PL INVOLUTIONS ON LENS SPACES AND OTHER 3-MANIFOLDS

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PL INVOLUTIONS ON LENS SPACES AND OTHER 3-MANIFOLDS

By

Paik Kee Kim

AN ABSTRACT OF A DISSERTATION

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ABSTRACT

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This thesis is to complete the classification problem for sense preserving PL involutions with non-empty fixed point sets on 3-dimensional lens spaces $L = L(p,q)$. The classification problem for PL involutions on the projective 3-space P^3 as well as that for PL involutions on $P^3 \# P^3$ will be settled. The principal results are the following theorems.

<u>Theorem 1</u>: If h is an orientation preserving PL involution on $L(p,q)$, p even, which preserves sense and has non-empty fixed point set F, then F is a disjoint union of two simple closed curves.

<u>Theorem 2</u>: Up to PL equivalences, there is exactly one orientation preserving PL involution on $L(p,q)$, p even, which preserves sense and has non-empty fixed point set.

<u>Corollary 3</u>: Up to PL equivalences, there is exactly one orientation preserving PL involution on P^3 with non-empty fixed point set and there is exactly one free involution on P^3.

Theorem 4: Let h be an orientation preserving PL involution on a lens space $L = L(p,q)$, p even, which preserves sense and has non-empty fixed point set. Then there exists a PL equivariant homeomorphism t on L such that t interchanges the two components of $\mathrm{Fix}(h)$ if and only if L is symmetric.

Corollary 5: Let h be an orientation preserving PL involution on $P^3 \# P^3$. If $\mathrm{Fix}(h) = \emptyset$ or $\mathrm{Fix}(h)$ is connected, h is the obvious involution which interchanges the two P^3. If $\mathrm{Fix}(h)$ is not connected, $\mathrm{Fix}(h)$ is a disjoint union of three simple closed curves and there is exactly one such h, up to PL equivalences.

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To my mother and Myung

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INTRODUCTION

An involution h of a lens space $L = L(p,q)$ is called sense preserving if h induces the identity of $H_1(L)$. The purpose of this thesis is to classify the orientation preserving PL involutions of L which preserve sense and have non-empty fixed point sets for p even. As results, this thesis will lead a complete classification of the PL involutions on the projective 3-space P^3 as well as that of the PL involutions on $P^3 \# P^3$.

We work in the piecewise linear (PL) category. All PL involutions are known on S^3 (see Livesay [5,6] and Waldhausen [20]) and on $S^1 \times S^2$ (see Fremon [3], Kwun [8], Tao [18], and Tollefson [19]). Therefore, in this thesis we will not consider S^3 and $S^1 \times S^2$ as lens spaces. Kwun [9,10] classified all orientation reversing PL involutions of L and all orientation preserving PL involutions of $L(p,q)$, p <u>odd</u>, which preserve sense and have non-empty fixed point sets. The classification problem of the (sense preserving) free involutions on $L(p,q)$, $p > 2$), is still open, but the problem on P^3 will be solved by using Rice's work [15]. It will be shown that, up to PL equivalences, there are exactly three PL involutions on P^3.

Let M_i $(i = 1,2)$ be oriented 3-manifolds and h_i be involutions on M_i. If there is a suitable invariant 3-cell in each M_i, by taking the connected sum $M_1 \# M_2$, along the 3-cells, one can define an involution, denoted by $h_1 \# h_2$, on $M_1 \# M_2$ induced by h_1 and h_2. The connected sum $M_1 \# M_2$ is obtained by removing the interior of a nice invariant 3-cell from each, and then matching the resulting boundaries using an orientation reversing equivariant homeomorphism. Notice that $h_1 \# h_2$ depends on the choice of the invariant 3-cells along whose boundaries the connected sum is constructed. All orientation reversing involutions on $L(p,q) \# L(\bar{p},\bar{q})$ are known ([7], [13], [16]). We will also investigate some orientation preserving PL involutions on $L(p,q) \# L(\bar{p},\bar{q})$. As a consequence, up to PL equivalences, there are exactly seven PL involutions on $P^3 \# P^3$.

CHAPTER I

FIXED POINT SETS

In this chapter we shall study the fixed point set of an orientation preserving involution on a lens space $L = L(p,q)$ which preserves sense and has non-empty fixed point set.

<u>Lemma 1.1</u>: Let X be a m-manifold which has a contractible universal covering space. If $H_1(X)$ is of rank $\geq n$ and there is a short exact sequence $0 \to A \xrightarrow{f} \Pi_1(X) \xrightarrow{g} Z_p \to 0$ where A is a free abelian group of rank n, then $\Pi_1(X)$ is a free abelian group of rank n.

<u>Proof</u>: Let $\{a_i\}_{i = 1,2,\cdots,n}$ be a basis for A. Since f is a monomorphism, we simply identify A with the image $f(A)$. Let t be an element of $\Pi_1(X)$ such that $g(t)$ generates Z_p. Then $\Pi_1(X)$ is generated by the a_i and t. Denote the image of an element e of $\Pi_1(X)$ by $\bar{e}$ under the natural homomorphism of Π_1 to H_1. Let Q be the rationals. $H_1(X;Q) = H_1(X) \otimes Q$ is generated by the $\bar{a}_i \otimes 1$ and $\bar{t} \otimes 1$. Since $g(t^p) = 0$, $t^p \in A$, and $\overline{t^p} = p\bar{t}$ is generated by the $\bar{a}_i$ (notice that we shift from the multiplication notation $\overline{t^p}$ to the additive notation $p\bar{t}$ as $H_1(X)$ is abelian). Hence $p\,\bar{t} \otimes 1 \in \langle \{\bar{a}_i \otimes 1\} \rangle$ which is Q-submodule of $H_1(X) \otimes Q$ generated by the $\bar{a}_i \otimes 1$. Hence, $\bar{t} \otimes 1 \in \langle \{\bar{a}_i \otimes 1\} \rangle$, and $\langle \{\bar{a}_i \otimes 1\} \rangle = H_1(X) \otimes Q$. Since

$H_1(X) \otimes Q$ is a vector space over Q of rank $\geq n$, $\{\bar{a}_i \otimes 1\}$ is a basis for $H_1(X) \otimes Q$. Since A is a normal subgroup of $\Pi_1(X)$, $t^{-1}a_i t = \prod_{j=1}^{n} a_j^{k_j}$ for some k_js'. Abelianizing it, $\bar{a}_i = \prod_{j=1}^{n} k_j\bar{a}_j$. Hence $\bar{a}_i \otimes 1 = \sum_{j=1}^{n} k_j(\bar{a}_j \otimes 1)$. Hence $k_i = 1$ and $k_j = 0$ if $j \neq i$. Therefore, $\Pi_1(X)$ is abelian. Since $0 \to A \to \Pi_1(X) \to Z_p \to 0$ is exact and Q is torsion free, $A \otimes Q = \Pi_1(X) \otimes Q$. Since $\Pi_1(X)$ is a finitely generated abelian group, $\Pi_1(X)$ is of rank n. But no non-trivial finite group can act freely on a finite dimensional contractible space (due to P.A. Smith [4], 287). Therefore, $\Pi_1(X)$ has no torsion subgroup. This completes the lemma.

<u>Definition 1.2</u>: Let M_1 and M_2 be PL manifolds. Two PL homeomorphisms h_i on M_i $(i = 1,2)$ are called PL equivalent if there is a PL homeomorphism t of M_1 onto M_2 such that $h_2 t = t h_1$. In this case t is called PL equivariant with respect to h_1 and h_2. We sometimes denote the fact by $h_1 \sim h_2$.

When h_i $(i = 1,2)$ happen to be involutions on M_i, obviously any equivariant map t sends the fixed point set of h_1 onto the fixed point set of h_2.

<u>Definition 1.3</u>: Let h be an involution on a space M. The quotient space M/Z_2 of M generated by h is called the orbit space of h and the projection $g\colon M \to M/Z_2$ is called the orbit map of h. We denote the fixed point set of h by Fix (h).

The following theorem is due to Stallings [17].

Theorem (Stallings): If M is a compact irreducible connected 3-manifold, and if $\Pi_1(M)$ has a finitely generated normal subgroup K different from Z_2, whose quotient group is Z, then M is the total space of a fiber space with base space a circle and with fiber a connected 2-manifold T embedded in M whose fundamental group is K.

Let D^2 be the unit disk in the Gaussian plane of complex numbers and S^1 its boundary. $D^2 \times S^1$ is a solid torus whose points can be denoted by $(\rho z_1, z_2)$ where $z_1, z_2 \in S^1$ and $0 \le \rho \le 1$.

<u>Lemma 1.4</u>: The orbit space of a free PL involution h on $D^2 \times S^1$ is homeomorphic to a disk bundle over S^1, and h is PL equivalent to an involution h_1 given by either $h_1(\rho z_1, z_2) = (\rho z_1, -z_2)$ or $h_1(\rho z_1, z_2) = (\rho \bar{z}_1, -z_2)$.

<u>Proof</u>: Since h is free, the orbit space $D^2 \times S^1/Z_2$ is a connected orientable compact 3-manifold with boundary. Hence the Betti numbers of $D^2 \times S^1/Z_2$ $\rho_3 = 0$ and $\rho_0 = 1$. Since $D^2 \times S^1/Z_2$ is covered by $D^2 \times S^1$, we have a short exact sequence $0 \to Z \to \Pi_1(D^2 \times S^1/Z_2) \to Z_2 \to 0$. Since $\chi(D^2 \times S^1) = 2 \cdot \chi(D^2 \times S^1/Z_2)$, $\chi(D^2 \times S^1/Z_2) = 0$. Hence $\rho_1 \ge 1$. By Lemma 1.1, $\Pi_1(D^2 \times S^1/Z_2) = Z$. On the other hand, $D^2 \times S^1/Z_2$ is irreducible as it is covered by $D^2 \times S^1$. Therefore by Stallings' theorem, $D^2 \times S^1/Z_2$ is homeomorphic to a disk bundle over S^1. That is, $D^2 \times S^1/Z_2$ is a solid torus or a non-orientable disk bundle over S^1, according

to h preserving or reversing the orientation. Let h and h' be any two orientation preserving (or reversing) free PL involutions of $D^2 \times S^1$. Let t' be a PL homeomorphism between the two orbit spaces of h and h'.

Consider the following diagram.

$$\begin{tikzcd} h \circlearrowright D^2 \times S^1 \arrow[r, "?"] \arrow[d, "g"] & D^2 \times S^1 \circlearrowleft h' \arrow[d, "g'"] \\ D^2 \times S^1/Z_2 \arrow[r, "t'"] & D^2 \times S^1/Z_2 \end{tikzcd}$$

where g and g' are the orbit maps of h and h' respectively.

Since $g_{\#}[\Pi_1(D^2 \times S^1)] = 2Z \subset \Pi_1(D^2 \times S^1/Z_2) = Z$, by the lifting theorem, we have a PL homeomorphism t of $D^2 \times S^1$ which makes the above diagram commutes. It follows that $th = h't$. This completes the Lemma.

<u>Remark 1.5</u>: Let h be a PL involution of a finite triangulated n-manifold M_1. It can be shown that h becomes simplicial after a suitable subdivision such that the fixed point set of h is a subcomplex of the subdivision M_2. Let M be the second barycentric subdivision of M_2. Then it is easy to check the following properties: (1) F is full subcomplex of M (2) the orbit map g of h and the orbit space of h are simplicial and g maps each simplex homeomorphically.

The following result seems to be well-known and freely used by various authors ([10], [19]). For the sake of completeness, we give a proof.

Lemma 1.6: Up to PL equivalences, there exists exactly one PL involution h of $D^2 \times S^1$ with the center circle as the fixed point set.

Proof: We first show that the orbit space of h is a solid torus. Let M be a triangulation of $D^2 \times S^1$ as in Remark 1.5 and U be the simplicial neighborhood of the center circle F in M. Then $U \approx D^2 \times S^1$ is an invariant neighborhood and $U' = g(U)$ is a simplicial neighborhood of $F' = g(F)$ where g is the orbit map of h. Since h is orientation preserving, the orbit space M' of h is an orientable manifold. Since U' is orientable, $(U',F') \approx (D^2 \times S^1, 0 \times S^1)$. We want $\Pi_1(M' - U') = Z \oplus Z$. Since $M - U \approx S^1 \times S^1 \times I$, $\chi(M-U) = 0$, and $\chi(M' - U') = 0$. But $H_2(M' - U';\, Q) \cong H_2(M - U;\, Q)^{Z_2}$ which is Z_2-invariant homology with rational coefficient Q (for the proof, see Floyd [2]). By the definition of Z_2-invariant homology, $H_2(M - U;\, Q)^{Z_2} = \{\alpha | h_*(\alpha) = \alpha,\ \alpha \in H_2(M - U;\, Q)\} \cong \{\alpha | h_*(\alpha) = \alpha,\ \alpha \in H_2(S^1 \times S^1;\, Q)\}$ where $S^1 \times S^1$ is the boundary of $M - U$ and $h' = h|S^1 \times S^1$. Since h' preserves the orientation, the induced isomorphism h'_* is the identity, and $H_2(M' - U';\, Q) = Q$. Hence the Betti number of $H_2(M' - U')$ is $\rho_2 = 1$. Since $\chi(M' - U') = 0$, $H_1(M' - U')$ is of rank 2. Since $M' - U'$ is covered by $M - U$, by Lemma 1.1,

$\Pi_1(M' - U') = Z \oplus Z$. Therefore, by Stallings' theorem, $L' - U'$ is fibered over a circle with fiber T and $\Pi_1(T) = Z$. Since T is a connected 2-manifold, T would be $S^1 \times I$ or möbius band. But since $L' - U'$ is orientable, T is orientable, and T must be $S^1 \times I$. Thus $L' - U'$ may be obtained from $S^1 \times I \times I$ by identifying each $(x,0)$ $(x \in S^1 \times I)$ with $(f(x),1)$ where f is a homeomorphism of $S^1 \times I$. Since the number of components of $M' - U'$ is two, f carries $S^1 \times i$ onto $S^1 \times i$ $(i = 0,1)$. Hence, since $M' - U'$ is orientable, f must preserve the orientation, and it can be shown that f is isotopic to the identity. Hence $M' - U' \approx S^1 \times S^1 \times I$. Since U' is a solid torus, M' must be a solid torus. Let $\tilde{h}$ be the PL involution of $D^2 \times S^1$ given by $\tilde{h}(\rho z_1, z_2) = (-\rho z_1, z_2)$. Let $\tilde{M}$ be a triangulation of $D^2 \times S^1$ with respect to $\tilde{h}$ as in Remark 1.5, $\tilde{U}$ the simplicial neighborhood of the center circle $\tilde{F}$ in $\tilde{M}$, $\tilde{M}'$ the orbit space of $\tilde{h}$ and $\tilde{g}$ the orbit map of $\tilde{h}$. By the above argument, $M' = D^2 \times S^1 \cup S^1 \times S^1 \times I = \tilde{M}'$ where $g(U) = D^2 \times S^1 = \tilde{g}(\tilde{U})$ and $S^1 \times S^1$ is the boundary of $D^2 \times S^1$. Since U and V are invariant simplicial neighborhoods of the fixed point sets, one can find invariant 2-cells C and D regularly embedded in U and V as subpolyhedra, respectively. Consider the following diagram.

$$\begin{tikzcd}
h_{\eta})\, M - F = (D^2 - O) \times S^1 \arrow[r, "?"] \arrow[d, "g"] & (D^2 - O) \times S^1 = \widetilde{M} - \widetilde{F} (_{\eta}\, h \arrow[d] \\
(D^2 - O) \times S^1 \cup S^1 \times S^1 \times I \arrow[r, "\ell"] & (D^2 - O) \times S^1 \cup S^1 \times S^1 \times I
\end{tikzcd}$$

where we will define a PL homeomorphism ℓ later. Let $(1,0)$ be generators of $\Pi_1(M - F)$ and $\Pi_1(\widetilde{M} - \widetilde{F})$ represented by the path ∂C and ∂D in M and $\widetilde{M}$, respectively. For the sake of briefness, again $(1,0)$ and $(0,1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ which are the fundamental groups of $M' - F'$ and $\widetilde{M}' - \widetilde{F}'$ where $\widetilde{F}' = g(\widetilde{F})$ and $S^1 \times S^1$ is the boundary of $U' = D^2 \times S^1 = \widetilde{U}'$. Without loss of generality, we may assume that $(1,0)$ are generated by $g(\partial C)$ and $g(\partial D)$ of $g(\partial U)$ and $\widetilde{g}(\partial\widetilde{U})$, respectively. The induced homomorphism $g_\#$ sends $(1,0)$ to $(2,0)$, so that $g_\#\ (Z \oplus Z) = 2Z + \langle(d,e)\rangle$ where $g_\#[(0,1)] = (d,e)$ for some integers d and e, and $\langle(d,e)\rangle$ is the subgroup of $\Pi_1(S^1 \times S^1)$ generated by (d,e). Since g is the double covering projection, $\langle(d,e)\rangle$ can not be contained in $\langle(1,0)\rangle$. Moreover, $\langle(1,0)\rangle \cap \langle(d,e)\rangle = \{0\}$ since otherwise $\langle(d,e)\rangle \subset \langle(1,0)\rangle$. Hence $\langle(1,0)\rangle + \langle(d,e)\rangle = \langle(1,0)\rangle \oplus \langle(d,e)\rangle$, and $g_\#(Z \oplus Z) = 2Z \oplus \langle(d,e)\rangle$. Since $[\Pi_1(S^1 \times S^1) : 2Z \oplus \langle(d,e)\rangle] = 2$, $[\Pi_1(S^1 \times S^1) : Z \oplus \langle(d,e)\rangle] = 1$, and $\Pi_1(S^1 \times S^1) = Z \oplus \langle(d,e)\rangle$. Since $g(C)$ and $\widetilde{g}(D)$ are regularly embedded in U' and $\widetilde{U}'$ as subpolyhedra, there exists a PL homeomorphism q' of U' onto V' carrying $g(C)$ onto $\widetilde{g}(D)$. Hence there exists an extended PL homeomorphism q of M' onto $\widetilde{M}'$. Therefore, $q_\#$ sends $(1,0)$ to $(1,0)$

and (d,e) to (a,b), and $q_\#(Z \oplus \langle(d,e)\rangle) = Z \oplus \langle(a,b)\rangle = \Pi_1(S^1 \times S^1)$. We may assume $b = 1$. Define a PL homeomorphism t' of $\tilde{V}' = D^2 \times S^1$ by $t'(\rho z_1, z_2) = (\rho z_1, z_2^{-a}, z_2)$. Since $\tilde{M}' = D^2 \times S^1 \cup S^1 \times S^1 \times I$, there exists an extension t on $\tilde{M}'$. Now define $\ell = tq$. Let φ be the nice path generating $(0,1)$ of $\Pi_1(S^1 \times S^1) = \Pi_1(\tilde{M}' - \tilde{F}')$. Considering the action of $\tilde{h}$ and the fact that V is the invariant simplicial neighborhood of $\tilde{F}$, $\tilde{g}^{-1}(\varphi)$ is a disjoint union of two simple closed curves. Denote one of them by φ'. Let α be an element of $\Pi_1(\tilde{M} - F)$ represented by the path φ'. Then $\tilde{g}_\#[(1,0)] = (2,0)$ and $\tilde{g}_\#(\alpha) = (0,1)$, and $\langle(1,0)\rangle \cap \langle\alpha\rangle = \{0\}$. Suppose the contrary that $\langle(1,0)\rangle \oplus \langle\alpha\rangle$ is a proper subgroup of $\Pi_1(\tilde{M} - \tilde{F})$. Then since $\tilde{g}_\#$ is monomorphism, $\langle(2,0)\rangle \oplus \langle(0,1)\rangle$ is a proper subgroup of $g_\#[\Pi_1(\tilde{M} - \tilde{F})]$. But $[\Pi_1(\tilde{g}(\tilde{M} - \tilde{F})) : \tilde{g}_\#(\Pi_1(\tilde{M} - \tilde{F}))] = 2$. Therefore $[\tilde{g}_\#(\Pi_1(\tilde{M} - \tilde{F}) : \langle(2,0)\rangle \oplus \langle(0,1)\rangle] = 1$. This is a contradiction. Let us look at the following diagram which is a concentration of the work done so far.

$$\begin{array}{ccccc}
\Pi_1(M - F) = Z \oplus Z & & \longrightarrow & & \Pi_1(\tilde{M} - \tilde{F}) = Z \oplus \langle\alpha\rangle \\
\quad\quad (1,0) \quad (0,1) & & & & \quad\quad (1,0) \quad \alpha \\
g_\# \downarrow \quad\quad \downarrow \quad\quad \downarrow & & & & \tilde{g}_\# \downarrow \quad\quad \downarrow \quad\quad \downarrow \\
\quad\quad (2,0) \quad (d,e) & & & & \quad\quad (2,0) \quad (0,1) \\
\Pi_1(S^1 \times S^1) = Z \oplus Z & & \xrightarrow{\ \ell_\#\ } & & \Pi_1(S^1 \times S^1 = Z \oplus Z
\end{array}$$

Since $t_\# = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$, one can check that $\ell_\# g_\#[(1,0)] = (2,0)$ and $\ell_\# g_\#(0,1) = (0,1)$. Hence by the lifting theorem, we have a PL homeomorphism f which makes the following diagram commute

$$\begin{tikzcd} h \circlearrowright (D^2 - 0) \times S^1 \arrow[r, "f"] \arrow[d, dashed, "g"] & (D^2 - 0) \times S^1 \circlearrowleft \tilde{h} \arrow[d, dashed, "\tilde{g}"] \\ (D^2 - 0) \times S^1 \arrow[r, "\ell"] & (D^2 - 0) \times S^1 \end{tikzcd}$$

where $(D^2 - 0) \times S^1 = g((D^2 - 0) \times S^1)$, $\tilde{g}((D^2 - 0) \times S^1)$. Hence $fh = \tilde{h}f$. One can extend f to $D^2 \times S^1$ in an obvious way such that $fh = \tilde{h}f$ on $D^2 \times S^1$. This completes the lemma.

Let h be an orientation preserving PL involution on a lens space $L = L(p,q)$ which preserves sense and has non-empty fixed point set F. By the dimensional parity theorem, each component F_0 of F is of 1-dimension, and F_0 is a simple closed curve. Let U be a regular neighborhood of F_0 such that $U \cap F = F_0$. Consider the usual covering projection $g: S^3 \to L$. By the lifting theorem, we have a PL involution $\tilde{h}: (S^3, y_0) \to (S^3, y_0)$ where $g(y_0) \in F_0$. Suppose h is sense preserving. Then $g^{-1}(F_0)$ is connected, and $\tilde{F} = g^{-1}(F_0)$ is the fixed point set of $\tilde{h}$. By Waldhausen [20], $\tilde{F}$ is an unknotted simple closed curve. Hence $\overline{S^3 - g^{-1}(U)}$ is a solid torus, and $\overline{L - U}$ is a solid torus. An explicit argument of the above may be found in [10].

<u>Theorem 1.7</u>: If h is an orientation preserving PL involution of $L = L(p,q)$, p even, which preserves sense and has non-empty fixed point set F, then F is a disjoint union of two simple closed curves.

<u>Proof</u>: By the above discussion, $L = D^2 \times S^1 \cup_k S^1 \times D^2$ such that $D^2 \times S^1$ is an invariant regular neighborhood of a component of F for an attaching map k of $S^1 \times S^1$. Denote $h|D^2 \times S^1$ and $h|S^1 \times D^2$ by h_1' and h_2', respectively. Suppose the contrary that $\mathrm{Fix}\,(h_2') = \emptyset$. Define h_1 on $D^2 \times S^1$ by $h_1(\rho z_1, z_2) = (-\rho z_1, z_2)$. By Lemma 1.6, there exists a PL homeomorphism t' on $D^2 \times S^1$ such that $h_1 = t'h_1't'^{-1}$. Define t of $D^2 \times S^1 \cup_k S^1 \times D^2$ onto $D^2 \times S^1 \cup_{kt'^{-1}} S^1 \times D^2$ by the following:

$$\begin{cases} t = t' & \text{on } D^2 \times S^1 \\ t = \text{identity} & \text{on } S^1 \times D^2 \end{cases}$$

It is a well defined PL homeomorphism. Define $\tilde{h}$ on $D^2 \times S^1 \cup_{kt'^{-1}} S^1 \times D^2$ by $\tilde{h} = tht^{-1}$. Since $h_1 = t'h_1't'^{-1}$, $\tilde{h}|D^2 \times S^1 = h_1$ and $\tilde{h}|S^1 \times D^2 = h_2'$. It is checked that $\tilde{h}$ is a PL involution, and h and $\tilde{h}$ are PL equivalent. Hence we may assume that $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$. By Lemma 1.4 and the similar argument as the above, we may further assume that $h(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$ since $\mathrm{Fix}(h) = \emptyset$. That is, we may assume that $L = D^2 \times S^1 \cup_f S^1 \times D^2$ for an appropriate attaching map f of $S^1 \times S^1$ and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$. Consider the following commutative diagram

$$
\begin{tikzcd}
 & S^1 \times S^1 \arrow[rr, "f"] \arrow[dl, "\varphi_1"'] \arrow[dd] \arrow[dr, "\varphi_2 f"] & & S^1 \times S^1 \arrow[dl, "\varphi_2"] \\
D^2 \times S^1 \arrow[dr] & & S^1 \times D^2 \arrow[dl] & \\
 & L(p,q) & &
\end{tikzcd}
$$

where φ_1 and φ_2 are the inclusion maps.

Let $(1,0)$ and $(0,1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ such that $f_{\#}(1,0) = (a,b)$ and $f_{\#}(0,1) = (c,d)$, where $f_{\#}$ is the isomorphism induced by f (we disregard the base point as $\Pi_1(S^1 \times S^1, *)$ is abelian). We may assume that $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$. One can show that by Van Kampen theorem, $\Pi_1(L) = \{\alpha,\beta \mid \beta^c = \alpha,\ \beta^a = 1\} = \{\beta \mid \beta^a = 1\}$ where α and β are the canonical generators of $\Pi_1(D^2 \times S^1)$ and $\Pi_1(S^1 \times D^2)$, respectively. Since $\Pi_1(L(p,q)) = Z_p$, $a = \pm p$, and a is even. Let g and $\hat{g}$ be the orbit maps of $h|D^2 \times S^1$ and $h|S^1 \times D^2$, respectively. Then by Lemma 1.4 and the proof of Lemma 1.6, $g(D^2 \times S^1)$ and $\hat{g}(S^1 \times D^2)$ are solid tori. Consider the following diagram

$$
\begin{tikzcd}
D^2 \times S^1 \supset S^1 \times S^1 \arrow[r, "f"] \arrow[d, "g'"] & S^1 \times S^1 \subset S^1 \times D^2 \arrow[d, "\hat{g}'"'] \\
D^2 \times S^1 \supset S^1 \times S^1 \arrow[r, "f"] & S^1 \times S^1 \subset S^1 \times D^2
\end{tikzcd}
$$

where g' and $\hat{g}'$ are induced by g and $\hat{g}$, respectively and f' is the induced attaching map in the orbit space of h. Notice that $g'_{\#}(r,s) = (2r,s)$ and $\hat{g}'_{\#}(r,s) = (2r,s)$ for any element $(r,s) \in \Pi_1(S^1 \times S^1)$. Let $f'_{\#}[(1,0)] = (a',b')$

and $f'_{\#}[(0,1)] = (c',d')$. By chasing the above commutative diagram, easy computation shows that $b = 2b'$, and b is even. Since a is even, we have a contradiction to the fact $ad - bc = 1$. Therefore, $\mathrm{Fix}(h|S^1 \times D^2)$ can not be empty. By Tollefson [19], $\mathrm{Fix}(h|S^1 \times D^2)$ is a simple closed curve. This completes the proof.

CHAPTER II

PL INVOLUTIONS ON SOME 3-MANIFOLDS

In this chapter, we will investigate all orientation preserving PL involutions on $L(p,q)$, p even, which preserve sense and have non-empty fixed point sets.

Kwun [9] considered all orientation reversing PL involutions on lens spaces, and proved that no lens space except the projective 3-space P^3 admits an orientation reversing PL involution and there exists exactly one orientation reversing PL involution on P^3, up to PL equivalences. In this case, the fixed point set is a projective plane P^2 plus an isolated point. Using this result applied to Kim and Tollefson's work [7] and Showers' work [16], the orientation reversing PL involutions on the connected sum of two lens spaces are easily classified. Myung [13] initiated this problem and gave a partial solution. We will also study all orientation preserving PL involutions on $P^3 \# P^3$ as well as those on P^3. Kwun [10] also showed that up to PL equivalences, there is exactly one orientation preserving PL involution h on $L = L(p,q)$, p <u>odd</u>, which preserves sense and has non-empty fixed point set if L is symmetric and there are exactly two such h if L is non-symmetric. In the latter case, the two different orbit spaces are $L(p,q')$ and $L(p,q'')$ where $2q' \equiv \pm q$ and $2q''q \equiv \pm 1 \bmod p$. In either case, $\mathrm{Fix}(h)$ is a simple closed curve.

Let $(1,0)$ and $(0,1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ and k be a PL homeomorphism on $S^1 \times S^1$ such that $k_{\#}[(1,0)] = (a,b)$ and $k_{\#}[(0,1)] = (c,d)$. We may assume $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$ and $a \geq 0$.

Definition 2.1: Define $L_k(a,c,b,d) = D^2 \times S^1 \cup_k S^1 \times D^2$ where $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$ and $a \geq 0$. We sometimes denote $L_k(a,c,b,d)$ by L_k if no confusion arises.

By Mangler [11], the isotopy classes of homeomorphisms of $S^1 \times S^1$ are precisely the automorphism classes of $\Pi_1(S^1 \times S^1)$. Hence, the integers a,b,c and d completely determine the isotopy class of k in Definition 2.1, and hence the homeomorphic type of $L_k(a,c,b,d)$. As Kwun [10] pointed out, if $a = 0$, L_k is homeomorphic to $S^1 \times S^2$, if $a = 1$, L_k is homeomorphic to S^3, and if $a > 1$, L_k is homeomorphic to $L(a,b)$. Recall that $L(p,q)$ is homeomorphic to $L(p,q')$ if and only if $q \equiv \pm q'$ or $qq' \equiv \pm 1 \bmod p$ [12,14].

Lemma 2.2: Let h be a PL involution of $L_k(a,c,b,d)$ such that $h(D^2 \times S^1) = D^2 \times S^1$ and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$. Then the orbit space of h is homeomorphic to $L_{k'}(\frac{a}{2}, c, b, 2d)$, and a is even, where k' is the attaching map induced by k.

Proof: By Lemma 1.6, the orbit space of $h|D^2 \times S^1$ and $h|S^1 \times D^2$ are solid tori. Hence the orbit space of

h is homeomorphic to $L_{k'}(p',c',b',d')$ for suitable k', p',b',c' and d'. Consider the following diagram.

$$\begin{array}{ccc} D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{\ k\ } & S^1 \times S^1 \subset S^1 \times D^2 \\ \big\downarrow g & & \big\downarrow g' \\ D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{\ k'\ } & S^1 \times S^1 \subset S^1 \times D^2 \end{array}$$

where g and g' are induced by the orbit maps of $h|D^2 \times S^1$ and $h|S^1 \times D^2$, respectively. Notice that $g_{\#}[(1,0)] = (2,0)$, $g_{\#}[(0,1)] = (0,1)$, $g'_{\#}[(1,0)] = (1,0)$ and $g'_{\#}[(0,1)] = (0,2)$. Easy computation shows that $p = 2p'$, $b = b'$, $c = c'$ and $2d = d'$. Hence the orbit space of h is homeomorphic to $L_{k'}(\frac{p}{2},c,b,2d)$. This completes the lemma.

<u>Definition 2.3</u>: Let p be even and a homeomorphism f of $S^1 \times S^1$ be given by $f(z_1,z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. Define an involution of $L_f(p,c,b,d)$ by $h(\rho z_1,z_2) = (-\rho z_1,z_2)$ on $D^2 \times S^1$ and $h(z_1,\rho z_2) = (z_1,-\rho z_2)$ on $S^1 \times D^2$. We denote the involution by $h(p,c,b,d)$.

In the above definition, since p is even and b is odd (recall that $pd-bc = 1$), one can easily check that $h = h(p,c,b,d)$ is compatible with the attaching map, i.e., $fh = hf$.

<u>Lemma 2.4</u>: Let $h_i\,(i = 1,2)$ be PL involutions on $L_i = L_{f_i}(a,c,b,d)$ such that $h_i(D^2 \times S^1) = D^2 \times S^1$, and $h_i(\rho z_1,z_2) = (-\rho z_1,z_2)$ on $D^2 \times S^1$ and $h_i(z_1,\rho z_2) = (z_1,-\rho z_2)$ on $S^1 \times D^2$. Then h_1 and h_2 are PL equivalent.

<u>Proof</u>: By Lemma 2.2, the orbit spaces of h_i are homeomorphic to $L_{f_i'}(\frac{a}{2},c,b,2d)$ for some attaching maps f_i'. Hence f_1' and f_2' are isotopic, and there exists a level preserving PL homeomorphism H of $S^1 \times S^1 \times I$ such that $H_1 f_1' = f_2'$ and $H_0 =$ identity. Since $S^1 \times S^1 \times I$ is a boundary collar of $S^1 \times D^2$, there exists a PL homeomorphism t' of $S^1 \times D^2 \subset L_{f_1'}$ onto $S^1 \times D^2 \subset L_{f_2'}$ such that $t'f_1' = f_2'$. Define a homeomorphism $t_1 \colon L_{f_1'} \to L_{f_2'}$ by $t_1(\rho z_1, z_2) = (\rho z_1, z_2)$ on $D^2 \times S^1$ and $t_1(z_1, \rho z_2) = t'(z_1, \rho z_2)$ on $S^1 \times D^2$. One can check that t_1 is well defined. Consider the following diagram.

$$\begin{tikzcd}
h_1 \circlearrowright L_1 - F_{11} - F_{12} \arrow[r, dashed, "?"] \arrow[d, "g_1"] & L_2 - F_{21} - F_{22} \circlearrowleft h_2 \arrow[d, dashed, "g_2"] \\
L_{f_1'} - F_{11}' - F_{12}' \arrow[r, dashed, "t_1"] & L_{f_2'} - F_{21}' - F_{22}'
\end{tikzcd}$$

where F_{ij} $(j = 1,2)$ are the components of the fixed point set F_i of h_i, g_i are the orbit maps of h_i and $g_i(F_{ij}) = F_{ij}'$.

Since $S^1 \times D^2$ is a regular neighborhood of F_{i_2}, $\Pi(L_i - F_{i_1} - F_{i_2}) = \Pi_1(D^2 \times S^1 - F_{i_1}) = \Pi_1((D^2 - 0) \times S^1)$. Since t_1 is identity on $D^2 \times S^1$, by the lifting theorem, we have a lifting t such that the above diagram computes. One can extend t to whole L_1 in an obvious way such that $th_1 = h_2 t$ on L_1. This completes the proof.

Remark 2.5: Let h be an orientation preserving PL involution h of $L = L(p,q)$, p even, which preserves sense and has non-empty fixed point set. By Tollefson [19], any orientation preserving PL involution on $S^1 \times D^2$ with non-empty fixed point set is PL equivalent to the involution h' given by $h'(z_1, \rho z_2) = (z_1, -\rho z_2)$. Therefore, by using same technique as in the proof of Theorem 1.7, we may assume that $L = D^2 \times S^1 \cup_f S^1 \times D^2$ for an appropriate attaching map f and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$. Hence, by Lemma 2.4, we may assume $h = h(p,c,b,d)$ on $L_f(p,c,b,d)$ where f is given by $f(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. Since $L_f(p,c,b,d) \approx L(p,b)$, $b \equiv \pm q$ or $bq \equiv \pm 1 \mod p$. By Lemma 2.2, the orbit space of h is homeomorphic to $L(\frac{p}{2}, b)$ where $b \equiv \pm q$ or $bq \equiv \pm 1 \mod p$.

Proposition 2.6: $h = h(p,c,b,d)$ can be extended to an effective circle action.

Proof: For each $z \in S^1$, define S^1-action by $z \cdot (\rho z_1, z_2) = (\rho z_1 z, z_2)$ on $D^2 \times S^1$ and $z \cdot (z_1, \rho z_2) = (z_1 z^p, \rho z_2 z^b)$ on $S^1 \times D^2$.

Remark 2.7: If an involution h of $L(p,q)$ can be extended to an effective circle action, h must be clearly sense preserving. By Proposition 2.6, $h(p,c,b,d)$ is sense preserving. Therefore, by Remark 2.5, the classification problem of orientation preserving PL involutions of $L(p,q)$,

p even, which preserve sense and have non-empty fixed point sets is the same problem as the classification of those $h(p,c,b,d)$ for various possible c,b,d with $pd - cb = 1$. The information that we have is that $bq \equiv \pm 1$ or $b \equiv \pm q$ mod p.

Now we analyze the involution $h(p,c,b,d)$. If $h(p,c,b,d)$ is equivalent to $h(p,c',b',d')$, we denote the fact by $h(p,c,b,d) \sim h(p,c',b',d')$.

Lemma 2.8: For any integers c,b,d with $pd - cb = 1$,

(1) $h(p,b,c,d) \sim h(p,c',b,d')$ for any integers c' and d' with $pd' - c'b = 1$.

(2) $h(p,c,b,d) \sim h(p,c,b+mp,d+mc)$

(3) $h(p,c,b,d) \sim h(p,-c,-b,d)$

(4) $h(p,c,b,d) \sim h(p,-b,-c,d)$

Proof: We will define a homeomorphism $t\colon L_f \to L_{f'}$ where $L_f = L_f(p,c,b,d)$ and $L_{f'}$ is the space corresponding to the equivalent involution claimed in (i), $i = 1,2,3,4$. In (1), since $pd - bc = 1 = pd' - bc'$, $c' = c+mp$, $d' = d+mb$ for some integer m. Define $t\colon L_f \to L_{f'}$ by $t(\rho z_1,z_2) = (\rho z_1 z_2^{-m},z_2)$ on $D^2 \times S^1$ and $t(z_1,\rho z_2) = (z_1,\rho z_2)$ on $S^1 \times D^2$. For (2), define $t\colon L_f \to L_{f'}$ by $t(\rho z_1,z_2) = (\rho z_1,z_2)$ on $D^2 \times S^1$ and $t(z_1,\rho z_2) = (z_1,\rho z_2 z_1^m)$ on $S^1 \times D^2$. For (3), define $t\colon L_f \to L_{f'}$ by $t(\rho z_1,z_2) = (\rho z_1,z_2^{-1})$ on $D^2 \times S^1$ and $t(z_1,\rho z_2) = (z_1,\rho z_2^{-1})$ on $S^1 \times D^2$. For (4), define $t\colon L_f \to L_{f'}$ by $t(\rho z_1,z_2) = (z_2,\rho z_1)$ on $D^2 \times S^1$ and $t(z_1,\rho z_2) = (\rho z_2,z_1)$ on $S^1 \times D^2$ such that

$t(D^2 \times S^1) = S^1 \times D^2$. It is checked that those t are well defined and equivariant homeomorphisms. This completes the proof.

Now we are in a position to state our main theorem.

Theorem 2.9: Up to PL equivalences, there is exactly one orientation preserving PL involution on $L(p,q)$, P even, which preserves sense and has non-empty fixed point set.

Proof: By Remark 2.7, we will consider two involutions $h_1 = h(p,c,b,d)$ and $h_2 = h(p,c',b',d')$. Let $L_1 = L_f(p,c,b,d)$ and $L_2 = L_{f'}(p,c',b',d')$ corresponding to h_1 and h_2, respectively. Since $L_1 \approx L(p,b)$ and $L_2 \approx L(p,b')$, $b \equiv \pm b'$ or $bb' \equiv \pm 1 \bmod p$. If $b \equiv \pm b' \bmod p$, $b = \pm b' + mp$ for some integer m. By (1), (2), and (3) $h_1 \sim h_2$. Suppose $bb' \equiv \pm 1 \bmod p$. Since $pd - bc = 1$, $b' = \pm c + mp$ for some m. By (1), (2), (3), and (4), again $h_1 \sim h_2$. This completes the theorem.

Now consider free Z_2-action h on p^3. The orbit space M of h is a closed 3-manifold. Since we have a universal covering projection $S^3 \to P^3 \to M$, the order of $\Pi_1(M)$ is 4, and $\Pi_1(M) = Z_2 \oplus Z_2$ on Z_4. Epstein [1] completely determined all possible abelian groups which can be fundamental groups of closed 3-manifolds; Z, $Z \oplus Z \oplus Z$, $Z \oplus Z_2$, and Z_r. Hence $\Pi_1(M)$ should be Z_4. Hence $S^3/Z_4 = M$. Rice [15] discussed free Z_4-action on S^3. As a consequence of the discussion, $M = L(4,1)$. Since every involution on P^3 is sense preserving, by Theorem 2.9 and the above discussion, we have the following Corollary.

Corollary 2.10: Up to PL equivalences, there is exactly one orientation preserving PL involution h on P^3 with $\mathrm{Fix}(h) \neq \emptyset$ and there is exactly one free involution on P^3.

Definition 2.11: $L(p,q)$ is called symmetric if $q^2 \equiv \pm 1 \bmod p$.

Since $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q' \equiv \pm q$ or $qq' \equiv \pm 1 \bmod p$, L being symmetric is a topological property.

Let h be an orientation preserving PL involution on a lens space $L(p,q)$, p even, which preserves sense and non-empty fixed point set. By Theorem 1.7, $\mathrm{Fix}(h)$ is a disjoint union of two simple closed curves F_1 and F_2. We ask that under what condition there exists a PL equivalent homeomorphism t with respect to h such that $t(F_1) = F_2$. Indeed, there exists such a t if $L(p,q)$ is symmetric. Furthermore, the converse is true. By Theorem 2.9, we may assume $h = h(p,c,b,d)$ on $L_f(p,c,b,d)$. If $L(p,q)$ is symmetric, $b^2 \equiv \pm 1 \bmod p$. Since $pd - cb = 1$, $c = \pm b + mp$ for some integer m. By Lemma 2.8, we have the following equivariant maps t_i. $h(p,c,b,d) = h(p, \pm b+mp, b,d) \overset{t_1}{\sim} h(p, b, \pm b+mp, d) \overset{t_2}{\sim} h(p, b, \pm b, d-mb) \overset{t_3}{\sim} h(p, \pm b, b, d-mb) \overset{t_4}{\sim} h(p, \pm b+mp, b, d) = h(p,c,b,d)$. Recall that $t_1(D^2 \times S^1) = S^1 \times D^2$ and $t_i(D^2 \times S^1) = D^2 \times S^1$. Let $t = t_4 t_3 t_2 t_1$.

Then t is a PL equivariant homeomorphism on $L_f(p,c,b,d)$ such that $t(O \times S^1) = S^1 \times O$. Conversely, suppose that there exists a PL equivariant homeomorphism t on $L(p,q)$ such that $t(F_1) = F_2$. By the proof of Theorem 1.7, $\Pi_1(L) = \{\alpha,\beta \mid \beta^c = \alpha,\ \beta^p = 1\}$ where α and β are represented by the loops F_1 and F_2, respectively. Let $t_\#$ be the automorphism induced by t. Then $t_\#(\alpha) = \beta^\epsilon$ and $t_\#(\beta) = \alpha^\delta$ where $\epsilon = \pm 1$ and $\delta = \pm 1$. Since $t_\#(\beta^c) = \alpha^{\delta c} = \beta^{\delta c^2}$ and $t_\#(\alpha) = \beta^\epsilon$, $\beta^\epsilon = \beta^{\delta c^2}$, and $c^2 \equiv \pm 1 \mod p$. Since $pd - bc = 1$, $b^2 \equiv \pm 1 \mod p$, which implies $L(p,q)$ is symmetric. Thus, we have the following theorem.

<u>Theorem 2.11</u>: Let h be an orientation preserving PL involution on a lens space $L = L(p,q)$, p even, which preserves sense and has non-empty fixed point set. Then there exists a PL equivariant homeomorphism t such that t interchanges the two components of $\mathrm{Fix}(h)$ if and only if L is symmetric.

Let M_i $(i = 1,2)$ be oriented, connected, closed, irreducible 3-manifolds. It is known [7] that a PL involution h on $M_1 \# M_2$ is either the obvious involution which interchanges M_1 and M_2 or of the form $h_1 \# h_2$ where each h_i is a PL involution on M_i. In the latter case, $\mathrm{Fix}(h)$ is not empty, and if $\dim \mathrm{Fix}(h) = 1$, the 2-sphere along which the M_1 and M_2 are joined meets F in general position. When each M_i happens to be a lens space and h is of the form $h_1 \# h_2$, it will be convenient to call decomposed sense preserving if h induces the identity of $H_1(M) = H_1(M_1) \oplus H_1(M_2)$. Obviously, if h is decomposed sense preserving, each h_i is sense preserving. In this case, if h is orientation preserving involution with $\mathrm{Fix}(h) \neq \emptyset$ and M_1 is symmetric, by Theorem 2.11, h does not depend on how an invariant 3-cell of M_1 is chosen to construct $h_1 \# h_2$. Therefore, the following corollary is obtained by using Kwun's result [10] and Theorem 2.9.

Corollary 2.12: Up to PL equivalences, there exists exactly one decomposed sense preserving PL involution h on $L(p,q) \# L(\bar{p},\bar{q})$, which preserves the orientation if $L(p,q)$ and $L(\bar{p},\bar{q})$ are symmetric ($p,\bar{p}$ are any integers). There exist exactly two such h if $L(p,q)$ is symmetric and $L(\bar{p},\bar{q})$ is non-symmetric lens space with $\bar{p}$ odd (p is any integer).

Since any involution h on $P^3 \# P^3$ of the form $h_1 \# h_2$ is decomposed sense preserving, we have the following corollary.

Corollary 2.13: Let h be an orientation preserving PL involution on $P^3 \# P^3$. If $\mathrm{Fix}(h) = \emptyset$ or $\mathrm{Fix}(h)$ is connected, h is the obvious involution which interchanges the two P^3. If $\mathrm{Fix}(h)$ is not connected, $\mathrm{Fix}(h)$ is a disjoint union of three simple closed curves and there is exactly one such h, up to PL equivalences.

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