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ABSTRACT

SOME TOPICS IN FINITE ELASTICITY

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This dissertation consists of two parts, both concerned with the investigation of problems in the theory of finite elastostatics.

In Part I an analytic approach for obtaining bounds on stress concentration factors in the theory of finite anti-plane shear is presented. The problem of an infinite slab with a traction free elliptical cavity subjected to a remotely applied finite simple shear deformation is considered. It is assumed that the slab is composed of a homogeneous incompressible elastic material. Explicit estimates are obtained for the stress concentration factor in terms of the dimensions of the cavity, the applied stress and the constitutive parameters. The limiting cases in which the cavity is circular or crack-shaped are also examined. The analysis is based on the application of maximum principles for second-order uniformly elliptic quasilinear partial differential equations.

In Part II the finite plane strain deformation of a circular tube of homogeneous compressible elastic material of harmonic type, subjected to simultaneous internal and external pressure, is considered. Explicit closed form solutions for the deformation and stress fields are obtained. The true stress distribution, expressed in terms of undeformed coordinates, is shown to be essentially independent of

material properties. The two cases of internal pressure only, and external pressure only, are examined in detail. In the former case there is a finite value of the applied pressure at which the maximum hoop stress in the tube, occurring at the inner surface, becomes unbounded. For the case of external pressure a finite value of the applied pressure exists for which the cavity closes.

Furthermore the stability of the equilibrium in the two special cases described above is investigated by employing a standard perturbation expansion. It is found that an internally pressurized tube is always stable whereas an externally pressurized tube buckles at a certain value of pressure. In the latter case the smallest buckling load is calculated and the existence of buckling loads corresponding to higher modes established.

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PART I

ESTIMATES FOR STRESS CONCENTRATION FACTORS IN FINITE ANTI-PLANE SHEAR

1. INTRODUCTION

Qualitative methods have been used in linear elasticity for a long time (see e.g. Villaggio (1977), Horgan (1982) and references cited therein). The objective of such studies is to find information about the solution of boundary-value problems without actually solving them. The desired results are generally in the form of a priori bounds for field quantities in terms of geometric, constitutive and boundary data. Analogous results in nonlinear elasticity are rare. Estimates of this type are especially important in the finite theory where exact solutions are seldom available. In addition to their inherent importance, such results are of value as guides in computational analyses.

Pointwise stress estimates are particularly important in problems involving stress concentration where localized stresses are of primary concern. In the present study, following on recent results of Abeyaratne and Horgan (1983), we shall consider the application of a priori estimation techniques to a stress concentration problem arising in finite elasticity theory.

We confine attention to the simplest possible setting within the exact theory of finite elasticity: finite anti-plane shear of an infinitely long cylinder composed of a homogeneous, isotropic, incompressible material. Such deformations have been extensively studied by Knowles (1976, 1977) and others. While of less practical interest than their analogs in plane stress or plane strain, these problems are much simpler to analyze analytically and serve a useful role as pilot problems.

We are concerned with the stress concentration arising in the problem of an infinite slab with a traction-free elliptic cavity subject to a state of finite simple shear deformation at infinity. A cross-section of the slab is shown in Figure 1. The constitutive law is assumed to belong to a special class of such laws for which nontrivial states of finite anti-plane shear do indeed exist.

The analogous problem for a circular cavity was treated recently by Abeyaratne and Horgan (1983). One of the motivations for the present study was to extend their techniques to the elliptical cavity problem, with particular interest in the limiting case modelling a straight crack. When the results of the present investigation are specialized to the case of a circular cavity, the bounds obtained are sharper than those found by Abeyaratne and Horgan (1983).

The boundary-value problem is formulated in Section 2. The maximum shearing stress, of principal interest here, is known to occur on the boundary of the cavity. Our purpose is to provide a means for estimating this quantity. The main results necessary for this task are given in Section 3. In Sections 4 and 5 these results are applied to find explicit bounds on the stress concentration factor for a wide class of materials in terms of the geometry, load and constitutive parameters. The results are illustrated for a particular constitutive law. We conclude with some general remarks in Section 6.

2. STATEMENT OF THE PROBLEM

2.1 Displacement Formulation

Let the three dimensional open region R be the exterior of an infinitely long right elliptical cylinder with semimajor axis A and semiminor axis B. Suppose that this open region is occupied by the interior of a body in its undeformed configuration. Choose the rectangular cartesian coordinates (x_1, x_2, x_3) with the x_3-axis parallel to the generator of the cylinder and the origin at the center. Let D be the cross section of R in the plane $x_3 = 0$, and denote by Γ the boundary of the elliptical cavity (Figure 1).

Suppose now that the body is subjected at infinity to a simple shear parallel to the (x_1, x_3) plane. The ensuing deformation maps a point with position vector $\underset{\sim}{x}$ in the undeformed configuration to a point with the position vector $\underset{\sim}{y}$:

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{u}\,(\underset{\sim}{x}) \quad \text{on } R. \tag{2.1}$$

The components of the displacement field are assumed to satisfy [1]

$$u_\alpha = 0, \quad u_3 = k_\infty x_2 \quad \text{as } x_\alpha x_\alpha \to \infty \;, \tag{2.2}$$

where $k_\infty(>0)$ is the amount of applied shear. The deformed surface of the cavity is assumed to be traction-free.

(1) The components of all vectors and tensors are taken with respect to the fixed rectangular coordinate system previously chosen. Greek subscripts have range (1,2) and summation convention is assumed throughout. A subscript preceded by a comma indicates partial differentiation with respect to the corresponding x-coordinate.

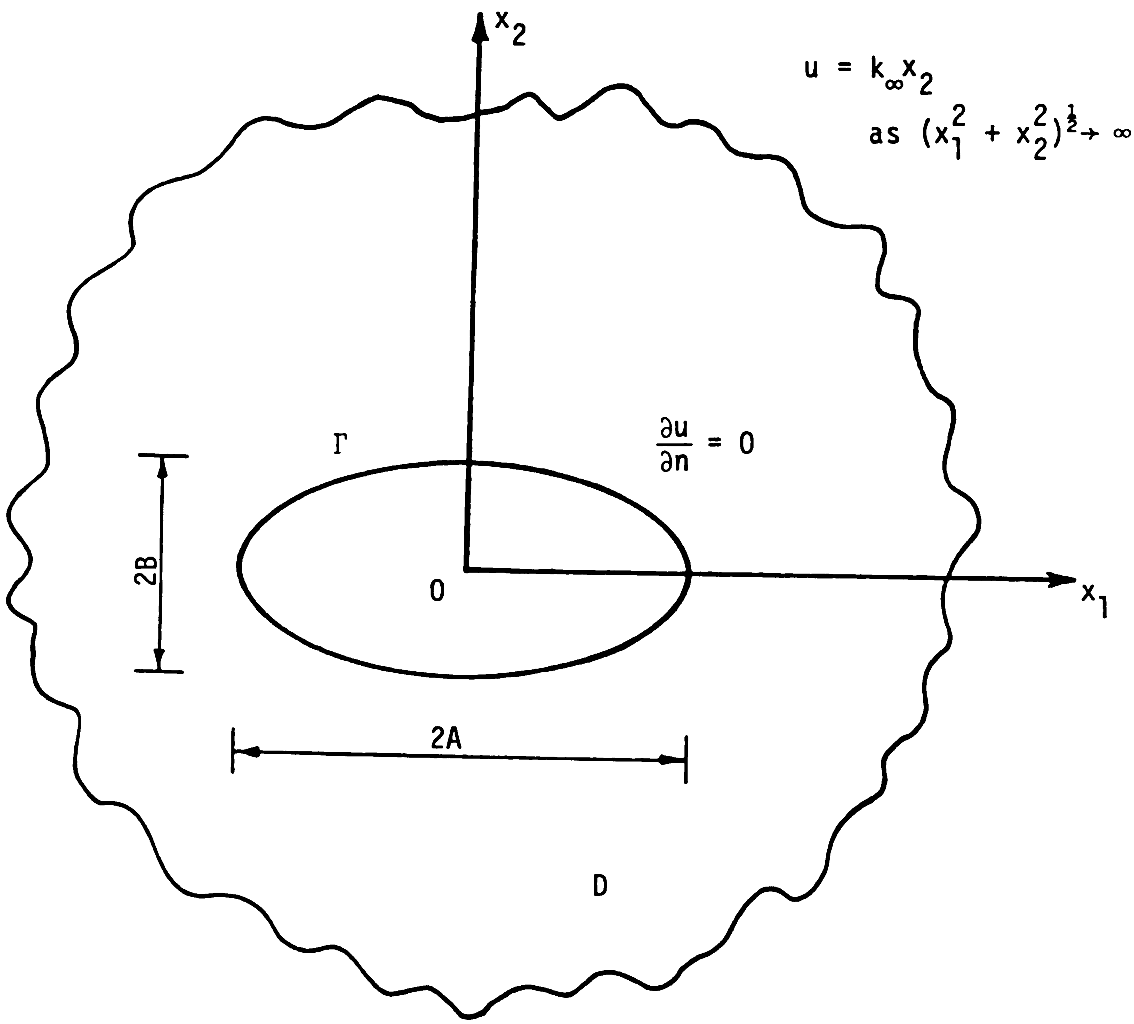

Figure 1. Cross-section of body, with cavity, coordinates and boundary conditions.

Suppose that the body is composed of a homogeneous, isotropic, incompressible elastic material with a strain-energy density function W. Denoting by I_1, I_2, and I_3 the fundamental invariants of the left (or right) Cauchy-Green deformation tensor we have $I_1 = I_2 = 3$ in the undeformed state and $I_1 \geq 3$, $I_2 \geq 3$ for all deformations. Since only locally volume preserving deformations are admissible $I_3 = 1$. The elastic potential W depends in general on I_1 and I_2, $W = \hat{W}(I_1, I_2)$. For reasons that will become apparent later it is convenient to confine attention to the restriction of $\hat{W}(I_1, I_2)$ to the line $I_1 = I_2$ (= I) and define $W(I)$ by

$$W = W(I) = \hat{W}(I, I), \; I \geq 3, \; W(3) = 0, \tag{2.3}$$

where W is assumed to be twice continously differentiable for $I \geq 3$.

The response of this material in simple shear is described by

$$\hat{\tau}(k) = 2kW'(3+k^2), \; -\infty < k < \infty, \tag{2.4}$$

where $\hat{\tau}(k)$ is the shear stress associated with an amount of shear k, and prime denotes differentiation with respect to the argument. The (secant) modulus of shear is now given by

$$M(k) = \frac{\hat{\tau}(k)}{k} = 2W'(3+k^2) \; (>0). \tag{2.5}$$

In order to satisfy the Baker-Ericksen inequality for the material under consideration we will assume that $M(k)>0$. (At infinitesimal deformations, we have from (2.5), $M(0) = 2W'(3)$ which we will denote by μ: the shear modulus.) Following Knowles (1977) such a material is said to be <u>softening in shear</u> if $M'(k) < 0$ and <u>hardening</u> if $M'(k) > 0$:

$$k\,\hat{\tau}'(k) < \hat{\tau}(k) \qquad \text{(softening)},$$
$$k\,\hat{\tau}'(k) > \hat{\tau}(k) \qquad \text{(hardening)}. \tag{2.6}$$

Knowles has shown that for a certain class of materials, the field equations and boundary conditions associated with the problem described above are consistent with the assumption that

$$u_\alpha = 0, \quad u_3 = u(x_1, x_2) \quad \text{on } R, \tag{2.7}$$

corresponding to a state of <u>anti-plane shear.</u> Two points should be noted. First, for all such deformations $I_1 = I_2$ $(=3+|\nabla u|^2)$. Secondly a material governed by an arbitrary strain-energy density function $\hat{W}(I_1, I_2)$ cannot sustain a nontrivial state of anti-plane shear. The entire class of materials which admit such a deformation has been determined by Knowles (1976) and it is only these materials that we consider here. (An example of such a material is the familiar neo-Hookean material with the elastic potential $W = \frac{1}{2}\mu(I_1-3)$, $\mu>0$.) The governing problem can then be shown, Knowles (1976, 1977), to reduce to the following two dimensional problem for u:

$$\operatorname{div}\,[W'(I)\,\operatorname{grad} u] = 0 \quad \text{on } D, \tag{2.8}$$

with

$$I = 3+|\nabla u|^2 \quad , \qquad \nabla u = \operatorname{grad} u, \tag{2.9}$$

$$u(x_1, x_2) = k_\infty x_2 \quad \text{as} \quad x_\alpha x_\alpha \to \infty, \tag{2.10}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma \;, \tag{2.11}$$

where $\partial u/\partial n$ denotes the outward normal derivative of u on Γ. The corresponding components of Cauchy stress τ_{ij} are given by

$$\tau_{\alpha 3} = \tau_{3\alpha} = 2W'(I)u,_{\alpha} , \tag{2.12}$$

$$\tau_{\alpha\beta} = 0, \qquad \tau_{33} = 2W'(I)|\nabla u|^2. \tag{2.13}$$

Since we have assumed that $M(k)>0$ it can easily be verified that the quasilinear partial differential equation (2.8) is <u>elliptic</u> at a solution u and at a point (x_1, x_2) if and only if

$$\hat{\tau}'(k)>0 \quad , \quad k = |\nabla u| , \tag{2.14}$$

where $\hat{\tau}(k)$ is given by (2.4) and the prime denotes differentiation. We shall impose a slightly stronger requirement: we assume that $\hat{\tau}(k)$ satisfies

$$b\hat{\tau}(k) \geq k\,\hat{\tau}'(k) \geq c\,\hat{\tau}(k) \quad \text{for all} \quad k \geq 0, \tag{2.15}$$

for some positive constants b and c. The right hand side of (2.15) together with (2.5) assures that (2.8) is uniformly elliptic (see Gilbarg and Trudinger (1977), p. 203) and implies in particular that $\hat{\tau}'(k)>0$ for all k as well as $\hat{\tau}(\infty)= \infty$. It follows that when (2.5) and (2.15) hold, $\tau = \hat{\tau}(k)$ can be inverted to give k as an odd, monotone strictly increasing function of τ: $k = \hat{k}(\tau)$ with $\hat{k}(\infty) = \infty$.
It will be seen later that the left hand side of (2.15) is equivalent to a uniform ellipticity assumption for a differential equation related to (2.8). Henceforth the <u>ellipticity constants</u> b and c are taken to be the smallest and largest constants respectively for which (2.15) holds.

In view of (2.6) we note that a softening material automatically satisfies the left inequality of (2.15) with $b = 1$, while a hardening material conforms to the right one with $c = 1$. Consequently, in the following we have

$$\begin{aligned} &\text{Softening:} \quad \hat{\tau}(k) > k\hat{\tau}'(k) \geq c\,\hat{\tau}(k) \qquad 0<c<1, \\ &\text{Hardening:} \quad b\hat{\tau}(k) \geq k\hat{\tau}'(k) > \hat{\tau}(k) \qquad b>1. \end{aligned} \tag{2.16}$$

The final results derived subsequently will be given in terms of the constitutive functions $m(s)$ and $n(s)$ which we define for all $s \geq 0$ in terms of the response function $\hat{\tau}(k)$ by

$$m(s) = \max_{0 \leq \tau \leq s} \left(\frac{\tau}{k\hat{\tau}'(k)} - 1 \right), \quad (s>0), \tag{2.17}$$

$$n(s) = \min_{0 \leq \tau \leq s} \left(\frac{\tau}{k\hat{\tau}'(k)} - 1 \right), \quad (s>0), \tag{2.18}$$

and $m(0) = n(0) = 0$, where $k = \hat{k}(\tau)$. From (2.4) we have $\hat{\tau}'(k) = 2W'(3) = \mu$ as $k \to 0$ and therefore $\lim m(s) = \lim n(s) = 0$ as $s \to 0$. This shows that the functions are continuous at $s = 0$. By their very definitions $m(s)$ is a non-decreasing and $n(s)$ is a non-increasing function; it then follows that $m(s) \geq 0$ and $n(s) \leq 0$.

Thus in view of (2.16) we have

$$0 \leq m(s) \leq \frac{1}{c} - 1, \tag{2.19}$$

$$0 \geq n(s) \geq \frac{1}{b} - 1. \tag{2.20}$$

In the following, the existence of a smooth solution $u(x_1, x_2)$ to the boundary value problem (2.8)-(2.11) will be assumed, where u is twice continuously differentiable on D and once so on Γ.

On linearizing the partial differential equation (2.8) formally by neglecting $|\nabla u|^2$ in comparison with 3, we recover the analogous problem in classical elasticity. This is a boundary-value problem for Laplace's equation which also describes the steady irrotational flow of an inviscid incompressible fluid past an elliptical cylinder. In the flow problem $\mathring{u}$ is identified with the velocity potential and k_∞ with the free stream speed. The solution $\mathring{u}$ (unique to within a constant) of the linearized problem may be found in standard text books; its explicit form need not concern us here. From (2.12) the corresponding linearized stresses are given by

$$\mathring{\tau}_{3\alpha} = \mu\mathring{u}_{,\alpha}, \quad \mathring{\tau} \equiv (\mathring{\tau}_{31}^2 + \mathring{\tau}_{32}^2)^{\frac{1}{2}} = \mu|\nabla\mathring{u}|. \tag{2.21}$$

It is well known that $\mathring{\tau}_{max}$ occurs on Γ.

For the linearized problem, the stress concentration factor $\mathring{K}$ is defined by

$$\mathring{K} = \mathring{\tau}_{max}/\ \mathring{\tau}_\infty \quad , \tag{2.22}$$

where $\mathring{\tau}_\infty = \mu k_\infty$ denotes the magnitude of the applied stress at infinity. It can be shown (see e.g. Milne-Thomson (1967), p. 171) that

$$\mathring{K} = 1 + A/B. \tag{2.23}$$

It should be noted that for a neo-Hookean material, the problem (2.8)-(2.11) specializes <u>exactly</u> (rather than merely by linearization) to the linear problem. Thus $\overset{\circ}{K}$ given by (2.23) is exact for this material.

Our main concern here is with the nonlinear problem (2.8)-(2.11). For this problem, we define a stress concentration factor K by

$$K = \tau_{max}/\tau_{\infty} \quad , \tag{2.24}$$

where

$$\tau_{max} = \max_{D \cup \Gamma} (\tau_{31}^2 + \tau_{32}^2)^{\frac{1}{2}} \, , \tag{2.25}$$

and $\tau_{\infty} = 2k_{\infty} \; W'(3+k_{\infty}^2)$ is the magnitude of the applied stress at infinity. Our objective is to develop techniques for obtaining bounds on τ_{max}, and so on K, which conform to the result (2.23) on linearization. The argument is based on maximum principles and comparison theorems for the second order quasilinear uniformly elliptic equation (2.8). (See Protter and Weinberger (1967), Gilbarg and Trudinger (1977)). Such maximum principles have been used (see e.g. Bers (1958),p. 41, Schiffer (1960),p. 95) to show that τ_{max} occurs on the boundary Γ and so our task is to estimate τ on Γ.

2.2 Reformulation in Terms of Stress Function

It is convenient for our purposes to convert the basic problem (2.8)-(2.11) to a problem of Dirichlet type. It follows from (2.8) that there exists a function v, twice continuously differentiable on D and once so on Γ such that

$$\tau_{3\alpha} = 2W'(3+ |\nabla u|^2)u,_{\alpha} = \epsilon_{\alpha\beta} \, v,_{\beta} \quad \text{on } D, \tag{2.26}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional alternator ($\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$). The function v is a stress function for the shear stresses $\tau_{3\alpha}$. From (2.26), (2.4) one infers that

$$\hat{\tau}\,(|\nabla u|) = |\nabla v| \,, \tag{2.27}$$

which upon inversion, yields

$$|\nabla u| = \hat{k}\,(|\nabla v|) \,. \tag{2.28}$$

We now define a function V by

$$V(\tau^2) = \frac{1}{2W'(3+\hat{k}^2(\tau))}\ (>0),\ -\infty<\tau<\infty \,, \tag{2.29}$$

and note that (2.26) may then be written as

$$u,_\alpha = -\,V(|\nabla v|^2)\varepsilon_{\alpha\beta}\, v,_\beta \quad . \tag{2.30}$$

It then follows that the stress function v satisfies the differential equation

$$L\ v \equiv \operatorname{div}\,[V(|\nabla v|^2)\ \operatorname{grad} v] = 0 \quad \text{on D.} \tag{2.31}$$

It can be verified that equation (2.31) is uniformly elliptic by virtue of the left-hand-side of (2.15). From (2.26), (2.10) and (2.11) v may be shown to satisfy the boundary conditions

$$v(x_1,\, x_2) = \tau_\infty\ x_1 \quad \text{as} \quad x_\alpha x_\alpha \to \infty \,, \tag{2.32}$$

$$v = 0 \quad \text{on} \quad \Gamma. \tag{2.33}$$

It is convenient in the subsequent analysis to restrict attention to D_+, the right half of D where $x_1>0$. The notation L_-, L_+ is also introduced for the line segments

$$\{(x_1, x_2) | \ x_1 = 0, -\infty < x_2 < -B\} \quad , \ \{(x_1, x_2) | \ x_1 = 0, \ B < x_2 < \infty\}$$

respectively; Γ_+ denotes the part of Γ where $x_1 \geq 0$. It follows from symmetry considerations that v vanishes on L_- and L_+. Thus the boundary value problem for v is

$$\mathcal{L} v = 0 \qquad \text{on } D_+, \tag{2.34}$$

$$v = 0 \qquad \text{on } \Gamma_+ \cup L_- \cup L_+ , \tag{2.35}$$

$$v = \tau_\infty x_1 \quad \text{as } x_\alpha x_\alpha \to \infty \quad \text{in } D_+. \tag{2.36}$$

From (2.26) we see that

$$\tau = (\tau_{31}^2 + \tau_{32}^2)^{\frac{1}{2}} = |\nabla v| \ , \tag{2.37}$$

and so by virtue of (2.35) we have

$$\tau = \left|\frac{\partial v}{\partial n}\right| \quad \text{on} \quad \Gamma_+ \ . \tag{2.38}$$

Thus to estimate τ on Γ_+ we need to estimate the outward normal derivative $\partial v/\partial n$ on Γ_+.

3. COMPARISON THEOREMS

The following theorems are fundamental to the rest of this study. They are quoted here without proof. The proofs can be found in Abeyaratne and Horgan (1983), and are consequences of standard comparison principles for the uniformly elliptic quasilinear partial differential equation (2.31).

Theorem 1. Let v be a solution to the Dirichlet Problem (2.34)-(2.36) on D_+. Then

$$\frac{\partial v}{\partial n} \leq 0 \quad \text{on} \quad \Gamma_+. \tag{3.1}$$

From (3.1) and (2.38) one concludes that

$$\tau = -\frac{\partial v}{\partial n} \quad \text{on } \Gamma_+ . \tag{3.2}$$

Theorem 2. Let v be a solution of the Dirichlet problem (2.34)-(2.36) on D_+. Suppose that a function w, with the same smoothness as v, exists on D_+ such that

$$L\, w \geq 0 \quad \text{on } D_+, \tag{3.3}$$

$$w = v \quad \text{on } \Gamma_+, \tag{3.4}$$

$$w \leq v \quad \text{on } L_-,\ L_+, \tag{3.5}$$

$$\lim w(x_1, x_2) < \tau_\infty x_1 \quad \text{as} \quad x_\alpha x_\alpha \to \infty. \tag{3.6}$$

Then

$$\frac{\partial v}{\partial n} \leq \frac{\partial w}{\partial n} \quad \text{on} \quad \Gamma_+. \tag{3.7}$$

By virtue of (3.2), the inequality (3.7) provides the lower bound

$$\tau_{max} \geq - \frac{\partial w}{\partial n} \quad \underline{\text{on}} \quad \Gamma_+ . \tag{3.8}$$

If the inequality signs in (3.3), (3.5) and (3.6) are reversed, then a conclusion similar to (3.7) follows with the inequality sign reversed, thus yielding an upper bound result.

In the next section we will consider the construction of a suitable comparison function w conforming to (3.3)-(3.6).

4. COMPARISON FUNCTION

In this section we will construct a comparison function satisfying (3.3)-(3.6) and then use it to find a lower bound for τ_{max} for a softening material. A similar analysis yields an upper bound for a hardening material and is briefly discussed in Section 6. It is convenient to work with the elliptic coordinates ξ and η defined (implicitly) by

$$x_1 = C \cosh \xi \ \cos \eta \qquad \xi \geq \xi_o ,$$
$$x_2 = C \sinh \xi \ \sin \eta \qquad -\pi \leq \eta < \pi , \tag{4.1}$$

where $\xi = \xi_o$ represents the boundary of the elliptic cavity Γ and 2C is the distance between its foci. In terms of the semimajor and semiminor axes A and B of the ellipse one has

$$\xi_o = .5 \ \ln [(A+B)/(A-B)],$$
$$C = (A^2 - B^2)^{\frac{1}{2}}. \tag{4.2}$$

The differential operator L appearing in (3.3) can be written as[1]

$$L\, w = 2V'(|\nabla w|^2) \ \{\frac{V(|\nabla w|^2)}{2V'(|\nabla w|^2)} (w_{\xi\xi} + w_{\eta\eta})$$
$$+ \frac{1}{h^2} [w_\xi^2 \ w_{\xi\xi} + 2w_\xi \ w_\eta \ w_{\xi\eta} + w_\eta^2 \ w_{\eta\eta}$$
$$- \frac{1}{h} (w_\xi^2 + w_\eta^2) \ (h_\xi w_\xi + h_\eta w_\eta)]\} \ , \tag{4.3}$$

(1) Here and in the sequel subscripts ξ and η denote partial differentiation with respect to ξ and η respectively.

where h is the "scale factor" for elliptic coordinates and $|\nabla w|$ is the magnitude of the gradient of w in these coordinates:

$$h = C\,(\sinh^2\xi + \sin^2\eta)^{\frac{1}{2}}\ ,$$
$$|\nabla w|^2 = (w_\xi^2 + w_\eta^2)/h^2\ . \tag{4.4}$$

Since V and V' have $|\nabla w|^2$ as their arguments, the form of the operator L given by (4.3) is quite complicated. However, we note that by (3.3), we merely require $L\ w \geq 0$, and it turns out that a simpler set of sufficient conditions is obtainable to ensure that this holds.

To see this, we first note that from (2.4) and (2.29) one has

$$\frac{2V'(\tau^2)}{V(\tau^2)} = \frac{1}{\tau^2}\left(\frac{\hat{\tau}}{k\hat{\tau}'(k)} - 1\right)\ , \qquad \hat{k} = k(\tau), \tag{4.5}$$

and so from the definition of m in (2.17) we have

$$m(s) = \max_{0\leq\tau\leq s} \frac{2\tau^2 V'(\tau^2)}{V(\tau^2)}\ . \tag{4.6}$$

Consequently, one can readily verify that, if for any positive number s_o, w satisfies

$$(w_\xi^2 + w_\eta^2)(w_{\xi\xi}+w_{\eta\eta}) + m(s_o)\ [w_\xi^2\, w_{\xi\xi} + 2w_\xi\, w_\eta\, w_{\xi\eta} + w_\eta^2\, w_{\eta\eta}$$
$$- \frac{1}{h}(w_\xi^2 + w_\eta^2)(h_\xi w_\xi + h_\eta w_\eta)] \geq 0 \quad \text{on } D, \tag{4.7}$$

$$w_{\xi\xi} + w_{\eta\eta} \geq 0, \quad 0 \leq |\nabla w| \leq s_o \qquad \text{on } D, \tag{4.8}$$

then $L\, w \geq 0$ on D_+. Therefore, we can in Theorem 2 replace (3.3) by the simpler requirements (4.7) and (4.8).

Consideration of (2.35) and (3.4)-(3.6) motivates us to seek a comparison function of the form

$$w(\xi,\eta) = f(\xi)\cos\eta \quad , \quad f(\xi) \geq 0. \tag{4.9}$$

It is seen that w vanishes on L_- and L_+ and so, in view of (2.35) satisfies (3.5) with equality. Substituting (4.9) into (4.7) shows that the inequality holds <u>if</u> the following four ordinary differential inequalities are satisfied

$$(m+1)\sinh\xi f'' - m\cosh\xi f' - \sinh\xi f \geq 0, \tag{4.10}$$

$$\sinh\xi f f'' + 2m\sinh\xi f'^2 - m\cosh\xi f f' - (m+1)\sinh\xi f^2 \geq 0, \tag{4.11}$$

$$(m+1)f'' + (m-1)f \geq 0, \tag{4.12}$$

$$ff'' + 2mf'^2 - f^2 \geq 0, \tag{4.13}$$

for $\xi > \xi_o$ where $m = m(s_o)$ and primes denote differentiation with respect to ξ. Substitution of (4.9) into the first of (4.8), (3.4) and (3.6) gives

$$f'' - f \geq 0 \quad \text{for } \xi > \xi_o \ , \tag{4.14}$$

$$f(\xi_o) = 0 \quad , \lim_{\xi\to\infty} (2e^{-\xi} f(\xi)/C) < \tau_\infty \ . \tag{4.15}$$

We now construct a function $f(\xi)$ conforming to (4.10)-(4.15) and then show that the second of (4.8) holds. To this end we solve (4.10) with equality subject to (4.15) to find

$$f(\xi) = C\tau_\infty \cosh\xi F(\xi), \tag{4.16}$$

where

$$F(\xi) = 1 - \frac{\displaystyle\int_{\xi}^{\infty} \frac{(\sinh\zeta)^{\frac{m}{m+1}}}{\cosh\zeta^{2}}\, d\zeta}{\displaystyle\int_{\xi_o}^{\infty} \frac{(\sinh\zeta)^{\frac{m}{m+1}}}{\cosh\zeta^{2}}\, d\zeta} \; (\geq 0) \; . \tag{4.17}$$

Direct computation shows that (4.16) satisfies (4.14) with inequality. We next rearrange (4.12) and using the fact that $f \geq 0$, $m \geq 0$, obtain

$$f'' + \frac{m-1}{m+1} f = f'' - f + \frac{2m}{m+1} f \geq f'' - f > 0, \tag{4.18}$$

the last inequality following from (4.14) with inequality. Thus (4.12) is seen to hold. To verify (4.13) we note that

$$ff'' + 2mf'^{2} - f^{2} \geq f(f'' - f) > 0, \tag{4.19}$$

where (4.14) has again been used.

Finally, we turn to the verification of (4.11). We observe that from (4.10) (with equality) and (4.14) we have

$$\cosh\xi f' \geq \sinh\xi f \qquad \text{for } \xi \geq \xi_o \; , \tag{4.20}$$

which in particular implies that

$$f'(\xi) \geq 0 \qquad \text{for } \xi \geq \xi_o \; . \tag{4.21}$$

Multiplying (4.11) by (1+m) and making use of (4.20) we find that (4.11) holds provided that

$$\sinh\xi f' - \cosh\xi f \geq 0 \quad \text{for } \xi > \xi_o \; . \tag{4.22}$$

To verify (4.22) let

$$\Phi(\xi) = \sinh\xi \, f' - \cosh\xi \, f \ , \quad \text{for} \quad \xi \geq \xi_o \ . \tag{4.23}$$

Differentiation yields

$$\Phi'(\xi) = (f'' - f) \sinh\xi \geq 0, \ \text{for} \ \xi \geq \xi_o \quad , \tag{4.24}$$

where the inequality holds by virtue of (4.14). $\Phi(\xi)$ is thus a nondecreasing function of ξ. But

$$\Phi(\xi_o) = \sinh\xi \quad f'(\xi_o) \geq 0, \tag{4.25}$$

by virtue of the first of (4.15) and (4.21). The inequality (4.22) now follows.

We now show that $|\nabla w|$ attains its maximum at the point $\xi = \xi_o, \eta = 0$. From (4.4) and (4.9) we have

$$|\nabla w|^2 = \frac{f'^2 \cos^2\eta + f^2 \sin^2\eta}{C^2(\sinh^2\xi + \sin^2\eta)} \quad . \tag{4.26}$$

Simple calculations show that, on using (4.20),

$$|\nabla w(\xi,\eta)|^2 = \frac{f'^2\cos^2\eta + f^2\sin^2\eta}{C^2(\sinh^2\xi + \sin^2\eta)} \leq \left(\frac{f'(\xi)}{C\sinh\xi}\right)^2 = |\nabla w(\xi, 0)|^2 \ , \tag{4.27}$$

It is readily shown that the right hand side of (4.27) decreases with ξ so that

$$\left(\frac{f'(\xi)}{C\sinh\xi}\right)^2 \leq \left(\frac{f'(\xi_o)}{C\sinh\xi_o}\right)^2 = |\nabla w(\xi_o, 0)|^2. \tag{4.28}$$

Thus from (4.27), (4.28), (4.16), (4.17) we have

$$0 \leq |\nabla w| \leq |\nabla w|_{max} = \tau_\infty \frac{(\sinh\xi_\circ)^{-\frac{1}{m+1}}}{\cosh\xi_\circ} \frac{1}{g(\xi_\circ;m)} , \tag{4.29}$$

where

$$g(\xi;m) = \int_\xi^\infty \frac{(\sinh\zeta)^{\frac{m}{m+1}}}{\cosh^2\zeta} d\zeta, \quad m = m\ (s_\circ). \tag{4.30}$$

Therefore if a positive number $s_\circ$ exists such that

$$\tau_\infty \frac{(\sinh\xi_\circ)^{-\frac{1}{m+1}}}{\cosh\xi_\circ} \frac{1}{g(\xi_\circ;m)} \leq s_\circ,\ m = m(s_\circ), \tag{4.31}$$

then the second of (4.8) is satisfied. Assuming for the moment the existence of $s_\circ$ (>0) we note that $w(\xi,\eta)$ given by (4.9), (4.16) and (4.17) satisfies all of the requirements for an admissible comparison function. We can therefore use it, in conjunction with (3.8), to find a lower bound on the stress concentration factor K:

$$K \geq \frac{(\sinh\xi_\circ)^{-\frac{1}{m+1}}}{\cosh\xi_\circ} \frac{1}{g(\xi_\circ;m)} , \tag{4.32}$$

where $m = m(s_\circ)$ is defined by (2.17), $g(\xi\ ;m)$ is given by (4.30), and $s_\circ$ is any positive number conforming to (4.31).

We now prove that such a number can always be found. Furthermore $s_\circ$ can be determined in such a way that the right hand side of (4.32) is maximized. In other words the optimum value for $s_\circ$ can be determined.

To show the existence of a number s_o conforming to (4.31), we define a function $y(t)$ by

$$y(t) = \frac{\tau_\infty}{\cosh\xi_o} \frac{(\sinh\xi_o)^{-\frac{1}{t+1}}}{g(\xi_o;t)} \quad \text{for} \quad t \geq 0 \; . \tag{4.33}$$

Differentiation with respect to t shows that $y'(t) \leq 0$.

Now let

$$z(s) = y(m(s)) - s \quad \text{for} \quad s \geq 0. \tag{4.34}$$

Using the chain rule and recalling that $m(s)$ is a non-decreasing function of s we find $z'(s) < 0$. Thus z is a decreasing function of s. Moreover $z(0) = y(0) > 0$ and $z(\infty) = -\infty < 0$. Therefore there exists a unique positive number s_* such that $z(s_*) = 0$, $z(s) < 0$ for $s > s_*$ and $z(s) > 0$ for $0 < s < s_*$. Thus any number $s > s_*$ satisfies (4.31). It can be easily shown that s_* is the optimum value of s. To see this we note that the "best value" of s is a positive number which conforms to (4.31) and maximizes the right hand side of (4.32) which means maximizing $y(m(s))$ given by (4.33). Since $y(m)$ is a decreasing function of m the optimum value must minimize $m(s)$. There is only one such value of s namely $s = s_*$. From (4.34) then we have that s_* is the unique positive root of

$$s_* = y(m(s_*)), \tag{4.35}$$

and

$$K = \tau_{max}/\tau_\infty \geq s_*/\tau_\infty \quad . \tag{4.36}$$

The lower bound on the stress concentration factor K given by (4.36) depends in particular on τ_∞, the applied stress at infinity.

A weaker (and simpler) "load independent" lower bound can be found which is independent of τ_∞. We simply recall that $y(m)$ is a decreasing function of m and $m \leq \frac{1}{c} - 1$ (see (2.19)). It then follows that $y(m) \geq y(\frac{1}{c} - 1)$ and so we have

$$K \geq \frac{s_*}{\tau_\infty} \geq \frac{y(\frac{1}{c} - 1)}{\tau_\infty} . \tag{4.37}$$

To summarize, an admissible comparison function has been constructed (see (4.9), (4.16) and (4.17)) and expressions for lower bounds on the stress concentration factor for a softening material derived ((4.37)).

5. RESULTS

In this section we first discuss the load independent lower bound given in (4.37). Then we provide an example of how the general load dependent lower bound given by the first of (4.37) can be found explicitly by considering a special constitutive law. We conclude with a brief discussion of the limiting case of a "thin" ellipse.

5.1 The Load Independent Lower Bound

Denoting the load independent lower bound in (4.37) by K_* we have

$$K_* = \frac{y(\frac{1}{c}-1)}{\tau_\infty} = \frac{(\sinh\xi_o)^{-c}}{\cosh\xi_o} \; \frac{1}{g(\xi_o;\frac{1}{c}-1)}, \tag{5.1}$$

where

$$g(\xi_o\,;\,\frac{1}{c}-1) = \int_{\xi_o}^{\infty} \frac{(\sinh\zeta)^{1-c}}{\cosh^2\zeta}\,d\zeta \quad , \qquad 0<c<1. \tag{5.2}$$

It appears that the integral on the right-hand-side of (5.2) cannot be evaluated analytically and so we seek an upper bound for it. It can be shown that (see Appendix A)

$$g(\xi_o;\frac{1}{c}-1) \leq \frac{1}{1+c}\,[2(1-\tanh\xi_o)]^{\frac{1+c}{2}} \quad . \tag{5.3}$$

Substituting into (5.1) and simplifying we find

$$K_* \geq (1+c)\left(\frac{B}{A}\right)^{\frac{1-c}{2}}\left(\frac{A+B}{2B}\right)^{\frac{1+c}{2}} \quad , \tag{5.4}$$

where A and B are the semimajor and semiminor axes of the ellipse respectively.

Specializing to the neo-Hookean material, we set $c = 1$ in (5.4) and obtain

$$K_* \geq 1 + A/B \ . \tag{5.5}$$

Recalling the exact result (2.23) we see that the lower bound is optimal in this case. We note also that for the special case of a circle ($A = B$) (5.4) reduces to

$$K_* \geq 1 + c \ , \tag{5.6}$$

which is the result found by Abeyaratne and Horgan (1983).

5.2 The Load Dependent Lower Bound; Results for a Ramberg-Osgood Material

A Ramberg-Osgood material is a material with a response function in simple shear given by

$$k = \hat{k}(\tau) = \tau + \frac{1}{Q}\tau^{\frac{1}{c}} \ , \qquad Q > 0, \quad 0<c<1, \tag{5.7}$$

where k is the amount of shear and τ is the corresponding nondimensionalized shear stress (τ/μ); $Q > 0$ is a material constant and c is a softening parameter $0 < c < 1$. From (4.6) we find that for this material

$$m(s) = \left(\frac{1}{c} - 1\right) \frac{1}{1+Q\, s^{1-\frac{1}{c}}} \ , \tag{5.8}$$

and by (4.35) the optimum value of s is the unique positive root of

$$s_* = \frac{\tau_\infty}{\cosh\xi_o} \frac{(\sinh\xi_o)^{-\frac{1}{m(s_*)+1}}}{g(\xi_o;\, m(s_*))} \qquad , \qquad (5.9)$$

where m(s) and g(ξ; m) are given by (5.8) and (4.30) respectively. Thus with a simple change of variable $u = e^{-\xi}$ we can write (5.9) in the form

$$\int_0^{e^{-\xi_o}} \left[\frac{s_*}{\tau_\infty} (\sinh\xi_o)^{\frac{1}{m+1}} \cosh\xi_o \left(\frac{1-u^2}{2}\right)^{\frac{m}{m+1}} \left(\frac{2}{1+u^2}\right)^2 u^{\frac{1}{m+1}} - e^{\xi_o}\right] du = 0 \,. \qquad (5.10)$$

Solving (5.8) and (5.10) numerically, the value of s_*, and hence by (4.37), a lower bound on the stress concentration factor K can be determined.

Obviously the lower bound depends on the geometry of the cavity and the material properties. In Figure 2 the effect of geometry is shown where we have drawn the graphs of s_*/τ_∞ versus τ_∞ for a material with the constant Q = 100 and the softening parameter c = .1 for different ellipses. Similar results are shown in Figures 3 and 4 for values of c = .2 and .5 respectively. Figure 5 on the other hand shows the effect of the softening parameter. Here we have drawn the lower bound versus τ_∞ for a circle. The material constant has again been taken as Q = 100 but the softening parameter varies.

It can be seen from these Figures that for small values of the applied stress, the graphs are almost horizontal with the load dependent lower bound nearly equal to the exact value for the linear case i.e. 1 + A/B , as one would expect. On increasing the applied stress the load dependent lower bound s_*/τ_∞, as expected,

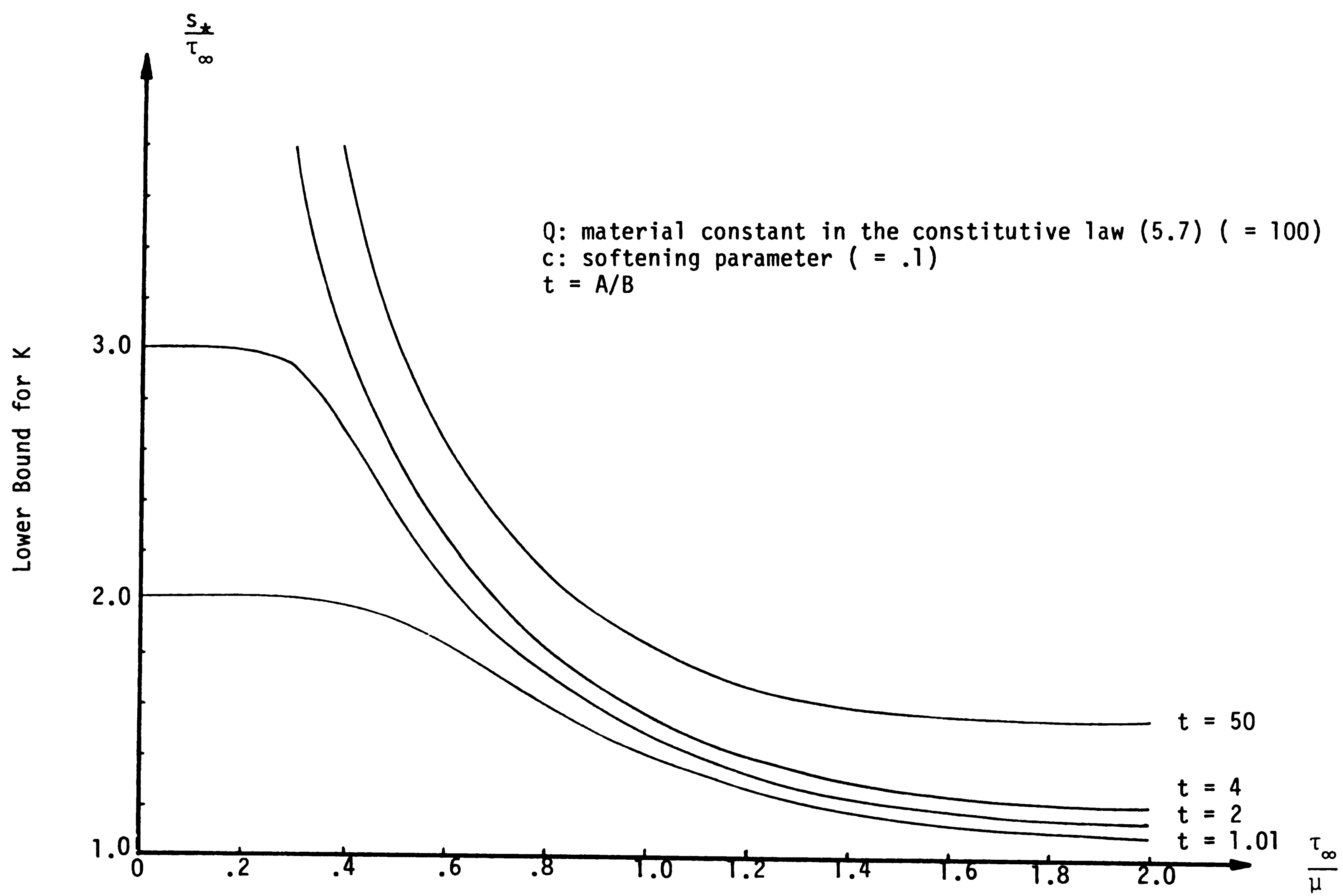

Figure 2. Lower bound for the stress concentration factor versus the applied stress at infinity for different ellipses.

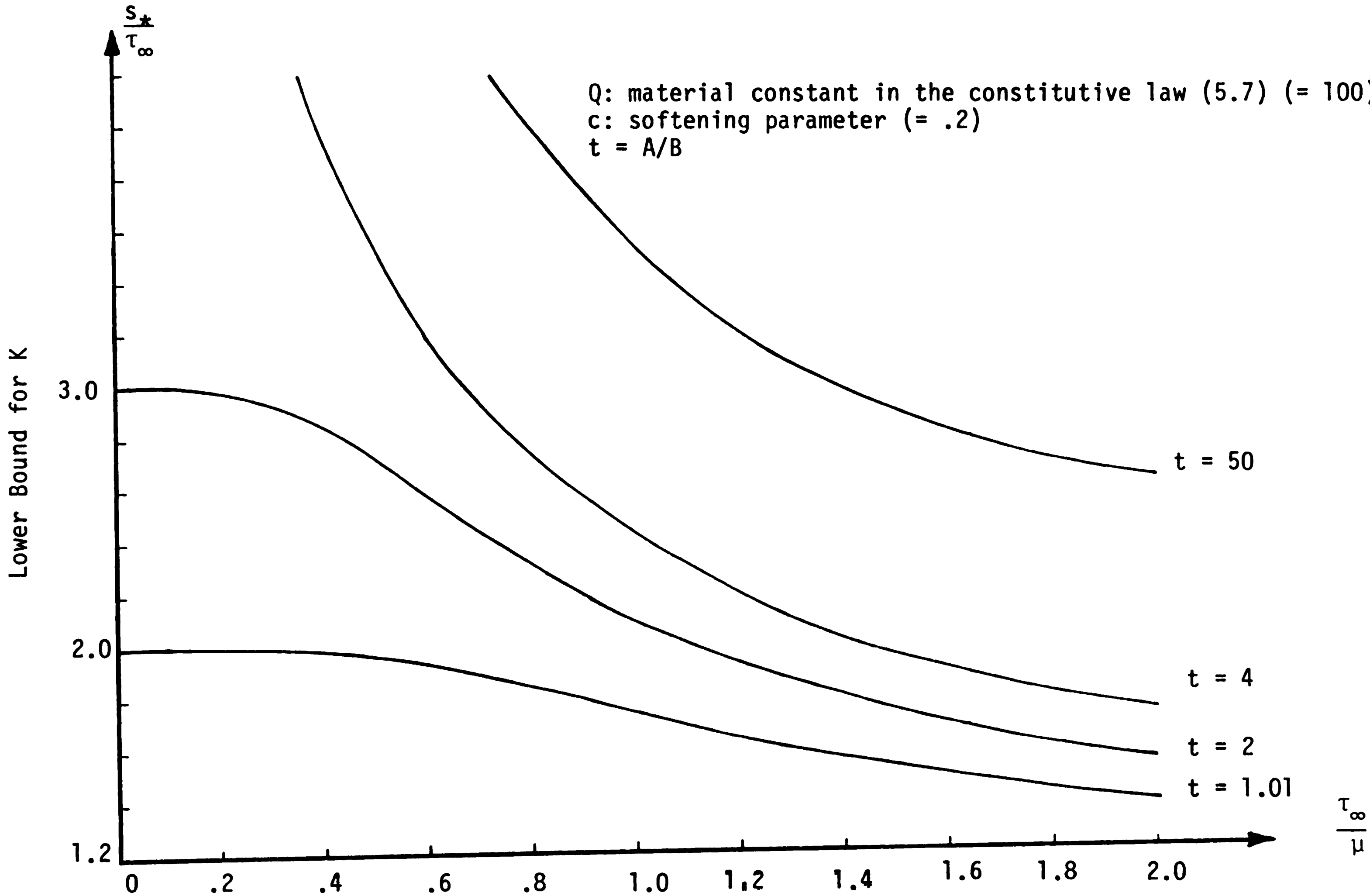

Figure 3. Lower bound for the stress concentration factor versus the applied stress at infinity for different ellipses.

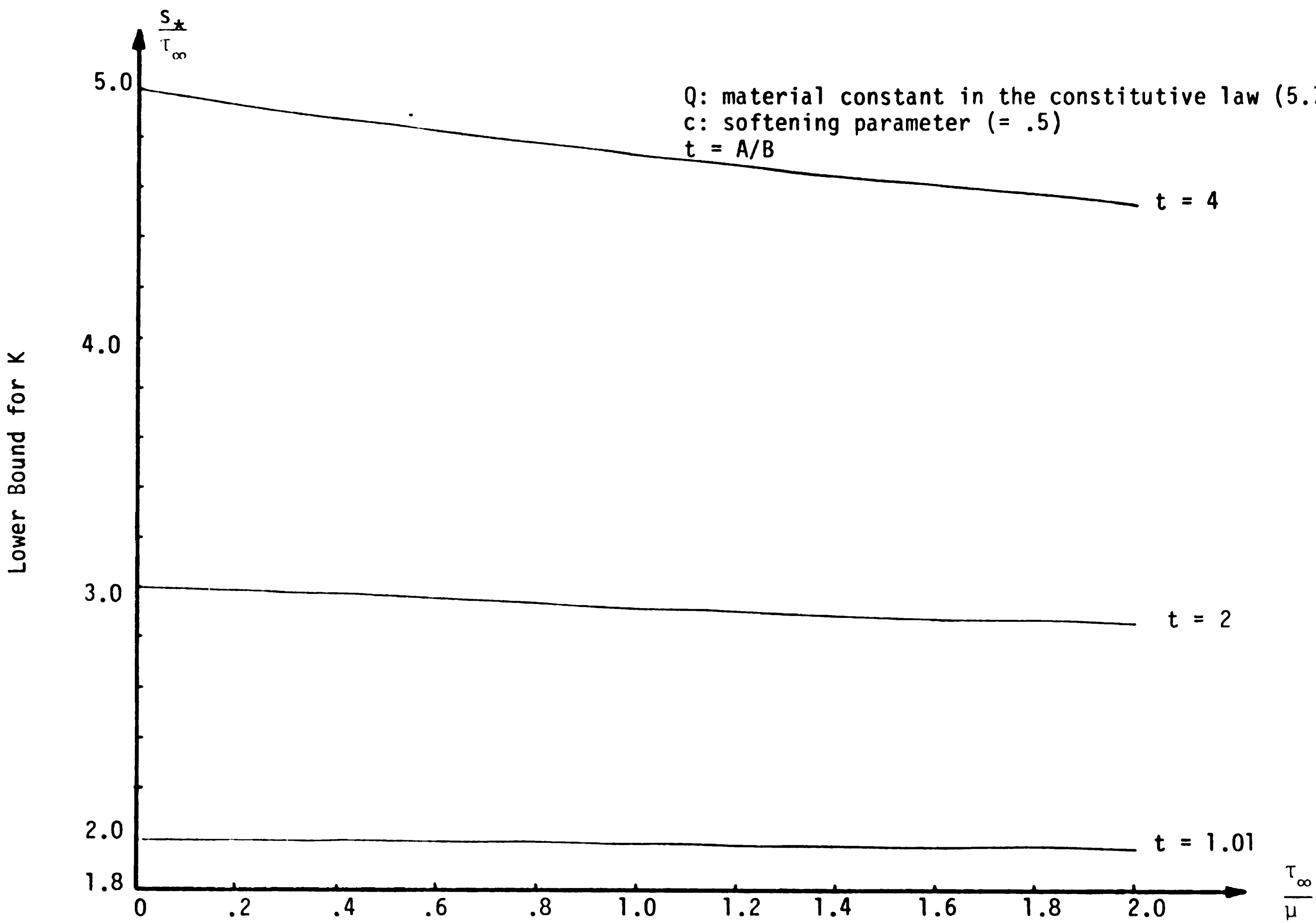

Figure 4. Lower bound for the stress concentration factor versus the applied stress at infinity for different ellipses.

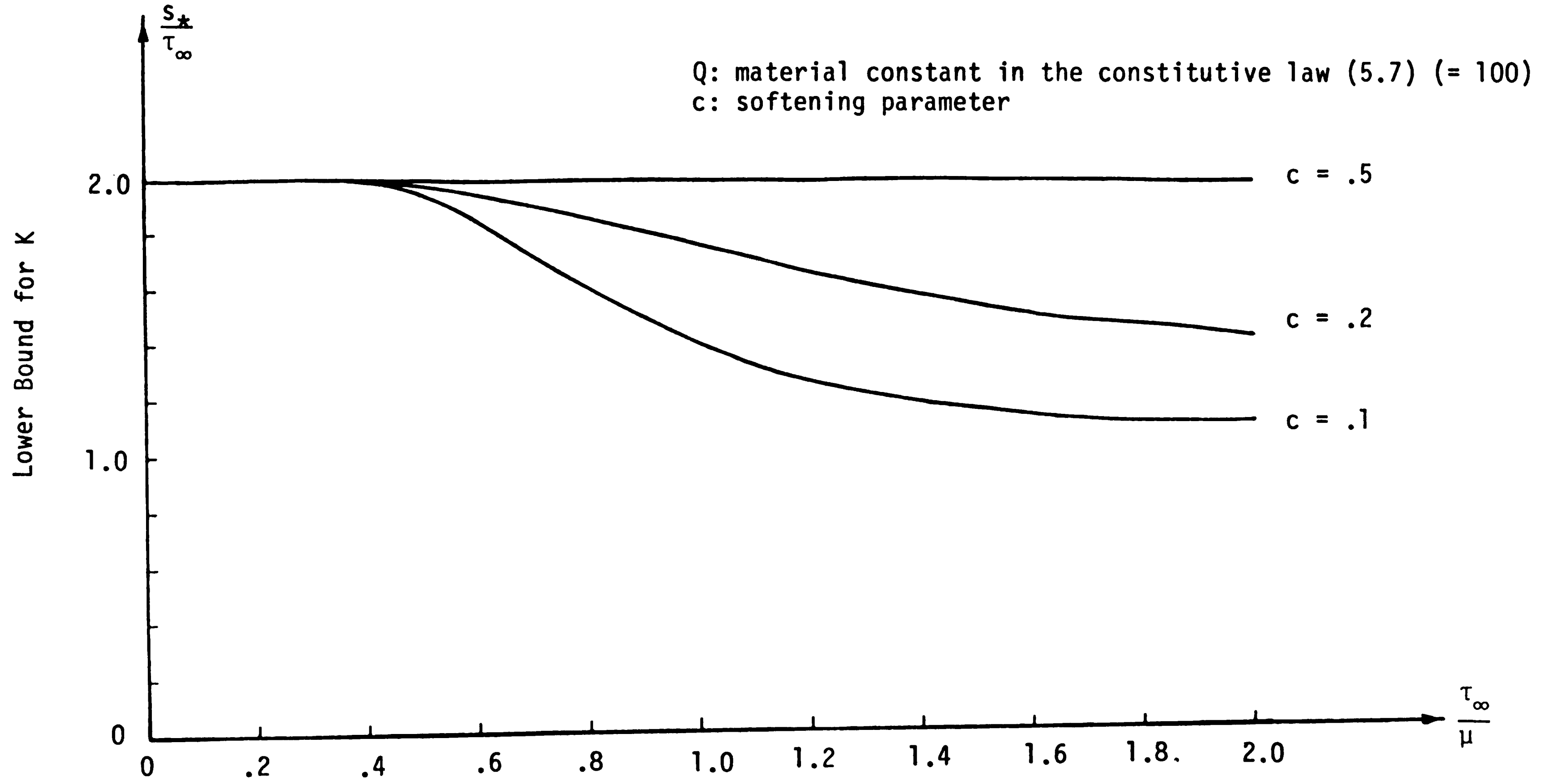

Figure 5. Lower bound for the stress concentration factor for a circular cavity versus the applied stress at infinity.

decreases, since the material is softening. The decrease in the lower bound at higher loads depends on the softening parameter as is shown in Figure 5.

5.3 Limiting Results for a Thin Ellipse

The case of a thin ellipse in which $B/A \ll 1$ is of interest since it can be used to model a straight crack. If in (5.4) we assume that $B/A \ll 1$ we find that

$$K_* \approx 2^{-\frac{1+c}{2}} (1+c) \left(\frac{B}{\rho}\right)^c , \tag{5.11}$$

where $\rho = B^2/A$ is the radius of curvature at the "tip" of the ellipse.

6. CONCLUDING REMARKS

In this section we first discuss the question of obtaining an upper bound for the stress concentration factor for softening materials. We will then briefly consider upper and lower bounds for hardening materials and conclude with some suggestions for further work.

6.1 Upper Bound for a Softening Material

We recall that Theorem 2, Section 3, will yield an upper bound on τ_{max} and hence on the stress concentration factor if one reverses the inequality signs in (3.3) and (3.5) (or equivalently in (4.7), the first of (4.8) and (3.5)). Again it is natural to seek comparison functions of the form (4.9) which leads to (4.10)-(4.14) with inequality signs reversed. Since by the first of (4.15) $f(\xi_o) = 0$, both (4.11) and (4.13) require that $2mf'^2(\xi_o) \leq 0$ (m is positive, see (4.6)), which yields $f'(\xi_o) = 0$. Consequently, comparison functions of this form are not of interest. Attempts to construct admissible comparison functions of a different type have so far proved unsuccessful. Since the stress response curve for a softening material always lies below the corresponding curve for a linear material, one might conjecture that $1+ A/B$ is a universal upper bound for K. We have not, however, been able to provide a proof for this conjecture.

6.2 Upper Bound for a Hardening Material

As was pointed out at the beginning of Section 4, the results found there for a softening material can be modified to yield an upper bound for a hardening material.

We first note that from (4.5) and (2.18) one has

$$n(s) = \min_{0 \leq \tau \leq s} \frac{2\tau^2 V'(\tau^2)}{V(\tau^2)} \quad . \tag{6.1}$$

Now if for any positive number s_o, w satisfies

$$(w_\xi^2 + w_\eta^2)(w_{\xi\xi} + w_{\eta\eta}) + n(s_o)\,[w_\xi^2 w_{\xi\xi} + 2w_\xi w_\eta w_{\xi\eta} + w_\eta^2 w_{\eta\eta} - \frac{1}{h}(w_\xi^2 + w_\eta^2)(h_\xi w_\xi + h_\eta w_\eta)] \leq 0 \quad \text{on } D, \tag{6.2}$$

$$w_{\xi\xi} + w_{\eta\eta} \leq 0 \quad , \quad 0 < |\nabla w| < s_o \qquad \text{on } D, \tag{6.3}$$

then $L\, w \leq 0$ on D_+. Therefore in the upper bound version of Theorem 2 we can replace (3.3) by (6.2) and (6.3).

It can be verified that comparison functions of the form (4.9) are admissable provided (4.10)-(4.14) hold with inequality signs reversed and m(s) replaced by n(s). Equation (6.3) must of course hold. For simplicity here, we confine attention to a load independent upper bound K^*. Here we replace n(s) by $\frac{1}{b} - 1$ (b>1) and after some simple calculations strictly analogous to those in the softening case we find

$$K^* = \frac{(\sinh\xi_o)^{-b}}{\cosh\xi_o} \; \frac{1}{g(\xi_o; \frac{1}{b} - 1)} \quad , \tag{6.4}$$

where $g(\xi_o; \frac{1}{b} - 1)$ is given by (5.2) with c replaced by b(>1). It can easily be shown that

$$g(\xi_o; \frac{1}{b} - 1) \geq \frac{1}{1+b} \left[2(1-\tanh\xi_o)\right]^{\frac{1+b}{2}} . \tag{6.5}$$

Substitution in (6.4) yields

$$K^* \leq (1+b) \left(\frac{B}{A}\right)^{\frac{1-b}{2}} \left(\frac{A+B}{2B}\right)^{\frac{1+b}{2}} , \tag{6.6}$$

where A and B are the semimajor and semiminor axes of the ellipse respectively. In the special case of a circle (A = B) we find $K^* \leq 1+b$ which is the result found by Abeyaratne and Horgan (1983).

6.3 Suggestions for Further Work

We have established lower bounds for softening materials (and upper bounds for hardening ones) (see (4.37), (5.4) and (6.6)). As noted previously, we have been unable to find upper bounds for softening materials (and lower bounds for the hardening case). This issue should be resolved if possible. There are also some places where the present work may possibly be improved. The differential inequalities (4.10)-(4.14) are sufficient conditions for $L\,w \geq 0$. While it is not difficult to establish necessary and sufficient conditions they are rather complicated. It would be worthwhile to investigate these and see if sharper results can be established. Another area where improvement may be possible is in connection with the integral on the right hand side of (5.2). Efforts have been made unsuccessfully to evaluate this integral analytically; furthermore, it does not appear to be evaluated explicitly in the standard integral tables. Finally, it would be of interest to use numerical methods (e.g. finite difference or finite element schemes) to compare with the results obtained here.

APPENDIX

APPENDIX A

An Upper Bound for $g(\xi_o;\ \frac{1}{c} - 1)$ Defined by (5.2)

We wish to find an upper bound for the integral

$$g(\xi_o;\ \frac{1}{c} - 1) = \int_{\xi_o}^{\infty} \frac{(\sinh\zeta)^{1-c}}{\cosh^2\zeta}\, d\zeta \quad ,\ \xi_o>0,\ 0<c<1. \tag{A.1}$$

Making the change of variable $z = e^{2\zeta} + 1$ one finds

$$g(\xi_o;\ \frac{1}{c} - 1) = 2^c \int_{1+e^{2\xi_o}}^{\infty} \frac{1}{z^2}\,(z-2)^{\frac{1-c}{2}} \left(\frac{z-2}{z-1}\right)^{\frac{1-c}{2}} dz. \tag{A.2}$$

Since $0<c<1$, $0<(z-2)/(z-1)<1$ it follows that

$$g(\xi_o;\ \frac{1}{c} - 1) \leq 2^c \int_{1+e^{2\xi_o}}^{\infty} \frac{(z-2)^{\frac{1-c}{2}}}{z^2}\, dz \leq 2^c \int_{1+e^{2\xi_o}}^{\infty} z^{-\frac{3+c}{2}}\, dz =$$

$$= \frac{2^{1+c}}{1+c}\,(1+e^{2\xi_o})^{\frac{1+c}{2}}\ . \tag{A.3}$$

On using the identity $1+e^{2\xi_o} = 2(1-\tanh\xi_o)$ we have

$$g(\xi_o;\ \frac{1}{c} - 1) \leq \frac{1}{1+c}\,[2(1-\tanh\xi_o)]^{\frac{1+c}{2}}\ . \tag{A.4}$$

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PART II

DEFORMATION AND STABILITY

OF

A PRESSURIZED TUBE OF HARMONIC MATERIAL

1. INTRODUCTION

The finite deformation of a circular tube of homogeneous, compressible, elastic material of harmonic type subject to simultaneous internal and external pressure is considered. The fundamental (plane strain, axisymmetric) solution is obtained. The stability of this solution in the two special cases of zero external pressure, and zero internal pressure is investigated.

The stability problem for a pressurized tube has been considered by a number of investigators but mostly for incompressible materials. See e.g. Hill (1975, 1976), Haughton and Ogden (1979a, 1979b) and references cited therein. This problem has also been studied numerically for elastic-plastic materials by Chu (1979), Larsson et al (1982) and Reddy (1982) among others. Sensenig (1964) has investigated the stability of a tube composed of a harmonic material of special type - the so called standard harmonic material - under external pressure. His work for this problem seems to be the only one to deal with a compressible elastic material.

Larsson et al (1982) have conducted a numerical and experimental investigation of the deformation of internally pressurized circular tubes composed of ductile metals with slight geometric imperfections. They examine in detail the onset and development of surface instabilities as well as the subsequent initiation and growth of shear bands until failure. The present study was undertaken in an attempt to examine this problem analytically.

In the following section the axisymmetric problem for the pressurized tube is formulated and a brief description of harmonic materials given. The deformation and stress fields are determined explicitly. We observe that the stress field is essentially independent of the constitutive details of the (harmonic) material under consideration. The examination of the deformation field shows that in an externally pressurized tube the pressure must be restricted to values less than 2μ so that no interpenetration occurs. Here μ is the shear modulus of the material at infinitesimal deformations. For an internally pressurized tube, one finds that the pressure must be restricted to values less than $\mu(1-a^2/b^2)$ (a and b being the inner and outer radii of the underformed tube respectively) because the hoop stress at the inner wall becomes unbounded at this pressure. This is an unexpected result since there are no discontinuities in either loading or geometry and is clearly a consequence of the nonlinearity of the constitutive relation.

In Section 3 the equilibrium problem for a tube with small geometrical imperfections in both internal and external boundaries is examined. It is assumed that the resulting plane strain nonaxisymmetric deformation field is a small perturbation of the axisymmetric deformation of a perfectly circular tube. A standard perturbation expansion is then employed to derive the equilibrium equations and the boundary conditions. The equilibrium equations in terms of displacements consist of a set of two linear homogeneous second order partial differential equations with variable coefficients.

In Section 4 we first solve the equilibrium equations and then consider the problem of stability.

In the case of internal pressure only, the analysis shows that no instability occurs. On the other hand, in the case of external pressure the occurrence of an instability is established and the corresponding smallest buckling load determined. Moreover, one finds that for sufficiently thick tubes, and under a rather mild restriction on the material behavior, the buckling loads corresponding to different modes are distinct and form a bounded monotone increasing sequence.

Finally, in Section 5 some numerical results for a special constitutive law are presented.

2. THE PRESSURIZED CYLINDRICAL TUBE; HARMONIC MATERIALS

2.1 Statement of Problem

Let the open region $D_o = \{(r,\phi) \mid a < r < b,\ o < \phi \leq 2\pi\}$ denote the cross section of a right circular cylinder with inner radius a, and outer radius b, in its undeformed configuration. The cylinder is subjected to internal and external pressure of magnitude p_i and p_o respectively. The ensuing deformation is a one-to-one mapping which takes the point with polar coordinates (r,ϕ) in the undeformed region D_o to the point (ρ,ψ) in the deformed region D. We assume that a state of plane strain prevails with appropriate tractions being applied to the ends of the cylinder.

In view of the symmetry of the problem, the deformation is axisymmetric with

$$\rho = rf(r), \quad \psi = \phi \qquad \text{on } D_o, \tag{2.1}$$

where the function f(r) is to be determined. The polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with (2.1) (see e.g. Malvern (1969), p. 652) are given by

$$F_{rr} = rf'(r) + f(r), \quad F_{\phi\phi} = f(r), \quad F_{r\phi} = F_{\phi r} = 0, \tag{2.2}$$

where the prime denotes differentiation with respect to the argument. The left Cauchy-Green deformation tensor is defined as $\underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T$ and its fundamental scalar invariants can be taken as

$$I = \mathrm{tr}\underset{\sim}{G}, \quad J = (\det \underset{\sim}{G})^{\frac{1}{2}}, \tag{2.3}$$

so that in the present problem

$$I = r^2f'^2 + 2rff' + 2f^2, \qquad J = f^2 + rff'. \tag{2.4}$$

The cylinder is assumed to be composed of a material of "harmonic type" as introduced by John (1960), and investigated in detail by Knowles and Sternberg (1975). The brief account of harmonic materials given in the following sub-section follows closely the work by Knowles and Sternberg cited above.

2.2 Harmonic Materials

Harmonic materials are compressible elastic materials with a strain-energy density function in plane strain given by

$$W(I,J) = 2\mu[H(R) - J]\ , \ R = (I+2J)^{\frac{1}{2}}, \tag{2.5}$$

where μ is a positive constant that can be identified with the infinitesimal shear modulus and H is a continuous function of R, defined for all $R > 0$, with continuous derivatives of all orders.

The Cauchy stress tensor $\underset{\sim}{\tau}$ associated with a plane deformation is given by

$$\begin{aligned} \underset{\sim}{\tau} &= \frac{2}{J}\ \frac{\partial W}{\partial I}\ \underset{\sim}{G} + \frac{\partial W}{\partial J}\ \underset{\sim}{1} \\ &= 2\mu\{\frac{1}{J}\ h(R)\underset{\sim}{G} + [H(R)-1]\ \underset{\sim}{1}\}, \text{ on } D, \end{aligned} \tag{2.6}$$

where we have set

$$h(R) = \frac{H'(R)}{R} \quad \text{for} \quad R > 0, \tag{2.7}$$

and $\underset{\sim}{1}$ is the second order identity tensor. The Piola stress field $\underset{\sim}{\sigma}$ associated with the Cauchy stress field $\underset{\sim}{\tau}$ is defined by

$$\underset{\sim}{\sigma} = J\underset{\sim}{\tau}\underset{\sim}{F}^{-T} = 2\mu\{h(R)\underset{\sim}{F} + J[h(R)-1]\ \underset{\sim}{F}^{-T}\}\ , \text{ on } D_o, \tag{2.8}$$

where $\underset{\sim}{F}^{-T}$ is the transpose of the inverse of $\underset{\sim}{F}$.

In order to ensure a physically reasonable response, one must impose certain restrictions on the constitutive function $H(R)$. Since Cauchy stress and the strain-energy density should vanish in the undeformed state, we must have

$$H(2) = 1, \quad H'(2) = 1. \tag{2.9}$$

Furthermore, the strain-energy density function (2.5) should be positive in every state, except the undeformed one. This requirement entails the inequality

$$H(R) > R^2/4 \text{ for all } R > 0,\ R \neq 2. \tag{2.10}$$

Next, from consideration of the true stress field induced in a plane isotropic deformation and the requirement, on physical grounds, that stress should be monotone increasing with the amount of stretch one deduces that $h(R) = H'(R)/R$ must be monotone increasing, i.e.

$$h'(R) > 0 \text{ for } 0<R<\infty . \tag{2.11}$$

Finally, we will suppose that the material admits a regular state of uniaxial tension in plane strain, for which it is necessary and sufficient that there exist a number $R_* \in (1,2)$ such that

$$h(R_*) = 0, \quad h(R) \to 1 \text{ as } R \to \infty \text{ and } H''(R)>1 \text{ for } R_* < R<\infty. \tag{2.12}$$

A more complete discussion can be found in the paper by Knowles and Sternberg (1975).

2.3 Deformation and Stress Fields

Returning to the problem under consideration we have from (2.4) and the second of (2.5) that

$$R = 2f + rf', \quad a< r<b. \tag{2.13}$$

On substituting from (2.2) into (2.8) and making use of (2.13) one finds that the components of Piola stress $\underset{\sim}{\sigma}$ are given by

$$\sigma_{rr} = 2\mu\,[H'(R) - f]\ , \quad \sigma_{\phi\phi} = 2\mu[H'(R) - (R - f)]\ ,$$

$$\sigma_{r\phi} = \sigma_{\phi r} = 0. \tag{2.14}$$

Also from (2.8) we find the Cauchy stress components

$$\tau_{\rho\rho} = \sigma_{rr}/f,\ \tau_{\psi\psi} = \sigma_{\phi\phi}/(f+rf'),\ \tau_{\rho\psi} = \tau_{\psi\rho} = 0. \tag{2.15}$$

In the absence of body forces, the equilibrium equations $\operatorname{div}\underset{\sim}{\sigma} = \underset{\sim}{0}$, in the present case reduce to the single equation

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0. \tag{2.16}$$

After substituting for the stresses from (2.14) and making use of (2.13), one finds that (2.16) reduces to $\frac{d}{dr}\,[H'(R)] = 0$ for $a<r<b$ which is equivalent to

$$\left.\frac{dH(R)}{dR}\right|_{R\,=\,2f+rf'} = \text{constant}. \tag{2.17}$$

Now by (2.12) $H''(R) > 0$ for $R_* < R < \infty$. Therefore, if for the deformation considered here,

$$R > R_*, \tag{2.18}$$

where R is given by (2.13), then $H'(R)$ is monotone increasing on the interval of interest and hence may be uniquely inverted. It will be shown later that (2.18) indeed holds. Here we <u>assume</u> that (2.18) holds and so deduce from (2.17) that

$$2f(r) + rf'(r) = R_o \text{ (constant) for } a<r<b. \tag{2.19}$$

Integration of (2.19) yields

$$f(r) = \frac{1}{2} R_o + \frac{C}{r^2} , \tag{2.20}$$

where R_o and C are constants to be determined.

The prescribed (pressure) boundary conditions are

$$\tau_{\rho\rho} = - p_i \text{ on } \rho = \alpha \quad , \tau_{\rho\rho} = - p_o \text{ on } \rho = \beta , \tag{2.21}$$

where $\alpha(=af(a))$ and $\beta(=bf(b))$ are the inner and outer radii respectively, in the deformed configuration. In view of (2.7), (2.14) and (2.15) the boundary conditions (2.21) may be written as

$$2\mu R_o h(R_o) = (2\mu - p_i) f(a), \tag{2.22}$$

$$2\mu R_o h(R_o) = (2\mu - p_o) f(b). \tag{2.23}$$

Upon substituting for $f(a)$ and $f(b)$ from (2.20) and rearranging, one has

$$\begin{bmatrix} 4\mu R_o a^2 & -2(2\mu - p_i) \\ 4\mu R_o b^2 & -2(2\mu - p_o) \end{bmatrix} \begin{bmatrix} h(R_o) \\ C \end{bmatrix} = \begin{bmatrix} (2\mu - p_i) a^2 R_o \\ (2\mu - p_o) b^2 R_o \end{bmatrix} , \tag{2.24}$$

which can be solved for $h(R_o)$ and C to yield

$$h(R_o) = \frac{(b^2 - a^2)(2\mu - p_o)(2\mu - p_i)}{4\mu[b^2(2\mu - p_i) - a^2(2\mu - p_o)]} , \tag{2.25}$$

$$C = \frac{a^2 b^2 (p_i - p_o) R_o}{2[b^2(2\mu - p_i) - a^2(2\mu - p_o)]} . \tag{2.26}$$

In view of (2.1), (2.20), and (2.26) it is evident that the deformation field is completely determined if (2.25) can be solved for $R_o(>0)$. By virtue of the assumed monotonocity of $h(R)$ the existence of a unique positive solution to (2.25) is guaranteed provided that its right-hand-side lies in the open interval (0,1) (see (2.11), (2.12), (2.18)). It will be shown later that this is always satisfied if the applied pressures are appropriately restricted.

The corresponding components of Cauchy stress are found from (2.7), (2.14), (2.15) in conjunction with (2.19), (2.25) and (2.26) to be

$$\tau_{\rho\rho} = 2\mu \left[\frac{R_o h(R_o)}{f(r)} - 1\right] = 2\mu \left[\frac{Q_1 r^2 - Q_2}{Q_3 r^2 + Q_2}\right], \tag{2.27}$$

$$\tau_{\psi\psi} = 2\mu \left[\frac{R_o h(R_o)}{R_o - f(r)} - 1\right] = 2\mu \left[\frac{Q_1 r^2 + Q_2}{Q_3 r^2 - Q_2}\right], \tag{2.28}$$

where

$$Q_1 = p_i a^2(2\mu - p_o) - p_o b^2(2\mu - p_i), \tag{2.29}$$

$$Q_2 = 2\mu a^2 b^2(p_i - p_o), \tag{2.30}$$

$$Q_3 = 2\mu [b^2(2\mu - p_i) - a^2(2\mu - p_o)]. \tag{2.31}$$

The deformed inner radius of the tube α is found from the first of (2.1), (2.20), (2.25) and (2.26) to be

$$\alpha = \frac{aR_o(b^2 - a^2)(2\mu - p_o)}{2[b^2(2\mu - p_i) - a^2(2\mu - p_o)]}. \tag{2.32}$$

Examination of (2.27) - (2.31) shows that <u>the true stress distribution in the cylinder is independent of the constitutive function $H(R)$ and depends on the (harmonic) material at most through its infinitesimal shear modulus</u> μ.

We now examine the preceeding results in the two cases of internal and external pressure separately:

(i) <u>Internal Pressure Only</u> ($p_i > 0$, $p_o = 0$)

Considering the hoop stress $\tau_{\psi\psi}$ we find that $-Q_2(Q_1+Q_3) < 0$, provided that $p_i < 2\mu$. Assuming temporarily this to be the case it then follows from (2.28) that $\tau_{\psi\psi}$ is monotone decreasing with r, so that the maximum hoop stress occurs at the inner surface $r = a$ and is

$$\tau_{\psi\psi} = \frac{\mu p_i (b^2 + a^2)}{\mu(b^2 - a^2) - p_i b^2} . \tag{2.33}$$

For this to be positive and bounded, one must have

$$p_i < \mu \; (1 - a^2/b^2) \quad (< 2\mu). \tag{2.34}$$

It is easy to check that (2.34) implies $0 < h(R_o) < 1$, i.e. the right-hand-side of (2.25) with $p_o = 0$ is a number in the open interval (0,1). Thus (2.25) can be solved for a unique value of R_o (> 0). Moreover since $h(R)$ is monotone increasing it follows that $R_o > R_*$ and (2.18) is thus verified.

(ii) <u>External Pressure Only</u> ($p_i = 0$, $p_o > 0$)

On consideration of (2.32) with $p_i = 0$, we see that as $p_o \to 2\mu$, $\alpha \to 0$ i.e. the cavity closes at $p_o = 2\mu$. (For values of $p_o > 2\mu$ one finds $\alpha < 0$!) We therefore require that

$$p_o < 2\mu . \tag{2.35}$$

This also ensures that $0 < h(R_o) < 1$, i.e. the right-hand-side of (2.25) with $p_i = 0$ lies in the open interval (0,1). The argument in the previous case applies and hence a unique solution R_o (> 0) for (2.25) is guaranteed and (2.18) is verified.

In the particular case when the applied pressures are small ($p_i/2\mu$, $p_o/2\mu \ll 1$) it can be easily verified that upon linearization, (2.27) - (2.31) yield the well known results according to the infinitesimal theory of elasticity (see e.g. Timoshenko and Goodier (1970), p 70)

$$\tau_{\rho\rho} = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\,\frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}\,, \tag{2.36}$$

$$\tau_{\psi\psi} = -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\,\frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}\,. \tag{2.37}$$

To summarize, in this section the plane axisymmetric deformation of a hollow cylinder subject to simultaneous internal and external pressure has been examined. The stress and deformation fields were determined (see (2.1), (2.20), (2.25), (2.27)-(2.31)) and various features of these fields were examined. In the following sections the stability of the equilibrium solution obtained here will be investigated. This investigation will determine whether the results found here pertaining to an infinite hoop stress (in the internal pressure case) and the closing of the cavity (in the external pressure case) are indeed attainable.

3. GEOMETRICALLY PERTURBED PROBLEM

3.1 Deformation and Stress Fields

In order to investigate the stability[1] of the equilibrium solution found in Section 2, we consider the pressurizing of a right hollow cylinder with "almost" circular boundaries

$$r = a + \varepsilon g_i(\phi), \quad r = b + \varepsilon g_o(\phi), \ |\varepsilon| << 1, \ 0 \leq \phi \leq 2\pi,$$

$$g_i(0) = g_i(2\pi), \ g_o(0) = g_o(2\pi), \tag{3.1}$$

where g_i, g_o are bounded fuctions on $[0,2\pi]$. The resulting deformation is assumed to be a slight perturbation of the purely radial deformation already discussed.

To this end we introduce

$$\rho = rf(r) + \varepsilon \tilde{u}(r,\phi), \ \psi = \phi + \frac{\varepsilon}{r} \tilde{v}(r,\phi), \tag{3.2}$$

where (r,ϕ) are the polar coordinates of a generic point in the undeformed configuration which is mapped to (ρ,ψ) by the deformation (3.2), and ε is a measure of the "imperfection" of the boundaries. Since it is assumed that ε is a small number, in all the developments that follow, terms which contain powers of ε higher than one are neglected.

(1) As mentioned in the Introduction the study of a geometrically perturbed problem was undertaken in the hope of calculating the buckling pressure for an internally pressurized tube as well as obtaining detailed information regarding the deformation field (from the prebifurcation state to beyond the occurrence of instability). The buckling pressure itself can of course be determined without the introduction of geometrical inhomogenities.

The polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with (3.2) are given by (see e.g. Malvern (1969), p. 651)

$$\begin{aligned} F_{rr} &= f + rf' + \varepsilon\tilde{u}_r \ , & F_{r\phi} &= \frac{\varepsilon}{r}(\tilde{u}_\phi - f\tilde{v}), \\ F_{\phi r} &= \varepsilon(f\tilde{v})_r \ , & F_{\phi\phi} &= f + \frac{\varepsilon}{r}[\tilde{u} + (f\tilde{v})_\phi], \end{aligned} \tag{3.3}$$

where a prime denotes differentiation with respect to r and subscripts r and ϕ on $\tilde{u}$ and $(f\tilde{v})$ denote partial differentiation with respect to r and ϕ respectively.

For the deformation given by (3.2) we find from (3.3) that

$$I = \mathrm{tr}\, \underset{\sim}{F}\underset{\sim}{F}^T = I_o + \varepsilon\tilde{I}, \quad J = \det \underset{\sim}{F} = J_o + \varepsilon\tilde{J}, \quad R = (I+2J)^{\frac{1}{2}} = R_o + \varepsilon\tilde{R}, \tag{3.4}$$

where I_o, J_o and R_o are the invariants in the axisymmetric case and are given by the right-hand-sides of (2.3), (2.4), and (2.19) respectively. The "first order invariants" $\tilde{I}$, $\tilde{J}$, and $\tilde{R}$ are found to be

$$\tilde{I} = 2\ \{(f+rf')\tilde{u}_r + \frac{f}{r}[\tilde{u}+(f\tilde{v})_\phi]\}\ , \tag{3.5}$$

$$\tilde{J} = f\tilde{u}_r + \frac{1}{r}(f+rf')\ [\tilde{u}+(f\tilde{v})_\phi]\ , \tag{3.6}$$

$$\tilde{R} = \tilde{u}_r + \frac{1}{r}[\tilde{u}+(f\tilde{v})_\phi]\ . \tag{3.7}$$

Next we expand $h(R) = h(R_o+\varepsilon\tilde{R})$ in powers of $\varepsilon\tilde{R}$ about $R = R_o$. To leading order we have

$$h(R) = h(R_o) + \varepsilon\tilde{h}, \quad \tilde{h} = \tilde{R}h'(R_o) = \frac{\tilde{R}}{R_o}[H''(R_o) - h(R_o)]. \tag{3.8}$$

Substituting for I, J, and $h(R)$ from (3.5), (3.6) and (3.8) into (2.8) we find

$$\begin{aligned} \sigma_{rr} &= \sigma^o_{rr} + \varepsilon\tilde{\sigma}_{rr} \ , & \sigma_{r\phi} &= \varepsilon\tilde{\sigma}_{r\phi} \\ \sigma_{\phi r} &= \varepsilon\tilde{\sigma}_{\phi r} \ , & \sigma_{\phi\phi} &= \sigma^o_{\phi\phi} + \varepsilon\tilde{\sigma}_{\phi\phi}\ , \end{aligned} \tag{3.9}$$

where σ°_{rr} and $\sigma^{\circ}_{\phi\phi}$ are the Piola stress components in the axisymmetric case and are given by the right-hand-sides of (2.14) and $\tilde{\sigma}_{rr}$, $\tilde{\sigma}_{r\phi}$, $\tilde{\sigma}_{\phi r}$, and $\tilde{\sigma}_{\phi\phi}$ are found to be

$$\tilde{\sigma}_{rr} = 2\mu \{ H''(R_\circ) [\tilde{u}_r + \frac{1}{r} (\tilde{u} + (f\tilde{v})_\phi)] - \frac{1}{r} [\tilde{u} + (f\tilde{v})_\phi]\} , \quad (3.10)$$

$$\tilde{\sigma}_{r\phi} = 2\mu \{\frac{1}{r} h(R_\circ) (\tilde{u}_\phi - f\tilde{v}) - [h(R_\circ) - 1] (f\tilde{v})_r \} , \quad (3.11)$$

$$\tilde{\sigma}_{\phi r} = 2\mu \{h(R_\circ) (f\tilde{v})_r - \frac{1}{r} [h(R_\circ) - 1] (\tilde{u}_\phi - f\tilde{v}) \} , \quad (3.12)$$

$$\tilde{\sigma}_{\phi\phi} = 2\mu \{H''(R_\circ) [\tilde{u}_r + \frac{1}{r} (\tilde{u} + (f\tilde{v})_\phi)] - \tilde{u}_r \} . \quad (3.13)$$

Having found expressions for the stresses, the equilibrium equations, div $\underset{\sim}{\sigma} = \underset{\sim}{0}$ in polar coordinates, can be written down immediately. (See e.g. Malvern (1969), p. 655). These are

$$\begin{aligned} &H''(R_\circ) \{ r^2 \tilde{u}_{rr} + r [\tilde{u}_r + (f\tilde{v})_{r\phi}] - \tilde{u} - (f\tilde{v})_\phi\} \\ &\qquad + h(R_\circ) [\tilde{u}_{\phi\phi} - (f\tilde{v})_\phi - r (f\tilde{v})_{r\phi}] = 0 , \\ &\qquad\qquad\qquad \text{on } D_\circ , \end{aligned} \quad (3.14)$$

$$\begin{aligned} &H''(R_\circ) [(r\tilde{u}_r + \tilde{u})_\phi + (f\tilde{v})_{\phi\phi}] \\ &- h'(R_\circ) [r^2(f\tilde{v})_{rr} - r(\tilde{u}_\phi - f\tilde{v})_r + \tilde{u}_\phi - f\tilde{v}] = 0 , \\ &\qquad\qquad\qquad \text{on } D_\circ . \end{aligned} \quad (3.15)$$

It should be noted that the equilibrium equations div $\underset{\sim}{\sigma} = \underset{\sim}{0}$ give rise to two groups of terms; one of which does not involve ε while the other one does. The former consists of one equation which is, of course, exactly the same as that found when considering the equilibrium of the unperturbed cylinder (see (2.16)). The latter consists of (3.14) and (3.15) above.

3.2 Pressure Boundary Conditions on a Perturbed Surface

We next formulate the boundary conditions for a perturbed boundary subject to pressure. Once this general form is derived, specialization to the particular cases of internal and external pressure will be immediate. To this end let the unit outward normal to a surface S at point M in the undeformed configuration be $\underset{\sim}{N}$. Under deformation the surface S wil be mapped to s and the point M to m. Let $\underset{\sim}{n}$ be the unit outword normal to s at m. Denote the area of a surface element surrounding M on S by dA and its image under the given deformation by da. Then (see e.g. Chadwick (1976), p.61)

$$\underset{\sim}{n} = J\underset{\sim}{F}^{-T} \underset{\sim}{N} \frac{dA}{da} . \tag{3.16}$$

If the deformed surface being considered is subject to a hydrostatic pressure p then the appropriate traction boundary condition is given by

$$\underset{\sim}{T} \underset{\sim}{n} = - p \underset{\sim}{n} \qquad \text{on s.} \tag{3.17}$$

It is often convenient to transform this condition into one which holds on the (known) undeformed surface S. To this end, we substitute for $\underset{\sim}{T}$ and $\underset{\sim}{n}$ in (3.17) from (2.8) and (3.16) respectively. This gives

$$\underset{\sim}{\sigma} \underset{\sim}{N} = - p J\underset{\sim}{F}^{-T} \underset{\sim}{N} \quad \text{on S.} \tag{3.18}$$

We now consider the particular case in which the surface S, in the undeformed configuration, is a right cylindrical surface, with the generator perpendicular to the (r,ϕ) plane. The intersection of S with this plane is given by

$$r = r_o + \varepsilon g(\phi), \ |\varepsilon|<<1, \ 0<\phi\leq 2\pi . \tag{3.19}$$

We approximate the boundary condition (3.18) to leading order in ε. In (3.19) above, r_o (>0) and ε are constants and $g(\phi)$ is a given bounded smooth function of ϕ such tht $\phi(0) = \phi(2\pi)$. (See Figure 6.) As mentioned previously, ε determines how much the curve under consideration differs from the circle $r = r_\circ$.

From elementary calculus one has

$$\tan\theta = r\left(\frac{dr}{d\phi}\right)^{-1}, \tag{3.20}$$

where θ is the angle between the radial line and tangent to the curve at any point. The polar components of the unit normal $\underset{\sim}{N}$ are $N_r = -\sin\theta$ and $N_\phi = \cos\theta$. Equation (3.20) in conjunction with (3.19) yields

$$\cot\theta = \varepsilon g'(\phi)/r_\circ + O(\varepsilon^2), \tag{3.21}$$

where a prime denotes differentiation with respect to ϕ. On recalling the trigonometric identities $\sin\theta = (1+\cot^2\theta)^{-\frac{1}{2}}$ and, $\cos\theta = \cot\theta\,(1+\cot^2\theta)^{-\frac{1}{2}}$, expanding their right-hand-sides by the binomial formula, and making use of (3.21) one finds that to first order in ε

$$N_r = -1 \quad , \quad N_\phi = \varepsilon g'(\phi)/r_\circ \; . \tag{3.22}$$

We turn now to the evaluations of the remaining quantities in (3.18). By way of illustration, expressions for one element of $\underset{\sim}{A} = J\underset{\sim}{F}^{-T}$ and $\underset{\sim}{\sigma}$ each will be derived in detail. The derivation of other elements is accomplished in a similar fashion.

From (3.3) and the second of (3.4) it follows that the element A_{11} is given by $f+\varepsilon\,[\tilde{u} + (f\tilde{v})_\phi]/r$ with $r = r_\circ + \varepsilon g(\phi)$. Now

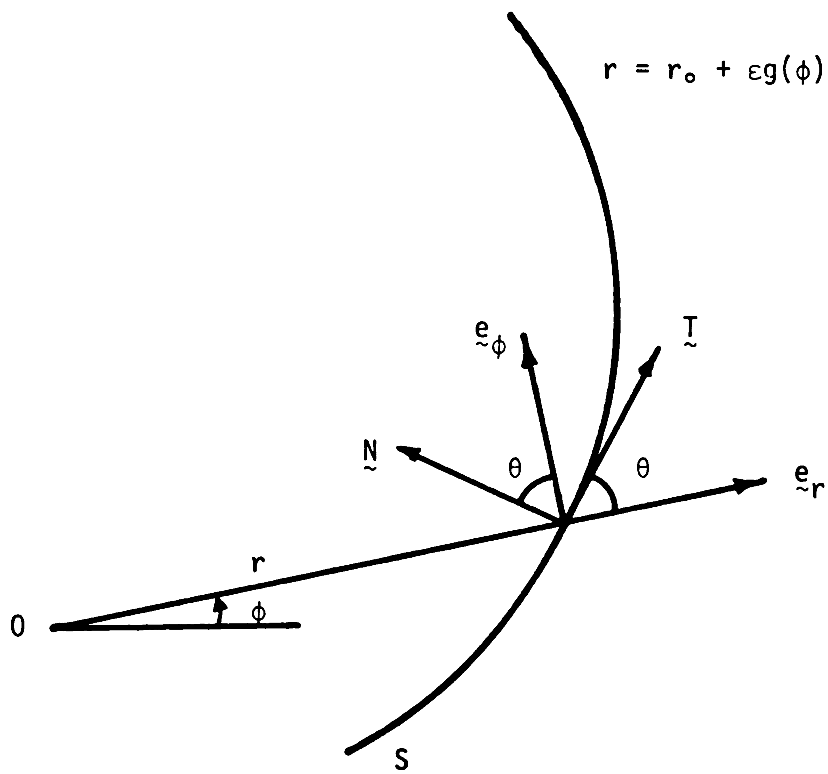

Figure 6. Geometry and coordinate system for the perturbed boundary.

$$\{f + \frac{\varepsilon}{r}\,[u + (fv)_\phi]\}\Big|_{r = r_o+\varepsilon g(\phi)} = f(r_o+\varepsilon g(\phi))$$

$$+ \frac{\varepsilon}{r_o + \varepsilon g(\phi)}\,[\tilde{u}(r_o + \varepsilon g(\phi),\phi)$$

$$+ f(r_o + \varepsilon g(\phi))\tilde{v}_\phi(r_o + \varepsilon g(\phi),\phi)\,]\,. \tag{3.23}$$

After expanding the right hand side in a power series and dropping terms of order higher than unity in ε one finds

$$\{f + \frac{\varepsilon}{r}\,[\tilde{u} + (f\tilde{v})_\phi]\}\Big|_{r = r_o + \varepsilon g(\phi)} = f(r_o)$$

$$+ \varepsilon\;\{f'(r_o)\,g(\phi) + \frac{1}{r_o}[\tilde{u}(r_o,\phi) + f(r_o)\tilde{v}_\phi(r_o,\phi)]\}\,. \tag{3.24}$$

In a similar fashion one may write

$$\sigma_{rr} = \sigma_{rr}(r_o + \varepsilon g(\phi),\phi) = \sigma_{rr}(r_o,\phi)$$

$$+ \frac{\partial\sigma_{rr}}{\partial r}\Big|_{r = r_o}\varepsilon g(\phi), \tag{3.25}$$

to leading order in ε. Substitution for $\sigma_{rr}(r_o,\phi)$ in the above from the first of (3.9) and (3.10) gives

$$\sigma_{rr}\Big|_{r = r_o + \varepsilon g(\phi)} = \sigma^{o}_{rr}\,(r_o) + \varepsilon[\,\tilde{\sigma}_{rr}(r_o,\phi) + \frac{\partial\sigma_{rr}}{\partial r}\Big|_{r=r_o} g(\phi)]\;. \tag{3.26}$$

Calculating the other elements of $J\underset{\sim}{F}^{-T}$ and $\underset{\sim}{\sigma}$, making use of (3.18) and (3.22), and after some algebraic manipulation, the order ε terms in the boundary conditions take the form

$$2\mu H''(R_\circ)\,[r_\circ\tilde{u}_r + \tilde{u}\,(f\tilde{v})_\phi] - (2\mu - p)\,[\tilde{u} + (f\tilde{v})_\phi] =$$

$$= (2\mu - p)r_\circ\, f'g \quad \text{on } r = r_\circ, \qquad (3.27)$$

and

$$2\mu\, h(R_\circ)\,[r_\circ(f\tilde{v})_r - \tilde{u}_\phi + f\tilde{v}] + (2\mu - p)(\tilde{u}_\phi - f\tilde{v}) =$$

$$[2\mu H'(R_\circ) - (R_\circ - f)\,(2\mu - p)]\, g' \text{ on } r = r_\circ. \quad (3.28)$$

The boundary condition arising from the term of zero order is of course $\sigma^{\circ}_{rr} + pf = 0$ as before.

3.3 Special Cases

We now specialize the results found in the previous subsection to the two cases of internal and external pressure loading. In both cases the inner and outer boundaries are assumed to have imperfections as described by (3.1). The two cases differ only in the loading.

Case (i) Internal Pressure Only ($p_i > 0$, $p_\circ = 0$). From (3.1), (3.27) and (3.28) with $g(\phi) = g_i(\phi)$, $p = p_i$, and $r_\circ = a$, the boundary conditions on the inner boundary are found to be

$$2\mu H''(R_\circ)\,[a\tilde{u}_r + \tilde{u} + (f\tilde{v})_\phi] - (2\mu - p_i)\,[\tilde{u} + (f\tilde{v})_\phi] =$$

$$= (2\mu - p_i)af'g_i \text{ on } r = a, \qquad (3.29)$$

$$2\mu h(R_\circ)\,[a(f\tilde{v})_r - \tilde{u} + f\tilde{v}] + (2\mu - p_i)\,(\tilde{u}_\phi - f\tilde{v}) =$$

$$= [2\mu H'(R_\circ) - (R_\circ - f)\,(2\mu - p_i)]g'_i \text{ on } r = a. \quad (3.30)$$

Using the same equations as above with $g(\phi) = g_\circ(\phi)$, $p = 0$, and $r_\circ = b$, one finds that the boundary conditions on the traction-free outer boundary are given by

$$H''(R_\circ)\ [b\tilde{u}_r + \tilde{u} + (f\tilde{v})_\phi] - u - (f\tilde{v})_\phi = bf'g_\circ \quad \text{on } r = b, \qquad (3.31)$$

$$h(R_\circ)\ [b(f\tilde{v})_r - \tilde{u}_\phi + f\tilde{v}] + \tilde{u}_\phi - f\tilde{v} =$$

$$= [H'(R_\circ) - R_\circ + f]\ g_\circ' \text{ on } r = b. \qquad (3.32)$$

Case (ii) External Pressure Only ($p_i = 0$, $p_\circ > 0$)

Again using (3.1), (3.27) and (3.28) with $g(\phi) = g_i(\phi)$, $p = 0$, and $r_i = a$, the boundary conditions on the traction-free inner boundary take the form

$$H''(R_\circ)\ [a\tilde{u}_r + \tilde{u} + (f\tilde{v})_\phi] - \tilde{u} - (f\tilde{v})_\phi = af'g_i \quad \text{on } r = a, \qquad (3.33)$$

$$h(R_\circ)\ [a(f\tilde{v})_r - \tilde{u}_\phi + f\tilde{v}] + \tilde{u}_\phi - f\tilde{v} = [H'(R_\circ) - R_\circ + f]\ g_i'$$
$$\text{on } r = a. \quad (3.34)$$

The above equations with $g(\phi) = g_\circ(\phi)$, $p = p_\circ$ and $r_\circ = b$ furnish the boundary conditions on the outer boundary as follows

$$2\mu H''(R_\circ)\ [b\tilde{u}_r + \tilde{u} + (f\tilde{v})_\phi] - (2\mu - p_\circ)\ [\tilde{u} + (f\tilde{v})_\phi] =$$

$$= (2\mu - p_\circ)\ f'g_\circ \quad \text{on } r = b, \qquad (3.35)$$

$$2\mu h(R_\circ)\ [b(f\tilde{v})_r - \tilde{u}_\phi + f\tilde{v}] + (2\mu - p_\circ)\ (\tilde{u}_\phi - f\tilde{v}) =$$

$$= [2\mu H'(R_\circ) - (R_\circ - f)\ (2\mu - p_\circ)]\ g_\circ'$$
$$\text{on } r = b. \qquad (3.36)$$

The formulation of the equilibrium problem for a pressurized cylinder with "imperfect" boundaries is thus complete. The solution of the equilibrium equations (3.14) and (3.15) subject to the boundary conditions (3.29) - (3.32) or (3.33) - (3.36) will be considered in the following section. A unique solution of this problem implies stability.

4. STABILITY

4.1 Solution of the Equilibrium Equations

We now turn to the solution of the equilibrium equations (3.14) and (3.15). Assume that the displacement components $\tilde{u}$ and $f\tilde{v}$ can be expanded in trigonometric series of the form

$$\tilde{u} = \sum_{n=o}^{\infty} a_n(r)\cos n\phi \quad , \tag{4.1}$$

$$f\tilde{v} = \sum_{n=o}^{\infty} r\, b_n(r)\sin n\phi \ , \tag{4.2}$$

where without loss of generality we take $b_o(r) \equiv 0$. Substituting into the equilibrium equations (3.14) and (3.15) we find that, for each $n \geq 0$,

$$H''(R_o)(a_n' + \frac{1}{r} a_n + nb_n)' - nh(R_o)\ (b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n) = 0, \tag{4.3}$$

$$h(R_o)\ [(b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n)' + \frac{1}{r}(b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n)]$$
$$- H''(R_o)\frac{n}{r^2}(a_n' + \frac{1}{r} a_n + b_n) = 0, \tag{4.4}$$

for $a<r<b$, where the primes denote differentiation with respect to r.

On setting

$$x_n(r) = a_n' + \frac{1}{r} a_n + nb_n \ , \qquad a<r<b, \tag{4.5}$$

$$y_n(r) = b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n, \qquad a<r<b, \tag{4.6}$$

we can write (4.3) and (4.4) as

$$H''(R_\circ)\, x_n' - nh(R_\circ)\, y_n = 0, \tag{4.7}$$

$$h(R_\circ)\, (y_n' + \frac{1}{r} y_n) - H''(R_\circ) \frac{n}{r^2}\, x_n = 0, \tag{4.8}$$

for $a<r<b$. This is a system of two first-order ordinary differential equations for $x_n(r)$ and $y_n(r)$. In solving this system we distinguish between the two cases $n = 0$ and $n \geq 1$.

Leaving the special case $n = 0$ for later consideration, we seek a solution of the form $x_n = Ar^m$, $y_n = Br^{m-1}$, where m, A and B are constants. Substitution into (4.7) and (4.8) yields the general solution

$$x_n(r) = h(A_n r^n + B_n r^{-n}) \quad , \tag{4.9}$$

$$y_n(r) = H''(a_n r^n - B_n r^{-n-1}) \quad , \tag{4.10}$$

$n = 1,2\ldots$ Here and in the following h, H', and H'' are always evaluated at $R_\circ$. One can now solve the nonhomogeneous differential equations (4.5) and (4.6) in conjunction with (4.9) and (4.10) for $a_n(r)$ and $b_n(r)$. This yields

$$a_1(r) = \frac{1}{8}\,(3h-H'')A_1 r^2 + \frac{1}{2}\,(h + H'')B_1 \ln r + C_1 + \frac{D_1}{r^2}\,, \tag{4.11}$$

$$b_1(r) = \frac{1}{8}(3H''-h)A_1 r + \frac{1}{2}\,(h-H'')\,\frac{B_1}{r} - \frac{1}{2}\,(h+H'')\,\frac{B_1}{r} \ln r - \frac{C_1}{r} + \frac{D_1}{r^3}\,, \tag{4.12}$$

$$a_n(r) = \frac{2h-n(H''-h)}{4(n+1)}\, A_n r^{n+1} - \frac{2h+n(H''-h)}{4(n-1)}\, \frac{B_n}{r^{n-1}} + C_n r^{n-1} + \frac{D_n}{r^{n+1}}\,, \tag{4.13}$$

$$b_n(r) = \frac{2H''+n(H''-h)}{4(n+1)} A_n r^n + \frac{2H''-n(H''-h)}{4(n-1)} \frac{B_n}{r^n} - C_n r^{n-2} + \frac{D_n}{r^{n+2}}, \quad (4.14)$$

$n = 2, 3, \ldots$

Turning now to the special case $n = 0$, one may readily integrate (4.7) to obtain

$$x_o(r) = A_o. \quad (4.15)$$

Equation (4.5) then yields

$$a_o(r) = \frac{1}{2} A_o r + \frac{D_o}{r}. \quad (4.16)$$

Integrating (4.8) one finds that $y_o(r) = B_o/r$. On recalling that $b_o(r) \equiv 0$, we find $B_o = 0$ and thus

$$y_o(r) = 0. \quad (4.17)$$

A formal solution to the displacement equations of equilibrium is thus given by (4.1), (4.2), (4.11) - (4.14) and (4.16). The coefficients A_o, D_o, A_n, B_n, C_n, and D_n ($n = 1, 2, \ldots$) are constants which are to be determined from the boundary conditions. When the imposition of the boundary conditions leads to unique values for A_o, D_o, and A_n - D_n ($n = 1, 2, \ldots$), the corresponding equilibrium state of the tube is stable; otherwise one has a bifurcation and stability is lost.

4.2 Buckling of an Internally Pressurized Tube

In this sub-section we consider the boundary conditions for the case of an imperfect cylinder subject to internal pressure only. We suppose that the functions $g_i(\phi)$ and $g_o(\phi)$ defining the boundaries of the cylinder can be written in the form of infinite series

$$g_i(\phi) = \sum_{n=o}^{\infty} q_n \cos n\phi \quad , \tag{4.18}$$

$$g_o(\phi) = \sum_{n=o}^{\infty} s_n \cos n\phi. \tag{4.19}$$

Introducing the displacements (4.1) and (4.2) along with (4.18) and (4.19) into the boundary conditions (3.29)-(3.32), making use of (4.5) and (4.6), and equating the coefficients of $\cos n\phi$ and $\sin n\phi$ leads to

$$2\mu H''ax_n - (2\mu - p_i)(a_n + nab_n) = (2\mu - p_i)f'aq_n \quad \text{on } r=a, \tag{4.20}$$

$$2\mu ha^2 y_n - (2\mu - p_i)(na_n + ab_n) =$$
$$= -[2\mu H' - (R_o - f)(2\mu - p_i)]nq_n \qquad \text{on } r = a, \tag{4.21}$$

$$H''bx_n - a_n - nbb_n = f'bs_n \text{ on } r = b, \tag{4.22}$$

$$hb^2 y_n - na_n - bb_n = -(H' - R_o + f)\, ns_n \quad \text{on } r = b, \tag{4.23}$$

$n = 0, 1, 2, \ldots$

The three cases $n = 0$, $n = 1$ and $n \geq 2$ must be treated separately.

First we consider the case $n = 0$. Substituting from (4.15) and (4.16) into (4.20) and (4.22) yields the following set of two linear nonhomogeneous algebraic equations for A_o and D_o.

$$\begin{bmatrix} 2\,H''-(2\mu-p_i)/2 & -(2\mu-p_i)/a^2 \\ H''-1/2 & -1/b^2 \end{bmatrix} \begin{bmatrix} A_o \\ D_o \end{bmatrix} = \begin{bmatrix} (2\mu-p_i)f'(a)q_o \\ f'(b)s_o \end{bmatrix}. \tag{4.24}$$

As mentioned at the end of Sub-section 4.1 the question of importance to us is whether a unique solution A_o, D_o to (4.24) exists. To investigate this we compute the determinant of the coefficient matrix in (4.24) to find

$$\Delta = \frac{1}{2b^2}[(2\mu-p_i)(1+\frac{2H''-1}{k})-4\mu H''], \tag{4.25}$$

where Δ denotes the determinant and $k = (a/b)^2$. Setting $\Delta = 0$ and solving for p_i we have $p_i = 2\mu(2H''-1)(1-k)/(2H''-(1-k))$. Recalling that $H''>1$, by (2.12), $p_i>2\mu(1-k)$. But for finite hoop stress we must have $p_i<\mu(1-k)$ (cf. (2.34)). It therefore follows that, for p_i in the admissible range, $\Delta \neq 0$ and hence a unique solution exists.

Next we turn to the case n = 1. Substituting from (4.9)-(4.12) into (4.20)-(4.23) and simplifying we find the following set of algebraic equations

$$\begin{bmatrix} [2\mu H''h-(2\mu-p_i)(H''+h)/4]a & [2\mu H''h+(2\mu-p_i)(H''-h)/2]a^{-1} & -2(2\mu-p_i)a^{-3} \\ 2\mu H''h-(2\mu-p_i)(H''+h)/4 & -[2\mu H''h-(2\mu-p_i)(H''-h)/2]a^{-2} & -2(2\mu-p_i)a^{-4} \\ [H''h-(H''+h)/4]b & [H''h+(H''-h)/2]b^{-1} & -2b^{-3} \\ H''h-(H''+h)/4 & -[H''h-(H''-h)/2]b^{-2} & -2b^{-4} \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ B_1 \\ D_1 \\ \end{bmatrix} = \begin{bmatrix} (2\mu-p_i)f'q_1 \\ -[2\mu H'-(R_o-f)(2\mu-p_i)]a^{-2}q_1 \\ f's_1 \\ -(H'-R_o+f)b^{-2}s_1 \end{bmatrix}. \tag{4.26}$$

The terms involving C_1 drop out of these equations and thus its value is arbitrary. It will be shown, however, that the expressions involving C_1 correspond to a rigid body translation and therefore C_1 can be taken to be zero. Two considerations concern us here as regards equations (4.26). We must first show that the four equations for the three unknowns A_1, B_1, and D_1 are consistent so that they can indeed be solved. We will then consider the uniqueness issue.

Algebraic manipulation of equations (4.26) and use of (2.25) and (2.26) with $p_o = 0$ gives $B_1 = 0$. Thus (4.26) reduces to a system of four equations in the two unknowns A_1 and D_1. Note that this amounts to the removal of the second column of the coefficient matrix in (4.26). In the new matrix thus found the first row is a multiple of the second and the third row a multiple of the fourth. To establish consistency it is necessary and sufficient to show that the same relations hold between the elements of the right hand side column. Simple calculations show that this is indeed the case. Thus we are left with the following system of two equations in two unknowns

$$\begin{bmatrix} 2\mu H''h-(2\mu-p_i)(H''+h)/4 & -2(2\mu-p_i)/a^4 \\ H''h-(H''+h)/4 & -2/b^4 \end{bmatrix} \begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} (2\mu-p_i)f'(a)q_1/a \\ f'(b)s_1/b_1 \end{bmatrix} . \qquad (4.27)$$

To investigate the uniqueness of solutions we calculate the determinant of the coefficient matrix. Denoting this by Δ we find

$$\Delta = \frac{(1-k)^2(p_i-2\mu)}{8\mu a^4}\left[2H'' + \frac{(k+1)(2\mu-p_i)-8\mu kH''}{p_i+2(k-1)\mu}\right] , \qquad (4.28)$$

where $k = (a/b)^2$. Setting $\Delta = 0$, one finds $p_i = 2\mu$ and $p_i = 2\mu\,(k+1)(2H''-1)/[(2H''-1)-k]$. The second of these is greater than 2μ (recall that by (2.12) $H''>1$). Since we require that $p_i<\mu\,(1-k)$ none of these values can be attained and therefore a unique solution to (4.27), in the range of interest, is guaranteed.

From (4.1) and (4.2) with $n = 1$, and (4.11), (4.12) the components of the displacement for the case under consideration are given by

$$\tilde{u} = a_1\cos\phi = [\frac{1}{8}(3h-H'')A_1 r^2 + \frac{D_1}{r^2}]\cos\phi + C_1\cos\phi\ , \tag{4.29}$$

$$f\tilde{v} = b_1 r\sin\phi = [\,\frac{1}{8}\,(3H''-h)A_1 r^2 + \frac{D_1}{r^2}]\sin\phi - C_1\sin\phi\ . \tag{4.30}$$

(Recall that $B_1 = 0$). Now $C_1\cos\phi$ and $-C_1\sin\phi$ are the radial and tangential components of a rigid body displacement parallel to the x_1 - axis of magnitude C_1, and therefore as noted earlier one can, without loss of generality, set $C_1 = 0$.

Finally, we turn to the case $n\geq 2$. Substituting from (4.9), (4.10), (4.13) and (4.14) into (4.20)-(4.23) we get for each $n\geq 2$, a set of four linear algebraic equations for $A_n - D_n$ which in matrix form reads

$$\begin{bmatrix} [2\mu H''h-(2\mu-p_i)(nH''-(n-2)h)/4]a^{n} & [2\mu H''h-(2\mu-p_i)((n+2)h-nH'')/4]a^{-n} & (2\mu-p_i)(n-1)a^{n-2} & -(2\mu-p_i)(1+n)a^{-n-2} \\ [2\mu H''h-(2\mu-p_i)(nh-(n-2)H'')/4]a^{n-1} & -[2\mu H''h+(2\mu-p_i)(nh-(n+2)H'')/4]a^{-n-1} & (2\mu-p_i)(1-n)a^{n-3} & -(2\mu-p_i)(1+n)a^{-n-3} \\ [H''h-(nH''-(n-2)h)/4]b^{n} & [H''h-((n+2)h-nH'')/4]b^{-n} & (n-1)b^{n-2} & -(1+n)b^{-n-2} \\ [H''h-(nh-(n-2)H'')/4]b^{n-1} & -[H''h+(nh-(n+2)H'')/4]b^{-n-1} & (1-n)b^{n-3} & -(1+n)b^{-n-3} \end{bmatrix}$$

$$\begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \begin{bmatrix} (2\mu-p_i)f'q_n \\ [2\mu H'-(R_o-f)(2\mu-p_i)]a^{-2}nq_n \\ f's_n \\ -(H'-R_o+f)b^{-2}ns_n \end{bmatrix} . \qquad (4.31)$$

We first compute the determinant of the coefficient matrix. It is convenient to introduce the parameter S defined by

$$S = \frac{2kH''p_i}{(2H''-1)(k-1)(2\mu-p_i)}, \quad k = (a/b)^2(<1). \tag{4.32}$$

Denoting the determinant of the four by four matrix in (4.31) by Δ, after some lengthy algebra, one finds that

$$\Delta = \frac{\mu^4(H''-h)^2}{4k^2(S+1)^2}(2\mu-p_i)\,[(h_n+k-2)\,S^2+2(h_n+1)\,S+h_n-k], \tag{4.33}$$

where we have set

$$h_n = \frac{n^2}{\lambda_n}(1-k)^2\ (>0), \quad \lambda_n = k^n + k^{-n}-2(>0). \tag{4.34}$$

For instability we must have $\Delta = 0$, which according to (4.33) yields a quadratic equation for S. Solving this, we find values of S given by S_n^+ and S_n^- where

$$S_n^{\pm} = \frac{k-h_n}{1+h_n \pm [(1-k)^2+4h_n]^{\frac{1}{2}}}. \tag{4.35}$$

Thus if there is a value of internal pressure $p_i(<\mu(1-k))$ for which the parameter S defined by (4.32) satisfies $S = S_n^{\pm}$, then the system of equations (4.31) becomes singular at that pressure and the corresponding equilibrium solution is unstable.

We now show that there does not exist such a value of p_i and thus an internally pressurized tube of harmonic material does <u>not</u> buckle. We will show this by demonstrating that S_n^+ and S_n^- given by (4.35) are always positive, whereas by definition, (4.32), S is negative ($H''>1$, $p_i<\mu(1-k)<2\mu$ by (2.12) and (2.34) respectively), and so $S\neq S_n^{\pm}$.

In order to show this, we need a result which we will state here, the proof of which is given in Appendix A, namely that

$$k - h_n >0, \quad n\geq 2. \tag{4.36}$$

Considering first the case where the positive sign is chosen in (4.33) it is obvious that S_n^+ is positive by virtue of (4.36). It is easy to see that the negative sign is also inadmissible. Suppose that S_n^- is negative. For this the denominator of (4.35) must be negative. Simple calculations show that this in turn implies that $k + h_n >2$. Now by (4.36) we have $k + h_n <2k<2$, which is a contradiction.

Thus we are led to the conclusion that the tube is stable at all values of the internal pressure $p_i <\mu(1-k)$.

4.3 Buckling of an Externally Pressurized Tube

The question of the stability of the equilibrium solution of an imperfect cylinder subject to external pressure will be considered in this sub-section. The treatment will parallel the case of internal pressure given in 4.2.

Introducing the displacements (4.1) and (4.2) in conjunction with (4.18) and (4.19) into the boundary conditions (3.33)-(3.36), making use of (4.5) and (4.6) and then equating the coefficients of $\cos n\phi$ and $\sin n\phi$ leads to, for all $n\geq 0$,

$$H''ax_n - a_n - nab_n = f'aq_n \quad , \qquad \text{on } r = a, \tag{4.37}$$

$$ha^2y_n - na_n - ab_n = -(H'-R_o + f)nq_n, \text{ on } r = a, \tag{4.38}$$

$$2\mu H''bx_n-(2\mu-p_o)(a_n+nbb_n) = (2\mu - p_o)\, f'bs_n, \text{ on } r = b, \tag{4.39}$$

$$2\mu h b^2y_n - (2\mu-p_o)(na_n+bb_n) = -[2\mu H'-(R_o-f)(2\mu-p_o)]\, ns_n, \text{ on } r = b. \tag{4.40}$$

Again the three cases n = 0, n = 1 and n ≥ 2 must be treated separately.

Considering the case n = 0 first, we substitute from (4.15) and (4.16) into (4.37) and (4.39) to get the following set of linear algebraic equations.

$$\begin{bmatrix} H'' - 1/2 & -1/a^2 \\ 2\mu H'' - (2\mu-p_o)/2 & -(2\mu-p_o)/b^2 \end{bmatrix} \begin{bmatrix} A_o \\ D_o \end{bmatrix} = \begin{bmatrix} f'(b)q_o \\ (2\mu-p_o)f'(b)s_o \end{bmatrix}. \tag{4.41}$$

As in the previous case it is the question of the existence of a unique solution of the above equations which is of special interest to us. Computing the determinant of the coefficient matrix in (4.41) we find that it vanishes if and only if

$$p_o = 2\mu \frac{(1-k)(1-2H'')}{1+k(2H''-1)}, \quad k = (a/b)^2. \tag{4.42}$$

Recalling that $H''>1$ (by (2.12)), it is evident that the value of p_o given by (4.42) is negative. Thus no loss of stability occurs for n = 0.

We next consider the case n = 1. Substituting from (4.9)-(4.12) into (4.37)-(4.40) and simplifying yields the following set of four linear equations in three unknowns

$$\begin{bmatrix} [H''h-(H''+h)/4]a & [H''h+(H''-h)/2]a^{-1} & -2a^{-3} \\ H''h-(H''+h)/4 & -[H''h-(H''-h)/2]a^{-2} & -2a^{-4} \\ [2\mu H''h-(2\mu-p_o)(H''+h)/4]b & [H''h+(2\mu-p_o)(H''-h)/2]b^{-1} & -2(2\mu-p_o)b^{-3} \\ 2\mu H''h-(2\mu-p_o)(H''+h)/4 & -[H''h-(2\mu-p_o)(H''-h)/2]b^{-2} & -2(2\mu-p_o)b^{-4} \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ B_1 \\ D_1 \\ \end{bmatrix} = \begin{bmatrix} f'q_1 \\ -(H'-R_o+f)a^{-2}q_1 \\ (2\mu-p_o)f's_1 \\ -[2\mu H'-(R_o-f)(2\mu-p_o)]b^{-2}s_1 \end{bmatrix}. \qquad (4.43)$$

We note that the terms containing C_1 have dropped out of the equations (4.43). Thus the value of C_1 is arbitrary. As was shown in the case of internal pressure the expressions containing C_1 corresponds to a rigid body translation and therefore one can set $C_1 = 0$. (The argument is exactly the same as before and will not be repeated).

As in the previous case there are two questions concerning (4.43) which are of importance: consistency, which is necessary for existence of a solution and uniqueness which implies stability. Before investigating these issues, we note that $B_1 = 0$ satisfies the equations. (The calculations leading to this are somewhat lengthy but straightforward.) We can thus remove the second column of the coefficient matrix in (4.43) to obtain a 4 by 2 matrix in which the first row is a multiple of the second and the third row a a multiple of the fourth. The consistency of the equations is established if the same relations hold between the elements of the right hand side column vector. It can be verified that this is indeed the case, hence of the four equations only two are independent and we have the following system

$$\begin{bmatrix} h''h-(H''+h)/4 & -2/a^4 \\ 2\mu H''h-(2\mu-p_\circ)(H''+h)/4 & -2(2\mu-p_\circ)/b^4 \end{bmatrix} \begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} -(H'-R_\circ+f)q_1/a^2 \\ -[2\mu H'-(R_\circ-f)(2\mu-p_\circ)s_1]/b^2 \end{bmatrix}. \tag{4.44}$$

To investigate the uniqueness of solution we look for values of $p_\circ$ for which the determinant of the 2 by 2 matrix above vanishes. If any of these values lie in the range $(0,\ 2\mu)$ it would be a buckling load. Denoting the determinant by Δ we find

$$\Delta = \frac{2(1-k)(2\mu-p_\circ)}{8a^4}\left[\frac{4H''[2\mu-(2\mu-p_\circ)k^2]-(1-k^2)(2\mu-p_\circ)}{2\mu-k(2\mu-p_\circ)} - 2H''(1+k)\right], \tag{4.45}$$

where $k = (a/b)^2$. Setting $\Delta = 0$ one finds $p_\circ = 2\mu$ and $p_\circ = 2\mu(1+k)(2H''-1)/[k(2H''-1)-1]$. Since we are interested in values of $p_0<2\mu$, we need consider the second expression only. By (2.12) the numerator is positive. As for the denominator there are two different cases: (i) the denominator is positive, in which case $p_\circ>2\mu$ which is inadmissible, (ii) the denominator is nonpositive which results in $p_\circ<0$ or $p_\circ$ infinite, both of which are unacceptable. We are thus led to the conclusion that for values of external pressure $p_\circ$ in the range of interest, no buckling will occur in the mode $n = 1$.

Finally, we take up the case $n \geq 2$. Upon introduction of (4.9), (4.10), (4.13) and (4.14) into (4.37)-(4.40) we obtain, for each $n \geq 2$, a set of four linear algebraic equations for A_n - D_n which in matrix form can be written as

$$\begin{bmatrix} [H''h-(nH''-(n-2)h)/4]a^{n} & [H''h-((n+2)h-nH'')/4]a^{-n} & (n-1)a^{n-2} & -(1+n)a^{-n-2} \\ [H''h-(nh-(n-2)H'')/4]a^{n-1} & -[H''h+(nh-(n-2)H'')/4]a^{-n-1} & (1-n)a^{n-3} & -(1+n)a^{-n-3} \\ [2\mu H''h-(2\mu-p_o)(nH''-(n-2)h)/4]b^{n} & [2\mu H''h-(2\mu-p_o)((n+2)h-nH'')/4]b^{-n} & (2\mu-p_o)(n-1)b^{n-2} & -(2\mu-p_o)(1+n)b^{-n-2} \\ [2\mu H''h-(2\mu-p_o)(nh-(n-2)H'')/4]b^{n-1} & -[2\mu H''h+(2\mu-p_o)(nh-(n+2H'')/4]b^{-n-1} & (2\mu-p_o)(1-n)b^{n-3} & -(2\mu-p_o)(1+n)b^{-n-3} \end{bmatrix}$$

$$\begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \begin{bmatrix} f'q_n \\ -(H'-R_o+f)a^{-2}nq_n \\ (2\mu-p_o)f's_n \\ -[2\mu H'-(R_o-f)(2\mu-p_o)]b^{-2}ns_n \end{bmatrix} . \qquad (4.46)$$

We again seek those values of p_o in the range $0<p_o<2\mu$ for which (4.46) becomes singular. To this end we introduce the parameter

$$T \equiv \frac{2H''p_o}{(2H''-1)(1-k)(2\mu-p_o)} , \tag{4.47}$$

in terms of which the determinant of the coefficient matrix in (4.46) is found to be

$$\Delta = \frac{\mu^4(H''-h)^2\lambda_n}{4(T+1)^2}(2\mu-p_o)\ \{[h_n+(1-2k)k]T^2+2(h_n+k^2)\ T+h_n-k\} , \tag{4.48}$$

where $k = (a/b)^2$ and h_n and λ_n are given by (4.34). Setting $\Delta = 0$ and solving the quadratic in T we find two values of T given by T_n^+ and T_n^- where

$$T_n^{\pm} = \frac{k-h_n}{k^2+h_n\pm[(1-k)^2+4h_n]^{\frac{1}{2}}} . \tag{4.49}$$

If for some value of pressure $p_o(<2\mu)$ the parameter T defined by (4.47) has the values given by (4.49) then the system of equations (4.46) is singular. We now address the question of the existence of a pressure for which $T = T_n^{\pm}$.

Since $H''>1$ and $p_o<2\mu$, by (2.12) and (2.35) respectively, T as defined by (4.47) is positive. Recalling that $k-h_n>0$, (4.36), it is evident that T_n^+ is positive. On the other hand, a simple calculation shows that for thick enough cylinders $(k<\frac{1}{2})$ T_n^- is negative and hence is not of interest. However, for values of k in the range $\frac{1}{2}<k<1$, one can verify that T_n^- is negative for certain values of n (e.g. $n = 2, 3, 4$) and positive for others. In what follows, wherever we consider T_n^- it is understood that we are restricting attention to those values of k and n which render it positive.

Next we rearrange (4.47) to give

$$p_o = \frac{2\mu}{1+\frac{1}{(1-k)T}\frac{2H''}{2H''-1}} . \tag{4.50}$$

Now define functions $\Psi_\pm(p_o)$ for all $p_o \in [0,2\mu]$ by

$$\Psi_\pm(p_o) \equiv p_o - \frac{2\mu}{1+\frac{1}{(1-k)T_n^\pm}\frac{2H''(R_o)}{2H''(R_o)-1}} , \tag{4.51}$$

where R_o is given in terms of p_o by (2.25) with $p_i = 0$. Calculating $\Psi_\pm(0)$ and $\Psi_\pm(2\mu)$ we find

$$\Psi_\pm(0) = - \frac{2\mu}{1+\frac{1}{(1-k)T_n^\pm}\frac{2H''(2)}{2H''(2)-1}} < 0 , \tag{4.52}$$

$$\Psi_\pm(2\mu) = 2\mu \left[1 - \frac{1}{1+\frac{1}{(1-k)T_n^\pm}\frac{2H''(R_o)}{2H''(R_o)-1}}\right] > 0. \tag{4.53}$$

The inequalities in (4.52) and (4.53) follow from (2.12) and $k < 1$. Since $\Psi_\pm$ depends continously on p_o it follows that there exist numbers $p_n^\pm \in (0,2\mu)$ such that $\Psi_+(p_n^+) = 0$, $\Psi_-(p_n^-) = 0$. This establishes the existence (of two sequences) of buckling pressures p_n^+, p_n^-. The former exists for tubes of arbitrary thickness ratio k and all modes $n \geq 2$, whereas the latter exists for tubes with the thickness ratio k in the range $(1/2, 1)$ and sufficiently large modes n. (See paragraph preceeding (4.50)).

We show next that the buckling pressures $p_n^{\pm}$ corresponding to a given value of n are unique. Clearly it is sufficient for this purpose that $\Psi_{\pm}(p_o)$ be monotone increasing functions on $(0, 2\mu)$. Differentiation of $\Psi_{\pm}(p_o)$ with respect to p_o yields

$$\Psi_{\pm}'(p_o) = 1 - \frac{4\mu(1-k)T_n^{\pm} H'''}{[2H''+(1-k)(2H''-1)T_n^{\pm}]^2} \frac{dR_o}{dp_o} , \tag{4.54}$$

where dR_o/dp_o may be calculated from (2.25) with $p_i = 0$ to be

$$\frac{dR_o}{dp_o} = - \frac{\mu(1-k)}{[2\mu-k(2\mu-p_o)]^2 h'(R_a)} < 0, \tag{4.55}$$

the inequality holds by virtue of (2.11) and $k<1$. Thus a <u>sufficient</u> condition for $\Psi_{\pm}(p_o)$ to be monotone increasing is that

$$H'''(R) \geq 0 \quad \text{for} \quad R>0. \tag{4.56}$$

It follows that if (4.56) holds the buckling pressures p_n^+, p_n^- corresponding to a given mode n are unique.

We now turn to determining the <u>smallest</u> buckling pressure. We will first show that the buckling pressures p_n^+ form a monotone increasing sequence (with respect to n) so that we then have $p_2^+ < p_n^+$ $(n>2)$. Furthermore we shall also demonstrate that $p_2^+ < p_n^-$ $(n>2)$ as well. Thus p_2^+ is the <u>smallest buckling</u> <u>pressure</u> for an externally pressurized tube.

To show the monotonocity of p_n^+ with respect to n we replace T by T^+ in (4.47) and differentiate with respect to p_o to find

$$\frac{dT^+}{dp_o} = \frac{1}{2H''^2(2\mu-p_o)^2}\left[2(2H''-1)H''-p_o(2\mu-p_o)H'''\frac{dR_o}{dp_o}\right] > 0. \quad (4.57)$$

The inequality follows from (2.12), (2.35), (4.55), and (4.56). Treating n as a continuous variable and differentiating (4.49) with respect to n gives $dT^+/dn > 0$. Now

$$\frac{dp_o}{dn} = \frac{dT^+/dn}{dT^+/dp_o} > 0, \quad (4.58)$$

which implies that $dp_n^+/dn > 0$, i.e. p_n^+ form a monotone increasing sequence.

To establish $p_2^+ < p_n^-$ we first observe that $T_n^- > T_n^+ > T_2^+ (n>2)$. The last inequality follows since, as already mentioned, T_n^+ is monotone increasing with n. Now writing $\Psi_-(p_o)$ from (4.51) we have

$$\Psi_-(p_o) = p_o - \frac{2\mu}{1+\frac{1}{(1-k)T_n^-}\frac{2H''}{2H''-1}}. \quad (4.59)$$

Calculating $\Psi_-(p_2^+)$ we find

$$\Psi_-(p_2^+) = 2\mu\left[\frac{1}{1+\frac{1}{(1-k)T_2^+}\frac{2H''}{2H''-1}} - \frac{1}{1+\frac{1}{(1-k)T_n^-}\frac{2H''}{2H''-1}}\right] < 0, \quad (4.60)$$

which together with (4.56) verifies our claim.

To summarize: a formal solution to the equilibrium equations has been found (see (4.1), (4.2), (4.11)-(4.14) and (4.16)). The stability of this solution in the two cases of internal and external pressure was examined. The investigation showed that for an internally pressurized cylinder the equilibrium is always stable in the range of interest, whereas under external pressure the tube becomes unstable. In particular, this implies that under external pressure, the tube buckles before the cavity closes. On the other hand, an internally pressurized tube reaches the "bursting pressure" before any instability is encountered. The existence of buckling pressures, in the former case, was proved (see remarks following (4.53)) and a sufficient condition for the uniqueness of the buckling loads corresponding to different modes established. Finally, the smallest buckling load for a cylinder subject to external pressure was determined. In the next section we illustrate some of these results using a particular constitutive relation.

5. ILLUSTRATIVE EXAMPLE

In this section a special (hypothetical) harmonic material is introduced to illustrate some of the results found previously.

We recall that for harmonic materials the strain energy density function is given by $W = 2\mu[H(R)-J]$. In the following we generalize a particular power-law form of $H(R)$ used by Knowles and Sternberg (1977) and suppose that

$$H(R) = \frac{1}{2}R^2 + \frac{R}{m-1}\left(\frac{2}{R}\right)^m + \frac{1+m}{1-m}, \quad m \geq 0,\ m \neq 1,\ R > 0. \tag{5.1}$$

Clearly $H(R)$ is continuous and it is easy to verify that the restrictions (2.9)-(2.12) are all satisfied.

Taking (2.25) with $p_i = 0$ in conjunction with (5.1), we find that the invariant R_o is given by

$$R_o = 2\left[\frac{2\mu-k(2\mu-p_o)}{2\mu+p_o-k(2\mu-p_o)}\right]^{\frac{1}{m+1}}, \tag{5.2}$$

where $k = (a/b)^2$. (Recall that a,b denote the inner and outer undeformed radii of the tube.) Now from the first of (2.1), (2.20), (2.25) and (2.26) with $p_i = 0$ one has

$$\frac{\beta}{b} = \frac{(1-k)\mu R_o}{2\mu-k(2\mu-p_o)}. \tag{5.3}$$

Equations (5.2) and (5.3) provide a relation between the applied external pressure p_o and the deformed outer radius β. Graphs of $p_o/2\mu$ versus β/b for different values of the hardening exponent m

and the geometric parameter $t = b/a$ are shown in Figures (7) and (8) respectively.

Moreover on using (4.34), (4.49), (4.50) and (5.2) (with positive sign chosen in (4.49) and the resulting T^+ used in (4.50)) we may calculate the smallest buckling load (p_2^+) of the externally pressurized tube. These buckling pressures are also marked on the graphs.

As can be seen from Figure 7, the material hardens with increasing values of m. Moreover the buckling pressure also increases as the hardening exponent increases. It is also evident from Figure 8 that the buckling pressure increases as the tube becomes thicker which is what one would expect.

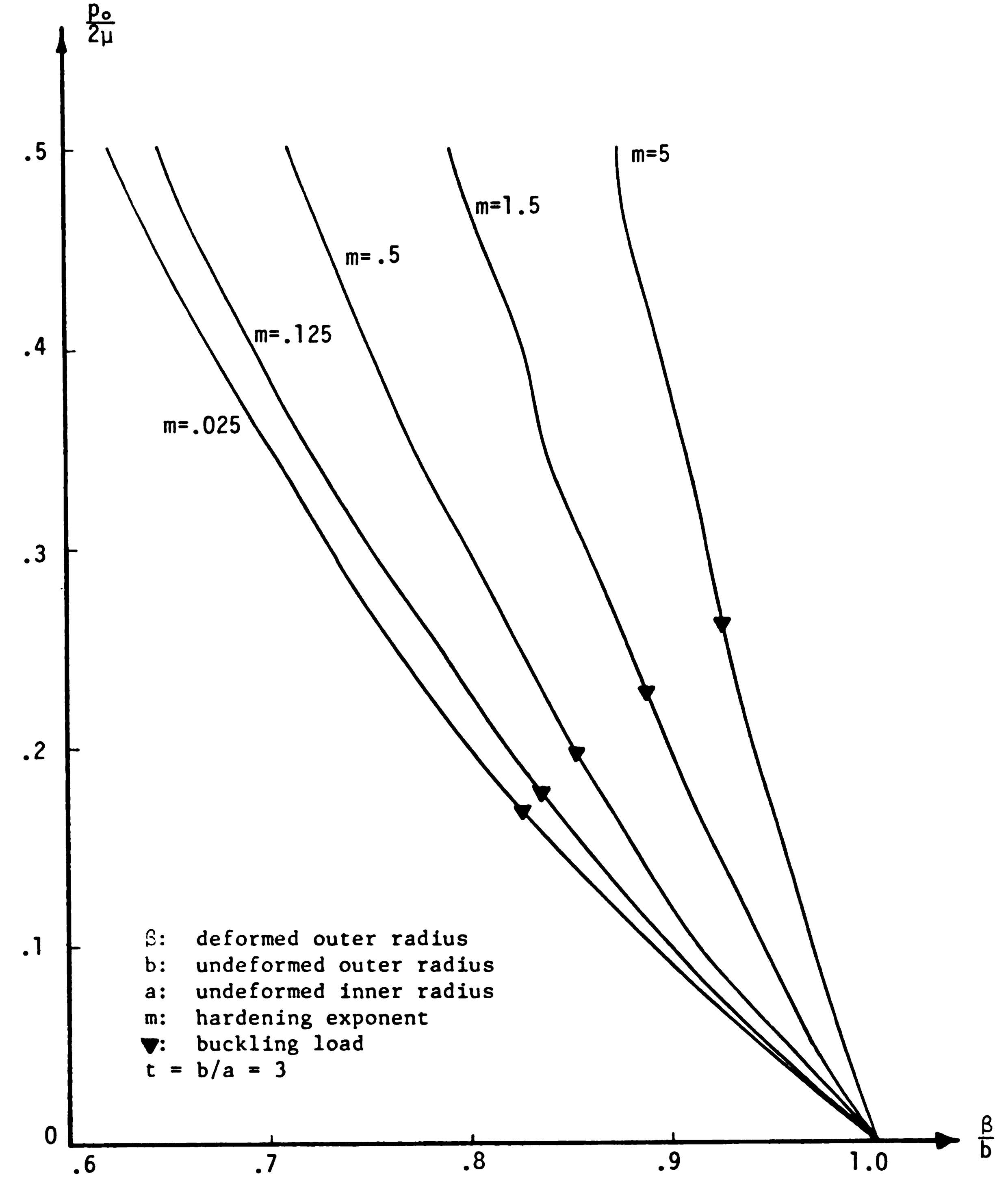

Figure 7. External pressure versus β/b for different materials.

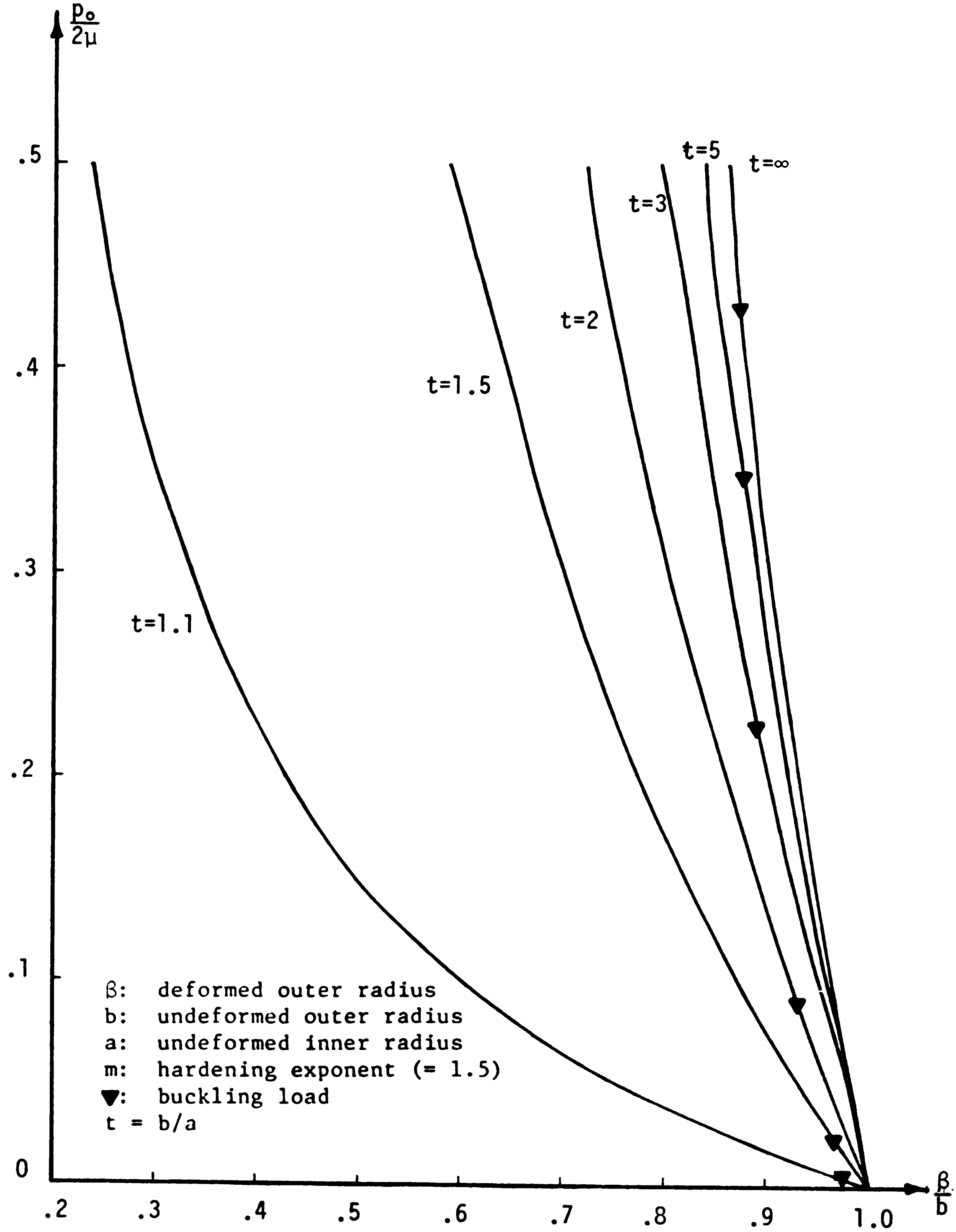

Figure 8. External pressure versus β/b for different tubes.

6. CONCLUDING REMARKS

The equilibrium of a pressurized tube of homogeneous, isotropic, compressible material of harmonic type was considered and the stability of the solutions investigated. In summary, the results show that a tube subjected to internal pressure will fail by bursting while an externally pressurized tube fails by buckling. For the latter case the smallest buckling load was calculated and the existence of buckling loads for higher modes proved. Furthermore, a sufficient condition for the uniqueness and monotonocity of these loads with respect to the mode number was established.

The results found in this study pertain to a special material (harmonic) and a particular geometry (circular tube). Ideally one would wish to establish results with few restrictions on the constitutive law or geometry. This, however, is a formidable undertaking. (Analytical determination of the stress field in a circular tube composed of a general compressible homogeneous isotropic hyperelastic material is a daunting task!) There are more modest goals that one may pursue which can shed some light on these issues. These can be summarized as follows:

(i) The effect of geometry: The question to be answered is whether the result found here for an internally pressurized tube (sudden bursting without any instability preceding it) depends on geometry or is a "property" of the material. Study of tubes with noncircular bores and/or noncircular outer boundaries would provide, at least, a partial answer to this. It would be interesting to see if externally pressurized tubes with noncircular outer boundaries always become unstable by buckling and if so whether all the modes are generated.

(ii) The effect of the material: Here there are two questions to be examined. First the effect of hardening. The fact that no buckling instability occurs in a circular tube subject to internal pressure is apparently due to the very rapid hardening of the harmonic material. Examination of other compressible materials (hypothetical or otherwise) could provide some insight into this question.. It would be interesting to see if a "critical" hardening rate can be found such that materials which harden at a lower rate will "permit buckling" while the ones which harden at a higher rate will not. Secondly, it would be interesting to examine whether similar phenomena arise when the material is incompressible.

APPENDIX

APPENDIX A:

Proof of Equation (4.36)

We wish to show that

$$k - h_n > 0 \quad , \qquad \text{for all} \geq n \ 2, \tag{A.1}$$

where k is a real number $0<k<1$ and h_n is given by

$$h_n = \frac{n^2(1-k)^2}{k^n+k^{-n}-2} \quad , \ n \geq 2. \tag{A.2}$$

The proof is by induction on n. We first note that for n = 2 one has

$$k-h_2 = \frac{k(1-k)^2}{(1+k)^2} > 0. \tag{A.3}$$

Now assuming that $k-h_n > 0$ for any $n>2$, we have to show that $k-h_{n+1} > 0$. It is sufficient for this to show that h_n is a monotone decreasing function of n. To this end we define the function h(x) for all $x>1$ by

$$h(x) = (1-k)^2/\phi(x), \tag{A.4}$$

where

$$\phi(x) = \frac{k^x+k^{-x}-2}{x^2} \quad . \tag{A.5}$$

It would be sufficient for us to show that $\phi(x)$ is a monotone increasing function of x.

Differentiating $\phi(x)$ with respect to x and making the substitution $y = k^{-x}$ we find

$$\phi'(x) = \frac{y^2-1}{x^3 y} \left(\ln y + 2\,\frac{1-y}{1+y}\right). \tag{A.6}$$

We note that $y = k^{-x} > 1$ and therefore the problem reduces to showing that

$$Y(y) = \ln y + 2\,\frac{1-y}{1+y} > 0, \quad \text{for all} \quad y > 1. \tag{A.7}$$

We observe that $Y(1) = 0$ and

$$Y'(y) = \frac{(1-y)^2}{y(1+y)^2} > 0 \text{ for all } y > 1. \tag{A.8}$$

Thus $\phi(x)$ is monotone increasing and hence $h(x)$ is monotone decreasing. Therefore, h_n is monotone decreasing and the result is thus established.

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