = \frac{1}{2}\gamma$. Otherwise, a, b, and c are non-linear and therefore a, m_{bc}, and c are non-linear and lie in V_p. Therefore

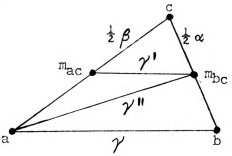
$$R_S(\gamma'^2, \tfrac{1}{8}\alpha^2 + \tfrac{1}{2}\gamma''^2 - \tfrac{1}{4}\beta^2)$$

holds, or

(4.7) $R_S(2\gamma'^2 - \frac{1}{4}\alpha^2 + \frac{1}{2}\beta^2, \gamma''^2)$

Also,

(4.8) $R_S(\gamma''^2, \frac{1}{2}\beta^2 + \frac{1}{2}\gamma^2 - \frac{1}{4}\alpha^2)$.



The transitivity of R_S then implies from (4.7) and (4.8)

$$R_S(2\gamma'^2 - \tfrac{1}{4}\alpha^2 + \tfrac{1}{2}\beta^2, \tfrac{1}{2}\beta^2 + \tfrac{1}{2}\gamma^2 - \tfrac{1}{4}\alpha^2)$$

or

$$R_S(\gamma'^2, \tfrac{1}{4}\gamma^2)$$

Since γ' and γ are positive then $R_S(\gamma', \frac{1}{2}\gamma)$ holds, which completes the proof.

Busemann proves in [6] pp. 269-70 that in a Riemannian space with classical curvature K defined at each point, $K_B \leqslant 0$ and $K \leqslant 0$ everywhere are equivalent conditions in the space, that also $K_B = 0$ and $K = 0$ at each point are equivalent, and that $K < 0$ at every point implies $K_B < 0$. It is now possible to give a similar comparison between K and K_M. This is done in the next two theorems.

Theorem 4.2: In a Riemannian space M, the following conditions are equivalent:

a. $K \leqslant 0$ at every point.

b. $K_M \leqslant 0$.

c. The space is locally ptolemaic.

Proof: (a) implies (b) since (a) implies the cosine inequality (3.2) and, as in the proof of Theorem 3.1, the cosine inequality implies $K_M \leqslant 0$. (b) implies (c) by Theorem (2.1), and (c) implies (a) by Theorem 3.2.

Remark: It follows that in a Riemannian space, non-positive Busemann curvature and non-positive median curvature are equivalent. Indeed, $K_B \leqslant 0$ and the three conditions of Theorem 4.2 are equivalent for a Riemannian space.

Theorem 4.3: In a Riemannian space M, $K = 0$ at every point iff $K_M = 0$, and if $K < 0$ at every point then $K_M < 0$ and M is locally strictly ptolemaic.

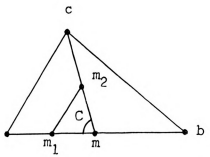
Proof: The first statement follows easily from the fact that $K = 0$ at every point makes M locally euclidean (isometric to E^n locally) so that $K_M = 0$ holds, and conversely it will be seen later that $K_M = 0$ implies M is locally euclidean which means $K = 0$ at every point. For the second assertion, let $K < 0$ at every point. Then $K_B < 0$ from Busemann's theorem, and the cosine inequality (3.2) holds locally. Let V_p be a neighborhood about an arbitrary point $p \in M$ such that for any non-linear triple x, y, and z in V_p, m_{xz} and m_{yz} exist and both inequalities

$$m_{xz}m_{yz} < \tfrac{1}{2}\, xy \tag{4.9}$$

$$xy^2 \geqslant yz^2 + xz^2 - 2\, yz \cdot xz \cdot \cos Z \tag{4.10}$$

are valid, where Z = angle (x,z,y). Let W_p be the neighborhood about p whose radius is one-fourth that of V_p and suppose a, b, and c are any three non-linear points in W_p. The midpoint m of

a and b lies in a neighborhood about p of radius one-half that of V_p, and therefore the midpoints m_1 and m_2 of m and a and of m and c respectively lie in V_p. The non-linearity of a, b, and c implies that of a, c, and m. Hence (4.9) implies



$$m_1m_2 < \tfrac{1}{2}\, ac. \tag{4.11}$$

But from (4.10), with C = angle (m_1, m, m_2),

$$m_1m_2^2 \geqslant m_1m^2 + m_2m^2 - 2\cdot m_1m\cdot m_2m\cdot \cos C$$

$$= \frac{1}{16}\cdot ab^2 + \frac{1}{4}\cdot mc^2 - \frac{1}{4}\cdot ab\cdot mc\cdot \cos C$$

and from (4.11),

$$ac^2 > \tfrac{1}{4}\, ab^2 + mc^2 - ab\cdot mc\cdot \cos C \tag{4.12}$$

Also from (4.10),

$$bc^2 \geqslant \tfrac{1}{4}\, ab^2 + mc^2 + ab\cdot mc\cdot \cos C \tag{4.13}$$

Summing,

$$ac^2 + bc^2 > \tfrac{1}{2}\, ab^2 + 2\, mc^2$$

or

$$mc^2 < \tfrac{1}{2}\, ac^2 + \tfrac{1}{2}\, bc^2 - \tfrac{1}{4}\, ab^2. \tag{4.14}$$

This proves that for any non-linear triple a, b, c in W_p the strict inequality (4.4) holds, hence $K_M < 0$. That the strict ptolemaic inequality holds for distances in W_p (for quadruples of distinct non-linear points) now follows from Corollary 2.1.

The above theorems show that any one of the concepts we have been considering serves to distinguish the non-positive sign of curvature in Riemannian geometry. This suggests the use of either K_M or the local

ptolemaic condition--both metric concepts--as a means of defining more generally non-positive curvature in G-spaces, analogous to Busemann's condition $K_B \leqslant 0$. This will be treated in the next section.

5. The Ptolemaic Inequality and G-Spaces. Metric spaces which satisfy the following list of axioms have been termed G-spaces, or geodesic spaces, by Busemann [6]:

a. The space is finitely compact.

b. The space is convex.

c. The space is locally prolongable, that is, for p any point of the space there exists a neighborhood V_p of p such that given q and r any two distinct points in V_p there exists a point s with (qrs) satisfied.

d. The space is uniquely prolongable, that is, if p and q are distinct points then (pqr_1) and (pqr_2) imply $r_1 = r_2$ whenever $qr_1 = qr_2$.

The immediate important consequences of these axioms are:

a. Each point has a neighborhood in which each pair of points can be joined by a unique segment.

b. Each segment can be prolonged to a unique geodesic of the space which contains that segment as a sub-arc. Accordingly, we shall denote the unique geodesic obtained in this manner by $L(p,q)$ where p and q are the end-points of the segment.

The stock of "standard" examples of G-spaces consists of Riemannian space, along with the special cases of hyperbolic, euclidean, and spherical space, Finsler space, along with the special case of

Minkowski space, Hilbert geometry, and Cartesian products of any of the above metrized by

$$d((x_1,x_2),(y_1,y_2)) = (x_1y_1^{\alpha} + x_2y_2^{\alpha})^{1/\alpha}$$

with $\alpha > 1$, where x_i and y_i are in M_i, a G-space with metric x_iy_i, $i = 1, 2$.

In this section we extend the study of G-spaces to include our concept of median curvature of space, with particular emphasis on G-spaces which are locally ptolemaic. Our original objective at this point was to establish or disprove the conjecture that a G-space with $K_B < 0$ is locally ptolemaic. In this we were unsuccessful but the investigation led to many interesting results which we include here. The argument presented in the proof of Theorem 4.1 is equally applicable to a G-space. It then follows that in any G-space $K_M \leqslant 0$ implies $K_B \leqslant 0$, while in a Riemannian space, these conditions are equivalent. This points up how desirable it would be to know the precise relationship between K_M and K_B in a G-space.

In [6] Busemann shows that a G-space with $K_B = 0$ is a locally Minkowskian space, the term "locally" being used in the strong sense indicating that each point has a neighborhood isometric to one in a Minkowskian G-space. Since the condition $K_M = 0$ implies both that $K_B = 0$ and that the space is locally ptolemaic, then it implies in a G-space that the space is a locally ptolemaic Minkowskian space. This, together with the fact (Schoenberg [17]) that a ptolemaic Banach space is an inner product space proves

Theorem 5.1: A G-space is locally euclidean iff $K_M = 0$.

It should be remarked here that this theorem is implicit in the work of Blumenthal [2,3] since the condition $K_M = 0$ is locally what he terms the feeble euclidean four-point property. Our

proof of the above theorem then provides an independent proof of some of Blumenthal's characterization theorems.

The proof of Theorem 5.1 given above employs Busemann's theorem which characterizes locally Minkowskian G-spaces as those for which $K_B = 0$. Since Busemann's argument establishing this fact is scattered throughout a number of other important results, it seems desirable to present a new and somewhat more direct proof of this fact here. Then, a more direct proof of Theorem 5.1 will be indicated, after which, as stated earlier, we shall explore the implications of the locally ptolemaic condition in certain G-spaces.

Preliminary to the proof of the Busemann result we sketch a few of the relevant facts concerning normed linear spaces which will be useful throughout the remainder of the thesis.

Definition 5.1: A vector space M over the real field F is said to be normed, or to have a norm, iff there is a mapping of M into the set of non-negative reals, denoted by $\|x\|$ for $x \in M$, satisfying the axioms

a. $\|x\| = 0$ iff $x = o$, the zero of M as a vector space.

b. $\|\lambda x\| = |\lambda| \|x\|$ for $x \in M$, $\lambda \in F$.

c. $\|x + y\| \leqslant \|x\| + \|y\|$ for $x \in M$, $y \in M$.

M is said to have a semi-norm $\|x\|$ iff just (a) and (b) hold.

It is easily seen that M becomes a metric space by defining the metric as $xy = \|x - y\|$. The topological terms connected with norms are with this understanding.

Definition 5.2: A space M is called a (real) Banach space iff it is a vector space with a complete norm, that is, a norm for which

every Cauchy sequence has a limit. The dimension of M is understood to mean its dimension as a vector space.

Definition 5.3: The unit sphere U of a semi-normed vector space is the set of all points x such that $\|x\| = 1$. A point x is called an interior or an exterior point of U according as either $\|x\| < 1$ or $\|x\| > 1$.

Definition 5.4: The unit sphere U is said to be convex iff for each $x \in U$, $y \in U$ then

$$\|\lambda x + (1 - \lambda)y\| \leq 1, \quad 0 < \lambda < 1, \tag{5.1}$$

and it is called strictly convex iff the strict inequality holds in (5.1) whenever $x \neq y$.

It follows automatically that a normed vector space, and therefore also a Banach space, has a convex unit sphere, for if $0 < \lambda < 1$ then since

$$\|x\| = \|y\| = 1,$$

$$\|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\| = |\lambda| \|x\| + |(1 - \lambda)| \|y\| = 1.$$

The set L_{xy}, sometimes called a "one-flat," defined as the set of all x in a Banach space M such that

$$z = z_\lambda = \lambda x + (1 - \lambda)y, \quad \lambda \in F,$$

is isometric to E^1. For, if $f(z_\lambda) = \lambda \cdot xy$ where xy is the metric of M defined as above, it follows that

$$\begin{aligned} z_{\lambda_1} z_{\lambda_2} &= \|z_{\lambda_1} - z_{\lambda_2}\| \\ &= \|(\lambda_1 - \lambda_2)x - (\lambda_1 - \lambda_2)y\| \\ &= |\lambda_1 - \lambda_2| \, \|x - y\| \\ &= |\lambda_1 - \lambda_2| \cdot xy \end{aligned}$$

while

$$\begin{aligned} f(z_{\lambda_1})f(z_{\lambda_2}) &= |f(z_{\lambda_1}) - f(z_{\lambda_2})| \\ &= |\lambda_1 \cdot xy - \lambda_2 \cdot xy| \\ &= |\lambda_1 - \lambda_2| \cdot xy \end{aligned}$$

or $z_{\lambda_1} z_{\lambda_2} = f(z_{\lambda_1})f(z_{\lambda_2})$. The subset S_{xy} of L_{xy} defined by restricting λ to $0 \leqslant \lambda \leqslant 1$ is therefore isometric to the euclidean segment $0 \leqslant \tau \leqslant xy$, $\tau \in E^1$ and is then a segment joining x and y. Segments are unique in M iff U is strictly convex (see Busemann [6] , p. 96). The only axiom for a G-space not satisfied by a Banach space with strictly convex unit sphere is therefore that of finite compactness.

Definition 5.5: A G-space is called a straight space iff it has the property that each two of its points can be joined by a unique segment.

Definition 5.6: A Minkowski space is a finite-dimensional Banach space.

It then follows that a Minkowski space M with strictly convex unit sphere is a straight G-space, with the unique segment joining a pair of points x and y in M and the unique geodesic through x and y being respectively the sets S_{xy} and L_{xy} defined above, and we may now employ our previous notation S(x,y) and L(x,y) to represent these sets. Now consider any three points x, y, and z in M and let m_{xz} and m_{yz} be the midpoints of S(x,z) and S(y,z) respectively. Then

$$m_{xz} = \tfrac{1}{2} x + (1 - \tfrac{1}{2}) z = \tfrac{1}{2} (x + z)$$

and similarly $m_{yz} = \frac{1}{2} (y + z)$ so that

$$m_{xz}m_{yz} = \| \tfrac{1}{2} (x + z) - \tfrac{1}{2} (y + z) \| = \| \tfrac{1}{2} (x - y) \| = \tfrac{1}{2} xy$$

and therefore $K_B = 0$, showing that this condition is necessary for a G-space to be locally Minkowskian.

It is useful to express the Minkowski metric differently and to view the space in a more familiar setting. Suppose a basis for an n-dimensional Banach space M consists of the points $b_1, b_2, \cdots, b_n$ in M. By the linear independence of the b_i each point $x \in M$ has the unique representation

$$x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$$

where the α_i are real numbers. Thus the map f of $M \longrightarrow E^n$ given by

$$f(x) = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

is a one-to-one onto correspondence. It moreover follows that

$$f(x + y) = f(x) + f(y)$$

and

$$f(\lambda x) = \lambda f(x)$$

for each x and y in M and all real λ, so that M and E^n in this way prove to be isomorphic vector spaces and thus their topologies are equivalent. From now on the distinction will not be made between the points of M and E^n although, naturally, their metrics differ in general. Now let x and y be two given points in M, $x \neq y$, and let the metric of E^n be denoted by $\overline{xy}$. Put $z = x - y$ and let $L(o,z)$ cut the unit sphere U in u and -u (since $\|u\| = \|-u\| = 1$). Then

(5.2) $$xy = \|x - y\| = \|z\| = oz$$

and since the points o, x, y, and z are vertices of a parallelogram,

(5.3) $$\overline{xy} = \overline{oz}.$$

There exists a real λ such that $z = \lambda u$ and therefore

(5.4) $\qquad oz = \|z\| = \|\lambda u\| = |\lambda|\,\|u\| = |\lambda| \ .$

It has been observed that scalar multiplication in M and E^n coincide. Hence, by definition of scalar multiplication in E^n,

(5.5) $\qquad \overline{oz} = |\lambda|\,\overline{ou}$

The result is

(5.6) $\qquad xy = \dfrac{\overline{xy}}{\overline{ou}},$

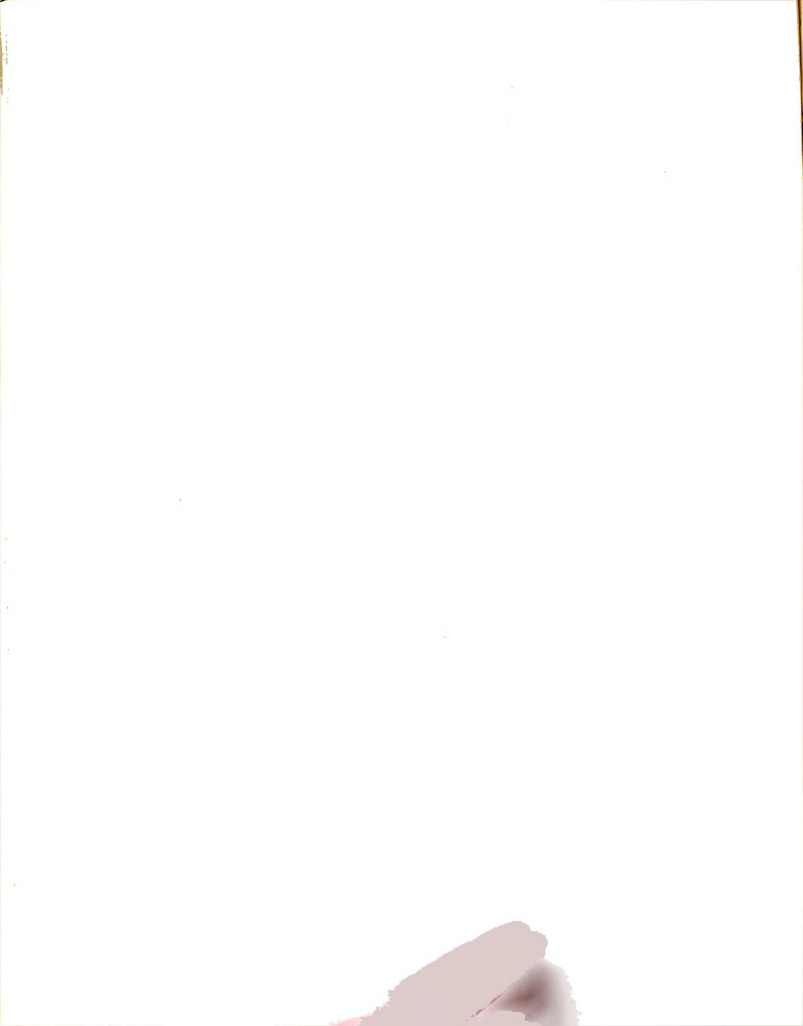
and so it follows that the metric xy of M is given by dividing the euclidean distance from x to y by the euclidean "radius" of the unit sphere that is parallel to $L(x,y)$.

It is clear from the preceding remarks that if U is any euclidean sphere, M is isometric to E^n. A much stronger result is: Two n-dimensional Minkowski spaces are isometric iff there exists an affine transformation which maps the unit sphere of one onto that of the other, from which it follows that a Minkowski metric is euclidean iff its unit sphere is an ellipsoid. A proof of this may be found in Busemann [6] .

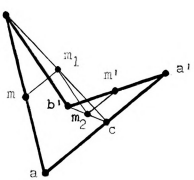
It is now possible to prove the theorem mentioned earlier:

Theorem 5.2: A straight G-space is Minkowskian iff $K_B = 0$ in the large.

Proof: The necessity of the condition was shown in the remarks following Definition 5.6. The proof of the sufficiency will be given in a sequence of lemmas. In each of these the assumption will be that M is a straight G-space in which the distance between the midpoints of two sides of any triangle equals one-half the third side.



Lemma 1: If a, a', b', and b are any four points of M with m and m' the respective midpoints of S(a,b) and S(a',b'), then $mm' \leqslant \frac{1}{2}(aa' + bb')$.



Proof: Let c be the midpoint of S(a,a') and m_1, m_2 the midpoints of S(b,c) and S(b',c) respectively. Then

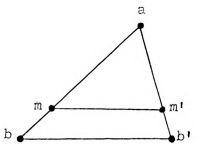
$$mm' \leqslant mm_1 + m_1m_2 + m_2m'$$
$$= \tfrac{1}{4}aa' + \tfrac{1}{2}bb' + \tfrac{1}{4}aa'$$
$$= \tfrac{1}{2}(aa' + bb').$$

Lemma 2: Suppose in triangle T(a,b,b') that $m \in S(a,b)$ and $m' \in S(a,b')$ are two points such that

$$\frac{am}{ab} = \frac{am'}{ab'}$$

holds. Then triangles T(a,m,m') and T(a,b,b') are similar, that is,

$$\frac{am}{ab} = \frac{am'}{ab'} = \frac{mm'}{bb'}.$$



Proof: Define m_τ and m_τ' respectively on S(a,b) and S(a,b') and S(a,b') with $(am_\tau)/ab = (am_\tau')/ab' = \tau$, $0 \leqslant \tau \leqslant 1$. It must be shown that for all real τ in this range

$$m_\tau m_\tau' = \tau \cdot bb'.$$

This will be shown first for the diadic rationals by induction on n where $R(n) = \left\{ \frac{1}{2^n}, \frac{3}{2^n}, \cdots, \frac{2^n - 1}{2^n} \right\}$ for n a positive integer. For n = 1, $m_{\frac{1}{2}}m_{\frac{1}{2}}' = \frac{1}{2}bb'$ follows by hypothesis. Assume it has been

proved for $\tau \in R(k)$, $k < n$. Let τ be in $R(n)$. Then it may be assumed that

$$\tau = \frac{2p - 1}{2^n},$$

where p is some integer, $1 \leqslant p \leqslant 2^{n-1}$. If $p = 1$ so that $\tau = \frac{1}{2^n}$, put $\sigma = 2\tau = \frac{1}{2^{n-1}}$ so that $\sigma \in R(k)$ for some $k < n$, which implies by the induction hypothesis,

$$m_\sigma m_\sigma' = \sigma \cdot bb'. \tag{5.7}$$

Also, m_τ and m_τ' are the respective midpoints of $S(a, m_\sigma)$ and $S(a, m_\sigma')$ so it follows that

$$m_\tau m_\tau' = \tfrac{1}{2} m_\sigma m_\sigma' = \tfrac{1}{2} \sigma \cdot bb' = \tau \cdot bb'$$

and the assertion holds for $p = 1$.

It may then be assumed that $2 \leqslant p \leqslant 2^{n-1}$. Write

$$\alpha = \frac{2p - 4}{2^n} = \frac{p - 2}{2^{n-1}}$$

$$\beta = \frac{2p - 3}{2^n}$$

$$\rho = \frac{2p - 2}{2^n} = \frac{p - 1}{2^{n-1}}$$

$$\left(\tau = \frac{2p - 1}{2^n} \right)$$

$$\sigma = \frac{2p}{2^n} = \frac{p}{2^{n-1}}$$

so that α, ρ, and σ are each in some $R(k)$ for $k < n$ and the induction hypothesis applies to $m_\alpha m_\alpha'$, $m_\rho m_\rho'$, and $m_\sigma m_\sigma'$:

$$m_\alpha m_\alpha' = \alpha \cdot bb', \quad m_\rho m_\rho' = \rho \cdot bb', \quad m_\sigma m_\sigma' = \sigma \cdot bb' \tag{5.8}$$

where $\alpha \neq 0$ and $\sigma \neq 1$. But if m_0 and m_0' are defined as a,

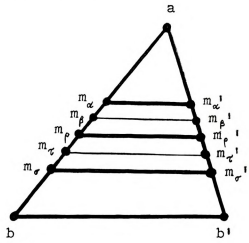
and m_1 and m_1' as b and b' respectively, then (5.8) holds even for $\alpha = 0$ and $\sigma = 1$, so that for all p, $2 \leqslant p \leqslant 2^{n-1}$, (5.8) holds. It is clear that m_β, m_ρ, and m_τ, divide $S(m_\alpha, m_\sigma)$ into equal segments, and similarly for m_β', m_ρ', and m_τ'. Thus Lemma 1 implies

$$m_\tau m_\tau' \leqslant \tfrac{1}{2}\,(m_\rho m_\rho' + m_\sigma m_\sigma')$$

$$= \tfrac{1}{2}\,(\rho + \sigma)\ bb'$$

$$= \tfrac{1}{2}\left(\frac{p-1}{2^{n-1}} + \frac{p}{2^{n-1}}\right)\ bb'$$

$$= \frac{2p-1}{2^n}\,.bb'$$



or,

(5.9) $$m_\tau m_\tau' \leqslant \tau \cdot bb'.$$

Lemma 1 also implies

(5.10) $$\rho \cdot bb' = m_\rho m_\rho' \leqslant \tfrac{1}{2}\,(m_\beta m_\beta' + m_\tau m_\tau')$$

and

(5.11) $$\rho \cdot bb' \leqslant \tfrac{1}{2}\left[\tfrac{1}{2}(m_\alpha m_\alpha' + m_\rho m_\rho') + m_\tau m_\tau'\right].$$

Using (5.8) this becomes

$$\rho \cdot bb' \leqslant (\tfrac{1}{4}\alpha + \tfrac{1}{4}\rho)\ bb' + \tfrac{1}{2}\, m_\tau m_\tau'$$

and therefore

$$m_\tau m_\tau' \geqslant (\tfrac{3}{2}\rho - \tfrac{1}{2}\alpha)\ bb'$$

$$= \left(\tfrac{3}{2} \cdot \frac{p-1}{2^{n-1}} - \tfrac{1}{2} \cdot \frac{p-2}{2^{n-1}}\right) bb'$$

$$= \frac{2p-1}{2^n}\,.bb'.$$

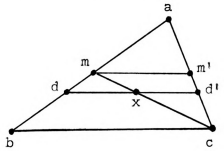
That is,

$$m_\tau m_\tau' \geqslant \tau \cdot bb',$$

and in view of (5.9) this proves that $m_\tau m_\tau' = \tau \cdot bb'$. Induction then carries, and the proposition has been proved for τ a diadic rational. The continuity of the metric then implies $m_\tau m_\tau' = \tau \cdot bb'$ for arbitrary $0 \leqslant \tau \leqslant 1$.

Lemma 3: The medians of a triangle are concurrent in a point which is two-thirds the distance on each median from the vertex to the midpoint of the opposite side.

Proof: Let $T(a,b,c)$ be any triangle and consider the median $S(m,c)$ from c. Let x be the point on $S(m,c)$ such that $xc = \frac{2}{3}mc$, and let d and d' be those points on $S(a,b)$ and $S(a,c)$ respectively such that $ad = \frac{2}{3}ab$ and $ad' = \frac{2}{3}ac$. The lemma will be proved by first showing that x is the midpoint of $S(d,d')$. By Lemma 2,



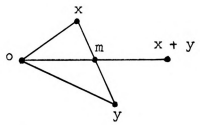
$$dd' = \frac{2}{3}bc. \tag{5.12}$$

But $bd = \frac{1}{3} \cdot 2\ bm = \frac{2}{3}bm$ and similarly $cd' = \frac{2}{3}cm'$, so again Lemma 2 implies $dx = \frac{1}{3}bc$ and $xd' = \frac{2}{3}mm' = \frac{2}{3} \cdot \frac{1}{2}bc = \frac{1}{3}bc$, or $dx = xd' = \frac{1}{3}bc$. From (5.12) it follows $dx + xd' = dd'$. Therefore x is the midpoint of $S(d,d')$.

Now if y is that point on the median $S(b,m')$ from b such that $by = \frac{2}{3}bm'$, it follows in exactly the same manner that y is the midpoint of $S(d,d')$. Therefore, $x = y$ thus proving that any two medians of a triangle intersect in a point x which is two-thirds the distance on each median from the vertex to the midpoint of the opposite side. This establishes the lemma.

Remark: For the proper interpretation of the next lemma, let the following conventions be established. Whenever $x = y$ the midpoint of $S(x,y)$ will be taken as x, and $L(x,y)$ will consist of the single point x. Also, (xyz) is interpreted loosely as meaning simply $xy + yz = xz$ without any demands on distinctness of x, y, and z.

Lemma 4: Let o be any fixed point and x and y arbitrarily given points in M. Define the point $z = x + y$ by locating the midpoint m of $S(x,y)$ and taking z as that point of $L(o,m)$ such that (omz) holds and $om = mz$. The operation $x + y$ thus defined makes M an additive abelian group.



Proof: It is clear from the definition that:

a. The operation is well-defined and closed by the fact that M is a straight space.

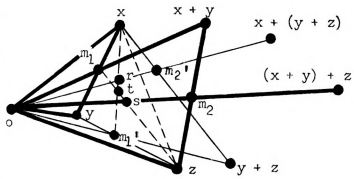
b. $x + y = y + x$ for $x \in M$, $y \in M$.

c. $o + x = x$ for all $x \in M$.

d. $-x$ is that point z on $L(o,x)$ such that (zox) holds and $zo = ox$.

All that remains to prove is the associative law. Let x, y, and z be three given distinct points, each distinct from o. This involves no loss of generality, for by the continuity of $x + y$ in M this case implies those cases where at least one pair of points coincide. In the definition of $(x + y) + z$ let m_1 be the midpoint of $S(x,y)$ thus locating $x + y$, and let m_2 be the midpoint of $S(x + y,z)$ which then defines $(x + y) + z$. Next, let m_1' be the midpoint of

$S(y,z)$ which defines $y + z$ and let m_2' be the midpoint of $S(x,y + z)$ thus locating $x + (y + z)$. By Lemma 3, in triangle $T(o,x, y + z)$ medians $S(o,m_2')$ and $S(x,m_1')$ intersect in a point r such that



$$xr = \tfrac{2}{3}\, xm_1' \tag{5.13}$$

and

$$or = \tfrac{2}{3}\, om_2'. \tag{5.14}$$

In triangle $T(o, x + y,z)$ medians $S(o,m_2)$ and $S(z,m_1)$ intersect in a point s such that

$$zs = \tfrac{2}{3}\, zm_1 \tag{5.15}$$

and

$$os = \tfrac{2}{3}\, om_2. \tag{5.16}$$

Finally, in triangle $T(x,y,z)$ medians $S(x,m_1')$ and $S(z,m_1)$ intersect in a point t such that

$$xt = \tfrac{2}{3}\, xm_1' \tag{5.17}$$

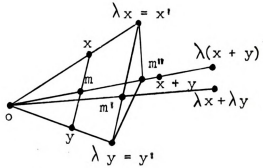
and

$$zt = \tfrac{2}{3}\, zm_1. \tag{5.18}$$

Then it follows that r and t are both on $S(x,m_1')$ with $xr = \frac{2}{3}\, xm_1'$ from (5.13) and $xt = \frac{2}{3}\, xm_1'$ from (5.17), hence $xr = xt$ and therefore $r = t$. Also, s and t are both on $S(z,m_1)$ with $zs = \frac{2}{3}\, zm_1$ from (5.15) and $zt = \frac{2}{3}\, zm_1$ from (5.18), hence $zs = zt$ and therefore $s = t$. Then $r = s$ and therefore m_2 and m_2' are such that (orm_2) and (orm_2') hold. But also $om_2 = om_2'$ from (5.14) and (5.16) which implies $rm_2 = rm_2'$. Therefore $m_2 = m_2'$ from which it follows that $(x + y) + z = x + (y + z)$.

Lemma 5: Let x be given and λ any real non-negative number. Define λx as that point y on $L(o,x)$ such that one of (oxy) or (oyx) holds and $oy = \lambda \cdot ox$. Extend the definition to negative λ by defining $\lambda x = (-\lambda)(-x)$. With this as scalar multiplication, M is a real vector space.

Proof: Directly from the definition it follows that $1 \cdot x = x$ and it is an elementary exercise to verify that $(\lambda\mu)x = \lambda(\mu x)$ and $(\lambda + \mu)x = \lambda x + \mu x$ for real λ, μ. To prove that $\lambda(x + y) = \lambda x + \lambda y$, for positive λ, let m be the midpoint of $S(x,y)$ which defines $x + y$ and $\lambda(x + y)$. Let m' be the midpoint of $S(\lambda x, \lambda y)$ which defines $\lambda x + \lambda y$. Put $m'' = \lambda m$, $x' = \lambda x$, and $y' = \lambda y$. It will be proved that m'' is the midpoint of $S(x',y')$. Now $(ox')/ox = (oy')/oy = \lambda$ so that $(x'y')/xy = \lambda$. Similarly, $(x'm'')/xm = \lambda$ and $(y'm'')/ym = \lambda$, or



$$\frac{x'm''}{xm} = \frac{y'm''}{ym} = \frac{x'y'}{xy} = \frac{x'y'}{2\,xm} = \frac{x'y'}{2\,ym}$$

and therefore $x'm'' = y'm'' = \frac{1}{2}\, x'y'$ so that $x'm'' + y'm'' = x'y'$. Hence m'' is the midpoint of $S(x',y')$ or $S(\lambda x, \lambda y)$ and it follows that m'' coincides with m'. Writing $w_1 = \lambda(x + y)$ and $w_2 = \lambda x + \lambda y$, both w_1 and w_2 lie on the single geodesic $L(o,m)$. But if $z = x + y$ then $oz = 2\,om$ so that $w_1 = \lambda z$ means $ow_1 = \lambda \cdot oz = 2\lambda \cdot om$. Also, $ow_2 = 2\,om' = 2\,om'' = 2\lambda \cdot om$ since m'' was defined as λm. Hence $ow_1 = ow_2$ and the betweenness relations then imply $w_1 = w_2$ or that

$$\lambda(x + y) = \lambda x + \lambda y.$$

For $\lambda = 0$, the axiom in question is trivial; for $\lambda < 0$, the above proof implies $(-\lambda)(-x-y) = (-\lambda)(-x) + (-\lambda)(-y)$, so by the definition of λx for this case,

$$\begin{aligned}\lambda(x + y) &= (-\lambda)\left[-(x + y)\right] \\ &= (-\lambda)(-x - y) \\ &= (-\lambda)(-x) + (-\lambda)(-y) \\ &= \lambda x + \lambda y\end{aligned}$$

which verifies the axiom for all real λ.

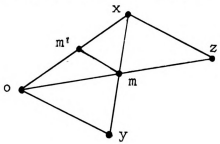
Lemma 6: Define $\|x\| = ox$ for $x \in M$. Then $\|x\|$ is a norm for M and under this norm, M is a Banach space.

Proof: That $\|x\|$ is a norm follows from:

a. $\|x\| = ox = 0$ iff $x = o$ by the fact that the metric xy behaves this way.

b. If $y = \lambda x$, $\|\lambda x\| = \|y\| = oy = |\lambda| ox = |\lambda| \|x\|$, for all real λ.

c. Let $x + y = z$. Then if m is the midpoint of $S(x,y)$ and m' that of $S(o,x)$,



$$m'm = \tfrac{1}{2}\, xz, \quad m'm = \tfrac{1}{2}\, oy$$

so that $xz = oy$. Hence from the triangle inequality of the metric it follows that

$$\|x + y\| = \|z\| = oz \leqslant ox + xz = ox + oy = \|x\| + \|y\|.$$

That the norm is complete is clear from the finite compactness of M. Therefore, M is a Banach space.

Lemma 7: M is finite dimensional, and therefore M is a Minkowski space.

Proof: This also follows by the finite compactness of M by standard arguments (see, for example, Taylor [19] p. 97).

These seven lemmas complete the proof of Theorem 5.2. The preceding arguments may all be localized to give:

Corollary 5.2: A G-space is locally Minkowskian iff $K_B = 0$.

It was stated earlier that a direct proof of Theorem 5.1 would be indicated. That proof proceeds as follows: The local condition $K_M = 0$ implies $K_B = 0$, so the proofs of Lemmas 1-7 follow locally, and it then follows that some neighborhood V of an arbitrary point $o \in M$ is Minkowskian. The following theorem of Schoenberg (see [17] for the proof) may now be employed, stated here as a lemma since it will be found useful later:

Lemma 5.1: A linear space M is an inner product space (and therefore euclidean) iff for each two points x and y on the unit sphere,

$$\|x + y\|^2 + \|x - y\|^2 \geqslant 4. \tag{5.19}$$

Since the neighborhood V either contains the unit sphere or is isometric to one which does, there is no loss in generality in assuming that V contains the unit sphere U. Now let x and y be two given points in V. Applying the condition $K_M = 0$ to the median $S(o, \frac{1}{2}(x+y))$ of triangle $T(o,x,y)$ it follows that

$$\|\tfrac{1}{2}(x + y)\|^2 = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \tfrac{1}{4}\|x - y\|^2$$

or

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(the familiar parallelogram law) which, in particular, implies (5.19)

if $\|x\| = \|y\| = 1$, that is, if x and y are on U. Therefore, the metric of V is euclidean and M is locally euclidean. Conversely, if M is locally euclidean, it is clear that $K_M = 0$.

The characterization theorems mentioned earlier will now be presented. Certain preliminary definitions and remarks are necessary for each theorem.

Definition 5.7: If the sphere S in a straight G-space is convex, then it is called differentiable (Busemann [6]) at x on S iff no proper subset of the set W, formed by the points on the supporting lines of S at x, decomposes the space.

The Parallel Axiom for Two Dimensions: The Parallel Axiom holds in a straight two-dimensional G-space iff for a given line (geodesic) L and a given point x not on L, exactly one line through x exists which does not intersect L.

The parallel axiom may be formulated in straight G-spaces of higher dimension whenever the space is Desarguesian or hyperplanes exist. In this connection Busemann [6] , p. 147, proves that a straight G-space of dimension greater than two in which the spheres are convex and differentiable and the parallel axiom holds, is a Minkowski space with differentiable spheres. He proves another theorem (p. 157) which handles the case for two dimensions: A two-dimensional straight G-space is Minkowskian iff it satisfies the parallel axiom and an $\alpha \geqslant 1$ exists such that $py^\alpha \leqslant \frac{1}{2}(px^\alpha + pz^\alpha)$ whenever y is the midpoint of $S(x,z)$. It is easy to combine these two theorems into the following characterization of E^n by means of the ptolemaic inequality.

Theorem 5.3: A straight G-space M in which the spheres are convex and differentiable is euclidean iff it is ptolemaic and satisfies the parallel axiom.

Proof: The necessity of the conditions are obvious; it suffices to prove for the sufficiency that M is Minkowskian, for then the ptolemaic inequality implies (5.19) from which it follows by Lemma 5.1 that M is euclidean. To obtain (5.19) from the ptolemaic inequality, let x and y be any two points on the unit sphere, and consider the quadruple x, y, $-y$, and $-x$. The ptolemaic inequality gives

$$xy \cdot (-y)(-x) + x(-y) \cdot y(-x) \geqslant x(-x) \cdot y(-y)$$

or

$$\| x - y \|^2 + \| x + y \|^2 \geqslant 2 \cdot ox \cdot 2 \cdot oy = 4.$$

Now if the dimension of M is higher than two the theorem of Busemann mentioned above already implies M is Minkowskian. For dimension two, it need only be shown that if y is the midpoint of $S(x,z)$ and p is any other point, there exists an $\alpha \geqslant 1$ with

$$py^{\alpha} \leqslant \tfrac{1}{2} (px^{\alpha} + pz^{\alpha}). \tag{5.20}$$

But the ptolemaic inequality applied to p, x, y, z gives

$$pz \cdot xy + px \cdot yz \geqslant py \cdot xz$$

or, since $xy = yz = \tfrac{1}{2} xz$,

$$\tfrac{1}{2} (pz + px) \geqslant py,$$

which is (5.20) with $\alpha = 1$, thus completing the proof of the theorem.

The next theorem concerns itself with Finsler spaces, the essential axioms of which are now presented.

a. The space (denoted from now on by M) is an n-manifold with a differentiable structure of class C^4. Neighborhoods of M are

coordinatized by means of the homeomorphisms into E^n (just as they are in Riemannian space). Thus, overlapping neighborhoods have more than one coordinate system, so the differentiability structure sets up a tensor calculus there.

b. Let a curve of class C^1 be represented by $x^i = x^i(\tau)$, $\alpha \leqslant \tau \leqslant \beta$. Then $\frac{dx^i}{d\tau}$ is a contravariant tensor. Let a function $F(x^1,x^2,\cdots,x^n,\xi^1,\xi^2,\cdots,\xi^n)$ be defined over the $2n$ variables $x^1, x^2, \cdots, x^n, \xi^1,\xi^2, \cdots,\xi^n$ and let it be of class C^3 where the domain of the x^i is the neighborhood where this coordinate system is valid, and that of ξ^i all of E^n. Further, let F satisfy

(1) $F(x,\xi) > 0$ for $\xi \neq 0$, where $x = (x^1,x^2, \cdots,x^n)$ and $\xi = (\xi^1,\xi^2,\cdots,\xi^n)$.

(2) $F(x, \lambda\xi) = |\lambda| F(x,\xi)$.

(3) The surface $F(x, \xi) = 1$ in ξ- space for each fixed x has everywhere positive Gauss curvature. Then the definition

$$\Lambda_\alpha^\beta = \int_\alpha^\beta F(x^1(\tau),x^2(\tau),\cdots x^n(\tau);\frac{dx^1}{d\tau},\frac{dx^2}{d\tau},\cdots,\frac{dx^n}{d\tau})\ d\tau$$

will be a suitable arc-length for curves locally (a classical result in the calculus of variations).

c. The metric of M is taken as

$$xy = \inf_{(x^i(\tau))\in \mathcal{C}_{xy}} \int_\alpha^\beta F(x,\frac{dx}{d\tau})\ d\tau$$

where $\mathcal{C}_{xy}$ is the class of C^1 curves in M with

$$x = (x^1(\alpha),x^2(\alpha),\cdots,x^n(\alpha)),\quad y = (x^1(\beta),x^2(\beta),\cdots,x^n(\beta)).$$

Theorem 5.4: A Finsler space is Riemannian with $K_B \leqslant 0$ iff it is locally ptolemaic.

Proof: The necessity is clear from Theorem 4.2 and the remark following it. A result of Busemann and Mayer [4] will be used to prove the sufficiency. Define for $p \in M$, $\xi \in E^n$,

$$F_p(\xi) = F_p(\xi^1,\xi^2,\cdots,\xi^n) = F(p^1,p^2,\cdots,p^n;\xi^1,\xi^2,\cdots,\xi^n).$$

Then for $p = (p^1,p^2,\cdots,p^n)$, a fixed point in M, the points in E^n satisfying $F_p(\xi) = 1$ define a strictly convex surface U. If x and y now be given in M and if $\xi = x - y$ then since $F_p(\lambda\xi) = |\lambda| F_p(\xi)$, it follows that

$$\overline{xy} = m_p(x,y) = F_p(x - y)$$

defines a Minkowski metric at p (since a Minkowski metric is determined uniquely up to an isometry by specifying its unit sphere). The result of Busemann and Mayer we shall use is:

$$\lim_{\substack{x \to p \\ y \to p}} \frac{xy}{\overline{xy}} = 1.$$

Suppose U is not an ellipsoid. Then $m_p(x,y)$ is not euclidean, and in view of Lemma 5.1 this means there exist a pair of points a and b in M lying on U satisfying

$$\|a + b\|^2 + \|a - b\|^2 < 4.$$

If $c = -b$ and $d = -a$, this is equivalent to

$$\overline{ab}\cdot\overline{cd} + \overline{ac}\cdot\overline{bd} < \overline{ad}\cdot\overline{bc},$$

which becomes, with $\alpha = (\overline{ab}\cdot\overline{cd})/(\overline{ad}\cdot\overline{bc})$ and $\beta = (\overline{ac}\cdot\overline{bd})/(\overline{ad}\cdot\overline{bc})$,

$$\alpha + \beta < 1.$$

Define the points $x = \lambda a$, $y = \lambda b$, $z = \lambda c$, and $w = \lambda d$, $0 < \lambda < 1$, in M. Then since $\overline{xy} = \lambda\cdot\overline{ab}$, $\overline{xz} = \lambda\cdot\overline{ac}$, and so on, we have

$$\frac{\overline{xy}\cdot\overline{zw}}{\overline{xw}\cdot\overline{yz}} = \alpha \qquad \frac{\overline{xz}\cdot\overline{yw}}{\overline{xw}\cdot\overline{yz}} = \beta,$$

where α and β are constant with respect to λ. Hence

$$\lim_{\lambda \to 0} \left(\frac{xy \cdot zw}{xw \cdot yz} + \frac{xz \cdot yw}{xw \cdot yz} \right)$$

$$= \lim_{\lambda \to 0} \left(\frac{xy \cdot zw}{\overline{xy} \cdot \overline{zw}} \cdot \frac{\overline{xw} \cdot \overline{yz}}{xw \cdot yz} \cdot \alpha + \frac{xz \cdot yw}{\overline{xz} \cdot \overline{yw}} \cdot \frac{\overline{xw} \cdot \overline{yz}}{xw \cdot yz} \cdot \beta \right)$$

$$= \alpha + \beta$$

$$< 1$$

which means there is a sequence of points x_i, y_i, z_i, and w_i converging to p for which

$$x_i y_i \cdot z_i w_i + x_i z_i \cdot y_i w_i < x_i w_i \cdot y_i z_i$$

holds, $i = 1, 2, \cdots$. This contradicts the fact that M is locally ptolemaic.

Therefore, U must be an ellipsoid with the equation (using summation convention)

$$g_{ij} \xi^i \xi^j = 1$$

where $\left[g_{ij} \right]$ is a symmetric positive definite $n \times n$ matrix defined at p. Therefore $F_p^2(\xi) \equiv g_{ij}(p)\, \xi^i \xi^j$ and in general

$$F^2(x^1, x^2, \cdots, x^n; dx^1, dx^2, \cdots, dx^n) = g_{ij}(x^1, x^2, \cdots, x^n)\, dx^i dx^j$$

thus proving that M is a Riemannian space, with $K_B \leq 0$.

The final theorem of this section requires the definition of Hilbert geometry, which will now be given.

Definition 5.8: A Hilbert geometry M corresponding to any convex body $\overline{U}$ with boundary U in E^n consists of the set of points $\overline{U} - U$ metrized as follows. If x and y be any two points of M, then the affine line through x and y cuts U in two points u and v; define

$$xy = \left| \log \frac{\overline{xv} \cdot \overline{yu}}{\overline{xu} \cdot \overline{yv}} \right|$$

where $\overline{pq}$ denotes the euclidean metric. A symmetric Hilbert geometry is one for which the convex surface U is symmetric, that is, a surface for which there exists a point o called its center such that for all $x \in U$, the affine line through o and x intersecting U also in x', $\overline{ox} = \overline{ox'}$.

It can be shown (see Busemann [6] , p. 96) that M is a straight G-space whenever U is strictly convex, with the lines and segments coinciding with the intersection of M with the affine lines and segments. The reader may notice that Hilbert geometry is a generalization of hyperbolic geometry when U is an ellipsoid. The convex surface U is often called the absolute.

Remark: The fact that a Hilbert geometry remains unaltered under affine transformations will be used later. This can be observed from the fact that the expression in the logarithm which defines the metric is a cross ratio, an invariant under affine transformations, so that although the points themselves have been transformed, the new space is isometric to the old.

Lemma: In any Hilbert geometry M with absolute U, if $\{x_i\}$ and $\{y_i\}$ are two sequences of points in M converging to $b \in M$ in such a way that the intersections u_i and v_i of $L(x_i, y_i)$ and U for each i converge to $a \in U$ and $c \in U$ respectively, then

$$\lim_{i \to \infty} \frac{x_i y_i}{\overline{x_i y_i}} = \frac{\overline{ac}}{\overline{ab} \cdot \overline{bc}} .$$

Proof: For i sufficiently large, $\overline{x_iy_i} < \overline{x_iu_i}$ and it may be assumed without loss of generality that for all such i x_i is between u_i and y_i. For any such i, put $\overline{x_iy_i} = \tau$, $\overline{x_iu_i} = \alpha$, and $\overline{x_iv_i} = \beta$. Then $\tau/\alpha < 1$ and $\tau/\beta < 1$, so by using a Taylor expansion,

$$\frac{x_iy_i}{\overline{x_iy_i}} = \frac{1}{\tau}\left|\log \frac{\overline{x_iv_i}\cdot\overline{y_iu_i}}{\overline{x_iu_i}\cdot\overline{y_iv_i}}\right|$$

$$= \frac{1}{\tau}\cdot \log \frac{\beta(\alpha+\tau)}{\alpha(\beta-\tau)}$$

$$= \frac{1}{\tau}\left[\log\left(1+\frac{\tau}{\alpha}\right) - \log\left(1-\frac{\tau}{\beta}\right)\right]$$

$$= \frac{1}{\tau}\left[\frac{\tau}{\alpha} - \frac{\tau^2}{2\alpha^2} + \frac{\tau^3}{3\alpha^3} - \cdots + \frac{\tau}{\beta} + \frac{\tau^2}{2\beta^2} + \frac{\tau^3}{3\beta^3} + \cdots\right]$$

$$= \frac{1}{\alpha} + \frac{1}{\beta} + \left(\frac{1}{2\beta^2} - \frac{1}{2\alpha^2}\right)\tau + \left(\frac{1}{3\beta^3} + \frac{1}{3\alpha^3}\right)\tau^2 + \cdots$$

Letting $i \longrightarrow \infty$, then $\tau \longrightarrow 0$, $\alpha \longrightarrow \overline{ab}$, and $\beta \longrightarrow \overline{bc}$, so the above limit is $(1/\overline{ab}) + (1/\overline{bc})$, proving the lemma.

Theorem 5.5: A symmetric Hilbert geometry is hyperbolic iff it is locally ptolemaic.

Proof: Again the necessity is clear; assume for the sufficiency that M is locally ptolemaic symmetric Hilbert geometry, but not hyperbolic. Then the symmetric surface U with center o is not an ellipsoid, and there accordingly exists a plane section C of U through o which is not an ellipse. The assertion is made: There exists an ellipse E inside C with center at o, and with four points in common with C. To prove this, first observe that there exists among all ellipses interior to C and with center at o, an ellipse E with maximal area.*

*This argument is due to Schoenberg [17] .

This follows from the fact that the set consisting of C and all its interior points is closed. It is clear that E must have at least one point a in common with C, or it would not have maximal area. By the fact that C is symmetric about o, then also -a is a common point. Now perform an affine transformation taking E into the unit circle E' (the maximal property of E is preserved under such a mapping), C into some convex curve C', a into a', -a into -a', and assume a coordinate system (ξ,η) such that o is (0,0), a' is (1,0) and -a' is (-1,0). Suppose E' and C' have no other points in common besides a' and -a', and consider the one-parameter family of ellipses E_α' passing through $\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$ given by

$$\alpha \xi^2 + (2 - \alpha)\eta^2 = 1, \qquad 1 < \alpha < 2$$

with conjugate radii of respective lengths $1/\sqrt{\alpha}$ and $1/\sqrt{2-\alpha}$, and hence with area

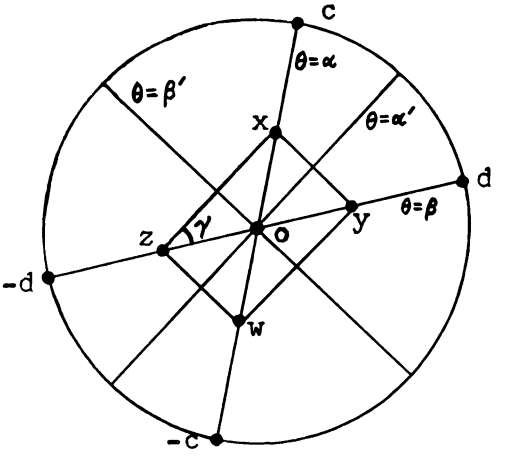
$$A_\alpha = \frac{\pi}{\sqrt{2\alpha - \alpha^2}}.$$

Thus, since $\alpha > 1$, $2\alpha - \alpha^2 < 1$ so that $A_\alpha > \pi$ for each ellipse in the family. But taking α close enough to unity, there exists an ellipse of this family completely interior to C', and with area greater than that of E'. This contradicts the maximal property of E' and proves that E' and C' must have at least a third point b', and therefore a fourth, -b' in common. If b and -b are the corresponding points under the mapping, then E has the four points a, -a, b, and -b in common with C.

It is convenient to argue in terms of the above transformed ellipse E' and curve C'. The assumption made above now implies that E' is interior to, and distinct from, C', with the four points of contact a', b', $-a'$, and $-b'$. Since the Hilbert geometry M' defined by C' is isometric to the original one defined by C, M' is also locally ptolemaic. Let $\rho = f(\theta)$ be the polar representation of C'. It follows that $f(\theta) \geqslant 1$, $0 \leqslant \theta \leqslant \pi$. The distinctness of E' and C' implies the existence of at least one θ for which $f(\theta) > 1$, and hence by continuity of $f(\theta)$ an interval $\alpha < \theta < \beta$ for which $f(\theta) > 1$. From the existence of the four points of contact, it follows that α and β may be taken so that $f(\alpha) = 1$, $f(\beta) = 1$, $0 \leqslant \alpha < \pi$, $0 \leqslant \beta < \pi$, and $0 < |\alpha - \beta| < \pi$. Let c and d be the points of C' corresponding to $\theta = \alpha$ and $\theta = \beta$, respectively, and let $\alpha' = \frac{1}{2}(\alpha + \beta)$ and $\beta' = \frac{1}{2}(\alpha + \beta) \pm \frac{\pi}{2}$, choosing the sign so that $0 \leqslant \beta' < \pi$. For some neighborhood of o, the ptolemaic inequality is valid; let x, y, z, and w be in that neighborhood and such that they are the vertices of a rectangle with diagonals along $L(o,c)$ and $L(o,d)$, with $x \in S(o,c)$ and $y \in S(o,d)$. Hence

$$xy \cdot zw + xz \cdot yw \geqslant xw \cdot yz$$

and, dividing by $\overline{xw} \cdot \overline{yz}$, this may be written in the form



$$(5.21)\qquad \frac{xy \cdot zw}{\overline{xy} \cdot \overline{zw}} \cdot \frac{\overline{xy} \cdot \overline{zw}}{\overline{xw} \cdot \overline{yz}} + \frac{xz \cdot yw}{\overline{xz} \cdot \overline{yw}} \cdot \frac{\overline{xz} \cdot \overline{yw}}{\overline{xw} \cdot \overline{yz}} \geqslant \frac{xw \cdot yz}{\overline{xw} \cdot \overline{yz}}.$$

But if γ is the angle between $S(x,z)$ and $S(z,y)$ it follows that

$$\frac{\overline{xy}}{\overline{xw}} = \frac{\overline{zw}}{\overline{yz}} = \sin\gamma, \qquad \frac{\overline{xz}}{\overline{xw}} = \frac{\overline{yw}}{\overline{yz}} = \cos\gamma$$

and (5.21) becomes

$$\frac{xy}{\overline{xy}} \cdot \frac{zw}{\overline{zw}} \sin^2\gamma + \frac{xz}{\overline{xz}} \cdot \frac{yw}{\overline{yw}} \cos^2\gamma \geqslant \frac{xw}{\overline{xw}} \cdot \frac{yz}{\overline{yz}}$$

which, in the limit as x, y, z, and w tend to o, becomes by the lemma,

$$\left(\frac{2\,f(\beta')}{f^2(\beta')}\right)^2 \cdot \sin^2\gamma + \left(\frac{2\,f(\alpha')}{f^2(\alpha')}\right)^2 \cdot \cos^2\gamma \geqslant \frac{2\,f(\alpha) \cdot 2\,f(\beta)}{f^2(\alpha) \cdot f^2(\beta)}$$

or,

$$\frac{\cos^2\gamma}{f^2(\alpha')} + \frac{\sin^2\gamma}{f^2(\beta')} \geqslant 1. \tag{5.22}$$

But $f(\alpha') > 1$ and $f(\beta') \geqslant 1$. provides the contradiction

$$1 \leqslant \frac{\cos^2\gamma}{f^2(\alpha')} + \frac{\sin^2\gamma}{f^2(\beta')} < \cos^2\gamma + \sin^2\gamma = 1. \tag{5.23}$$

Therefore U must be an ellipsoid which means that M is a hyperbolic space.

PART II

6. Preliminaries. We turn now from the study of ptolemaic spaces to a consideration of the class of metric spaces in which geodesics are characterized by the identical vanishing of one of the metric curvatures thus far introduced. We have had a measure of success in extending the fundamental theorem to spaces which are not locally ptolemaic. In particular, we extend the theorem to include a restricted class of strictly convex Banach spaces--Banach spaces whose unit spheres are strictly convex--but we are still unable to settle the question with that restriction removed. However, we introduce another quite natural definition of curvature, κ_T, and for this definition geodesics in strictly convex Banach spaces are characterized by the vanishing of κ_T at every point.

In considering Menger curvature, κ_M, an alternative definition κ_S suggested itself which seemed equivalent to κ_M. In Section 7 we prove this equivalence and utilize this result to effect a new proof of the fundamental theorem in ptolemaic spaces, based on Haantjes' original argument. Then in the final sections we introduce the curvature κ_T, study the relationship between κ_T in a "smooth" Minkowski space and the classical curvature relative to the associated euclidean metric, and prove the fundamental theorem for it in a broad class

of metric spaces, which includes strictly convex Banach spaces.

We introduce the notation to be used in connection with curves, arcs, and arc-length. A curve in a metric space M was defined previously as a continuous image of E^1, while an arc is a homeomorphism of an interval. Rectifiability is defined here in terms of arcs and not curves.

Definition 6.1: A lattice P of an arc A with (distinct) endpoints a and b, henceforth denoted A_a^b, is any finite set of points $p_i \in A_a^b$, $i = 1, 2, \cdots, n$, ordered by $<$ by means of the natural ordering of the interval of which A_a^b is a homeomorphism, such that

$$a = p_0 < p_1 < p_2 < \cdots < p_n = b.$$

The norm of P is defined as

$$\max \left[p_{i-1}p_i \mid i = 1, 2, \cdots, n \right]$$

and the length of P is

$$p_0p_1 + p_1p_2 + p_2p_3 + \cdots + p_{n-1}p_n$$

where pq is the metric of M.

Definition 6.2: If P_Δ is any lattice of arc A_a^b with norm Δ, $L(P_\Delta)$ is its length, and R is the set of positive reals, consider the number $\operatorname{Sup}_{\Delta \in R} L(P_\Delta)$. The arc is said to be rectifiable with arc-length Λ_a^b iff this number is finite.

An important theorem used later is

Theorem 6.1: If an arc A_a^b is rectifiable of length Λ_a^b then

$$\lim_{\Delta \to 0} L(P_\Delta) = \Lambda_a^b$$

(A proof may be found in Busemann [6], pp. 19-20.)

It is clear that if an arc A_a^b is rectifiable then each of its subarcs A_x^y must also be rectifiable. If Λ_x^y is defined as

$-\Lambda_y^x$ for $y \leqslant x$, it can be shown that for arbitrary points x, y, and z on a rectifiable arc A_a^b,

$$\Lambda_x^y + \Lambda_y^z = \Lambda_x^z .$$

From this it follows that if $x(\tau)$ is the homeomorphism of the interval $\alpha \leqslant \tau \leqslant \beta$ into M defining A_a^b, then $\Lambda_a^{x(\tau)}$ is a continuous, one-to-one function of τ, and therefore of $x \in A_a^b$. A parametrization $y(\sigma)$, $0 \leqslant \sigma \leqslant \Lambda_a^b$, of A_a^b by arc-length σ can now be given. Define $y(0) = a$, and for each σ, $0 < \sigma \leqslant \Lambda_a^b$, define $y(\sigma)$ as that point y on A_a^b such that

$$\Lambda_a^y = \sigma .$$

A geodesic was previously defined as a curve which was "locally" a metric segment. For greater ease of exposition we shall confine our attention to "geodesic arcs" which are now given a precise definition:

Definition 6.3: A geodesic arc is an arc which has the property that each point on it has a neighborhood in which that arc is a metric segment.

It is clear that an arc A_a^b parametrized by arc-length $y(\sigma)$, $0 \leqslant \sigma \leqslant \Lambda_a^b$, is a metric segment iff $\Lambda_a^b = ab$ with $y(\sigma)$ as the isometry of the interval into M which defines A_a^b. It follows that a rectifiable arc A is a geodesic arc iff $\Lambda_x^y = xy$ holds locally on A.

Another theorem which we shall use is the "n-lattice theorem" of Schoenberg:

Theorem 6.2: For any arc A_a^b and each positive integer n there exists an n-lattice, that is, a lattice

$$\{a = p_0, p_1, p_2, \cdots, p_n = b\}$$

of A_a^b such that

$$p_0p_1 = p_1p_2 = p_2p_3 = \cdots = p_{n-1}p_n.$$

(Schoenberg's proof of this is found in Blumenthal [3] , pp. 73-4.)

7. Menger Curvature and Haantjes' Proof of the Fundamental Theorem. Following Menger [14] , Menger curvature for curves is defined, then a simplified version of it is given which is later proved to be equivalent to it.

Definition 7.1: Let p be any point on an arc $\mathbf{A}$ and let q, r, and s be any triple of distinct points on $\mathbf{A}$. Define

$$\kappa_M(q,r,s) = \frac{\sqrt{(qr + rs + sq)(qr + rs - sq)(qr - rs + sq)(-qr + rs + sq)}}{qr \cdot rs \cdot sq}$$

Then $\mathbf{A}$ is said to have at p Menger curvature $\kappa_M(p)$ iff to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that for pq, pr, and ps each less than δ,

$$|\kappa_M(q,r,s) - \kappa_M(p)| < \epsilon.$$

Note that there are no difficulties arising due to permutations of q, r, and s, since the expression $\kappa_M(q,r,s)$ is symmetric in q, r, and s.

Definition 7.2: Let p be a given point of an arc $\mathbf{A}$ with q, r, and s any triple of distinct points of $\mathbf{A}$, with sq maximal among the values qr, rs, and sq. Define

$$\kappa_S(q,r,s) = \sqrt{\frac{8(qr + rs - sq)}{qr \cdot rs \cdot sq}}.$$

Then A is said to have at p simple Menger curvature $\kappa_S(p)$ iff to each $\epsilon>0$ there corresponds a $\delta>0$ such that for pq, pr, and ps each less than δ,

$$|\kappa_S(q,r,s) - \kappa_S(p)| < \epsilon.$$

Since the restriction qr = rs in the definition of $\kappa_M(q,r,s)$ above effects a close resemblance between Menger curvature and another curvature considered by Haantjes and Finsler, from now on referred to simply as Haantjes curvature, its definition is stated here.

Definition 7.3: Let A be a rectifiable arc and p any point on it, with q and r a pair of points of A such that $q < r$. Writing arc-length Λ_x^y on A as $\Lambda(x,y)$, define

$$\kappa_H(q,r) = 24 \sqrt{\frac{\Lambda(q,r) - qr}{\Lambda^3(q,r)}} .$$

Then the arc is said to have Haantjes curvature $\kappa_H(p)$ iff to each $\epsilon>0$ there corresponds a $\delta>0$ such that for pq and pr each less than δ,

$$|\kappa_H(q,r) - \kappa_H(p)| < \epsilon.$$

Pauc [15] has proved that the existence of Menger curvature at a point of a curve C in an arbitrary metric space implies some sub-arc of C containing that point is a rectifiable arc, so the two curvatures κ_M and κ_H are comparable, that is, they apply to the same class of curves--those which possess a sub-arc at one of their points which is a rectifiable arc. Haantjes [10] proved that in abstract metric spaces the existence of κ_M implies that of κ_H at a point on an arc but not conversely, and that in E^2 the two are equivalent. It is well known that for curves in E^n of differentiability class C^4 or higher if the classical curvature $\kappa(p) \neq 0$ at

a point p of the curve, the curvatures κ_M and κ_H each exist there and agree with the classical curvature. Note that in E^n the expression $1/\kappa_M(q,r,s)$ is the radius of the circle $C(q,r,s)$ passing through q, r, and s, provided q, r, and s are non-linear. For curves of differentiability class C^4 or higher with $\kappa(p) > 0$, then if q, r, and s each approach p along the curve, $C(q,r,s)$ approaches the (free) osculating circle at p with $1/\kappa(p)$ as radius. We now prove

Theorem 7.1: In any metric space and for an arbitrary arc A, $\kappa_M(p)$ exists for $p \in A$ iff $\kappa_S(p)$ exists, and the two are equal.

Proof: (1) Suppose $\kappa_S(p)$ exists. Let $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ be three sequences of points on A which converge to p, with q_i, r_i, and s_i distinct for $i = 1, 2, \cdots$, and consider $\kappa_M(q_i,r_i,s_i)$. It may be assumed without loss of generality that the ordering is such that for each i, s_iq_i is maximal in q_ir_i, r_is_i, s_iq_i. Then $\kappa_S(q_i,r_i,s_i)$ converges to $\kappa_S(p)$ and therefore $r_is_i \cdot s_iq_i \cdot \kappa_S^2(q_i,r_i,s_i)$ $q_ir_i \cdot s_iq_i \cdot \kappa_S^2(q_i,r_i,s_i)$, and $q_ir_i \cdot r_is_i \cdot \kappa_S^2(q_i,r_i,s_i)$ each converge to zero, hence

$$\lim \frac{q_ir_i+r_is_i+s_iq_i}{s_iq_i} = \lim \left[\frac{q_ir_i+r_is_i-s_iq_i}{s_iq_i} + 2 \right] = 2$$

$$\lim \frac{q_ir_i-r_is_i+s_iq_i}{q_ir_i} = \lim \left[- \frac{q_ir_i+r_is_i-s_iq_i}{q_ir_i} + 2 \right] = 2 \tag{7.1}$$

$$\lim \frac{-q_ir_i+r_is_i+s_iq_i}{r_is_i} = \lim \left[- \frac{q_ir_i+r_is_i-s_iq_i}{r_is_i} + 2 \right] = 2$$

But

$$\kappa_M(q_i,r_i,s_i) =$$

$$\sqrt{\frac{q_ir_i+r_is_i+s_iq_i}{s_iq_i}}\cdot\frac{\kappa_S(q_i,r_i,s_i)}{\sqrt{8}}\cdot\sqrt{\frac{q_ir_i-r_is_i+s_iq_i}{q_ir_i}}\cdot\sqrt{\frac{-q_ir_i+r_is_i+s_iq_i}{r_is_i}}$$

which converges to

$$\sqrt{2}\cdot\frac{\kappa_S(p)}{\sqrt{8}}\cdot\sqrt{2}\cdot\sqrt{2} = \kappa_S(p)$$

proving that $\kappa_M(p)$ exists and equals $\kappa_S(p)$.

(2) Let $\kappa_M(p)$ exist. Consider any three sequences $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ of points on A which converge to p with s_iq_i maximal in q_ir_i, r_is_i, s_iq_i, and with q_i, r_i, q_i distinct, for $i = 1, 2, \cdots$. Since $\kappa_M(q_i,r_i,s_i)$ converges to $\kappa_M(p)$ then $(s_iq_i)^2\cdot\kappa_M^2(q_i,r_i,s_i)$ converges to zero. That is,

$$\frac{q_ir_i+r_is_i+s_iq_i}{q_ir_i}\cdot\frac{q_ir_i+r_is_i-s_iq_i}{r_is_i}\cdot\frac{q_ir_i-r_is_i+s_iq_i}{q_ir_i}\cdot\frac{-q_ir_i+r_is_i+s_iq_i}{r_is_i}$$

converges to zero. In the four above expressions, the first, third, and fourth are bounded away from zero, therefore the second must converge to zero. For, in the first expression $r_is_i+s_iq_i \geqslant q_ir_i$ implies that

$$\frac{q_ir_i+r_is_i+s_iq_i}{q_ir_i} \geqslant \frac{q_ir_i+q_ir_i}{q_ir_i} = 2,$$

in the third expression, $s_iq_i \geqslant r_is_i$ implies

$$\frac{q_ir_i-r_is_i+s_iq_i}{q_ir_i} \geqslant \frac{q_ir_i+0}{q_ir_i} = 1,$$

and in the last, $s_iq_i \geqslant q_ir_i$ implies

$$\frac{-q_ir_i+r_is_i+s_iq_i}{r_is_i} \geqslant \frac{0+r_is_i}{r_is_i} = 1.$$

Therefore

$$\lim \frac{q_i r_i + r_i s_i - s_i q_i}{r_i s_i} = 0. \tag{7.2}$$

Now, since $s_i q_i / r_i s_i \geqslant 1$, then from

$$\lim \frac{s_i q_i}{r_i s_i} \cdot \frac{q_i r_i + r_i s_i - s_i q_i}{s_i q_i} = \lim \frac{q_i r_i + r_i s_i - s_i q_i}{r_i s_i} = 0$$

it follows that

$$\lim \frac{q_i r_i + r_i s_i - s_i q_i}{s_i q_i} = 0. \tag{7.3}$$

Next it will be shown that

$$\lim \frac{q_i r_i + r_i s_i - s_i q_i}{q_i r_i} = 0. \tag{7.4}$$

Suppose (7.4) is false. Then there exists a subsequence $\{j\}$ of $\{i\}$ and some $\epsilon > 0$ (since the expression in (7.4) is non-negative by the triangle inequality for every i), such that for all j in the subsequence,

$$\frac{q_j r_j + r_j s_j - s_j q_j}{q_j r_j} > \epsilon. \tag{7.5}$$

Consider

$$\frac{q_j r_j}{q_j r_j + r_j s_j - s_j q_j} \cdot \frac{q_j r_j + r_j s_j - s_j q_j}{s_j q_j} = \frac{q_j r_j}{s_j q_j},$$

for each j. From (7.5) the first expression on the left is bounded while the second converges to zero by (7.3). Hence

$$\lim_j \frac{q_j r_j}{s_j q_j} = 0, \tag{7.6}$$

which implies, also from (7.3),

$$\lim_{j} \frac{r_j s_j}{s_j q_j} = 1. \tag{7.7}$$

Hence there exists a j_o such that for all $j > j_o$

$$\frac{r_j s_j}{s_j q_j} > \tfrac{1}{2} \tag{7.8}$$

Next consider, for $j > j_0$, $(r_j s_j)^2 \cdot \kappa_M^2(q_j, r_j, s_j)$, or

$$\frac{q_j r_j + r_j s_j + s_j q_j}{s_j q_j} \cdot \frac{q_j r_j + r_j s_j - s_j q_j}{q_j r_j} \cdot \frac{q_j r_j - r_j s_j + s_j q_j}{q_j r_j} \cdot \frac{-q_j r_j + r_j s_j + s_j q_j}{s_j q_j}.$$

By the triangle inequality,

$$\frac{q_j r_j + r_j s_j + s_j q_j}{s_j q_j} \geqslant \frac{s_j q_j + s_j q_j}{s_j q_j} = 2,$$

by (7.5),

$$\frac{q_j r_j + r_j s_j - s_j q_j}{q_j r_j} > \epsilon,$$

from $s_j q_j \geqslant r_j s_j$,

$$\frac{q_j r_j - r_j s_j + s_j q_j}{q_j r_j} \geqslant \frac{q_j r_j + 0}{q_j r_j} = 1,$$

and by (7.8), for $j > j_0$, and using $s_j q_j \geqslant q_j r_j$,

$$\frac{-q_j r_j + r_j s_j + s_j q_j}{s_j q_j} \geqslant \frac{0 + r_j s_j}{s_j q_j} = \tfrac{1}{2}.$$

Therefore,

$$(r_j s_j)^2 \cdot \kappa_M^2(q_j, r_j, s_j) > 2 \cdot \epsilon \cdot 1 \cdot \tfrac{1}{2} = \epsilon$$

for all $j > j_0$. But this contradicts the fact that $\kappa_M(q_j, r_j, s_j)$ converges, hence (7.4) holds.

Making use of the limits (7.2), (7.3), and (7.4) it follows that

$$\kappa_S(q_i, r_i, s_i) =$$

$$= \sqrt{8} \cdot \sqrt{\frac{q_i r_i + r_i s_i - s_i q_i}{q_i r_i \cdot r_i s_i \cdot s_i q_i}}$$

$$= \frac{\sqrt{8} \cdot \kappa_M(q_i, r_i, s_i)}{\sqrt{\frac{q_i r_i + r_i s_i + s_i q_i}{s_i q_i} \cdot \frac{q_i r_i - r_i s_i + s_i q_i}{q_i r_i} \cdot \frac{-q_i r_i + r_i s_i + s_i q_i}{r_i s_i}}}$$

$$= \frac{\sqrt{8} \cdot \kappa_M(q_i, r_i, s_i)}{\sqrt{\left[2 + \frac{q_i r_i + r_i s_i - s_i q_i}{s_i q_i}\right]\left[2 - \frac{q_i r_i + r_i s_i - s_i q_i}{q_i r_i}\right]\left[2 - \frac{q_i r_i + r_i s_i - s_i q_i}{r_i s_i}\right]}}$$

converges to

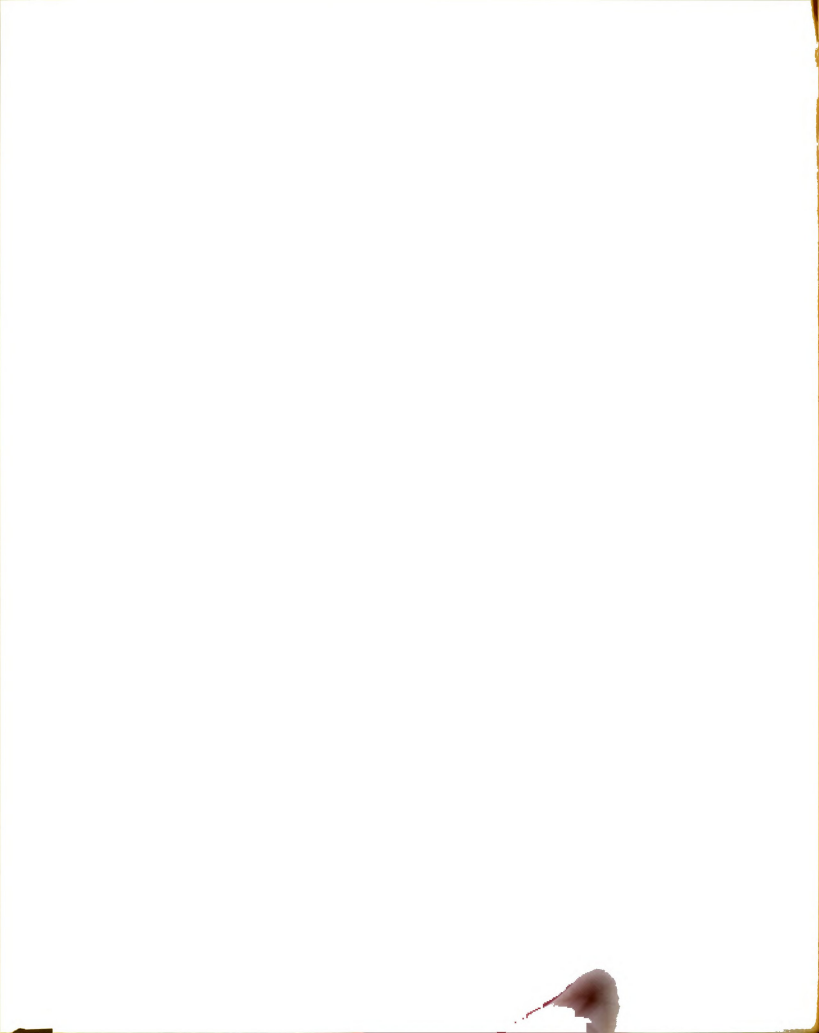
$$\frac{\sqrt{8} \cdot \kappa_M(p)}{\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}} = \kappa_M(p)$$

proving that $\kappa_S(p)$ exists and equals $\kappa_M(p)$. This completes the proof of Theorem 7.1.

Remark: Suppose that $\kappa_M(p)$ exists for $p \in C$. Then let $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ be three sequences of points on C which converge to p in such a way that for each $i = 1, 2, \cdots$

$$q_i r_i = r_i s_i.$$

Hence



$$(r_i s_i)^2 \cdot \kappa_M^2(q_i, r_i, s_i) = \frac{2\, q_i r_i + s_i q_i}{q_i r_i} \cdot \frac{2\, q_i r_i - s_i q_i}{q_i r_i},$$

which converges to zero. Since $(2\, q_i r_i + s_i q_i)/(q_i r_i) \geqslant 2$ for each i, it follows that

$$0 = \lim \frac{2\, q_i r_i - s_i q_i}{q_i r_i} = 2 - \lim \frac{s_i q_i}{q_i r_i},$$

and therefore

$$\lim \frac{q_i r_i}{s_i q_i} = \lim \frac{r_i s_i}{s_i q_i} = \tfrac{1}{2}. \tag{7.9}$$

It follows that there exists an i_0 such that for all $i > i_0$ $q_i r_i = r_i s_i < s_i q_i$, and hence by Theorem 7.1,

$$\kappa_M(p) = \lim \kappa_S(q_i, r_i, s_i).$$

If $\lambda(q_i, s_i)$ is defined as $2\, q_i r_i$, then by (7.9) $\lim \dfrac{\lambda(q_i, s_i)}{q_i s_i} = 1$ and therefore from

$$\kappa_S(q_i, r_i, s_i) = \sqrt{\frac{8\,(2\, q_i r_i - s_i q_i)}{(q_i r_i)^2 \cdot s_i q_i}} = \sqrt{\frac{32\,(\lambda(q_i, s_i) - s_i q_i)}{\lambda^3(q_i, s_i)} \cdot \frac{\lambda(q_i, s_i)}{s_i q_i}}$$

the result is

$$\kappa_M(p) = \lim_{i \to \infty} \sqrt{32} \cdot \sqrt{\frac{\lambda(q_i, s_i) - q_i s_i}{\lambda^3(q_i, s_i)}}. \tag{7.10}$$

This expression bears a striking resemblance to the one for Haantjes curvature and suggests the possibility of a proof of the fundamental theorem for Menger curvature that closely parallels the proof Haantjes gave.

For the purpose of simplicity, throughout the remainder of this thesis we shall confine our attention to the characterization of geodesic arcs by identically vanishing curvature among all rectifiable arcs.

Theorem 7.2: In any locally ptolemaic metric space M a rectifiable arc $\mathbf{A}$ is a geodesic arc iff $\kappa_M(p) = 0$ for each point $p \in \mathbf{A}$.

Proof: The necessity of the condition being obvious, suppose $\mathbf{A}$ is a rectifiable arc with $\kappa_M(p) = 0$ for $p \in \mathbf{A}$. Restricting the arc to a neighborhood V_p of p in which M is ptolemaic, there exists an arc $\mathbf{A}'$ with endpoints a and b, $a<b$, such that $\mathbf{A}' \subset \mathbf{A}$ and $\mathbf{A}' \subset V_p$. It suffices to show that $\Lambda(a,b) = ab$ proving that $\mathbf{A}'$ is a metric segment. Let $\epsilon > 0$ be given and consider $x \in \mathbf{A}'$. Since $\kappa_M(x) = 0$, by (7.10) there exists a $\delta_x > 0$ such that for every triple of points q, r, and s in $\mathbf{A}'$ with $qr = rs$ and $xr < \delta_x$, $xs < \delta_x$,

$$(7.11) \qquad 0 \leqslant \frac{\lambda(q,s) - qs}{\lambda^3(q,s)} < \frac{\epsilon}{16\,\Lambda^3}$$

holds, where $\lambda(q,s) = 2qr$ and $\Lambda = \Lambda(a,b) > 0$. But if $W_x = \{w \mid wx < \delta_x\}$ then $\{W_x\}_{x \in \mathbf{A}'}$ is an open covering of the compact set $\mathbf{A}'$, whence there is a finite subcovering $W_{x_1}, W_{x_2}, \cdots, W_{x_m}$. It is at once clear that there exists a $\delta > 0$ such that if now q, r, and s are any three points of $\mathbf{A}'$ with $qr = rs$ and $qr < \delta$ then q, r, and s all three will lie in at least one of the W_{x_i} and hence will satisfy (7.11). By Theorem 6.2 there is an n-lattice of $\mathbf{A}'$

$$a = p_0 < p_1 < p_2 < \cdots < p_n = b$$

where $p_0p_1 = p_1p_2 = \cdots = p_{n-1}p_n = \lambda_n$. By the rectifiability of $\mathbf{A}'$ and Theorem 6.1 the limit of the lengths of these n-lattices, $n\lambda_n$, as $n \to \infty$ is $\Lambda(a,b)$, hence there is an n_1 such that for all $n > n_1$, $0 \leqslant \Lambda(a,b) - n\lambda_n < \frac{\epsilon}{2}$, and since $\lambda_n \leqslant (1/n)\,\Lambda(a,b)$ there is an

n_2 such that for all $n > n_2$, $\lambda_n < \delta$. Let n be any integer larger than either n_1 and n_2 and consider the corresponding n-lattice $\{p_k\}$, $k = 0, 1, 2, \cdots, n$. It then follows that

$$0 \leqslant \Lambda(a,b) - n\lambda_n < \frac{\epsilon}{2} \tag{7.12}$$

and for $k = 1, 2, \cdots, n-1$,

$$0 \leqslant \frac{\lambda(p_{k-1},p_{k+1}) - p_{k-1}p_{k+1}}{\lambda^3(p_{k-1},p_{k+1})} < \frac{\epsilon}{16\Lambda^3},$$

which reduces to

$$0 \leqslant 2\lambda_n - p_{k-1}p_{k+1} < \frac{\lambda_n^3 \epsilon}{2\Lambda^3}, \quad k = 1, 2, \cdots, n-1 \tag{7.13}$$

from $\lambda(p_{k-1},p_{k+1}) = 2\lambda_n$.

The points p_0, p_{k-1}, p_k, and p_{k+1} for each k, $k=1, 2, \cdots, n-1$, are ptolemaic, hence

$$p_0p_{k-1}\cdot p_kp_{k+1} + p_0p_{k+1}\cdot p_{k-1}p_k \geqslant p_0p_k\cdot p_{k-1}p_{k+1}$$

which reduces by (7.13) and $p_kp_{k+1} = p_{k-1}p_k = \lambda_n$ to

$$p_0p_{k-1} + p_0p_{k+1} > p_0p_k \left(2 - \frac{\lambda_n^2 \epsilon}{2\Lambda^3}\right)$$

or,

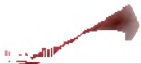
$$2\,p_0p_k < p_0p_{k-1} + p_0p_{k+1} + p_0p_k \cdot \frac{\lambda_n^2 \epsilon}{2\Lambda^3}$$

and using $p_0p_k \leqslant \Lambda(a,p_k) < \Lambda(a,b) = \Lambda$ this becomes

$$2\,p_0p_k < p_0p_{k-1} + p_0p_{k+1} + \frac{\lambda_n^2 \epsilon}{2\Lambda^2}. \tag{7.14}$$

Define

$$\alpha_k = k\lambda_n - p_0p_k, \qquad k = 0, 1, 2, \cdots, n,$$



and substitute into (7.14) The result is the set of n-1 inequalities

$$(7.15) \qquad \alpha_{k+1} - \alpha_k < \alpha_k - \alpha_{k-1} + \frac{\lambda_n^2 \epsilon}{2\Lambda^2}, \quad k = 1, 2, \cdots, n-1.$$

For each integer $j = 1, 2, \cdots, n-1$ sum (7.15) over $k = 1, 2, \cdots, j$ to obtain the n-1 inequalities

$$\alpha_{j+1} - \alpha_1 < \alpha_j - \alpha_0 + \frac{j\lambda_n^2 \epsilon}{2\Lambda^2} .$$

But $\alpha_1 = \alpha_0 = 0$ by definition and $j\lambda_n < n\lambda_n \leqslant \Lambda$ so

$$(7.16) \qquad \alpha_{j+1} < \alpha_j + \frac{\lambda_n \epsilon}{2\Lambda}, \quad j = 1, 2, \cdots, n-1.$$

Summing (7.16) over $j = 1, 2, \cdots, n-1$,

$$\alpha_2 + \alpha_3 + \cdots + \alpha_n < \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \frac{(n-1)\lambda_n \epsilon}{2\Lambda}$$

and again since $(n-1)\lambda_n < n\lambda_n \leqslant \Lambda$ this reduces to

$$\alpha_n < \frac{\epsilon}{2} .$$

From (7.12) it follows that

$$\begin{aligned} 0 &\leqslant \Lambda(a,b) - ab \\ &= \Lambda(a,b) - n\lambda_n + n\lambda_n - p_0 p_n \\ &= (\Lambda(a,b) - n\lambda_n) + \alpha_n \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

and therefore

$$0 \leqslant \Lambda(a,b) - ab < \epsilon.$$

Since $\Lambda(a,b)$ and ab are constants and $\epsilon > 0$ was arbitrary then $\Lambda(a,b) - ab = 0$ must hold which proves that A' is a metric

segment and that the given arc A is a geodesic arc.

8. Haantjes' Proof in Non-Ptolemaic Metric Spaces. The attempt was made to adapt the Haantjes proof of the fundamental theorem (essentially the proof just presented in the preceding section) to Banach spaces with strictly convex unit spheres. It is clear that if strict convexity is not required, then the two-dimensional space (x,y) with unit sphere as $|x| + |y| = 1$ affords a counterexample: the curve $y = x^4$ is not a geodesic but it has zero κ_M and κ_H at all its points. The resulting partial solution to this problem applies to certain non-linear metric spaces.

Definition 8.1: A metric space M will be said to have weakly non-positive median (space) curvature iff there exists a positive constant $\gamma_p < 1$ and a neighborhood V_p for each point $p \in M$ such that if a, b, and c are any three points in V_p and m is any midpoint of a and b, then

$$\gamma_p \cdot cm \leqslant \sqrt{\tfrac{1}{2} ac^2 + \tfrac{1}{2} bc^2 - \tfrac{1}{4} ab^2}\,. \tag{8.1}$$

Definition 8.2: A metric space M is said to have feebly non-positive median (space) curvature iff there exists a neighborhood V_p of each point $p \in M$ such that if a, b, and c are any three points in V_p with m any midpoint of a and b, then

$$cm \leqslant \tfrac{1}{2}(ac + bc). \tag{8.2}$$

Unfortunately, the condition for "feeble" does not seem to be a special case of the condition for "weakly" non-positive median curvature. Both conditions are needed in the proof of the fundamental theorem given in this section, as is the existence of midpoints, so

there M will be assumed to be (metrically) convex and complete. If this is not assumed, then (8.1) and (8.2) could be satisfied trivially without imposing any condition on the space.

It is interesting that in Banach spaces (8.1) is a form of what will be called the "weak" parallelogram law.

Definition 8.3: The norm $\|x\|$ of a Banach space M for $x \in M$ is said to satisfy the weak parallelogram law iff there exists a positive constant $\gamma < 1$ such that for every x and y in M,

$$\|x + y\|^2 + \gamma^2 \|x - y\|^2 \leqslant 2\|x\|^2 + 2\|y\|^2 . \tag{8.3}$$

To see that (8.3) implies (8.1) in any strictly convex Banach space, given a, b, and c, set $x = a - c$ and $y = -(b - c)$. Since $m = \frac{1}{2}(a + b)$ is the only midpoint of a and b in this case, it follows from (8.3) that

$$\|a - b\|^2 + 4\gamma^2 \|\tfrac{1}{2}(a + b) - c\|^2 \leqslant 2\|a - c\|^2 + 2\|b - c\|^2$$

or,

$$ab^2 + 4\gamma^2 \cdot cm^2 \leqslant 2ac^2 + 2bc^2$$

which is (8.1). In euclidean space, trivially (8.1), (8.2), and (8.3) all three are satisfied in the large with $\gamma = 1$, while for arbitrary metric spaces, M has weakly non-positive median curvature if it has non-positive median curvature, has feebly non-positive median curvature if it is locally ptolemaic (see the proof to Theorem 5.3), and in a Banach space, the weak parallelogram law holds if the parallelogram law holds. Following the proof of the next theorem, the question of what non-ptolemaic metric spaces have weakly and feebly non-positive curvature will be discussed.

Theorem 8.1: In any complete, convex metric space M having both weak and feeble non-positive median space curvature, a rectifiable arc **A** is a geodesic arc iff it has vanishing Menger curvature at each of its points, and iff it has vanishing Haantjes curvature at each of its points.

Proof: The necessity of the condition is clear. For the sufficiency, let κ_M and κ_H vanish identically along a rectifiable arc **A**. There exists a sub-arc **A'** of **A** containing an arbitrary point $p \in A$ which is contained in a neighborhood V_p of p in which (8.1) and (8.2) are valid. As in the proof of Theorem 7.2, let a and b, $a < b$, be the endpoints of **A'**; it suffices to prove $\Lambda(a,b) = ab$. Let $0 < \epsilon < \Lambda(a,b)$ be given. In order to handle both κ_M and κ_H simultaneously, the following notation will be used. Let

(8.4) $$a = p_0 < p_1 < p_2 < \cdots < p_n = b$$

be an n-lattice of **A'** in the case of κ_M with $\lambda_n = p_{k-1}p_k$ for $k = 1, 2, \cdots, n$, where n is so large that both

(8.5) $$\frac{2\lambda_n + p_{k-1}p_{k+1}}{(2\lambda_n)^3} < \epsilon', \quad k = 1, 2, \cdots, n-1,$$

the choice of ϵ' to be made later, and

(8.6) $$0 \leqslant \Lambda(a,b) - n\lambda_n < \frac{\epsilon}{2}.$$

In the case of κ_H, let (8.4) be a lattice of **A'** such that

$$\Lambda(p_0,p_1) = \Lambda(p_1,p_2) = \cdots = \Lambda(p_{n-1},p_n)$$

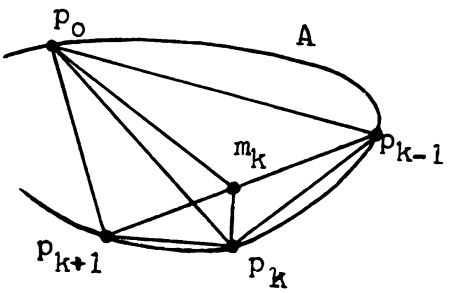
with $\lambda_n = \Lambda(p_{k-1},p_k)$ for $k = 1, 2, \cdots, n$ where n is so large that

(8.7) $$\kappa_H(p_{k-1},p_{k+1}) < \epsilon', \quad k = 1, 2, \cdots, n-1,$$

and, in this case for any choice of n,

$$(8.8) \qquad \Lambda(a,b) - n\lambda_n = 0 < \frac{\epsilon}{2} .$$

But (8.7) is the same inequality as (8.5) and (8.8) and (8.6) are the same, so from this point on the proofs for κ_M and κ_H will be identical. Moreover, it is sufficient to show how to obtain the set of n-1 inequalities of (7.15) for, from that point on the proof will be identical to that of Theorem 7.2.



For each $k = 1, 2, \cdots, n-1$ let m_k be any midpoint of p_{k-1} and p_{k+1}, the existence of at least one being assured by the fact that M is convex and complete. From $p_0p_k \leqslant p_0m_k + m_kp_k$ and by (8.1) and (8.2), it follows that

$$p_0p_k \leqslant \tfrac{1}{2}(p_0p_{k-1} + p_0p_{k+1}) + (1/\gamma)\sqrt{\tfrac{1}{2}(p_{k-1}p_k)^2 + \tfrac{1}{2}(p_kp_{k+1})^2 - \tfrac{1}{4}(p_{k-1}p_{k+1})^2}$$

while $p_{k-1}p_k \leqslant \lambda_n$ for $k = 1, 2, \cdots, n$ reduces this to

$$2\,p_0p_k \leqslant p_0p_{k-1} + p_0p_{k+1} + (1/\gamma)\sqrt{(2\lambda_n - p_{k-1}p_{k+1})(2\lambda_n + p_{k-1}p_{k+1})}$$

But by (8.5)

$$2\lambda_n - p_{k-1}p_{k+1} < 8\lambda_n^3\,\epsilon'$$

and

$$2\lambda_n + p_{k-1}p_{k+1} \leqslant 2\lambda_n + p_{k-1}p_k + p_kp_{k+1} \leqslant 2\lambda_n + \lambda_n + \lambda_n = 4\lambda_n$$

so therefore,

$$2\,p_0p_k < p_0p_{k-1} + p_0p_{k+1} + (1/\gamma)\sqrt{32\lambda_n^4\,\epsilon'} .$$

Now ϵ' is chosen as $(\gamma^2\epsilon^2)/(128\,\Lambda^4)$ and $\alpha_k = k\lambda_n - p_0p_k$ for $k = 0, 1, 2, \cdots, n$ are defined, with $\alpha_2 = 2\lambda_n - p_0p_2 < 8\lambda_n^3\epsilon' < \frac{\lambda_n^3\epsilon}{16\Lambda^3}$. By substitution,

$$2k\lambda_n - 2\alpha_k < (k-1)\lambda_n - \alpha_{k-1} + (k+1)\lambda_n - \alpha_{k+1} + \frac{\lambda_n^2 \epsilon}{2\Lambda^2}$$

which simplifies to

$$(8.9) \qquad \alpha_{k+1} - \alpha_k < \alpha_k - \alpha_{k-1} + \frac{\lambda_n^2 \epsilon}{2\Lambda^2}, \quad k = 1, 2, \cdots, n-1,$$

identical to (7.15). The remainder of the proof proceeds as in Theorem 7.2, hence **A** is a geodesic arc.

The following discussion will serve as a sketch of the proof that any Riemannian surface has both feebly and weakly non-positive median curvature. It is clear that in view of Theorem 4.2 a Riemannian <u>space</u> with non-positive classical curvature has non-positive median curvature and is locally ptolemaic, and therefore will have both feebly and weakly non-positive curvature.

Consider any 2-sphere S_ρ with radius ρ in E^3. Letting $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ be any three sequences of points on S_ρ converging to a point $p \in S_\rho$, for all sufficiently large i the following is true:

(a) $m_i = m_{a_i b_i}$ exists.

(b) The formula from spherical trigonometry

$$\cos \frac{c_i m_i}{\rho} = \frac{\cos \frac{a_i c_i}{\rho} + \cos \frac{b_i c_i}{\rho}}{2 \cos \frac{a_i b_i}{2\rho}}$$

is valid.

Let $\alpha_i = \frac{b_i c_i}{\rho}$, $\beta_i = \frac{a_i c_i}{\rho}$, and $\gamma_i = \frac{a_i b_i}{\rho}$. Again, for i large enough, the following manipulation with Taylor series is permissible:

$$2\left(1-\cos\frac{c_i m_i}{\rho}\right) = 2 - \frac{\cos\alpha_i+\cos\beta_i}{\cos\frac{1}{2}\gamma_i}$$

$$= 2 - \frac{2 - \frac{1}{2}\alpha_i^2 - \frac{1}{2}\beta_i^2 + o(\alpha_i^3) + o(\beta_i^3)}{1 - \frac{1}{8}\gamma_i^2 + o(\gamma_i^3)}$$

$$= 2 - \left[2-\tfrac{1}{2}\alpha_i^2-\tfrac{1}{2}\beta_i^2+o(\alpha_i^3)+o(\beta_i^3)\right]\left[1+\tfrac{1}{8}\gamma_i^2+o(\gamma_i^3)\right]$$

$$= \tfrac{1}{2}\alpha_i^2+\tfrac{1}{2}\beta_i^2-\tfrac{1}{4}\gamma_i^2+\frac{\alpha_i^2\gamma_i^2}{16}+\frac{\beta_i^2\gamma_i^2}{16}+o(\alpha_i^3)+o(\beta_i^3)+o(\gamma_i^3)$$

Therefore,

$$\frac{\frac{c_i m_i^2}{\rho^2}}{\frac{1}{2}\alpha_i^2+\frac{1}{2}\beta_i^2-\frac{1}{4}\gamma_i^2} = \frac{\frac{c_i m_i^2}{\rho^2}}{2\left(1-\cos\frac{c_i m_i}{\rho}\right)}\cdot\left[1+\frac{\alpha_i^2\gamma_i^2+\beta_i^2\gamma_i^2+o(\alpha_i^3)+o(\beta_i^3)+o(\gamma_i^3)}{8\alpha_i^2+8\beta_i^2-4\gamma_i^2}\right]$$

Now

$$\lim_{i\to\infty}\frac{\frac{c_i m_i^2}{\rho^2}}{2\left(1-\cos\frac{c_i m_i}{\rho}\right)} = 1$$

and it is easily verified that

$$\lim_{i\to\infty}\frac{\alpha_i^2\gamma_i^2+\beta_i^2\gamma_i^2+o(\alpha_i^3)+o(\beta_i^3)+o(\gamma_i^3)}{8\alpha_i^2+8\beta_i^2-4\gamma_i^2} = 0$$

regardless of the way in which the values $\alpha_i, \beta_i, \gamma_i$ converge to zero. Hence there cannot exist three sequences of points converging to p for which

$$\frac{\frac{c_i m_i^2}{\rho^2}}{2\left(1-\cos\frac{c_i m_i}{\rho}\right)}\cdot\left[1+\frac{\alpha_i^2\gamma_i^2+\beta_i^2\gamma_i^2+o(\alpha_i^3)+o(\beta_i^3)+o(\gamma_i^3)}{8\alpha_i^2+8\beta_i^2-4\gamma_i^2}\right] > 4$$

so there evidently exists a δ_ρ depending only on ρ such that if a, b, and c are any three points of S_ρ with $ap < \delta_\rho$, $bp < \delta_\rho$, $cp < \delta_\rho$ and m the midpoint of a and b writing as before $\alpha = \frac{bc}{\rho}$, $\beta = \frac{ac}{\rho}$, and $\gamma = \frac{ab}{\rho}$, then the above expression in α, β, γ, c, and m is not greater

than 4, that is,

$$\frac{\frac{cm^2}{\rho^2}}{\frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - \frac{1}{4}\gamma^2} \leqslant 4$$

or

$$\frac{cm^2}{\frac{1}{2}bc^2 + \frac{1}{2}ac^2 - \frac{1}{4}ab^2} \leqslant 4$$

which is (8.1) with $\gamma_p = \frac{1}{2}$. It is well known from spherical geometry that the median of every triangle in a small enough neighborhood is less than half the sum of the adjacent sides, which is (8.2). This, together with the preceding paragraph shows that non-ptolemaic spherical geometry has both weakly and feebly non-positive median curvature.

The result of Wald [20] for Riemannian surfaces is now employed. Define S_k for each real number k to be the 2-sphere in E^3 of radius $1/k$ if $k > 0$, to be the euclidean plane E^2 if $k = 0$, and to be the hyperbolic plane of curvature k if $k < 0$. For a Riemannian surface M, let V be any neighborhood of an arbitrary point p having unique segments. The theorem of Wald states that each four points of V can be isometrically imbedded in some S_k, where k lies between the upper and lower bounds of the Gauss curvature K in V. Then already (8.2) is clear. For (8.1), if $k \leqslant 0$ there is no problem; if $k > 0$, using the above remarks, if the imbedded points lie outside the neighborhood of radius $\delta_{1/k}$ then the neighborhood V is chosen smaller, but k is changed (perhaps). However, $\delta_{1/k}$ is evidently a decreasing function of k so if k' is the least upper bound of k, it suffices to take the radius of V

to be $\delta_{1/k'}$, for then the imbedded points will lie in a neighborhood of radius $\delta_{1/k'} \leqslant \delta_{1/k}$. That is, those points lie in the desired neighborhood on the sphere $S_{1/k}$ and therefore (8.1) holds. It is natural to conjecture from the locally euclidean character of any Riemannian space, that a Riemannian n-manifold has both feebly and weakly non-positive median curvature.

There are some strictly convex Banach spaces which satisfy (8.3) and therefore (8.1). The condition (8.2) is of course always satisfied by a simple application of the triangle inequality for the norm. The following theorem gives a sufficient condition for (8.3).

Theorem 8.2: Let M be a Minkowski space with strictly convex unit sphere U with center o. If there exists an upper bound ρ for the diameters of all ellipses with center o and passing through three distinct points of U, then the norm of M satisfies the weak parallelogram law.

Proof: Let $\overline{pq}$ denote an associated euclidean metric of M. Then put $\sigma = \underset{u \in U}{\text{Min}}\ \overline{ou}$ and take

$$\gamma = \text{Min}\ (\tfrac{1}{2}, \sigma/\rho),$$

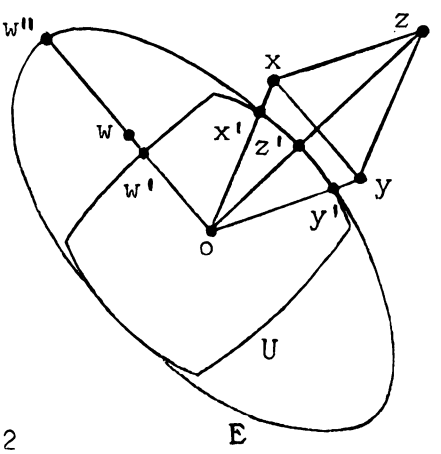
hence, $0 < \gamma < 1$. The inequality

$$\|x + y\|^2 + \gamma^2 \cdot \|x - y\|^2 \leqslant 2\|x\|^2 + 2\|y\|^2$$

is trivially true for the cases $x = o$, $y = o$, and $y = \lambda x$, λ real. Assume $y \neq \lambda x$ for all real λ and that $x \neq o$ and $y \neq o$. If x', y', and z' are the points where the rays from o through x, y, and $z = x + y$ respectively intersect U, then these will be three distinct points of U. By the strict convexity of U, there is a unique

ellipse E through x', y', and z' with center o. The argument may now be restricted to the plane of E. Put $w = x - y$ and let w' be the point of intersection of the ray from o through w and U, and w'' the point where that ray intersects E. Letting $\|x\|_E$ represent the norm with unit sphere E, since E is an ellipse it follows that $\|x\|_E$ defines an inner-product space and hence satisfied the parallelogram law

$$\|x + y\|_E^2 + \|x - y\|_E^2 = 2\|x\|_E^2 + 2\|y\|_E^2.$$



But as the unit spheres U and E of the two spaces coincide at x', y', and z', it follows that

$$\|x + y\|^2 + \|x - y\|_E^2 = 2\|x\|^2 + 2\|y\|^2. \tag{8.10}$$

Now $\overline{ow''} < \text{Diameter } E \leqslant \rho$ so that $1/\rho < 1/\overline{ow''}$, while $\sigma \leqslant \overline{ow'}$ implies that $\sigma/\rho < (\overline{ow'} / \overline{ow''})$. That is,

$$\gamma < \frac{\overline{ow'}}{\overline{ow''}} . \tag{8.11}$$

Then from (8.10),

$$\|x + y\|^2 + \gamma^2 \cdot \|x - y\|^2 < \|x + y\|^2 + \left(\frac{\overline{ow'}}{\overline{ow''}}\right)^2 \left(\frac{\overline{xy}}{\overline{ow'}}\right)^2$$

$$= \|x + y\|^2 + \left(\frac{\overline{xy}}{\overline{ow''}}\right)^2$$

$$= \|x + y\|^2 + \|x - y\|_E^2$$

$$= 2\|x\|^2 + 2\|y\|^2 .$$

Example: The two-dimensional Minkowski spaces $\{(x,y)\}$ given parametrically by taking $U_{\lambda,\mu}$ as the curve

$$\begin{cases} x^2 + \lambda xy + y^2 = 1, \text{ for } (x,y) \text{ in first and third quadrants} \\ x^2 - \mu xy + y^2 = 1, \text{ for } (x,y) \text{ in second and fourth quadrants} \end{cases}$$

where $0 < \lambda < 2$ and $0 < \mu < 2$, clearly satisfy the hypothesis of the above theorem, and so is a class of non-ptolemaic Banach spaces in which the fundamental theorem holds.

It follows in a similar manner that a two-dimensional l_p space, $1 < p \leqslant 2$, satisfies the condition of Theorem 8.2 since its unit sphere is given by

$$|x|^p + |y|^p = 1$$

which has everywhere positive curvature, infinite curvature at $(\pm 1, 0)$, $(0, \pm 1)$ if $p \neq 2$. Also by curvature considerations it can be shown--although it will not be undertaken here--that finite dimensional l_p spaces, $1 < p \leqslant 2$ in general satisfy that condition.

A Banach space which does <u>not</u> satisfy the weak parallelogram law is the set $\{(x,y)\}$ metrized by taking as unit sphere $|y| + x^4 = 1$, $|x| \leqslant 1$. Consider the points on the unit sphere

$$a = (\tau, 1 - \tau^4) \qquad b = (-\tau, 1 - \tau^4) \qquad 0 < \tau < 1.$$

$a + b = (0, 2 - 2\tau^4)$ and $a - b = (2\tau, 0)$ imply that

$$\|a + b\| = 2 - 2\tau^4, \qquad \|a - b\| = 2\tau.$$

But then

$$\frac{2\|a\|^2 + 2\|b\|^2 - \|a + b\|^2}{\|a - b\|^2} = \frac{4 - (2 - 2\tau^4)^2}{4\tau^2} = 2\tau^2 - \tau^6$$

which converges to zero as $\tau \longrightarrow 0$.

9. <u>Transverse Curvature</u>. Attention now turns to a different concept of curvature for arcs from those considered earlier. The

definition will be given in as general a metric space as possible, and for as general a curve as is possible, then emphasis will be on applying the definition to arcs in G-spaces.

Definition 9.1: Let p be any point of a curve C in a metric space having unique local segments, and suppose V_p is a neighborhood of p in which M has unique segments. Given q, r, and s three points on C and in V_p such that $q<r<s$ and $qr = rs$, define

$$\kappa_T(q,r,s) = \frac{8\, m_{qs}\, r}{qs^2} .$$

Then the curve C is said to have transverse curvature $\kappa_T(p)$ at the point p iff to each $\epsilon>0$ there corresponds a $\delta>0$ such that whenever qp, rp, and sp are each less than δ then

$$\left|\kappa_T(q,r,s) - \kappa_T(p)\right| < \epsilon .$$

It will be apparent that this curvature has some advantages over the others, although it is not known precisely how it compares with them. We do not explore that problem here. However, we do make a few comparisons of this curvature with so-called classical curvature for euclidean space. We take as the definition of classical curvature the following:

Definition 9.2: For any arc in E^n the classical curvature $\kappa(p)$ exists at a point p of that arc iff for any sequence of circles $\{C_i\}$ which have three points q_i, r_i, and s_i in common with the arc for $i = 1, 2, \cdots$ such that $\lim q_i = \lim r_i = \lim s_i = p$, then the sequence $\{1/\alpha_i\}$ of inverses of the radii α_i of C_i, for

each i, converges to $\kappa(p)$. (A straight line in E^n is defined to be a circle with zero as the inverse of its radius.)

This strictly geometric definition of curvature actually coincides with that of Menger curvature when applied to the metric of E^n.

Now suppose C is a circle in E^2 with radius α. To find the transverse curvature of C at a point $p \in C$, let $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ be three sequences of points on C with $q_i r_i = r_i s_i$ and each sequence converging to p. Then if o is the center of C, writing m_i for $m_{q_i s_i}$ for each i, it follows that

$$\kappa_T(q_i, r_i, s_i) = \frac{8\, m_i r_i}{(q_i s_i)^2}$$

$$= \frac{8(\alpha - om_i)}{4(m_i s_i)^2}$$

$$= \frac{2(\alpha - om_i)}{\alpha^2 - om_i^2}$$

$$= \frac{1}{\alpha - \frac{1}{2} m_i r_i}$$

Since $\lim_{i \to \infty} m_i r_i = 0$ then

$$\lim_{i \to \infty} \kappa_T(q_i, r_i, s_i) = \frac{1}{\alpha} = \kappa(p) = \kappa_T(p)$$

More generally, let A be any arc in E^n whose classical curvature exists at a point $p \in A$. Let $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ be three sequences of points on A each converging to p such that for each $i = 1, 2, \cdots$, q_i, r_i, s_i are distinct points and $q_i r_i = r_i s_i$. Put $m_i = m_{q_i s_i}$. Let $\{j\}$ be that subsequence of $\{i\}$ obtained

by considering those triples q_i, r_i, s_i which are linear, and let $\{k\}$ be the sequence remaining. Consider the case when both subsequences $\{j\}$ and $\{k\}$ are infinite. By the definition of $\{j\}$, $m_j r_j = 0$ for every j and $q_j s_j > 0$ so $\kappa_T(q_j, r_j, s_j) = 0$, and therefore

$$\lim_j \kappa_T(q_j, r_j, s_j) = 0.$$

For each k, the radius α_k of the circle through q_k, r_k, and s_k exists, and the calculations above show that

$$\kappa_T(q_k, r_k, s_k) = \frac{1}{\alpha_k - \frac{1}{2} m_k r_k}. \tag{9.1}$$

The linearity of an infinite subsequence of triples q_i, r_i, s_i implies that $\kappa(p) = 0$, and its existence implies for the subsequence $\{k\}$ that $\alpha_k \longrightarrow \infty$, hence

$$\lim_k \kappa_T(q_k, r_k, s_k) = 0$$

proving that $\kappa_T(p)$ exists and equals $\kappa(p)$. The case in which $\{k\}$ is a finite subsequence leads to the same result. Finally, when $\{j\}$ is a finite sequence, it may be assumed without loss of generality that $\{i\} = \{k\}$ and if $\alpha_k \longrightarrow \infty$ then again $\kappa_T(p)$ exists and equals $\kappa(p) = 0$. Otherwise, $\lim \alpha_k$ exists and since $m_k r_k \leqslant r_k s_k + m_k s_k = r_k s_k + \frac{1}{2} q_k s_k$ then $\lim m_k r_k = 0$ so (9.1) again shows that $\kappa_T(p)$ exists and equals $\kappa(p)$. This proves the theorem:

Theorem 9.1: For arcs in E^n the transverse curvature exists at a point of the arc whenever the classical curvature exists, and the two are equal.

In order for Definition 9.1 to have any meaning in a Banach space, it must have unique segments and therefore strictly convex unit sphere. The following property is needed for the next theorem.

Definition 9.3: A Minkowski space is said to be smooth iff its unit sphere has at each of its points a unique hyperplane of support.

In a smooth Minkowski space, if p is any point in M let U_p be the unit sphere with center p. The sphere U_p will have the same "smoothness" properties as U since they are congruent bodies in E^n. Suppose plane π_p passes through p and cuts U_p in C_p. Then C_p is a strictly convex curve in π_p with unique support lines at each point u_p of C_p. The diameter $S(u_p,-u_p)$ of C_p is uniquely determined by u_p as is the conjugate diameter $S(u_p^{\perp}, -u_p^{\perp})$ where $u_p^{\perp}$ and $-u_p^{\perp}$ are defined as the points of intersection of the line through p parallel to the unique line of support at u_p of C_p. If L_p is any line through p cutting C_p at the points $\pm u_p$ the line $L(u_p^{\perp},-u_p^{\perp})$ will be denoted $L_p^{\perp}$.

The euclidean area of a parallelogram with adjacent sides of unit Minkowski length along two distinct lines L and L' will be denoted $A(L,L')$, while $A(L)$ will denote the euclidean area of a square with side of Minkowski unit length along line L. It is clear that $A(L,L')$ and $A(L)$ are respectively equal to $A(K,K')$ and $A(K)$ whenever K is a line parallel to L and K' is one parallel to L'.

Finally, we need definitions for the existence of tangents and osculating planes. For greater simplicity these concepts will be

stated in algebraic terms, the first definition after Kelly and Ewald [12].

Definition 9.4: If for any two sequences $\{q_i\}$ and $\{r_i\}$ of points on an arc A converging to $p \in A$ and $q_i < r_i$ holds for each i the limit

$$\lim_{i\to\infty} \left[p + \frac{r_i - q_i}{r_i q_i} \right] = t_p$$

exists, the line $L(p,t_p)$ is called the free tangent to A at p.

Definition 9.5: If for any three sequences $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$ of points on an arc A converging to $p \in A$ and $q_i < r_i < s_i$ holds for each i with x_p any limit point of the sequence $\left\{ p + \frac{s_i - q_i}{s_i q_i} \right\}$ and y_p any limit point of the sequence $\left\{ p + \frac{m_i - r_i}{m_i r_i} \right\}$, where $m_i = \frac{1}{2}(q_i + s_i)$, the plane $\pi_p = \pi(p,x_p,y_p)$ determined by p, x_p, y_p is unique, then that plane is called the free osculating plane to A at p.

It is easily shown that the free tangent and osculating planes of an arc are geometric objects of E^n, independent of the particular Minkowski metric defining them. That is, the free tangent of an arc exists in E^n iff it exists in M, and the two are equal. A similar statement holds for the free osculating plane. Standard examples show that the existence of the free tangent does not guarantee that of the free osculating plane, nor does the existence of the free osculating plane ensure the existence of the free tangent, as a plane curve without a tangent shows.

Remark: A result of L. Kelly and G. Ewald [12] which will be implicitly assumed in the statement of the following theorem states that (as applied to our case) if the classical curvature of an arc

exists at a point of that arc, then it has a free tangent there.

Theorem 9.2: If M is any smooth Minkowski space with strictly convex unit sphere and A is an arc in M for which both the classical curvature and the free osculating plane exist at some point $p \in A$ regarded as an arc in E^n, then the transverse curvature $\kappa_T(p)$ exists at p. Moreover, if the free tangent to A at p is $T_p = L(p,t_p)$, then $\kappa_T(p)$ is given by the formula

$$\kappa_T(p) = \frac{\kappa(p)\, A^{\frac{3}{2}}(T_p)}{A(T_p, T_p^{\perp})} .$$

Proof: Without loss of generality it may be assumed that $p = o$. Let $\{q_i\}$, $\{r_i\}$, $\{s_i\}$ be any three sequences of points on A converging to p such that for each i, $q_i < r_i < s_i$ and $q_i r_i = r_i s_i$. From the existence of the free tangent at p (and since $p = o$), if

$$a_i = \frac{r_i - q_i}{r_i q_i} , \qquad b_i = \frac{s_i - r_i}{s_i r_i} , \qquad c_i = \frac{s_i - q_i}{s_i q_i} ,$$

then the sequences $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ each converge to t_p, some point on U. It is now shown that if $t_p^{\perp}$ and $-t_p^{\perp}$ are the endpoints of the diameter of U which is conjugate to $S(t_p, -t_p)$, the sequence of elements

$$\bar{m}_i = \frac{m_i - r_i}{m_i r_i} \qquad i = 1, 2, \cdots$$

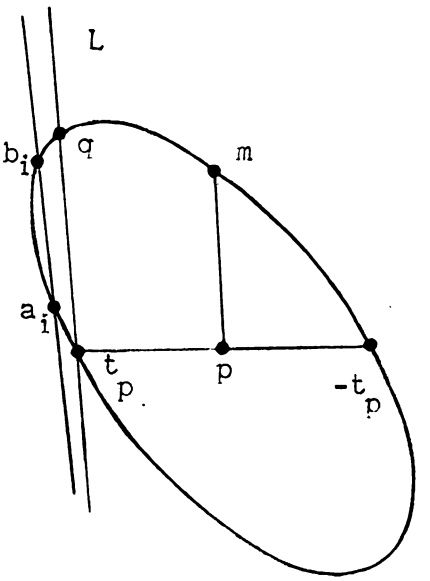
either converges to $t_p^{\perp}$, converges to $-t_p^{\perp}$, or can be divided into two subsequences, one converging to $t_p^{\perp}$, the other to $-t_p^{\perp}$. Since $\bar{m}_i \in U$ for each i, $\bar{m}_i$ is bounded and thus has at least one limit point. Consider m any such limit point and let $\{j\}$ be any

subsequence of $\{i\}$ such that

$$\lim_j \bar{m}_j = m.$$

The line L through t_p parallel to $L(p,m)$ is the set of points

$$\{t_p + \lambda m \mid \lambda \text{ real}\}.$$



Suppose L is not a line of support at t_p. Then there exists a second point $q \neq t_p$ in common between U and L. Then for some λ,

$$q = t_p + \lambda m$$

which means that

$$\|\lambda m\| = |\lambda| = \|q - t_p\| = qt_p$$

or

$$\lambda = \pm qt_p.$$

Since the argument for both cases is similar, it may be assumed that $\lambda = qt_p$ and that therefore

$$q = t_p + (qt_p)\, m.$$

Put $r = \frac{1}{2}(t_p + q) = t_p + \frac{1}{2}(qt_p)\, m$. By the strict convexity of U, r is an interior point of U. Therefore $\|r\| < 1$. Consider the sequence of points

$$\bar{r}_j = b_j + \tfrac{1}{2}(qt_p)\, \bar{m}_j.$$

This converges to r since $\lim_j b_j = t_p$ and $\lim_j \bar{m}_j = m$. But since

$$\bar{m}_j = \frac{r_j - \frac{1}{2}(q_j + s_j)}{r_j m_j}$$

$$= \tfrac{1}{2} \cdot \frac{r_j q_j}{r_j m_j} \cdot \frac{r_j - q_j}{r_j q_j} - \tfrac{1}{2} \cdot \frac{s_j r_j}{r_j m_j} \cdot \frac{s_j - r_j}{s_j r_j}$$

$$= \tfrac{1}{2} \cdot \frac{q_j r_j}{m_j r_j} (a_j - b_j)$$

$$= \frac{a_j - b_j}{a_j b_j}$$

then

$$\bar{r}_j = b_j + \tfrac{1}{2} (qt_p) \frac{a_j - b_j}{a_j b_j} = \mu_j a_j + (1 - \mu_j) b_j$$

where $\mu_j = (qt_p)/2\, a_j b_j$. Since $\lim_j a_j b_j = t_p t_p = 0$, for all j sufficiently large μ_j will be greater than unity and the corresponding $\bar{r}_j$ will be exterior to the segment $S(a_j, b_j)$ and therefore to U, hence $\|\bar{r}_j\| > 1$ which results in the contradiction

$$1 > \|r\| = \lim_j \|\bar{r}_j\| \geqslant 1.$$

Therefore, L is a line of support.

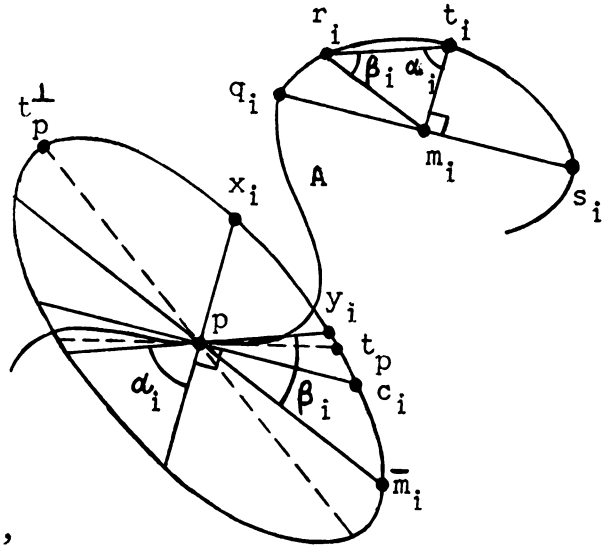
Next, L is in π_p, the free osculating plant of A at p, for by definition π_p is the plane determined by p, $\lim_j a_j$, and $\lim_j \bar{m}_j$, or p, t_p, and m. Hence $t_p + m$ is also in that plane. But t_p and $t_p + m$ are then two distinct points of L lying on π_p, so that L is in π_p. L must accordingly be the unique support line of $U \cap \pi_p$ at t_p. Since L(p,m)=L coincides with $T_p^{\perp}$ where T_p is the tangent to A at p, m is either $t_p^{\perp}$ or $-t_p^{\perp}$. Thus the only limit points of the sequence $\{\bar{m}_i\}$ are $+t_p^{\perp}$ and $-t_p^{\perp}$, proving the assertion. It follows that since $pt_p = p(-t_p^{\perp})$, then **if $\overline{xy}$ is a euclidean metric for M**

$$\lim_{i \to \infty} \overline{p\bar{m}}_i = \overline{pt}_p^{\perp}. \tag{9.2}$$

Also, it follows from $\lim_{i \to \infty} c_i = t_p$ that

$$\lim_{i \to \infty} \overline{pc}_i = \overline{pt}_p . \tag{9.3}$$

Now if t_i is a point on A for each i such that $\overline{t_i s_i} = \overline{t_i q_i}$, let



$$x_i = \frac{t_i - m_i}{t_i m_i}, \quad y_i = \frac{r_i - t_i}{r_i t_i}$$

and suppose that $\alpha_i = m(r_i, t_i, m_i)$ where $m(x,y,z)$ denotes the (euclidean) measure of angle (x,y,z), and $\beta_i = m(t_i, r_i, m_i)$. α_i equals either $m(x_i,p,y_i)$ or $\pi - m(x_i,p,y_i)$ so that $\sin \alpha_i = \sin m(x_i,p,y_i)$; also β_i equals either $m(y_i,p,\bar{m}_i)$ or $\pi - m(y_i,p,\bar{m}_i)$ and therefore $\sin \beta_i = \sin m(y_i,p,\bar{m}_i)$. Again by the existence of the free tangent to A at p, the only limit points of $\{y_i\}$ are $\pm t_p$. Also, since $L(p,x_i)$ is always orthogonal to $L(p,c_i)$ then $\{x_i\}$ has at most two limit points, x and -x such that $L(x,-x)$ is orthogonal to T_p. Since

$$\sin m(x,p,t_p) = \sin m(\pm x,p,\pm t_p)$$

this proves the only limit point for $\{\sin m(x_i,p,y_i)\}$ is $\sin m(x,p,t_p)$, that is

$$\lim_{i\to\infty} \sin \alpha_i = \sin m(x,p,t_p) = \sin \tfrac{1}{2}\pi = 1 \tag{9.4}$$

and similarly,

$$\lim_{i\to\infty} \sin \beta_i = \sin m(t_p,p,t_p^{\perp}). \tag{9.5}$$

Therefore,

$$\frac{8\, m_i r_i}{q_i s_i^2} = \frac{8 \cdot \dfrac{\overline{m_i r_i}}{\overline{p\, m_i}}}{\dfrac{\overline{q_i s_i}^2}{\overline{pc_i}^2}}$$

$$= \frac{8\, \overline{m_i t_i}}{\overline{q_i s_i}^2} \cdot \frac{\overline{m_i r_i}}{\overline{m_i t_i}} \cdot \frac{\overline{pc_i}^2}{\overline{pm_i}}$$

$$\frac{8\, \overline{m_i t_i}}{\overline{q_i s_i}^2} \quad \frac{\sin \alpha_i}{\sin \beta_i} \quad \frac{\overline{pc_i}^2}{\overline{pm_i}}$$

with the limit, since $\kappa(p) = \lim_{i \to \infty} (8\, \overline{m_i t_i}) \,/\, (\overline{q_i s_i})^2$,

$$\kappa(p) \cdot \frac{1}{\sin m(t_p, p, t_p^{\perp})} \cdot \frac{\overline{pt}_p^2}{\overline{pt}_p^{\perp}}$$

$$= \frac{\kappa(p)\, (\overline{pt}_p)^3}{\overline{pt}_p \cdot \overline{pt}_p^{\perp} \cdot \sin m(t_p, p, t_p^{\perp})}$$

proving that $\kappa_T(p)$ exists at p and that

$$\kappa_T(p) = \frac{\kappa(p)\, \mathbf{A}^{\frac{3}{2}}(T_p)}{\mathbf{A}(T_p, T_p^{\perp})}$$

The resemblance of the above formula to one obtained by Busemann [5] is close. His formula involved a different definition of curvature $x(S)$ where S is Minkowski arc length and the same curvature $\bar{x}(\bar{S})$ under the associated euclidean metric. The formula is

$$x(s) = \bar{x}(\bar{S})\, \sigma(t_2)\, \sigma^{-3}(t_1)$$

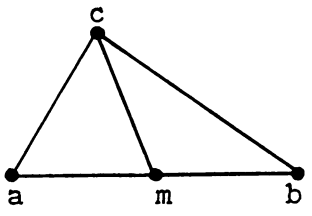
where $\sigma(t_2)$ is the area of a certain two-dimensional region and

$\sigma(t_1)$ is the length of a certain line segment. The corresponding factors in our formula are therefore inverted.

10. Transverse Curvature and the Fundamental Theorem. By a theorem of Busemann proved in [6] a G-space having non-positive Busemann curvature also has feebly non-positive median curvature; that is, each point of the space has a neighborhood in which every triangle has the property that any median is less than or equal to one-half the sum of the adjacent sides. For, Busemann proves that in a G-space with $K_B \leqslant 0$, each point has a neighborhood V with unique segments and in which, if $y(\tau)$ is the arc-length parametrization of a geodesic arc in V and x is any fixed point in V not on that geodesic, the function $xy(\tau)$ is convex. Specifically, this means that for any two values τ_1 and τ_2 of τ,

$$xy\left(\frac{\tau_1 + \tau_2}{2}\right) \leqslant \tfrac{1}{2}\, xy(\tau_1) + \tfrac{1}{2}\, xy(\tau_2).$$

Applying this to a triangle $T(a,b,c)$ with vertices and sides in V with the midpoint m of $S(a,b)$ proves



(10.1) $\qquad cm \leqslant \tfrac{1}{2}\, ac + \tfrac{1}{2}\, bc.$

The following theorem applies more generally to all metric spaces with locally unique segments and having feebly non-positive median curvature. This class of metric spaces is indeed large, for it includes

(1) G-spaces with non-positive Busemann curvature.

(2) Banach spaces with strictly convex unit spheres.

(3) Locally ptolemaic metric spaces with locally unique segments.

(4) Riemannian surfaces, and, apparently, Riemannian space.

(5) The three classical geometries, hyperbolic, euclidean, and spherical spaces.

Theorem 10.1: If M is a metric space with locally unique segments and having feebly non-positive median space curvature, then any rectifiable arc of M is a geodesic iff its transverse curvature vanishes identically along the arc.

Proof: For the necessity, let A be a geodesic arc and p any point on A. There exists a neighborhood V_p of p in which M has unique segments and in which A is a metric segment. Let $\{q_i\}$, $\{r_i\}$, and $\{s_i\}$, be three sequences of points on A converging to p, with $q_i < r_i < s_i$ and $q_i r_i = r_i s_i$ for each i. For all sufficiently large i q_i, r_i, and s_i lie in V_p and hence, as distinct points on a metric segment, one of $(q_i r_i s_i)$, $(r_i q_i s_i)$, or $(r_i s_i q_i)$ must hold. But $q_i < r_i < s_i$ implies that $(q_i r_i s_i)$ is the only possibility and hence r_i is the midpoint of q_i and s_i. Therefore $r_i m_{q_i s_i} = r_i r_i = 0$ and $\kappa_T(q_i, r_i, s_i) = 0$ for all sufficiently large i, proving that $\kappa_T(p)$ exists and equals zero, for $p \in A$.

Now let A be a rectifiable arc with $\kappa_T(p) = 0$ for $p \in A$. Let V_p be a neighborhood of p in which M has unique segments and (10.1) holds for every triangle with vertices in V_p. Let A' be any sub-arc of A contained in V_p with endpoints a and b, with $\Lambda(x,y)$, $x<y$, as arc-length along A. Suppose $\epsilon>0$ be given. Following the method of proof of Theorem 7.2, choose an n-lattice of A'

$$a = p_0 < p_1 < p_2 < \cdots < p_n = b$$

with n so large that

$$(10.2) \qquad 0 \leqslant \Lambda(a,b) - n\lambda_n < \frac{\epsilon}{2}, \quad \lambda_n = p_{k-1}p_k$$

and

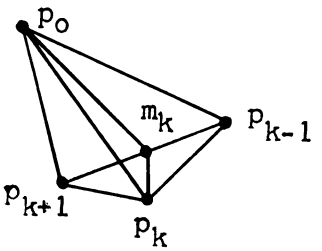
$$(10.3) \qquad \kappa_T(p_{k-1},p_k,p_{k+1}) < \frac{\epsilon}{2\Lambda^2}, \quad k = 1, 2, \cdots, n-1,$$

where $\Lambda = \Lambda(a,b)$. Again, it will be sufficient to obtain the set of $n-1$ inequalities of (7.15). With m_k as the midpoint of p_{k-1} and p_{k+1}, (10.3) becomes

$$8\, m_k p_k < \frac{(p_{k-1}p_{k+1})^2\epsilon}{2\Lambda^2} \leqslant \frac{(p_{k-1}p_k + p_kp_{k+1})^2\epsilon}{2\Lambda^2} = \frac{4\lambda_n^2\epsilon}{2\Lambda^2}$$

or,

$$(10.4) \qquad m_k p_k < \frac{\lambda_n^2\epsilon}{4\Lambda^2}, \quad k = 1, 2, \cdots n-1.$$



substituting (10.1) and (10.4) in the triangle inequality $p_0p_k \leqslant p_0m_k + m_kp_k$, it follows that

$$(10.5) \qquad 2\, p_0p_k \leqslant p_0p_{k-1} + p_0p_{k+1} + \frac{\lambda_n^2\epsilon}{2\Lambda^2}, \quad k = 1, 2, \cdots, n-1.$$

Put

$$\alpha_k = k\lambda_n - p_0p_k, \qquad k = 0, 1, 2, \cdots, n,$$

and substitute into (10.5):

$$2k\lambda_n - 2\alpha_k < (k-1)\lambda_n - \alpha_{k-1} + (k+1)\lambda_n - \alpha_{k+1} + \frac{\lambda_n^2\epsilon}{2\Lambda^2}$$

which simplifies to

$$(10.6) \qquad \alpha_{k+1} - \alpha_k < \alpha_k - \alpha_{k-1} + \frac{\lambda_n^2\epsilon}{2\Lambda^2}, \quad k = 1, 2, \cdots, n-1,$$

the desired set of inequalities (7.15). The remainder of the proof follows exactly as before, proving that $\Lambda(a,b) = ab$. Therefore A'

is a metric segment and **A** is a geodesic arc.

The result of this theorem is that transverse curvature not only affords the same characterization of geodesics in practically the same class of metric spaces, namely ptolemaic spaces, as the Menger and Haantjes curvatures originally afforded, but widens the class to include such general spaces as Banach spaces with strictly convex unit spheres--spaces which were not covered even by Haantjes' theorems covering certain non-ptolemaic spaces such as spherical space. The fact that the transverse curvature agrees with classical curvature when the space is euclidean makes it a "reasonable" concept of curvature. Moreover, the existence of a readily applied formula in the case of Minkowski spaces makes it a useful generalization of euclidean curvature as far as those spaces are concerned. (The Menger and Haantjes curvatures are not so readily calculated in Minkowski spaces.) In conclusion, many interesting questions regarding this concept of curvature await their answers.

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