

ABSTRACT

TOPOLOGICAL SYNTHESIS OF N-PORT RESISTIVE NETWORKS FROM SHORT CIRCUIT CONDUCTANCE MATRICES THAT ARE REALIZABLE WITH TWO-TREE PORT-STRUCTURES

by

C. G. Jambotkar

Procedures are available in the literature for synthesizing resistive networks from short circuit conductance matrices which are realizable with connected (one-tree) port-structures. Little is known, however, regarding synthesis of resistive networks from short circuit conductance matrices which are realizable only with separated (k-tree) port-structures.

In this thesis, a procedure is established for the synthesis of resistive networks from short circuit conductance matrices which are realizable with two-tree port-structures. The formulation presented in the thesis enables the problem to be reduced, in fact, to the well-known synthesis of resistive networks having linear port-structures. For a complete resistive network with $(n+2)$ nodes, the number of the constituent two-terminal resistors is $(n+1)(n+2)/2$. In the devised procedure, the conductance values of $(n+1)$ of these constituent resistors - which are incident at a particular node - have been considered as parameters. It is established that the indicated parameters are subject to certain bounds, which, in fact, facilitate the desired realization of matrices. The problem then is to obtain

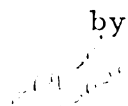
a suitable set of values of these parameters within the established bounds. This is done with the help of a digital computer. Once the values of the parameters are decided, the complete realization of the short circuit conductance matrix follows immediately. In general, the set of values of the parameters is not unique, so that many equivalent realizations are obtainable by means of the devised procedure.

Certain short circuit conductance matrices which belong to a certain special class are also considered in the thesis. They are realizable with a special version of the above procedure, which has one important feature of providing two distinct "minimal" realizations. Furthermore, the necessity of machine computations is avoided in this case.

It is proved in this thesis that every paramount matrix of order three belongs to the above special class. Thus, a new, straightforward procedure is established for the realization of any third-order paramount matrix which is considered as either a short circuit conductance matrix or an open circuit resistance matrix.

Finally, some ideas are included in this thesis on a possible approach to the problem of realization of short circuit conductance matrices which are realizable with k -tree port-structures ($3 \leq k \leq n$).

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1967

ACKNOWLEDGMENTS

The author wishes to express, first of all, his deep gratitude to Dr. Y. Tokad for providing excellent direction to the research reported in this thesis. The author is also grateful to Dr. H. G. Hedges for his every interest in the author's doctoral studies in the capacity of his academic adviser, guidance committee chairman, and the acting chairman of the department. Sincere thanks are due to Prof. T. W. Culpepper, Dr. J. Kateley, and Dr. N. L. Hills for their willing participation in the author's guidance committee. Finally, it is acknowledged that this research was financially supported by the Division of Engineering Research at Michigan State University.

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INTRODUCTION

1.1. Motivation

Synthesis of a transformerless n -port resistive network from its short circuit (s.c.) conductance matrix is an important topic in network theory. In his classical book, Cauer [CA 1] presented a complete solution to the relatively less important problem of the synthesis of a network which includes ideal transformers. However, the main problem of transformerless synthesis remained almost totally unsolved until recently.

The progress made in recent years in this area reveals that consideration of a network from the topological point of view offers much insight into the problem. This thesis, also, is based upon topological considerations while dealing with one important part of the whole problem, viz., the synthesis of transformerless n -port resistive networks from s.c. conductance matrices that are realizable with two-tree port-structures though not realizable with connected (one-tree) port-structures.

The general problem of transformerless synthesis is a basic theoretical problem. It can be looked upon as the inverse of the problem of analysis of resistive networks that was solved long ago by Kirchhoff and Maxwell. Knowledge in the area of synthesis of resistive networks has important applications in fields such as contact, communication, and probabilistic networks or sequential machines inasmuch as the weights assigned to edges

of the pertinent graphs in these fields are normally non-negative real numbers. It is known [PA 1] that the solution to the problem of synthesis of resistive networks does not lend itself to direct extension to the synthesis of the broader class of networks, viz., the RLC networks, as characterized in the s -domain. However, it is believed [DE 1], [LE 1], [RA 1] that if the RLC networks are characterized by their state equations, then an application of the topological synthesis of resistive networks may very well provide such an extension. Moreover, even if the techniques of synthesis of resistive networks cannot be extended directly to the synthesis of RLC networks characterized in the s -domain, those techniques do have value in regard to the latter problem. For example, an RLC network displays the properties of a resistive network for positive, real values of the complex frequency, s , so that one of the obvious necessary conditions for the synthesis of an RLC network is the R-realizability of its s.c. admittance matrix for all positive, real values of s .

The general problem may be defined precisely as the "problem of synthesis of a resistive network, if existent, from an n th-order, real, symmetric matrix considered as a s.c. conductance matrix." The solution to this problem must consist of two parts: (1) a formulation of the necessary and sufficient conditions for a matrix to be the s.c. conductance matrix corresponding to one or more resistive networks, and (2) a statement of a procedure for synthesizing at least one of those networks without using any transformers. The latter includes:

(i) stating the network configuration(s), (ii) specifying the locations and orientations of ports, and (iii) stating the values of network elements in terms of entries of the s.c. conductance matrix.

It is fitting at this stage to review briefly the recent progress in the area of synthesis of resistive networks. It has been established [CE 1], [CE 2] that a matrix must be paramount in order that it may be a s.c. conductance matrix, but that paramountcy is not always sufficient for a matrix to have the said electrical significance when the matrix order exceeds three. A procedure is known [SL 1] for realization of a very special class of matrices, viz., the dominant matrices. Satisfactory necessary and sufficient conditions are known on matrices if they are to correspond to networks having two special types of port-structures, viz., those that form linear (path) trees and starlike (Lagrangian) trees. In the case of a linear tree, the necessary and sufficient condition is that a matrix be uniformly tapered [BI 1], [GU 1]; in the case of a starlike tree, the necessary and sufficient condition is that a matrix be dominant with non-positive off-diagonal entries [BO 1]. The articles dealing with these two special port-structures also state the corresponding realization procedures, which are straight-forward. It is possible that a matrix is the s.c. conductance matrix of a resistive network having a port-structure that does form a connected graph - the graph must be a tree in such a case - but then the graph is neither a linear tree nor a starlike tree. Necessary and sufficient conditions for a matrix to belong to this broader class have been stated by several

research workers [GU 1], [BI 2], [CE 3], [HA 1], [BO 2], though, as commented by one of the contributors himself [PA 1], none of these sets of necessary and sufficient conditions is fully satisfactory. Their deficiency lies in the fact that each one of them requires execution of the complete process of building up the network; as such, they are all operational in character. The possibility of establishing proper conditions which can be tested without resorting to a building-up process appears, at present, rather remote to many research workers, and the few articles that have been published in the area of R-network synthesis during the past four years have been directed exclusively toward the solution of the next and the much more challenging problem of matrices that may be s.c. conductance matrices realizable with separated port-structures alone. At the time the research reported in this thesis was started, the problem remained far from being solved.

Some elementary ideas were offered by Guillemin [GU 1] for the first time in 1960 regarding matrices that might be realizable with separated port-structures. A year later, he elaborated on these ideas [GU 2] without any claim of having introduced a practical procedure. A good illustration of this formidable "augmentation" procedure incorporating trial-and-error was supplied by Brown and Tokad in one of their articles [BR 1], which also presented some further fundamental concepts in the area. The same idea of augmentation has recently been pursued by Swaminathan [SW 1], who finally formulates some

necessary conditions - named the "supremacy" conditions - that are applicable in the case of piecewise linear, separated port-structures. Five more articles [BI 3], [BI 4], [CE 4], [CE 5], [CI 2] and four short notes [BI 5], [BI 6], [NA 1], [NA 2] are available in the literature, which considerably aid our understanding of various important aspects of resistive networks having more than $(n+1)$ nodes. None of these articles and notes directly cover, however, the problem of establishing some practical techniques for synthesizing resistive networks from s.c. conductance matrices that are realizable with separated port-structures alone. The more recent one of the two articles by Lupo and Halkias [HA 2], [LU 1] is of value in the sense just mentioned. It is directed toward presenting a new method that may apply to a class of matrices which are realizable with known two-tree port-structures, the class being defined by the applicability of the method itself.

The above survey of all the existent literature reveals that the problem of synthesis of n -port resistive networks is, in fact, only partially solved. At the same time, as indicated earlier, the solution to the problem is of much significance, especially in the context of transformerless synthesis of RLC networks characterized by their state equations. It was through the realization of the overall significance of the problem that the author was motivated to carry out further research in the area of R-network synthesis.

1.2. Some Basic Concepts and Definitions

1. Throughout the thesis, a resistive network will be represented by a linear graph such that the vertices correspond to the nodes of the network, the edges correspond to the resistors, and the edge-weights, to the conductance values.

2. A "port" is defined as a pair of nodes of a network accessible for excitations and measurements. A port will be indicated by an oriented edge in heavy line, the orientation indicating the polarity of the excitation source. The linear graph constituted by these edges contains no circuits and is termed as the "port-structure" (or the "terminal graph") of the pertinent network.

3. Let Q denote the s.c. conductance matrix for an n -port resistive network. Further, let Q' denote the s.c. conductance matrix for the same network after its original port-numbers $1, 2, \dots, i, \dots, j, \dots, n$ are replaced, respectively, by $k, 1, \dots, m, \dots, c, \dots, f$ ($k, 1, m, c, f \leq n$). Then matrices Q and Q' are related by:

$$Q' = E_1^{(k, 1, \dots, m, \dots, c, \dots, f)} Q E_1^{(k, 1, \dots, m, \dots, c, \dots, f)} \quad (1.2.1)$$

where $E_1^{(k, 1, \dots, m, \dots, c, \dots, f)}$ denotes an n th-order matrix derived from the unit matrix by rearranging its columns such that the entries in positions $(k, 1), (1, 2), \dots, (m, i), \dots, (c, j), \dots$, and (f, n) are unit entries and the entries in remaining positions are zero entries.

4. Let Q denote the s.c. conductance matrix for an n -port resistive network. Further, let Q' denote the s.c. conductance matrix for the same network after the orientations of some of its ports $i, j, \dots, k$ ($i, j, k \leq n$) are reversed. Then matrices Q and Q' are related by:

$$Q' = U_s^{(i, j, \dots, k)} Q U_s^{(i, j, \dots, k)} \quad (1.2.2)$$

where $U_s^{(i, j, \dots, k)}$ denotes the matrix which results from reversing the signs of the entries in positions (i, i) , (j, j) , $\dots$, and (k, k) of the n th-order unit matrix. The pre- and post-multiplication of matrix Q by $U_s^{(i, j, \dots, k)}$ is referred to as the "cross-sign change operation" on matrix Q .

5. Let $Y = [y_{ij}]_n$ be a real, symmetric matrix and let a matrix, $T = [t_{ij}]_n$, be defined by:

$$t_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

Then matrix Y is said to be in the uniformly tapered form if each $y_{ij} \geq 0$ and further, with $y_{0,j} = y_{i,n+1} = 0$, each entry, ${}^{(t)}y_{ij}^{(t)}$, in the upper triangular portion[†] of matrix TYT , viz.,

$${}^{(t)}y_{ij}^{(t)} = y_{ij} - y_{i,j+1} - y_{i-1,j} + y_{i-1,j+1} \geq 0 \quad \text{for } j \geq i$$

$$(i, j=1, 2, \dots, n) \quad (1.2.3)$$

[†] Throughout the thesis, the "upper triangular portion" of a matrix will be considered to include all diagonal entries.

A uniformly tapered matrix, $Y = [y_{ij}]_n$, can always be realized as a s.c. conductance matrix in the manner shown in Figure 2.2.1, where the conductance value of a resistor across the positive terminal of a port, i , and the negative terminal of a port, j ($i, j = 1, 2, \dots, n$), is given by ${}^{(t)}y_{ij}^{(t)}$ in (1.2.3) [GU 1].

6. Let

$$Q^* = \begin{bmatrix} {}^{(n)}Q_{11}^* & {}^{(k)}Q_{12}^* \\ Q_{12}^{*T} & {}^{(k)}Q_{22}^* \end{bmatrix} \begin{matrix} (n) \\ (k) \end{matrix}$$

be the s.c. conductance matrix of an $(n+k)$ -port resistive network. If k of its ports are no longer of interest for any reason, then the s.c. conductance matrix, Q , corresponding to the first n ports is given for the same network by:

$$Q = Q_{11}^* - Q_{12}^* Q_{22}^{*-1} Q_{12}^{*T} \quad (1.2.4)$$

provided Q_{22}^* is non-singular [BR 1], [KR 1].

7. Let $Q^{(1)}$ and $Q^{(2)}$ be the s.c. conductance matrices corresponding to two different connected (tree) port-structures of an n -port resistive network. Then matrices $Q^{(1)}$ and $Q^{(2)}$ are related by:

$$Q^{(1)} = C^T Q^{(2)} C$$

where C is a unimodular matrix [BR 1].

Consider, now, the s.c. conductance matrices, $Q^{(1)}$ and $Q^{(2)}$, corresponding to two different separated (k -tree) port-structures of an n -port network such that both port-structures

have the same number of parts and, further, the i -th parts ($i = 1, 2, \dots, k$) of both port-structures contain the same set of nodes. Then a congruent relationship holds between matrices $Q^{(1)}$ and $Q^{(2)}$ even in this case as established in the following.

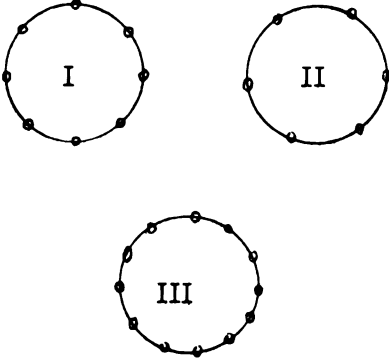


Figure 1.2.1

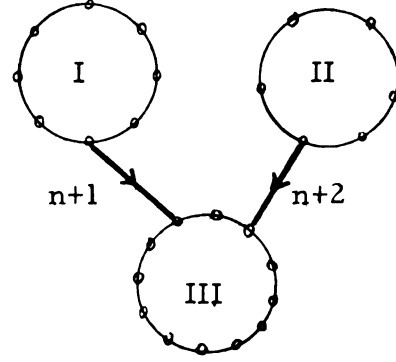


Figure 1.2.2

For the sake of definiteness, suppose that the two port-structures have three separated parts each as shown in Figure 1.2.1. Let $Q_I^{(1)}$ ($Q_I^{(2)}$) denote the s.c. conductance matrix for the network when measurements are made at ports belonging to part I alone of the first (second) k -tree port-structure. Since $Q_I^{(1)}$ and $Q_I^{(2)}$ correspond to two different connected port-structures of the same network, they must be related by:

$$Q_I^{(1)} = C_I^T Q_I^{(2)} C_I$$

where C_I is a unimodular matrix.

After defining $Q_{II}^{(1)}$, $Q_{II}^{(2)}$, $Q_{III}^{(1)}$, $Q_{III}^{(2)}$, C_{II} and C_{III} in an analogous manner, the following relations can be written:

$$Q_{II}^{(1)} = C_{II}^T Q_{II}^{(2)} C_{II}$$

$$Q_{III}^{(1)} = C_{III}^T Q_{III}^{(2)} C_{III}$$

Suppose, now, ports (n+1) and (n+2) are added, as shown in Figure 1.2.2, to each of the original two k-tree port-structures so that two connected (one-tree) port-structures are generated. Note that in both cases ports (n+1), as also (n+2), are connected between identical pairs of nodes. Let $Q^{(1)*}$ and $Q^{(2)*}$ denote the s.c. conductance matrices corresponding to these new connected port-structures. Then matrices $Q^{(1)*}$ and $Q^{(2)*}$ must be related by:

$$Q^{(1)*} = \begin{bmatrix} C^T & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} Q_{11}^{(2)*} & Q_{12}^{(2)*} \\ Q_{12}^{(2)*T} & Q_{22}^{(2)*} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & U_2 \end{bmatrix}$$

where

$$C = \begin{bmatrix} C_I & 0 & 0 \\ 0 & C_{II} & 0 \\ 0 & 0 & C_{III} \end{bmatrix} \quad \text{and} \quad Q^{(2)*} = \begin{bmatrix} Q_{11}^{(2)*} & Q_{12}^{(2)*} \\ Q_{12}^{(2)*T} & Q_{22}^{(2)*} \end{bmatrix}$$

Therefore,

$$Q^{(1)*} = \begin{bmatrix} C^T Q_{11}^{(2)*} C & C^T Q_{12}^{(2)*} \\ Q_{12}^{(2)*T} C & Q_{22}^{(2)*} \end{bmatrix}$$

Now if ports (n+1) and (n+2) are considered to be no longer of interest, then s.c. conductance matrices $Q^{(1)}$ and $Q^{(2)}$ corresponding to the two original k-tree port-structures will

be respectively given by:

$$\begin{aligned} Q^{(1)} &= C^T Q_{11}^{(2)*} C - C^T Q_{12}^{(2)*} Q_{22}^{(2)*^{-1}} Q_{12}^{(2)*T} C \\ &= C^T \left(Q_{11}^{(2)*} - Q_{12}^{(2)*} Q_{22}^{(2)*^{-1}} Q_{12}^{(2)*T} \right) C \end{aligned}$$

and

$$Q^{(2)} = Q_{11}^{(2)*} - Q_{12}^{(2)*} Q_{22}^{(2)*^{-1}} Q_{12}^{(2)*T}$$

Comparison of the above two expressions establishes the congruent relationship between matrices $Q^{(1)}$ and $Q^{(2)}$.

8. A realization of an n th-order s.c. conductance matrix is termed a "minimal realization" if it contains, at most, $n(n+1)/2$ resistors.

Chapter 2

MATRICES OF THE N-TH ORDER

2.1. Introduction

The present chapter deals, mainly, with the synthesis of resistive networks from n th-order s.c. conductance matrices that are realizable with specified two-tree port-structures. In the beginning, some pertinent analytical aspects of resistive networks are investigated. The systematic procedure for realizing the indicated class of s.c. conductance matrices is then established on the basis of those analytical aspects. Some machine computations form an integral part of the realization procedure. Theoretical considerations which facilitate these computations are presented in one of the sections, followed by essential details of the method of computations itself. An interesting version of the above procedure is discussed next. It has the significant feature of yielding minimal realizations in the case of a special class of s.c. conductance matrices. Furthermore, it can be applied easily to solve the pertinent synthesis problem analytically without the necessity of any machine computations. Matrices of the fourth order are considered subsequently in order to illustrate all the foregoing theory, which covers, in fact, the complete solution to the problem of realization of matrices with specified two-tree port-structures. In addition to the above solution, some ideas are included in this chapter on a possible approach to the problem of realization of matrices when the port-structure consists of k trees ($3 \leq k \leq n$).

2.2. Some Analytical Considerations

In order to investigate certain analytical aspects of resistive networks, consider a uniformly tapered matrix, $Y = [y_{ij}]_n$. The graphs of its $(n+1)$ -node realization as a s.c. conductance matrix is shown in Figure 2.2.1. The edges indicated by heavy lines in this figure represent the n ports, and these edges constitute a linear tree. With reference to this tree, let c_{ii} ($i = 1, 2, \dots, n$) denote the sum of conductance values of the edges which belong to the cut-set defined by branch i , and let c_{ij} ($i, j = 1, 2, \dots, n; i \neq j$) denote the sum of conductance values of the edges which are common to the cut-sets defined by branches i and j . Then, as is well-known,

$$y_{ij} = c_{ij} \quad (i, j = 1, 2, \dots, n) \quad (2.2.1)$$

Consider another resistive network derived from the one in Figure 2.2.1 by adding, as shown in Figure 2.2.2, $(n+1)$ resistors which have their respective non-negative conductance values $= g_k$ ($k = 1, 2, \dots, n+1$).

Let $\bar{Q} = [\bar{q}_{ij}]_n$ denote the s.c. conductance matrix for this derived network corresponding to the port-structure indicated in Figure 2.2.2 itself. The following discussion will be directed, then, toward establishing a relation between matrices Y and $\bar{Q}$.

For the derived network, consider a port-structure which is obtained by augmentation of the original port-structure with a port, h^* , as shown in Figure 2.2.3. Let $Q^* = [q_{ij}^*]_{n+1}$ denote the uniformly tapered s.c. conductance matrix corresponding to this augmented port-structure. Then, the entries of matrix Q^* can be written in terms of the entries of matrix $\bar{Q}$ as follows:

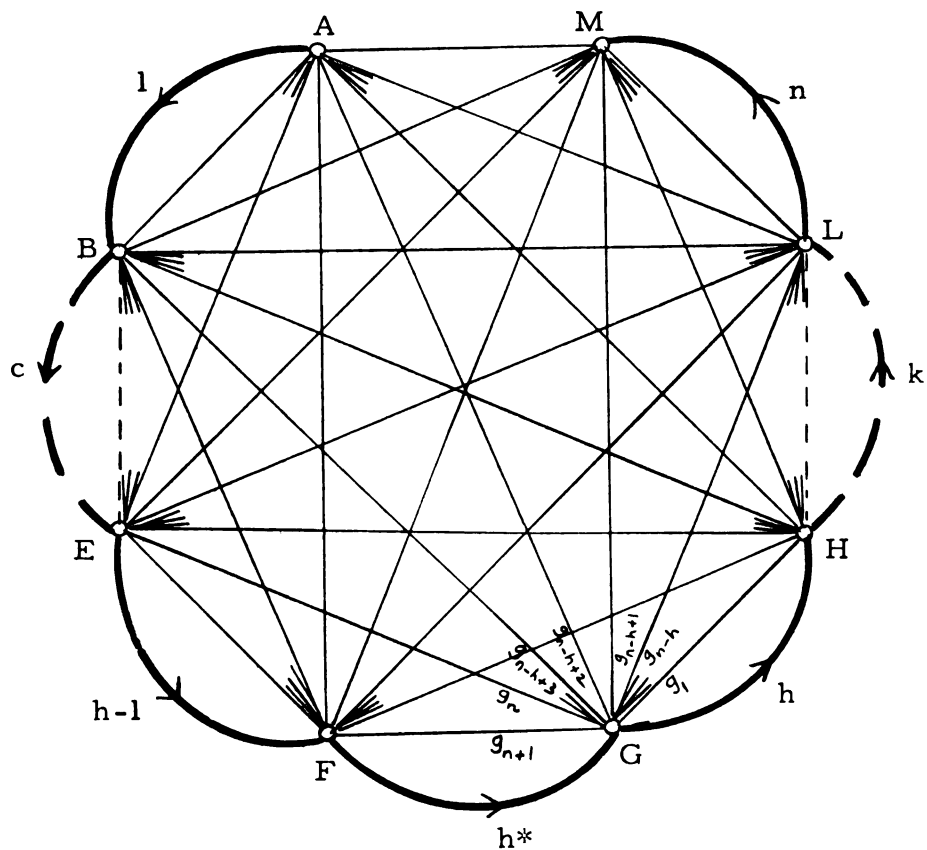


Figure 2.2.3

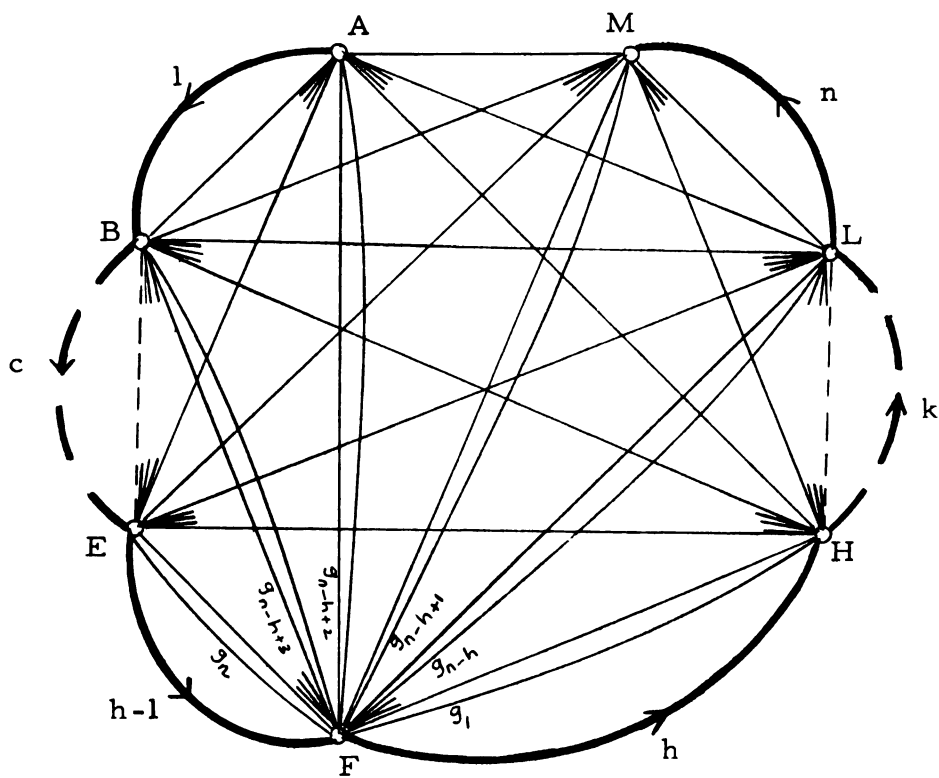


Figure 2.2.4

$$Q^* = \begin{bmatrix} \overline{q_{11}} + p_1^2 & \overline{q_{12}} + p_1 p_2 & \cdots & \overline{q_{1, h-1}} + p_1 p_{h-1} & p_1 p & \overline{q_{1h}} + p_1 p_h & \overline{q_{1, h+1}} + p_1 p_{h+1} & \cdots & \overline{q_{1n}} + p_1 p_n \\ \overline{q_{22}} + p_2^2 & \cdots & \overline{q_{2, h-1}} + p_2 p_{h-1} & p_2 p & \overline{q_{2h}} + p_2 p_h & \overline{q_{2, h+1}} + p_2 p_{h+1} & \cdots & \overline{q_{2n}} + p_2 p_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{q_{h-1, h-1}} + p_{h-1}^2 & \cdots & \overline{q_{h-1, h-1}} + p_{h-1} p_{h-1} & p_{h-1} p & \overline{q_{h-1, h}} + p_{h-1} p_h & \overline{q_{h-1, h+1}} + p_{h-1} p_{h+1} & \cdots & \overline{q_{h-1, n}} + p_{h-1} p_n \\ p^2 & \cdots & p p & p p_h & p p_{h+1} & \cdots & p p_n \\ \overline{q_{hh}} + p_h^2 & \cdots & \overline{q_{h, h+1}} + p_h p_{h+1} & \cdots & \overline{q_{hn}} + p_h p_n \\ \overline{q_{h+1, h+1}} + p_{h+1}^2 & \cdots & \overline{q_{h+1, h+1}} + p_{h+1} p_{h+1} & \cdots & \overline{q_{h+1, n}} + p_{h+1} p_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{q_{nn}} + p_n^2 \end{bmatrix} \quad (2.2.2)^{\dagger}$$

[†] In this thesis, the entries in the lower triangular portion of a symmetric matrix will often be omitted in the representation of the matrix.

$$\begin{aligned}
&= \begin{bmatrix} \bar{q}_{11} & \bar{q}_{12} & \cdots & \bar{q}_{1,h-1} & 0 & \bar{q}_{1h} & \bar{q}_{1,h+1} & \cdots & \bar{q}_{1n} \\ \bar{q}_{22} & & & \bar{q}_{2,h-1} & 0 & \bar{q}_{2h} & \bar{q}_{2,h+1} & \cdots & \bar{q}_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{q}_{h-1,h-1} & & & \bar{q}_{h-1,h-1} & 0 & \bar{q}_{h-1,h} & \bar{q}_{h-1,h+1} & \cdots & \bar{q}_{h-1,n} \\ 0 & & & & 0 & 0 & 0 & \cdots & 0 \\ & & & & & \bar{q}_{hh} & \bar{q}_{h,h+1} & \cdots & \bar{q}_{h,n} \\ & & & & & \bar{q}_{h+1,h+1} & \bar{q}_{h+1,h+1} & \cdots & \bar{q}_{h+1,n} \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & \bar{q}_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_{h-1} \\ p \\ p_h \\ p_{h+1} \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \\
&\quad [p_1 \ p_2 \ \cdots \ p_{h-1} \ p \ p_h \ p_{h+1} \ \cdots \ p_n] \\
&\quad (2.2.2a)
\end{aligned}$$

where

$$0 \leq p_1 \leq p_2 \leq \dots \leq p_{h-1} \leq p \leq p_h \geq p_{h+1} \geq \dots \geq p_n \geq 0 \quad (2.2.3)$$

$$\text{i. e., } q_{ij}^* = \bar{q}_{ij} + p_i p_j \quad (i, j = 1, 2, \dots, h-1)$$

$$q_{ij}^* = \bar{q}_{i, j-1} + p_i p_{j-1} \quad (i = 1, 2, \dots, h-1; j = h+1, h+2, \dots, n+1)$$

$$q_{ij}^* = \bar{q}_{i-1, j-1} + p_{i-1} p_{j-1} \quad (i, j = h+1, h+2, \dots, n+1)$$

$$q_{hj}^* = p p_j \quad (j = 1, 2, \dots, h-1)$$

$$q_{hj}^* = p p_{j-1} \quad (j = h+1, h+2, \dots, n+1)$$

$$q_{hh}^* = p^2.$$

Consider a network which is obtained by shorting port h^* in Figure 2.2.3. The s.c. conductance matrix, $\tilde{Q} = [\tilde{q}_{ij}]_n$, for this new network can be obtained by deleting the h -th row and the h -th column of matrix Q^* .

Thus, from (2.2.2a),

$$\tilde{Q} = \bar{Q} + P P^T$$

where (2.2.4)

$$P^T = [p_1 \ p_2 \ \dots \ p_{h-1} \ p_h \ p_{h+1} \ \dots \ p_n]$$

This network is shown in Figure 2.2.4. The edges corresponding to the n ports are shown, as before, in heavy lines, and they constitute a linear tree. Let $\tilde{c}_{ii}$ ($i = 1, 2, \dots, n$) denote the sum of conductance values of the edges which belong to the cut-set defined by branch i , and let $\tilde{c}_{ij}$ ($i, j = 1, 2, \dots, n; i \neq j$) denote the sum of conductance values of the edges which are common to the cut-sets defined by branches i and j . Then,

$$\tilde{q}_{ij} = \tilde{c}_{ij} \quad (i, j = 1, 2, \dots, n) \quad (2.2.5)$$

By comparing Figures 2.2.1 and 2.2.4 at this stage, one can write the relation:

$$\tilde{c}_{ij} = c_{ij} + a_{ij} \quad (i, j = 1, 2, \dots, n)$$

where

$$a_{ij} = \sum_{k=2}^{i+1} g_{n-h+k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=j}^n g_{k-h+1} \quad (j = h, h+1, \dots, n; i=h, h+1, \dots, j)$$

After substituting y_{ij} for c_{ij} and $\tilde{q}_{ij}$ for $\tilde{c}_{ij}$ in view of (2.2.1) and (2.2.5), one can write the above relation in a matrix form as follows:

$$\tilde{Q} = Y + A \quad (2.2.6)$$

where

$$A = [a_{ij}]_n$$

Thus, from (2.2.4) and (2.2.6), we obtain the following result:

$$Y = \overline{Q} - A + P P^T \quad (2.2.8)$$

In order to establish expressions for the entries of matrix $P P^T$ in terms of g_k ($k = 1, 2, \dots, n+1$) and $\overline{q}_{ij}$, consider a matrix, $T = [t_{ij}]_{n+1}$, defined by:

$$t_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

Pre- and post-multiplying the uniformly tapered matrix Q^* in (2.2.2) by T , we obtain matrix TQ^*T , whose entries yield (cf. section 1.2.5) the following expressions for the conductance values g_k ($k = 1, 2, \dots, n+1$) in Figure 2.2.3:

From entry (1, h),

$$g_{n-h+2} = -\overline{q}_{1h} + p_1 (p-p_h) \quad (2.2.9)$$

From entry (c, h),

$$g_{n-h+c+1} = -\overline{q}_{ch} + \overline{q}_{c-1, h} + (p_c - p_{c-1}) (p-p_h) \quad (c = 2, 3, \dots, h-1) \quad (2.2.10)$$

From entry (h, h),

$$g_{n+1} = \overline{q}_{h-1, h} + (p-p_{h-1}) (p-p_h) \quad (2.2.11)$$

From entry (h+1, k),

$$g_{k-h} = \overline{q}_{h, k-1} - \overline{q}_{hk} + (p_k - p_{k-1}) (p-p_h) \quad (k = h+1, h+2, \dots, n) \quad (2.2.12)$$

From entry $(h+1, n+1)$,

$$g_{n-h+1} = \bar{q}_{hn} - p_n (p-p_h) \quad (2.2.13)$$

(Note: (2.2.10) is to be deleted if $h = 2$).

Let the relation in (2.2.13) be rewritten as:

$$p_n (p-p_h) = \bar{q}_{hn} - g_{n-h+1} \quad (2.2.14)$$

As inspection of Figure 2.2.2 will indicate that $(\bar{q}_{hn} - g_{n-h+1}) \neq 0$ except in a highly degenerate case, so that, in general, $p_n \neq 0$ and $(p-p_h) \neq 0$.

Assuming, then, $p_n \neq 0$,[†] (2.2.14) can be rewritten as:

$$p - p_h = \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} \quad (2.2.15)$$

We proceed to prove the following relation using mathematical induction:

$$p_k = (\bar{q}_{hk} - \sum_{i=k}^n g_{i-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.16)$$

$(k = h+1, h+2, \dots, n)$

For that, we shall require the trivial identity:

$$p_n = (\bar{q}_{hn} - g_{n-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.17)$$

Assuming (2.2.16), and substituting (2.2.15) - (2.2.16) into (2.2.12),

[†] Even in the highly degenerate case, where $\bar{q}_{hn} - g_{n-h+1} = 0$, the same final results which we propose to establish can, in fact, be arrived at by starting with another suitable equation out of (2.2.9) - (2.2.12), rather than (2.2.13), and by modifying the whole treatment appropriately.

we have, for $k = h+1, h+2, \dots, n$,

$$g_{k-h} = \bar{q}_{h, k-1} - \bar{q}_{hk} + \left([\bar{q}_{hk} - \sum_{i=k}^n g_{i-h+1}] \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} - p_{k-1} \right) \cdot \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} \quad (2.2.18)$$

$$= \bar{q}_{h, k-1} - \bar{q}_{hk} + \bar{q}_{hk} - \sum_{i=k}^n g_{i-h+1} - p_{k-1} \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n}$$

or

$$p_{k-1} = (\bar{q}_{h, k-1} - \sum_{i=k-1}^n g_{i-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.19)$$

($k = h+1, h+2, \dots, n$)

In view of (2.2.17) and (2.2.19), the hypothesis of (2.2.16) is, in fact, proved.

As a result of the relation just proved, (2.2.19) holds for $k = h+1, h+2, \dots, n$. In particular, for $k = h+1$, (2.2.19) yields:

$$p_h = (\bar{q}_{hh} - \sum_{i=h}^n g_{i-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.20)$$

We are justified, therefore, in extending the lowest possible value of k from $(h+1)$ to h in (2.2.16) so as to cover (2.2.20). Thus,

$$p_k = (\bar{q}_{hk} - \sum_{i=k}^n g_{i-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.21)$$

$$(k = h, h+1, \dots, n)$$

We prove, next, the following relation using mathematical induction once again:

$$p_k = (\bar{q}_{kh} + \sum_{i=1}^k g_{n-h+i+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.22)$$

$$(k = 1, 2, \dots, h-2)$$

Substituting (2.2.15) into (2.2.9), we obtain:

$$g_{n-h+2} = -\bar{q}_{lh} + p_1 \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n}$$

or

$$p_1 = (\bar{q}_{lh} + g_{n-h+2}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.23)$$

From (2.2.10), we have, for $c = k + 1$, ($c = 2, 3, \dots, h-1$),

$$g_{n-h+k+2} = -\bar{q}_{k+1,h} + \bar{q}_{kh} + (p_{k+1} - p_k)(p - p_h) \quad (2.2.24)$$

Further, assuming (2.2.22) and substituting (2.2.15) and (2.2.22) into (2.2.24), we obtain:

$$\begin{aligned} g_{n-h+k+2} &= -\bar{q}_{k+1,h} + \bar{q}_{kh} + (p_{k+1} - [\bar{q}_{kh} + \sum_{i=1}^k g_{n-h+i+1}]) \\ &\quad \cdot \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} + \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} \\ &= -\bar{q}_{k+1,h} + \bar{q}_{kh} + p_{k+1} \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} - \bar{q}_{kh} - \sum_{i=1}^k g_{n-h+i+1} \end{aligned}$$

or

$$p_{k+1} = (\bar{q}_{k+1,h} + \sum_{i=1}^{k+1} g_{n-h+i+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.25)$$

$$(k = 1, 2, \dots, h-2)$$

In view of (2.2.23) and (2.2.25), the hypothesis of (2.2.22) is, in fact, proved.

As a result of the relation just proved, (2.2.25) holds for $k = 1, 2, \dots, h-2$. In particular, for $k = h-2$, (2.2.25) yields;

$$p_{h-1} = (\bar{q}_{h-1, h} + \sum_{i=1}^{h-1} g_{n-h+i+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.26)$$

We are justified, therefore, in extending the highest possible value of k from $(h-2)$ to $(h-1)$ in (2.2.22) so as to cover (2.2.26).

Thus,

$$p_k = (\bar{q}_{kh} + \sum_{i=1}^k g_{n-h+i+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.27)$$

($k = 1, 2, \dots, h-1$)

Substituting, next, (2.2.20) into (2.2.15), we obtain:

$$p = (\bar{q}_{hh} - \sum_{i=h}^n g_{i-h+1}) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} + \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} \quad (2.2.28)$$

$$= \left(\bar{q}_{hh} - \sum_{i=h}^n g_{i-h+1} + \frac{(\bar{q}_{hn} - g_{n-h+1})^2}{p_n^2} \right) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \quad (2.2.29)$$

In view of (2.2.15), (2.2.26), and (2.2.29), (2.2.11) can be written as:

$$g_{n+1} = \bar{q}_{h-1, h} + \left(\bar{q}_{hh} - \sum_{i=h}^n g_{i-h+1} + \frac{(\bar{q}_{hn} - g_{n-h+1})^2}{p_n^2} - \bar{q}_{h-1, h} - \sum_{i=1}^{h-1} g_{n-h+i+1} \right) \frac{p_n}{\bar{q}_{hn} - g_{n-h+1}} \cdot \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n}$$

or

$$\frac{(\bar{q}_{hn} - g_{n-h+1})^2}{p_n^2} = \sum_{i=1}^{n+1} g_i - \bar{q}_{hh}$$

$$\therefore \frac{\bar{q}_{hn} - g_{n-h+1}}{p_n} = \pm \left(\sum_{i=1}^{n+1} g_i - \bar{q}_{hh} \right)^{1/2} \quad (2.2.30)$$

Note, in passing, that $(\bar{q}_{hn} - g_{n-h+1}) > 0$ and $p_n > 0$ implies:

$$\sum_{i=1}^{n+1} g_i - \bar{q}_{hh} > 0 \quad (2.2.31)$$

Substituting the positive root in (2.2.30) into (2.2.29), (2.2.27), and (2.2.21), we have, respectively:

$$p = \left(\sum_{i=1}^h g_{n-h+i+1} \right) \left(\sum_{i=1}^{n+1} g_i - \bar{q}_{hh} \right)^{-1/2} \quad (2.2.32)$$

$$p_k = \left(\bar{q}_{kh} + \sum_{i=1}^k g_{n-h+i+1} \right) \left(\sum_{i=1}^{n+1} g_i - \bar{q}_{hh} \right)^{-1/2} \quad (2.2.33)$$

$$(k = 1, 2, \dots, h-1)$$

$$p_k = \left(\bar{q}_{hk} - \sum_{i=k}^n g_{i-h+1} \right) \left(\sum_{i=1}^{n+1} g_i - \bar{q}_{hh} \right)^{-1/2} \quad (2.2.34)$$

$$(k = h, h+1, \dots, n)$$

The negative root in (2.2.30) is neglected since p and p_k ($k = 1, 2, \dots, n$) must be non-negative (cf. (2.2.3)).

By introducing the above expressions for p_k ($k = 1, 2, \dots, n$) into (2.2.8), the following significant relation between matrices Y and $\bar{Q}$ is finally established:

$$Y = \bar{Q} - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{hh} \right)^{-1} \bar{Q}'_{\cdot h} \bar{Q}'_h. \quad (2.2.35)^\dagger$$

where matrix A is as defined in (2.2.7), and

[†]The notations $M_{\cdot h}$ and M_h denote, respectively, the h -th column and h -th row of a matrix, M .

$$[\overline{Q}'_h.]^T = \overline{Q}'_{.h} = \begin{bmatrix} \overline{q}_{1h} + g_{n-h+2} \\ \overline{q}_{2h} + g_{n-h+2} + g_{n-h+3} \\ \vdots \\ \overline{q}_{h-1,h} + g_{n-h+2} + g_{n-h+3} + \dots + g_n \\ \hline \overline{q}_{hh} - g_1 - g_2 - \dots - g_{n-h+1} \\ \overline{q}_{h+1,h} - g_2 - g_3 - \dots - g_{n-h+1} \\ \vdots \\ \overline{q}_{nh} - g_{n-h+1} \end{bmatrix} \quad (2.2.36)$$

Note that the above relation (2.2.35) between matrices Y and $\overline{Q}$ can be reproduced directly by referring to Figure 2.2.2.

An Important Special Case

When the conductance values g_k ($k = 1, 2, \dots, n$) of the corresponding resistors are constrained to zero, the network in Figure 2.2.2 reduces to the one in Figure 2.2.5. For convenience of notation, we shall let $g_o \equiv g_{n+1}$. Now, an examination of the foregoing analytical considerations will reveal that the desired relation between matrices Y and $\overline{Q}$ for this special case can, in fact, be obtained from (2.2.35) by setting $g_k = 0$ ($k = 1, 2, \dots, n$) therein. That is, for this special case, we have:

$$Y = \overline{Q} + (1/\overline{x}) \overline{Q}_{.h} \overline{Q}_h. \quad (2.2.37)$$

where $\overline{x} = g_o - \overline{q}_{hh}$ is positive (cf. (2.2.31)), and $[\overline{Q}_h.]^T = \overline{Q}_{.h}$ denotes the h -th column of matrix $\overline{Q}$.

2.3. A Theorem in Algebra

In this section, we shall establish a theorem in algebra, which will find its application in the subsequent discussion.

Theorem: For a given real, symmetric matrix, $Y = [y_{ij}]_n$, and a set of real numbers, a_k ($k = 0, 1, 2, \dots, n$), there exists a unique real, symmetric matrix, $Q = [q_{ij}]_n$, which satisfies the relation:

$$Y = Q + (a_o - q_{hh})^{-1} Q'_{.h} Q'_{.h}. \quad (2.3.1)$$

where $q_{hh} \neq a_o$ and

$$[Q'_{.h}]^T = Q'_{.h} = [q_{1h} + a_1 \quad q_{2h} + a_2 \quad \dots \quad q_{nh} + a_n], \quad h \leq n$$

Proof: Let there exist another real, symmetric matrix, $\bar{Q} = [\bar{q}_{ij}]_n$, satisfying the relation:

$$Y = \bar{Q} + (a_o - \bar{q}_{hh})^{-1} \bar{Q}'_{.h} \bar{Q}'_{.h}. \quad (2.3.2)$$

where $\bar{q}_{hh} \neq a_o$ and

$$[\bar{Q}'_{.h}]^T = \bar{Q}'_{.h} = [\bar{q}_{1h} + a_1 \quad \bar{q}_{2h} + a_2 \quad \dots \quad \bar{q}_{nh} + a_n], \quad h \leq n$$

Equating, then, the entries in positions (h, h) of the right-hand sides of (2.3.1) and (2.3.2), we have:

$$\begin{aligned} q_{hh} + (a_o - q_{hh})^{-1} (q_{hh} + a_h)^2 &= \bar{q}_{hh} + (a_o - \bar{q}_{hh})^{-1} (\bar{q}_{hh} + a_h)^2 \\ \therefore \frac{a_o q_{hh} - q_{hh}^2 + q_{hh}^2 + 2 a_h q_{hh} + a_h^2}{a_o - q_{hh}} &= \frac{a_o \bar{q}_{hh} - \bar{q}_{hh}^2 + \bar{q}_{hh}^2 + 2 a_h \bar{q}_{hh} + a_h^2}{a_o - \bar{q}_{hh}} \end{aligned}$$

$$\therefore (a_o - \bar{q}_{hh}) ([a_o + 2 a_h] q_{hh} + a_h^2) = (a_o - q_{hh}) ([a_o + 2 a_h] \bar{q}_{hh} + a_h^2)$$

or

$$\begin{aligned}
& a_o(a_o + 2a_h)q_{hh} + a_o a_h^2 - (a_o + 2a_h)q_{hh}\bar{q}_{hh} - a_h^2 \bar{q}_{hh} \\
& = a_o(a_o + 2a_h)\bar{q}_{hh} + a_o a_h^2 - (a_o + 2a_h)q_{hh}\bar{q}_{hh} - a_h^2 q_{hh} \\
& \therefore a_o(a_o + 2a_h)(q_{hh} - \bar{q}_{hh}) + a_h^2(q_{hh} - \bar{q}_{hh}) = 0 \\
& \text{i.e., } [a_o(a_o + 2a_h) + a_h^2](q_{hh} - \bar{q}_{hh}) = 0 \quad (2.3.3)
\end{aligned}$$

Since, in general, $a_o(a_o + 2a_h) + a_h^2 \neq 0$, the equation (2.2.3) implies that:

$$q_{hh} = \bar{q}_{hh} \quad (2.3.4)$$

Equating, now, the entries in position (i, h) , $i \neq h$, of the right-hand sides of equations (2.3.1) and (2.3.2), we have:

$$q_{ih} + (a_o - q_{hh})^{-1}(q_{ih} + a_i)(q_{hh} + a_h) = \bar{q}_{ih} + (a_o - \bar{q}_{hh})^{-1}(\bar{q}_{ih} + a_i)(\bar{q}_{hh} + a_h)$$

Noting that $q_{hh} = \bar{q}_{hh}$ and multiplying both sides of the above equation by $(a_o - q_{hh})$, we further obtain:

$$\begin{aligned}
& a_o q_{ih} - q_{hh} q_{ih} + q_{hh} q_{ih} + a_h q_{ih} + a_i q_{hh} + a_i a_h \\
& = a_o \bar{q}_{ih} - q_{hh} \bar{q}_{ih} + q_{hh} \bar{q}_{ih} + a_h \bar{q}_{ih} + a_i q_{hh} + a_i a_h \\
& \therefore (a_o + a_h)(q_{ih} - \bar{q}_{ih}) = 0 \quad (2.3.5)
\end{aligned}$$

Since, in general, $a_o + a_h \neq 0$, the equation (2.3.5) implies that:

$$q_{ih} = \bar{q}_{ih}, \quad i \neq h \quad (2.3.6)$$

Thus, from (2.3.4) and (2.3.6), we can write the identity:

$$Q'_{.h} Q'_{h.} \equiv \bar{Q}'_{.h} \bar{Q}'_{h.} \quad (2.3.7)$$

The matrix identity

$$Q \equiv \overline{Q}$$

follows when the right-hand sides of (2.3.1) and (2.3.2) are compared, keeping in view the equalities established in (2.3.4) and (2.3.7).

Having proved thus the uniqueness of matrix Q which satisfies the relation stated in (2.3.1), we are now ready to establish the procedure for synthesizing resistive networks from certain s.c. conductance matrices.

2.4. Conductance-Parameter Procedure for Realization of n-th Order Matrices with Two-Tree Port-Structures

Let $Q^{(1)} = [q_{ij}^{(1)}]_n$ denote a paramount matrix to be realized as the s.c. conductance matrix with a specified two-tree port-structure. If both the trees are not linear, then, as explained in section 1.2.7, an appropriate congruent transformation can be applied to matrix $Q^{(1)}$. This appropriate congruent transformation can yield a matrix, $Q = C^T Q^{(1)} C$, which would correspond to the bilinear port-structure shown in Figure 2.4.1. Having obtained matrix Q , we can proceed to realize the same; for the realizations of matrices Q and $Q^{(1)}$ are identical except for their port-structures.



The port-numbering is in the natural order.

Figure 2.4.1

Consider the matrix equation

$$Y = Q - A + \left(\sum_{k=1}^{n+1} g_k - q_{hh} \right)^{-1} Q'_{.h} Q'_{h.} \quad (2.4.1)$$

where matrix A is as defined in (2.2.7), and

$$[Q'_{h.}]^T = Q'_{.h} = \begin{bmatrix} q_{1h} + g_{n-h+2} \\ q_{2h} + g_{n-h+2} + g_{n-h+3} \\ \vdots \\ q_{h-1,h} + g_{n-h+2} + g_{n-h+3} + \dots + g_n \\ \text{-----} \\ q_{hh} - g_1 - g_2 - \dots - g_{n-h+1} \\ q_{h+1,h} - g_2 - g_3 - \dots - g_{n-h+1} \\ \vdots \\ q_{nh} - g_{n-h+1} \end{bmatrix} \quad (2.4.2)$$

Suppose a set of non-negative parameters g_k ($k = 1, 2, \dots, n+1$) is found, which, when substituted in (2.4.1), gives a uniformly tapered form to matrix Y. This uniformly tapered matrix Y can be realized readily as shown in Figure 2.2.1. Suppose, from this realization of matrix Y, another network is derived, as shown in Figure 2.2.2, by adding $(n+1)$ resistors which have their respective conductance values equal to the parameters g_k ($k = 1, 2, \dots, n+1$). Then, if $\overline{Q}$ denotes the s.c. conductance matrix for the derived network, these matrices Y and $\overline{Q}$ must be related by (2.2.35).

Further, by virtue of the theorem proved in section 2.3, equations (2.2.35) and (2.4.1) together must imply the identity of matrices Q and $\overline{Q}$. This leads to the significant conclusion that the above derived network must, in fact, have been a realization of matrix Q itself.

The above considerations at once indicate the different steps of the "conductance-parameter" procedure being devised for synthesizing resistive networks from s.c. conductance matrices which are realizable with specified two-tree port-structures. The procedure is almost evident already; it is presented below explicitly for sake of completeness.

Let $Q^{(1)} = [q_{ij}^{(1)}]_n$ denote a paramount matrix to be realized as the s.c. conductance matrix with a specified two-tree port-structure. If both the trees are not linear, apply an appropriate congruent transformation to matrix $Q^{(1)}$ and obtain a matrix, $Q = C^T Q^{(1)} C$, which corresponds to the bilinear port-structure shown in Figure 2.4.1.

Find a set of non-negative parameters g_k ($k = 1, 2, \dots, n+1$) which, when substituted in (2.4.1), gives a uniformly tapered form to matrix Y defined therein.[†]

Realize the uniformly tapered matrix Y with $(n+1)$ nodes as shown in Figure 2.2.1.

Consider $(n+1)$ resistors which have their respective conductance values equal to the parameters g_k ($k = 1, 2, \dots, n+1$),

[†] A method for finding such a set of parameters by machine is indicated later on.

and add these resistors to the above realization of matrix Y in the manner shown in Figure 2.2.2. The new network thus obtained is, in fact, the desired realization of matrix $Q^{(1)}$, the accompanying port-structure to be considered being, of course, the original two-tree port-structure.

It can be noticed that the network shown in Figure 2.2.2 forms a full polygon of $(n+2)$ nodes. The network is thus of the most general form for that many nodes. Further, whenever a set (or sets) of non-negative parameters g_k ($k = 1, 2, \dots, n+1$) exists so as to provide a uniformly tapered form to matrix Y in (2.4.1), it can be computed by machine as indicated later. In view of these facts, if a certain matrix, $Q^{(1)}$, is, indeed, realizable with a specified two-tree port-structure, it can always be realized by the procedure stated above. In other words, given the two-tree port-structure, computation of a set of non-negative conductance-parameters g_k ($k = 1, 2, \dots, n+1$) so as to provide a uniformly tapered form to matrix Y in (2.4.1) is the necessary and sufficient condition for realizability of any given n th-order matrix, $Q^{(1)}$.

Before proceeding to the considerations of the conductance parameters, let us reiterate here one well-known result, viz., for an n th-order matrix, Q , which is realizable with a bilinear port-structure shown in Figure 2.4.1, the submatrices† $Q^{(1, 2, \dots, h-1)}$ and $Q^{(h, h+1, \dots, n)}$ must both be uniformly tapered. Now, given

† In this thesis, the principal submatrix formed by rows and columns $i, j, \dots, k$ ($i, j, k \leq n$) of an n th-order symmetric matrix, Q , will be denoted by the symbol $Q^{(i, j, \dots, k)}$.

a matrix, $Q^{(1)}$, this property of the related matrix Q can be checked easily; as such, we note that it is desirable to do so first of all by way of checking a fundamental necessary condition whenever a matrix, $Q^{(1)}$, is posed for realization with a specified two-tree port-structure.

2.5. The Conductance Parameters

In this section we shall establish upper and lower bounds on the conductance values which have been regarded as parameters in the previous section. These bounds are valuable in the machine computations of the parametric conductance values themselves.

Let a matrix, $T = [t_{ij}]_n$, be defined by:

$$t_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

By definition, matrix Y in (2.4.1) would be in uniformly tapered form if, and only if, the $n(n+1)/2$ entries in the upper triangular portion of matrix TYT are non-negative. Let these entries be denoted by ${}^{(t)}y_{ij}^{(t)}$. From (2.4.1),

$$TYT = TQT - TAT + \left(\sum_{k=1}^{n+1} g_k - q_{hh} \right)^{-1} TQ'_{.h} Q'_{h.} T \quad (2.5.1)$$

Let ${}^{(t)}q_{ij}^{(t)}$ and ${}^{(t)}a_{ij}^{(t)}$ denote, respectively, the entries in positions (i, j) of the upper triangular portions of matrices TQT and TAT , and let ${}^{(t)}q'_{ih}$ and $q'_{hj}^{(t)}$ denote, respectively, the entries in positions $(i, 1)$ and $(1, j)$ of column and row matrices $TQ'_{.h}$ and $Q'_{h.} T$. Then, through matrix-multiplications,

$${}^{(t)}q_{ij}^{(t)} = q_{ij} - q_{i,j+1} - q_{i-1,j} + q_{i-1,j+1} \quad (2.5.2)$$

$$(j > i; i, j = 1, 2, \dots, n)$$

where

$$q_{i, n+1} = q_{0, j} = 0 \quad (i, j = 0, 1, 2, \dots, n+1)$$

Also,

$${}^{(t)}a_{ij}^{(t)} = g_{n-h+i+1} \quad (i = 1, 2, \dots, h-1; j = h-1)$$

$${}^{(t)}a_{ij}^{(t)} = \varepsilon_{j-h+1} \quad (i = h; j = h, h+1, \dots, n)$$

$$a_{ij}^{(t)} = 0 \quad \text{otherwise} \quad (\text{for } j \geq i; \quad i, j = 1, 2, \dots, n)$$

or, in matrix form,

Diagram illustrating the structure of a tensor TAT (labeled $(2.5.3)^\dagger$).

The diagram shows a large square frame containing a dashed line representing a tensor structure. The label (h) is on the left, and $(h-1)$ is at the top center.

The tensor structure is defined by the following components:

- A vertical dashed line segment labeled with g_{n-h+2} , g_{n-h+3} , $\vdots$, g_{n-1} , and g_n .
- A horizontal dashed line segment labeled with g_1 , g_2 , $\dots$, and g_{n-h+1} .

There are two 'O' symbols: one in the upper right and one in the lower right.

[†]The symbol $\times$ is used to imply that the entries in the lower triangular portion of the matrix are of no interest in the discussion and hence have been omitted.

Further,

$${}^{(t)}q'_{ih} = q'_{ih} - q'_{i-1, h} \quad (i = 1, 2, \dots, n)$$

and

$$q'^{(t)}_{hj} = q'_{hj} - q'_{h, j+1} \quad (j = 1, 2, \dots, n)$$

where, by definition, $q_{0, h} = q_{h, n+1} = 0$.

With reference to (2.4.2),

$$\left. \begin{aligned} {}^{(t)}q'_{ih} &= q_{ih} - q_{i-1, h} + g_{n-h+i+1} & (i = 1, 2, \dots, h-1) \\ & & (q_{0, h} = 0 \text{ by definition}) \\ {}^{(t)}q'_{hh} &= q_{hh} - q_{h-1, h} - \sum_{k=1}^n g_k \\ {}^{(t)}q'_{ih} &= q_{ih} - q_{i-1, h} + g_{i-h} & (i = h+1, h+2, \dots, n) \end{aligned} \right\} \quad (2.5.4)$$

and

$$\left. \begin{aligned} q'^{(t)}_{hj} &= q_{hj} - q_{h, j+1} - g_{n-h+j+2} & (j = 1, 2, \dots, h-2) \\ q'^{(t)}_{h, h-1} &= q_{h, h-1} - q_{hh} + \sum_{k=1}^n g_k \\ q'^{(t)}_{hj} &= q_{hj} - q_{h, j+1} - g_{j-h+1} & (j = h, h+1, \dots, n) \\ & & (q_{h, n+1} = 0 \text{ by definition}) \end{aligned} \right\} \quad (2.5.5)$$

Comparing (2.5.4) and (2.5.5), we may observe that:

$$q'^{(t)}_{hj} = -{}^{(t)}q'_{j+1, h} \quad (j = 1, 2, \dots, n-1) \quad (2.5.6)$$

Obviously, entries in the upper triangular portion of matrix $TQ'_{\cdot, h} Q'_{h, \cdot} T$ can be obtained from (2.5.4) and (2.5.5) by considering the products ${}^{(t)}q'_{ih} q'^{(t)}_{hj}$ for $j \geq i$; $i, j = 1, 2, \dots, n$.

Now, from (2.5.1), matrix Y would be uniformly tapered if, and only if,

$$(t)_{q_{ij}}(t) - (t)_{a_{ij}}(t) + \left(\sum_{k=1}^{n+1} g_k - q_{hh} \right)^{-1} (t)_{q'_{ih}} q'_{hj}(t) \geq 0 \quad (2.5.7)$$

$$(j \geq i; i, j = 1, 2, \dots, n)$$

For sake of illustration, let us consider the particular case $i = j = 1$ and $h > 2$. Then,

$$(t)_{q_{11}}(t) - (t)_{a_{11}}(t) + \left(\sum_{k=1}^{n+1} g_k - q_{hh} \right)^{-1} (t)_{q'_{1h}} q'_{h1}(t) \geq 0 \quad (2.5.8)$$

where, through (2.5.2) - (2.5.5),

$$(t)_{q_{11}}(t) = q_{11} - q_{12}$$

$$(t)_{a_{11}}(t) = 0$$

(2.5.9)

$$(t)_{q'_{1h}} = q_{1h} + g_{n-h+2}$$

$$q'_{h1}(t) = q_{h1} - q_{h2} - g_{n-h+3}$$

Substituting (2.5.9) into (2.5.8), we have:

$$q_{11} - q_{12} + \left(\sum_{k=1}^{n+1} g_k - q_{hh} \right)^{-1} (q_{1h} + g_{n-h+2})(q_{h1} - q_{h2} - g_{n-h+3}) \geq 0 \quad (2.5.10)$$

This inequality is given by consideration of entry $(t)_{y_{11}}(t)$. We

observe that each of the $n(n+1)/2$ entries $(t)_{y_{ij}}(t)$ in the upper triangular portion of matrix TYT offers a similar inequality.

Thus, from what has been stated in section 2.4, if a matrix, Q , is realizable with a specified bilinear port-structure, the problem of realization reduces, now, to solving these $n(n+1)/2$ nonlinear simultaneous inequalities involving the $(n+1)$ non-negative parameters

g_k ($k=1, 2, \dots, n+1$). A computer method is indicated in section 2.6 for solving these inequalities. Note, incidentally, that more than one set of non-negative parameters g_k will exist, in general, which will satisfy the indicated simultaneous inequalities.

We proceed, next, to establish upper and lower bounds on the parameters g_k .

Recall relation (2.2.3), viz.,

$$0 \leq p_1 \leq p_2 \leq \dots \leq p_{h-1} \leq p \leq p_h \leq p_{h+1} \leq \dots \leq p_n \leq 0.$$

We shall have occasion to refer to this relation a few times in what follows.

From (2.2.9),

$$g_{n-h+2} = -\bar{q}_{1h} + p_1 (p - p_h)$$

Since $p_1 \geq 0$ and $(p - p_h) \geq 0$, therefore,

$$g_{n-h+2} \geq -\bar{q}_{1h} \quad (2.5.11)$$

From (2.2.10),

$$g_{n-h+c+1} = -\bar{q}_{ch} + \bar{q}_{c-1,h} + (p_c - p_{c-1}) (p - p_h) \\ (c = 2, 3, \dots, h-1)$$

Since $(p_c - p_{c-1}) \geq 0$ and $(p - p_h) \geq 0$, therefore,

$$g_{n-h+c+1} \geq -\bar{q}_{ch} + \bar{q}_{c-1,h} \quad (2.5.12) \\ (c = 2, 3, \dots, h-1)$$

From (2.2.11),

$$g_{n+1} = \bar{q}_{h-1,h} + (p - p_{h-1}) (p - p_h)$$

Since $(p - p_{h-1}) \geq 0$ and $(p - p_h) \geq 0$, therefore,

$$g_{n+1} \geq \bar{q}_{h-1, h} \quad (2.5.13)$$

From (2.2.12),

$$g_{k-h} = \bar{q}_{h, k-1} - \bar{q}_{hk} + (p_k - p_{k-1})(p - p_h) \\ (k = h+1, h+2, \dots, n)$$

Since $(p_{k-1} - p_k) \geq 0$ and $(p - p_h) \geq 0$, therefore,

$$g_{k-h} \leq \bar{q}_{h, k-1} - \bar{q}_{hk} \quad (k = h+1, h+2, \dots, n) \quad (2.5.14)$$

From (2.2.13),

$$g_{n-h+1} = \bar{q}_{hn} - p_n (p - p_h)$$

Since $p_n \geq 0$ and $(p - p_h) \geq 0$, therefore,

$$g_{n-h+1} \leq \bar{q}_{hn} \quad (2.5.15)$$

Also, from (2.2.31),

$$\sum_{k=1}^{n+1} g_k > \bar{q}_{hh} \quad (2.5.16)$$

Recall having seen in section 2.4 that in the context of realization of a matrix, Q , by the conductance-parameter procedure, matrices Q and $\bar{Q}$ are, in fact, identical. Hence we may rewrite the above bounds in terms of entries q_{ij} of matrix Q , rather than in terms of entries $\bar{q}_{ij}$ of matrix $\bar{Q}$. Thus, from (2.5.11) - (2.5.16), we have, respectively:

$$g_{n-h+2} \geq -q_{1h} \quad (2.5.17)$$

$$g_{n-h+c+1} \geq -q_{ch} + q_{c-1,h} \quad (c = 2, 3, \dots, h-1) \quad (2.5.18)$$

$$g_{n+1} \geq q_{h-1,h} \quad (2.5.19)$$

$$g_{k-h} \leq q_{h,k-1} - q_{hk} \quad (k = h+1, h+2, \dots, n) \quad (2.5.20)$$

$$g_{n-h+1} \leq q_{hn} \quad (2.5.21)$$

$$\sum_{k=1}^{n+1} g_k > q_{hh} \quad (2.5.22)$$

Note: (2.5.18) is to be deleted if $h = 2$.

We shall proceed to establish further significant bounds on the conductance parameters g_k (cf. (2.5.27) - (2.5.30) below). For that, we note that consideration of (2.5.4) along with the above inequalities (2.5.17) - (2.5.22) leads to the following information concerning column matrix $TQ'_{\cdot h}$:

$$\begin{aligned} [\text{From (3.5.17) - (3.5.18)}] \quad & (t) \, q'_{ih} \geq 0 \quad (i = 1, 2, \dots, h-1) \\ [\text{From (3.5.20)}] \quad & (t) \, q'_{ih} \leq 0 \quad (i = h+1, h+2, \dots, n) \\ & (t) \, q'_{hh} \geq 0 \end{aligned} \quad (2.5.23)$$

Again, considering (2.5.5) along with (2.5.17) - (2.5.22), we get the following information concerning row matrix $Q'_{\cdot h}T$:

$$\begin{aligned} [\text{From (3.5.18)}] \quad & q'^{(t)}_{hj} \leq 0 \quad (j = 1, 2, \dots, h-2) \\ [\text{From (3.5.20) and (3.5.21)}] \quad & q'^{(t)}_{hj} \geq 0 \quad (j = h, h+1, \dots, n) \\ & q'^{(t)}_{h, h-1} \leq 0 \end{aligned} \quad (2.5.24)$$

Recall, in passing, the result in (2.5.6), viz.,

$$q_{h,j}^{(t)} = -^{(t)}q_{j+1,h} \quad (j = 1, 2, \dots, n-1)$$

Now, by virtue of (2.5.23) - (2.5.24), the sign-pattern matrix for $TQ_{\cdot h}^i Q_{\cdot h}^i T$ is established as;

$$\begin{array}{c}
\left[\begin{array}{c}
\oplus \\
\oplus \\
\vdots \\
\vdots \\
\oplus \\
\hline
\oplus \\
\hline
\ominus \\
\ominus \\
\vdots \\
\vdots \\
\ominus
\end{array} \right]_{(h)} \left[\begin{array}{ccccccc}
\ominus & \ominus & \dots & - & \overset{(h-1)}{\oplus} & \oplus & \oplus & \dots & \dots & \oplus
\end{array} \right] \quad (2.5.25)^{\dagger}
\end{array}$$

i. e.,

[†] The symbols $\oplus$ and $\ominus$ denote, respectively, a non-negative and a non-positive entry. The symbol $\pm$ implies that the pertinent entry may be positive, negative, or zero.

In the above sign-pattern matrix of $TQ'_{\cdot h} Q'_h T$, the entries in positions $(i, h-1)$ ($i = 1, 2, \dots, h-1$) are non-positive and the entries in positions (h, j) ($j = h, h+1, \dots, n$) are non-negative when $q_{hh} - q_{h-1, h} - \sum_{k=1}^n g_k \geq 0$. On the other hand, the entries in positions $(i, h-1)$ ($i = 1, 2, \dots, h-1$) are non-negative and the entries in positions (h, j) ($j = h, h+1, \dots, n$) are non-positive when $q_{hh} - q_{h-1, h} - \sum_{k=1}^n g_k \leq 0$. Note also that the sign-pattern matrix for $(\sum_{k=1}^{n+1} g_k - q_{hh})^{-1} TQ'_{\cdot h} Q'_h T$ is the same as that for $TQ'_{\cdot h} Q'_h T$ since, according to (2.5.22), $\sum_{k=1}^{n+1} g_k > q_{hh}$.

Consider the entries in positions $(i, h-1)$ ($i = 1, 2, \dots, h-1$) and (h, j) ($j = h, h+1, \dots, n$) of matrix TYT given by (2.5.1). We recall that the non-negative conductance parameters g_k ($k = 1, 2, \dots, n+1$) are to be chosen such that the indicated entries - along with the rest of the entries in the upper triangular portion of matrix TYT - are non-negative. Then, with reference to (2.5.2), (2.5.3), (2.5.7), and (2.5.26), we can infer the following:

(1) Whenever, in the given matrix, Q ,

$${}^{(t)}q_{i, h-1} = q_{i, h-1} - q_{i, h} - q_{i-1, h-1} + q_{i-1, h}$$

is non-positive for even one i amongst $i = 1, 2, \dots, h-1$, conductance parameters g_k ($k = 1, 2, \dots, n$) must be subject to the lower bound:

$$q_{hh} - q_{h-1, h} - \sum_{k=1}^n g_k \leq 0$$

i. e.,

$$\sum_{k=1}^n g_k \geq q_{hh} - q_{h-1, h} ; \quad (2.5.27)$$

further, conductance parameters g_k ($k = 1, 2, \dots, n-h+1$) must be subject to the upper bounds:

$$q_{h,j} - q_{h,j+1} - q_{h-1,j} + q_{h-1,j+1} \geq g_{j-h+1} \quad (2.5.28)$$

$$(j = h, h+1, \dots, n)$$

Bounds in (2.5.28) are, of course, additional to those stated in (2.5.20) - (2.5.21) for the same parameters. These bounds are easily derived by noting that (2.5.27) implies, as stated earlier, that the entries in positions (h, j) ($j = h, h+1, \dots, n$) of matrix $(\sum_{k=1}^{n+1} g_k - q_{hh})^{-1} T Q'_h Q'_h T$ are non-positive. Bounds in (2.5.28) follow when (2.5.7) is considered along with (2.5.2) and (2.5.3) for $i = h$; $j = h, h+1, \dots, n$.

(2) Whenever, in the given matrix, Q ,

$$q_{hj}^{(t)} = q_{h,j} - q_{h,j+1} - q_{h-1,j} + q_{h-1,j+1}$$

is non-positive for even one j amongst $j = h, h+1, \dots, n$, conductance parameters g_k ($k = 1, 2, \dots, n$) must be subject to the upper bound:

$$q_{hh} - q_{h-1,h} - \sum_{k=1}^n g_k \geq 0$$

i.e.,

$$\sum_{k=1}^n g_k \leq q_{hh} - q_{h-1,h}; \quad (2.5.29)$$

further, parameters g_k ($k = n-h+2, n-h+3, \dots, n$) must be subject to the upper bounds:

$$q_{i,h-1} - q_{i,h} - q_{i-1,h-1} + q_{i-1,h} \geq g_{n-h+i+1} \quad (2.5.30)$$

$$(i = 1, 2, \dots, h-1)$$

The inequality in (2.5.29) obviously suggests the bounds:

$$q_{hh} - q_{h-1, h} \geq g_k \quad (k = 1, 2, \dots, n) \quad (2.5.31)$$

Thus, for g_k ($k = n-h+2, n-h+3, \dots, n-1$), we have the bounds in (2.5.30) as additional to those in (2.5.31). For the derivation of (2.5.30), we only need to note one of the implications of (2.5.29) stated earlier, viz., the entries in positions $(i, h-1)$ ($i = 1, 2, \dots, h-1$) of matrix $(\sum_{i=1}^{n+1} g_i - q_{hh})^{-1} TQ'_{.h} Q'_h T$ are non-positive. Bounds in (2.5.30) follow when (2.5.7) is considered along with (2.5.2) and (2.5.3) for $i = 1, 2, \dots, h-1$; $j = h-1$.

In addition to the bounds established thus far, some further conditional upper bounds stated in (2.5.37) below can be established for conductance parameters g_k ($k = n-h+2, n-h+3, \dots, n$). But let us first enunciate one significant necessary condition, which follows directly from the above discussion, for the realizability of a given matrix, Q , accompanied by a specified bilinear port-structure:

If, for a given matrix, Q ,

$$q_{i, h-1} - q_{i, h} - q_{i-1, h-1} + q_{i-1, h} < 0$$

for even one i amongst $i = 1, 2, \dots, h-1$, then the matrix Q is realizable with the bilinear port-structure only if

$$q_{h, j} - q_{h, j+1} - q_{h-1, j} + q_{h-1, j+1} \geq 0$$

for each j amongst $j = h, h+1, \dots, n$. On the other hand, if

$$q_{h, j} - q_{h, j+1} - q_{h-1, j} + q_{h-1, j+1} < 0$$

for even one j amongst $j = h, h+1, \dots, n$, then the matrix Q is realizable with the bilinear port-structure only if

$$q_{i, h-1} - q_{i, h} - q_{i-1, h-1} + q_{i-1, h} \geq 0$$

for each i amongst $i = 1, 2, \dots, h-1$.

To establish the bounds stated in (2.5.37) below, let us consider entries ${}^{(t)}y_{i, h-1}^{(t)}$ ($i = 1, 2, \dots, h-1$) as given by (2.5.7). Recall that it is our aim to select conductance-parameters g_k ($k = 1, 2, \dots, n+1$) such that these entries in the upper triangular portion of matrix TYT are non-negative. Thus, we must have:

$${}^{(t)}q_{i, h-1}^{(t)} - {}^{(t)}a_{i, h-1}^{(t)} + \left(\sum_{k=1}^{n+1} g_k - q_{hh}\right)^{-1} {}^{(t)}q'_{ih} q'_{h, h-1}^{(t)} \geq 0$$

Therefore, from (2.5.2) - (2.5.5),

$$\begin{aligned} q_{i, h-1} - q_{ih} - q_{i-1, h-1} + q_{i-1, h} - g_{n-h+i+1} \\ + \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(q_{h, h-1} - q_{hh} + \sum_{k=1}^n g_k)}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0 \end{aligned}$$

where, we recall, $q_{oh} = q_{o, h-1} = 0$ by definition. Or,

$$\begin{aligned} q_{i, h-1} - q_{i-1, h-1} + \frac{-(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(\sum_{k=1}^{n+1} g_k - q_{hh})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \\ + \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(q_{h, h-1} - q_{hh} + \sum_{k=1}^n g_k)}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0 \end{aligned}$$

or

$$q_{i, h-1} - q_{i-1, h-1} + \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(q_{h, h-1} - g_{n+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0$$

Hence,

$$q_{i,h-1} - q_{i-1,h-1} \geq \frac{(q_{i,h} - q_{i-1,h} + g_{n-h+i+1})(g_{n+1} - q_{h,h-1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \quad (2.5.32)$$

(i = 1, 2, \dots, h-1)

Consider entry ${}^{(t)}y_{hj}^{(t)}$ (j = h, h+1, \dots, n) as given by (2.5.7).

Since it is desired to have each entry ${}^{(t)}y_{hj}^{(t)}$ (j = h, h+1, \dots, n) non-negative, we can write through (2.5.2) - (2.5.5):

$$q_{hj} - q_{h,j+1} - q_{h-1,j} + q_{h-1,j+1} - g_{j-h+1} + \frac{(q_{hh} - q_{h-1,h} - \sum_{k=1}^n g_k)(q_{hj} - q_{h,j+1} - g_{j-h+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0$$

where, we recall, $q_{h-1,n+1} = q_{h,n+1} = 0$ by definition. Therefore,

$$- q_{h-1,j} + q_{h-1,j+1} + \frac{(\sum_{k=1}^{n+1} g_k - q_{hh})(q_{hj} - q_{h,j+1} - g_{j-h+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} + \frac{(q_{hh} - q_{h-1,h} - \sum_{k=1}^n g_k)(q_{hj} - q_{h,j+1} - g_{j-h+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0$$

or

$$- q_{h-1,j} + q_{h-1,j+1} + \frac{(g_{n+1} - q_{h-1,h})(q_{hj} - q_{h,j+1} - g_{j-h+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq 0$$

Hence,

$$\frac{(g_{n+1} - q_{h-1, h})(q_{hj} - q_{h, j+1} - g_{j-h+1})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \geq q_{h-1, j} - q_{h-1, j+1} \quad (j = h, h+1, \dots, n) \quad (2.5.33)$$

Suppose, in the given matrix, Q , $q_{h-1, j} - q_{h-1, j+1} > 0$ for at least one j amongst $j = h, h+1, \dots, n$. Let $\hat{j}$ denote such a j ; i.e., $q_{h-1, \hat{j}} - q_{h-1, \hat{j}+1} > 0$. Then, with reference to (2.5.19) - (2.5.22), we see that the term $(q_{h\hat{j}} - q_{h, \hat{j}+1} - g_{\hat{j}-h+1})$ - as well as each of the other two terms on the left-hand side of (2.5.33) - is strictly positive. Further, from (2.5.17) - (2.5.18), the term $(q_{ih} - q_{i-1, h} + g_{n-h+i+1}) \geq 0$ for $i = 1, 2, \dots, h-1$. Multiplying, then, both sides of (2.5.33) by the non-negative quotient $(q_{ih} - q_{i-1, h} + g_{n-h+i+1}) / (q_{h\hat{j}} - q_{h, \hat{j}+1} - g_{\hat{j}-h+1})$, we have:

$$\begin{aligned} & \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(g_{n+1} - q_{h-1, h})}{\sum_{k=1}^{n+1} g_k - q_{hh}} \\ & \geq \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(q_{h-1, \hat{j}} - q_{h-1, \hat{j}+1})}{(q_{h\hat{j}} - q_{h, \hat{j}+1} - g_{\hat{j}-h+1})}, \quad (2.5.34) \\ & \quad (i = 1, 2, \dots, h-1) \end{aligned}$$

We note that the right-hand side of (2.5.32) is the same as the left-hand side of (2.5.34), so that

$$q_{i, h-1} - q_{i-1, h-1} \geq \frac{(q_{ih} - q_{i-1, h} + g_{n-h+i+1})(q_{h-1, \hat{j}} - q_{h-1, \hat{j}+1})}{(q_{h\hat{j}} - q_{h, \hat{j}+1} - g_{\hat{j}-h+1})}$$

Therefore,

$$\frac{(q_{i,h-1} - q_{i-1,h-1})(q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1} - g_{\hat{j}-h+1}^{\wedge})}{(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1})} \geq (q_{ih} - q_{i-1,h} + g_{n-h+i+1}) \quad (2.5.35)$$

Recall having seen toward the end of section 2.4 that in matrix Q , submatrices $Q^{(1,2,\dots,h-1)}$ and $Q^{(h,h+1,\dots,n)}$ must be uniformly tapered. This implies, among other things, that $q_{i,h-1} - q_{i-1,h-1} \geq 0$ ($i = 1, 2, \dots, h-1$; $q_{0,h-1} \equiv 0$) and $q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1} \geq 0$, $h \leq \hat{j} \leq n$, ($q_{h,n+1} \equiv 0$). Next, by our choice, conductance parameters $g_{\hat{j}-h+1}^{\wedge} \geq 0$; further, through (2.5.20) - (2.5.21), $q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1} - g_{\hat{j}-h+1}^{\wedge} \geq 0$; and finally, as hypothesized earlier, $q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1} > 0$. Hence, we can write:

$$\begin{aligned} & \frac{(q_{i,h-1} - q_{i-1,h-1})(q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1})}{(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1})} \\ & \geq \frac{(q_{i,h-1} - q_{i-1,h-1})(q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1} - g_{\hat{j}-h+1}^{\wedge})}{(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1})} \end{aligned} \quad (2.5.36)$$

The left-hand side of (2.5.35) being equal to the right-hand side of (2.5.36), we have:

$$\frac{(q_{i,h-1} - q_{i-1,h-1})(q_{h\hat{j}}^{\wedge} - q_{h,\hat{j}+1})}{(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1})} \geq (q_{ih} - q_{i-1,h} + g_{n-h+i+1})$$

or,

$$\frac{(q_{i,h-1} - q_{i-1,h-1})(q_{h,\hat{j}} - q_{h,\hat{j}+1})}{(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1})} - q_{ih} + q_{i-1,h} \geq g_{n-h+i+1},$$

$$i = 1, 2, \dots, h-1;$$

$$h \leq \hat{j} \leq n, \text{ such that}$$

$$(q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1}) > 0 \quad (2.5.37)$$

Note: If there does not exist even one value of index $\hat{j}$ between h and n such that $q_{h-1,\hat{j}} - q_{h-1,\hat{j}+1} > 0$, then, of course, the bounds in (2.5.37) will not hold.

2.6. The Machine Computations of the Conductance Parameters

It is seen in the previous section that in the realization of n th-order matrices with two-tree port-structures, a set of $n(n+1)/2$ nonlinear simultaneous inequalities involving the $(n+1)$ conductance parameters g_k ($k = 1, 2, \dots, n+1$) must be solved. In the absence of an analytical method to solve such a set of inequalities in general, a numerical method must be used to obtain one or more solutions. The idea which forms the basis of the numerical method can be explained as follows:

The $n(n+1)/2$ simultaneous inequalities are of the form:

$${}^{(t)}y_{ij}^{(t)} \geq 0$$

where ${}^{(t)}y_{ij}^{(t)}$ is an algebraic expression involving $(n+1)$ parameters g_k ($k = 1, 2, \dots, n+1$) (cf. (2.5.10)). We select a random set of the $(n+1)$ parameters within the bounds established earlier and evaluate each expression ${}^{(t)}y_{ij}^{(t)}$. If each ${}^{(t)}y_{ij}^{(t)} \geq 0$ for the selected set of parameters, the set is, indeed, a solution

satisfying the simultaneous inequalities. In general, however, there will be some ${}^{(t)}y_{ij}^{(t)}$ such that ${}^{(t)}y_{ij}^{(t)} < 0$ for the randomly selected set of parameters. Let these particular expressions be denoted by ${}^{(t)}\tilde{y}_{ij}^{(t)}$. Then a systematic search for a suitable set of parameters g_k is started with the aim of minimizing the expression $\sum |{}^{(t)}\tilde{y}_{ij}^{(t)}|$. A suitable set of the parameters will have been located when $\sum |{}^{(t)}\tilde{y}_{ij}^{(t)}|$ attains, in fact, zero value. The logic diagram of a typical program capable of the aforesaid systematic search within the bounds is given in Appendix 3.

2.7. Realization of a Special Class of nth-Order Matrices

So far, we considered different aspects of the conductance-parameter procedure capable of realizing s.c. conductance matrices with two-tree port-structures. We shall discuss, now, a special class of nth-order matrices which can be realized through a special version of the conductance-parameter procedure.

One important feature of the special version[†] is that it avoids the necessity of computations by machine. It is also significant that the procedure readily offers infinitely many equivalent realizations of matrices belonging to the special class, two of these realizations being assured to be minimal.

[†] For convenience, we shall often refer to the special version of the conductance-parameter procedure by the term "the special procedure."

To enunciate the special procedure, let $Q^{(1)} = [q_{ij}^{(1)}]_n$ denote a paramount matrix to be realized as the s.c. conductance matrix with a specified two-line port-structure. If the port-structure is not bilinear, apply an appropriate congruent transformation to matrix $Q^{(1)}$ and obtain a matrix, $Q = C^T Q^{(1)} C$, which corresponds to the bilinear port-structure shown in Figure 2.7.1.



The port-numbering is in the natural order.

Figure 2.7.1

Letting $Q_{\cdot h}$ ($Q_{h \cdot}$) represent the h -th column (row) of matrix Q , find a positive parameter, x , such that a matrix, Y , defined by:

$$Y = Q + (1/x) Q_{\cdot h} Q_{h \cdot} \quad (2.7.1)$$

assumes a uniformly tapered form.

Realize the uniformly tapered matrix Y with $(n+1)$ nodes as shown in Figure 2.2.1.

To the above realization of matrix Y , add, in the manner shown in Figure 2.2.5, a resistor which has the conductance value, $g_o = q_{hh} + x$. Then the resulting network is a realization of matrix $Q^{(1)}$ if the original two-line port-structure is considered instead of the bilinear one shown in Figure 2.2.5.

Evidently, a matrix $Q^{(1)}$ will belong to the special class of matrices realizable with the above procedure if, and only if, a positive value can be assigned to the parameter x in (2.7.1) such that the matrix Y defined therein assumes a uniformly tapered form.

We observe that the port h and the particular resistor having its conductance value g_0 are in series in Figure 2.2.5. As such, their positions can always be interchanged without disturbing the network electrically. Thus, in the realization of a matrix, $Q^{(1)}$, by the above procedure, the port-structure corresponding to matrix Q can freely be considered to be as either in Figure 2.7.1 or Figure 2.7.2.

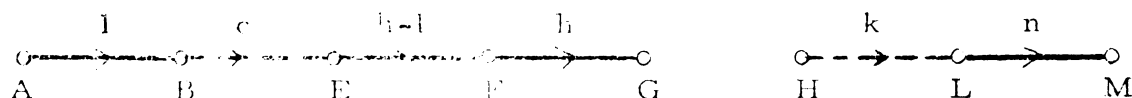


Figure 2.7.2

One implication of this fact is that both the submatrices $Q^{(1,2,\dots,h)}$ and $Q^{(h,h+1,\dots,c)}$ of matrix Q must be uniformly tapered. Indeed, given a matrix, $Q^{(1)}$, this property of the related matrix Q can be checked easily; as an h , we note that it is desirable to do so first by way of checking a fundamental necessary condition for realizability of the matrix with the above special procedure.

The condition just mentioned implies, among other things, the following relation:

$$0 \leq q_{1h} \leq \dots \leq q_{0n} \leq \dots \leq q_{1h} \leq q_{h+1,h} \leq \dots \leq q_{kh} \leq \dots \leq q_{nh} \leq 0$$

(2.7.2)

This relation will be exploited later on.

Let a matrix, $T = [t_{ij}]_n$, be defined by

$$t_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i = j + 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.7.3)$$

Pre- and post-multiplying the matrix equation (2.7.1) by T , we have:

$$TYT = TQT + (1/x) TQ_{.h}Q_{.h}^T \quad (2.7.4)$$

The entries, ${}^{(t)}q_{ij}^{(t)}$, in the upper triangular portion of matrix TQT are given by:

$${}^{(t)}q_{ij}^{(t)} = q_{ij} - q_{i,j+1} - q_{i-1,j} + q_{i-1,j+1} \quad (2.7.5)$$

($j \geq i$; $i, j = 1, 2, \dots, n$)

where, by definition,

$$q_{i,n+1} = q_{0,j} = 0 \quad (i, j = 0, 1, 2, \dots, n+1)$$

Further, the entries, $(1/x) {}^{(t)}q_{ih}q_{hj}^{(t)}$, in the upper triangular portion of the matrix $(1/x) TQ_{.h}Q_{.h}^T$ can be obtained from:

$$\frac{1}{x} T Q_{\cdot h} Q_{h \cdot}^T = \frac{1}{x} \begin{bmatrix} q_{1h} \\ q_{2h} - q_{1h} (\equiv u) \\ q_{3h} - q_{2h} (\equiv v) \\ \vdots \\ q_{kh} - q_{k-1,h} (\equiv w) \\ \vdots \\ q_{nh} - q_{n-1,h} (\equiv z) \end{bmatrix} \begin{bmatrix} -u & -v & \dots & -w & \dots & -z & q_{nh} \end{bmatrix} \quad (2.7.6)$$

As discussed earlier, a matrix, $Q^{(1)}$, belongs to the special class being considered if, and only if, a positive value can be assigned to parameter x such that matrix Y in (2.7.1) is uniformly tapered, i.e., each entry in the upper triangular portion of matrix TYT , viz.,

$${}^{(t)}y_{ij}^{(t)} = {}^{(t)}q_{ij}^{(t)} + (1/x) {}^{(t)}q_{ih} q_{hj}^{(t)} \quad (2.7.7)$$

$$(j \geq i, \quad i, j = 1, 2, \dots, n)$$

becomes non-negative.

Now, in view of (2.7.2), the sign-pattern matrix for $(1/x) T Q_{\cdot h} Q_{h \cdot}^T$ in (2.7.6) can be written for $x > 0$ as follows:

$$\begin{aligned}
(h) \quad & \left[\begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \vdots \\ \oplus \\ \hline \ominus \\ \ominus \\ \vdots \\ \ominus \end{array} \right] \quad \left[\begin{array}{ccccccc} \ominus & \ominus & \dots & \overset{(h-1)}{\ominus} & \vdots & \oplus & \oplus & \dots & \oplus \end{array} \right] \quad (2.7.8)
\end{aligned}$$

condition is actually implied by the necessary condition already established, viz., the submatrices $Q^{(1,2,\dots,h)}$ and $Q^{(h,h+1,\dots,n)}$ of matrix Q must be uniformly tapered. Now, in view of (2.7.9), consideration of each entry ${}^{(t)}y_{ij}^{(t)}$, as given by (2.7.7), in the upper triangular portions of submatrices $(TYT)^{(1,2,\dots,h-1)}$ and $(TYT)^{(h+1,h+2,\dots,n)}$ establishes an individual lower bound on the positive parameter x , while consideration of each of the remaining entries ${}^{(t)}y_{ij}^{(t)}$ in the upper triangular portion of matrix TYT establishes an individual upper bound on the parameter x . As an example, consider the entries ${}^{(t)}y_{11}^{(t)}$ and ${}^{(t)}y_{1h}^{(t)}$. From (2.7.5) - (2.7.7), and (2.7.9),

$${}^{(t)}y_{11}^{(t)} = (q_{11} - q_{12}) + (1/x) q_{1h} (q_{1h} - q_{2h}) \geq 0$$

i.e.,

$$(q_{11} - q_{12}) \geq (1/x) q_{1h} (q_{2h} - q_{1h})$$

or

$$x \geq \frac{q_{1h} (q_{2h} - q_{1h})}{(q_{11} - q_{12})} \quad (2.7.10)$$

Again,

$${}^{(t)}y_{1h}^{(t)} = (q_{1h} - q_{1,h+1}) + (1/x) q_{1h} (q_{hh} - q_{h+1,h}) \geq 0 \quad (2.7.11)$$

The entry ${}^{(t)}y_{1h}^{(t)}$ would, in fact, be automatically non-negative without imposing any bound whatsoever on the positive parameter x if the given matrix, $Q^{(1)}$, is such that

$${}^{(t)}q_{1h}^{(t)} = (q_{1h} - q_{1,h+1}) \geq 0$$

This follows from noting the non-negative character of the term

$$(1/x) {}^{(t)}q_{1h} {}^{(t)}q_{hl} = (1/x) q_{1h} (q_{hh} - q_{h+1,h}). \text{ (cf. 2.7.9)).}$$

It is clear that similar statements can be made with regard to

individual bounds imposed on x through consideration of entries

$${}^{(t)}y_{ij}^{(t)} \quad (i = 1, 2, \dots, h; j = h, h+1, \dots, n).$$

To continue the illustration, suppose the given matrix, $Q^{(1)}$, is such that:

$${}^{(t)}q_{1h}^{(t)} = q_{1h} - q_{1,h+1} < 0.$$

Accordingly, from (2.7.11), we have:

$$(1/x) q_{1h} (q_{hh} - q_{h+1,h}) \geq q_{1,h+1} - q_{1h}$$

or

$$\frac{q_{1h} (q_{hh} - q_{h+1,h})}{q_{1,h+1} - q_{1h}} \geq x \quad (2.7.12)$$

For a given matrix, $Q^{(1)}$, let x_1 and x_2 denote,

respectively, the least upper bound and the greatest lower bound

on parameter x . Then, in view of the statement immediately

following (2.7.6), the inequality

$$x_1 \geq x_2 \quad (2.7.13)$$

is a necessary and sufficient condition for matrix $Q^{(1)}$ to belong

to the special class considered in this section. If the inequality

in (2.7.13) is satisfied, parameter x can evidently be assigned

any value within the bounds x_1 and x_2 , and each value of x

will offer a different equivalent realization of matrix $Q^{(1)}$. All

these infinitely many equivalent realizations would contain, at most, $\frac{1}{2} n(n+1) + 1$ resistors. In fact, if x is assigned a value either equal to x_1 or x_2 , a distinct minimal realization containing exactly $\frac{1}{2} n(n+1)$ resistors would be obtained.

2.8. Matrices of the Fourth Order

The various considerations stated previously are the most general in the sense that they hold for any particular order, n , of a s.c. conductance matrix which is realizable with a specified two-tree port-structure. Now we shall apply some of these considerations to matrices of the fourth order mainly with the intention of illustrating them.

It was indicated earlier that the problem of matrix realization with a two-tree port-structure is easily reducible to the one of realization with a bilinear port-structure. Hence we shall be considering only the bilinear port-structures in what follows. As shown in Figure 2.8.1, only two distinct bilinear

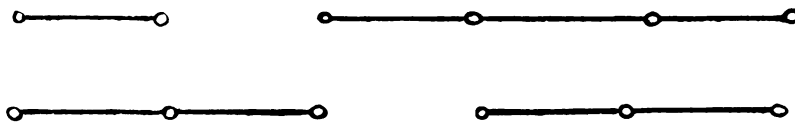


Figure 2.8.1

port-structures are possible in the case of four-port networks. We shall consider either of these port-structures separately so that the problem of realization of fourth-order matrices with two-tree port-structures will have been dealt with completely.

Case I. Suppose, for a given fourth-order matrix, $Q^{(1)}$,

the port-structure is specified as indicated in Figure 2.8.2.

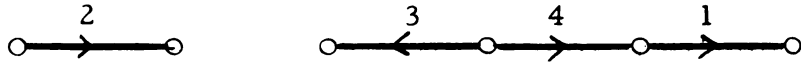


Figure 2.8.2

To obtain a matrix, Q , corresponding to the port-structure shown in Figure 2.8.3, first we pre- and post-multiply matrix $Q^{(1)}$ by a suitable E_1 matrix, viz., $E_1^{(2,3,4,1)}$ in the present example, and then pre- and post-multiply the resultant matrix by a suitable U_s matrix, viz., $U_s^{(2)}$ in the present example, so that $Q = U_s^{(2)} E_1^{(2,3,4,1)} Q^{(1)} E_1^{(2,3,4,1)} U_s^{(2)}$. We note that the realizations of matrices $Q^{(1)}$ and Q are identical except for the port-numbering and port-orientations.

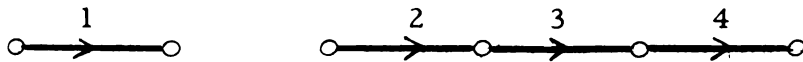


Figure 2.8.3

Let $Y = [y_{ij}]_4$ denote a fourth-order uniformly tapered matrix. Its realization as a s.c. conductance matrix with five nodes is shown in Figure 2.8.4, where the edges indicated by heavy lines represent the ports. Let another network be derived, as shown in Figure 2.8.5, from the one in Figure 2.8.4 by adding

five resistors which have their respective non-negative conductance-values equal to g_k ($k = 1 - 5$).

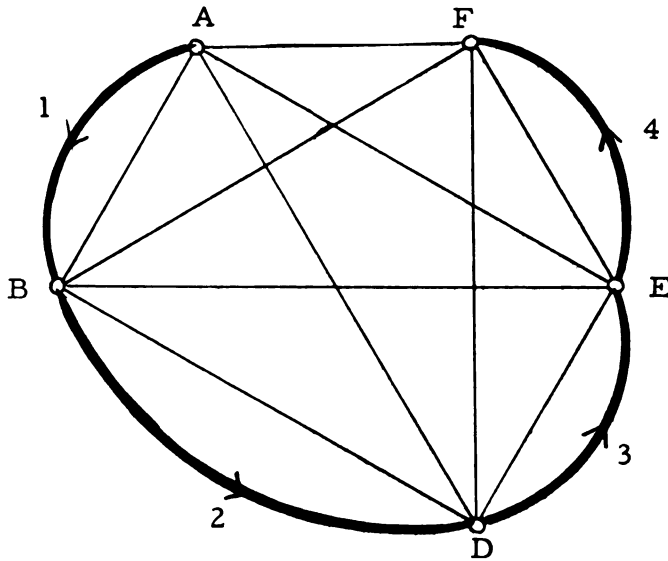


Figure 2.8.4

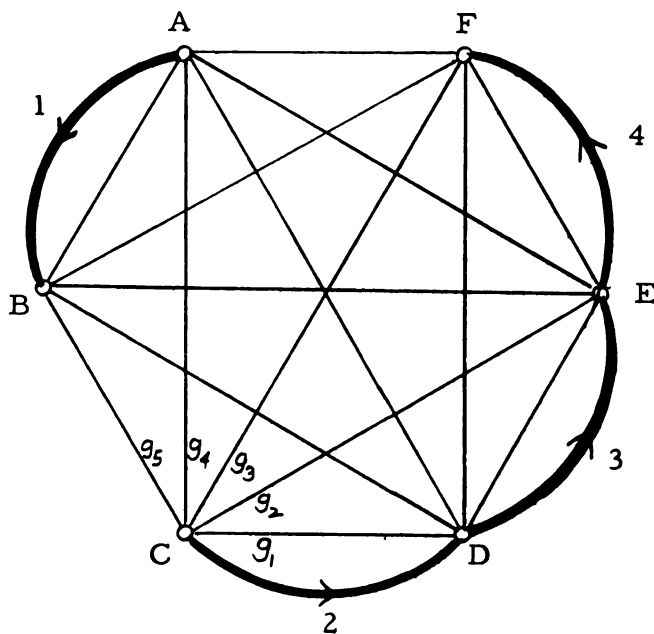


Figure 2.8.5

If matrix $\overline{Q} = [\overline{q}_{ij}]_4$ denotes the s.c. conductance matrix for the derived network, then by noting that $n = 4$ and $h = 2$ and by applying the final result of the generalized discussion in section 2.2, we can write the following relation between matrices Y and $\overline{Q}$. That is, from (2.2.7), (2.2.35), and (2.2.36),

$$Y = \overline{Q} - A + \left(\sum_{k=1}^5 g_k - q_{22} \right)^{-1} \overline{Q}'_{.2} \overline{Q}'_{2.} \quad (2.8.1)$$

where

$$A = \begin{bmatrix} g_4 & 0 & 0 & 0 \\ & g_1 + g_2 + g_3 & g_2 + g_3 & g_3 \\ & & g_2 + g_3 & g_3 \\ & & & g_3 \end{bmatrix} \quad (2.8.2)$$

and

$$\overline{Q}'_{.2} \overline{Q}'_{2.} = \begin{bmatrix} \overline{q}_{12} + g_4 \\ \overline{q}_{22} - g_1 - g_2 - g_3 \\ \overline{q}_{32} - g_2 - g_3 \\ \overline{q}_{42} - g_3 \end{bmatrix} \begin{bmatrix} \overline{q}_{12} & \overline{q}_{22} - g_1 & \overline{q}_{32} & \overline{q}_{42} \\ + g_4 & -g_2 - g_3 & -g_2 - g_3 & -g_3 \end{bmatrix} \quad (2.8.3)$$

We shall proceed, now, to apply the important results of the discussion in section 2.5 to the conductance-parameters g_1 through g_5 . Thus, from (2.5.2),

$$TQT = \begin{bmatrix} (q_{11} - q_{12}) & (q_{12} - q_{13}) & (q_{13} - q_{14}) & (q_{14}) \\ \text{---} & (q_{22} - q_{23} - q_{12} + q_{13}) & (q_{23} - q_{24} - q_{13} + q_{14}) & (q_{24} - q_{14}) \\ & \text{---} & (q_{33} - q_{34} - q_{23} + q_{24}) & (q_{34} - q_{24}) \\ & & X & (q_{44} - q_{34}) \end{bmatrix} \quad (2.8.4)$$

From (2.5.3),

$$TAT = \begin{bmatrix} g_4 & 0 & 0 & 0 \\ \text{---} & g_1 & g_2 & g_3 \\ & \text{---} & 0 & 0 \\ & & X & 0 \end{bmatrix} \quad (2.8.5)$$

From (2.5.4) - (2.5.5),

$$TQ'_{.2}Q'_{2.T} = \begin{bmatrix} q_{12} + g_4 \\ q_{22} - q_{12} - g_1 - g_2 - g_3 - g_4 \quad (\equiv u) \\ q_{32} - q_{22} + g_1 \quad (\equiv v) \\ q_{42} - q_{32} + g_2 \quad (\equiv w) \end{bmatrix} \begin{bmatrix} -u & -v & -w & q_{42} - g_3 \end{bmatrix} \quad (2.8.6)$$

Applying, next, the results in (2.5.17) - (2.5.22) to the present case, the following bounds are established on g_k ($k = 1 - 5$):

$$g_4 \geq -q_{12} \quad (2.8.7)$$

$$g_5 \geq q_{12} \quad (2.8.8)$$

$$g_1 \leq q_{22} - q_{23} \quad (2.8.9)$$

$$g_2 \leq q_{23} - q_{24} \quad (2.8.10)$$

$$g_3 \leq q_{24} \quad (2.8.11)$$

$$\sum_{k=1}^5 g_k > q_{22} \quad (2.8.12)$$

From (2.5.26), the sign-pattern matrix for $TQ'_2 Q'_2.T$ is obtained as

$$\begin{bmatrix} \oplus & \oplus & \oplus & \oplus \\ & \oplus & \oplus & \oplus \\ & & \ominus & \ominus \\ & & & \ominus \end{bmatrix} \quad (2.8.13)$$

X

In the above sign-pattern matrix, the entry in position (1,1) is non-positive and the entries in positions (2,2), (2,3), and (2,4) are non-negative when $q_{22} - q_{12} \geq g_1 + g_2 + g_3 + g_4$. On the other hand, the entry in position (1,1) is non-negative and the entries in positions (2,2), (2,3), and (2,4) are non-positive when $q_{22} - q_{12} \leq g_1 + g_2 + g_3 + g_4$. Further, by virtue of (2.8.12), the sign-pattern matrix for $(\sum_{k=1}^5 g_k - q_{23})^{-1} TQ'_2 Q'_2.T$ is the same as that for matrix $TQ'_2 Q'_2.T$.

From (2.5.29) - (2.5.30) and the associated generalized statement, if, in matrix Q , $q_{22} - q_{23} - q_{12} + q_{13} \leq 0$ and/or $q_{23} - q_{24} - q_{13} + q_{14} \leq 0$ and/or $q_{24} - q_{14} \leq 0$, then conductance-parameters g_1 through g_4 are subject to the upper bounds:

$$g_1 + g_2 + g_3 + g_4 \leq q_{22} - q_{12}; \quad (2.8.14)$$

$$g_4 \leq q_{11} - q_{12} \quad (2.8.15)$$

A conditional upper bound on g_4 is further obtained through (2.5.37) as follows:

$$\frac{(q_{11})(q_{2\hat{j}} - q_{2,\hat{j}+1})}{(q_{1\hat{j}} - q_{1,\hat{j}+1})} - q_{12} \geq g_4 \quad (2.8.16)$$

where $\hat{j}$ is any index between 2 and 4 such that, in matrix Q ,

$$q_{1\hat{j}} - q_{1,\hat{j}+1} > 0.$$

As an example, suppose, in matrix Q , $q_{13} - q_{14} > 0$; then the upper bound on g_4 is given by:

$$\frac{(q_{11})(q_{23} - q_{24})}{q_{13} - q_{14}} - q_{12} \geq g_4$$

Further, if, say, $q_{12} - q_{13} > 0$, then an additional upper bound is:

$$\frac{(q_{11})(q_{22} - q_{23})}{q_{12} - q_{13}} - q_{12} \geq g_4.$$

Case II. Suppose, for a given fourth-order matrix, $Q^{(1)}$, the port-structure is specified as indicated in Figure 2.8.6.



Figure 2.8.6

To obtain a matrix, Q , corresponding to the port-structure shown in Figure 2.8.7, we first pre- and post-multiply matrix $Q^{(1)}$ by a suitable E_1 matrix, viz., $E^{(3,1,4,2)}$ in the present example, and then pre- and post-multiply the resultant matrix by a suitable U_s matrix, viz., $U_s^{(2)}$ in the present example, so that $Q = U_s^{(2)} E_1^{(3,1,4,2)} Q^{(1)} E_1^{(3,1,4,2)} U_s^{(2)}$.



Figure 2.8.7

Let $Y = [y_{ij}]_4$ denote a fourth-order uniformly tapered matrix. Its realization as a s.c. conductance matrix with five nodes is shown in Figure 2.8.4. Let another network be derived, as shown in Figure 2.8.8, from the one in Figure 2.8.4 by adding five resistors which have their respective non-negative conductance values equal to g_k ($k = 1 - 5$). If matrix $\overline{Q} = [\overline{q}_{ij}]_4$ denotes the s.c. conductance matrix for the derived network, then by noting that $n = 4$ and $h = 3$ and by applying the final result of the generalized discussion in section 2.2, we can write the following relation between matrices Y and $\overline{Q}$.

That is, from (2.2.7), (2.2.35), and (2.2.36),

$$Y = \overline{Q} - A + \left(\sum_{k=1}^5 g_k - q_{33} \right)^{-1} \overline{Q}'_{.3} \overline{Q}'_{.3}. \quad (2.8.17)$$

where

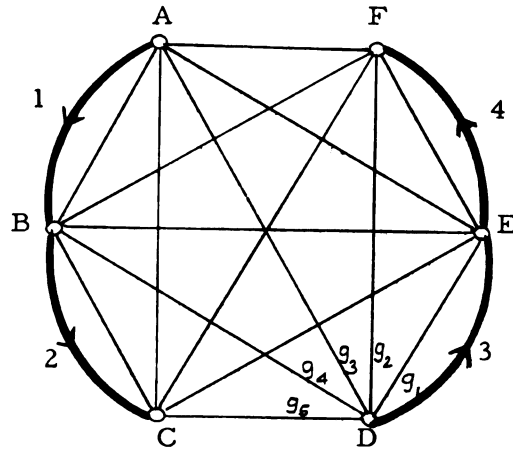


Figure 2.8.8

$$A = \left[\begin{array}{cc|cc} g_3 & g_3 & & \\ & g_3 + g_4 & & \\ & & & 0 \\ & & g_1 + g_2 & g_2 \\ & & & g_2 \end{array} \right] \quad (2.8.18)$$

and

$$\overline{Q}'_{.3} \overline{Q}'_{.3} = \left[\begin{array}{c} \overline{q}_{13} + g_3 \\ \overline{q}_{23} + g_3 + g_4 \\ \hline \overline{q}_{33} - g_1 - g_2 \\ \overline{q}_{43} - g_2 \end{array} \right] \left[\begin{array}{cc|cc} \overline{q}_{13} & \overline{q}_{23} & & \overline{q}_{33} & \overline{q}_{43} \\ +g_3 & +g_3 + g_4 & & -g_1 - g_2 & -g_2 \end{array} \right] \quad (2.8.19)$$

We shall proceed, now, to apply the important results of the discussion in section 2.5 to the conductance parameters g_1 through g_5 . Thus from (2.5.3),

$$TAT = \begin{bmatrix} 0 & g_3 & 0 & 0 \\ & g_4 & 0 & 0 \\ & & g_1 & g_2 \\ X & & & 0 \end{bmatrix} \quad (2.8.20)$$

From (2.5.4) - (2.5.5),

$$TQ'_3 Q'_3 T = \begin{bmatrix} q_{13} + g_3 \\ q_{23} - q_{13} + g_4 \quad (\equiv u) \\ q_{33} - q_{23} - g_1 - g_2 - g_3 - g_4 \quad (\equiv v) \\ q_{43} - q_{33} + g_1 \quad (\equiv w) \end{bmatrix} \begin{bmatrix} -u & -v & -w & q_{43} - g_2 \end{bmatrix} \quad (2.8.21)$$

Applying, next, the results in (2.5.17) - (2.5.22), the following bounds are established on g_k ($k = 1 - 5$):

$$g_3 \geq -q_{13} \quad (2.8.22)$$

$$g_4 \geq -q_{23} + q_{13} \quad (2.8.23)$$

$$g_5 \geq q_{23} \quad (2.8.24)$$

$$g_1 \leq q_{33} - q_{34} \quad (2.8.25)$$

$$g_2 \leq q_{34} \quad (2.8.26)$$

$$\sum_{k=1}^5 g_k > q_{33} \quad (2.8.27)$$

From (2.5.26), the sign-pattern matrix for $TQ'_3 Q'_3 T$ is obtained as:

$$\begin{bmatrix}
 \ominus & \oplus & \oplus & \oplus \\
 & \oplus & \oplus & \oplus \\
 & & \oplus & \oplus \\
 & & & \ominus
 \end{bmatrix} \quad (2.8.28)$$

In the above sign-pattern matrix, the entries in positions (1, 2) and (2, 2) are non-positive and the entries in positions (3, 3) and (3, 4) are non-negative when $q_{33} - q_{23} \geq g_1 + g_2 + g_3 + g_4$. On the other hand, the entries in positions (1, 2) and (2, 2) are non-negative and the entries in positions (3, 3) and (3, 4) are non-positive when $q_{33} - q_{23} \leq g_1 + g_2 + g_3 + g_4$. Further, by virtue of (2.8.27), the sign-pattern matrix for $(\sum_{k=1}^5 g_k - q_{33}) TQ'_{.3}Q'_3.T$ is the same as that for matrix $TQ'_{.3}Q'_3.T$.

From (2.5.27) - (2.5.28) and the associated generalized statement, if, in matrix Q , $q_{12} - q_{13} \leq 0$ and/or $q_{22} - q_{23} - q_{12} + q_{13} \leq 0$, then conductance-parameters g_1 through g_4 are subject to the bounds:

$$g_1 + g_2 + g_3 + g_4 \geq q_{33} - q_{23} \quad (2.8.29)$$

$$g_1 \leq q_{33} - q_{34} - q_{23} + q_{24} \quad (2.8.30)$$

and

$$g_2 \leq q_{34} - q_{24} \quad (2.8.31)$$

Again, from (2.5.29) - (2.5.30) and the associated statement, if, in matrix Q , $q_{33} - q_{34} - q_{23} + q_{24} \leq 0$ and/or $q_{34} - q_{24} \leq 0$, then parameters g_1 through g_4 are subject to the bounds:

$$g_1 + g_2 + g_3 + g_4 \leq q_{33} - q_{23} \quad (2.8.32)$$

$$g_3 \leq q_{12} - q_{13} \quad (2.8.33)$$

$$g_4 \leq q_{22} - q_{23} - q_{12} + q_{13} \quad (2.8.34)$$

It is evident that a matrix Q is, in fact, not realizable with the bilinear port-structure shown in Figure 2.8.7 if (i) $q_{12} - q_{13} < 0$ simultaneously with $q_{33} - q_{34} - q_{23} - q_{24} < 0$ and/or $q_{34} - q_{24} < 0$, and/or if (ii) $q_{22} - q_{23} - q_{12} + q_{13} < 0$ simultaneously with $q_{33} - q_{34} - q_{23} - q_{24} < 0$ and/or $q_{34} - q_{24} < 0$.

Next, application of (2.5.37) offers the following bounds on parameters g_3 and g_4 . Thus, with $i = 1$,

$$\frac{(q_{12})(q_{3\hat{j}} - q_{3,\hat{j}+1})}{(q_{2\hat{j}} - q_{2,\hat{j}+1})} - q_{13} \geq g_3 \quad (2.8.35)$$

With $i = 2$,

$$\frac{(q_{22} - q_{12})(q_{3\hat{j}} - q_{3,\hat{j}+1})}{(q_{2\hat{j}} - q_{2,\hat{j}+1})} - q_{23} + q_{13} \geq g_4 \quad (2.8.36)$$

where $\hat{j}$ is either equal to 3 or 4 such that, in matrix Q ,

$$q_{2\hat{j}} - q_{2,\hat{j}+1} > 0.$$

As an example, suppose, in matrix Q , $q_{23} - q_{24} > 0$; then the upper bounds on g_3 and g_4 are given by:

$$\frac{(q_{12})(q_{33} - q_{34})}{(q_{23} - q_{24})} - q_{13} \geq g_3 \quad (2.8.37)$$

and

$$\frac{(q_{22} - q_{12})(q_{33} - q_{34})}{(q_{23} - q_{24})} - q_{23} + q_{13} \geq g_4. \quad (2.8.38)$$

However, if matrix Q is such that $q_{23} - q_{24} \leq 0$ as well as $q_{24} \leq 0$, then the bounds in (2.8.35) - (2.8.36) do not hold.

Example 1. Consider the s.c. conductance matrix

$$Q^{(1)} = \begin{bmatrix} 12 & 1 & -6 & 0 \\ 1 & 10 & 1 & 4 \\ -6 & 1 & 11 & 2 \\ 0 & 4 & 2 & 9 \end{bmatrix} \quad (2.8.39)$$

the port-structure being specified as shown in Figure 2.8.9.



Figure 2.8.9

We apply the conductance-parameter procedure for synthesizing a six-terminal resistive network from this s.c. conductance matrix. The matrix, Q , which corresponds to the bilinear port-structure shown in Figure 2.8.10 is given by:

$$Q = U_s^{(2)} E_1^{(3, 1, 4, 2)} Q^{(1)} E_1^{(3, 1, 4, 2)} U_s^{(2)}$$



Figure 2.8.10

Thus,

$$Q = \begin{bmatrix} 11 & 6 & 2 & 1 \\ 6 & 12 & 0 & -1 \\ 2 & 0 & 9 & 4 \\ 1 & -1 & 4 & 10 \end{bmatrix} \quad (2.8.40)$$

We observe that both the submatrices $Q^{(1,2)}$ and $Q^{(3,4)}$ are uniformly tapered so that one fundamental necessary condition for realizability of the original matrix, $Q^{(1)}$, is indeed satisfied.

We also observe that this example is a direct illustration of Case II discussed above, so that the realization of matrix Q must assume the form shown in Figure 2.8.8.

With reference to (2.8.17), we can write:

$$TYT = TQT - TAT + \left(\sum_{k=1}^5 g_k - q_{33} \right)^{-1} TQ'_3 Q'_3 T$$

Applying (2.8.4), (2.8.20), and (2.8.21) to the present example, we have:

$$TYT = \begin{bmatrix} 5 & 4 & 1 & 1 \\ & 8 & 0 & -2 \\ & & 4 & 5 \\ X & & & 6 \end{bmatrix} - \begin{bmatrix} 0 & g_3 & 0 & 0 \\ & g_4 & 0 & 0 \\ & & g_1 & g_2 \\ X & & & 0 \end{bmatrix} + \frac{1}{\sum_{k=1}^5 g_k - 9} \begin{bmatrix} 2 + g_3 \\ -2 + g_4 \quad (\equiv u) \\ 9 - g_1 - g_2 - g_3 - g_4 \quad (\equiv v) \\ -5 + g_1 \quad (\equiv w) \end{bmatrix} \begin{bmatrix} -u & -v & -w & 4 - g_2 \end{bmatrix} \quad (2.8.41)$$

The problem of realization of matrix $Q^{(1)}$ now depends upon finding a suitable set of parameters g_k ($k = 1 - 5$) such that each entry in the upper triangular portion of the above matrix, TYT , is non-negative. The following bounds are obtained on g_k through (2.8.22) - (2.8.38):

$$[\text{From (2.8.24)}] : \quad g_4 \geq 2$$

$$[\text{From (2.8.26)}] : \quad g_1 \leq 5$$

$$[\text{From (2.8.27)}] : \quad g_2 \leq 4$$

$$[\text{From (2.8.37)}] : \quad g_3 \leq 28$$

$$[\text{From (2.8.38)}] : \quad g_4 \leq 32$$

A machine search within these bounds, using the principles indicated in section 2.6, yields one suitable set of parameters g_k as follows:

$$g_1 = 1.005252 , \quad g_2 = 0.000000 , \quad g_3 = 0.209919 ,$$

$$g_4 = 8.070859 , \text{ and } g_5 = 11.802151 .$$

Another suitable set with integral values can be easily obtained from the above set as:

$$g_1 = 1 , \quad g_2 = 0 , \quad g_3 = 0 , \quad g_4 = 8 , \text{ and } g_5 = 12 .$$

Substituting these values in (2.8.41), we get:

$$\begin{aligned}
 \text{TYT} &= \begin{bmatrix} 5 & 4 & 1 & 1 \\ & 8 & 0 & -2 \\ & & 4 & 5 \\ \text{X} & & & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 8 & 0 & 0 \\ & & 1 & 0 \\ \text{X} & & & 0 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} 2 \\ 6 \\ 0 \\ -4 \end{bmatrix} \begin{bmatrix} -6 & 0 & 4 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 4 & 1\frac{2}{3} & 1\frac{2}{3} \\ & 0 & 2 & 0 \\ & & 3 & 5 \\ \text{X} & & & 4\frac{2}{3} \end{bmatrix}
 \end{aligned}$$

The realization of matrix $Q^{(1)}$ is readily obtained as shown in Figure 2.8.11 (cf. Figures 2.8.8 and 2.8.9).

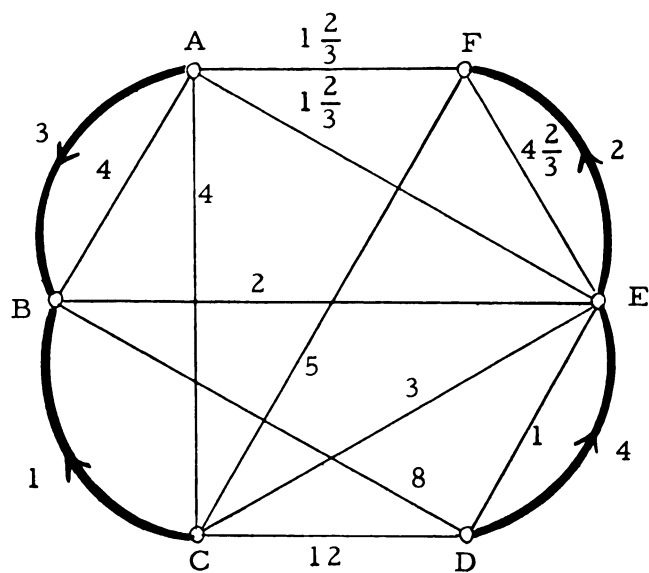


Figure 2.8.11

We shall consider, now, another matrix and apply to it the special version of the conductance-parameter procedure discussed in section 2.7. It will be seen that the matrix does, in fact, belong to the special class of matrices realizable with the special procedure, with the result that its two distinct minimal realizations can be obtained very easily.

Example 2: Consider the s.c. conductance matrix [SL 1]

$$Q^{(1)} = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 12 & 4 & 5 \\ 2 & 4 & 15 & 6 \\ 3 & 5 & 6 & 18 \end{bmatrix} \quad (2.8.42)$$

with the port-structure shown in Figure 2.8.12.

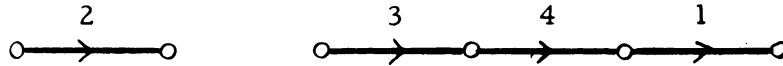


Figure 2.8.12

Pre- and post-multiplying $Q^{(1)}$ by $E_1^{(2,3,4,1)}$, we obtain matrix $Q = E_1^{(2,3,4,1)} Q^{(1)} E_1^{(2,3,4,1)}$ corresponding to the bilinear port-structure shown in Figure 2.8.13. Thus,

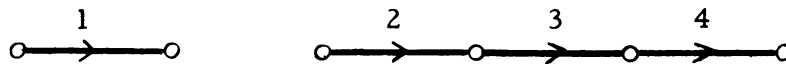


Figure 2.8.13

$$Q = \begin{bmatrix} 12 & 4 & 5 & 1 \\ 4 & 15 & 6 & 2 \\ 5 & 6 & 18 & 3 \\ 1 & 2 & 3 & 7 \end{bmatrix} \quad (2.8.43)$$

We note that here $n = 4$ and $h = 2$. We also note that the submatrices $Q^{(1,2)}$ and $Q^{(2,3,4)}$ are both in the uniformly tapered form so that the fundamental necessary condition for realizability of $Q^{(1)}$ with the special procedure is satisfied.

Applying the relation in (2.7.4) to the present example, we have:

$$\begin{aligned} TYT &= TQT + (1/x) TQ_2 Q_2^T \\ &= \begin{bmatrix} 8 & -1 & 4 & 1 \\ & 10 & 0 & 1 \\ & & 11 & 1 \\ X & & & 4 \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 4 \\ 11 \\ -9 \\ -4 \end{bmatrix} \begin{bmatrix} -11 & 9 & 4 & 2 \end{bmatrix} \end{aligned}$$

It is our aim to assign, if possible, a positive value to the parameter x so that each entry, $y_{ij}^{(t)}$, in the upper triangular position of matrix TYT becomes non-negative. Signwise, the above equation may be written as:

$$TYT = \begin{bmatrix} + & 1 & - & + & + \\ & & + & + & + \\ & & & - & - \\ & & & & + \\ X & & & & + \end{bmatrix} + \begin{bmatrix} - & 1 & + & + & + \\ & & + & + & + \\ & & & - & - \\ & & & & - \\ X & & & & - \end{bmatrix}$$

Evidently, the upper bound on x is imposed only through consideration of entry ${}^{(t)}y_{12}^{(t)}$, while an individual lower bound on x is imposed through consideration of entries ${}^{(t)}y_{11}^{(t)}$, ${}^{(t)}y_{33}^{(t)}$, ${}^{(t)}y_{34}^{(t)}$, and ${}^{(t)}y_{44}^{(t)}$. Thus,

$$[{}^{(t)}y_{12}^{(t)}] : \quad -1 + \frac{4 \cdot 9}{x} \geq 0, \text{ or } x \leq 36$$

$$[{}^{(t)}y_{11}^{(t)}] : \quad 8 - \frac{4 \cdot 11}{x} \geq 0, \text{ or } x \geq \frac{11}{2}$$

$$[{}^{(t)}y_{33}^{(t)}] : \quad 11 - \frac{9 \cdot 4}{x} \geq 0, \text{ or } x \geq \frac{36}{11}$$

$$[{}^{(t)}y_{34}^{(t)}] : \quad 1 - \frac{9 \cdot 2}{x} \geq 0, \text{ or } x \geq 18$$

$$[{}^{(t)}y_{44}^{(t)}] : \quad 4 - \frac{4 \cdot 2}{x} \geq 0, \text{ or } x \geq 2$$

Therefore, we have the least upper bound on $x (\equiv x_1) = 36$

and the greatest lower bound on $x (\equiv x_2) = 18$.

The compatibility of these two bounds directly implies that matrix $Q^{(1)}$ can, in fact, be realized by the "special procedure." Infinitely many equivalent realization of matrix $Q^{(1)}$ can be obtained by assigning different values to x within the bounds 18 and 36. We shall consider here three realizations corresponding to the values 18, 36, and 24; the first two realizations will be the minimal ones containing exactly ten resistors.

For $x = 18$, we have:

$$\text{TYT} = \begin{bmatrix} 5\frac{5}{9} & 1 & 4\frac{8}{9} & 1\frac{4}{9} \\ & 15\frac{1}{2} & 2\frac{4}{9} & 2\frac{2}{9} \\ & & 9 & 0 \\ X & & & 3\frac{5}{9} \end{bmatrix} \quad (2.8.44)$$

For $x = 36$, we have:

$$\text{TYT} = \begin{bmatrix} 6\frac{7}{9} & 0 & 4\frac{4}{9} & 1\frac{2}{9} \\ & 12\frac{3}{4} & 1\frac{2}{9} & 1\frac{11}{18} \\ & & 10 & \frac{1}{2} \\ X & & & 3\frac{7}{9} \end{bmatrix} \quad (2.8.45)$$

For $x = 24$, we have:

$$\text{TYT} = \begin{bmatrix} 6\frac{1}{6} & \frac{1}{2} & 4\frac{2}{3} & 1\frac{1}{3} \\ & 14\frac{1}{8} & 1\frac{5}{6} & 1\frac{11}{12} \\ & & 9\frac{1}{2} & \frac{1}{4} \\ X & & & 3\frac{2}{3} \end{bmatrix} \quad (2.8.46)$$

The realizations shown in Figures 2.8.14 - 2.8.16 follow when we recall from sections 2.2 and 2.7 that the conductance value, g_o , of the pertinent resistor is given by $(q_{22} + x)$, i.e., $(15 + x)$.

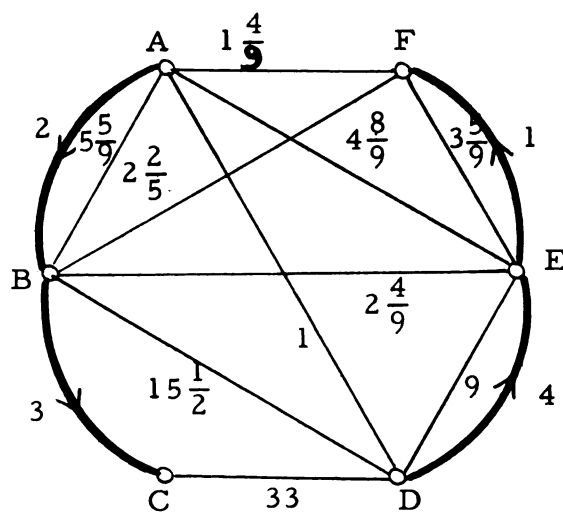


Figure 2.8.14

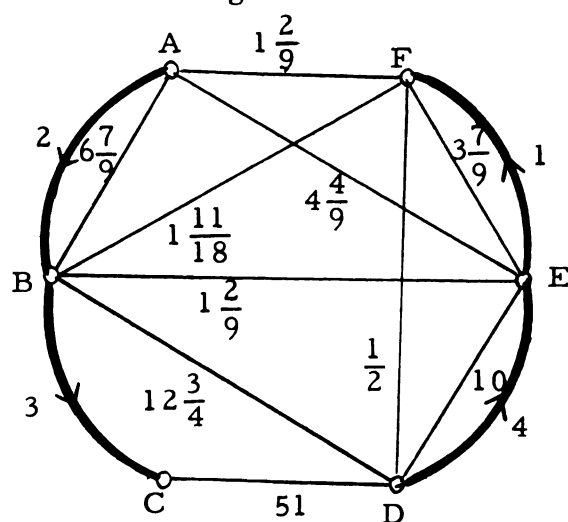


Figure 2.8.15

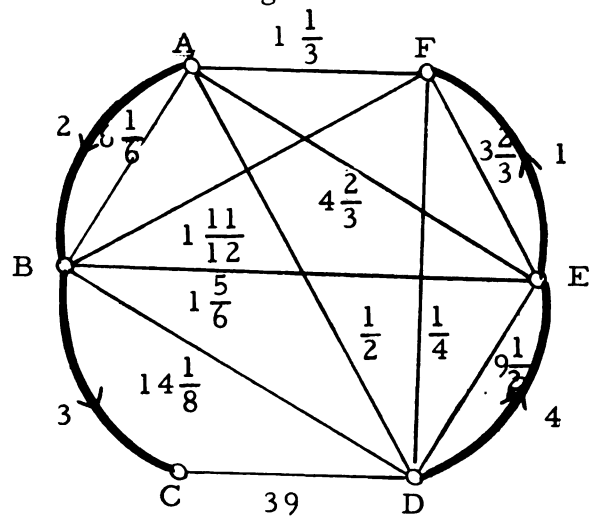


Figure 2.8.16

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2.9. Realization of Matrices with k-Tree Port-Structures

The problem of synthesis of resistive networks from s.c. conductance matrices which are realizable with two-tree port-structures was dealt with thoroughly in the preceding sections. However, certain matrices may be realizable, exclusively or otherwise, with port-structures forming k-trees ($n \geq k \geq 3$). Some ideas on a possible approach to the problem of realization of these matrices are presented in the following.

We shall establish first one useful result in matrix algebra.

Theorem 1: Let real, non-singular matrices $Y = [y_{ij}]_n$ and $Q = [q_{ij}]_n$ be related by:

$$Y = (Q^{-1} - R)^{-1} \quad (2.9.1)$$

where all entries of matrix $R = [r_{ij}]_n$ are zero excepting certain diagonal entries $r_{bb}, r_{ff}, \dots, r_{ll}$ ($b < f < \dots < l < n$). Then the relation in (2.9.1) has the alternate form:

$$Y = Q + \begin{bmatrix} & & & \\ Q_{.b} & Q_{.f} & \dots & Q_{.l} \\ & & & \end{bmatrix} \begin{bmatrix} \frac{1}{r_{bb}} - q_{bb} & -q_{bf} & \dots & -q_{bl} \\ -q_{fb} & \frac{1}{r_{ff}} - q_{ff} & \dots & -q_{fl} \\ \vdots & \vdots & \ddots & \vdots \\ -q_{lb} & -q_{lf} & \dots & \frac{1}{r_{ll}} - q_{ll} \end{bmatrix}^{-1} \begin{bmatrix} Q_{b.} \\ Q_{f.} \\ \vdots \\ Q_{l.} \end{bmatrix}, \quad (2.9.2)$$

provided the inverse matrix on the right-hand side of the equation (2.9.2) exists.[†]

Proof: Let d denote the number of non-zero entries of the diagonal matrix R .

Let the $n \times d$ matrix $P = [p_{ij}]$ be such that $p_{b1} = p_{f2} = \dots = p_{ld} = 1$, the rest of the entries being zero.

Let the $n \times n$ diagonal matrix $R^{(s)} = [r_{ij}^{(s)}]$ be such that $r_{bb}^{(s)} = r_{bb}^{-1}$, $r_{ff}^{(s)} = r_{ff}^{-1}$, ..., $r_{ll}^{(s)} = r_{ll}^{-1}$, the rest of the entries being zero. (Matrix $R^{(s)}$ represents thus the semi-inverse of matrix R .)

Let $\tilde{R} = [\tilde{r}_{ij}]$ denote the $d \times d$ non-singular diagonal matrix $P^T R P$. Evidently, $\tilde{r}_{11} = r_{bb}$, $\tilde{r}_{22} = r_{ff}$, ..., $\tilde{r}_{dd} = r_{ll}$. Further, we have $\tilde{R}^{-1} = P^T R^{(s)} P$ and $P \tilde{R} P^T = R$.

Consider matrix $P^T Q Y^{-1}$.

$$\begin{aligned}
 P^T Q Y^{-1} &= P^T Q (Q^{-1} - R) \\
 &= P^T - P^T Q R \\
 &= \tilde{R}^{-1} \tilde{R} P^T - P^T Q P \tilde{R} P^T \\
 &= (\tilde{R}^{-1} - P^T Q P) \tilde{R} P^T \\
 &= (P^T R^{(s)} P - P^T Q P) \tilde{R} P^T \\
 &= (P^T [R^{(s)} - Q] P) \tilde{R} P^T
 \end{aligned}$$

Therefore,

$$P^T Q = (P^T [R^{(s)} - Q] P) \tilde{R} P^T Y \quad (2.9.3)$$

[†] The alternate form in (2.9.2) is valuable in that it facilitates computations.

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$$P^T [R^{(s)} - Q] P = \begin{bmatrix} \frac{1}{r_{bb}} - q_{bb} & -q_{bf} & \cdots & -q_{b1} \\ -q_{fb} & \frac{1}{r_{ff}} - q_{ff} & \cdots & -q_{f1} \\ \vdots & \vdots & \ddots & \vdots \\ -q_{1b} & -q_{1f} & \cdots & \frac{1}{r_{11}} - q_{11} \end{bmatrix} \quad (2.9.4)$$

Let this be non-singular (hypothesis). Note, further, that

$Y = (Q^{-1} - R)^{-1}$ implies $R = (Q^{-1} - Y^{-1})$, so that

$$Q R Y = (Y - Q) \quad (2.9.5)$$

Pre-multiplying both sides of the relation in (2.9.3) by the non-singular matrix $(P^T [R^{(s)} - Q] P)^{-1}$ and interchanging the sides, we have:

$$\tilde{R} P^T Y = (P^T [R^{(s)} - Q] P)^{-1} P^T Q$$

Therefore,

$$Q P \tilde{R} P^T Y = Q P (P^T [R^{(s)} - Q] P)^{-1} P^T Q$$

i.e.,

$$Q R Y = Q P (P^T [R^{(s)} - Q] P)^{-1} P^T Q$$

From (2.9.5), then,

$$Y - Q = Q P (P^T [R^{(s)} - Q] P)^{-1} P^T Q$$

or

$$Y = Q + Q P (P^T [R^{(s)} - Q] P)^{-1} P^T Q. \quad (2.9.6)$$

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Note that

$$Q P = \begin{bmatrix} Q_{.b} & Q_{.f} & \cdots & Q_{.1} \end{bmatrix} \quad (2.9.7)$$

and

$$P^T Q = \begin{bmatrix} Q_{b.} \\ Q_{f.} \\ \vdots \\ Q_{1.} \end{bmatrix} \quad (2.9.8)$$

In view of (2.9.4), (2.9.7), and (2.9.8), the relation in (2.9.6) is, in fact, the same as that in (2.9.2). This proves the theorem.

An obvious corollary of the above theorem is that if every diagonal entry of matrix R is non-zero, then the relation $Y = (Q^{-1} - R)^{-1}$ has the alternate form:

$$Y = Q + Q(R^{-1} - Q)^{-1} Q \quad (2.9.9)$$

Consider, now, an n -port network, A , such that its s.c. conductance matrix, $Y = [y_{ij}]_n$, corresponding to some particular port-structure is non-singular. Let E_s ($s = 1, 2, \dots, n$) denote the n voltage-generators exciting the network.

Consider another n -port network, N , obtained from the above network by adding non-negative resistors, r_{ss} , in series with generators E_s ($s = 1, 2, \dots, n$). Let $\bar{Q} = [\bar{q}_{ij}]_n$ denote the s.c. conductance matrix for the derived network N , and let a diagonal matrix, R , be formed with resistor values r_{ss} as its

diagonal entries in the same sequence. If matrix $\bar{Q}$ is non-singular, matrices $\bar{Q}^{-1}$ and Y^{-1} are obviously the open-circuit resistance matrices for networks N and A respectively. Now, in view of the above discussion, it is clear that

$$\bar{Q}^{-1} = Y^{-1} + R, \quad (2.9.10)$$

so that

$$Y = (\bar{Q}^{-1} - R)^{-1}. \quad (2.9.11)$$

We observe that s.c. input conductance $\bar{q}_{ss}$ ($s = 1, 2, \dots, n$) is given by s.c. input conductance y_{ss} itself if $r_{ss} = 0$. Again, $\bar{q}_{ss}$ is given by the series combination of conductances y_{ss} and r_{ss}^{-1} whenever $r_{ss} > 0$. As a consequence, the following relation holds:

$$r_{ss}^{-1} > \bar{q}_{ss} \quad (s = 1, 2, \dots, n; r_{ss} \neq 0). \quad (2.9.12)$$

Suppose, now, that network A has exactly $(n+1)$ nodes, so that the n voltage-generators E_s ($s = 1, 2, \dots, n$) constitute a tree. On the other hand, if network N is obtained from network A by adding a positive resistor, r_{mm} , in series with at least one generator, E_m ($1 \leq m \leq n$), which is represented by an internal branch, m , then the n voltage-generators exciting network N must necessarily constitute a forest. This concept is illustrated in Figure 2.9.1, where figures (a) and (b) depict

networks A and N respectively.

After establishing a simple result in matrix algebra at this point, we shall be in a position to consider the problem proper of realization of s.c. conductance matrices with k-tree port-structures.

Theorem 2: Let an nth-order real, non-singular matrix Y be related to nth-order, real, non-singular matrices Q and $\overline{Q}$ by:

$$Y = (Q^{-1} - R)^{-1} \quad (2.9.13)$$

and

$$Y = (\overline{Q}^{-1} - R)^{-1} . \quad (2.9.14)$$

Then $Q \equiv \overline{Q} . \quad (2.9.15)$

Proof: Equating the right-hand sides of equations (2.9.13) and (2.9.14), the identity in (2.9.15) follows immediately.

Let $Q = [q_{ij}]_n$ denote a paramount matrix to be realized as a s.c. conductance matrix. If matrix Q is not realizable with either a connected or a two-tree port-structure, then realization must be tried with a k-tree port-structure ($n \geq k \geq 3$). Excepting in the very special case where matrix Q is dominant, no technique is available in the literature for realizing matrix Q as specified above. In this situation, the trial-and-error technique established below has some value. Besides the fact that by means of this technique we may be able to realize certain matrices with k-tree port-structure, it could perhaps lead to a precise method of realization of all s.c. conductance matrices which are realizable with k-tree port-structures.

Suppose, then, that a matrix, $R = [r_{ij}]_n$, is found with all entries zero excepting certain diagonal entries $r_{bb}, r_{ff}, \dots, r_{ll}$, ($b < f < \dots < l < n$) such that a matrix, Y , as given by:

$$Y = Q + \begin{bmatrix} Q_{.b} & Q_{.f} & \dots & Q_{.l} \end{bmatrix} \begin{bmatrix} \frac{1}{r_{bb}} - q_{bb} & -q_{bf} & \dots & -q_{bl} \\ -q_{fb} & \frac{1}{r_{ff}} - q_{ff} & \dots & -q_{fl} \\ \vdots & \vdots & \ddots & \vdots \\ -q_{lb} & -q_{lf} & \dots & \frac{1}{r_{ll}} - q_{ll} \end{bmatrix}^{-1} \begin{bmatrix} Q_{b.} \\ Q_{f.} \\ \vdots \\ Q_{l.} \end{bmatrix} \quad (2.9.16)$$

is realizable, through the known techniques, as a s.c. conductance matrix with one or two-tree port-structures. Evidently, entries $r_{bb}, r_{ff}, \dots, r_{ll}$ of matrix R must be such that the inverse matrix on the right-hand side of (2.9.16) exists; further, as will become clear below, they are subject to the condition $r_{ss}^{-1} > q_{ss} > 0$ ($s = b, f, \dots, l$). If matrices Q and Y are non-singular, then, through Theorem 1, the relation in (2.9.16) is, in fact, an alternate form of the relation:

$$Y = (Q^{-1} - R)^{-1} \quad (2.9.17)$$

Let an n -port network, A , be the realization of s.c. conductance matrix Y with E_s ($s = 1, 2, \dots, n$) denoting the n voltage-generators. Let another n -port network, N , be obtained from network A by adding non-negative resistors having the above values r_{ss} , in series with the corresponding generators E_s .

If $\overline{Q} = [\overline{q}_{ij}]$ denotes the s.c. conductance matrix for network N , then, as seen earlier, matrices Y and $\overline{Q}$ must be related by (2.9.11). Further, as stated in Theorem 2 above, the equations (2.9.11) and (2.9.17) together imply the identity of matrices Q and $\overline{Q}$. Hence, network N must, in fact, be a realization of matrix Q . It should be noted that the relation in (2.9.12) and the relevant discussion offer a useful guideline for the selection of the diagonal matrix R . Stated explicitly, r_{ss} ($s = b, f, \dots, l$) must be chosen such that $r_{ss}^{-1} > q_{ss}$.

Chapter 3

MATRICES OF THE THIRD ORDER

3.1. Introduction

Tellegen has proved [TE 1] that a matrix of order ≤ 3 is realizable either as a s.c. conductance matrix or as an o.c. resistance matrix if, and only if, the matrix is paramount. He has also given canonical structures of realization in each case. Recently, Cederbaum has shown [CE 4] that every paramount matrix of order three can also be realized with a network which is topologically optimal in accordance with the criteria specified in his paper. The problem of synthesis of resistive networks from matrices of the second and the third order can thus be regarded as, in essence, solved. However, the problem is reinvestigated in this chapter with an entirely fresh approach, which besides being interesting in itself, has the feature of offering two distinct minimal realizations amongst infinitely many continuously equivalent realizations for any third-order paramount matrix considered as either a s.c. conductance matrix or an o.c. resistance matrix. Further, extremely simple computations are involved in the application of the new procedure as is illustrated by means of an example toward the end of the chapter.

We shall begin by establishing certain useful properties of third-order paramount matrices in the following.

3.2. Some Properties of Third-Order Paramount Matrices

Lemma: Let $Q^{(1)} = [q_{ij}^{(1)}]_3$ be a real, symmetric matrix such that the entries $q_{13}^{(1)}$ and $q_{31}^{(1)}$ are negative, the rest of the entries being non-negative. Then by applying a cross-sign change operation to, and/or by interchanging some rows and the corresponding columns of matrix $Q^{(1)}$, it is always possible to obtain a matrix, $Q = [q_{ij}]_3$, such that the relations stated in (3.2.1) and (3.2.2) below hold simultaneously:

$$\left. \begin{aligned} q_{11} q_{23} + q_{12} |q_{13}| &\geq \\ q_{33} q_{12} + q_{23} |q_{13}| &\geq \end{aligned} \right\} \begin{aligned} & \\ & q_{12} q_{23} + q_{22} |q_{13}| \end{aligned} \quad \begin{aligned} (3.2.1) \\ (3.2.2) \end{aligned}$$

Proof: Consider the following four possible cases separately:

Case 1:

$$\left. \begin{aligned} q_{11}^{(1)} q_{23}^{(1)} + q_{12}^{(1)} |q_{13}^{(1)}| &\geq \\ q_{33}^{(1)} q_{12}^{(1)} + q_{23}^{(1)} |q_{13}^{(1)}| &\geq \end{aligned} \right\} \begin{aligned} & \\ & q_{12}^{(1)} q_{23}^{(1)} + q_{22}^{(1)} |q_{13}^{(1)}| \end{aligned} \quad \begin{aligned} (3.2.3) \\ (3.2.4) \end{aligned}$$

Case 2:

$$\left. \begin{aligned} q_{11}^{(1)} q_{23}^{(1)} + q_{12}^{(1)} |q_{13}^{(1)}| &\geq \\ q_{33}^{(1)} q_{12}^{(1)} + q_{23}^{(1)} |q_{13}^{(1)}| &< \end{aligned} \right\} \begin{aligned} & \\ & q_{12}^{(1)} q_{23}^{(1)} + q_{22}^{(1)} |q_{13}^{(1)}| \end{aligned} \quad \begin{aligned} (3.2.5) \\ (3.2.6) \end{aligned}$$

Case 3:

$$\left. \begin{aligned} q_{11}^{(1)} q_{23}^{(1)} + q_{12}^{(1)} |q_{13}^{(1)}| &< \\ q_{33}^{(1)} q_{12}^{(1)} + q_{23}^{(1)} |q_{13}^{(1)}| &\geq \end{aligned} \right\} \begin{aligned} & \\ & q_{12}^{(1)} q_{23}^{(1)} + q_{22}^{(1)} |q_{13}^{(1)}| \end{aligned} \quad \begin{aligned} (3.2.7) \\ (3.2.8) \end{aligned}$$

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Case 4:

$$\left. \begin{aligned} q_{11}^{(1)} q_{23}^{(1)} + q_{12}^{(1)} |q_{13}^{(1)}| &< \\ q_{33}^{(1)} q_{12}^{(1)} + q_{23}^{(1)} |q_{13}^{(1)}| &< \end{aligned} \right\} \begin{aligned} & q_{12}^{(1)} q_{23}^{(1)} + q_{22}^{(1)} |q_{13}^{(1)}| \\ & \end{aligned} \quad \begin{aligned} (3.2.9) \\ (3.2.10) \end{aligned}$$

Case 1: Letting $\mathbf{Q} = \mathbf{Q}^{(1)}$, the relations in (3.2.1) - (3.2.2) follow trivially without the necessity of any cross-sign change operation or of interchanging any rows and columns of matrix $\mathbf{Q}^{(1)}$.

Case 2: With the notations defined in section 1.2, let

$$\mathbf{Q} = \mathbf{U}_s^{(1)} \mathbf{E}_1^{(1,3,2)} \mathbf{Q}^{(1)} \mathbf{E}_1^{(1,3,2)} \mathbf{U}_s^{(1)}$$

i.e.,

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ & & q_{33} \end{bmatrix} = \begin{bmatrix} q_{11}^{(1)} & |q_{13}^{(1)}| & -q_{12}^{(1)} \\ & q_{33}^{(1)} & q_{23}^{(1)} \\ & & q_{22}^{(1)} \end{bmatrix} \quad (3.2.11)$$

Noting that $|q_{13}| = q_{12}^{(1)}$ and applying (3.2.11) to (3.2.5) - (3.2.6), we get:

$$\left. \begin{aligned} q_{11} q_{23} + |q_{13}| q_{12} &\geq \\ q_{22} |q_{13}| + q_{23} q_{12} &< \end{aligned} \right\} \begin{aligned} & |q_{13}| q_{23} + q_{33} q_{12} \\ & \end{aligned} \quad \begin{aligned} (3.2.12) \\ (3.2.13) \end{aligned}$$

The relation in (3.2.13) is the same as that in (3.2.2), while the one in (3.2.1) follows from comparison of the left-hand sides of (3.2.12) and (3.2.13). Clearly, both the inequalities in (3.2.1) and

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(3.2.2) hold as strict inequalities in the present case.

Case 3: Let $Q = U_s^{(3)} E_1^{(2,1,3)} Q^{(1)} E_1^{(2,1,3)} U_s^{(3)}$

i.e.,

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ & & q_{33} \end{bmatrix} = \begin{bmatrix} q_{22}^{(1)} & q_{12}^{(1)} & -q_{23}^{(1)} \\ & q_{11}^{(1)} & |q_{13}^{(1)}| \\ & & q_{33}^{(1)} \end{bmatrix} \quad (3.2.14)$$

Noting that $|q_{13}| = q_{23}^{(1)}$ and applying (3.2.14) to (3.2.7) - (3.2.8),

we get:

$$\left. \begin{aligned} q_{22} |q_{13}| + q_{12} q_{23} &< \end{aligned} \right\} \quad (3.2.15)$$

$$\left. \begin{aligned} q_{33} q_{12} + |q_{13}| q_{23} &\geq \end{aligned} \right\} \begin{aligned} q_{12} |q_{13}| + q_{11} q_{23} \end{aligned} \quad (3.2.16)$$

The relation in (3.2.15) is the same as that in (3.2.1), while the one in (3.2.2) follows from the comparison of the left-hand sides of (3.2.15) and (3.2.16). As in Case 2, both the inequalities in (3.2.1) and (3.2.2) hold as strict inequalities in the present case.

Case 4: Let

$$Q^{(2)} = U_s^{(1)} E_1^{(1,3,2)} Q^{(1)} E_1^{(1,3,2)} U_s^{(1)}$$

i. e.,

$$\begin{bmatrix} q_{11}^{(2)} & q_{12}^{(2)} & q_{13}^{(2)} \\ & q_{22}^{(2)} & q_{23}^{(2)} \\ & & q_{33}^{(2)} \end{bmatrix} = \begin{bmatrix} q_{11}^{(1)} & |q_{13}^{(1)}| & -q_{12}^{(1)} \\ & q_{33}^{(1)} & q_{23}^{(1)} \\ & & q_{22}^{(1)} \end{bmatrix} \quad (3.2.17)$$

Noting that $|q_{13}^{(2)}| = q_{12}^{(1)}$ and applying (3.2.17) to (3.2.9) - (3.2.10), we get:

$$\left. \begin{aligned} q_{11}^{(2)} q_{23}^{(2)} + |q_{13}^{(2)}| q_{12}^{(2)} &< \\ q_{22}^{(2)} |q_{13}^{(2)}| + q_{23}^{(2)} q_{12}^{(2)} &< \end{aligned} \right\} \begin{aligned} &|q_{13}^{(2)}| q_{23}^{(2)} + q_{33}^{(2)} q_{12}^{(2)} \\ &|q_{13}^{(2)}| q_{23}^{(2)} + q_{33}^{(2)} q_{12}^{(2)} \end{aligned} \quad \begin{aligned} (3.2.18) \\ (3.2.19) \end{aligned}$$

Two possible cases will be considered separately:

Case 4(a):

$$q_{11}^{(2)} q_{23}^{(2)} + |q_{13}^{(2)}| q_{12}^{(2)} \geq q_{22}^{(2)} |q_{13}^{(2)}| + q_{23}^{(2)} q_{12}^{(2)} \quad (3.2.20)$$

Case 4(b):

$$q_{11}^{(2)} q_{23}^{(2)} + |q_{13}^{(2)}| q_{12}^{(2)} < q_{22}^{(2)} |q_{13}^{(2)}| + q_{23}^{(2)} q_{12}^{(2)} \quad (3.2.21)$$

Case 4(a): Letting $Q^{(2)} = Q$, we observe that the relation (3.2.20) is the same as that in (3.2.1), while the one in (3.2.19) is the same as that in (3.2.2) holding as a strict inequality.

Case 4(b): Let $Q = U_s^{(3)} E_1^{(2,1,3)} Q^{(2)} E_1^{(2,1,3)} U_s^{(3)}$

Therefore,

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ & & q_{33} \end{bmatrix} = \begin{bmatrix} q_{22}^{(2)} & q_{12}^{(2)} & -q_{23}^{(2)} \\ & q_{11}^{(2)} & |q_{13}^{(2)}| \\ & & q_{33}^{(2)} \end{bmatrix} \quad (3.2.22)$$

Noting that $|q_{13}| = q_{23}^{(2)}$ and applying (3.2.22) to (3.2.19) and (3.2.21), we get:

$$q_{11} q_{23} + |q_{13}| q_{12} < q_{23} |q_{13}| + q_{33} q_{12} \quad (3.2.23)$$

$$q_{22} |q_{13}| + q_{23} q_{12} < q_{11} q_{23} + |q_{13}| q_{12} \quad (3.2.24)$$

The inequality in (3.2.24) is the same as the strict inequality in (3.2.1). Again, the strict inequality in (3.2.2) follows from observing the identity of the left- and the right-hand sides of (3.2.23) and (3.2.24) respectively and then comparing the right- and the left-hand sides of the same two inequalities.

The lemma is proved thus in all the four possible cases.

Before proceeding to the main theorem, we consider one special case where the original matrix, $Q^{(1)}$, is such that

$$q_{11} = q_{12} \quad \text{and/or} \quad q_{33} = q_{23}$$

in the matrix Q which is obtained as indicated in the lemma. In this special case, an inspection of (3.2.1) - (3.2.2) will reveal that $q_{11} = a_{12}$ implies:

$$q_{22} = q_{12} \quad (= q_{11}) \quad (\text{since } |q_{13}| \neq 0) \quad (3.2.25)$$

while $q_{33} = q_{23}$ implies:

$$q_{22} = q_{23} \quad (= q_{33}) \quad (\text{since } |q_{13}| \neq 0) \quad (3.2.26)$$

We shall have occasion to refer to this special case later on.

Theorem: Let $Q^{(1)} = [q_{ij}^{(1)}]_3$ be a paramount matrix such that all its entries $q_{ij}^{(1)}$ are non-negative excepting, possibly, the entries $q_{13}^{(1)}$ and $q_{31}^{(1)}$. Then by applying a suitable cross-sign change operation to, and/or by interchanging some rows and the corresponding columns of matrix $Q^{(1)}$, it is always possible to obtain a matrix, Q , such that a matrix, Y , defined by:

$$Y = Q + (1/x) Q_{.2} Q_2. \quad (3.2.27)$$

assumes a uniformly tapered form for some positive value of the parameter x .

Proof: Let

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (3.2.28)$$

Then, from (3.2.27),

$$TYT = TQT + (1/x)TQ_2Q_2^T$$

$$= \begin{bmatrix} q_{11} - q_{12} + \frac{q_{12}(q_{12} - q_{22})}{x} & q_{12} - q_{13} + \frac{q_{12}(q_{22} - q_{32})}{x} & q_{13} + \frac{q_{12}q_{32}}{x} \\ X & q_{22} - q_{32} - q_{12} + q_{13} + \frac{(q_{22} - q_{12})(q_{22} - q_{32})}{x} & q_{23} - q_{13} + \frac{(q_{22} - q_{12})q_{32}}{x} \\ X & X & q_{33} - q_{32} + \frac{(q_{32} - q_{22})q_{32}}{x} \end{bmatrix}$$

(3.2.29)

Consider the following two possible cases separately:

Case 1: All the entries in matrix $Q^{(1)}$ are non-negative.

Case 2: The entries $q_{13}^{(1)}$ and $q_{31}^{(1)}$ are negative, the rest of the entries being non-negative in matrix $Q^{(1)}$.

Case 1: Let $Q = E_1 Q^{(1)} E_1$ such that $q_{13} = \min_{i,j} (q_{ij})$

Then

$$q_{11} \geq q_{12} \geq q_{13} \leq q_{23} \leq q_{33}$$

and

$$q_{12} \leq q_{22} \geq q_{23}, \quad (3.2.30)$$

so that:

$$\begin{aligned} q_{11} - q_{12} \geq 0, \quad q_{12} - q_{13} \geq 0, \quad q_{23} - q_{13} \geq 0 \\ q_{33} - q_{23} \geq 0, \quad q_{22} - q_{12} \geq 0, \text{ and } q_{22} - q_{23} \geq 0 \end{aligned} \quad (3.2.31)$$

Consider two further possible cases separately:

$$\text{Case 1(a): } q_{22} - q_{32} - q_{12} + q_{13} \geq 0 \quad (3.2.32)$$

$$\text{Case 1(b): } q_{22} - q_{32} - q_{12} + q_{13} < 0 \quad (3.2.33)$$

Case 1(a): In this case, we observe incidentally that matrix Q itself is in the uniformly tapered form.

In view of (3.2.31)-(3.2.32), the entries in positions (1, 2), (1, 3), (2, 2), and (2, 3) of the matrix TYT in (3.2.29) are non-negative for every positive value of x . Thus, matrix Y is uniformly tapered if:

$$\left. \begin{array}{l} \text{from the entry} \\ \text{in position (1, 1)} \\ \text{in (3.2.29)} \end{array} \right\} \quad \begin{array}{l} q_{11} - q_{12} + (1/x) q_{12} (q_{12} - q_{22}) \geq 0 \\ \text{or} \quad x \geq \frac{q_{12} (q_{22} - q_{12})}{q_{11} - q_{12}} \end{array} \quad (3.2.34)$$

$$\left. \begin{array}{l} \text{from the entry} \\ \text{in position (3, 3)} \\ \text{in (3.2.29)} \end{array} \right\} \quad \begin{array}{l} q_{33} - q_{32} + (1/x) (q_{32} - q_{22}) q_{32} \geq 0 \\ \text{or} \quad x \geq \frac{(q_{22} - q_{32}) q_{32}}{q_{33} - q_{32}} \end{array} \quad (3.2.35)$$

It is evident that a positive parameter x can always be chosen such that the conditions in both (3.2.34) and (3.2.35) are satisfied. ($x \rightarrow \infty$ as $q_{11} - q_{12} \rightarrow 0$ and/or $q_{33} - q_{32} \rightarrow 0$.)

Case 1(b): In view of (3.2.31), the entries in positions (1, 2), (1, 3), and (2, 3) of the matrix TYT in (3.2.29) are always non-negative for every positive value of x . Thus, matrix Y is in the uniformly tapered form if a positive value can be assigned to x such that the entries in positions (1, 1), (3, 3), and (2, 2) in (3.2.29) are non-negative. Consideration of the entries in positions (1, 1) and (3, 3) implies the same constraints on x as stated in (3.2.34) and

and (3.2.35), while from the entry in position (2, 2), we have:

$$q_{22} - q_{32} - q_{12} + q_{13} + (1/x)(q_{22} - q_{12})(q_{22} - q_{32}) \geq 0$$

or

$$\frac{(q_{22} - q_{12})(q_{22} - q_{32})}{-q_{22} + q_{32} + q_{12} - q_{13}} \geq x \quad (3.2.36)$$

In order to establish the compatibility of this upper bound on x with the lower bounds in (3.2.34) - (3.2.35), we observe that the paramountcy of the original matrix, $Q^{(1)}$, implies the paramountcy of matrix Q so that:

$$q_{11} q_{22} - q_{12}^2 \geq q_{11} q_{23} - q_{12} q_{13} \quad (3.2.37)$$

and

$$q_{22} q_{33} - q_{23}^2 \geq q_{33} q_{12} - q_{32} q_{13} \quad (3.2.38)$$

Adding the quantity $(-q_{22} q_{12} + q_{32} q_{12} - q_{32} q_{11} + q_{12}^2)$ to both sides of (3.2.37), we get:

$$q_{11} q_{22} - q_{22} q_{12} + q_{32} q_{12} - q_{32} q_{11} \geq -q_{22} q_{12} + q_{32} q_{12} + q_{12}^2 - q_{12} q_{13}$$

i. e.,

$$(q_{22} - q_{32})(q_{11} - q_{12}) \geq (-q_{22} + q_{32} + q_{12} - q_{13})q_{12} \quad (3.2.39)$$

Note that, in view of (3.2.37), the hypothesis $q_{22} - q_{23} - q_{12} - q_{13} < 0$ implies $q_{11} \neq q_{12}$; for if $q_{11} = q_{12}$, then from (3.2.37),

$$q_{11} (q_{22} - q_{12}) \geq q_{11} (q_{23} - q_{13}).$$

Because of paramountcy, q_{11} must be positive and hence

$$q_{22} - q_{12} - q_{23} + q_{13} \geq 0.$$

From (3.2.39), then,

$$\frac{q_{22} - q_{32}}{-q_{22} + q_{32} + q_{12} - q_{13}} \geq \frac{q_{12}}{q_{11} - q_{12}}$$

Multiplying both sides of this inequality by the non-negative quantity

$(q_{22} - q_{12})$, we have:

$$\frac{(q_{22} - q_{12})(q_{22} - q_{32})}{-q_{22} + q_{32} + q_{12} - q_{13}} \geq \frac{q_{12}(q_{22} - q_{12})}{q_{11} - q_{12}} \quad (3.2.40)$$

Adding the quantity $(-q_{12}q_{33} - q_{22}q_{32} + q_{12}q_{32} + q_{32}^2)$ to both sides of (3.2.38), we get:

$$q_{22}q_{33} - q_{12}q_{33} - q_{22}q_{32} + q_{12}q_{32} \geq -q_{22}q_{32} + q_{32}^2 + q_{12}q_{32} - q_{32}q_{13}$$

i. e.,

$$(q_{22} - q_{12})(q_{33} - q_{32}) \geq (-q_{22} + q_{32} + q_{12} - q_{13}) q_{32} \quad (3.2.41)$$

Note that, in view of (3.2.38), the hypothesis $q_{22} - q_{23} - q_{12} + q_{13} < 0$ implies $q_{33} \neq q_{32}$; for if $q_{33} = q_{32}$, then from (3.2.38),

$$q_{33}(q_{22} - q_{23}) \geq q_{33}(q_{12} - q_{13}).$$

Because of paramountcy, q_{33} must be positive and hence

$$q_{22} - q_{23} - q_{12} + q_{13} \geq 0.$$

From (3.2.41), then,

$$\frac{q_{22} - q_{12}}{-q_{22} + q_{32} + q_{12} - q_{13}} \geq \frac{q_{32}}{q_{33} - q_{32}}$$

Multiplying both sides of this inequality by the non-negative quantity

$$(q_{22} - q_{32}),$$

$$\frac{(q_{22} - q_{12})(q_{22} - q_{32})}{-q_{22} + q_{32} + q_{12} - q_{13}} \geq \frac{(q_{22} - q_{32}) q_{32}}{q_{33} - q_{32}} \quad (3.2.42)$$

The relations in (3.2.40) and (3.2.42) establish the compatibility of the upper bound on x in (3.2.36) with the lower bounds on x in (3.2.34) - (3.2.35). This proves the theorem in the present case, 1(b).

Case 2: Let Q denote the matrix obtained from $Q^{(1)}$ such that the inequalities in (3.2.1) and (3.2.2) hold simultaneously (cf. Lemma). We observe that, as in matrix $Q^{(1)}$, the entries in positions (1, 3) and (3, 1) of matrix Q are negative, the rest of the entries being non-negative. This can be readily ascertained by inspecting the relations in (3.2.11), (3.2.14), (3.2.17), and (3.2.22). Therefore, we have:

$$q_{11} \geq q_{12} \geq q_{13} \leq q_{23} \leq q_{33}$$

and

$$(3.2.43)$$

$$q_{12} \leq q_{22} \geq q_{33},$$

so that:

$$q_{11} - q_{12} \geq 0, \quad q_{12} - q_{13} \geq 0, \quad q_{23} - q_{13} \geq 0 \quad (3.2.44)$$

$$q_{33} - q_{23} \geq 0, \quad q_{22} - q_{12} \geq 0, \quad q_{22} - q_{23} \geq 0$$

In view of (3.2.44), the entries in positions (1, 2) and (2, 3) of the matrix TYT in (3.2.29) are always non-negative for every positive value of x . Thus, matrix Y is in the uniformly tapered

form if a positive value can be assigned to x such that the entries in positions (1, 1), (2, 2), (3, 3), and (1, 3) of TYT are non-negative. In fact, the entry in position (2, 2) is non-negative for every positive value of x whenever $q_{22} - q_{32} - q_{12} + q_{13} \geq 0$. Only if $q_{22} - q_{32} - q_{12} + q_{13} < 0$, the entry in position (2, 2) yields the bound stated in (3.2.36). The entries in positions (1, 1) and (3, 3) respectively imply the bounds stated in (3.2.34) and (3.2.35) provided $q_{11} \neq q_{12}$ and $q_{33} \neq q_{23}$. If $q_{11} = q_{12}$, then through (3.2.25), $q_{22} = q_{12} (= q_{11})$; for, by hypothesis, q_{13} is negative; i.e., $|q_{13}| \neq 0$. Now, an inspection of (3.2.34) will reveal that the same relation does hold as an equality for all positive values of x . Again, if $q_{33} = q_{23}$, then through (3.2.26), $q_{22} = q_{23} (= q_{33})$. An inspection of (3.2.35) will reveal that the same relation does hold as an equality for all positive values of x .

Consideration of the entry in position (1, 3) of TYT in (3.2.29) yields the following constraint:

$$- |q_{13}| + \frac{q_{12} q_{32}}{x} \geq 0$$

or

$$\frac{q_{12} q_{32}}{|q_{13}|} \geq x \quad (3.2.45)$$

Matrix Y can be made uniformly tapered if, and only if, a positive value can be assigned to x such that the upper bounds on x as given by (3.2.36) and (3.2.45) are compatible with the lower bounds given by (3.2.34) and (3.2.35). Inasmuch as the

proof for case 1(b) is independent of the sign of q_{13} , it applies directly to the present case, proving thus the compatibility of the bound in (3.2.36) with those in (3.2.34) and (3.2.35).

Consider, now, the bound in (3.2.45) in relation with the bounds in (3.2.34) and (3.2.35). If either $q_{11} = q_{12}$ or $q_{33} = q_{23}$, then, in view of the earlier discussion, the compatibility of the upper bound in (3.2.45) with the lower bounds in (3.2.34) and (3.2.35), respectively, is established trivially. In the following discussion, we shall assume, therefore, $q_{11} \neq q_{12}$ and $q_{33} \neq q_{23}$.

Rearranging (3.2.1) - (3.2.2), we have:

$$q_{11} q_{32} - q_{12} q_{32} \geq q_{22} |q_{13}| - q_{12} |q_{13}| \quad (3.2.46)$$

$$q_{33} q_{12} - q_{12} q_{32} \geq q_{22} |q_{13}| - q_{32} |q_{13}| \quad (3.2.47)$$

From (3.2.46),

$$(q_{11} - q_{12}) q_{32} \geq (q_{22} - q_{12}) |q_{13}|$$

or

$$\frac{q_{32}}{|q_{13}|} \geq \frac{q_{22} - q_{12}}{q_{11} - q_{12}}$$

Multiplying both sides of this inequality by the non-negative factor q_{12} , we get

$$\frac{q_{12} q_{32}}{|q_{13}|} \geq \frac{q_{12} (q_{22} - q_{12})}{q_{11} - q_{12}} \quad (3.2.48)$$

From (3.2.47),

$$(q_{33} - q_{32}) q_{12} \geq (q_{22} - q_{32}) |q_{13}|$$

or

$$\frac{q_{12}}{|q_{13}|} \geq \frac{q_{22} - q_{32}}{q_{33} - q_{32}}$$

Multiplying both sides of this inequality by the non-negative factor q_{32} , we get:

$$\frac{q_{12} q_{32}}{|q_{13}|} \geq \frac{(q_{22} - q_{32}) q_{32}}{q_{33} - q_{32}} \quad (3.2.49)$$

The relations in (3.2.48) and (3.2.49) establish the compatibility of the upper bound on x in (3.2.45) with the lower bounds on x in (3.2.34) - (3.2.35). This proves the theorem in the present case, 2.

3.3. Realization of Third-Order Paramount Matrices

We shall establish first a new proof for the fact that the property of paramountcy is, indeed, sufficient for realizing a third-order matrix as either a s.c. conductance matrix or an o.c. resistance matrix.

Without loss of generality, let the zero entries in a matrix, if any, be regarded as positive. Then, we observe that any third-order symmetric matrix must have one of the eight sign-patterns shown below:

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$$\begin{array}{cccc}
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 (1) & (2) & (3) & (4) \\
 \begin{bmatrix} + & - & - \\ & + & + \\ & & + \end{bmatrix} & \begin{bmatrix} - & - & + \\ & + & - \\ & & + \end{bmatrix} & \begin{bmatrix} + & + & - \\ & + & - \\ & & + \end{bmatrix} & \begin{bmatrix} + & - & - \\ & + & - \\ & & + \end{bmatrix} \\
 (5) & (6) & (7) & (8)
 \end{array} \tag{3.3.1}$$

Let the sign-patterns numbered (1), (5), (6), and (7) be considered to belong to a group, I, and the rest of the sign-patterns in (3.3.1), to another group, II. Now, if a matrix, $Q^{(2)}$, has a sign-pattern which belongs to Group I (Group II), then it is always possible to reduce the problem of realization of that matrix $Q^{(2)}$ to the problem of realization of a matrix, $Q^{(1)}$, whose sign-pattern is of type (1) (type (2)) and whose every entry has the absolute value equal to that of the corresponding entry in matrix $Q^{(2)}$. This follows from the fact that any sign-pattern belonging to Group I (Group II) can be converted to that of type (1) (type (2)) by a proper cross-sign change operation; and we have already seen that a cross-sign change operation has the electrical equivalence of port-reorientations.

It is sufficient, therefore, to consider third-order matrices which have the sign-patterns of types (1) and (2) only. Let $Q^{(1)}$ denote a third-order paramount matrix of either type (1) or (2). In the previous section it was seen that by applying a suitable cross-

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sign change operation to, and/or by interchanging some rows and the corresponding columns of matrix $Q^{(1)}$, it is always possible to obtain a matrix, Q , such that a matrix, Y , defined by

$$Y = Q + (1/x) Q_{.2} Q_2. \quad (3.3.2)$$

assumes a uniformly tapered form for some positive value of the parameter x .

Let, now, the uniformly tapered matrix Y be realized as a s.c. conductance matrix with four nodes (Figure 3.3.1). Let a new network be derived from this realization of matrix Y by adding, in the manner shown in Figure 3.3.2, a resistor which has its conductance-value = g_o .

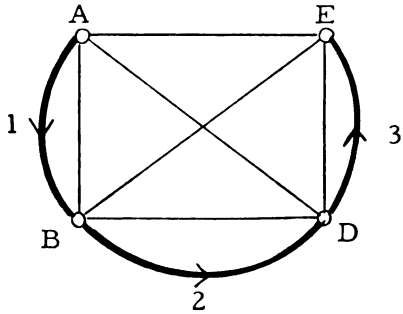


Figure 3.3.1

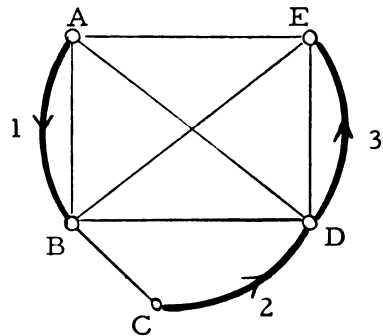


Figure 3.3.2

If $\bar{Q}$ denotes the s.c. conductance matrix for the derived network, matrices Y and $\bar{Q}$ must be related by:

$$Y = \bar{Q} + (1/\bar{x}) \bar{Q}_{.2} \bar{Q}_2, \quad (\bar{x} = g_o - \bar{q}_{22} > 0) \quad (3.3.3)$$

(cf. the discussion on pp.27-28). Further, through the theorem proved in section 2.3, the relations in (3.3.2) and (3.3.3) together

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must imply the identity of matrices Q and $\overline{Q}$. Hence the network in Figure 3.3.2 must have been a realization of matrix Q itself considered as the s.c. conductance matrix, realization of the original matrix $Q^{(2)}$ following immediately. This establishes the fact that the property of paramountcy is, indeed, sufficient for realizing a third-order matrix considered as the s.c. conductance matrix.

Note that the networks in Figures 3.3.1 and 3.3.2 are planar; as such, their duals are existent [WH1]. These dual networks are shown in Figures 3.3.3 and 3.3.4 respectively. Now, it is established above that every third-order paramount matrix can be realized as a s.c. conductance matrix in the manner shown in Figure 3.3.2, with appropriate re-numbering and/or re-orienting of some ports. It is obvious, then, that every third-order paramount matrix can also be realized as an o.c. resistance matrix in the manner shown in Figure 3.3.4, with appropriate re-numbering and/or re-orienting of some ports.

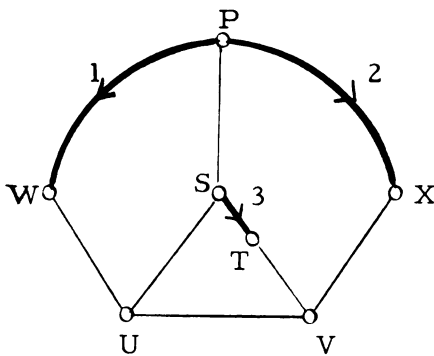


Figure 3.3.3

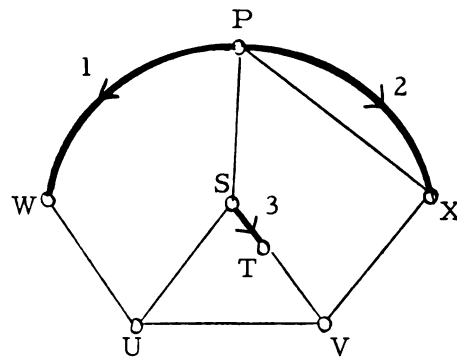


Figure 3.3.4

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The earlier discussions, including those in the previous section, indicate the new procedure for realizing any third-order paramount matrix. The procedure is stated below explicitly in four steps:

1. Let $Q^{(2)}$ denote a third-order paramount matrix to be realized as either a s.c. conductance matrix or an o.c. resistance matrix.

Apply, if necessary, a suitable cross-sign change operation to matrix $Q^{(2)}$ so as to obtain a matrix, $Q^{(1)}$, which has the sign-pattern either of type (1) or type (2) indicated below:

$$\begin{array}{cc} \left[\begin{array}{ccc} + & + & + \\ & + & + \\ & & + \end{array} \right] & \left[\begin{array}{ccc} + & + & - \\ & + & + \\ & & + \end{array} \right] \\ (1) & (2) \end{array} \quad (3.3.4)$$

(cf. 3.3.1). This can always be done easily. If the sign-pattern of matrix $Q^{(1)}$ belongs to type (2), proceed to step 2(b) below.

2(a). Interchange suitable rows and the corresponding columns of matrix $Q^{(1)}$ so as to obtain a matrix, Q , which has the smallest entry in its position (1, 3). Proceed to step 3 below.

2(b). Compute the quantities $A = q_{11}^{(1)} q_{23}^{(1)} + q_{12}^{(1)} |q_{13}^{(1)}|$, $B = q_{33}^{(1)} q_{12}^{(1)} + q_{23}^{(1)} |q_{13}^{(1)}|$, and $C = q_{12}^{(1)} q_{23}^{(1)} + q_{22}^{(1)} |q_{13}^{(1)}|$.

(i) If $\left. \begin{array}{l} A \geq \\ B \geq \end{array} \right\} C$, let $Q = Q^{(1)}$ and proceed to step 3.

(ii) If $\left. \begin{array}{l} A \geq \\ B < \end{array} \right\} C$, let $Q = U_s^{(1)} E_1^{(1,3,2)} Q^{(1)} E_1^{(1,3,2)} U_s^{(1)}$ and proceed to step 3.

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(iii) If $\left. \begin{array}{l} A < \\ B \geq \end{array} \right\} C$, let $Q = U_s^{(3)} E_1^{(2,1,3)} Q^{(1)} E_1^{(2,1,3)} U_s^{(3)}$
and proceed to step 3.

(iv) If $\left. \begin{array}{l} A < \\ B < \end{array} \right\} C$, let $\hat{Q} = U_s^{(1)} E_1^{(1,3,2)} Q^{(1)} E_1^{(1,3,2)} U_s^{(1)}$
and compute:

$$D = \hat{q}_{11} \hat{q}_{23} + |\hat{q}_{13}| \hat{q}_{12}$$

$$\text{and } E = \hat{q}_{22} |\hat{q}_{13}| + \hat{q}_{23} \hat{q}_{12} .$$

(iva) If $D \geq E$, let $Q = \hat{Q}$ and proceed to step 3.

(ivb) If $D < E$, let $Q = U_s^{(3)} E_1^{(2,1,3)} \hat{Q} E_1^{(2,1,3)} U_s^{(3)}$
and proceed to step 3.

3. Let $Y = Q + (g_o - q_{22})^{-1} Q_{.2} Q_{2.}$. Select any one value of $g_o > q_{22}$ such that matrix Y assumes a uniformly tapered form. (This is always possible.)

4. Realize the uniformly tapered matrix Y as shown in Figure 3.3.1 (Figure 3.3.3). From this realization of Y , obtain the realization of matrix Q as shown in Figure 3.3.2 (Figure 3.3.4). Realization of the original matrix, $Q^{(2)}$, as the s.c. conductance matrix (o.c. resistance matrix) will follow when cross-sign change operations and/or interchanges of the rows and the corresponding columns carried out in the earlier steps are taken into consideration for the purpose of assigning proper numberings and orientations to the ports.

3.4. Example

Consider the realization of the paramount matrix

$$Q^{(2)} = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 12 & 3 \\ 1 & 3 & 5 \end{bmatrix} \quad (3.4.1)$$

both as a s.c. conductance matrix and an o.c. resistance matrix by the procedure established in the previous section.

1. We observe that reversing the signs of the first row and the first column results in a sign-pattern of type (2) defined in (3.3.4). Let, therefore, $Q^{(1)} = U_s^{(1)} Q^{(2)} U_s^{(1)}$; i.e.,

$$Q^{(1)} = \begin{bmatrix} 7 & 2 & -1 \\ 2 & 12 & 3 \\ -1 & 3 & 5 \end{bmatrix} \quad (3.4.2)$$

$$\begin{aligned} 2(b). \quad A &= (7)(3) + (2)(1) = 23 \\ B &= (5)(2) + (3)(1) = 13 \\ C &= (2)(3) + (12)(1) = 18 \end{aligned}$$

Since $A > C > B$, let

$$\begin{aligned} Q &= U_s^{(1)} E_1^{(1,3,2)} Q^{(1)} E_1^{(1,3,2)} U_s^{(1)} \\ &= \begin{bmatrix} 7 & 1 & -2 \\ 1 & 5 & 3 \\ -2 & 3 & 12 \end{bmatrix} \end{aligned} \quad (3.4.3)$$

$$3. \quad Y = Q + (g_o - q_{22})^{-1} Q_{.2} Q_{2.} \quad (3.4.4)$$

In order to obtain a parameter $g_o > q_{22}$ such that this matrix Y assumes a uniformly tapered form, pre- and post-multiply the above relation by:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (3.4.5)$$

so that

$$\begin{aligned} TYT &= TQT + (g_o - q_{22})^{-1} TQ_{.2}Q_{2.}T \\ &= \begin{bmatrix} q_{11} - q_{12} & q_{12} - q_{13} & q_{13} \\ x & q_{22} - q_{23} - q_{12} + q_{13} & q_{23} - q_{13} \\ x & x & q_{33} - q_{23} \end{bmatrix} + \frac{1}{(g_o - q_{22})} \begin{bmatrix} q_{12} \\ q_{22} - q_{12} \\ q_{32} - q_{22} \end{bmatrix} \begin{bmatrix} q_{12} & q_{22} & q_{32} \\ -q_{22} & -q_{32} & \end{bmatrix} \end{aligned} \quad (3.4.6)$$

$$= \begin{bmatrix} 6 & 3 & -2 \\ x & -1 & 5 \\ x & x & 9 \end{bmatrix} + \frac{1}{(g_o - 5)} \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & -2 \\ x & -1 & 5 \\ x & x & 9 \end{bmatrix} + \frac{1}{(g_o - 5)} \begin{bmatrix} -4 & 2 & 3 \\ x & 8 & 12 \\ x & x & -6 \end{bmatrix} \quad (3.4.7)$$

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$$\begin{aligned}
\left[\begin{array}{l} \text{From the entry} \\ \text{in position (1,1)} \end{array} \right] : & \quad (g_o - 5) \geq \frac{4}{6} \quad , \quad \text{i.e., } g_o \geq 5\frac{2}{3} \\
\left[\begin{array}{l} \text{From the entry} \\ \text{in position (3,3)} \end{array} \right] : & \quad (g_o - 5) \geq \frac{6}{9} \quad , \quad \text{i.e., } g_o \geq 5\frac{2}{3} \\
\left[\begin{array}{l} \text{From the entry} \\ \text{in position (1,3)} \end{array} \right] : & \quad (g_o - 5) \leq \frac{3}{2} \quad , \quad \text{i.e., } g_o \leq 6\frac{1}{2} \\
\left[\begin{array}{l} \text{From the entry} \\ \text{in position (2,2)} \end{array} \right] : & \quad (g_o - 5) \leq \frac{8}{1} \quad , \quad \text{i.e., } g_o \leq 13
\end{aligned}$$

It follows, therefore, that matrix Y in (3.4.4) will assume a uniformly tapered form if g_o is assigned any value within the lower and upper bounds of $5\frac{2}{3}$ and $6\frac{1}{2}$ respectively. We shall assign the limiting values to g_o and obtain two distinct minimal realizations of matrix $Q^{(2)}$ considered both as a s.c. conductance matrix and an o.c. resistance matrix. Figures 3.4.1 and 3.4.2 depict the realizations when $Q^{(2)}$ is considered as a s.c. conductance matrix. Figures 3.4.3 and 3.4.4 depict the realizations when $Q^{(2)}$ is considered as an o.c. resistance matrix.

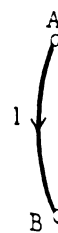
For $g_o = 5\frac{2}{3}$, (3.4.7) yields:

$$TYT = \begin{bmatrix} 0 & 6 & 2\frac{1}{2} \\ x & 11 & 23 \\ x & x & 0 \end{bmatrix} \quad (3.4.8)$$

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For $g_0 = 6\frac{1}{2}$, (3.4.7) yields:

$$TYT = \begin{bmatrix} 3\frac{1}{3} & 4\frac{1}{3} & 0 \\ x & 4\frac{1}{3} & 13 \\ x & x & 5 \end{bmatrix} \quad (3.4.9)$$

For assigning the proper numberings and orientations to the ports, we have only to observe that:

$$\begin{aligned} Q &= U_s^{(1)} E_1^{(1,3,2)} Q^{(1)} E_1^{(1,3,2)} U_s^{(1)} \\ &= U_s^{(1)} E_1^{(1,3,2)} U_s^{(1)} Q^{(2)} U_s^{(1)} E_1^{(1,3,2)} U_s^{(1)} \end{aligned}$$

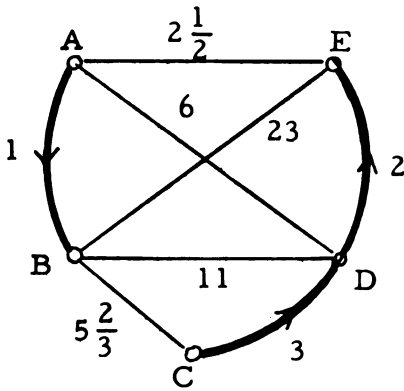


Figure 3.4.1

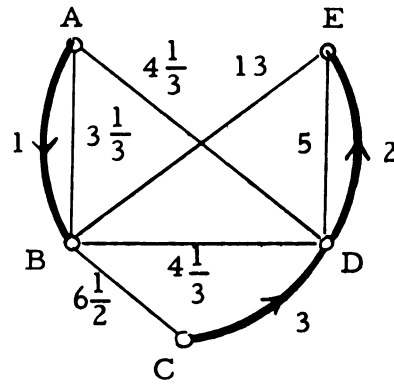


Figure 3.4.2

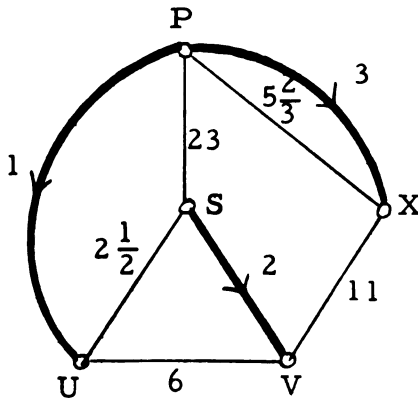


Figure 3.4.3

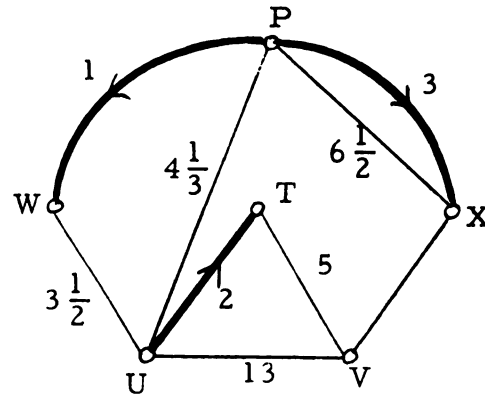


Figure 3.4.4

Chapter 4

CONCLUSION

A complete solution is presented in this thesis to the problem of synthesis of n -port resistive networks from short circuit conductance matrices which are realizable with two-tree port-structures. The fact that the conductance values of $(n+1)$ constituent resistors themselves are considered as parameters provides excellent control over the maximum number of resistors which constitute the network. This is directly illustrated by the attainment of "minimal" realizations in the case of a certain class of matrices described in section 2.7 of the thesis.

It is possible that certain n th-order short circuit conductance matrices may be realizable only with k -tree port-structures ($3 \leq k \leq n$). It would be an interesting problem for further research to investigate whether the "conductance-parameter" approach presented in the thesis can be extended to the realization of these matrices.

APPENDIX 1

An interesting "special case" was discussed on pp. 27-28. If matrices $Y = [y_{ij}]_n$ and $\overline{Q} = [\overline{q}_{ij}]_n$ as considered there are non-singular, then we can observe that the above special case can also be regarded as a special case of the considerations on p. 82. As a consequence, matrices Y and $\overline{Q}$ must be related by:

$$Y = (\overline{Q}^{-1} - R)^{-1} \quad (\text{A.1.1})$$

where matrix $R = [r_{ij}]_n$ is defined by:

$$r_{ij} \begin{cases} > 0 & \text{for } i = j = h \quad h \text{ constant} < n. \\ = 0 & \text{otherwise} \end{cases}$$

That the relation in (2.2.37) is, in fact, an alternate form of the above relation, (A.1.1), can be easily proved by applying Theorem 1 on p. 79. However, we present below an interesting direct proof for the indicated fact by exploiting the theory of functions of matrices:

Proof: The relation in (A.1.1) can be written as:

$$\begin{aligned} Y &= (\overline{Q}^{-1} [U - \overline{Q} R])^{-1} \\ &= (U - \overline{Q} R)^{-1} \overline{Q} \\ &= (U - B)^{-1} \overline{Q} \end{aligned} \quad (\text{A.1.2})$$

where

$$B = \overline{Q}R = r_{hh} \begin{bmatrix} 0 \dots 0 & \overline{q}_{1h} & 0 \dots 0 \\ 0 \dots 0 & \overline{q}_{2h} & 0 \dots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 \dots 0 & \overline{q}_{nh} & 0 \dots 0 \end{bmatrix} \quad (\text{A.1.3})$$

$$\text{Let} \quad \Phi(B) = (U - B)^{-1} \quad (\text{A.1.4})$$

so that the relation in (A.1.2) assumes the form:

$$Y = \Phi(B) \overline{Q} \quad (\text{A.1.5})$$

Note that (n-1) eigenvalues of matrix B are zero, the n-th eigenvalue being $= r_{hh} q_{hh}$.

$$\text{Let} \quad t = r_{hh} \overline{q}_{hh}.$$

By definition, the minimal polynomial of matrix $B = m(\lambda) = |\lambda U - B|/p(\lambda)$ where $p(\lambda)$ denotes the highest common factor amongst all the entries of the adjoint of matrix $(\lambda U - B)$ [KO 1]. In the present case, it can be easily seen that $p(\lambda) = \lambda^{n-2}$. Therefore,

$$m(\lambda) = \lambda^{n-1} (\lambda - t)/\lambda^{n-2} = \lambda (\lambda - t)$$

Hence the Lagrange-Sylvester interpolation polynomial [KO 1] for

$$\begin{aligned} \Phi(\lambda) \text{ on the spectrum of matrix } B &= h(\lambda) = \frac{\lambda - t}{-t} \cdot \Phi(0) + \frac{\lambda}{t} \cdot \Phi(t) \\ &= \frac{\lambda - t}{-t} \cdot 1 + \frac{\lambda}{t} \cdot (1 - t)^{-1} \\ &= \frac{\lambda}{-t} + 1 + \frac{\lambda}{t(1-t)} \\ &= 1 + (1 - t)^{-1} \lambda \end{aligned}$$

Therefore,

$$h(B) = \underline{\Phi}(B) = U + (1 - t)^{-1} B$$

so that, from (A.1.2) - (A.1.5),

$$\begin{aligned} Y &= [U + (1 - t)^{-1} B] \bar{Q} \\ &= \bar{Q} + (1 - t)^{-1} B \bar{Q} \\ &= \bar{Q} + (1 - r_{hh} \bar{q}_{hh})^{-1} r_{hh} \begin{bmatrix} 0 \dots 0 & \bar{q}_{1h} & 0 \dots 0 \\ 0 \dots 0 & \bar{q}_{2h} & 0 \dots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 \dots 0 & \bar{q}_{nh} & 0 \dots 0 \end{bmatrix} \bar{Q} \\ &= \bar{Q} + (1/r_{hh} - \bar{q}_{hh})^{-1} \bar{Q}_{.h} \bar{Q}_h. \end{aligned} \quad (A.1.6)$$

Since $g_o = 1/r_{hh}$ and $g_o - \bar{q}_{hh} = \bar{x}$, the relation in (A.1.6) is, in fact, identical to the one in (2.2.37). Q. E. D.

APPENDIX 2

It is seen that the conductance-parameter procedure established in section 2.4 has, at its basis, the relation (2.2.35). Several similar relations can be discovered so as to build the procedure upon them. The derivations of these relations are fundamentally of the same nature as in section 2.2. We shall state these relations directly in the following with the purpose of making them available for ready reference.

- 1 -

Ref.: Figures A.2.1 and A.2.2.

$$Y = \overline{Q} - A + \left(\sum_{k=1}^{n+1} g_k - \overline{q}_{h-1, h-1} \right)^{-1} \overline{Q}'_{\cdot, h-1} \overline{Q}'_{h-1}.$$

where

$$\left[\overline{Q}'_{h-1} \right]^T = \overline{Q}'_{\cdot, h-1} = \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{il}^{(1)}] \quad \text{and} \quad Q_2 = [q_{il}^{(2)}] \quad \text{with}$$

$$q_{il}^{(1)} = \overline{q}_{i, h-1} - \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1)$$

$$q_{il}^{(2)} = \overline{q}_{i, h-1} + \sum_{k=1}^{n-i+1} g_{h+k-1} \quad (i = h, h+1, \dots, n)$$

Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$\begin{aligned}
 a_{ij} &= 0 & (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n) \\
 a_{ij} &= \sum_{k=1}^{n-j+1} g_{h+k-1} & (j = h, h+1, \dots, n; i = h, h+1, \dots, j)
 \end{aligned}$$

- 2 -

Ref.: Figures A.2.3 - A.2.5.

Let matrix $Y = [y_{ij}]_n$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same resistive network which is shown in Figure A.2.3, the port-numberings being altered as indicated in Figure A.2.4. Matrix Y is uniformly tapered and $Y = E_1 \bar{Y} E_1$, where the exact nature of matrix E_1 can be readily decided by comparing the port-structures associated with matrices Y and $\bar{Y}$. Then

$$Y = E_1 \bar{Q} E_1 - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{11} \right)^{-1} \bar{Q}'_{.1} \bar{Q}'_1.$$

where

$$[\bar{Q}'_1]^T = \bar{Q}'_{.1} = \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ (n-h+1) \end{matrix} \quad (1)$$

$$Q_1 = [q_{il}^{(1)}] \quad \text{and} \quad Q_2 = [q_{il}^{(2)}] \quad \text{with}$$

$$q_{il}^{(1)} = \bar{q}_{h-i, 1} - \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1)$$

$$q_{il}^{(2)} = \bar{q}_{n+h-i, 1} + \sum_{k=1}^{n-i+1} g_{h+k-1} \quad (i = h, h+1, \dots, n)$$

Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{n-j+1} g_{h+k-1} \quad (j = h, h+1, \dots, n; i = h, h+1, \dots, j)$$

- 3 -

Ref.: Figures A.2.6 - A.2.8.

Let matrix $Y = [y_{ij}]_n$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same resistive network which is shown in Figure A.2.6, the port-numberings being altered as indicated in Figure A.2.7. Matrix Y is uniformly tapered and $Y = E_1 \bar{Y} E_1$, where the exact nature of matrix E_1 can be readily decided by comparing the port-structures associated with matrices Y and $\bar{Y}$. Then

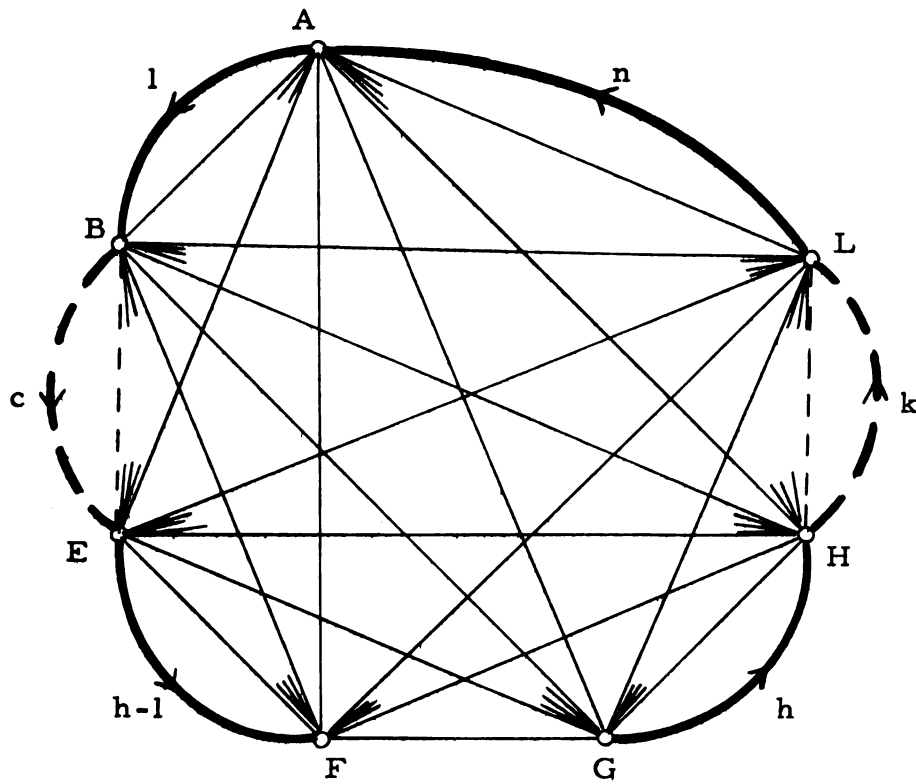
$$Y = E_1 \bar{Q} E_1 - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{nn} \right)^{-1} \bar{Q}'_{.n} \bar{Q}'_n.$$

where

$$[\bar{Q}'_n]^T = \bar{Q}'_{.n} = \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{i1}^{(1)}] \quad \text{and} \quad Q_2 = [q_{i1}^{(2)}] \quad \text{with}$$

$$q_{i1}^{(1)} = \bar{q}_{h-i, n} + \sum_{k=1}^i g_{n-h+1+k} \quad (i = 1, 2, \dots, h-1)$$



$$\bar{Y} = [\bar{y}_{ij}]_n$$

Figure A.2.6

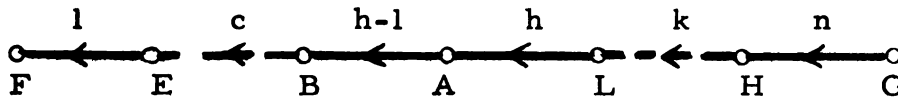
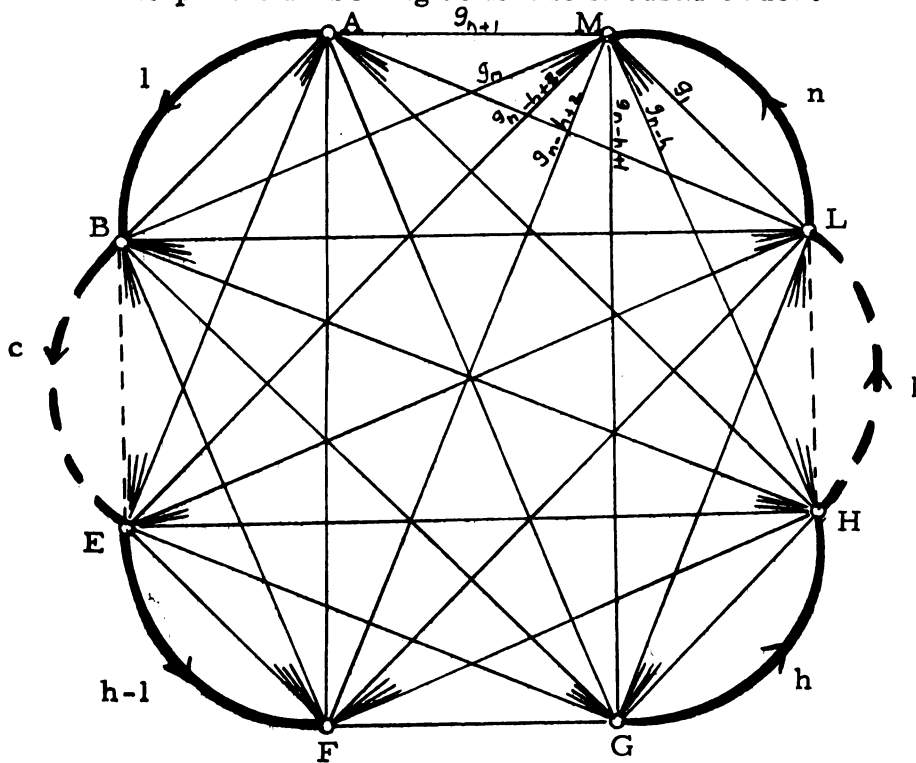


Figure A.2.7

The port-numbering is in the natural order.



$$\bar{Q} = [\bar{q}_{ij}]_n$$

Figure A.2.8

$$q_{il}^{(2)} = \bar{q}_{n+h-i, n} - \sum_{k=1}^{n-i+1} g_{n-h-k+2} \quad (i = h, h+1, \dots, n)$$

Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{n-h+1+k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{n-j+1} g_{n-h-k+2} \quad (j = h, h+1, \dots, n; i = h, h+1, \dots, j)$$

- 4 -

Ref.: Figures A.2.9 - A.2.13.

The derivation of the relation stated in this section incorporates some ideas that are somewhat different from those used in section 2.2. These ideas will be applicable even in deriving the relations stated in the following three sections, and have been described in sufficient details below.

Let $\bar{Y} = [\bar{y}_{ij}]_n$ denote the s.c. conductance matrix for the network shown in Figure A.2.9. Let matrix $Y = U_s E_1 \bar{Y} E_1 U_s$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same network after the port-numberings and port-orientations are altered as indicated in Figure A.1.10. Matrix Y is uniformly tapered and the exact forms of matrices U_s and E_1 can be readily determined by comparing the port-structures associated with matrices Y and $\bar{Y}$.

Let $\bar{Q} = [\bar{q}_{ij}]_n$ denote the s.c. conductance matrix for the network derived from the one in Figure A.2.9 as shown in Figure A.2.11. Then in establishing the relation between matrices Y and $\bar{Q}$, it is necessary to consider the s.c. conductance matrix $\hat{Q}^*$ corresponding to the

derived network whose port-structure is modified as shown in Fig. A.2.12. A uniformly tapered matrix, Q^* , can be easily obtained from $\hat{Q}^*$ by suitably interchanging some of its rows and the corresponding columns and by applying cross-sign change operations. Let the network shown in Figure A.2.13 be obtained from the one in Figure A.1.12 by shorting port h^* . Then, as in section 3.2, the s.c. conductance matrix, $\tilde{Q}$, corresponding to this last network is required to be considered in the derivation of the relation, which is stated below:

$$Y = U_s E_1 \bar{Q} E_1 U_s - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{hh} \right)^{-1} \bar{Q}'_{.h} \bar{Q}'_h.$$

where

$$[\bar{Q}'_h]^T = \bar{Q}'_{.h} = \begin{bmatrix} Q_1 \\ - \\ Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{i1}^{(1)}] \quad \text{and} \quad Q_2 = [q_{i1}^{(2)}] \quad \text{with}$$

$$q_{i1}^{(1)} = -\bar{q}_{h-i, h} + \sum_{k=1}^i g_{n-k+2} \quad (i = 1, 2, \dots, h-1)$$

$$q_{i1}^{(2)} = -\bar{q}_{i, h} - \sum_{k=1}^{i-n+1} g_{n-h+k} \quad (i = h, h+1, \dots, n)$$

Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{n-k+2} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{j-n+1} g_{n-h+k} \quad (j = h, h+1, \dots, n; i = h, h+1, \dots, j)$$

Ref.: Figures A.2.14 - A.2.16.

Let matrix $Y = [y_{ij}]_n$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same resistive network which is shown in Figure A.2.14, the port-numberings and port-orientations being altered as indicated in Figure A.2.15. Matrix Y is uniformly tapered and $Y = U_s E_1 \bar{Y} E_1 U_s$, where the exact nature of matrices U_s and E_1 can be readily determined by comparing the port-structures associated with matrices Y and $\bar{Y}$. Then

$$Y = U_s E_1 \bar{Q} E_1 U_s - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{h-1, h-1} \right)^{-1} \bar{Q}'_{\cdot, h-1} \bar{Q}'_{h-1}.$$

where

$$[\bar{Q}'_{h-1, \cdot}]^T = \bar{Q}'_{\cdot, h-1} = \begin{bmatrix} Q_1 \\ - \\ Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{il}^{(1)}] \quad \text{and} \quad Q_2 = [q_{il}^{(2)}] \quad \text{with}$$

$$q_{il}^{(1)} = \bar{q}_{i, h-1} - \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1)$$

$$q_{il}^{(2)} = \bar{q}_{n+h-i, h-1} + \sum_{k=1}^{i-n+1} g_{n+k} \quad (i = h, h+1, \dots, n)$$

Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{j-n+1} g_{n+k} \quad (j = h, h+1, \dots, n; j = h, h+1, \dots, j)$$

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Ref.: Figures A.2.17 - A.2.18.

Let matrix $Y = [y_{ij}]_n$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same resistive network which is shown in Figure A.2.17, the port-numberings and port-orientations being altered as indicated in Figure A.2.18. Matrix Y is uniformly tapered and $Y = U_s E_1 \bar{Y} E_1 U_s$, where the exact nature of matrices U_s and E_1 can be readily determined by comparing the port-structures associated with matrices Y and $\bar{Y}$. Then

$$Y = U_s E_1 \bar{Q} E_1 U_s - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{11} \right)^{-1} \bar{Q}'_{.1} \bar{Q}'_1.$$

where

$$[\bar{Q}'_1]^T = \bar{Q}'_{.1} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{i1}^{(1)}] \quad \text{and} \quad Q_2 = [q_{i1}^{(2)}] \quad \text{with}$$

$$q_{i1}^{(1)} = -\bar{q}_{h-i,1} + \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1)$$

$$q_{i1}^{(2)} = -\bar{q}_{i,1} - \sum_{k=1}^{i-n+1} g_{n+k} \quad (i = h, h+1, \dots, j)$$

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Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{h-k} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{j-n+1} g_{n+k} \quad (j = h, h+1, \dots, n; i = h, h+1, \dots, j)$$

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Ref.: Figures A.2.20 - A.2.22.

Let matrix $Y = [y_{ij}]_n$ be obtained from $\bar{Y}$ such that the former is the s.c. conductance matrix for the same network which is shown in Figure A.2.20, the port-numberings and port-orientations being altered as indicated in Figure A.2.21. Matrix Y is uniformly tapered and $Y = U_s E_1 \bar{Y} E_1 U_s$, where the exact nature of matrices U_s and E_1 can be readily determined by comparing the port-structures associated with matrices Y and $\bar{Y}$. Then

$$Y = U_s E_1 \bar{Q} E_1 U_s - A + \left(\sum_{k=1}^{n+1} g_k - \bar{q}_{nn} \right)^{-1} \bar{Q}'_{.n} \bar{Q}'_n.$$

where

$$[\bar{Q}'_{n.}]^T = \bar{Q}'_{.n} = \begin{bmatrix} Q_1 \\ \text{---} \\ Q_2 \end{bmatrix} \begin{matrix} (h-1) \\ \\ (n-h+1) \end{matrix}$$

$$Q_1 = [q_{i1}^{(1)}] \quad \text{and} \quad Q_2 = [q_{i1}^{(2)}] \quad \text{with}$$

$$q_{i1}^{(1)} = -\bar{q}_{i,n} + \sum_{k=1}^i g_{n-k+2} \quad (i = 1, 2, \dots, h-1)$$

$$q_{il}^{(2)} = -\bar{q}_{n+h-i, n} - \sum_{k=1}^{i-n+1} g_{n-h+k} \quad (i = h, h+1, \dots, n)$$

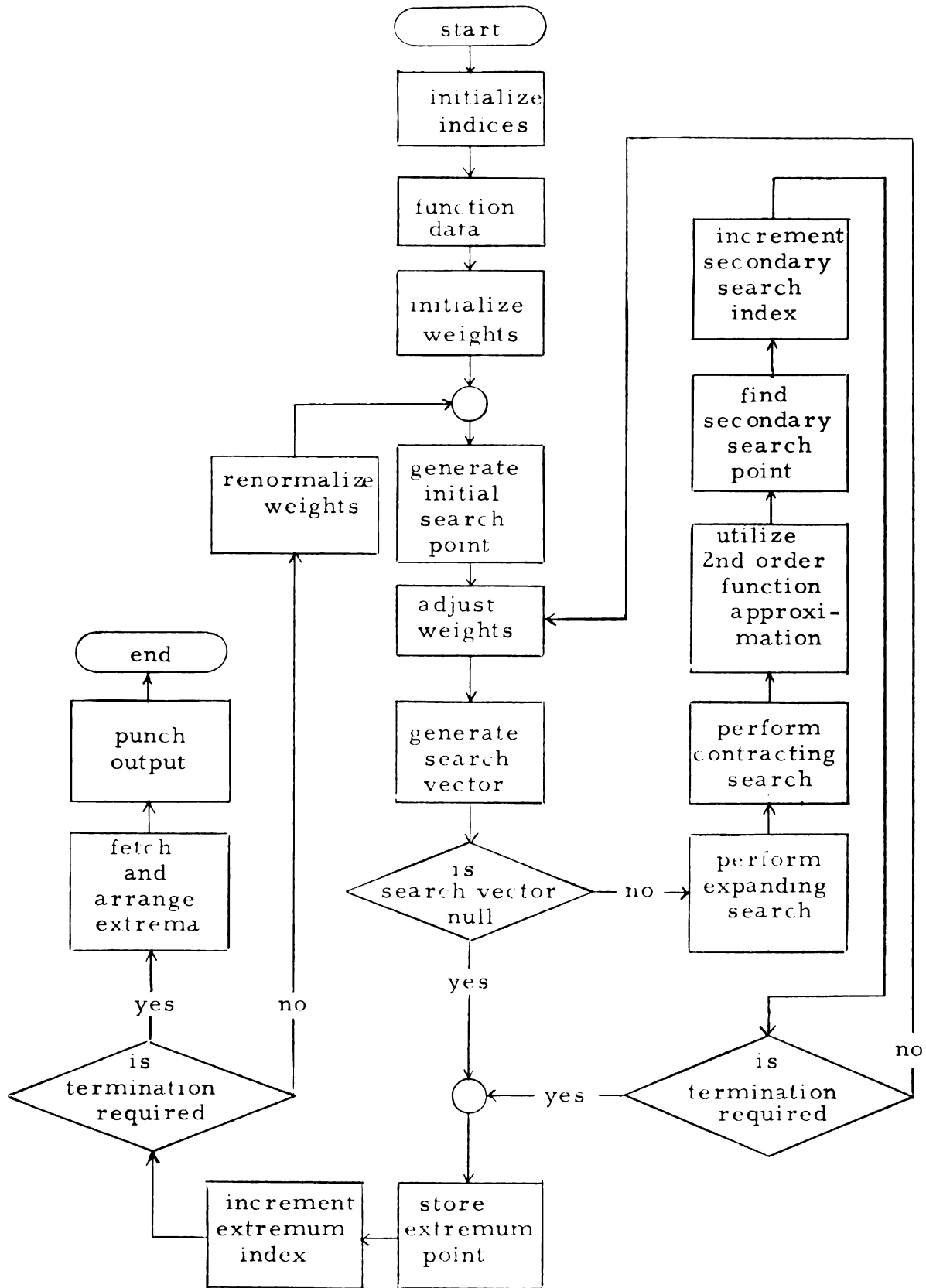
Further, $A = [a_{ij}]_n$ with

$$a_{ij} = \sum_{k=1}^i g_{n-k+2} \quad (i = 1, 2, \dots, h-1; j = i, i+1, \dots, h-1)$$

$$a_{ij} = 0 \quad (i = 1, 2, \dots, h-1; j = h, h+1, \dots, n)$$

$$a_{ij} = \sum_{k=1}^{j-n+1} g_{n-h+k} \quad (j = h, h+1, \dots, n; i = h, h+1, \dots, j)$$

APPENDIX 3



Logic Diagram of the Computer Program
 "Generalized Random Extremum Analysis Technique"
 (Courtesy of Instrument Division, Lear Siegler, Inc.,
 Grand Rapids, Michigan)

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