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THE NATURE OF THE SOLUTIONS OF DAMPED LINEAR DYNAMIC SYSTEMS

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ABSTRACT

THE NATURE OF THE SOLUTIONS OF DAMPED LINEAR DYNAMIC SYSTEMS

By

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An analysis of the qualitative nature of the solutions of viscously damped linear dynamic systems is presented. Both lumped parameter and distributed parameter systems are considered. Conditions illustrating whether or not a given system will oscillate are derived. These conditions can be checked without having to solve the governing differential equations.

The conditions applied to the lumped parameter case are shown to imply certain closed form solutions for arbitrarily forced systems. Several examples are given illustrating how these conditions may be used to design a specified system so that it will either oscillate or not, as desired.

The theory developed here is compared with previous results by other authors. In the case of distributed parameter theory, the results derived here are compared to specific problems from the literature. New information is provided about certain classes of damped beams and plates.

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Chapter 1

INTRODUCTION

1.1 Motivation

The simplest linear system with damping is a single degree of freedom system with a mass suspended by a parallel spring and damper arrangement. Such a system can be described by a second order ordinary differential equation with constant coefficients corresponding to the physical parameters; mass, spring constant and damping constant. The resulting initial value problem is easily solved. It is well known that the qualitative nature of the solution can be determined by examining certain ratios of these coefficients.

The intent of this dissertation is to define and investigate the concepts of critical damping, overdamping and underdamping associated with the single degree of freedom problem, for more general damped linear systems. The purpose being to find conditions which are easy to apply and which will indicate the qualitative nature of the solution of a complicated damped linear system without having to solve the governing differential equations.

1.2 Physical Systems

The work here examines certain classes of both lumped parameter systems and distributed parameter systems. Lumped parameter systems are written in matrix form and the problem is to derive conditions for the resulting coefficient matrices which, when satisfied, will insure that each mode of the system will have the desired damping characteristic. The physical problems may stem from complicated arrangements of stiffness, inertia and damping elements, and represent unusual geometry. The restrictions are that the system is asymptotically stable and that the coefficient matrices satisfy the unusual symmetry and definiteness conditions.

The distributed parameter systems considered here are those described by linear partial differential equations with appropriate boundary and initial conditions. The spatial coefficient operators of the governing differential equation must be self-adjoint, positive definite and possess Hilbert-Schmidt inverses. The physical problems to which the theory applies includes various problems concerning strings, beams, membranes and plates with either internal and/or external damping of the type that may be caused by immersion in a fluid.

1.3 Previous Work

There has been some interest in the lumped parameter case. In previous work Duffin [1] defined an overdamped system in terms of a function of quadratic forms of the coefficient matrices. Later Lancaster [2] developed and added to Duffin's work. Concurrently Meirovitch [3] commented briefly on a matrix concept of overdamping and underdamping in terms of characteristic roots for the special case when the equations of motion are decoupled.

More recently, Nicholson [4] defined an underdamped system in terms of the eigenvalues of the mass, stiffness and damping matrices. Müller [5] responded to Nicholson's attempts and defined an underdamped system in terms similar to those of Lancaster and Duffin. He then derived a sufficient condition for a system to be underdamped. The condition is stated in terms of the definiteness of a certain combination of the coefficient matrices.

The conditions offered by the above authors fall short of a complete theory. Each author has considered only one possibility. Some of the results apply only in special cases or are difficult to check. These results have not been applied to example problems.

Extensions of the damping problem to distributed parameter systems has been untouched in the literature. However, most analysis of forced problems tacitly assumes the nature of the damped time response [2], [3]. Stakgold [6] discusses the nature of the time response in terms of periodic and aperiodic solutions. Conditions, which when satisfied, guarantee that the time response can be uncoupled from the spatial eigenvalue problem were presented by Caughey and O'Kelly [7] as an extension of the lumped parameter case. More recently, Strenkowski and Pilkey [8] have discussed a closed form solution for the time response of a general forced distributed parameter system with damping, without any restrictions on the stiffness operator. However, they require the damping operator to be free of derivatives and they make no attempt to characterize the nature of the solution. Other authors who have addressed damped distributed parameter systems recently have assumed a constant damping term in their analysis (see for example Leissa [9]). This approach is common but excludes such examples as a beam or string vibrating in a fluid.

1.4 New Results

In this dissertation conditions are found which can be easily applied to the coefficients of the governing differential equations and which when satisfied will guarantee that the resulting solutions will be damped in a specified manner. The presentation of the results is as follows:

Chapter 2 presents a review of basic mathematical definitions and theorems needed to treat both the lumped parameter case (linear algebra) and the distributed parameter case (functional analysis).

In Chapter 3, the definitions and problem for the lumped parameter case are formulated. Sufficient conditions are stated and derived in terms of the definiteness of various combinations of the coefficient matrices. Examples are offered illustrating the correctness and use of the conditions. This chapter is concluded with a comparison to previous results of other authors.

In Chapter 4, the results of Chapter 3 are applied to the general theory of forced lumped parameter systems. In addition, examples are given illustrating that the results can be used as a design tool for systems of low order.

In Chapter 5 the problem and definitions for distributed parameter systems are formulated. Proofs of sufficient conditions to determine the nature of the time solutions are offered and examples are solved illustrating the correctness and use of the conditions.

Chapter 6 presents a brief summary of the dissertation as well as an indication of further areas of investigation.

Chapter 2

MATHEMATICAL PRELIMINARIES

This chapter presents basic definitions and theorems from linear algebra and functional analysis which will be used in presenting the results in subsequent chapters. Here, background theorems are stated without proof except those which are not found in the references.

2.1 Concepts from Linear Algebra

The material in this section can be found in any number of references [2, 10] and is presented here for completeness.

In the subsequent presentation, the following notation will be used. Any n-dimensional column vector will be denoted by $\underline{x}$, $\underline{y}$, $\underline{z}$, etc.; the transpose and complex conjugate of the transpose of the vector $\underline{x}$ will be denoted by $\underline{x}^T$ and $\underline{x}^*$, respectively. The i^{th} element of a vector $\underline{x}$ will be denoted by x_i. This is not to be confused with the vector $\underline{x}_i$ which denotes the i^{th} vector in a sequence of indexed vectors. Any nxn real matrix will be denoted by A, B, C, etc.; the transpose of a matrix A will be denoted by A^T. The element in the i^{th} row and j^{th} column of the matrix A is denoted by a_{ij}.

Definition (2.1-1): The inner product of two vectors $\underline{x}$ and $\underline{y}$, denoted $\underline{x}^*\underline{y}$, is defined by

$$\underline{x}^*\underline{y} = \sum_{i=1}^{n} x_i^* y_i .$$

Definition (2.1-2): The norm of the vector $\underline{x}$, denoted by $||x||$, is defined by

$$||\underline{x}|| = (\underline{x}^* \underline{x})^{1/2}.$$

Theorem (2.1-3): The norm and inner product are related by the Cauchy-Schwarz inequality

$$|(\underline{x}^* \underline{y})| \leq ||\underline{x}|| \; ||\underline{y}||,$$

with equality if and only if $\underline{y} = \alpha \underline{x}$ where α is a scalar.

Definition (2.1-4): Two vectors $\underline{x}$ and $\underline{y}$ are said to be orthogonal if $\underline{x}^* \underline{y} = 0$.

Definition (2.1-5): A scalar λ is called an eigenvalue or a characteristic value of a matrix A and the vector $\underline{x} \neq 0$ is called the corresponding eigenvector of A if $A\underline{x} = \lambda \underline{x}$.

A necessary and sufficient condition for λ to be an eigenvalue of A is that λ satisfy det $(A-\lambda I) = 0$, where det$(\cdot)$ indicates the determinant of $(\cdot)$ and I is identity matrix of appropriate dimension.

In the rest of this dissertation all matrices are considered to be real symmetric arrays unless otherwise stated.

Definition (2.1-6): The matrix A is said to be positive definite if $\underline{x}^T A \underline{x} > 0$ for all non-zero real vectors $\underline{x}$. This is denoted $A > 0$.

Definition (2.1-7): The matrix A is said to be positive semi-definite if $\underline{x}^T A \underline{x} \geq 0$ for all non-zero real vectors $\underline{x}$. This is denoted $A \geq 0$.

Theorem (2.1-8): All the eigenvalues of a real symmetric matrix are real numbers.

Theorem (2.1-9): The matrix A is positive definite if and only if all of its eigenvalues are positive.

Theorem (2.1-10): The matrix A is positive semi-definite if and only if all of its eigenvalues are non-negative.

Theorem (2.1-11): If the matrix A is positive definite, then $\underline{x}^* A \underline{x} > 0$ for all non-zero complex vectors $\underline{x}$.

Let $|A_i|$ denote the determinant of minors as follows:

$$|A_1| = a_{11}$$

$$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\vdots$$

$|A_n| = \det(A).$

Theorem (2.1-12): The matrix A is positive definite if and only if $|A_i| > 0$ for all i=1,2...n.

Definition (2.1-13): The inverse of the matrix A, denoted A^{-1} is a matrix such that $A^{-1}A = I$. If A^{-1} exists, A is said to be non-singular.

Theorem (2.1-14): The matrix A^{-1} exists if and only if $\det(A) \neq 0$.

Theorem (2.1-15): If $A > 0$ and $B > 0$ then $A+B > 0$.

Theorem (2.1-16): If $A > 0$ and $B > 0$ then the product matrix AB is positive definite if and only if $AB = BA$.

Definition (2.1-17): A set of vectors $\{\underline{x}_j\}_{j=1}^{n}$ are said to be linearly independent if

$$\sum_{j=1}^{n} \alpha_j \underline{x}_j = 0$$

implies that $\alpha_j = 0$ for all j, where the α_j are scalars.

Definition (2.1-18): A set of vectors $\underline{x}_j$ is said to be orthonormal if

$$\underline{x}_i^* \underline{x}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} .$$

Theorem (2.1-19): (Gram-Schmidt) Any set of n linearly independent vectors can be used to generate a set of n orthonormal vectors.

Theorem (2.1-20): An nxn real symmetric matrix has n linearly independent eigenvectors associated with its eigenvalues regardless of eigenvalue multiplicity.

Definition (2.1-21): The rank of a matrix A is the largest number of linearly independent rows (columns) of A.

Theorem (2.1-22): If $A > 0$, the rank of A is n.

Definition (2.1-23): The matrix A is diagonal if $a_{ij} = 0$ for all i and j such that $i \neq j$.

Definition (2.1-24): A matrix S is orthogonal if $S^T S = I$.

Theorem (2.1-25): Every real symmetric matrix A can be diagonalized by an orthogonal matrix S, consisting of the eigenvectors of A, such that the resulting diagonal matrix has the eigenvalues of A, λ_i, as its elements. Notationally $\Lambda = S^T A S$, where $S^T S = I$, and

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Theorem (2.1-26): Let Λ_A denote the diagonal matrix of eigenvalues of A and Λ_B the diagonal matrix of eigenvalues of B. Then there exists an orthogonal matrix S such that $\Lambda_A = S^T AS$, and $\Lambda_B = S^T BS$, if and only if $AB = BA$.

Definition (2.1-27): The matrix A is said to be orthogonally similar to a matrix B, denoted by $A \sim B$, if there exists a matrix U such that $U^T AU = B$ and $U^T U = I$.

Theorem (2.1-28): If $A \sim B$, then A and B have the same eigenvalues.

Definition (2.1-29): The square of a matrix A, denoted A^2, is defined by $A^2 = AA$.

Definition (2.1-30): The square root of a matrix A is defined as a matrix, denoted $A^{1/2}$, such that $A = A^{1/2}A^{1/2}$.

Theorem (2.1-31): If the matrices A and B commute so do A and $B^{1/2}$.

Theorem (2.1-32): If $A > 0$ then there exists a unique positive definite square root of A. The eigenvalues of $A^{1/2}$ are the positive square roots of those of A. If λ_i are the eigenvalues of A then $A^{1/2} = S^T \Lambda_A^{1/2} S$ where

$$\Lambda_A^{1/2} = \begin{bmatrix} \lambda_1^{1/2} & & & 0 \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{1/2} \end{bmatrix}$$

and S is the orthogonal matrix of eigenvectors of A.

If A-B is a positive definite matrix this is denoted by $A > B$.

Theorem (2.1-33): [11] If $A > B > 0$, then $A^{1/2} > B^{1/2}$.

It should be noted that if $A > B$ it does not necessarily follow that $A^2 > B^2$.

Theorem (2.1-34): [12] If $A > 0$, $B > 0$ and $AB = BA$ then $A > B$ implies $A^k > B^k$ where k is a positive integer.

2.2 Concepts from the Theory of Lambda Matrices

The definitions and theorems stated here can be found in Lancaster's excellent text [2] except where otherwise indicated.

Definition (2.2-1): A lambda matrix, denoted $D_\ell(\lambda)$, is a polynomial in the scalar λ with matrix coefficients where ℓ denotes the highest power of λ.

The lambda matrix that is of interest here is

$$D_2(\lambda) = A\lambda^2 + B\lambda + C$$

where A, B and C are nxn real symmetric matrices such that A and C are positive definite while B is positive semi-definite.

Definition (2.2-2): A vector $\underline{x} \neq 0$ is an eigenvector (latent vector) of $D_2(\lambda)$ and λ is the associated eigenvalue (latent root) if $D_2(\lambda)\underline{x} = 0$. The eigenvalues are thus the 2n roots of the scalar equation $\det[D_2(\lambda)] = 0$.

Definition (2.2-3): The lambda matrix $D_2(\lambda)$ is said to be simple if the rank of $D_2(\lambda_i)$ is equal to $n-\alpha$ where λ_i is an eigenvalue of multiplicity α, for each λ_i.

This definition implies that there exist α linearly independent vectors associated with the repeated root λ_i. Hence, $D_2(\lambda)$ is simple if there exist n linearly independent eigenvectors of $D_2(\lambda)$. Otherwise $D_2(\lambda)$ is said to be degenerate.

Note that in section (2.1) the simplicity of $D_1(\lambda)$ is guaranteed by the symmetry (theorem 2.1-20). This is not the case for $D_2(\lambda)$. Lancaster has settled the question with the following theorem.

Theorem (2.2-4): The lambda matrix $D_2(\lambda)$ is degenerate if and only if there exists an eigenvalue λ with associated eigenvector $\underline{q}$ such that

$$\underline{r}^T [2\lambda A+B]\underline{q} = 0,$$

for all left eigenvectors $\underline{r}$ associated with λ.

If the lambda matrix $D_2(\lambda)$ is simple, then its eigenvectors can be used to generate an n-dimensional vector space.

Definition (2.2-5): The lambda matrix $D_2(\lambda)$ is said to be asymptotically stable if the real parts of all the eigenvalues are nega tive numbers.

Theorem (2.2-6): [13] The lambda matrix $D_2(\lambda)$ is asymptotically stable if and only if the rank of

$$\begin{bmatrix} B \\ BC \\ BC^2 \\ \cdot \\ \cdot \\ \cdot \\ BC^{n-1} \end{bmatrix}$$

is equal to n.

2.3 Concepts from Functional Analysis

The majority of the material in this section can be found in any text on functional analysis or linear operator theory, see for example [14], [15]. Kato [16] should be consulted for information on operator square roots.

The underlying Hilbert space is taken to be $\mathcal{L}_2(\Omega)$, the space of all functions which are square integrable in the Lebesgue sense over the bounded region Ω. The functions u(x) and v(x) are real valued functions depending on x which may be vector valued in Ω. The symbol L denotes a real linear operator whose domain is dense in $L_2(\Omega)$.

Definition (2.3-1): The inner product of two functions u(x), v(x) in $L_2(\Omega)$, denoted $\langle u,v\rangle$, is defined by

$$\langle u,v\rangle = \int_\Omega u(x)\, v(x)\, dx.$$

Definition (2.3-2): The norm of a function u(x) is denoted and defined by $||u|| = \langle u,u\rangle^{1/2}$.

The Cauchy-Schwarz inequality (2.1-3) stated for vectors holds in terms of the norm and inner-product defined here.

Definition (2.3-3): The operator L is bounded if there exists a finite constant $c > 0$ such that $||Lu|| \leq c||u||$ for all u in the domain of L, denoted D(L).

If such a constant does not exist, L is said to be unbounded.

Definition (2.3-4): The numerical range of L, denoted W(L), is defined to be the set

$$W(L) = \{\langle Lu,u\rangle \mid u \in D(L) \text{ and } ||u|| = 1\}.$$

Definition (2.3-5): The operator L is positive definite if $<Lu,u> > 0$ for all u in D(L).

Theorem (2.3-6): If every element of W(L) is positive, then L is positive definite.

Proof: Suppose $<Lu,u> < 0$ for some u in D(L). Then

$$<L \frac{u}{||u||}, \frac{u}{||u||}> < 0$$

for some u in D(L) or $<Lv,v> < 0$ for v in D(L) with $||v|| = 1$, i.e., for v in W(L). Thus, L must be positive definite if W(L) is positive.

Let $\overline{D(L)}$ denote the closure of the set D(L).

Definition (2.3-7): The set D(L) is dense in $L_2(\Omega)$, denoted $\overline{D(L)} = L_2(\Omega)$, if for every u in $L_2(\Omega)$ and every $\varepsilon > 0$ there exists a v in D(L) such that $||u-v|| < \varepsilon$.

Theorem (2.3-8): Let L be a linear operator with $\overline{D(L)} = L_2(\Omega)$, then L has a unique adjoint operator denoted L^*. The domain of L^*, denoted $D(L^*)$ is defined as follows: v(x) is in $D(L^*)$ if and only if there exists a function g such that $<Lu,v> = <u,g>$ for all u in D(L). The adjoint operator is defined by $g = L^*v$ so that if u is in D(L) and v is in $D(L^*)$ then $<Lu,v> = <u,L^*v>$.

Definition (2.3-9): The operator L, with $\overline{D(L)} = \mathcal{L}_2(\Omega)$, is self-adjoint if $D(L^*) = D(L)$ and $Lu = L^*u$ for all u in D(L).

Definition (2.3-10): The square of the operator is defined by $L^2u = L(Lu)$ for all u in $D(L^2)$.

Theorem (2.3-11): [16] If L is a positive self-adjoint operator then there exists a unique positive self-adjoint operator $L^{1/2}$, called the square root of L, such that $L^{1/2}(L^{1/2}u) = Lu$ for all u in D(L).

Definition (2.3-12): Let A and B be two self-adjoint operators, then A > B if

i) $\langle Au,u\rangle > \langle Bu,u\rangle$ for all u in D(A); and

ii) D(B) D(A).

Definition (2.3-13): A complex scalar λ is in the point spectrum of L, denoted $\sigma_p(L)$, if $(\lambda - L)^{-1}$ does not exist.

Definition (2.3-14): A non-zero function u such that, $Lu = \lambda u$, is called an eigenfunction of the operator L, and λ in $\sigma_p(L)$ is called the eigenvalue of L associated with u.

Theorem (2.3-15): If L is self-adjoint, $\sigma_p(x)$ is real.

Definition (2.3-16): An operator K defined by

$$Ku(x) = \int_\Omega k(x,\zeta)u(\zeta)d\zeta, \text{ where } \int_\Omega\int_\Omega |k(x,\zeta)|^2 dx d\zeta < \infty$$

is called a Hilbert-Schmidt operator.

Theorem (2.3-17): [17] If the inverse of the self-adjoint operator L is a Hilbert-Schmidt operator, then every function u in D(L) can be written as the uniformly convergent series

$$u(x) = \sum_{n=1}^{\infty} <\phi_n,u> \phi_n(x)$$

where the $\phi_n(x)$ are the complete set of eigenfunctions of L.

Theorem (2.3-18): If the operator L satisfies the conditions of (2.3-17) and has a positive point spectrum, then L is a positive definite operator.

Proof: Since the convergence in (2.3-16) is uniform, (2.3-17) yields ($c_n = <\zeta_n,u>$)

$$<Lu,u> = <Lu, \sum_{n=1}^{\infty} c_n \zeta_n>$$

$$= {}_\Omega\int \Sigma c_n \zeta_n (Lu)\, dx$$

$$= \Sigma\, c_n {}_\Omega\int \zeta_n Lu\, dx$$

$$= \Sigma\, c_n <\zeta_n, Lu>$$

$$= \Sigma \lambda_n c_n^2 > 0$$

since each λ_n is positive.

Lemma (2.3-19): If A and B are two self-adjoint linear operators with common domains such that A is positive and the inverse of A-B is a Hilbert-Schmidt operator, then $A^2 > B^2$ implies $A > B$.

Proof: Choose $\mu < 0$ and let u be an arbitrary element in $W(A^2)$. Consider

$$||(A-\mu)u||^2 - ||Bu||^2$$

$$= <(A^2-B^2)u,u> - 2\mu<Au,u> + \mu^2.$$

Since $A^2-B^2 > 0$ and $-2\mu > 0$ this becomes

$$||(A-\mu)u||^2 - ||Bu||^2 = \{||(A-\mu)u||-||Bu||\}\{||(A-\mu)u||+||Bu||\} \geq \mu^2$$

or

$$||(A-\mu)u||-||Bu|| \geq \frac{\mu^2}{||(A-\mu)u||+||Bu||} > 0.$$

The triangle inequality combined with this last inequality yields

$$||(A-B-\mu)u||=||(A-B)u-\mu u||\geq||(A-\mu)u||-||Bu|| > 0.$$

Hence $(A-B)u \neq \mu u$ for any u in D(A). Thus $\mu < 0$ is not in $\sigma_p(A-B)$, so that $\sigma_p(A-B) > 0$ and, by theorem (2.3-18), $A > B$.

The above proof follows the method used by Shiu [18] in a proof of the same theorem for bounded operators.

Theorem (2.3-20): Let L_1 and L_2 be self-adjoint commuting operators with no repeated eigenvalues where a and b are two real scalars, then the eigenvalues of $(aL_1 + bL_2^2)$ are

$$a\lambda_i + b\mu_i^2$$

where λ_i is the i^{th} eigenvalue of L_1 and μ_i is the i^{th} eigenvalue of L_2.

Proof: Let α be an eigenvalue of aL_1 and bL_2^2 with eigenfunction u, then

$$aL_1u + bL_2^2u = \alpha u$$

or

$$(aL_1 + bL_2^2)(L_1u) = \alpha(L_1u)$$

so that L_1u is an eigenfunction of $(aL_1+bL_2^2)$ with eigenvalue α. Since there are no repeated roots this yields

$$L_1u = \gamma u,$$

and u is also an eigenfunction of L_1. That u is an eigenfunction of L_2 follows similarly. Hence the above becomes

$$(a\lambda + b\mu^2)u = \alpha u$$

$u \neq 0$ and the associated eigenvalue is

$$\alpha = a\lambda + b\mu^2$$

where λ is an eigenvalue of L_1 and μ an eigenvalue of L_2.

Chapter 3

SOME RESULTS FOR LUMPED PARAMETER SYSTEMS

3.1 Problem Description

In this chapter a special class of lumped parameter systems, i.e., systems with multiple degrees of freedom which can be described by a set of coupled second order ordinary differential equations with constant coefficients is examined. Interest will be focused on equations of the following form

$$(3.1\text{-}1) \qquad M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} = 0$$

with arbitrary initial conditions, where M, C and K are real nxn symmetric matrices, and $\underline{x}$ is an n-dimensional column vector whose elements represent the displacement from an established equilibrium position. The independent variable, denoted by t, is time. It is assumed that M and K are positive definite and that C is positive semi-definite. It is also assumed that the system is asymptotically stable in the sense that all motions eventually decay to zero [13].

The intent here will be to derive conditions which, when satisfied, will guarantee certain qualitative aspects of the solution $\underline{x}(t)$, i.e., oscillation or non-oscillation. The conditions should require less calculation to check than that required to find the actual solution $\underline{x}(t)$.

3.2 One Degree of Freedom Systems

The solution to the problem posed above is well known for the case of a single degree of freedom system with viscous damping. In fact the nature of the solution is determined by a straightforward examination of the coefficients.

The describing scalar differential equation is

$$(3.2\text{-}1) \quad m\ddot{x} + c\dot{x} + kx = 0$$

where $x = x(t)$ is the displacement from equilibrium, m is the mass, c is the viscous damping constant and k is the spring constant. The solution of this equation is of the form e^{rt} where r satisfies the scalar binomial equation

$$(3.2\text{-}2) \quad mr^2 + cr + k = 0.$$

Since m, c and k are all positive real numbers, the solution $x(t)$ can be characterized in terms of the critical damping constant (denoted c_c) defined by $c_c = 2\sqrt{km}$. It is then common to classify the one degree of freedom system by the nature of the roots of (3.2-2) which are determined by the sign of $(c-c_c)$ in the following way:

Definition (3.2-3): If $c = c_c$ the system in (3.2-1) is said to be critically damped.

Theorem (3.2-4): If (3.2-1) is critically damped then (3.2-2) has one repeated negative real root and the solution $x(t)$ does not oscillate and decays exponentially.

Definition (3.2-5): If $c-c_c>0$ the system in (3.2-1) is said to be overdamped.

Theorem (3.2-6): If (3.2-1) is overdamped then (3.2-2) has two negative real roots and the solution x(t) does not oscillate and decays exponentially.

Definition (3.2-7): If $c_c-c>0$ the system in (3.2-1) is said to be underdamped.

Theorem (3.2-8): If (3.2-1) is underdamped then (3.2-2) has a pair of complex conjugate roots with negative real parts and the solution x(t) is an exponentially decaying oscillation.

Hence, the problem for one degree of freedom is easily solved. It is this fact that has motivated the research presented in this dissertation. The premise that the physical nature of the solution is determined by the coefficients of the describing differential equation is explored in the following section.

3.3 Multiple Degree of Freedom Systems

Before considering the multiple degree of freedom case, it is convenient to make the simplifying transformation

$$\underline{x} = M^{-1/2}\underline{y}$$

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where the (-1/2) indicates the inverse of the positive definite square root of the matrix M. This exists since M is positive definite. Substitution of this transformation into (3.1-1) yields

$$M^{-1/2}\ddot{\underline{y}} + CM^{-1/2}\dot{\underline{y}} + KM^{-1/2}\underline{y} = 0.$$

Pre-multiplying this expression by $M^{-1/2}$ yields

$$(3.3\text{-}1) \qquad \ddot{\underline{y}} + \tilde{C}\dot{\underline{y}} + \tilde{K}\underline{y} = \underline{0}$$

where $\tilde{C} = M^{-1/2}CM^{-1/2}$ and $\tilde{K} = M^{-1/2}KM^{-1/2}$. Note that $\tilde{C}$ and $\tilde{K}$ have retained the same symmetry and definiteness as C and K. The expression (3.3-1) is more tractable notationally than (3.1-1) but still reflects the same geometry and coefficients as the original system.

The characteristic equation associated with (3.3-1) is

$$(3.3\text{-}2) \qquad |\lambda^2 I + \lambda\tilde{C} + \tilde{K}| = 0$$

where $|\cdot|$ denotes the determinant of the indicated matrix and I is the identity matrix. The eigenvalues of (3.3-1) are the roots of (3.3-2).

Motivated by the definitions of section 3.2, the following definitions are stated:

Definition (3.3-3): The critical damping matrix denoted C_c is defined as

$$C_c = 2\tilde{K}^{1/2}.$$

Definition (3.3-4): The system described by (3.3-1) is critically damped if $\tilde{C} = C_c$.

Definition (3.3-5): The system described by (3.3-1) is overdamped if the matrix $\tilde{C}-C_c$ is positive definite.

Definition (3.3-6): The system described by (3.3-1) is underdamped if the matrix $C_c-\tilde{C}$ is positive definite.

Definition (3.3-7): The system described by (3.3-1) is said to exhibit mixed damping if the matrix $\tilde{C}-C_c$ is indefinite.

These definitions are consistent with the one degree of freedom case and it will be shown that they have similar implications. Note, that for the matrix case a fourth possibility presents itself in the form of definition (3.3-7). The following theorems and proofs complete the analogy.

Theorem (3.3-8): If (3.3-1) is critically damped then (3.3-1) has at most n negative real eigenvalues and no complex eigenvalues. Each of the modes of (3.3-1) behaves in a critically damped fashion and none of them oscillate.

Proof: Since $\tilde{C} = 2\tilde{K}^{1/2}$, (3.3-1) becomes

$$\ddot{\underline{y}} + 2\tilde{K}^{1/2}\dot{\underline{y}} + \tilde{K}\underline{y} = 0.$$

Let S be the orthogonal matrix that diagonalizes $\tilde{K}$ (note that S is the undamped modal matrix) and apply the transformation $\underline{y} = S\underline{x}$ which yields, after pre-multiplication by S^T,

$$\ddot{\underline{x}} + 2\, S^T\, \tilde{K}^{1/2} S\, \dot{\underline{x}} + S^T \tilde{K} S\, \underline{x} = 0$$

From theorems (2.1-25) and (2.1-32), it follows that

$$\Lambda = S^T\, \tilde{K}\, S,$$

and

$$\Lambda^{1/2} = S^T\, \tilde{K}^{1/2}\, S,$$

where Λ is the diagonal matrix of eigenvalues of $\tilde{K}$. Hence, the above becomes

$$\ddot{\underline{x}} + 2\Lambda^{1/2}\dot{\underline{x}} + \Lambda\underline{x} = 0$$

which is a diagonal system of n ordinary differential equations, the $i^{\underline{th}}$ equation of which is the scalar equation

$$\ddot{x}_i + 2(\Lambda^{1/2})_{ii}\, \dot{x}_i + (\Lambda)_{ii}\, x_i = 0.$$

Here Λ_{ij} denotes the i-j^{th} element of the matrix Λ. The discriminant of the characteristic equation associated with the above is

$$(2\Lambda_{ii}^{1/2})^2 - 4\Lambda_{ii} = 0.$$

Therefore, each of the n equations yields a repeated real eigenvalue. Similarity transformations such as S preserve eigenvalues. Hence, equation (3.3-1) has at most n negative real eigenvalues and no complex eigenvalues. Each mode will behave in a critically damped fashion.

Lemma (3.3-9): [7] A necessary and sufficient condition for the orthogonal modal matrix of $\tilde{K}$ to diagonalize (3.3-1) is that $\tilde{K}\tilde{C} = \tilde{C}\tilde{K}$.

Proof: This follows as a corollary of theorem (2.1-26) with $A = \tilde{C}$ and $B = \tilde{K}$ where S denotes the orthogonal modal matrix of $\tilde{K}$.

Corollary (3.3-10): If (3.3-1) is critically damped then it is diagonalized by the undamped modal matrix.

Proof: Since $\tilde{C} = 2\tilde{K}^{1/2}$, then $\tilde{C}$ and $\tilde{K}$ commute and the result follows from lemma (3.3-9).

Theorem (3.3-11): If (3.3-1) is overdamped then the eigenvalues of (3.3-1) are all negative real numbers and each mode behaves in an overdamped manner with no oscillation.

Proof: Consider the eigenvalue problem associated with (3.3-1)

$$\lambda^2\underline{y} + \lambda\tilde{C}\underline{y} + \tilde{K}\underline{y} = 0.$$

Premultiplying by $\underline{y}^*$ and solving for λ yields

(3.3-12) $$2\underline{y}^*\underline{y}\lambda = -\underline{y}^*\tilde{C}\underline{y} \pm ((\underline{y}^*\tilde{C}\underline{y})^2-4\underline{y}^*\underline{y}\underline{y}^*\tilde{K}\underline{y})^{1/2}.$$

Clearly the discriminant in this expression determines the nature of the eigenvalues λ, as $\underline{y}$ ranges through the set of eigenvectors. Motivated by this, define the form

$$D = (\underline{x}^*\tilde{C}\underline{x})^2 - 4(\underline{x}^*\underline{x}\ \underline{x}^*\tilde{K}\underline{x}),$$

for all non-zero complex vectors $\underline{x}$. Since, $\tilde{C}-2\tilde{K}^{1/2}$ is positive definite there exists a positive definite matrix P such that $\tilde{C}-2\tilde{K}^{1/2} = \varepsilon_0 P$, where ε_0 is a positive constant. Substitution of $\tilde{C}=2\tilde{K} + \varepsilon_0 P$ into this expression defines the scalar function $D(\varepsilon_0)$ by

$$D(\varepsilon_0) = [\underline{x}^*(2\tilde{K}^{1/2} + \varepsilon_0 P)\underline{x}]^2 - 4\underline{x}^*\underline{x}\ \underline{x}^*\tilde{K}\underline{x}$$

for all non-zero complex vectors $\underline{x}$. Now fix ε_0 and P. Then $\tilde{C}-2\tilde{K}^{1/2} = \varepsilon_0 P > \varepsilon P$, for all ε such that $\varepsilon_0 > \varepsilon > 0$. Now define $D(\varepsilon)$ by

$$D(\varepsilon) = 4(\underline{x}^*\tilde{K}^{1/2}\underline{x})^2 + 4\varepsilon\underline{x}^*\tilde{K}^{1/2}\underline{x}\ \underline{x}^*P\underline{x} + \varepsilon^2\ (x^*P\underline{x})^2 - 4\underline{x}^*\underline{x}\ \underline{x}^*\tilde{K}\underline{x}.$$

Note that $D(\varepsilon_0) > D(\varepsilon)$ for all ε such that $\varepsilon_0 > \varepsilon > 0$. Thus if $D(\varepsilon)$ is positive, $D(\varepsilon_0)$ will also be positive. Differentiating with respect to ε for fixed P yields

$$D'(\varepsilon) = 4\underline{x}^*\tilde{K}^{1/2}\underline{x}\ \underline{x}^*P\underline{x} + 2\varepsilon\underline{x}^*P\underline{x}.$$

Now note that if $\tilde{C}-2\tilde{K}^{1/2}$ is positive definite then $D'(\varepsilon) > 0$ for all non-zero complex vectors $\underline{x}$ since $\tilde{K}^{1/2}$ and P are positive definite and $\varepsilon > 0$. In particular, $D'(\varepsilon) > 0$ for all eigenvectors of (3.3-1).

Now consider $D(\varepsilon)$ defined on the set of all eigenvectors of (3.3-1) denoted by $\underline{y}$. When $\varepsilon=0$, $\tilde{C}=2\tilde{K}^{1/2}$ and from Corollary (3.3-10) and [7] the damped modal vectors are the eigenvectors of $\tilde{K}$. Thus

$$\begin{aligned} D(0) &= 4[(\underline{y}^*\tilde{K}^{1/2}\underline{y})^2 - \underline{y}^*\underline{y}\ \underline{y}^*\tilde{K}\underline{y}] \\ &= 4[(\underline{y}^*\lambda^{1/2}\underline{y})^2 - \underline{y}^*\underline{y}\ \underline{y}^*\lambda\underline{y}] \\ &= 4[\lambda(\underline{y}^*\underline{y})^2 - \lambda(\underline{y}^*\underline{y})^2] \\ &= 0, \end{aligned}$$

where λ is the eigenvalue associated with the eigenvector $\underline{y}$. Hence, $D(\varepsilon)$ defined on the set of eigenvectors of (3.3-1) is positive for all ε such that $\varepsilon_0 > \varepsilon > 0$ and $D(\varepsilon_0)$ is positive on this set. But ε_0 may be arbitrarily large so that the discriminant of (3.3-12) is always positive for overdamped systems.

Therefore, the eigenvalues of (3.3-1) are all negative real numbers and the eigenvectors must all be real ($\underline{y}^* = \underline{y}^T$). The sign of the eigenvalues follows from the assumption that $\tilde{C}$ is positive definite. For the overdamped case none of the modes will oscillate.

Note that if (3.3-1) is overdamped the assumption of asymptotic stability can be relaxed. Since $\tilde{C}>2\tilde{K}^{1/2}>0$, $\tilde{C}$ is positive definite and hence all of the eigenvalues will have negative real parts and the system is already asymptotically stable.

Theorem (3.3-13): If (3.3-1) is underdamped then the eigenvalues of (3.3-1) are all complex conjugate pairs with negative real parts and each of the modes oscillates in damped harmonic motion.

Proof: The definition of underdamping implies that

$$\underline{x}^T(2\tilde{K}^{1/2}-C)\underline{x} > 0$$

for all real $\underline{x}$. By theorem (2.1-11) this becomes

$$2\underline{x}^* \tilde{K}^{1/2}\underline{x} > \underline{x}^* \tilde{C}\underline{x}$$

for all complex non-zero vectors $\underline{x}$. $\tilde{C}$ is positive semi-definite so squaring the above yields

$$4(\underline{x}^* \tilde{K}^{1/2}\underline{x})^2 > (\underline{x}^* C\underline{x})^2$$

The Cauchy-Schwarz inequality (2.1-3) with $\underline{x}=\underline{x}$ and $\underline{y}=\tilde{K}^{1/2}\underline{x}$ yields

$$\underline{x}^*\underline{x}\ \underline{x}^*\ \tilde{K}\underline{x} > (\underline{x}^*\ \tilde{K}^{1/2}\underline{x})^2.$$

Combining the last two inequalities yields

$$(\underline{x}^*C\underline{x})^2 - 4\underline{x}^*\underline{x}\ \underline{x}^*\ \tilde{K}\underline{x} < 0$$

for all complex vectors $\underline{x}$. Thus the discriminant of (3.3-13) is always negative and all the eigenvalues of (3.3-1) must appear in complex conjugate pairs. The real part of λ is negative or zero via the definiteness condition on $\tilde{C}$. However, zero is excluded as a possibility by the assumption of asymptotic stability. Hence, each mode will be a damped oscillation.

Note that the above theorem shows that the eigenvectors of an underdamped system are complex. For a physical interpretation of these eigenvectors as modes consider the real part of $\underline{y}$ in the following sense. Suppose the eigenvector $\underline{y}$ is of the form

$$\underline{y} = \underline{\zeta}e^{\lambda t}$$

where $\underline{\zeta}$ is a vector of constants and λ is a complex scalar i.e.,

$$\lambda = \mu + j\omega,$$

where $j = \sqrt{-1}$. The physical displacement from equilibrium is given by the real part of $\underline{y}$ denoted $R(\underline{y})$. The velocity is given by $R(\dot{\underline{y}})$, and the acceleration by $R(\ddot{\underline{y}})$.

Theorem (3.3-14): Let $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$, then (3.3-1) exhibits mixed damping if and only if there is at least one real eigenvalue and at least one complex eigenvalue. At least one mode will oscillate and one will mode will not.

Proof: Since $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$ there exists an orthogonal transformation S such that

$$\Lambda_c = S^T\tilde{C}S$$

and

$$\Lambda_K = S^T\tilde{K}S$$

are both diagonal (lemma 3.3-9). Hence, the substitution $\underline{y}=S\underline{x}$ in (3.3-1) followed by premultiplying by S^T yields the diagonal system

$$\ddot{\underline{x}} + \Lambda_c\dot{\underline{x}} + \Lambda_k\underline{x} = 0.$$

Since $(\tilde{C}-C_c) \sim (\Lambda_c-2\Lambda_k^{1/2})$, and $(\Lambda_c-2\Lambda_K^{1/2})$ is diagonal, $(\tilde{C}-C_c)$ is indefinite if and only if there is one value of i such that

$$(\Lambda_c - 2\Lambda_k^{1/2})_{ii} < 0$$

and another value of i such that

$$(\Lambda_c - 2\Lambda_k^{1/2})_{ii} > 0.$$

Now note that the i^{th} pair of eigenvalues is found from

$$2\lambda_i = -(\Lambda_c)_{ii} \pm [(\Lambda_c)_{ii}^2 - 4(\Lambda_k)_{ii}]^{1/2}.$$

There will be one complex λ and one real λ if and only if there is one value of i such that

$$(\Lambda_c)_{ii}^2 - 4(\Lambda_k)_{ii} < 0$$

and one value of i such that

$$(\Lambda_c)_{ii}^2 - 4(\Lambda_k)_{ii} > 0.$$

That these last two inequalities are the same as the previous pair follows from the fact that each matrix is diagonal with nonnegative elements, since $\tilde{C}$ is positive semi-definite and $\tilde{K}$ is positive definite.

Except for this last theorem, the definiteness of $(\tilde{C}-2\tilde{K}^{1/2})$ is in general only sufficient to determine the nature of the solution of (3.3-1). However, all of the above theorems become both necessary and sufficient for the special class of problems in which the damping matrix can be diagonalized by the undamped modal matrix transformation. Thus for systems such that $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$ the theory is complete. This special case is of practical importance in as much as in many cases the damping matrix is unknown and hence modeled as a matrix which can be diagonalized by the undamped modal matrix.

3.4 Examples

Several two degree of freedom examples serve to illustrate the validity of the above results.

Consider the theorem for critically damped systems. In equation (3.3-1) let

$$\tilde{K} = \begin{bmatrix} 2.5 & 1.25 \\ 1.25 & 1.25 \end{bmatrix}$$

which is positive definite. Then

$$C_c = 2\tilde{K}^{1/2} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

which is also positive definite. Choosing $\tilde{C}=C_c$ the associated eigenvalue problem is

$$\left\{ \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} + \begin{bmatrix} 3\lambda & \lambda \\ \lambda & 2\lambda \end{bmatrix} + \begin{bmatrix} 2.5 & 1.25 \\ 1.25 & 1.25 \end{bmatrix} \right\} \underline{x} = 0$$

The characteristic equation is

$$\lambda^4 + 5\lambda^3 + 8.75\lambda^2 + 6.25\lambda + 1.5625 = 0$$

which has the following roots

$$\lambda_{1,2} = -.690983005,$$
$$\lambda_{3,4} = -1.809016994.$$

Thus, there are at most n=2 negative real roots as predicted by theorem (3.3-8). Also note that $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$.

Next consider the theorem for underdamped systems. Here the most interesting examples are those which do not diagonalize by the undamped modal matrix transformation. Hence, this example is chosen so that $\tilde{C}\tilde{K}\neq\tilde{K}\tilde{C}$. In (3.3-1) let

$$\tilde{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$\tilde{K} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$C_c = 2\tilde{K}^{1/2} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that $\tilde{C}$ is positive semi-definite while $\tilde{K}$ and C_c are positive definite. The system is underdamped since

$$C_c - \tilde{C} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} ,$$

which is positive definite. The characteristic equation becomes

$$\lambda^4 + 2\lambda^3 + 5\lambda^2 + 5\lambda + 4 = 0$$

which has roots

$$\lambda_{1,2} = -.754042874 \pm .911291349j$$

and

$$\lambda_{3,4} = -.245957125 \pm 1.672908736j$$

where $j = \sqrt{-1}$. All the roots are complex conjugate pairs in agreement with the results stated in theorem (3.3-15).

For the last example consider the overdamped case. Again a system is considered such that $\tilde{C}\tilde{K} \neq \tilde{K}\tilde{C}$. Let

$$\tilde{K} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \tilde{C} = \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}$$

so that

$$C_c = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$\tilde{C}-C_c = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

which is positive definite and theorem (3.3-12) indicates that the eigenvalues should all be negative real numbers. The characteristic equation is

$$\lambda^4 + 9\lambda^3 + 22\lambda^2 + 18\lambda + 4 = 0$$

which has roots

$$\lambda_1 = -.354248688,$$
$$\lambda_2 = -1.0,$$
$$\lambda_3 = -2.0,$$
$$\lambda_4 = -5.645751311,$$

all negative real numbers as predicted.

In using the theorems of this chapter the following comments may be useful. The definiteness of $(\tilde{C}-C_c)$ can easily be checked by examining the determinant of each of its minors as indicated in theorem (2.1-12). Another check is to calculate the eigenvalues of the matrix $(\tilde{C}-C_c)$ (theorem 2.1-9). This involves calculating the eigenvalues of an nxn array as opposed to a 2n x 2n array needed to solve the full problem. The square root of $\tilde{K}$ can be found by finding the eigenvalues and eigenvectors of $\tilde{K}$ and using theorem (2.1-32), Newton's method or a generalization of Newton's method given by [19]. For an easier first check which avoids calculating $\tilde{K}^{1/2}$, one can look at the definiteness of

$(\tilde{C}^2-4\tilde{K})$ since an application of theorem (2.1-33) yields that; $(\tilde{C}^2-4\tilde{K})$ positive definite implies $(\tilde{C}-2\tilde{K}^{1/2})$ is positive definite and; $(4\tilde{K}-\tilde{C}^2)$ positive definite implies that $(2\tilde{K}^{1/2}-\tilde{C})$ is positive definite. Also, $\tilde{C}^2 = 4\tilde{K}$ if and only if $\tilde{C} = 2\tilde{K}^{1/2}$. However, if $(\tilde{C}^2 - 4\tilde{K})$ is indefinite $(\tilde{C}-C_c)$ should still be checked since it yields stronger results than those based on the definiteness of $(\tilde{C}^2-4\tilde{K})$.

More examples are considered in the next chapter where the theorems stated here are used as a design tool.

3.5 Comparison with Previous Work

Duffin's definition states that an overdamped system is one such that

$$(\underline{x}^T\tilde{C}\underline{x})^2 > 4\underline{x}^T\underline{x}\ \underline{x}^T\tilde{K}\underline{x}$$

for all real non-zero vectors $\underline{x}$. However, it was shown in the proof of theorem (3.3-12) that this condition, when restricted to the set of eigenvectors of (3.3-1), follows from the definition of overdamping stated here. It should also be noted that Duffin's condition is difficult to check given a specific system.

Nicholson's definition of underdamping states that a system is underdamped if all the modes of (3.3-1) are underdamped. He then states that a sufficient condition for (3.3-1) to be underdamped is for

$$c_1 \leq 2\sqrt{k_m}$$

where c_1 is the largest eigenvalue of the matrix $\tilde{C}$ and k_m is the smallest eigenvalue of the matrix $\tilde{K}$. This test requires substantial calculation since it involves finding the eigenvalues of both $\tilde{C}$ and $\tilde{K}$. Müller improves this result and extends it by showing that a sufficient condition for (3.3-1) to be underdamped is for $(4\tilde{K}-\tilde{C}^2)$ to be positive definite. If $\tilde{C}\tilde{K} = \tilde{K}\tilde{C}$, Müller's condition is equivalent to the one stated here. To see this in one direction substitute $\tilde{C}$ and $\tilde{K}$ into theorem (2.1-33). The other direction follows from theorem (2.1-34) with $k=2$.

Each of the above mentioned authors discusses only one type of damping, i.e., either the overdamped case or the underdamped case, but not both. The results reported here are complete and stronger than those of other researchers.

Chapter 4

APPLICATIONS OF LUMPED PARAMETER THEORY

In this chapter the results of the previous chapter are applied in two ways. The first section applies the theorems of section 3.3 to the theory of forced vibrations of lumped parameter systems. The second section consists of an example illustrating how the theorems of section 3.3 can be used to design a given structure so that it will have the desired type of eigenvalues.

4.1 Implications for the Forced Problem

The theorems of chapter 3 have some interesting implications for systems of the form

$$(4.1\text{-}1) \qquad \ddot{\underline{x}}(t) + \tilde{C}\dot{\underline{x}}(t) + \tilde{K}\,\underline{x}(t) = \underline{f}(t)$$

where $\underline{x}(t)$, $\tilde{C}$ and $\tilde{K}$ are as defined in chapter 3, and $\underline{f}(t)$ represents an n-vector of applied external forces which are arbitrary.

Lancaster [2] as well as others have shown that if

$$(4.1\text{-}2) \qquad D_2(\lambda) = \lambda^2 I + \tilde{C}\lambda + \tilde{K}$$

is simple then the solution of (4.1-1) is given by

$$(4.1\text{-}3) \qquad \underline{x}(t) = \sum_{j=1}^{2s} \underline{q}_j e^{-\lambda_j t} \int_0^t e^{-\lambda_j \tau}\, \underline{q}^T{}_j\, \underline{f}(\tau)d\tau$$

$$+ 2 \sum_{j=2s+1}^{n+s} \int_0^t R\{e^{-\lambda_j(t-\tau)}\, \underline{q}_j \underline{q}_j^*\}\underline{f}(\tau)d\tau$$

where λ_j are the 2n eigenvalues of $D_2(\lambda)\underline{x} = 0$; the real roots of $\det(D_2(\lambda)) = 0$ are numbered 1 through 2s, the complex conjugate pairs are numbered 2s + 1 through 2n, $\underline{q}_j$ indicates the eigenvector associated with λ_j and $R\{\cdot\}$ indicates the real part of $\{\cdot\}$.

The intent of this section is to investigate when statements similar to (4.1-3) can be made without assuming a simple structure for $D_2(\lambda)$. To this end consider the following lemma which is needed to prove the main result for overdamped systems.

Lemma (4.1-4): If the homogeneous system corresponding to (4.4-1) is overdamped then $D_2(\lambda)$ is simple.

Proof: From theorem (2.2-4), $D_2(\lambda)$ is degenerate if and only if there exists an eigenvector $\underline{q}$ associated with the eigenvalue λ such that

$$\underline{r}^T[2\lambda I + \tilde{C}]\underline{q} = 0$$

for all latent vectors r associated with λ. Since I, $\tilde{C}$ and $\tilde{K}$ are symmetric $\underline{r} = \underline{q}$ and this becomes

$$\underline{q}^T[2\lambda I + \tilde{C}]\underline{q} = 0$$

for all eigenvectors $\underline{q}$ associated with the eigenvalue λ. Since λ is an eigenvalue it must satisfy

$$\lambda^2\underline{q}^T\underline{q} + \lambda\underline{q}^T\tilde{C}\underline{q} + \underline{q}^T\tilde{K}\underline{q} = 0.$$

Solving this for λ yields

$$(4.1\text{-}5) \quad 2\underline{q}^T\underline{q}\lambda + \underline{q}^T\tilde{C}\underline{q} = \pm[(\underline{q}^T\tilde{C}\underline{q})^2 - 4\underline{q}^T\underline{q}\ \underline{q}^T\tilde{K}\underline{q}]^{1/2}$$

Since the system here is overdamped theorem (3.3-11) shows that λ and $\underline{q}$ are both real and that the right hand side of (4.1-4) is non-zero. Thus, theorem (2.2-4) is violated and $D_2(\lambda)$ is simple.

The following theorem can now be stated for arbitrarily forced overdamped systems.

Theorem (4.1-6): If the homogeneous equation associated with (4.1-1) is overdamped then the solution for any $\underline{f}(t)$ of (4.1-1) is given by

$$\underline{x}(t) = \sum_{j=1}^{2n} \underline{q}_j e^{-\lambda_j t} \int_0^t e^{-\lambda_j \tau} \underline{q}_j^T \underline{f}(\tau) d\tau.$$

Proof: Since (4.4-1) is overdamped all its eigenvalues are real by theorem (3.3-12) so that s=n in equation (4.1-3). Lemma (4.1-4) yields the required simplicity, thus this result follows from (4.1-3).

Next consider the (special) case that occurs when the undamped modal matrix diagonalizes the damping matrix, i.e., when $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$. Again a lemma is needed before the result for underdamping can be stated.

Lemma (4.1-7): If $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$, then $D_2(\lambda)$ is simple.

Proof: If $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$, then by lemma (3.3-9), the eigenvectors of $\tilde{K}$ are syst

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Proof

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those of $D_2(\lambda)$ and they form n linearly independent vectors since $\tilde{K}$ is symmetric. Therefore theorem (2.1-20) applies and $D_2(\lambda)$ is simple.

Now a concise statement can be made about forced underdamped systems in the special case where the undamped modal vectors decouple the system.

Theorem (4.1-8): If the homogeneous equation associated with (4.1-1) is underdamped and if $\tilde{C}\tilde{K}=\tilde{K}\tilde{C}$, then the solution of (4.1-1) for any $\underline{f}(t)$ is given by

$$\underline{x}(t) = 2\sum_{j=1}^{n} {}_0\int^t R\{e^{-\lambda_j(t-\tau)} \underline{q}_j\underline{q}_j^*\underline{f}(\tau)\}d\tau.$$

Proof: Lemma (4.1-7) yields the simplicity of $D_2(\lambda)$ and expression (4.1-3) yields the solution. The underdamping condition implies that all the eigenvalues are complex by theorem (3.3-15). Hence, s=0 and the theorem follows.

One is tempted to try and prove theorem (4.1-8) for the case where $\tilde{C}\tilde{K}\neq\tilde{K}\tilde{C}$ by mimicking the proof of lemma (4.4-4). However a counter example exists to this conjecture as pointed out by Lancaster [20]. For completeness, it is presented here.

Consider $D_2(\lambda)$ with

$$\tilde{C} = \begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 2 \end{bmatrix}$$

and

$$\tilde{K} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

so that

$$2\tilde{K}^{1/2} - \tilde{C} = \begin{bmatrix} 1 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 2 \end{bmatrix}$$

which is positive definite, so that this $D_2(\lambda)$ is underdamped. Also, note that $\tilde{C}\tilde{K} \neq \tilde{K}\tilde{C}$. The characteristic equation is

$$\det(D_2(\lambda)) = \lambda^4 + 3\lambda^3 + \frac{25}{4}\lambda^2 + 6\lambda + 4$$

$$= (\lambda + 3 + \sqrt{23}j)^2(\lambda + 3 - \sqrt{23}j)^2/4^4$$

Now consider the repeated root $\lambda_1 = (-3 + \sqrt{23}j)/4$. Then

$$D_2(\lambda_1) = \frac{1}{16} \begin{bmatrix} -10-2\sqrt{23}j & -6\sqrt{3}+2\sqrt{69}j \\ -6\sqrt{3}+2\sqrt{69}j & 26+2\sqrt{23}j \end{bmatrix}$$

so that

$$\det D_2(\lambda_1) = -184-38\sqrt{23}j \neq 0.$$

Thus, the rank of $D_2(\lambda_1)$ is 2. From the definition of simple, $D_2(\lambda)$ is simple if and only if the rank of $D_2(\lambda)$ is $n-\alpha = 2-2 = 0$. Hence, the underdamping condition by itself is not enough to guarantee that $D_2(\lambda)$ has a simple structure.

4.2 System Design

The well known result that a matrix is positive definite if and only if the determinant of each principal minor is positive allows the theorems of chapter 3 to be used as design tools. That is, the conditions stated here can be used to choose the values of the masses, spring constants and damping coefficients so that all the systems modes will oscillate or not as desired.

To illustrate this use, consider the following two degree of freedom system.

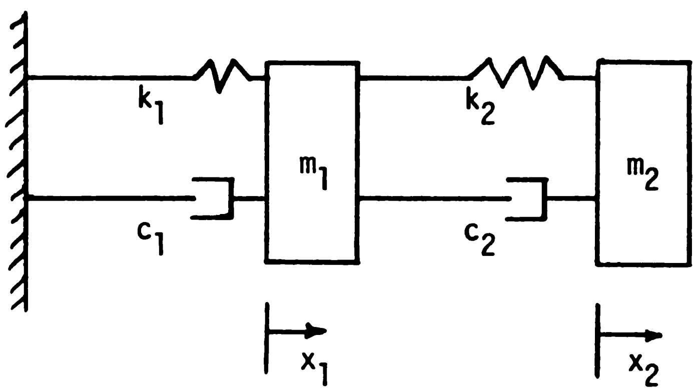

Fig. 1

The equations of motion are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\underline{x}} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \dot{\underline{x}} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \underline{x} = 0.$$

For the sake of simplicity assume $m_1 = m_2 = 1$. Then

$$\tilde{C}^2 = \begin{bmatrix} 2c_1^2+2c_1c_2+c_2^2 & -2c_1^2-c_1c_2 \\ -2c_1^2-c_1c_2 & 2c_1^2 \end{bmatrix}$$

and

$$\tilde{C}^2-4\tilde{K} = \begin{bmatrix} 2c_1^2+2c_1c_2+c_2^2-4(k_1+k_2) & 4k_1-2c_1^2-c_1c_2 \\ 4k_1-2c_1^2-c_1c_2 & 2(c_1^2-2k_1) \end{bmatrix}$$

Now attempt to choose c_1, c_2, k_1 and k_2 so that $\tilde{C}^2-4\tilde{K}$ is positive definite, i.e., so that $\tilde{C}-2\tilde{K}^{1/2}$ is positive definite. Then the system will be overdamped. The determinant condition yields the following inequalities which must be satisfied

$$2c_1^2 + 2c_1c_2 + c_2^2 > 4(k_1+k_2)$$

and

$$2(c_1^2 -2k_1)(2c_1^2 + 2c_1c_2+c_2^2) > (4k_1-2c_1^2-c_1c_2)^2,$$

which cannot be satisfied unless

$$c_1^2 > 2k_1.$$

One possible solution is

$$c_1 = 4 \qquad k_1 = 1$$
$$c_2 = 5 \qquad k_2 = 2.$$

This yields

$$\tilde{C} = \begin{bmatrix} 9 & -4 \\ -4 & 4 \end{bmatrix} \quad \text{and } \tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} .$$

It is interesting to note that for this choice

$$\tilde{C}\tilde{K} = \begin{bmatrix} 31 & -13 \\ -16 & 8 \end{bmatrix} \neq \begin{bmatrix} 31 & -16 \\ -13 & 8 \end{bmatrix} = \tilde{K}\tilde{C}.$$

Also note that

$$\tilde{C}^2 - 4\tilde{K} = \begin{bmatrix} 85 & -48 \\ -48 & 28 \end{bmatrix}$$

which is certainly positive definite. Thus, according to the definitions of chapter 3 this system is overdamped.

The eigenvalues for this system are found from

$$\det[D_2(\lambda)] = \lambda^4 + 13\lambda^3 + 24\lambda^2 + 13\lambda + 2 = 0,$$

which yields

$\lambda_1 = -0.2662$

$\lambda_2 = -0.5323$

$\lambda_3 = -1.2941$

$\lambda_4 = -10.9074.$

Thus, applying the overdamping condition allows the system in figure 1 to be designed in such a way that each of its modes are overdamped.

Suppose now that it is desired to design the system in figure 1 in such a way that each mode will oscillate. Enforcing the underdamping condition yields the following inequalities for c_1, c_2, k_1 and k_2.

$$4(k_1+k_2) > 2c_1^2 + 2c_1c_2 + c_2^2$$

and

$$4(k_1-2c_1^2-c_1c_2)^2 > 2(c_1^2-2k_1)(2c_1^2+2c_1c_2+c_2^2).$$

One solution of these inequalities is

$$c_1=1,\ c_2=2,\ k_1=4 \text{ and } k_2=1.$$

The underdamping condition is satisfied since

$$4\tilde{K}-\tilde{C}^2= \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$$

which is positive definite so that $2\tilde{K}^{1/2}-\tilde{C}$ is positive definite.
Again, note that

$$\tilde{C}\tilde{K} = \begin{bmatrix} 16 & -5 \\ -6 & 2 \end{bmatrix} \neq \begin{bmatrix} 16 & -6 \\ -5 & 2 \end{bmatrix} = \tilde{K}\tilde{C}.$$

The eigenvalues for this system are found from

$$\det[D_2(\lambda)] = 0 = \lambda^4 + 4\lambda^3 + 8\lambda^2 + 6\lambda + 4 = 0,$$

and have the values

$$\lambda_{1,2} = -0.338 \pm 0.8327j,$$

$$\lambda_{3,4} = -1.6662 \pm 1.4813j,$$

so that each mode will oscillate as desired.

Now consider an attempt to design the system in figure 1 so that each mode is critically damped. Note that in this case since the damping and stiffness matrix commute the definition becomes both necessary and sufficient. It is more interesting to allow the masses to be chosen. Hence

$$\tilde{C} = M^{-1/2}CM^{-1/2} = \begin{bmatrix} \dfrac{c_1+c_2}{m_1} & \dfrac{-c_2}{\sqrt{m_1m_2}} \\ \dfrac{-c_2}{\sqrt{m_1m_2}} & \dfrac{c_2}{m_2} \end{bmatrix}$$

and

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{\sqrt{m_1 m_2}} \\ \frac{-k_2}{\sqrt{m_1 m_2}} & \frac{k_2}{m_2} \end{bmatrix} .$$

Then

$$\tilde{C}^2 = \begin{bmatrix} \frac{(c_1+c_2)^2}{m_1^2} + \frac{c_2^2}{m_1 m_2} & \frac{-c_1c_2-c_2^2}{m_1\sqrt{m_1m_2}} - \frac{c_2^2}{m_2\sqrt{m_1m_2}} \\ \frac{-c_1c_2-c_2^2}{m_1\sqrt{m_1m_2}} - \frac{c_2^2}{m_2\sqrt{m_1m_2}} & \frac{c_2^2}{m_2^2} + \frac{c_2^2}{m_1 m_2} \end{bmatrix}$$

Demanding that $\tilde{C}^2=4K$ yields the following three equations for the six design parameters

(4.2-1) $$\frac{c_1^2+c_2^2+2c_1c_2}{m_1} + \frac{c_2^2}{m_2} = 4(k_1+k_2),$$

(4.2-2) $$\frac{c_2^2}{m_2} + \frac{c_2^2+c_1c_2}{m_1} = 4k_2,$$

and

(4.2-3) $$\frac{c_2^2}{m_2} + \frac{c_2^2}{m_1} = 4k_2.$$

provided both m_1 and m_2 are non-zero. Also note that there are six other conditions on the parameters, namely that they all be positive. In total the six parameters must satisfy nine conditions.

The last two equations imply that $c_1c_2 = 0$. Suppose $c_1=0$. Then (4.2-1) yields

$$\frac{c_2^2}{m_1} + \frac{c_2^2}{m_2} = 4(k_1+k_2).$$

But (4.2-2) yields

$$\frac{c_2^2}{m_1} + \frac{c_2^2}{m_2} = 4k_2$$

so that k_1 must also be zero. This eliminates one degree of freedom, hence, $c_1 \neq 0$.

Now suppose $c_2 = 0$. Then equation (4.2-3) requires $k_2=0$. Thus the system in figure 1 cannot be critically damped in both modes at once.

One might expect that this should not be the case since a critically damped system can be decoupled by the undamped modal matrix. However, while the undamped modal matrix transformation yields separate equations for each mode, the coefficients do not uncouple. For example the equation for $x_1(t)$ will have coefficients involving all of the parameters m_1, m_2, k_1, k_2, c_1 and c_2. Thus forcing x_1 to be critically damped puts further constraints on m_2, k_2 and c_2.

The design process illustrated here is limited by ones ability to solve n nonlinear algebraic inequalities in 3n variables subject to 3n constraints (each parameter must be positive).

Chapter 5

SOME RESULTS FOR DISTRIBUTED PARAMETER SYSTEMS

5.1 Problem Description

This chapter examines distributed parameter systems which can be described by linear partial differential equations of the following form

$$(5.1\text{-}1)\qquad u_{tt}(x,t) + L_1 u_t(x,t) + L_2 u(x,t) = 0, \text{ in } \Omega,$$

where Ω is a bounded open region and $u(x,t)$ is a function of the spatial coordinate $x = (x_1,x_2,x_3)$. The independent variable is time, denoted by t, and the subscript t indicates partial differentiation with respect to time. The operators L_1 and L_2 are self-adjoint spatial differential operators independent of t. In addition, the solution to (5.1-1) is subject to spatial boundary conditions denoted by

$$(5.1\text{-}2)\qquad B(u) = 0 \text{ on } \partial\Omega$$

where $\partial\Omega$ is the boundary of Ω, and initial conditions denoted by

$$(5.1\text{-}3)\qquad I(u) = f(x) \text{ at } t = 0.$$

Many physical problems can be described by this general form. For instance with L_1 a positive constant and L_2 defined by

$$L_2 = -\nabla^2 = -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}),$$

equation (5.1-1) describes the transverse vibrations of a membrane with external damping such as air resistance. Other examples are available from string, beam, membrane and plate theory.

The problem of interest here is to derive conditions on L_1 and L_2 which are easily checked and which indicate whether or not the solution of (5.1-1) will be oscillatory in time.

5.2 Basic Assumptions

Results similar to those derived for the lumped parameter case can be formulated for the distributed parameter case if certain restrictions are imposed. The operators L_1 and L_2 are taken to be positive definite self-adjoint operators with inverses that are Hilbert-Schmidt operators. Furthermore, they must have these properties on the same domain. Let n_1, be the order of L_1 and n_2 the order of L_2 and define this domain as follows:

Definition (5.2-1): The domain D(L) is defined to be the set of all functions u(x) in $\mathcal{L}_2(\Omega)$ satisfying the spatial boundary conditions (5.1-3) and having derivatives in $\mathcal{L}_2(\Omega)$ of order $\max(2n_1, n_2)$.

Note that the functions in D(L) may be smoother than the solution of (5.1-1) requires. The set W(L) will denote all those functions in D(L) with $||u|| = 1$.

It is further assumed that any solution of (5.1-1) can be written as

$$(5.2\text{-}2) \qquad u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

where the set of spatial functions $\{\phi_n(x)\}_{n=1}^{\infty}$ form a complete orthonormal set of real functions in $L_2(\Omega)$.

5.3 Definitions

Motivated by the definitions for the lumped parameter case, it is tempting to define critical damping, overdamping, underdamping and mixed damping in terms of the operator $(L_1 - 2L_2^{1/2})$. However, the operator L_2 may be unbounded and the literature does not offer useful techniques to compute the positive definite square root of a positive definite unbounded operator. Thus, the following definitions are formulated in terms of the operator $(L_1^2 - 4L_2)$, which is straightforward to compute.

Definition (5.3-1): The system described by (5.1-1) is said to be critically damped if $L_1^2 = 4L_2$ on D(L).

Definition (5.3-2): The system described by (5.1-1) is said to be overdamped if the operator $(L_1^2 - 4L_2)$ is positive definite on D(L).

Definitior (5.3-3): The system described by (5.1-1) is said to be underdamped if the operator $(4L_2 - L_1^2)$ is positive definite on D(L).

Definition (5.3-4): The system described by (5.1-1) is said to exhibit mixed damping if the operator $(L_1^2 - 4L_2)$ is indefinite on D(L).

As in the lumped parameter case there is a special class of problems that occur when the eigenfunctions of the undamped problem ($L_1 \equiv 0$) are also the eigenfunctions of the damped problem. This class of problems was termed "classical" by Caughey and O'Kelley [7] and is stated as a definition here.

Definition (5.3-5): The system described by (5.1-1) is said to possess classical normal modes if the eigenfunctions for the related undamped problem ($L_1 = 0$) are also eigenfunctions of (5.1-1).

If the system described by (5.1-1) does not possess classical normal modes it will be referred to as a non-classical system. Likewise, if definition (5.3-5) is satisfied the system will be called classical.

5.4 Results

In this section results similar to those of section (3.3) will be derived that will indicate something about the nature of the functions $a_n(t)$ in (5.2-2). First, consider two results due to Caughey and O'Kelley. The proofs can be found in [7].

Theorem (5.4-1): The system described by (5.1-1) possesses classical normal modes if and only if the operators L_1 and L_2 commute on

$D(L)$, if and only if L_1 and L_2 have a common set of eigenfunctions.

Theorem (5.4-2): If the system described by (5.1-1) possesses classical normal modes, then the functions $a_n(t)$ in (5.2-2) are solutions of the initial value problems given by

$$\ddot{a}_n(t) + \langle \phi_n, L_1\phi_n \rangle \, \dot{a}_n(t) + \langle \phi_n, L_2\phi_n \rangle \, a_n(t) = 0$$

$$I(a_n) = 0 \text{ at } t = 0, \text{ for } n = 1, 2 \ldots$$

where $\phi_n = \phi_n(x)$ are the spatial eigenfunctions of the undamped problem, and "$\cdot$" indicates differentiation with respect to time.

Thus, when a system possesses classical normal modes, the solution can be calculated term by term from the eigenfunctions of the undamped spatial eigenvalue problem. The following theorem illustrates a special situation for systems with classical normal modes.

Theorem (5.4-3): If the system described by (5.1-1) is critically damped then each of the functions $a_n(t)$ are critically damped and the solution $u(x,t)$ decays exponentially in time without oscillation.

Proof: Suppose $L_1^2 = 4L_2$ on $D(L)$. Then L_1 and L_2 of (5.1-1) commute on $D(L)$ and by theorem (5.4-2), each $a_n(t)$ is a solution of the initial value problem

$$\ddot{a}_n(t) + \langle\phi_n, L_1\phi_n\rangle\, \dot{a}_n(t) + \frac{1}{4}\langle\phi_n, L_1^2\,\phi_n\rangle\, a_n(t) = 0.$$

By theorem (5.2-1), the ϕ_n are eigenfunctions of L_1 so that

$$\langle\phi_n, L_1\,\phi_n\rangle = \lambda_n\,\langle\phi_n, \phi_n\rangle = \lambda_n$$

since the $\phi_n(x)$ are orthonormal. Also since L_1 is self-adjoint

$$\begin{aligned}\langle\phi_n, L_1^2\phi_n\rangle &= \langle L_1\,\phi_n, L_1\phi_n\rangle \\ &= \langle\lambda_n\,\phi_n, \lambda_n\phi_n\rangle \\ &= \lambda_n^2.\end{aligned}$$

Substitution of these last two calculations into the expression for $a_n(t)$ yields

$$\ddot{a}_n(t) + \lambda_n\dot{a}_n(t) + \frac{\lambda_n^2}{4}\, a_n(t) = 0,$$

for each $a_n(t)$. Trivially,

$$a_n(t) = e^{-(\lambda_n/2)t}\,(A_n t + B_n)$$

where A_n and B_n are constants determined by the initial conditions $I(u) = 0$, at $t = 0$. Since the operator L_1 is assumed to be positive definite the eigenvalues, λ_n, are all positive.

Thus, each of the functions $a_n(t)$ will be critically damped and $u(x,t)$ is an exponentially decaying function of time without ocillation.

Something can be said about the oscillatory nature of the functions $a_n(t)$ even when the system does not possess classical normal modes. To this end consider solutions to (5.1-1) of the form

$$u_n(x,t) = a_n(t)\ \phi_n(x)$$

where the $\phi_n(x)$ are taken from the complete set of orthonormal functions of (5.2-2), and in general are not necessarily classical normal modes. Substitution of this form into (5.1-1) yields

$$\text{(5.4-4)} \quad \ddot{a}_n(t)\phi_n(x) + \dot{a}_n(t)\ L_1\phi_n(x) + a_n(t)\ L_2\phi_n(x) = 0$$

Multiplying this expression by $\phi_n(x)$ and integrating over Ω yields, assuming $||\phi_n|| = 1$,

$$\text{(5.4-5)} \quad \ddot{a}_n(t) + \dot{a}_n(t)\ \langle\phi_n,\ L_1\phi_n\rangle + a_n(t)\ \langle\phi_n,\ L_2\ \phi_n\rangle = 0.$$

Note that the functions $\phi_n(x)$ in (5.4-5) are not necessarily eigenfunctions of L_1 and L_2 but rather functions taken from a complete set of orthonormal functions. The product function $a_n(t)\phi_n(x)$ is a solution of (5.1-1) if $a_n(t)$ satisfies (5.4-5). Since the system is linear the function

$$u(x,t) = \sum_{n}^{\infty} a_n(t)\ \phi_n(x)$$

is also a solution of (5.1-1). Convergence is guaranteed by the completeness of the $\phi_n(x)$.

Theorem (5.4-6): If the system described by (5.1-1) is overdamped then each of the functions $a_n(t)$ are overdamped and the solution $u(x,t)$ decays exponentially in time without oscillation.

Proof: Consider equation (5.4-5). The related characteristic equation has roots

$$\text{(5.4-7)} \quad 2r_{1,2} = -\langle\phi_n, L_1\phi_n\rangle \pm [\langle\phi_n, L_1\phi_n\rangle^2 - 4\langle\phi_n, L_2\phi_n\rangle]^{1/2}$$

The values of r_1 and r_2 clearly determine the oscillatory nature of $a_n(t)$. In order to analyze the nature of $r_{1,2}$ consider the scalar d defined on the set of all functions $\phi(x)$ in W(L) by

$$\text{(5.4-8)} \quad d(\phi) = \langle\phi, L_1\phi\rangle^2 - 4\langle\phi, L_2\phi\rangle$$

and investigate the sign of $d(\phi)$.

Since $(L_1^2 - 4L_2)$ is positive definite on W(L), the operator $(L_1 - 2L_2^{1/2})$ is also positive definite on W(L) by lemma (2.3-19). Thus, there exists a positive scalar ε_0 and a positive definite operator P defined on W(L) such that

$$\text{(5.4-9)} \quad L_1 - 2L_2^{1/2} = \varepsilon_0 P$$

on W(L). Now suppose ε_0 and P are chosen so that ε_0 is arbitrarily large and consider any ε such that $\varepsilon_0 > \varepsilon > 0$. Then for all ϕ in W(L)

$$<\phi,(L_1-2L_2^{1/2})\phi< \; = \varepsilon_0<\phi,P\Phi> \; > \varepsilon <\phi,P\phi>$$

where P is fixed and ε takes any value in the interval $(0,\varepsilon_0)$. Rearranging this inequality yields

$$<\phi,L_1\phi> \; > 2<\phi,L_2^{1/2}\phi> + \varepsilon<\phi,P\phi> \; > 0.$$

Squaring this inequality and subtracting $4<\phi,L_2\phi>$ yields upon comparison with (5.4-8)

$$d(\phi) = \varepsilon^2<\phi,P\phi> + 4\varepsilon<\phi,P\phi><\phi, L_2^{1/2}\phi> + 4<\phi, L_2^{1/2}\phi>^2$$

$$- 4 <\phi, L_2\phi>.$$

Define $d_\varepsilon(\phi)$ as the expression on the right and the above becomes

$$d(\phi) > d_\varepsilon(\phi) \; .$$

for all ϕ in W(L) and ε such that $\varepsilon_0 > \varepsilon > 0$. If $d_\varepsilon(\phi)$ is positive for all ϕ in W(L) then $d(\phi)$ will be since ε_0 is arbitrary. To this end consider the sign of $d_\varepsilon(\phi)$.

The derivative of $d_\varepsilon(\phi)$ with respect to ε for fixed P is

$$d'_\varepsilon(\phi) = 2\varepsilon\langle\phi,P\phi\rangle + 4\langle\phi,P\phi\rangle\langle\phi,L_2^{1/2}\phi\rangle$$

which is obviously positive for all functions $\phi(x)$ in $W(L)$, for all $\varepsilon > 0$. In particular $d'_\varepsilon(\phi) > 0$ for all functions $\phi_n(x)$ in the expansion of (5.2-2).

Now consider the evaluation of d_ε at $\varepsilon = 0$. By (5.4-9), $L_1^2 = 4L_2$ on $W(L)$ when $\varepsilon = 0$. Then by theorem (5.4-1) the functions $\phi_n(x)$ become the eigenfunctions of L_1 and L_2. Therefore,

$$\begin{aligned} d_o(\phi_n) &= 4\{\langle\phi_n,\ L_2^{1/2}\phi_n\rangle^2 - \langle\phi_n,\ L_2\phi_n\rangle\} \\ &= 4\{\lambda_n\langle\phi_n,\ \phi_n\rangle^2 - \lambda_n\ \langle\phi_n,\ \phi_n\rangle\} \\ &= 4(\lambda_n-\lambda_n) \\ &= 0 \end{aligned}$$

where λ_n are the eigenvalues of L_2 associated with $\phi_n(x)$. The function $d_\varepsilon(\phi_n)$ defined on the set of functions $\phi_n(x)$ of (5.2-2) has a positive derivative and passes through zero and hence is positive for all $\varepsilon > 0$. Since ε_o is arbitrary, $D(\phi)$ is always positive and the discriminant in (5.4-7) is positive if $(L_1^2 - 4L_2)$ is positive definite, and the theorem easily follows.

Theorem (5.4-10): If the system described by (5.1-1) is underdamped then each of the functions $a_n(t)$ are underdamped and the solution $u(x,t)$ decays exponentially with oscillation in time.

Proof: Since $(4L_2-L_1^2)$ is positive definite, the following holds

$$<\phi,\ L_1^2\phi> \ < 4 <\phi,L_2\phi>$$

for all ϕ in W(L). The Cauchy-Schwarz inequality for the two functions $\phi(x)$ and $L_1\phi(x)$ yields

$$<\phi,\ L_1\phi>^2 \ \leq \ <\phi,\ L_1^2\phi>$$

for ϕ in W(L). The above two inequalities imply

$$<\phi,L_1\phi>^2 \ < 4<\phi,L_2\phi>$$

so that

$$<\phi,\ L_1\phi>^2 - 4<\phi,\ L_2\phi> \ < 0,$$

for all ϕ in W(L). In particular, this is true for the set of functions $\phi_n(x)$ so that the discriminant in equation (5.4-7) is always negative, and the theorem easily follows.

The next two theorems again deal with the special case of systems which possess classical normal modes, i.e, for systems which have solutions that can be expanded in the eigenfunctions of the undamped system. Stronger results can be derived for these systems than those for non-classical systems.

Theorem (5.4-11): If the system described by (5.1-1) possesses classical normal modes then the solution of (5.1-1) exhibits mixed damping if and only if there exists at least one value of n such that $a_n(t)$ is overdamped and at least one value of n such that $a_n(t)$ is underdamped.

Proof: Since (5.1-1) possesses classical normal modes, theorem (5.4-2) yields the following expression for each $a_n(t)$:

$$\ddot{a}_n(t) + \langle \phi_n, L_1\phi_n \rangle \, \dot{a}_n(t) + \langle \phi_n, L_2\phi_n \rangle \, a_n(t) = 0,$$

where $\phi_n(x)$ is an eigenfunction of both L_1 and L_2. Let $\lambda_n^{(1)}$ be the n^{th} eigenvalue of L_1 associated with the eigenfunction $\phi_n(x)$ and let $\lambda_n^{(2)}$ be the n^{th} eigenvalue of L_2. Then the above expression reduces to

(5.4-12) $$\ddot{a}_n(t) + \lambda_n^{(1)} \dot{a}_n(t) + \lambda_n^{(2)} a_n(t) = 0.$$

for n = 1, 2, 3... Hence, a particular $a_n(t)$ will be overdamped if and only if

(5.4-13) $$(\lambda_n^{(1)})^2 > 4\lambda_n^{(2)}$$

and underdamped if and only if

(5.4-14) $$4\lambda_n^{(2)} > (\lambda_n^{(1)})^2.$$

Next consider the operator L defined by

$$L = L_1^2 - 4L_2$$

on D(L) and note that it is self-adjoint. By theorem (2.3-20), the eigenvalues of L, denoted μ_n are

$$\mu_n = (\lambda_n^{(1)})^2 - 4\lambda_n^{(2)}$$

and the eigenfunctions are, obviously, $\phi_n(x)$. Furthermore, from theorem (2.3-17) any function u(x) in D(L) can be written as the uniformly converging series

$$u(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

where $c_n = \langle\phi_n, u\rangle$. Since the convergence is uniform, for all u(x) in D(L),

$$\text{(5.4-15)} \quad \langle Lu,u\rangle = \langle Lu, \sum_{n=1}^{\infty} c_n\phi_n\rangle$$

$$= \Sigma\ \mu_n\ c_n^2$$

following the proof of theorem (2.3-18).

To prove sufficiency, assume L is indefinite. Then there is a function u in D(L) such that

$$\text{(5.4-16)} \quad \langle Lu,u\rangle = \Sigma\ \mu_n\ c_n^2 < 0$$

and a function v in D(L) such that

$$\langle Lv,v\rangle = \Sigma\mu_n\, b_n^2 > 0, \tag{5.4-17}$$

where $b_n = \langle v,\phi_n\rangle$, via (5.4-15). Since $c_n^2 > 0$ for all n there must be at least one value of n such that $\mu_n < 0$. Then (5.4-14) is satisfied and there is at least one $a_n(t)$ that is underdamped. Since $b_n^2 > 0$ for all n there exists at least one value of n such that $\mu_n > 0$ and (5.4-13) is satisfied. Thus, there is also at least one $a_n(t)$ that is overdamped.

To prove necessity, suppose there is a value of n such that $a_n(t)$ is overdamped and another value of n such that $a_n(t)$ is underdamped. Then from (5.4-13) there is one $\mu_n < 0$ and another eigenvalue of L say μ_m, such that $\mu_m > 0$. Thus, there exists a function $v = \phi_m(x)$ in D(L) such that

$$\langle Lv,v\rangle = \langle \mu_m\phi_m, \phi_m\rangle$$

$$= \mu_m > 0$$

and a function $u(x) = \phi_n(x)$ in D(L) such that

$$\langle Lu,u\rangle = \mu_n < 0.$$

Hence, the operator $L = L_1^2 - 4L_2$ is indefinite on D(L).

Theorem (5.4-18): If the system described by (5.1-1) possesses classical normal modes, then (5.1-1) is

(I) critically damped if and only if each of the $a_n(t)$ are critically damped,

(II) overdamped if and only if each of the $a_n(t)$ are overdamped,

(III) underdamped if and only if each of the $a_n(t)$ are underdamped.

Proof: This set of theorems follows directly from the proof of the previous theorem. The oscillatory nature of $a_n(t)$ is determined by the sign of μ_n. The sufficiency of each of these results follow as corollaries to theorems (5.4-3), (5.4-6) and (5.4-10).

To see necessity for (I), suppose that each $a_n(t)$ is critically damped. Then each $\mu_n = 0$, so that $L_1^2 = 4L_2$ on D(L). For (II) suppose each $a_n(t)$ is overdamped, then $\mu_n > 0$ for all n and it follows from equation (5.4-15) that $L = L_1^2 - 4L_2$ is positive definite on D(L). Likewise for (III), if each $a_n(t)$ is underdamped, each $\mu_n < 0$ and (5.4-15) shows that the operator (-L) is positive definite, so that $4L_2 - L_1^2$ is positive definite on D(L).

5.5 Examples of Systems with Classical Modes

In this section some examples with known solutions are examined to illustrate the validity of the results in section 5.4. The problems here all satisfy the commutivity and self-adjoint conditions so that closed form solutions can be found by separation of variables or series solution techniques.

Example 5.5-1

Consider the longitudinal free vibration of a bar with internal damping. The describing equation of motion is

$$u_{tt} = au_{xx} + 2bu_{txx} \text{ on } (0,1)$$

with $a > 0$ and $b > 0$ and where $u = u(x,t)$ is the axial displacement of the bar. The subscripts t and x indicate partial derivatives with respect to these variables. For a clamped bar the boundary conditions are

$$u(0,t) = 0$$
$$u(1,t) = 0.$$

It is well known that the eigenfunctions are

$$\phi_n(x) = \sin n\pi x, \quad n=1,2,3...$$

and the eigenvalues are

$$\lambda_n = n\pi, \quad n=1,2,3...$$

for this problem. The time functions $a_n(t)$ are determined from

$$\ddot{a}_n(t) + 2b\lambda_n^2 \dot{a}_n(t) + a\lambda_n^2 a_n(t) = 0$$

Demanding that the discriminant of the resulting characteristic equation be positive yields

$$b > \frac{\sqrt{a}}{n\pi}, \quad n=1,2,3...$$

Since $n \geq 1$, if $b > \sqrt{a}/\pi$ all of the solutions will be decaying exponentials without oscillation. Even if b is small, as n increases the corresponding $a_n(t)$ will eventually be overdamped functions. Hence, the internally damped beam is inherently overdamped.

The above analysis of the closed form solution of this example is exactly predicted by theorem (5.4-6). To see this note that

$$L_1 = -2b \frac{\partial^2}{\partial x^2},$$

$$L_2 = -a \frac{\partial^2}{\partial x^2},$$

and

$$D(L) = \{u(x) | u(0) = u(1) = 0 \text{ and } u, u_x, u_{xx}, u_{xxx} \text{ and } u_{xxxx} \text{ are in } L_2(0,1)\}$$

These operators are both positive definite and self-adjoint. To see this requires some simple integration by parts.

$$\langle u, L_1 v\rangle = -2b \int_0^1 uv''dx$$

$$= -2b\ \{uv'\Big|_0^1 - {}_0\int^1 u'v'dx\}$$

$$= 2b\ {}_0\int^1 u'v'dx$$

which shows that L_1 is positive, since with $v = u(x)$ this becomes

$$<u,L_1u> \;=\; 2b\ {}_0\int^1 (u')^2 dx > 0$$

for all u in D(L). Proceeding with the computation yields

$$<u,L_1v> \;=\; 2b\{u''v\Big|_0^1 - {}_0\int^1 u''vdx\}$$

$$= <L_1u,v>$$

where the adjoint boundary conditions become $v(0) = v(1) = 0$. Hence, L_1 is self-adjoint. The calculation showing that L_2 is positive and self-adjoint is identical.

In order to apply theorem (5.4-6), note that

$$(L_1^2 - 4L_2) = 4\{b^2 \frac{\partial^4}{\partial x^4} + a \frac{\partial^2}{\partial x^2}\}.$$

The eigenvalues of $b^2 \frac{\partial^4}{\partial x^4}$ on D(L) are $n^4\pi^4b^2$. Those of $a \frac{\partial^2}{\partial x^2}$ are $-n^2\pi^2a$. From theorem (2.3-19) the eigenvalues of $(L_1^2 - 4L_2)$ are then

$$4(n^4\pi^4b^2 - n^2\pi^2a).$$

All the eigenvalues of $(L_1^2 - 4L_2)$ will be positive if $b > \sqrt{a}/\pi$. Thus, if $L_1^2 - 4L_2$ is positive definite then $b > \frac{\sqrt{a}}{\pi}$ and theorem (3.4-6) agrees with the actual solution. Note, that b would have to be a function of n in order for this system to be critically damped.

Example 5.4-2

Consider the transverse free vibration of a membrane in a surrounding medium furnishing resistance to the motion that is proportional to the velocity. The equation of motion is

$$u_{tt} + 2\gamma u_t - \nabla^2 u = 0 \text{ on } \Omega$$

with boundary conditions

$$u(x,t) = 0 \text{ on } \partial\Omega,$$

where ∇^2 is the two dimensional harmonic operator, $x = (x_1, x_2)$ and $u = u(x_1, x_2, t)$ is the deflection of the membrane in the direction perpendicular to the x_1 x_2 plane. Here Ω is a plane in two dimensional space and $\partial\Omega$ represents one or more curves in that space. If λ_n are the eigenvalues of ∇^2, then the known time solution [6, page 258, Vol. 2] is

$$a_n(t) = e^{-\gamma t} \frac{\cos[(\lambda_n-\gamma^2)^{1/2}t]}{(\lambda_n-\gamma^2)^{1/2}},$$

if $\lambda_k > \gamma^2$. Thus for $\lambda_k > \gamma^2$ all the time solutions are underdamped.

In terms of the theory presented here the operators of interest are

$$L_1 = 2\gamma$$

and

$$L_2 = -\nabla^2.$$

L_1 is obviously a positive self-adjoint operator on $D(L)$ since $\gamma > 0$. To see that L_2 is, note that

$$\begin{aligned} \langle u, L_2 u\rangle &= -\langle u, \nabla^2 u\rangle \\ &= -\int_\Omega u\nabla^2 u \, dx. \end{aligned}$$

Using Green's identity and the fact that $u(x) = 0$ on $\partial\Omega$ yields

$$\langle u, L_2 u\rangle = \int_\Omega (\nabla u)^2 dx$$

so that L_2 is positive definite. Also, from Green's identity

$$\begin{aligned} \langle v, L_2 u\rangle &= \int_\Omega (\nabla u)(\nabla v) \, dx - \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds \\ &= \langle L_2 v, u\rangle \end{aligned}$$

as long as $v(x)$ is taken to be zero on $\partial\Omega$. Thus L_2 is self-adjoint on $D(L)$.

Theorem (2.3-20) yields the eigenvalues of

$$4L_2 - L_1^2 = -4(\nabla^2 + \gamma^2)$$

to be $4(\lambda_n - \gamma^2)$. This will be positive, making $4L_2 - L_1^2$ positive definite, if $\lambda_n > \gamma^2$ for all n. Thus theorem (5.4-10) correctly predicts when each of the functions $a_n(t)$ will be underdamped.

Example 5.4-3

Consider the longitudinal free vibration of a bar with both internal and external damping. The equation of motion is

$$u_{tt} + 2[\gamma - b\frac{\partial^2}{\partial x^2}]u_t - au_{xx} = 0 \text{ on } (0,1)$$

where $u = u(x,t)$ is the displacement of the bar and the constants γ, b and a are positive. The boundary conditions for a clamped bar are

$$u(0,t) = u(1,t) = 0.$$

One method of solution is to assume a series expansion for $u(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

where

$$-a\phi_n''(x) = \lambda_n\phi_n(x),\ \lambda_n > 0$$

and $<\phi_n, \phi_m> = \delta_{mn}$, the Kronecker delta. Substitution into the equation of motion yields

$$\sum_{n=1}^{\infty} \{\ddot{a}_n(t)\phi_n(x) + 2\gamma\dot{a}_n(t)\phi_n(x) + \frac{2b}{a}\dot{a}_n(t)\lambda_n\phi_n(x)$$

$$+ a_n(t)\lambda_n\phi_n(x)\} = 0.$$

Multiplying this expression by $\phi_m(x)$ and forming an inner product yields

$$\ddot{a}_m(t) + 2[\gamma + \frac{b}{a}\lambda_m]\ \dot{a}_m(t) + \lambda_m a_m(t) = 0,$$

for m = 1,2,3... The discriminant of the corresponding characteristic equation is then

$$4[(\gamma + \frac{b}{a}\lambda_n)^2 - \lambda_n].$$

The obvious conclusions are as follows

(i) each $a_n(t)$ is critically damped if $(\gamma + \frac{b}{a}\lambda_n)^2 = \lambda_n$ for all n,

(ii) each $a_n(t)$ is overdamped if $(\gamma + \frac{b}{a}\lambda_n)^2 > \lambda_n$ for all n, and

(iii) each $a_n(t)$ is underdamped if $(\gamma + \frac{b}{a}\lambda_n)^2 < \lambda_n$ for all n.

Now consider this same problem in terms of the theory presented here. The operator $L_2 = - a \frac{\partial^2}{\partial x^2}$ is positive and self-adjoint as shown in example 5.5-1. Consider

$$L_2 = \gamma - b \frac{\partial^2}{\partial x^2} .$$

It is self-adjoint since it is the sum of two self-adjoint operators and positive definite for the same reason. Thus the theorems apply.

Note that

$$L_1^2 = 4(\gamma^2 - 2\gamma b \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^4}{\partial x^4})$$

so that

$$L_1^2 - 4L_2 = 4(\gamma^2 - 2\gamma b \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^4}{\partial x^4} + a \frac{\partial^2}{\partial x^2}).$$

A repeated application of theorem (2.3-20) yields the eigenvalues of $L_1^2 - 4L_2$ which are

$$4(\gamma^2 + 2\gamma \frac{b}{a} \lambda_n + \frac{b^2}{a^2} \lambda_n^2 - \lambda_n).$$

Demanding that these eigenvalues, and hence the appropriate operator, be positive, negative or zero leads to exactly the same conditions stated in (i), (ii) and (iii) above.

An analysis of these three conditions leads to some physical insight. Note that condition (i) cannot be met for all n because γ is not a function of n. Also since

$$\lambda_n = \sqrt{a}\, n\pi > \sqrt{a}\pi,$$

the only condition that can possibly be met for all n is (ii). Hence this system is either overdamped or exhibits mixed damping with the higher modes all being overdamped.

Example 5.4-4

As an illustration of the importance of the theorems stated in this work, consider the usual attack on solving a damped vibration problem. Commonly one assumes a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\, e^{-\zeta_n t} \sin \omega_n t, \quad \zeta_n > 0, \ \omega_n > 0.$$

However, theorem (5.3-13) states necessary and sufficient conditions under which $a_n(t)$ will have this specific form so that one does not have to make any such assumption (which may or may not be correct).

As an example of this pitfall consider the free flexural vibrations of a damped plate. This problem was solved by Murthy and Sherbourne [21]. They give the equation of motion as

$$\nabla^4 u(x,t) + \frac{mha^4}{D} u_{tt}(x,t) + \frac{ka^4}{D} u_t(x,t) = 0$$

where ∇^4, the biharmonic operator, and m, h, a, k and D are various physical parameters of the problem and are all positive constants. The boundary conditions are those of a clamped support

$$u(x,t) = 0 \text{ on } \partial\Omega,$$

and

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\frac{\partial}{\partial n}$ indicates the derivative normal to $\partial\Omega$.

Proceeding in terms of the theory presented here, note that

$$L_1 = \frac{k}{mh} = \beta^{1/2}$$

which is obviously a positive definite self-adjoint operator. The operator L_2 is of the form

$$L_2 = \gamma\nabla^4, \qquad \gamma = \frac{D}{mha^4} > 0.$$

To see that this is positive and self-adjoint, recall Green's identity for ∇^4:

$$\int_\Omega v\nabla^4 u = \int_\Omega \nabla^2 v \nabla^2 u \, dx - \int_{\partial\Omega} \left(u\frac{\partial}{\partial n}(\nabla^2 v) - \nabla^2 v\frac{\partial u}{\partial n}\right) ds.$$

Since the last integral vanishes from the application of the boundary conditions, L_2 is obviously positive definite. A second application of Green's identity yields

$$\langle v, L_2 u\rangle = \int_\Omega v\nabla^4 u = \int_\Omega \nabla^2 v \nabla^2 u$$

$$= \int_\Omega (\nabla^4 v)u - \int_{\partial\Omega} (v\frac{\partial}{\partial n} \nabla^2 u)ds + \int_{\partial\Omega} \nabla^2 u \frac{\partial v}{\partial n} ds$$

$$= \langle L_2 v, u\rangle$$

where the adjoint boundary conditions become $v = \frac{\partial v}{\partial n} = 0$ on $\partial\Omega$. Thus, L_2 is self-adjoint as well.

To apply the overdamping condition note that

$$L_1^2 - 4L_2 = \beta - 4\gamma\nabla^4$$

so that an application of Green's identity yields

$$\langle u, (L_1^2-4L_2)u\rangle = \beta||u||^2 - 4\gamma||\nabla^2 u||$$

It is not clear from this expression what the definiteness of $(L_1^2-4L_2)$ is. However, the eigenvalues of $(L_1^2-4L_2)$ can be computed using theorem (2.3-20). They are

$$\beta - 4\gamma\lambda_n$$

where λ_n are the eigenvalues of ∇^2 on D(L). These eigenvalues will be positive or negative depending on the value of n and the physical parameters k, m, D, h and a. From theorem (5.4-11) the values of $(\beta-4\gamma\lambda_n)$ for a specific n determine whether or not $a_n(t)$ for that n oscillates. Thus, while it is clear that this system will have time solutions of the form $(\beta<4\gamma\lambda_n)$

$$a_n(t) = e^{-\zeta_n t}\sin \omega_n t$$

for large values of n, the actual solution may have terms of the form $(\beta>4\gamma\lambda_n)$

$$a_n(t) = e^{-\zeta_n t}\sinh\omega_n t, \ \zeta_n > \omega_n$$

for lower values of n.

Murthy and Sherbourn consider, in their numerical treatment of this problem, the first ten modes of the spatial eigenvalue problem. It is possible that the damping constant k is large enough in comparison to the thickness of the plate h and the width of the plate a, that

$$\frac{k^2}{mh^2} > 4\,\frac{D}{mha^4}\,\lambda_n,$$

for the first ten modes $(n \leq 10)$. Hence, the results presented by the above authors should be modified to include the possibility of mixed damping.

5.5 Non Classical Mode Example

Systems with non-commuting operators are difficult to solve. This is reflected in the lack of such examples in texts and current literature. As an example of such a system, consider the bending vibrations of a beam with non-uniform area moment of inertia (denoted I(x)) and with a damping mechanism which yields a velocity dependent force resisting the bending moment.

The equation of motion of such a non-uniform beam without damping is well known [3] and given by (E is the elastic modulus)

$$u_{tt}(x,t) + (EI(x)u_{xx}(x,t))_{xx} = 0.$$

Attaching a damping force resisting the bending moment yields

$$u_{tt}(x,t) - 2cu_{txx}(x,t) + (EI(x)u_{xx}(x,t))_{xx} = 0.$$

where c is a damping constant assumed to be positive.

For this example consider a hinged-hinged configuration. The boundary conditions are that the deflection at the boundaries be zero so that

$$u(0,t) = u(1,t) = 0$$

and that the bending moments at the boundaries be zero

$$I(0)\ u''(0,t) = I(1)\ u''(1,t) = 0.$$

Suppose also that the area moment of inertia is bounded. That is that

$$0 < m \leq I(x) \leq M < \infty$$

for all x in the interval [0,1] where m and M are known constants. In particular then I(0) and I(1) are non-zero so that the last set of boundary conditions becomes

$$u''(0,t) = u''(1,t) = 0.$$

These boundary conditions define the domain D(L) described in (5.2-1). The domain D(L), in this example, is the set of all functions satisfying the above boundary conditions and having derivatives through the fourth order. The operator L_1 is then defined on D(L) by

$$L_1 = -2c \frac{\partial^2}{\partial x^2}$$

which was shown in example (5.5-1) to be self-adjoint and positive definite on D(L). The operator L_2 defined on D(L) is

$$L_2 = E \frac{\partial^2}{\partial x^2} (I(x) \frac{\partial^2}{\partial x^2})$$

which is also a self-adjoint positive operator. To see this, again integrate by parts

$$\langle v, L_1 u\rangle = E \int_0^1 v(Iu'')''dx$$

$$= v(\underset{\to 0}{\cancel{Iu''}})'\Big|_0^1 - E \int_0^1 v'(Iu'')'dx$$

$$= v'\cancelto{0}{(Iu'')}\Big|_0^1 + E \int_0^1 (Iv'')u''dx$$

$$= E \int_0^1 Iv''u''dx, \quad (>0 \text{ for } v=u)$$

$$= \cancelto{0}{(Iv'')}u'\Big|_0^1 - E \int_0^1 (Iv'')'u'dx$$

$$= -(Iv'')''\cancelto{0}{u}\Big|_0^1 + E \int_0^1 (Iv'')''u\, dx$$

$$= \langle L_1 v, u\rangle$$

Thus, the adjoint boundary conditions are $v(0) = v(1) = 0$ and $v''(0) = v''(1) = 0$ so that L_1 is self adjoint and positive definite. Hence, the theorems of section 5.4 can be applied to this problem.

A calculation quickly shows that $L_1L_2 \neq L_2L_1$ and Caughey and O'Kelley's work [7] suggest that attempts to solve this problem by modal analysis will fail. However, the theorems of section (5.4) can still be applied to see if the nature of the solutions can be determined. To this end consider the operator $L = (L_1^2 - 4L_2)$, which becomes in this example

$$L(\cdot) = 4 \{c^2 \frac{\partial^4}{\partial x^4} (\cdot) - E \frac{\partial^2}{\partial x^2} (I(x) \frac{\partial^2}{\partial x^2} (\cdot))\}.$$

Letting L act on $v(x)$ an arbitrary element in $D(L)$ and forming the inner product with $v(x)$ yields

$$\langle v, Lv\rangle = 4\{c^2 \int_0^1 v\, v^{IV}dx - E \int_0^1 v(I(x)v'')''dx\}$$

$$= 4 \{c^2 \int_0^1 (v'')^2 dx - E \int_0^1 I(x)(v'')^2 dx\},$$

after integrating by parts and using the boundary conditions. Here v^{IV} indicates the fourth derivative of $v(x)$, v'' the second, etc. Thus the condition of overdamping or underdamping will depend on the sign of

$$c^2 \int_0^1 (v'')^2 dx - E \int_0^1 I(x)(v'')^2 dx.$$

Suppose it is desired to have this system be overdamped. It is well known from calculus that if $g(x) \leq f(x)$ for all x in [a,b] then

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx.$$

If the operator L is to be positive, then

$$c^2 \int_0^1 (v'')^2 dx - E \int_0^1 I(x)(v'')^2 dx > 0$$

must hold. But $(v'')^2 > 0$, so by the assumption on $I(x)$

$$M(v'')^2 > I(x)\ (v'')^2 > m(v'')^2,$$

for all x in [0,1] and by the above result

$$EM \int_0^1 (v'')^2 dx \geq E \int_0^1 I(x)(v'')^2\, dx \geq Em \int_0^1 (v'')^2 dx.$$

Combining these inequalities yields

$$\langle v,Lv\rangle = c^2 \int_0^1 (v'')^2 dx - E \int_0^1 I(x)(v'')^2 dx$$

$$> c^2 \ {}_0\int^1 (v'')^2 dx - EM \ {}_0\int^1 (v'')^2 dx$$

$$= (c^2 - EM) \ {}_0\int^1 (v'')^2 dx > 0$$

if $c^2 - EM > 0$. Hence, if the damping constant c, the elastic modulus E and the largest value of the area moment of inertia M are such that

$$c^2 > EM$$

then all of the time solutions of this problem will be overdamped by theorem (5.5-6) and the solution will not oscillate with time.

Next suppose it is desired to have this system be underdamped so that each of the time solutions will oscillate. Since it is assumed that $I(x) \geq m$ for all x in [0,1] the computation above can be repeated with the sign reversed. That is

$$\langle v,(4L_2-L_1^2)v\rangle \ \geq (Em-c^2) \ {}_0\int^1 (v'')^2 dx > 0$$

for all v(x) in the domain D(L) if

$$Em > c^2$$

Theorem (5.4-10) then yields that the time solutions of this system are all damped oscillations.

In summary, a problem has been considered which does not have a solution which can be computed in closed form by the usual methods. It was then shown that if the area moment of inertia is bounded the systems time solutions will oscillate if

$$Em > c^2$$

and will not oscillate if

$$c^2 > EM.$$

Thus, the nature of the solution of this system is determined by the elastic modulus E, the value ("size") of the damping constant and the maximum and minimum values of the area moment of inertia. This is information about the solution of this example that has not been previously available.

Chapter 6

SUMMARY AND FURTHER STUDY

6.1 Summary

A set of conditions have been stated and derived which determine whether the solution of a given linear dynamic system with damping will oscillate. Several classes of both lumped parameter and distributed parameter systems were treated. These conditions involve checking the definiteness of certain matrices or operators and are in general easier to check than solving the governing differential equations.

It was further shown how these conditions can be used to design systems in such a manner that their solutions will either oscillate or not, as desired. In the case of lumped parameter systems, some of the conditions were shown to imply completeness of the eigenvectors associated with the damped system, thus allowing a closed form solution of a system with arbitrary forcing functions.

A selection of example problems were presented illustrating the use of the conditions to predict the qualitative nature of the free response to arbitrary initial conditions. These examples serve to verify the theorems. For the lumped parameter case examples using the conditions as design tools were also presented. For the distributed parameter case, the conditions were applied to several problems with known solutions to verify the theorems. Two examples which do not have known closed form solutions were also considered. For these examples, inequalities were found from enforcing the conditions which indicate the qualitative nature of the time response.

Comparisons were made to results found by other researchers. For the lumped parameter case, several results by other authors were available for comparison. In the distributed parameter case, no results related to the problem stated here have appeared in the literature. Thus, comparisons were made to specific examples treated in the literature.

A complete theory has been presented with illustrative examples and applications for viscously damped linear systems possessing the usual symmetry (self adjointness) and definiteness.

6.2 Suggestions for Further Research

There are several topics of study which may emanate from this dissertation. From the lumped parameter case there is motivation to examine the theorems of chapter 3 when the definiteness and real symmetric conditions on M, C and K are relaxed so that these coefficient matrices are just square arrays of numbers. In fact a discrete model of an acoustically coupled panel leads to a complex matrix.

Another possible topic is to formulate the method of system design following the examples in section 4.2. This would involve looking for an algorithm to solve a system of non-linear algebraic inequalities subject to constraints. Existence of a solution is an important question and it may be possible to insure uniqueness by imposing some optimality criteria. The result would be a practical method of designing systems which will not vibrate when perturbed.

The distributed parameter case also lends itself to further study in terms of relaxing the assumptions on L_1 and L_2. The most restrictive assumption is the self-adjointness. Many problems with varying moment or cross-sectional area do not have self-adjoint boundary conditions.

Another interesting extension of the distributed parameter theory is to consider systems having internal boundary conditions or so called "in-span" conditions. An example is a beam supported on a uniform viscous foundation. These problems involve both distributed and lumped parameter elements.

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