



This is to certify that the

thesis entitled

INVERSION FORMULAE FOR THE PROBABILITY
MEASURES ON BANACH SPACES

presented by

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has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Statistics and
Probability

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ABSTRACT

INVERSION FORMULAE FOR THE PROBABILITY MEASURES ON BANACH SPACES

By

Gholamhossein Gharagoz Hamedani

Let B be a real separable Banach space, and let μ be a probability measure on $\mathcal{B}(B)$, the Borel sets of B . The characteristic functional (Fourier transform) φ of μ defined by $\varphi(y) = \int_B \exp\{i(y,x)\} d\mu(x)$ for $y \in B^*$ (the topological dual of B) uniquely determines μ .

In order to determine μ on $\mathcal{B}(B)$, it suffices to obtain the value of $\int_B G(s) d\mu(s)$ for every real valued bounded continuous function G on B . Hence an inversion formula for μ in terms of φ is obtained if one can uniquely determine the value of $\int_B G(s) d\mu(s)$ for all real valued bounded continuous functions G on B in terms of φ and G . The main efforts of this thesis will be to prove such inversion formulae of various types.

For the Orlicz space E_α of real sequences we establish inversion formulae (Main Theorem II) which properly generalize the work of L. Gross and derive as a Corollary the extension of the Main Theorem of L. Gross to E_α spaces (Corollary 2.2.12).

In Chapter One we prove a Theorem (Main Theorem I) which is Banach space generalization of the Main Theorem of L. Gross which differs from the Main Theorem II in the sense that the class

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of probability measures for which inversion formulae hold is smaller than that of the Main Theorem II.

Finally in Chapter Three we assume our Banach space to have a shrinking Schander basis to prove inversion formulae (Main Theorem III) which express the measure directly in terms of φ and G without the use of extension of φ as required in the Main Theorems I and II. Furthermore this is achieved without using Lévy Continuity Theorem and hope that one can use this Theorem to obtain a direct proof for the Lévy Continuity Theorem.

INVERSION FORMULAE FOR THE PROBABILITY
MEASURES ON BANACH SPACES

By

Gholamhossein Gharagoz Hamedani

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0. INTRODUCTION

Let $(B, \|\cdot\|_B)$ be a real Banach space with Schauder basis $\{b_n\}$. Let $\mathcal{B}(B)$ denote the Borel sets of B , that is, the σ -field generated by the open sets. The characteristic functional (Fourier transform) φ of a probability measure μ on $\mathcal{B}(B)$ defined by $\varphi(y) = \int_B \exp \{i(y, x)\} d\mu(x)$ for $y \in B^*$ (the topological dual space of B) uniquely determines μ .

In order to determine μ on $\mathcal{B}(B)$, it suffices to obtain the value of $\int_B G(s) d\mu(s)$ for every real valued bounded continuous function G on B . Hence an inversion formula for μ in terms of φ is obtained if one can uniquely determine the value of $\int_B G(s) d\mu(s)$ for all real valued bounded continuous functions G on B in terms of φ and G . The main effort of this Thesis will be to prove such inversion formulae of various type for different Banach spaces B . The Main Theorems I, II, III give inversion formulae which express $\int_B G(s) d\mu(s)$ in terms of φ and G .

In order to motivate these formulae let us consider first a finite dimensional space R_k , a probability measure μ on R_k and a real valued bounded continuous function G on R_k . If G were not only bounded but also in $L^1(R_k, dx)$ and if μ were absolutely continuous with respect to Lebesgue measure with an L^2 derivative then putting

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$$(0.1) \quad \hat{G}(y) = (2\pi)^{-k/2} \int_{R_k} G(x) \exp[-i(x,y)] dx$$

the Plancherel Theorem asserts that

$$(0.2) \quad \int_{R_k} G(s) d\mu(s) = (2\pi)^{-k/2} \int_{R_k} \varphi(y) \hat{G}(y) dy .$$

In the absence of the validity of the two assumptions made above neither of the two right hand integrals in (0.1) and (0.2) need exist. However, upon inserting the convergence factors $\exp[-\|x\|^2/2\alpha^2]$ and $\exp[-\|y\|^2/2t^2]$ in these two integrals one may obtain an inversion formula in the following well known manner. Assuming merely that G is real valued, bounded, continuous we put

$$\hat{G}_\alpha(y) = (2\pi)^{-k/2} \int_{R_k} G(x) \exp[-\|x\|^2/2\alpha^2] \exp[-i(x,y)] dx .$$

Then it can easily be shown that for any probability measure μ

$$\int_{R_k} G(s) d\mu(s) = \lim_{\alpha, t \rightarrow \infty} (2\pi)^{-k/2} \int_{R_k} \varphi(y) \hat{G}_\alpha(y) \exp[-\|y\|^2/2t^2] dy .$$

Upon rewriting these equations in terms of the canonical normal distribution (cf., Remark 1.1.8) on R_k (with variance parameter one) we obtain by a change of variables

$$\int_{R_k} G(s) d\mu(s) = \lim_{\alpha, t \rightarrow \infty} \alpha^k t^k E_y E_x \{ \varphi(ty) G(\alpha x) \exp[-it\alpha(x,y)] \}$$

where E_x and E_y denote expectation in the indicated variable with respect to the canonical normal distribution. Before extending this formula to a real separable Hilbert space H three difficulties must be overcome in its formulation. First of all in the limit as $k \rightarrow \infty$, $(\alpha t)^k$ approaches either zero, one or ∞ .

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Secondly the expectation with respect to the canonical normal distribution on H will not make sense for a general bounded continuous function G ([8], Theorem 1) and thirdly even if G were, for example, uniformly τ -continuous (cf., Definition 1.1.11) so that $E_x\{G(\alpha x)\exp[it\alpha(x,y)]\}$ makes sense the resulting function of y need not define a measurable function with respect to the canonical normal distribution so that $E_y\{\dots\}$ will not be defined. These problems were first pointed out and handled by L. Gross in [7] by modifying the finite dimensional formula as follows. Instead of using the convergence factor $\exp[-\|x\|^2/2\alpha^2]$ one uses $\exp[-\|A^{-1}x\|^2/2\alpha^2]$ where A is an invertible operator on R_k . Furthermore, let us observe that $(t\alpha)^k |\det A|$ is asymptotically equal to $[\det(I + (t\alpha)^2 AA^*)]^{1/2}$ in the sense that their ratio approaches one as t and $\alpha \rightarrow \infty$. After an obvious change of variables one can then obtain the following correct inversion formula in R_k

$$(0.3) \quad \int_{R_k} G(s) d\mu(s) = \lim_{t, \alpha \rightarrow \infty} [\det(I + (t\alpha)^2 AA^*)]^{1/2} E_y E_x \{ \varphi(ty) G(\alpha Ax) \exp[-it\alpha(Ax, y)] \}.$$

In formulating this equation in a Hilbert space one notes that the determinant factor makes sense provided AA^* is trace class operator, that is, provided A is a Hilbert-Schmidt operator. Furthermore, if A is a Hilbert-Schmidt operator then it may be seen in view of Theorem 1 and Corollary 3.2 of [7] that the second and third difficulties mentioned above also vanish, since $E_x\{G(\alpha Ax)\exp[-it\alpha(Ax, y)]\}$ may also be written as

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$\int_H \exp[-it\alpha(x,y)] \cdot G(\alpha A) d\nu A^{-1}$ where ν is the canonical normal distribution on H . Since νA^{-1} is a measure on H ([7], Corollary 3.2) the last integral not only makes sense for an arbitrary bounded continuous function G but also defines a uniformly τ -continuous function of y ([7], Theorem 1).

A fourth difficulty now arises. An estimate of the difference between the left hand side of (0.3) and the expression under the limit sign in (0.3) shows that although this difference goes to zero for each dimension k as t and α go to ∞ independently, the rate at which the difference goes to zero depends more and more critically as the dimension k gets larger on the relative manner in which α and t go to ∞ . In the limiting case of a Hilbert space it results that the relative growth rates of α and t must be restricted. In the Theorem below which is a restatement of Theorem 4 of [7], this is effected by putting $\alpha = f(t)$ as is done in [7] following earlier work of Cameron and Donsker mentioned there.

0.4. Theorem. Let A be a Hilbert-Schmidt operator with dense range on a real separable Hilbert space H . Let μ be a probability measure on H and $f(t)$ a positive admissible function (cf., Definition 1.4.2) on $(0, \infty)$. Let $h(t)$ be a positive function on $(0, \infty)$ and denote by ν the measure νA^{-1} where ν is the canonical normal distribution on H . Let φ be the characteristic function of μ and denote by C_t the positive square root of $I + t^2 f(t)^2 A A^*$. Let E_n denote expectation with respect to the canonical normal distribution. In order for the inversion formula

$$(0.5) \quad \int_H G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t) E_n \{ \varphi(ty) \sim (\int_H G(f(t)x) \exp[-itf(t)(x,y)] dv(x)) \sim \}$$

to hold for all real valued bounded continuous functions G the following two conditions are necessary and sufficient

$$(0.6) \quad f(t)^2 \text{ trace } (C_t^{-2} A A^*) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

$$(0.7) \quad \text{The measures } h(t) \exp[-t^2 \|C_t^{-1} s\|^2 / 2] d\mu(s) \text{ converge weakly to } \mu \text{ as } t \rightarrow \infty.$$

Furthermore if (0.6) and (0.7) hold then (0.3) also holds for any bounded measurable function G which is strongly continuous almost everywhere with respect to μ .

The condition (0.6) of L. Gross although valid for Hilbert space seems to depend heavily on the symmetry structure of the space. We re-interpret this condition for a general Banach space in terms of convergence of certain Gaussian measures (cf., Lemma 2.3.4). In terms of this re-interpretation the Theorem can then be extended to a Banach space with Schauder basis as follows. Using the fact that B has a Schauder basis, we can, following ideas of J. Kuelbs [12], imbed B measurably in a real separable Hilbert space H_λ , whose norm is weaker than the Banach norm $\|\cdot\|_B$. We then treat the probability measure μ on B as a probability measure on H_λ . This enables us to get the necessary and sufficient conditions for the inversion of μ regarded as a measure on H_λ using essentially ideas of L. Gross [7]. However this method allows one to obtain such a formula only for G bounded and continuous on H_λ , which is a proper subclass of the G 's required.

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To circumvent the problem we have to use essentially the notion of λ -family introduced by J. Kuelbs and V. Mandrekar [13], which exhibits the detailed structure of the probability measure μ on H_λ which is actually supported on B . In Chapter I, such an inversion formula is obtained for any Banach space with Schauder basis.

Our initial objective in Chapter II is to prove a Theorem (Main Theorem II) which generalizes Theorem 4 of L. Gross [7] in his form. For this purpose we will need to restrict ourselves to Orlicz space E_α of real sequences since in this case the form of characteristic functional of a Gaussian measure is known (See, [13]). The Main Theorem II is stronger than the Main Theorem I, in the sense that, in case of E_α spaces, the class of measures for which the inversion formulae can be obtained from the Main Theorem II is larger than that of the Main Theorem I. We further assume that the function $\alpha(\cdot)$ associated with E_α possesses a particular property relative to one-dimensional Gaussian measures to get Corollary 2.2.12 which gives us analytic condition for the inversion formulae and also gives precise generalization of the main inversion formulae of L. Gross ([7], Theorem 4) to Orlicz spaces of real sequences.

Finally in the third Chapter we let $(B, \|\cdot\|_B)$ be a real Banach space with shrinking Schauder basis $\{b_n\}$. Since $\{b_n\}$ is shrinking, the coordinate functionals on B form a basis for B^* , and hence we may consider B^* as a Borel measurable subset of ℓ , the vector space of all sequences of real numbers with topology of coordinatewise convergence. Also we shall let n

be the canonical normal distribution on H_λ so that for each $x \in H_\lambda$, $n(x)$ is a random variable on B^* , and let P_λ be the countably additive (will be shown) cylinder set measure on B^* induced by the above family. Then we shall prove a Theorem (Main Theorem III) which gives a class of inversion formulae different from that of the Main Theorem I. In the Main Theorem I we have extension of characteristic functional to $\mathcal{L}$ whereas in the Main Theorem III we have extension of characteristic functional to B^* . Hence (3.2.2)_(a) is stronger than (1.4.4)_(a) since for $\lambda \in \mathcal{L}_1^+$, P_λ is countably additive on B^* . Furthermore since $\{b_n\}$ is shrinking we are able to give a proof for the Theorem without using Lévy Continuity Theorem and hope that one might be able to use this Theorem to obtain a proof for the Lévy Continuity Theorem.

CHAPTER I

INVERSION FORMULAE OF THE CHARACTERISTIC FUNCTIONAL OF A PROBABILITY MEASURE ON BANACH SPACES WITH A SCHAUDER BASIS

§1.0. Introduction.

Let $(B, \|\cdot\|_B)$ be a real Banach space with a Schauder basis denoted by $\{b_n\}$. Let $\mathcal{B}(B)$ be the σ -field generated by the open subsets of B . Every element of $\mathcal{B}(B)$ will be called a Borel set. In order to determine a probability measure μ on $\mathcal{B}(B)$, it suffices to obtain the value of $\int_B G(s) d\mu(s)$ for every real valued bounded continuous function G on B . Hence an inversion formula for μ in terms of its characteristic functional, φ is obtained if one can determine uniquely the value of $\int_B G(s) d\mu(s)$ for all real valued bounded continuous functions G on B in terms of φ and G . The main effort of this Chapter will be to prove such inversion formulae for μ on $\mathcal{B}(B)$.

Following [12], we shall first define a particular inner product on B which generates a norm weaker than the Banach norm $\|\cdot\|_B$. Upon completing B with respect to this norm we will obtain a real separable Hilbert space H_λ with the prescribed inner product. Since $\|x\|_B$ is measurable with respect to the norm $\|\cdot\|_\lambda$ on H_λ , it follows that $\mathcal{B}(B)$ is contained in the Borel subsets of H_λ which we denote by $\mathcal{B}(H_\lambda)$. Thus any probability measure μ on $(B, \mathcal{B}(B))$ induces a probability measure on $(H_\lambda, \mathcal{B}(H_\lambda))$ by defining subsets of $H_\lambda - B$ to be of μ -measure zero.

Now if μ is a probability measure on $(B, \mathcal{B}(B))$ with characteristic functional, φ , then μ can be defined to be a probability measure on $(H_\lambda, \mathcal{B}(H_\lambda))$ with characteristic functional, $\psi(\cdot)$. Note that $\psi(\cdot)$ is actually the restriction of φ to H_λ^* . L. Gross describes various inversion formulae for $\psi(\cdot)$ ([7], §4). We will use Gross' result ([7], Theorem 4), the notion of " λ -family" and the idea of "stochastic linear functional" first occurring in [13] and [14], to prove a class of inversion formulae for φ .

We start by introducing some preliminaries required in the remainder of this Chapter.

§1.1. Basic Definitions.

In this Section we present for the sake of completeness some standard concepts and definitions. For further details the reader is referred to [1] and [16].

1.1.1. Definition (a). Let S be a complete separable metric space and let $\mathcal{M}$ be the space of positive finite measures defined on the σ -field generated by the open subsets of S . A sequence μ_n of measures in $\mathcal{M}$ is said to converge weakly to a measure μ in $\mathcal{M}$ if $\int_S f d\mu_n \rightarrow \int_S f d\mu$ for every bounded continuous function f on S . We will denote this convergence by $\mu_n \xrightarrow{W} \mu$.

If $\{\mu_t : t \in (0, \infty)\}$ is a family of measures in $\mathcal{M}$, then we say $\mu_t \xrightarrow{W} \mu$ as $t \rightarrow \infty$, if for any sequence $\{t_n\}$ approaching infinity, $\mu_{t_n} \xrightarrow{W} \mu$.

(b) A sequence μ_n of measures in $\mathcal{M}$ is said to be conditionally compact (tight), if for every $\epsilon > 0$ there exists a compact set K^ϵ in S such that $\mu_n(K^\epsilon) > 1 - \epsilon$ for all n .

1.1.2. Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable on Ω taking values in S . Then X is said to be distributed as ν if $\nu = \text{Po}X^{-1}$. A family of S -valued random variables $\{X_t : t \in (0, \infty)\}$ is said to converge in distribution to an S -valued random variable X as $t \rightarrow \infty$ if $\text{Po}X_t^{-1} \xrightarrow{W} \text{Po}X^{-1}$ as $t \rightarrow \infty$. We will denote this convergence by $X_t \xrightarrow{D} X$ as $t \rightarrow \infty$. The following definitions are due to I. Segal and are taken here from [7].

1.1.3. Definition. A weak distribution on a topological linear space L is an equivalence class of linear mappings F from the (topological) dual space L^* to real-valued random variables on a probability space (depending on F) where two such mappings F_1 and F_2 are equivalent if for every finite set of vectors $y_1, \dots, y_k$ in L^* the sets $\{F_1(y_1), \dots, F_1(y_k)\}$ have the same distribution in k space for $i = 1$ or 2 .

Here L^* denotes the space of continuous linear functionals on L .

In a finite dimensional space a weak distribution coincides with the notion of a measure, that is, if L is finite dimensional then for any given weak distribution there exists a unique Borel probability measure on L such that the identity map on L^* is a representative of the given weak distribution ([9], p. 372).

1.1.4. Definition (a). A weak distribution m on a Banach space B is said to be continuous if for any sequence $\{y_k\} \subseteq B^*$, $\|y_k\|_{B^*}^1 \rightarrow 0$ implies $m(y_k)$ converges to zero in probability.

1) For $y \in B^*$, $\|y\|_{B^*} = \sup_{\|x\|_B \leq 1} |y(x)|$ (See, e.g. [18], p. 160).

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(b) A weak distribution m on a topological linear space L is a measure if there exists a probability measure μ defined on the σ -field S generated by weakly open subsets of L such that the identity map on L^* is a representative of m .

1.1.5. Definition. If m is a weak distribution on a locally convex topological linear space L and A is a continuous linear operator on L with adjoint A^* , then the weak distribution $y \rightarrow m(A^*y)$ will be denoted by mA^{-1} .

1.1.6. Definition. A measure μ on a locally convex topological linear space L is defined to be Gaussian if, for every continuous linear functional T on L , $T(x)$ has a Gaussian distribution. μ is called Gaussian with mean zero if, in addition $T(x)$ has mean zero for every T .

1.1.7. Definition. The characteristic functional (Fourier transform) of a probability measure μ on the Borel subsets of a linear topological space L is the function $\varphi(\cdot)$ on L^* (the topological dual of L) given by

$$\varphi(y) = \int_L \exp\{i(y,x)\} d\mu(x), \text{ for each } y \in L^*.$$

1.1.8. Remark. One special example of a weak distribution on a real separable Hilbert space H is the canonical normal distribution (with variance parameter one). This weak distribution is that unique weak distribution which assigns to each vector y in H^* a normally distributed random variable with mean zero and variance $\|y\|^2$. It follows from the preceding property that the canonical normal distribution carries orthogonal vectors into independent random variables ([7], p. 4). It is known that some of

the theory of integration with respect to a measure can also be carried out with respect to a weak distribution. For details we refer the reader to [9] and the bibliography given there. We shall also need the following definition from ([20], p. 190).

1.1.9. Definition. An operator from a real separable Hilbert space H into H , which is, linear, symmetric, nonnegative, compact, and having finite trace is called an S -operator.

If T is an S -operator on H , then it is well known that T has the representation

$$(1.1.10) \quad Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n$$

where $\{e_n\}$ is some orthonormal subset of H , $\lambda_n \geq 0$, and

$$\sum_{n=1}^{\infty} \lambda_n < \infty.$$

The S -operator T on ℓ_2 has a representation as an infinite symmetric, nonnegative-definite matrix $T = \{t_{ij}\}$ where by non-negative-definite it is meant that $\sum_{i,k=1}^n t_{ik} x_i x_k \geq 0$ for any integer n and any $(x_1, \dots, x_n) \in R_n$. Furthermore, $t_{ik} = (Tf_i, f_k)$ where f_j is the vector in ℓ_2 of all zeros except one in the j^{th} position and hence $\sum_{i=1}^{\infty} t_{ii} = \sum_{j=1}^{\infty} \lambda_j < \infty$ where λ_j 's are as in (1.1.10). From the representation in (1.1.10) it is easy to verify that $(Tcx, cx)^{\frac{1}{2}} = |c| (Tx, x)^{\frac{1}{2}}$ for any real number c and $(T(x+y), x+y)^{\frac{1}{2}} \leq (Tx, x)^{\frac{1}{2}} + (Ty, y)^{\frac{1}{2}}$. Thus $(Tx, x)^{\frac{1}{2}}$ is a semi-norm on ℓ_2 . Let Σ be the class of all S -operators.

1.1.11. Definition of τ -topology. The τ -topology on ℓ_2 is the smallest locally convex topology generated by the family of semi-norms $p_T(x) = (Tx, x)^{\frac{1}{2}}$ on ℓ_2 as T varies through Σ ([18], p. 172).

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1.1.12. Definition. Let H_1, H_2 be Hilbert spaces with orthonormal systems $\{e_n\}, \{f_n\}$ respectively. Then a continuous linear operator A from H_1 into H_2 is called Hilbert-Schmidt operator if there exists an orthonormal system $\{g_n\}$ in H_1 such that $\sum_{n=1}^{\infty} \|Ag_n\|_{H_2}^2$ is finite ([5], p. 34).

1.1.13. Remarks. (a) Let H be a real separable Hilbert space, then H is isomorphic to ℓ_2 . If T is an S-operator on H then T possesses a unique nonnegative, symmetric square root, which we denote by $T^{\frac{1}{2}}$ ([17], Theorem, p. 265). Now using the fact (See; [5], Theorem 4, p. 39) that the square roots of S-operators are Hilbert-Schmidt operators one can easily show that the topology τ on H is the weakest topology on H for which all Hilbert-Schmidt operators are continuous from τ to strong topology on H . Thus a basic open neighborhood of x_0 is $\{x : \|A(x-x_0)\| < \epsilon\}$ whenever A is a Hilbert-Schmidt operator on H . Therefore our definition of τ -topology coincides with that of L. Gross ([7], p. 5).

(b) By Corollary 3.2 of [7], if m is a continuous weak distribution on a real separable Hilbert space H and A is a Hilbert-Schmidt operator, then mA^{-1} is a measure on H . Hence noA^{-1} is a measure on H if n is the canonical normal distribution on H , since in this case n is clearly a continuous weak distribution on H . We will use the same notation, namely, noA^{-1} for the weak distribution noA^{-1} and its corresponding probability measure μ (See; Definition 1.1.4).

1.1.14. Definition. A tame function on a real Hilbert space H is a function of the form $f(x) = \phi(Px)$ where P is a

finite dimensional projection on H and ϕ is a Baire function on the finite dimensional space PH .

For such a function we have $f(x) = \psi((x, x_1), \dots, (x, x_k))$ where $x_1, \dots, x_k$ is a basis of PH and ψ is a Baire function of k real variables. If F is a representative of a weak distribution then the random variable $f \sim = \psi(F(x_1), \dots, F(x_k))$ depends only on the function f and the mapping F while integration properties of $f \sim$ such as the integral of $f \sim$, the distribution of $f \sim$, convergence in probability of sequences $f_n \sim$, etc. depend only on f and the f_n 's and on the weak distribution of which F is a representative. Let us denote by $\mathcal{F}$ the directed set of finite dimensional projections on H directed under inclusion of the ranges. For a given continuous function f on H and a given weak distribution one may consider whether the net $(f \circ P) \sim$ of the random variables where P ranges over directed set $\mathcal{F}$, converges in probability as P approaches the identity through $\mathcal{F}$. If so then the limit which we shall denote by $f \sim$ is a random variable whose integration properties are completely determined by the function f and the weak distribution. In [8] and [9] classes of continuous functions are described for which the limit defining the random variable $f \sim$ exists when the weak distribution in question is the canonical normal distribution, and some explicit evaluations are also given. The part of this integration required for our purpose is given below and is directly taken from ([9], p. 374).

It is clear that a function f on H is a tame function if and only if there is a finite dimensional projection P on H

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such that $f(x) = f(Px)$ for all x and such that f restricted to the finite dimensional space PH is a Baire function. Then f is said to be based on PH . If f is based on PH then it is clearly also based on QH whenever $Q \geq P$ and Q is a projection. If f is based on the finite dimensional subspace PH then we note that its expectation with respect to the canonical normal distribution (with variance parameter one) is given by

$$E(f^{\sim}) = (2\pi)^{-k/2} \int_{PH} f(x) \exp[-\|x\|^2/2] dx$$

when the integral exists where k is the dimension of PH and dx is Lebesgue measure on PH .

1.1.15. Remark. Let H be a real separable Hilbert space, and let $\{P_j\}$ be any sequence of finite dimensional projections converging strongly to the identity operator. If a complex-valued function f on H is uniformly continuous in the topology τ then $\lim_{j \rightarrow \infty} \text{in prob. } (f \circ P_j)^{\sim}$ exists with respect to the canonical normal distribution and equals $f^{\sim}$ ([7], Theorem, p. 5). Now if for each j , $E[(f \circ P_j)^{\sim}]$ exists and if $\lim_{j \rightarrow \infty} E[(f \circ P_j)^{\sim}]$ exists then following L. Gross [8], we say that f is integrable with respect to the canonical normal distribution. That is, for a uniformly τ -continuous function f on H which is integrable we denote by

$$E_n(f^{\sim}) = \lim_{j \rightarrow \infty} E[(f \circ P_j)^{\sim}] .$$

§1.2. Measures on Banach spaces with a Schauder basis.

We shall study here Banach spaces with Schauder basis. We need some preliminary results for measures on such spaces. We start with the following definition from ([3], p. 67).

1.2.1. Definition. Let B be a Banach space. A Schauder basis $\{b_i\}$ in B is a sequence of elements of B such that for each x in B there is a unique sequence of real numbers $\{a_i\}$, depending on x , such that

$$\lim_{n \rightarrow \infty} \|x - \sum_{i=1}^n a_i b_i\|_B = 0 ;$$

the series $\sum_{i=1}^{\infty} a_i b_i$ is called the expansion of x in the basis $\{b_i\}$, and the coefficient $a_i = \beta_i(x)$ is the i^{th} coordinate of x in the basis $\{b_i\}$.

Throughout this Chapter B will denote a real Banach space with Schauder basis $\{b_n\}$ such that without loss of generality $\|b_n\|_B = 1$ ([3], p. 68). We will write the expansion of x as $\sum_{n=1}^{\infty} \beta_n(x) b_n$ and this emphasizes that the coefficients generate coordinate functionals on B . It is clear that these coordinate functionals are linear and it is well known that they are continuous as well ([3], p. 68). Further it is possible to assume without loss of generality ([3], Theorem 1, p. 67) that

$$(1.2.2) \quad \|x\|_B = \sup_k \left\| \sum_{n=1}^k \beta_n(x) b_n \right\|_B .$$

Following ideas in [12], we introduce a Hilbert space associated with B .

For $\lambda \in \ell_1^+$ and $x, y \in B$ define

$$(1.2.3) \quad (x, y) = \sum_{n=1}^{\infty} \lambda_n \beta_n(x) \beta_n(y) \quad (\text{convergence follows from page 17, line 2})$$

where β_n 's are coordinate functionals. Then $(,)$ is an inner product on B and B is a pre-Hilbert space with the norm

$$\|x\|_{\lambda} = (x, x)^{\frac{1}{2}} \quad \text{and}$$

$$\|x\|_{\lambda}^2 \leq \|\lambda\|_1 \sup_n |\beta_n^2(x)| = C_1 \sup_n |\beta_n^2(x)|.$$

We know ([3], p. 68) that $\sup_n |\beta_n^2(x)| \leq C_2 \|x\|_B^2$, and hence

$$\|x\|_{\lambda}^2 \leq C_1 C_2 \|x\|_B^2 = C \|x\|_B^2.$$

This implies that the topology on B induced by $\|\cdot\|_{\lambda}$ is weaker than the norm topology on B .

Let H_{λ} denote the completion of B under $\|\cdot\|_{\lambda}$. Then clearly $B \subseteq H_{\lambda}$. Upon replacing y by b_k in (1.2.3) we get $(x, b_k) = \lambda_k \beta_k(x)$. Since $\lambda_k > 0$, $\beta_k(x)$ is uniformly continuous in x in $\|\cdot\|_{\lambda}$ -topology on B , and since B is dense in H_{λ} , $\beta_k(\cdot)$ can be extended uniquely to a continuous linear functional $\hat{\beta}_k(\cdot)$ on H_{λ} . Furthermore it can easily be seen that for $x, y \in H_{\lambda}$,

$$(1.2.4) \quad (x, y) = \sum_{n=1}^{\infty} \lambda_n \hat{\beta}_n(x) \hat{\beta}_n(y).$$

From (1.2.2) and the fact that β_n is a $\|\cdot\|_{\lambda}$ -continuous function on B , it follows that $\|x\|_B$ is a measurable function in $\|\cdot\|_{\lambda}$ -topology, and hence B is a $\|\cdot\|_{\lambda}$ -measurable subset of H_{λ} . Therefore if μ is a measure on B , it can be regarded as a measure on H_{λ} via $\mu(A) = \mu(A \cap B)$ for all $A \in \mathcal{B}(H_{\lambda})$.

Let ν be a Gaussian measure with mean zero on $(B, \mathcal{B}(B))$. Then by argument similar to Lemma 2.2. of [12], ν is a Gaussian measure on H_{λ} , and therefore there exists a nonnegative, symmetric trace class operator (that is, an S-operator) T_{ν} on H_{λ} such that

$$(T_{\nu} x, x)_{H_{\lambda}} = \int_{H_{\lambda}} (y, x)^2 d\nu(y)$$

for $x \in H_{\lambda}$, and that ν is uniquely determined on H_{λ} by the operator T_{ν} . These results are well known and appear, for example,

in [20]. Furthermore, T_ν has the representation (1.1.10), that is,

$$(1.2.5) \quad T_\nu(\cdot) = \sum_{k=1}^{\infty} \eta_k(\cdot, g_k) g_k$$

on H_λ where $\{g_k\}$ is an orthonormal sequence in H_λ and $\eta_k \geq 0$, $\sum_k \eta_k < \infty$.

1.2.6. Remarks. (a) Since B is separable, $\mathcal{B}(B)$ is the same as σ -field generated by the weakly open sets and the latter one is the same as σ -field generated by the field of the cylinder sets.

(b) Since T_ν is an S-operator, $T_\nu^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H_λ , and hence $\text{no}T_\nu^{-\frac{1}{2}}$ is a measure on H_λ . But T_ν uniquely determines ν , so by definition 1.1.4, ν is the probability measure on H_λ corresponding to the weak distribution $\text{no}T_\nu^{-\frac{1}{2}}$.

(c) Since cylinder sets in H_λ and B are the same, from (a) and (b) above, it follows that $\text{no}T_\nu^{-\frac{1}{2}}$ is countably additive on $(B, \mathcal{B}(B))$ and $\text{no}T_\nu^{-\frac{1}{2}}(B) = 1$.

(d) We note that (c) could also be obtained from the fact that the Borel subsets of B are also Borel subsets of H_λ (because $\|x\|_B$ is $\|\cdot\|_\lambda$ -measurable) and therefore every countably additive measure on H_λ is countably additive on B .

Following ideas are motivated by [13].

1.2.7. Definition. If $\lambda \in \ell_\infty$ and $\{\mu_t : t \in A\}$ is a family of probability measures on B such that

$$\mu_t\{x \in B : \sum_{n=1}^{\infty} \lambda_n \beta_n^2(x) < \infty\} = 1$$

for each $t \in A$ we say λ is sufficient for the family $\{\mu_t : t \in A\}$.

Now for each $x \in B$ we have $\sup_n |\beta_n(x)| < \infty$, thus it follows that any $\lambda \in \ell_1^+$ is sufficient for any family of probability measure on B .

1.2.8. Definition. A family of probability measures $\{\mu_t : t \in A\}$ on B is a λ -family for some $\lambda \in \ell_\infty^+$ if λ is sufficient for $\{\mu_t : t \in A\}$ and for every $\epsilon, \delta > 0$ there is a sequence $\{\epsilon_N\}$ such that

$$\mu_t\{x \in B : \sum_{n=N+1}^{\infty} \lambda_n \beta_n^2(x) < \delta\} > 1 - \epsilon$$

implies

$$\mu_t\{x \in B : \|\sum_{n=N+1}^{\infty} \beta_n(x) b_n\|_B < k(\delta)\} > 1 - (\epsilon + \epsilon_N) \text{ for all } t$$

where $\lim_N \epsilon_N = 0$ and k is a strictly increasing continuous function on $[0, \infty)$ with $k(0) = 0$.

A family of probability measures $\{\mu_t : t \in (0, \infty)\}$ on B is said to be a λ -family as $t \rightarrow \infty$, if for any sequence $\{t_n\}$ approaching infinity the family $\{\mu_{t_n} : n = 1, 2, \dots\}$ is a λ -family.

It is quite clear that any family of probability measures on a real separable Hilbert space is a λ -family with $\lambda = (1, 1, \dots)$ and $k(\delta) = \delta^{\frac{1}{2}}$.

For $x \in B$, $N = 1, 2, \dots$ we define

$$P_N x = \sum_{k=1}^N \beta_k(x) b_k$$

$$Q_N x = \sum_{k=N+1}^{\infty} \beta_k(x) b_k$$

and for $y \in B^*$, $N = 1, 2, \dots$ we define

$$P_N y(\cdot) = \sum_{k=1}^N \beta_k(\cdot) y(b_k).$$

1.2.9. Definition. If $\{\mu_t : t \in A\}$ is a family of probability measures on B and $\lambda \in \mathcal{L}_\infty^+$ then we say the family of c.f.'s $\{\varphi_{\mu_t}(\cdot); t \in A\}$ is λ -continuous at zero in B^* if:

(i) for every integer N the family $\{\varphi_{\mu_t}(\cdot); t \in A\}$ are equicontinuous at zero in $P_N(B^*)$, and

(ii) $\lim_N \sup_t \lim_k J_{N,k} [1 - \operatorname{Re} \varphi_{\mu_t}(\cdot)] = 0$ where

$$J_{N,k}[\dots] = \int (P_{N+k} - P_N)_{B^*} [\dots] \xi_\lambda(N, k, dy)$$

and $\xi_\lambda(N, k, \cdot)$ is the Gaussian product measure on $(P_{N+k} - P_N)_{B^*}$ with each coordinate $y(b_i)$, $N+1 \leq i \leq N+k$, Gaussian with mean zero and variance λ_i .

1.2.10. Lemma. Let

(i) $\lambda \in \mathcal{L}_1^+$,

(ii) $\{\mu_t : t \in (0, \infty)\}$ be a λ -family as $t \rightarrow \infty$ of probability measures on B ,

(iii) $\mu_t \xrightarrow{W} \mu$ on H_λ as $t \rightarrow \infty$ where μ is a probability measure on B .

Then $\mu_t \xrightarrow{W} \mu$ on B as $t \rightarrow \infty$.

Proof. Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then (iii) implies that $\mu_{t_n} \xrightarrow{W} \mu$ on H_λ , that is, $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact on H_λ . Since $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact on H_λ and P_N is continuous it follows that $\{\mu_{t_n} \circ P_N^{-1} : n = 1, 2, \dots\}$ is compact on H_λ for all $N = 1, 2, \dots$, which is equivalent to saying $\{\mu_{t_n} \circ P_N^{-1} : n = 1, 2, \dots\}$ is compact on B for all $N = 1, 2, \dots$. Thus the c.f.'s $\{\varphi_{\mu_{t_n} \circ P_N^{-1}}(\cdot) : n = 1, 2, \dots\}$

are equicontinuous at zero on $P_N(B^*)$ ([15], Corollary 2, p. 193) and since

$$\varphi_{\mu_{t_n} \circ P_n^{-1}}(x) = \varphi_{\mu_{t_n}}(P_n x)$$

the equicontinuity at zero on $P_N(B^*)$ of $\{\varphi_{\mu_{t_n}}(\cdot) : n = 1, 2, \dots\}$ follows. Hence condition (i) in Definition 1.2.9 is satisfied.

Let $\lambda \in \mathcal{L}_1^+$ and let $\epsilon > 0$ be given and choose a compact set K^ϵ in H_λ such that $\mu_{t_n}(K^\epsilon) > 1 - \epsilon/2$ for $n = 1, 2, \dots$. Since $\varphi_{\mu_{t_n}}(\cdot)$ is c.f. of μ_{t_n} and $\xi_\lambda(N, k, \cdot)$ is symmetric about zero it follows that

$$J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(y)] = \int_B [1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{N+k} \lambda_i \beta_i^2(x)\}] d\mu_{t_n}(x).$$

Since $\mu_{t_n}(B) = 1$ and $\beta_i = \hat{\beta}_i$ on B , it follows that

$$J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(y)] = \int_{H_\lambda} [1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{N+k} \lambda_i \hat{\beta}_i^2(x)\}] d\mu_{t_n}(x).$$

Since $1 - e^{-\zeta} \leq 1$ for $\zeta \geq 0$ we have

$$\leq \int_{K^\epsilon} [1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{N+k} \lambda_i \hat{\beta}_i^2(x)\}] d\mu_{t_n}(x) + \epsilon/2.$$

We note that $\sup_{x \in K^\epsilon} \sup_i |\hat{\beta}_i(x)| < M$ since $\sup_i |\hat{\beta}_i(x)| \leq C\|x\|_\lambda$

([3], p. 68) and $\|x\|_\lambda$ is a continuous function. Hence

$$J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(\cdot)] \leq 1 - \exp\{-\frac{M^2}{2} \sum_{i=N+1}^{\infty} \lambda_i\} + \epsilon/2$$

for all $n = 1, 2, \dots$, and letting N approach infinity and using the fact that $\lambda \in \mathcal{L}_1^+$ we have the right hand term dominated by $\epsilon/2$. Thus condition (ii) of Definition 1.2.9 is satisfied, and

hence $\{\varphi_{\mu_t}(\cdot); n = 1, 2, \dots\}$ is λ -continuous for $\lambda \in \mathcal{L}_1^+$. This together with the assumption that $\{\mu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$ imply ([13], Lemma 3.2, p. 11) that $\{\mu_t : n = 1, 2, \dots\}$ is conditionally compact on B . Hence $\mu_t \xrightarrow{W} \mu$ on B since $\{\mu_t : n = 1, 2, \dots\}$ is conditionally compact on B and $\mu_{t_n} \circ P_N^{-1} \xrightarrow{W} \mu \circ P_N^{-1}$ for all $N = 1, 2, \dots$ ([1], p. 35). Since this is true for any sequence $\{t_n\}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ we get $\mu_t \xrightarrow{W} \mu$ on B as $t \rightarrow \infty$.

1.2.11. Remark. We remark that from the proof of Lemma 1.2.10 one can derive the stronger statement: Let $\{\mu_t : t \in A\}$ be a λ -family of probability measures on B . Then μ_t conditionally compact on H_λ implies that $\{\mu_t : t \in A\}$ is conditionally compact on B . However, since we shall be needing only the statement in Lemma 1.2.10 for further easy reference we have not stated the Lemma in all its generality.

§1.3. Extensions of characteristic functional.

Suppose $\mathcal{L}$ is the space of all real sequences with the topology of coordinatewise convergence and $P_\lambda(\cdot)$ is the product probability measure on $\mathcal{L}$ such that the i^{th} coordinate is Gaussian with mean zero and variance λ_i where $\lambda = \{\lambda_i\} \in \mathcal{L}_1^+$. If μ is a probability measure on B then $\lambda \in \mathcal{L}_1^+$ is sufficient for μ and for $y \in \mathcal{L}$ we define the "stochastic linear functional" on B in the following manner:

$$(y, x)^\sim = \lim_N \sum_{i=1}^N \hat{\beta}_i(x) y_i = \lim_N \sum_{i=1}^N \beta_i(x) y_i.$$

The following Lemma is proved in [13], page 22.

1.3.1. Lemma. The stochastic linear functional

$$(1.3.2) \quad (y, x)^{\sim} = \lim_N \sum_{i=1}^N \hat{\beta}_i(x) y_i$$

is Borel measurable on $\mathcal{L} \times B$ and if

$$F = \{(y, x) : (y, x)^{\sim} \text{ exists and is finite}\},$$

then $P_{\lambda} \times \mu(F) = 1$.

1.3.3. Definition. If μ is a probability measure on the Borel subsets of B with the c.f. φ , we define the extended characteristic functional $\tilde{\varphi}(\cdot)$ on $\mathcal{L}$ as follows

$$\tilde{\varphi}(y) = \int_B \exp\{i(y, x)^{\sim}\} d\mu(x) \quad (y \in \mathcal{L})$$

([13], p. 23).

Then $\tilde{\varphi}(\cdot)$ is a Borel measurable function on $\mathcal{L}$ which is defined almost everywhere with respect to the measure P_{λ} . Furthermore, since each $y \in B^*$ generates the unique sequence of real numbers $\tilde{y} = \{y(b_1), \dots, y(b_k), \dots\}$ we may consider B^* as a linear subset of $\mathcal{L}$ under J. Kampe de Fériet map ([11], pp. 123-127), and hence the terminology extended c.f. since for $y \in B^*$ and $x \in B$, $(y, x)^{\sim} = (y, x)$ which implies that $\varphi(y) = \tilde{\varphi}(\tilde{y})$.

1.3.4. Remark. Let μ be a probability measure on B with c.f. φ , then as was shown earlier, μ can be regarded as a probability measure on H_{λ} . Let ψ be the c.f. of μ when μ is regarded as a measure on H_{λ} , then

$$\psi(y) = \int_{H_{\lambda}} \exp\{i(y, x)\} d\mu(x) \quad \forall y \in H_{\lambda}^* \subseteq B^* \subseteq \mathcal{L}.$$

By Theorem 1 of ([7], p. 7), ψ is uniformly τ -continuous on H_{λ}^* ,

and hence by Theorem of ([7], p. 5), the random variable $\psi^{\sim}$ (that is, the Gross extension of ψ) is well-defined with respect to the canonical normal distribution n on H_{λ} . Finally from (1.3.2), the fact that $\mu(B) = 1$, and Lemma 4.3 of [14], it follows that $\psi(y)^{\sim} = \tilde{\varphi}(y)$ almost everywhere with respect to P_{λ} .

§1.4. General inversion formulae.

Let ν be a Gaussian measure on $(B, \mathcal{B}(B))$ with mean zero. Then $\nu(A) = \nu(A \cap B)$, $A \in \mathcal{B}(H_{\lambda})$ is a Gaussian measure on $(H_{\lambda}, \mathcal{B}(H_{\lambda}))$, and there exists a nonnegative, symmetric trace class operator T_{ν} on H_{λ} corresponding to ν . Let $f(t)$ be a real valued function defined on $(0, \infty)$ and denote by C_t the positive square root of $I + t^2 f(t)^2 T_{\nu}$ on H_{λ} (See; 1.1.13 (a)). Let μ be a probability measure on $(B, \mathcal{B}(B))$, and define

$$\mu_t(A) = \frac{1}{a_t} \int_A h(t) \exp[-t^2 \|C_t^{-1} s\|_{\lambda}^2 / 2] d\mu(s) \quad A \in \mathcal{B}(B)$$

where $h(t)$ is a positive function on $(0, \infty)$, and

$$(1.4.1) \quad a_t = \int_B h(t) \exp[-t^2 \|C_t^{-1} s\|_{\lambda}^2 / 2] d\mu(s).$$

1.4.2. Definition. A real valued function $f(t)$ defined on $(0, \infty)$ will be called admissible if $t f(t) \rightarrow \infty$ as $t \rightarrow \infty$.

We are now ready to state and prove the Main Theorem I.

1.4.3. Main Theorem I. Let

- (i) B be a Banach space with Schauder basis $\{b_i\}$,
- (ii) μ be a probability measure on $(B, \mathcal{B}(B))$ with the c.f. φ ,
- (iii) $f(t)$ be a positive admissible function on $(0, \infty)$, and $h(t)$ be a positive function on $(0, \infty)$,

- (iv) X be a B -valued random variable distributed as ν where ν is a Gaussian measure on $(B, \mathcal{B}(B))$ with mean zero and the property that T_ν is positive-definite,
- (v) Y_t and Y be B -valued random variables distributed as μ_t and μ respectively,
- (vi) E_{P_λ} denote the integral with respect to P_λ on $\mathcal{L}$.

Then for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B the following are equivalent:

$$(1.4.4) \quad \begin{cases} (a) & \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t) E_{P_\lambda} \{ \varphi^\sim(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)^\sim] d\nu(x)) \} \\ (b) & \{ \mu_t : t \in (0, \infty) \} \text{ is a } \lambda\text{-family as } t \rightarrow \infty \end{cases}$$

$$(1.4.5) \quad \begin{cases} (a) & f(t) C_t^{-1} X \xrightarrow{\mathcal{D}} 0 \text{ as } t \rightarrow \infty \\ (b) & Y_t \xrightarrow{\mathcal{D}} Y \text{ as } t \rightarrow \infty. \end{cases}$$

Proof. We shall first show (1.4.4) implies (1.4.5). Let G be a real valued, bounded, $\|\cdot\|_B$ -continuous function on B , and observe that

$$\begin{aligned} & E_{P_\lambda} \{ \varphi^\sim(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)^\sim] d\nu(x)) \} \\ &= \int_{\mathcal{L}} (\int_B e^{it(y,s)^\sim} d\mu(s)) (\int_B G(f(t)x) e^{-itf(t)(y,x)^\sim} d\nu(x)) dP_\lambda(y). \end{aligned}$$

We note that $e^{it(y,s)^\sim} e^{-itf(t)(y,x)^\sim} G(f(t)x)$ is jointly measurable since it is a product of a jointly measurable function of s, x and y with a $\|\cdot\|_B$ -continuous function of x namely $G(f(t)x)$. Since it is bounded and all the measures are probability measures by Fubini's Theorem ([19], p. 140) we have

$$\begin{aligned}
& E_{P_\lambda} \{ \varphi^\approx(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)^\approx] dv(x)) \} \\
&= \int_B \int_B G(f(t)x) (\int_{\mathcal{L}} e^{it(y, s-f(t)x)^\approx} d_{P_\lambda}(y)) dv(x) d\mu(s) \\
&= \int_B \int_B G(f(t)x) (\int_{\mathcal{L}} e^{it \lim_{N \rightarrow \infty} \sum_{i=1}^N \beta_i(s-f(t)x)y_i} d_{P_\lambda}(y)) dv(x) d\mu(s).
\end{aligned}$$

By Dominated Convergence Theorem we get

$$\begin{aligned}
&= \int_B \int_B G(f(t)x) (\lim_{N \rightarrow \infty} \int_{\mathcal{L}} e^{it \sum_{i=1}^N \beta_i(s-f(t)x)y_i} d_{P_\lambda}(y)) dv(x) d\mu(s) \\
&= \int_B \int_B G(f(t)x) (\lim_{N \rightarrow \infty} \exp\{-\frac{t^2}{2} \sum_{i=1}^N \lambda_i \beta_i^2(s-f(t)x)\}) dv(x) d\mu(s) \\
&= \int_B \int_B G(f(t)x) \exp\{-\frac{t^2}{2} \sum_{i=1}^\infty \lambda_i \beta_i^2(s-f(t)x)\} dv(x) d\mu(s) \\
&= \int_B \int_B G(f(t)x) \exp[-t^2 \|s-f(t)x\|_\lambda^2 / 2] dv(x) d\mu(s).
\end{aligned}$$

Now from 1.2.6 (c) and the fact that $\mu(B) = 1$, it follows that

$$= \int_{H_\lambda} \int_{H_\lambda} G(f(t)x) \exp[-t^2 \|s-f(t)x\|_\lambda^2 / 2] dnoT_\nu^{-\frac{1}{2}}(x) d\mu(s).$$

We note that G is $\|\cdot\|_\lambda$ -measurable (that is, measurable in the $\|\cdot\|_\lambda$ -topology) since the norm $\|x\|_B$ is $\|\cdot\|_\lambda$ -measurable and G is $\|\cdot\|_B$ -continuous; and $T^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H_λ .

Hence by Lemma 4.1 of [7] we have

$$= \frac{1}{(\det C_t)} \int_{H_\lambda} \int_{H_\lambda} G(f(t)C_t^{-1}x + s - C_t^{-2}s) dnoT_\nu^{-\frac{1}{2}}(x) h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s).$$

Again from 1.2.6 (c) and the fact that $\mu(B) = 1$ we get

$$= \frac{1}{(\det C_t)} \int_B \int_B G(f(t)C_t^{-1}x + s - C_t^{-2}s) dv(x) h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s).$$

We may now start with the assumption that for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B we have

$$(1.4.6) \quad \begin{cases} (a) & \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} \int_B [\int_B G(f(t)C_t^{-1}x + s - C_t^{-2}s) dv(x)] h(t) \\ & \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s) \\ (b) & \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \rightarrow \infty. \end{cases}$$

Putting $G \equiv 1$ in (1.4.6)_(a) we get

$$1 = \lim_{t \rightarrow \infty} \int_B h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s).$$

From (1.4.1), it follows that

$$(1.4.7) \quad 1 = \lim_{t \rightarrow \infty} a_t.$$

Using (1.4.7) we obtain

$$(1.4.8) \quad \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} \int_B \int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s) dv(x) d\mu_t(s)$$

for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B .

From (1.4.8), it follows that

$$(1.4.9) \quad \nu \circ (f(t)C_t^{-1})^{-1} * \mu_t \circ (I - C_t^{-2})^{-1} \xrightarrow{W} \mu \text{ on } B \text{ as } t \rightarrow \infty.$$

Since G is bounded on B and ν is a probability measure, the measure $G(f(t)x)dv(x)$ is a measure of bounded variation on B , and hence a measure of bounded variation on H_λ . Therefore by Theorem 1 of ([7], p. 7) the Fourier transform of $G(f(t)x)dv(x)$ is uniformly τ -continuous, and hence the Gross extension of its Fourier transform is well-defined ([7], Theorem, p. 5). Similarly the Gross extension of the Fourier transform (c.f.) $\psi(\cdot)$ of μ

when regarding μ as a measure on $(H_\lambda, \mathcal{B}(H_\lambda))$ is well-defined.

Now from Remarks 1.3.4 and 1.2.6 (c), it follows that

$$(1.4.10) \quad E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)] dv(x)) \} \\ = E_{P_\lambda} \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} .$$

Let $\{P_j\}$ be a sequence of finite dimensional projections on H_λ converging strongly (that is, in $\|\cdot\|_\lambda$ -topology) to the identity operator. Then using the fact that P_j 's are continuous together with Lebesgue Dominated Convergence Theorem we obtain

$$E_{P_\lambda} \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} \\ = \lim_{j \rightarrow \infty} E_{P_\lambda} \{ \tilde{\psi}(tP_j y) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} .$$

Now using the fact that the integral of a tame function with respect to the product measure P_λ is the same as its integral with respect to the canonical normal distribution n on H_λ , we get

$$E_{P_\lambda} \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} \\ = \lim_{j \rightarrow \infty} E_n \{ \tilde{\psi}(tP_j y) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} .$$

where by E_n we mean integral with respect to the canonical normal distribution n on H_λ .

From Remark 1.1.15, it follows that

$$\lim_{j \rightarrow \infty} E_n \{ \tilde{\psi}(tP_j y) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} \\ = E_n \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y, x)] d\mu_{T_\lambda^{-\frac{1}{2}}}(x)) \} .$$

Hence from (1.4.10) we get

$$\begin{aligned}
& E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)] d\nu(x)) \} \\
& = E_n \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\nu_{\lambda}^{-\frac{1}{2}}(x)) \} .
\end{aligned}$$

Now from (1.4.4) (a) and the fact that $\|\cdot\|_\lambda$ -topology is weaker than $\|\cdot\|_B$ -topology on B we obtain

$$\begin{aligned}
\int_{H_\lambda} G(s) d\mu(s) &= \lim_{t \rightarrow \infty} h(t) (\det C_t) E_n \{ \tilde{\psi}(ty) (\int_{H_\lambda} G(f(t)x) \\
&\quad \exp[-itf(t)(y,x)] d\nu_{\lambda}^{-\frac{1}{2}}(x)) \}
\end{aligned}$$

for all real valued, bounded, $\|\cdot\|_\lambda$ -continuous functions G on H_λ .

Therefore by Theorem 4 of [7] we have

$$(1.4.11) \quad \begin{cases} (a) & f(t)^2 \text{ trace } (C_t^{-2} T_\nu) \rightarrow 0 \text{ as } t \rightarrow \infty \\ (b) & \text{The measures } h(t) \exp[-t^2 \|C_t^{-1} s\|_\lambda^2 / 2] d\mu(s) \text{ converge} \\ & \text{weakly to } \mu \text{ on } H_\lambda \text{ as } t \rightarrow \infty . \end{cases}$$

Now (1.4.11) (a) implies ([7], Corollary 3.4) that

$$(1.4.12) \quad f(t) C_t^{-1} X \xrightarrow{P} 0 \text{ on } H_\lambda \text{ as } t \rightarrow \infty ,$$

and (1.4.11) (b) together with (1.4.7) imply (Definition 1.1.2 (b)) that

$$(1.4.13) \quad Y_t \xrightarrow{P} Y \text{ on } H_\lambda \text{ as } t \rightarrow \infty .$$

By the assumption $\{\mu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$. This together with (1.4.13) satisfy the hypotheses of Lemma 1.2.10, and hence the conclusion of the Lemma which is $Y_t \xrightarrow{P} Y$ on B as $t \rightarrow \infty$ gives us (1.4.5) (b).

To get (1.4.5)_(a) we note that, it is easy to verify using the fact that T_ν is positive-definite (and therefore T_ν has positive eigenvalues) and $\text{tr}(t) \rightarrow \infty$ as $t \rightarrow \infty$ that C_t^{-2} converges strongly to zero operator as $t \rightarrow \infty$. One needs only express $\|C_t^{-2}x\|_\lambda^2$ in terms of an orthonormal basis in H_λ which diagonalizes T_ν . Hence $I - C_t^{-2}$ converges strongly to I , and clearly

$$(1.4.14) \quad \mu\{x \in B : (I - C_t^{-2})x_n \not\rightarrow x \text{ when } x_n \rightarrow x\} = 0.$$

From (1.4.14) and (1.4.5)_(b), it follows ([1], Theorem 5.5, p. 34) that

$$(1.4.15) \quad (I - C_t^{-2})Y_t \xrightarrow{\mathcal{D}} Y \text{ on } B \text{ as } t \rightarrow \infty.$$

Let $f(t)C_t^{-1}X$ be distributed as ν_t , then (1.4.9) and (1.4.15) imply ([16], Theorem 2.1, p. 58) that for any sequence t_n approaching infinity, $\{\nu_{t_n} : n = 1, 2, \dots\}$ is conditionally compact on B . Now by Lemma 3.1 of [13], $\{\nu_{t_n}\}$ is a λ -family for any $\lambda \in \mathcal{L}_\infty^+$ which is sufficient for $\{\nu_{t_n} : n = 1, 2, \dots\}$. But by assumption $\lambda \in \mathcal{L}_1^+$ which is sufficient for any family of probability measures on B , and hence $\{\nu_{t_n}\}$ is a λ -family on B . Since this is true for any sequence $\{t_n\}$ approaching infinity we conclude that $\{\nu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$ of probability measures on $(B, \mathcal{B}(B))$. From this and (1.4.12) it follows (Lemma 1.2.10) that

$$f(t)C_t^{-1}X \xrightarrow{\mathcal{D}} 0 \text{ on } B \text{ as } t \rightarrow \infty,$$

which is (1.4.5)_(a).

We now proceed to the proof of that (1.4.5) implies (1.4.4).

(1.4.5)_(b) implies (Definition 1.1.2) that for any sequence $t_n \rightarrow \infty$, $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact, and hence it is a λ -family for any $\lambda \in \mathcal{L}_1^+$ ([13], Lemma 3.1). Thus $\{\mu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$, hence (1.4.4)_(b) holds.

Furthermore, from (1.4.5)_(b), it follows that

$$(1.4.16) \quad \lim_{t \rightarrow \infty} a_t = 1.$$

Let G be a real valued, bounded, $\|\cdot\|_B$ -continuous function on B , and let

$$\beta_t = h(t) (\det C_t) E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_B G(f(t)x) \exp[-itf(t)(y, x) \tilde{\varphi}] dv(x)) \} - \int_B G(s) d\mu(s).$$

From (1.4.16) we get

$$\lim_{t \rightarrow \infty} \beta_t = \lim_{t \rightarrow \infty} \frac{\beta_t}{a_t} = \lim_{t \rightarrow \infty} \frac{1}{a_t} h(t) (\det C_t) E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_B G(f(t)x) \exp[-itf(t)(y, x) \tilde{\varphi}] dv(x)) \} - \int_B G(s) d\mu(s).$$

By the argument used to obtain (1.4.8) we have

$$(1.4.17) \quad \lim_{t \rightarrow \infty} \beta_t = \lim_{t \rightarrow \infty} \int_B \int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s) dv(x) d\mu_t(s) - \int_B G(s) d\mu(s).$$

We note that (1.4.15) and (1.4.5)_(a) imply ([16], Lemma 1.1 and Theorem 1.1, p. 57)

$$v \circ (f(t)C_t^{-1})^{-1} * \mu_t \circ (I - C_t^{-2})^{-1} \xrightarrow{W} \mu \quad \text{on } B \quad \text{as } t \rightarrow \infty,$$

which can equivalently be written as

$$(1.4.18) \quad \int_B \int_B G(f(t)C_t^{-1}x + (I-C_t^{-2})s) d\nu(x) d\mu_t(s) \rightarrow \int_B G(s) d\mu(s) \text{ as } t \rightarrow \infty$$

for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B .

From (1.4.17) and (1.4.18) we get $\lim_{t \rightarrow \infty} \beta_t = 0$ which completes the proof.

1.4.19. Corollary. Let

- (i) B be a Banach space with Schauder basis $\{b_i\}$,
- (ii) μ be a finite positive measure on $(B, \mathcal{B}(B))$ with Fourier transform φ ,
- (iii) $f(t)$ be a positive admissible function on $(0, \infty)$,
- (iv) X be a B -valued random variable distributed as ν where ν is a Gaussian measure on B with mean zero and the property that T_ν is positive-definite,
- (v) μ_t be a probability measure on $(B, \mathcal{B}(B))$ where

$$\mu_t(A) = \frac{1}{\int_B \exp[-t^2 \|C_t^{-1}s\|_\lambda^2/2] d\mu(s)} \int_A \exp[-t^2 \|C_t^{-1}s\|_\lambda^2/2] d\mu(s)$$

for all $A \in \mathcal{B}(B)$,

- (vi) E_{P_λ} denote the integral with respect to P_λ on $\mathcal{L}$.

Then for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B the following are equivalent:

$$(1.4.20) \quad \left\{ \begin{array}{l} \text{(a)} \quad \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} (\det C_t) E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_B G(f(t)x) \\ \exp[-itf(t)(y, x)^\sim] d\nu(x)) \} \\ \text{(b)} \quad \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \rightarrow \infty \end{array} \right.$$

$$(1.4.21) \quad \left\{ \begin{array}{l} \text{(a)} \quad f(t)C_t^{-1}X \xrightarrow{d} 0 \text{ as } t \rightarrow \infty \\ \text{(b)} \quad t^2(C_t^{-2}s, s) \rightarrow 0 \text{ in measure with respect to } \mu \\ \text{as } t \rightarrow \infty. \end{array} \right.$$

Proof. Theorem 1.4.3 is clearly true for a finite positive measure μ . Putting $h(t) = 1$ in Theorem 1.4.3 we see that

(1.4.5)_(b) implies $\int_B (1 - \exp[-t^2(C_t^{-2}s, s)/2]) d\mu(s) \rightarrow 0$ as $t \rightarrow \infty$, that is, $\exp[-t^2(C_t^{-2}s, s)/2]$ converges to one in $L^1(B, \mu)$ and in fact this is clearly equivalent to (1.4.5)_(b) when $h(t) = 1$. It is also equivalent to (1.4.21)_(b). (1.4.20) and (1.4.21)_(a) are restatement of (1.4.4) and (1.4.5)_(a) respectively.

CHAPTER II

OPERATOR THEORETIC CONDITIONS FOR THE INVERSION FORMULAE ON F-SPACES POSSESSING A SCHAUDER BASIS AND A QUASI-NORM WHICH IS ACCESSIBLE IN BOTH DIRECTIONS

§2.0. Introduction.

Let E be an F -space with Schauder basis $\{b_n\}$ and a quasi-norm $\|\cdot\|$ which is accessible in both directions ([13], p. 39). In Section 2.1, we will observe that, if the function $\alpha(\cdot)$ given in the definition of accessible quasi-norm $\|\cdot\|$ possesses a particular property, then E and the Orlicz space E_α of real sequences are homeomorphic and isomorphic ([13], Theorem 6.3). In Section 2.3 we reduce for the case of Orlicz space E_α condition (1.4.5)_(a) in the form similar to that of L. Gross ([7], (10), p. 36). This restriction on the space is needed since in this case the form of characteristic functional of a Gaussian measure is known (See; [13]).

We shall first give a Theorem (Main Theorem II) which is Orlicz space generalization of Theorem 4 of Gross [7] and is stronger than the Main Theorem I, in the sense that, in case of Orlicz spaces, the Theorem holds for λ belonging to a set containing ℓ_1^+ . Then we assume that the function $\alpha(\cdot)$ possesses another particular property to be able to get Corollary 2.3.12 which is our main objective. From Corollary 2.3.12 we get inversion formulae for $(\ell_p, p > 2)$ spaces. Finally as a special

case when $E_\alpha = \ell_2$, we take $\lambda = (1, 1, \dots)$, then $H_\lambda = \ell_2$, and Theorem 4 of [7] follows from Corollary 2.3.12.

§2.1. Preliminaries and Definitions.

Let $\alpha(s)$ be a convex function on $[0, \infty)$ such that $\alpha(0) = 0$, $\alpha(s) > 0$ for $s > 0$. Further, assume

$$(2.1.1) \quad \alpha(2s) \leq M \alpha(s)$$

for all $s \geq 0$ and some $M < \infty$.

Now we define E_α as the space of all sequences satisfying $\sum_{i=1}^{\infty} \alpha(x_i^2) < \infty$. Since $\alpha(\cdot)$ is convex, it follows that E_α is a vector space over the reals ([13], p. 49).

Let $\Gamma(s) = \alpha(s^2)$, then $\Gamma(\cdot)$ has the same properties as $\alpha(\cdot)$ and, it follows that

$$\Gamma(s) = \int_0^s \rho(x) dx$$

where $\rho(0) = 0$ and $\rho(s)$ is nondecreasing on $[0, \infty)$. We assume, without loss of generality, that $\rho(s)$ is left continuous.

2.1.2. Definition. By $u = \Phi(v)$ the inverse function of $\rho(u)$ is defined, on the understanding that if $\rho(u)$ makes a jump at $u = a$, then $\Phi(v) = a$ for $\rho(a-) < v \leq \rho(a+)$, while, if $\rho(u) = c$ for $a < u \leq b$, but $\rho(u) < c$ for $u < a$, then $\Phi(c) = a$. Furthermore $\Phi(0) = 0$, and, if $\lim_{u \rightarrow \infty} \rho(u) = \ell$ is finite, then $\Phi(v) = +\infty$ for $v > \ell$. With these conventions $u = \Phi(v)$ is evidently nondecreasing for $v \geq 0$, and left continuous for those values of v at which $\Phi(v)$ is finite. It follows that $\Phi(v)$ is also Borel measurable on $[0, \infty)$.

We now define

$$\Lambda(s) = \int_0^s \Phi(x) dx .$$

Then Γ and Λ are complementary in the sense of Young ([21], p. 77), and by δ_Γ^* we mean all real sequences $\{x_i\}$ such that

$$(2.1.3) \quad \sum_{i=1}^{\infty} \Gamma(|x_i|) = \sum_{i=1}^{\infty} \alpha(x_i^2) < \infty .$$

Similarly, δ_Λ^* is all sequences such that

$$\sum_{i=1}^{\infty} \Lambda(|x_i|) < \infty .$$

From (2.1.3), it follows that δ_Γ^* contains the same sequences as E_α .

If $x = \{x_i\}$ is a sequence we define

$$\|x\|_\Gamma = \sup_y \left\{ \sum_{i=1}^{\infty} |x_i y_i| : \sum_{i=1}^{\infty} \Lambda(|y_i|) \leq 1 \right\}$$

and

$$\|x\|_\Lambda = \sup_y \left\{ \sum_{i=1}^{\infty} |x_i y_i| : \sum_{i=1}^{\infty} \Gamma(|y_i|) \leq 1 \right\} .$$

2.1.4. Definition. The Orlicz space δ_Γ (δ_Λ) is the collection of all real sequences such that $\|x\|_\Gamma$ ($\|x\|_\Lambda$) is finite.

2.1.5. Remarks. (a) Since $\alpha(s)$ satisfies (2.1.1), it follows that

$$\Gamma(2s) = \alpha(4s^2) \leq M \alpha(2s^2) \leq M^2 \alpha(s^2) = M^2 \Gamma(s),$$

and hence we know ([21], Corollary, p. 81) that δ_Γ^* (and, therefore E_α) contains the same sequences as δ_Γ . Further, it is known that δ_Γ is a real separable Banach space in the norm $\|x\|_\Gamma$ and since

$\Gamma(2s) \leq M^2 \Gamma(s)$ for $s \geq 0$ we also have ([20], Lemma α , p. 83)

that $\{p_n\} \subseteq \delta_\Gamma$ converges to $p \in \delta_\Gamma$ in norm provided

$$\lim_n \sum_{i=1}^{\infty} \Gamma(|x_{i,n} - x_i|) = \lim_n \sum_{i=1}^{\infty} \alpha[(x_{i,n} - x_i)^2] = 0$$

where by $x_{i,n}, x_i$ we mean the i^{th} elements of p_n, p respectively.

(b) By Theorem 6.2 of [13], $(E_\alpha, \|\cdot\|_\Gamma)$ is a Banach space with a Schauder basis $\{b_n\}$ where b_n is the vector with one as the n^{th} coordinate and other coordinates zero.

Following [13], we now introduce the notion of F-space with a Schauder basis and an accessible quasi-norm in both directions.

Let E be an F-space with quasi-norm $\|\cdot\|$ (See; [13], p. 2) and Schauder basis $\{b_n\}$. We assume further that the following assumptions (A) are satisfied:

(A.1) the basis elements $\{b_n\}$ can be adjusted so that $\|b_n\| \leq 1$ (this is always possible),

(A.2) if K is any compact subset of E then $\sup_{n, x \in K} |\beta_n(x)| < \infty$ where β_n 's are the coordinate functionals on E ,

(A.3) the σ -field generated by the weakly open subsets of E is identical with the Borel subsets of E .

It can easily be seen that in case $\|\cdot\|$ is actually a norm and E is then a Banach space assumptions (A) are always satisfied.

Let $\alpha(\cdot)$ be a convex function on $[0, \infty)$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ if $s > 0$. Further, assume for every compact subset K of E there exists an $r > 0$ such that $x \in K$ implies $\Delta(x) = \sum_{i=1}^{\infty} \alpha[\beta_i^2(x)] < r$, and for every $r > 0$ there exists an $M > 0$ such that $\Delta(x) < r$ implies

$$\sum_{i=1}^{\infty} \alpha[\beta_i^2(x)] \leq M \gamma(\|x\|)$$

where $\gamma(\cdot)$ is a continuous function on $[0, \infty)$ such that $\gamma(0) = 0$.

2.1.6. Definition. If the quasi-norm $\|\cdot\|$ on E admits the existence of functions $\alpha(\cdot)$ and $\gamma(\cdot)$ having the above properties we will say that it is accessible.

We also note that if $\alpha(\cdot)$ and $\gamma(\cdot)$ satisfy the conditions indicated then $\alpha(\cdot)$ is continuous and strictly increasing on $[0, \infty)$, $\gamma(s) > 0$ for $s > 0$, and that $\gamma(\cdot)$ can be taken to be increasing on $[0, \infty)$.

2.1.7. Definition. The quasi-norm $\|\cdot\|$ on E is said to be accessible in both directions if there exist functions $\alpha, \gamma_1, \gamma_2$ such that $\|\cdot\|$ is accessible with respect to α and γ_2 , and for every $x \in E$

$$\gamma_1(\|x\|) \leq \sum_{i=1}^{\infty} \alpha[\beta_i^2(y)] .$$

Here $\gamma_1(\cdot)$ is an increasing continuous function on $[0, \infty)$ such that $\gamma_1(0) = 0$, $\gamma_1(s) > 0$ for $s > 0$, and α, γ_2 satisfy the conditions required in the Definition 2.1.6.

In this Section we shall be concerned with an F -space E with Schauder basis satisfying assumptions (A) for which the quasi-norm $\|\cdot\|$ is accessible in both directions and the associated $\alpha(\cdot)$ satisfies (2.1.1)

We now recall the following Theorem from ([13], p. 57).

Theorem. If E has a quasi-norm which is accessible in both directions with respect to the functions $\alpha, \gamma_1, \gamma_2$ and $\alpha(\cdot)$ satisfies condition (2.1.1) then E and the Orlicz space

E_α are homeomorphic and isomorphic.

2.1.8. Remark. In view of the above Theorem we can (and will) identify the F-space E with E_α and restrict ourselves to sequence space E_α .

§2.2. Associated Hilbert space.

Following [13], we shall denote by E_α the Hilbert space ℓ_2 or an E_α space where $\alpha(\cdot)$ satisfies (2.1.1). We assume that $\alpha_c(\cdot)$, the complementary function of $\alpha(\cdot)$ in the sense of Young ([21], p. 77), satisfy (2.1.1). Notice that if $E_\alpha = \ell_2$ then a natural choice for the function α is $\alpha(s) \equiv s$. Hence $\alpha_c(s) = 0$ on $[0,1]$ but $\alpha_c(s) = \infty$ for $s > 1$. Thus $\alpha_c(\cdot)$ does not satisfy (2.1.1) when $E_\alpha = \ell_2$ and this is a special case which is easily handled.

In terms of the notations we have used in Section 2.1, E_α is equivalent (isometrically isomorphic) to the Orlicz space δ_Γ where $\Gamma(s) = \alpha(s^2)$. We will let S_α, S_{α_c} denote the Orlicz spaces given by $\alpha(\cdot)$ and $\alpha_c(\cdot)$, respectively. Then the dual space of S_α can be identified as S_{α_c} and since $\alpha_c(\cdot)$ also satisfies (2.1.1), except when $E_\alpha = \ell_2$, it follows that the dual of S_{α_c} is S_α ([21], p. 150).

For each vector $\lambda = (\lambda_1, \lambda_2, \dots)$ in the positive cone of S_{α_c} , we define the space H_λ as all sequences $x = (x_1, x_2, \dots)$ such that $\sum_{i=1}^{\infty} \lambda_i x_i^2 < \infty$. Then H_λ is a Hilbert space with $\|x\|_\lambda = \left(\sum_{i=1}^{\infty} \lambda_i x_i^2 \right)^{\frac{1}{2}}$ and the inner product

$$(x, y) = \sum_{i=1}^{\infty} \lambda_i x_i y_i .$$

In the special case $E_\alpha = \ell_2$ we have $S_{\alpha_c} = \ell_\infty$ and for simplicity we take $\lambda = (1, 1, \dots)$. Then $H_\lambda = \ell_2$ and we shall assume without loss of generality that $\alpha(s) \equiv s$.

The following Lemma is proved in ([13], p. 62).

2.2.1. Lemma. E_α is a Borel subset of H_λ for each λ in the positive cone of S_{α_c} . Furthermore, every Borel subset of E_α is a Borel subset of H_λ .

We note that from Lemma 2.2.1, every probability measure μ on E_α can be regarded as a probability measure on H_λ . Furthermore every countably additive measure on H_λ is countably additive on E_α . Now having this observation in mind we shall prove the following Lemma, which is similar to Lemma 1.2.10.

2.2.2. Lemma. Let

- (i) $\lambda \in S_{\alpha_c}^+$,
- (ii) $\{\mu_t : t \in (0, \infty)\}$ be a λ -family as $t \rightarrow \infty$ of probability measures on E_α ,
- (iii) $\mu_t \xrightarrow{W} \mu$ on H_λ as $t \rightarrow \infty$ where μ is a probability measure on E_α .

Then $\mu_t \xrightarrow{W} \mu$ on E_α as $t \rightarrow \infty$.

Proof. Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then (iii) implies that $\mu_{t_n} \xrightarrow{W} \mu$ on H_λ , that is, $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact on H_λ . Using exactly the same argument given in Lemma 1.2.10 the equicontinuity at zero on $P_N(E_\alpha^*)$ of $\{\varphi_{\mu_{t_n}}(\cdot) : n = 1, 2, \dots\}$ follows. Hence condition (i) in Definition 1.2.9 is satisfied.

Let $\lambda \in S_{\alpha_c}^+$. Since $\varphi_{\mu_{t_n}}(\cdot)$ is c.f. of μ_{t_n} and $\xi_\lambda(N, k, \cdot)$ is symmetric about zero it follows that

$$J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(y)] = \int_{E_\alpha} [1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{N+k} \lambda_i x_i^2\}] d\mu_{t_n}(x).$$

By Young inequality ([21], p. 77)

$$\lambda_i x_i^2 \leq \alpha_c(\lambda_i) + \alpha(x_i^2) \quad \text{for each } i.$$

Hence

$$\begin{aligned} J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(y)] &\leq \int_{E_\alpha} [1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{N+k} \alpha_c(\lambda_i) - \frac{1}{2} \sum_{i=N+1}^{N+k} \alpha(x_i^2)\}] d\mu_{t_n}(x) \\ &\leq 1 - \exp\{-\frac{1}{2} \sum_{i=N+1}^{\infty} \alpha_c(\lambda_i)\} \int_{E_\alpha} \exp\{-\frac{1}{2} \sum_{i=N+1}^{\infty} \alpha(x_i^2)\} d\mu_{t_n}(x). \end{aligned}$$

We note that as $N \rightarrow \infty$, $\sum_{i=N+1}^{\infty} \alpha_c(\lambda_i) \rightarrow 0$ since $\lambda \in S_{\alpha_c}$, and hence

$\exp\{-\frac{1}{2} \sum_{i=N+1}^{\infty} \alpha_c(\lambda_i)\} \rightarrow 1$. Also for $x \in E_\alpha$ we have $\{x_i^2\} \in S_\alpha$,

that is $\sum_{i=1}^{\infty} \alpha(x_i^2) < \infty$. From this it follows that the functions $f_N(x) = \exp\{-\frac{1}{2} \sum_{i=N+1}^{\infty} \alpha(x_i^2)\}$ converge pointwise to one. Further-

more f_N 's are nondecreasing, so by Monotone Convergence Theorem we get

$$\int_{E_\alpha} \exp\{-\frac{1}{2} \sum_{i=N+1}^{\infty} \alpha(x_i^2)\} d\mu_{t_n}(x) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

Therefore

$$J_{N,k}[1 - \operatorname{Re} \varphi_{\mu_{t_n}}(y)] \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{for } n = 1, 2, \dots.$$

Thus condition (ii) of Definition 1.2.9 is also satisfied, and

hence $\{\varphi_{\mu_{t_n}}(\cdot) : n = 1, 2, \dots\}$ is λ -continuous for $\lambda \in S_{\alpha_c}^+$.

This together with the assumption that $\{\mu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$ imply ([13], Lemma 3.2, p. 11) that

$\{\mu_{t_n} : n = 1, 2, \dots\}$ is conditionally compact on E_α .

Hence $\mu_{t_n} \xrightarrow{W} \mu$ on E_α since $\{\mu_{t_n} : n = 1, 2, \dots\}$ is conditionally

compact on E_α and $\mu_{t_n} \circ p_N^{-1} \stackrel{W}{\Rightarrow} \mu \circ p_N^{-1}$ for all $N = 1, 2, \dots$ ([1], p. 35). Since this is true for any sequence $\{t_n\}$ approaching infinity we get $\mu_t \stackrel{W}{\Rightarrow} \mu$ on E_α as $t \rightarrow \infty$.

We use the fact that every linear operator on E_α^* into E_α can be represented as an infinite dimensional matrix to give the following definition.

2.2.3. Definition. A linear operator from E_α^* into E_α is an α -operator if the matrix of the operator, $\{t_{ij}\}$ is symmetric, nonnegative definite with $\sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty$.

2.2.4. Lemma. Let T be an infinite dimensional matrix $\{t_{ij}\}$ such that T is symmetric, nonnegative-definite and $\sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty$. Then T is an α -operator on E_α^* into E_α .

Proof. Let $y \in E_\alpha^*$, and define

$$(Ty)_i = \sum_{j=1}^{\infty} t_{ij} y(b_j)$$

where $\{b_n\}$ is the basis for E_α given in 2.1.5 (b). Clearly T is linear, and the proof will be completed as soon as we show T is well-defined, that is, $Ty \in E_\alpha$ for each $y \in E_\alpha^*$. Since T is symmetric, nonnegative-definite, it follows that

$$|(Ty)_i| \leq \sum_{j=1}^{\infty} |t_{ij}| |y(b_j)| \leq \sum_{j=1}^{\infty} t_{ii}^{\frac{1}{2}} t_{jj}^{\frac{1}{2}} |y(b_j)| = t_{ii}^{\frac{1}{2}} \sum_{j=1}^{\infty} t_{jj}^{\frac{1}{2}} |y(b_j)|.$$

By assumption $\sum_{j=1}^{\infty} \alpha[(t_{jj}^{\frac{1}{2}})^2] = \sum_{j=1}^{\infty} \alpha(t_{jj})$ is finite, hence $\{t_{jj}^{\frac{1}{2}}\} \in E_\alpha$. But $y \in E_\alpha^*$, $\{t_{jj}^{\frac{1}{2}}\} \in E_\alpha$ imply ([21], Theorem 3, p. 82) that

$$\sum_{j=1}^{\infty} t_{jj}^{\frac{1}{2}} |y(b_j)| = A < \infty,$$

hence $|(Ty)_i| \leq A t_{ii}^{\frac{1}{2}}$.

Since $\alpha(\cdot)$ is increasing we get

$$\alpha(|(Ty)_i|^2) \leq \alpha(A^2 t_{ii}) .$$

Now using the fact that $\alpha(\cdot)$ satisfies (2.1.1), for some k

$$\alpha(A^2 t_{ii}) \leq M^k \alpha(t_{ii}) .$$

Therefore

$$\sum_{i=1}^{\infty} \alpha(|(Ty)_i|^2) \leq M^k \sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty ,$$

which implies that $Ty \in E_{\alpha}$, and hence the proof is completed.

We know (Lemma 2.2.1) that E_{α} is a Borel subset of H_{λ} and the $\|\cdot\|_{\Gamma}$ -topology is stronger than $\|\cdot\|_{\lambda}$ -topology on E_{α} . Hence it follows that H_{λ}^* is a subset of E_{α}^* . We now identify H_{λ}^* by H_{λ} and prove the following Lemma.

2.2.5. Lemma. Every α -operator T on E_{α}^* is a trace class operator on H_{λ} .

Proof. Since T is an α -operator on E_{α}^* we get $\sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty$ which implies $\{t_{ii}\} \in S_{\alpha}$, and since $\lambda = \{\lambda_i\} \in S_{\alpha_c}$ it follows ([21], Theorem 3, p. 82) that $\sum_{i=1}^{\infty} \lambda_i t_{ii}$ is finite.

Observe that

$$\text{trace}_{H_{\lambda}} T = \sum_{i=1}^{\infty} \lambda_i t_{ii} ,$$

which is finite, hence T is a trace class operator on H_{λ} .

2.2.6. Remark. It is known (See; Remark 1.1.13) that, if T is a nonnegative, symmetric trace class operator (that is an S -operator) on H_{λ} then $T^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on

H_λ . Hence by 1.1.13 (b), $\nu = nT^{-\frac{1}{2}}$ is a probability measure on H_λ where n is the canonical normal distribution on H_λ . Since T actually takes its values in E_α the measure $\nu = nT^{-\frac{1}{2}}$ has its support in E_α , and since every Borel subset of E_α is a Borel subset of H_λ , the probability measure $nT^{-\frac{1}{2}}$ is countably additive on $(E_\alpha, \mathcal{B}(E_\alpha))$ (See also Remark 1.1.13).

§2.3. Inversion formulae for Orlicz space of real sequences.

In this Section we prove a Theorem (Main Theorem II) which gives a class of inversion formulae for a probability measure on the space E_α . It differs from the Main Theorem I in the sense that, in case of E_α spaces, (2.3.2) (b) is stronger than (1.4.4) (b). This can easily be seen since ℓ_1 is a subset of S_{α_c} .

Let $P_\lambda(\cdot)$ be the product probability measure on ℓ such that the i^{th} coordinate is Gaussian with mean zero and variance λ_i where $\lambda = \{\lambda_i\} \in S_{\alpha_c}$. If μ is a probability measure on E_α then $\lambda \in S_{\alpha_c}$ is sufficient for μ and as before for $y \in \ell$ we define the "stochastic linear functional" on E_α in the following manner:

$$(y, x)^\sim = \lim_N \sum_{i=1}^N x_i y_i.$$

The stochastic linear functional $(y, x)^\sim = \lim_N \sum_{i=1}^N x_i y_i$ is Borel measurable on $\ell \times E_\alpha$ and $P_\lambda \times \mu\{(y, x) : (y, x)^\sim \text{ exists and is finite}\} = 1$ (See; [13], p. 22). If μ is a probability measure on the Borel subsets of E_α with c.f. φ , then we define the extended c.f. $\tilde{\varphi}(\cdot)$ as in Definition 1.3.3. We also note that we can repeat Remark 1.3.4 to get $\psi(y)^\sim = \tilde{\varphi}(y)$ a.e. P_λ where

ψ is c.f. of μ when μ is regarded as a measure on $(H_\lambda, \mathcal{B}(H_\lambda))$.

Now let T be an α -operator on E_α^* , then by Lemma 2.2.5, T is a trace class operator on H_λ . Denote by C_t the positive square root of $I + t^2 f(t)^2 T$ on H_λ where $f(t)$ is a positive admissible function on $(0, \infty)$ ([17], Theorem, p. 265). As in Section 1.4 let μ be a probability measure on $(E_\alpha, \mathcal{B}(E_\alpha))$, and define

$$\mu_t(A) = \frac{1}{a_t} \int_A h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s) \quad A \in \mathcal{B}(E_\alpha)$$

where $h(t)$ is a positive function on $(0, \infty)$, and

$$a_t = \int_{E_\alpha} h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2 / 2] d\mu(s).$$

2.3.1. Main Theorem II. Let

- (i) E_α be as in 2.1.5 (b),
 - (ii) μ be a probability measure on $(E_\alpha, \mathcal{B}(E_\alpha))$ with c.f. φ ,
 - (iii) $f(t)$ be a positive admissible function on $(0, \infty)$, and $h(t)$ be a positive function on $(0, \infty)$,
 - (iv) T be a positive-definite α -operator on E_α^* ,
 - (v) X be an E_α -valued random variable distributed as $v = n\sigma T^{-\frac{1}{2}}$ where n is the canonical normal distribution on H_λ ,
 - (vi) Y_t and Y be E_α -valued random variables distributed as μ_t and μ respectively,
 - (vii) E_{P_λ} denote the integral with respect to P_λ on $\mathcal{L}$.
- Then for all real valued, bounded, $\|\cdot\|_\Gamma$ -continuous functions G on E_α the following are equivalent

$$(2.3.2) \quad \left\{ \begin{array}{l} (a) \quad \int_{E_\alpha} G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t)_{H_\lambda} E_{P_\lambda} \{ \varphi^\sim(ty) (\int_{E_\alpha} G(f(t)x) \\ \exp[-itf(t)(y,x)^\sim] dv(x)) \} \\ (b) \quad \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \rightarrow \infty \end{array} \right.$$

$$(2.3.3) \quad \left\{ \begin{array}{l} (a) \quad f(t) C_t^{-1} X \xrightarrow{D} 0 \text{ as } t \rightarrow \infty \\ (b) \quad Y_t \xrightarrow{D} Y \text{ as } t \rightarrow \infty . \end{array} \right.$$

Proof. The necessary Lemmas (Lemma 2.2.2, Lemma 2.2.5) and Remark (Remark 2.2.6) are given in Section 2.2 and the proof can be carried out in exactly the same way as in the Main Theorem I with no difficulties.

We now put another condition on α , and prove the following Lemma to reduce (2.3.3)_(a) in the form similar to that of L. Gross ([7], (10), p. 36).

2.3.4. Lemma. Let

- (i) E_α be as in 2.1.5 (b),
- (ii) T be a positive α -operator on E_α^* ,
- (iii) X be distributed as $v = n\sigma T^{-\frac{1}{2}}$ where n is the canonical normal distribution on H_λ ,
- (iv) assume further that there exists a constant C such that

$$(2.3.5) \quad \int_{-\infty}^{+\infty} \alpha(u^2) d\mu(u) \leq C \alpha \left[\int_{-\infty}^{+\infty} u^2 d\mu(u) \right]$$

for all Gaussian measure μ on $(-\infty, +\infty)$ with mean zero. Then the following are equivalent:

$$(2.3.6) \quad f(t) C_t^{-1} X \xrightarrow{D} 0 \text{ as } t \rightarrow \infty$$

$$(2.3.7) \quad \sum_{i=1}^{\infty} \alpha(f(t)^2 (C_t^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where $(C_t^{-2} T)_{ii} = (C_t^{-2} T b_i, b_i)_{H_\lambda}$.

Proof. We shall first prove that (2.3.6) implies (2.3.7).

Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$.

Let $f(t_n) C_{t_n}^{-1} X$ be distributed as v_{t_n} , then v_{t_n} is defined on $(E_\alpha, \mathcal{B}(E_\alpha))$

and by (2.3.6), $\{v_{t_n} : n = 1, 2, \dots\}$ converges weakly on E_α to

δ_0 . Since E_α is a Borel subset of H_λ , $\{v_{t_n} : n = 1, 2, \dots\}$ can

be regarded as probability measures on H_λ , and since topology of

H_λ is weaker than that of E_α , v_{t_n} converges weakly on H_λ to

δ_0 . The c.f. of v_{t_n} when regarded as a probability measure on

H_λ is $\exp\{-\frac{1}{2}(f(t_n)^2 C_{t_n}^{-2} T \cdot, \cdot)_{H_\lambda}\}$. Hence $v_{t_n} \xrightarrow{W} \delta_0$ on H_λ as

$n \rightarrow \infty$ implies ([7], Theorem 2, p. 8) that

$$\sum_{i=1}^{\infty} \lambda_i (f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) = \text{trace}_{H_\lambda} (f(t_n)^2 C_{t_n}^{-2} T) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

From this, it follows that

$$\lambda_i (f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for all } i.$$

Since $\lambda_i > 0$, we get

$$f(t_n)^2 (C_{t_n}^{-2} T)_{ii} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for each } i,$$

which implies that

$$\alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for each } i.$$

Hence, for every N

$$(2.3.8) \quad \sum_{i=1}^N \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Under condition (2.3.5) and the fact that $\{v_{t_n}\}$ is compact on E_α we get ([13], Theorem 9.1) that for each $\epsilon > 0$ there exists an N_0 such that

$$(2.3.9) \quad \sum_{i=N_0}^{\infty} \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) < \epsilon \quad \text{for all } t_n.$$

Therefore (2.3.8) and (2.3.9) imply that

$$\sum_{i=1}^{\infty} \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus (2.3.7) holds.

For the converse, let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then (2.3.7) implies that, for sufficiently large n , say n_0 ,

$$\sum_{i=1}^{\infty} \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) < \infty \quad n \geq n_0.$$

Hence for $n \geq n_0$, $\{f(t_n)^2 (C_{t_n}^{-2} T)_{ii}\} \in S_\alpha$.

Now using the fact that $\lambda = \{\lambda_i\} \in S_{\alpha_c}^+$, for $n \geq n_0$ we get ([21], Theorem 3, p. 82) that

$$\sum_{i=1}^{\infty} \lambda_i \{f(t_n)^2 (C_{t_n}^{-2} T)_{ii}\} \leq \|\lambda\|_{\alpha_c} \|\{f(t_n)^2 (C_{t_n}^{-2} T)_{ii}\}\|_{\alpha}.$$

From (2.3.7) and Lemma α of ([21], p. 83), it follows that

$$\|\{f(t_n)^2 (C_{t_n}^{-2} T)_{ii}\}\|_{\alpha} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\text{trace}_{H_\lambda} (f(t_n)^2 C_{t_n}^{-2} T) = \sum_{i=1}^{\infty} \lambda_i (f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, by Corollary 3.4 of [7] we have

$$(2.3.10) \quad v_{t_n} \xrightarrow{W} \delta_0 \quad \text{on } H_\lambda \quad \text{as } n \rightarrow \infty.$$

In order to complete the proof it suffices to show that

$\{\nu_{t_n} : n = 1, 2, \dots\}$ is a λ -family for $\lambda \in S_{\alpha_c}^+$. First observe that $\lambda \in S_{\alpha_c}^+$ is sufficient for $\{\nu_{t_n} : n = 1, 2, \dots\}$ since

$$\sum_{i=1}^{\infty} \lambda_i \alpha(x_i^2) < \infty \quad \text{for } x \in E_{\alpha}.$$

Let $\epsilon, \delta > 0$, and let γ_1 be any continuous strictly increasing function on $[0, \infty)$ satisfying (6.6) of ([13], p. 50). Then from ([13]; (6.6), (b.7), p. 51) it follows that

$$\begin{aligned} \nu_{t_n} \{x : \|Q_N x\|_{\Gamma} > \delta\} &= \nu_{t_n} \{x : \gamma_1(\|Q_N x\|_{\Gamma}) > \gamma_1(\delta)\} \\ &\leq \nu_{t_n} \{x : \sum_{i=N+1}^{\infty} \alpha[\beta_i^2(x)] > \gamma_1(\delta)\} \end{aligned}$$

where $\beta_i(x) = x_i$ is the i^{th} coordinate functional on E_{α} .

Using (2.3.5) we get

$$\begin{aligned} &\leq \frac{C}{\gamma_1(\delta)} \sum_{i=N+1}^{\infty} \alpha \left[\int_{E_{\alpha}} \beta_i^2(x) d\nu_{t_n}(x) \right] \\ &= \frac{C}{\gamma_1(\delta)} \sum_{i=N+1}^{\infty} \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}). \end{aligned}$$

From (2.3.7), it follows that, for each $\epsilon > 0$ there exists an N_0 such that for all $n = 1, 2, \dots$

$$\sum_{i=N_0}^{\infty} \alpha(f(t_n)^2 (C_{t_n}^{-2} T)_{ii}) < \frac{\gamma_1(\delta)}{C} \epsilon.$$

Therefore, for all $n = 1, 2, \dots$, and all $N \geq N_0$

$$(2.3.11) \quad \nu_{t_n} \{x : \|Q_N x\|_{\Gamma} > \delta\} < \epsilon.$$

Hence, $\{\nu_{t_n} : n = 1, 2, \dots\}$ is a λ -family for any $\lambda \in S_{\alpha_c}^+$. Since (2.3.10) and (2.3.11) hold for any sequence $\{t_n\}$ approaching

1

infinity, it follows (Lemma 2.2.2) that

$$v_t \xrightarrow{W} \delta_0 \quad \text{on } E_\alpha \quad \text{as } t \rightarrow \infty,$$

which is (2.3.6), and hence the proof is completed.

2.3.12. Corollary. If, in addition to (i) - (vii) of Theorem 2.3.1, the function $\alpha(\cdot)$ satisfies (2.3.5), then the following are equivalent:

$$(2.3.13) \quad \begin{cases} (a) & \int_{E_\alpha} G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t)_{H_\lambda E_P \lambda} \{ \tilde{\varphi}(ty) (\int_{E_\alpha} G(f(t)x) \\ & \exp[-itf(t)(y,x) \tilde{\varphi}] dv(x)) \} \\ (b) & \{ \mu_t : t \in (0, \infty) \} \text{ is a } \lambda\text{-family as } t \rightarrow \infty \end{cases}$$

$$(2.3.14) \quad \begin{cases} (a) & \sum_{i=1}^{\infty} \alpha(f(t)^2 (C_t^{-2} T)_{ii}) \rightarrow 0 \quad \text{as } t \rightarrow \infty \\ (b) & Y_t \xrightarrow{D} Y \quad \text{as } t \rightarrow \infty. \end{cases}$$

Proof. By Lemma 2.3.4, condition (2.3.14)_(a) is equivalent to (2.3.3)_(a), hence the proof follows from Theorem 2.3.1.

2.3.15. Remark. In case the matrix $\{t_{ij}\}$ of the α -operator T is diagonal, condition (2.3.14)_(a) would be replaced by

$$(2.3.16) \quad \sum_{i=1}^{\infty} \alpha\left(\frac{f(t)^2 t_{ii}}{1+t^2 f(t)^2 t_{ii}}\right) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

2.3.17. Definition. If $2 \leq p < \infty$ then a linear operator T from ℓ_q ($\frac{1}{p} + \frac{1}{q} = 1$) into ℓ_p is an S_p -operator if T can be represented as an infinite symmetric, nonnegative definite matrix

$\{t_{ij}\}$ such that $\sum_{i=1}^{\infty} (t_{ii})^{p/2}$ is finite. Here, by nonnegative definite, we mean that $\sum_{i,j=1}^n t_{ij} x_i x_j \geq 0$ for all $(x_1, x_2, \dots, x_n) \in R_n$

and all integer n . Thus for $p = 2$, S_2 -operator is S -operator.

For $p > 2$, ℓ_p is an E_α space (Orlicz space) with $\alpha(s) = s^{p/2}$ ([21], p. 78). Furthermore, for this $\alpha(\cdot)$, condition (2.3.5) is satisfied. The α -operators on $E_\alpha^* = \ell_p^*$ are the same as S_p -operators, and $S_{\alpha_c}^+ = (\ell_{p/2}^*)^+$. Now Corollary 2.3.12 gives us a proof for the following Corollary.

2.3.18. Corollary. Let $p > 2$, and let

- (i) T be a positive-definite S_p -operator on ℓ_p^* ,
- (ii) μ be a probability measure on $(\ell_p, \mathcal{B}(\ell_p))$ with c.f., φ ,
- (iii) $f(t)$ be a positive admissible function on $(0, \infty)$, and $h(t)$ be a positive function on $(0, \infty)$,
- (iv) $\nu = n\sigma T^{-\frac{1}{2}}$ where n is the canonical normal distribution on H_λ ,
- (v) E_{P_λ} denote the integral with respect to P_λ on ℓ .

Then for all real valued, bounded, ℓ_p -continuous (that is, continuous in ℓ_p norm) functions G on ℓ_p the following are equivalent:

$$(2.3.19) \quad \left\{ \begin{array}{l} (a) \quad \int_{\ell_p} G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t)_{H_\lambda} E_{P_\lambda} \{ \tilde{\varphi}(ty) (\int_{\ell_p} G(f(t)x) \\ \exp[-itf(t)(y, x)] d\nu(x)) \} \\ (b) \quad \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \rightarrow \infty \end{array} \right.$$

$$(2.3.20) \quad \left\{ \begin{array}{l} (a) \quad \sum_{i=1}^{\infty} [f(t)]^p [(C_t^{-2} T)_{ii}]^{p/2} \rightarrow 0 \text{ as } t \rightarrow \infty \\ (b) \quad \text{The measures } \{\mu_t : t \in (0, \infty)\} \text{ converge weakly} \\ \text{to } \mu \text{ on } \ell_p \text{ as } t \rightarrow \infty. \end{array} \right.$$

2.3.21. Remark. (a) If the matrix $\{t_{ij}\}$ of the S_p -operator T is diagonal, then (2.3.20)_(a) would be replaced by

$$(2.3.22) \quad \sum_{i=1}^{\infty} [f(t)]^p \left(\frac{t_{ii}}{1+t^2 f(t)^2 t_{ii}} \right)^{p/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

(b) It can easily be shown that in Theorem 4 of L. Gross [7] one can, without loss of generality, assume that the Hilbert-Schmidt operators in (10) and (11) of ([7], p. 36) to be diagonal.

(c) In the spacial case $E_{\alpha} = \ell_2$ we have $S_{\alpha_c} = \ell_{\infty}$ and for simplicity we take $\lambda = (1, 1, \dots)$, then $H_{\lambda} = \ell_2$. Now using Lemma 4.3 of [14], and the fact that $T^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on ℓ_2 whenever T is an S_2 -operator on ℓ_2 we get Theorem 4 of [7].

(d) From (b), it follows that in case of Hilbert space condition (2.3.22) is restatement of condition (10) of ([7], p. 36).

CHAPTER III

INVERSION FORMULAE OF THE CHARACTERISTIC FUNCTIONAL OF A PROBABILITY MEASURE ON BANACH SPACES WITH A SHRINKING SCHAUDER BASIS

§3.0. Introduction.

In the first Chapter B was taken to be a Banach space with Schauder basis $\{b_n\}$. In this Chapter we assume that the Schauder basis $\{b_n\}$ is also shrinking. Since $\{b_n\}$ is shrinking, the coordinate functionals β_n 's form a basis for B^* , and hence we can use results of J. Kampe de Fériet [11] to identify B^* with a Borel subset of ℓ . Thus any probability measure on B^* can be defined to be a probability measure on ℓ through this identification.

We shall let n be the canonical normal distribution on H_λ so that for each $x \in H_\lambda$, $n(x)$ is a random variable on B^* , and let P_λ be the cylinder set measure on B^* induced by the above family. Then we shall show P_λ is countably additive on the σ -field of tame sets of B^* . Finally we prove a Theorem (Main Theorem III) which gives a class of inversion formulae different from that of the Main Theorem I. In the Main Theorem I we have extension of characteristic functional to ℓ whereas in the Main Theorem III we have extension of characteristic functional to B^* . Hence (3.2.2)_(a) is stronger than (1.4.4)_(a) since for $\lambda \in \ell_1^+$, P_λ is countably additive on B^* . Furthermore

since $\{b_n\}$ is shrinking (and, therefore B^* has a basis) we are able to give a proof for the Theorem without using Lévy Continuity Theorem and hope that one might be able to use this Theorem to obtain a proof for the Lévy Continuity Theorem.

§3.1. Preliminaries and Definitions.

A tame (cylinder) set in a real separable Hilbert space H can be described as a set of the form $C = P^{-1}(E)$ where P is a finite dimensional orthogonal projection on H with range $\mathcal{R}$, say, and E is a Borel set in $\mathcal{R}$. The cylinder set measure ν (See; [6], p.32) associated with the canonical normal distribution is called Gauss measure on H , and for the above tame set C we have

$$\nu(C) = (2\pi)^{-k/2} \int_E e^{-\|x\|^2/2} dx$$

where k is the dimension of $\mathcal{R}$ and dx is Lebesgue measure on R_k .

3.1.1. Definition. A semi-norm $\|x\|_1$ on H is called a measurable semi-norm if for every real number $\epsilon > 0$ there exists a finite dimensional projection P_0 such that for every finite dimensional projection P orthogonal to P_0 we have

$$(3.1.2) \quad \text{Prob}(\|Px\|_1^{\sim} > \epsilon) < \epsilon$$

where $\|Px\|_1^{\sim}$ denotes the random variable on the probability space (Ω, m) corresponding to the tame function $\|Px\|_1$ and Prob. refers to the probability of the indicated event with respect to the probability measure m associated with the canonical normal distribution.

Observe that the condition (3.1.2) can also be written

$$\nu(\{x : \|Px\|_1 > \epsilon\}) < \epsilon$$

where ν is Gauss measure on H (See; [6], p. 33).

We note that a measurable norm is a measurable semi-norm which is a norm.

3.1.3. Definition ([3], p. 69). A Schauder basis $\{b_i\}$ in a Banach space B is called shrinking basis for B if for each β in B^* , $\lim_{n \rightarrow \infty} p_n(\beta) = 0$ where $p_m(\beta) =$ norm of β restricted to the range of $x - \sum_{i=1}^m \beta_i(x)b_i$; that is,

$$p_m(\beta) = \sup\{\beta(x) : \sum_{i=1}^m \beta_i(x)b_i = 0 \text{ and } \|x\|_B \leq 1\}.$$

Throughout this Chapter B will denote a real Banach space with a shrinking Schauder basis $\{b_n\}$ such that $\|b_n\|_B = 1$. As before we will write the expansion of x as $\sum_{n=1}^{\infty} \beta_n(x)b_n$, and

$$\|x\|_B = \lim_{k \rightarrow \infty} \left\| \sum_{n=1}^k \beta_n(x)b_n \right\|_B.$$

For $\lambda \in \mathcal{L}_1^+$, H_λ denotes the completion of B under the inner product (1.2.3).

Let n be the canonical normal distribution on H_λ^* into the set of all random variables defined on B^* , that is, for each $x \in H_\lambda^*$, $n(x)$ is a random variable on B^* which is distributed normally with mean zero and variance $\|x\|_{H_\lambda^*}^2$. We identify H_λ^* by H_λ , hence for each $x \in H_\lambda$, $n(x)$ is a random variable on B^* distributed normally with mean zero and variance $\|x\|_\lambda^2$.

The basis elements b_i 's can be considered as coordinate functionals on B^* ([3], Lemma 1, p. 70). Then $n(b_i) = (b_i, \cdot)$ is a random variable defined on B^* which is distributed normally with mean zero and variance $\|b_i\|_\lambda^2 = (b_i, b_i) = \lambda_i$.

Let P_λ be the cylinder set measure on the field $\mathcal{C}$ generated by tame sets of B^* induced by the above canonical normal distribution on H_λ .

3.1.4. Lemma. P_λ is countably additive on $\mathcal{C}$.

Proof. Without loss of generality we assume $\|\beta_i\|_{B^*} = 1$. By Lemma 1 of ([3], p. 70) $\{\beta_i\}$ is a basis for B^* , hence we have $\|y\|_{B^*} = \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n b_i(y) \beta_i \right\|_{B^*}$. Let $\|y\|_n = \left\| \sum_{i=1}^n b_i(y) \beta_i \right\|_{B^*}$ and observe that

$$\begin{aligned} P_\lambda \{y : \|y\|_{B^*} < \epsilon\} &= P_\lambda \{y : \lim_{n \rightarrow \infty} \|y\|_n < \epsilon\} \\ &> P_\lambda \{y : \lim_{n \rightarrow \infty} \sum_{i=1}^n |b_i(y)| < \epsilon\}. \end{aligned}$$

Let $X(y) = \lim_{n \rightarrow \infty} \sum_{i=1}^n |b_i(y)|$, then X is a random variable on B^* since $E\{\sum_{i=1}^n [b_i(y)]^2\} = \sum_{i=1}^n \lambda_i < \sum_{i=1}^\infty \lambda_i < \infty$, and the series is a series of independent random variables ([15], p. 234; and [2], Theorem 9.5.5). In view of the property of Laplace transform, Theorem 6.6.2 of [2], we observe that the distribution of X is absolute normal with mean $\sum_{i=1}^\infty E[|b_i(y)|]$ and variance $\sum_{i=1}^\infty \lambda_i$. Hence the distribution of X puts mass around zero, and therefore

$$(3.1.5) \quad P_\lambda \{y : \|y\|_{B^*} < \epsilon\} > P_\lambda \{X < \epsilon\} > 0.$$

From the definition of H_λ we see that H_λ is a dense subset of B^* , and hence B^* is the completion of H_λ in the Banach norm $\|\cdot\|_{B^*}$ on B^* . Furthermore, $\|y\|_n$ is a tame function on H_λ and hence it is a measurable norm (See; Definition 3.1.1).

From (3.1.5) and the fact that $\|y\|_n$ is a nondecreasing sequence of measurable norms on H_λ , it follows that $\|y\|_{B^*}$ is

a measurable norm on H_λ ([9], Corollary 4.4).

So far we have shown that $\|y\|_B^*$ is a measurable norm on H_λ and B^* is the completion of H_λ in this norm. Hence by Theorem 1 of [6], P_λ is countably additive on $\mathcal{C}$.

Let $x \in B$, $y \in B^*$, then

$$\begin{aligned} (y, x) = y(x) &= y\left(\sum_{i=1}^{\infty} \beta_i(x) b_i\right) = \sum_{i=1}^{\infty} \beta_i(x) y(b_i) \\ &= \sum_{i=1}^{\infty} \beta_i(x) b_i(y). \end{aligned}$$

It follows that $(\cdot, x) = \sum_{i=1}^{\infty} \beta_i(x) b_i(\cdot)$ is a random variable defined on B^* which is distributed normally with mean zero and variance $\sum_{i=1}^{\infty} \lambda_i \beta_i^2(x)$ under P_λ .

Now let $x \in H_\lambda$, $y \in B^*$ and define the "stochastic linear functional" $(y, x)^{\hat{}}$ as follows

$$(3.1.6) \quad (y, x)^{\hat{}} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \hat{\beta}_i(x) b_i(y)$$

where as before $\hat{\beta}_i$ is extension of β_i to H_λ .

From (3.1.6) we have $(y, x)^{\hat{}} = y(x)$ for $x \in B$, $y \in B^*$.

3.1.7. Lemma. The stochastic linear functional

$$(y, x)^{\hat{}} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \hat{\beta}_i(x) b_i(y)$$

is Borel measurable on $B^* \times H_\lambda$ and if

$$F = \{(y, x) : (y, x)^{\hat{}} \text{ exists and is finite}\},$$

then $P_\lambda \times \mu(F) = 1$ where μ is a probability measure on H_λ .

Proof. That $(y, x)^{\hat{}}$ is jointly Borel measurable follows easily since it is the limit of jointly Borel measurable functions.

Fix x and consider F_x (that is, the x -section of F ; See [10]). For this fixed x , $(y, x)^{\hat{\approx}}$ is the limit of sums of independent random variables, and the variance of the sum $\sum_{i=1}^{\infty} \lambda_i \hat{\beta}_i^2(x)$ is finite. This implies ([15], p. 234; and [2], Theorem 9.5.5) that $(y, x)^{\hat{\approx}}$ is finite almost everywhere with respect to P_λ . Thus $P_\lambda(F_x) = 1$ for each fixed $x \in H_\lambda$. Hence

$$P_\lambda \times \mu(F) = \int_{H_\lambda} P_\lambda(F_x) d\mu(x) = \int_{H_\lambda} 1 d\mu(x) = 1.$$

Let μ be a probability measure on B , then μ can be regarded as a probability measure on H_λ . Let ψ be the c.f. of μ when μ is regarded as a probability measure on H_λ , then

$$\psi(y) = \int_{H_\lambda} e^{i(y, x)} d\mu(x) \quad \text{for all } y \in H_\lambda^*.$$

Now ψ is a function defined on $H_\lambda^* \subseteq B^*$, and we would like to extend ψ to be defined on B^* .

For each $x \in H_\lambda$, $(y, x)^{\hat{\approx}}$ is defined on B^* a.e. P_λ , and is equal to $y(x)$ with μ measure one for each $y \in H_\lambda^*$. We call

$$\int_{H_\lambda} e^{i(y, x)^{\hat{\approx}}} d\mu(x) \quad (y \in B^*)$$

the extension of ψ to B^* .

Clearly on H_λ^* we have $\int_{H_\lambda} e^{i(y, x)^{\hat{\approx}}} d\mu(x) = \int_{H_\lambda} e^{i(y, x)} d\mu(x) = \psi(y)$.

Since μ is actually defined on B we have

$$\int_{H_\lambda} e^{i(y, x)^{\hat{\approx}}} d\mu(x) = \int_B e^{i(y, x)^{\hat{\approx}}} d\mu(x) \quad \text{a.e. } P_\lambda.$$

But $(y, x)^{\hat{\approx}} = (y, x)$ for all $x \in B$, $y \in B^*$, hence

$$(3.1.8) \quad \int_{H_\lambda} e^{i(y, x)^{\hat{\approx}}} d\mu(x) = \int_B e^{i(y, x)} d\mu(x) = \varphi(y) \quad \text{a.e. } P_\lambda$$

where φ is the c.f. of μ when μ is regarded as a probability measure on B . Thus c.f. of a probability measure μ on B when considered as a random variable on B^* is equal almost everywhere P_λ to the extension of the c.f. of μ when μ is regarded as a probability measure on H_λ .

3.1.9. Remarks. (a) Let ℓ be as before. Since $\{b_n\}$ is a shrinking basis for B , the coordinate functionals β_n 's form a basis for B^* ([3], Lemma 1, p. 70). Hence there exists an isomorphism U^* from B^* to a Borel measurable subset of ℓ , say Ω^* ([11], Section 2, pp. 123-127). Therefore B^* can be identified with a Borel measurable subset of ℓ , and hence P_λ can be regarded as a countably additive cylinder set measure on ℓ through this identification.

(b) By (a), Lemma 4.3 of [14] and the fact that P_λ sits actually on B^* we get $\psi(y)^\sim = \varphi(y)$ a.e. P_λ where $\psi(\cdot)^\sim$ is Gross extension of the uniformly τ -continuous function $\psi(\cdot)$ with respect to the canonical normal distribution n on H_λ .

We will close this section by proving the following Lemma. The hypotheses of this Lemma are the same as of Lemma 1.2.10, however the proof is completely different. In the proof of Lemma 1.2.10 we used Lévy Continuity Theorem which we will not use in the proof of the following Lemma. Instead we use the fact that $\{b_n\}$ is shrinking and the ideas of [4].

3.1.10. Lemma. Let

- (i) λ be in ℓ_1^+ ,
- (ii) $\{\mu_t : t \in (0, \infty)\}$ be a λ -family as $t \rightarrow \infty$ of probability measures on B ,

(iii) μ be a probability measure on B ,

(iv) $\{\mu_t : t \in (0, \infty)\}$ converges weakly to μ on H_λ as $t \rightarrow \infty$.

Then $\{\mu_t : t \in (0, \infty)\}$ converges weakly to μ on B as $t \rightarrow \infty$.

Proof. Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then (iv) implies that $\{\mu_{t_n} : n = 1, 2, \dots\}$ is tight on H_λ ([1], Theorem 6.2, p. 37). Hence, for each $\epsilon > 0$ there exists a compact subset of H_λ , say K^ϵ such that

$$\mu_{t_n}(K^\epsilon) > 1 - \epsilon \quad \text{for all } n.$$

K^ϵ is compact implies that K^ϵ is bounded, that is, for each $\delta > 0$ there exists an N such that $\sum_{i=N+1}^{\infty} \lambda_i \hat{\beta}_i^2(x) < \delta$ uniformly in $x \in K^\epsilon$. Thus $\mu_{t_n}\{x \in H_\lambda : \sum_{i=N+1}^{\infty} \lambda_i \hat{\beta}_i^2(x) < \delta\} > 1 - \epsilon$ for all n . Since λ_i 's are positive, for all $N' \geq N$

$$\{x \in H_\lambda : \sum_{i=N+1}^{\infty} \lambda_i \hat{\beta}_i^2(x) < \delta\} \subset \{x \in H_\lambda : \sum_{i=N'+1}^{\infty} \lambda_i \hat{\beta}_i^2(x) < \delta\}.$$

Hence, for all n and all $N' \geq N$

$$\mu_{t_n}\{x \in H_\lambda : \sum_{i=N'+1}^{\infty} \lambda_i \hat{\beta}_i^2(x) < \delta\} > 1 - \epsilon.$$

But for each n , μ_{t_n} is defined on $(B, \mathcal{B}(x))$, so we get

$$(3.1.11) \quad \mu_{t_n}\{x \in B : \sum_{i=N'+1}^{\infty} \lambda_i \beta_i^2(x) < \delta\} > 1 - \epsilon \quad \text{for all } n$$

and all $N' \geq N$.

Since $\{\mu_{t_n} : n = 1, 2, \dots\}$ is a λ -family, it follows that there exists $\{\epsilon_N\}$ such that

$$\mu_{t_n}\{x \in B : \|\sum_{i=N+1}^{\infty} \beta_i(x) b_i\|_B < k(\delta)\} > 1 - (\epsilon + \epsilon_N).$$

Now using (3.1.11) we have, for all n and all $N' \geq N$

$$\mu_{t_n} \{x \in B : \left\| \sum_{i=N'+1}^{\infty} \beta_i(x) b_i \right\|_B < k(\delta)\} > 1 - (\epsilon + \epsilon_N).$$

Let N_0 be sufficiently large so that, for all n

$$\mu_{t_n} \{x \in B : \left\| \sum_{i=N_0+1}^{\infty} \beta_i(x) b_i \right\|_B < k(\delta)\} \geq 1 - \epsilon.$$

We now let S be the subspace of B generated by $\{b_1, \dots, b_{N_0}\}$,

and let $S^{k(\delta)} = \{x : \inf\{\|x - z\|_B : z \in S\} \leq k(\delta)\}$. For $x \in B$ we have $P_{N_0} x = \sum_{i=1}^{N_0} \beta_i(x) b_i \in S$, hence

$$S^{k(\delta)} \supset \{x \in B : \left\| \sum_{i=N_0+1}^{\infty} \beta_i(x) b_i \right\|_B < k(\delta)\}.$$

Thus $\mu_{t_n}(S^{k(\delta)}) \geq 1 - \epsilon$ for all n .

Let T be the subspace of B^* generated by $\{\beta_1, \dots, \beta_{N_0}\}$.

Then $S^\perp \oplus T = B^*$ since β_i 's are coordinate functionals and form a basis for B^* (See; Remark 3.1.9 (a)).

We now show there exists an $r > 0$ such that for all n

$$\mu_{t_n} \{x \in B : |\beta_i(x)| < r, i = 1, 2, \dots, N_0\} \geq 1 - \epsilon.$$

Let $r_1 = \sup_{i, K^\epsilon} |\hat{\beta}_i(x)|$, then r_1 is finite since K^ϵ is a compact subset of $H_\lambda^{K^\epsilon}$, and $\sup_i |\hat{\beta}_i(x)| \leq M \|x\|_\lambda$ ([3], p. 68). Now let $r > r_1$, then

$$\begin{aligned} \mu_{t_n} \{x \in B : |\beta_i(x)| < r, i = 1, 2, \dots, N_0\} \\ &= \mu_{t_n} \{x \in B : |\hat{\beta}_i(x)| < r, i = 1, 2, \dots, N_0\} \\ &\geq \mu_{t_n} \{x \in B : \sup_i |\hat{\beta}_i(x)| < r, i = 1, 2, \dots, N_0\} \\ &\geq \mu_{t_n}(K^\epsilon) \\ &\geq 1 - \epsilon \quad \text{for all } n. \end{aligned}$$

Therefore by Theorem 2.1 of ([4], p. 11), $\{\mu_{t_n} : n = 1, 2, \dots\}$ is conditionally compact on B .

Since $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact on H_λ and P_N is continuous, it follows that $\{\mu_{t_n} \circ P_N^{-1} : n = 1, 2, \dots\}$ is compact on H_λ for all $N = 1, 2, \dots$. Hence $\{\mu_{t_n} \circ P_N^{-1} : n = 1, 2, \dots\}$ is compact on B for all $N = 1, 2, \dots$ and $\mu_{t_n} \circ P_N^{-1} \xrightarrow{W} \mu \circ P_N^{-1}$. This together with the fact that $\{\mu_{t_n} : n = 1, 2, \dots\}$ is conditionally compact on B imply that $\mu_{t_n} \xrightarrow{W} \mu$ on B ([1], p. 35). Since this is true for any sequence $\{t_n\}$ approaching infinity we have $\mu_t \xrightarrow{W} \mu$ on B as $t \rightarrow \infty$.

§3.2. Main Theorem III.

The following Theorem (Main Theorem III) gives inversion formulae for a probability measure on a Banach space B with a shrinking Schauder basis. It differs from the Main Theorem I in the sense that (3.2.2) (a) is stronger than (1.4.4) (a). This can easily be seen since $P_\lambda(B^*) = 1$ for $\lambda \in \ell_1^+$. Furthermore, we will not use Lévy Continuity Theorem in the proof, and hope that one might be able to use this Theorem to get a proof for the Lévy Continuity Theorem.

Let ν, T_ν, C_t, μ_t and a_t be as in Section 1.4.

3.2.1. Main Theorem III. Let

- (i) B be a Banach space with shrinking Schauder basis $\{b_i\}$,
- (ii) - (v) be as in Theorem 1.4.3,
- (vi) E_{P_λ} denote the integral with respect to P_λ on B^* (See; Lemma 3.1.4).

Then for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G

on B the following are equivalent:

$$(3.2.2) \quad \left\{ \begin{array}{l} (a) \quad \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t) E_{P_\lambda} \{ \varphi(ty) (\int_B G(f(t)x) \\ \exp[-itf(t)(y,x)] dv(x)) \} \\ (b) \quad \{ \mu_t : t \in (0, \infty) \} \text{ is } \lambda\text{-family as } t \rightarrow \infty \end{array} \right.$$

$$(3.2.3) \quad \left\{ \begin{array}{l} (a) \quad f(t) C_t^{-1} X \xrightarrow{D} 0 \text{ as } t \rightarrow \infty \\ (b) \quad Y_t \xrightarrow{D} Y \text{ as } t \rightarrow \infty . \end{array} \right.$$

We note that the proof is similar to the proof of Theorem 1.4.3.

Proof. Suppose (3.2.2) holds, and let G be a real valued, bounded, $\|\cdot\|_B$ -continuous function on B . Then

$$\begin{aligned} & E_{P_\lambda} \{ \varphi(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)] dv(x)) \} \\ &= \int_B \star \{ \varphi(ty) (\int_B G(f(t)x) \exp[-itf(t)(y,x)] dv(x)) \} dP_\lambda(y). \end{aligned}$$

Using the fact that $G(f(t)x) dnoT_V^{-\frac{1}{2}}(x)$ is a measure of bounded variation together with Remark 1.2.6 (c) and (3.1.8) we get

$$= \int_B \star (\int_{H_\lambda} e^{it(y,s)} \hat{\hat{d\mu}}(s)) (\int_{H_\lambda} G(f(t)x) e^{-itf(t)(y,x)} \hat{\hat{dnoT_V^{-\frac{1}{2}}}}(x)) dP_\lambda(y).$$

The function $e^{it(y,s)} \hat{\hat{e^{-itf(t)(y,x)}}} \hat{\hat{G(f(t)x)}}$ is jointly measurable and all the measures are probability measures so we may use Fubini's Theorem ([19], p. 140) to interchange some integrals to obtain

$$\begin{aligned}
& \int_B \star \left(\int_{H_\lambda} e^{it(y,x)} \hat{\approx} d\mu(s) \right) \left(\int_{H_\lambda} G(f(t)x) e^{-itf(t)(y,x)} \hat{\approx} dnoT_v^{-\frac{1}{2}}(x) \right) dP_\lambda(y) \\
&= \int_{H_\lambda} \int_{H_\lambda} G(f(t)x) \left(\int_B \star e^{it(y,x-f(t)x)} \hat{\approx} dP_\lambda(y) \right) dnoT_v^{-\frac{1}{2}}(x) d\mu(s) \\
&= \int_{H_\lambda} \int_{H_\lambda} G(f(t)x) \exp\left\{-t^2 \sum_{i=1}^{\infty} \lambda_i [\hat{\beta}_i(s-f(t)x)]^2/2\right\} dnoT_v^{-\frac{1}{2}}(x) d\mu(s) \\
&= \int_{H_\lambda} \int_{H_\lambda} G(f(t)x) \exp[-t^2 \|s-f(t)x\|_\lambda^2/2] dnoT_v^{-\frac{1}{2}}(x) d\mu(s).
\end{aligned}$$

Since G is $\|\cdot\|_\lambda$ -measurable and $T^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H_λ , it follows ([7], Lemma 4.1) that

$$\begin{aligned}
&= \frac{1}{(\det C_t)} \int_{H_\lambda} \int_{H_\lambda} G(f(t)C_t^{-1}x + s - C_t^{-2}s) dnoT_v^{-\frac{1}{2}}(x) h(t) \\
&\quad \exp[-t^2 \|C_t^{-1}s\|_\lambda^2/2] d\mu(s) \\
&= \frac{1}{(\det C_t)} \int_B \int_B G(f(t)C_t^{-1}x + s - C_t^{-2}s) dv(x) h(t) \exp[-t^2 \|C_t^{-1}s\|_\lambda^2/2] d\mu(s)
\end{aligned}$$

where we have used Remark 1.2.6 (c) and the fact that $\mu(B) = 1$.

Hence we may start with the assumption that for all real-valued, bounded, $\|\cdot\|_B$ -continuous functions G on B we have

$$(3.2.4) \quad \begin{cases} (a) & \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} \int_B \left[\int_B G(f(t)C_t^{-1}x + s - C_t^{-2}s) dv(x) \right] h(t) \\ & \quad \exp[-t^2 \|C_t^{-1}s\|_\lambda^2/2] d\mu(s) \\ (b) & \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \rightarrow \infty. \end{cases}$$

Putting $G \equiv 1$ in (3.2.4)_(a) and using the same argument given in the proof of Theorem 1.4.3, condition (3.2.4)_(a) can be written as follows

$$(3.2.5) \quad \int_B G(s) d\mu(s) = \lim_{t \rightarrow \infty} \int_B \int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s) dv(x) d\mu_t(s)$$

for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B .

From (3.2.5), it follows that

$$(3.2.6) \quad \nu \circ (f(t)C_t^{-1})^{-1} * \mu_t \circ (I - C_t^{-2}) \stackrel{W}{\Rightarrow} \mu \text{ on } B \text{ as } t \rightarrow \infty.$$

Since $G(f(t)x)d\nu(x)$ is a measure of bounded variation on H_λ , the Fourier transform of $G(f(t)x)d\nu(x)$ is uniformly τ -continuous ([7], p. 7). Hence the Gross extension of its Fourier transform is well-defined ([7], Theorem, p. 5). Similarly the Gross extension of the Fourier transform (c.f.) $\psi(\cdot)$ of μ when regarding μ as a probability measure on $(H_\lambda, \mathcal{B}(H_\lambda))$ is well-defined.

Now from Remarks 3.1.9 (b) and 1.2.6 (c), it follows that

$$(3.2.7) \quad E_{P_\lambda} \left\{ \varphi(ty) \left(\int_B G(f(t)x) \exp[-itf(t)(y,x)] d\nu(x) \right) \right\} \\ = E_{P_\lambda} \left\{ \psi(ty) \sim \left(\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\nu T_V^{-\frac{1}{2}}(x) \right) \sim \right\}.$$

Let $\{P_j\}$ be a sequence of finite dimensional projections on H_λ converging strongly to the identity operator. Then it is easy to see using Lemma 3.1.5, Lebesgue Dominated Convergence Theorem and the fact that P_j 's are continuous that

$$E_{P_\lambda} \left\{ \psi(ty) \sim \left(\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\nu T_V^{-\frac{1}{2}}(x) \right) \sim \right\} \\ = \lim_{j \rightarrow \infty} E_{P_\lambda} \left\{ \psi(tP_j y) \sim \left(\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\nu T_V^{-\frac{1}{2}}(x) \right) \sim \right\}.$$

Now as before, using the fact that the integral of a tame function with respect to the probability measure P_λ is the same as its integral with respect to the canonical normal distribution n on H_λ , we get

$$\begin{aligned}
& E_{P_\lambda} \{ \psi(ty) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y,x)] d\nu_V^{-\frac{1}{2}}(x)) \sim \} \\
& = \lim_{j \rightarrow \infty} E_n \{ \psi(tP_j y) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\nu_V^{-\frac{1}{2}}(x)) \sim \}
\end{aligned}$$

where E_n is the integral with respect to n on H_λ .

From Remark 1.1.15, it follows that

$$\begin{aligned}
& \lim_{j \rightarrow \infty} E_n \{ \psi(tP_j y) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(P_j y, x)] d\nu_V^{-\frac{1}{2}}(x)) \sim \} \\
& = E_n \{ \psi(ty) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y, x)] d\nu_V^{-\frac{1}{2}}(x)) \sim \} .
\end{aligned}$$

Hence from (3.2.7), we have

$$\begin{aligned}
& E_{P_\lambda} \{ \varphi(ty) (\int_B G(f(t)x) \exp[-itf(t)(y, x)] d\nu(x)) \} \\
& = E_n \{ \psi(ty) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(y, x)] d\nu_V^{-\frac{1}{2}}(x)) \sim \} .
\end{aligned}$$

Now from (3.2.2)_(a) and the fact that $\|\cdot\|_B$ -topology is stronger than $\|\cdot\|_\lambda$ -topology on B we obtain

$$\int_{H_\lambda} G(s) d\mu(s) = \lim_{t \rightarrow \infty} h(t) (\det C_t) E_n \{ \psi(ty) \sim (\int_{H_\lambda} G(f(t)x) \exp[-itf(t)(x, y)] d\nu_V^{-\frac{1}{2}}(x)) \sim \}$$

for all real valued, bounded, $\|\cdot\|_\lambda$ -continuous functions G on H_λ .

Therefore by Theorem 4 of [7] we have

$$(3.2.8) \quad \begin{cases} (a) & f(t)^2 \text{ trace } (C_t^{-2} T_V) \rightarrow 0 \text{ as } t \rightarrow \infty \\ (b) & \text{The measures } h(t) \exp[-t^2 \|C_t^{-1} s\|_\lambda^2 / 2] d\mu(s) \text{ converge} \\ & \text{weakly to } \mu \text{ on } H_\lambda \text{ as } t \rightarrow \infty . \end{cases}$$

Now (3.2.8)_(a) implies ([7], Corollary 3.4) that

$$(3.2.9) \quad f(t)C_t^{-1}X \xrightarrow{D} 0 \text{ on } H_\lambda \text{ as } t \rightarrow \infty ,$$

and (3.2.8) (b) implies (Definition 1.1.2) that

$$(3.2.10) \quad Y_t \xrightarrow{D} Y \text{ on } H_\lambda \text{ as } t \rightarrow \infty .$$

From (3.2.2) (b) and (3.2.10), it follows (Lemma 3.1.10) that

$$Y_t \xrightarrow{D} Y \text{ on } B \text{ as } t \rightarrow \infty .$$

Thus condition (3.2.3) (b) holds.

To get (3.2.3) (a) we note that

$$(3.2.11) \quad \mu\{x \in B : (I-C_t^{-2})x_n \not\rightarrow x \text{ when } x_n \rightarrow x\} = 0 .$$

From (3.2.11) and (3.2.3) (b), it follows ([1], Theorem 5.5, p. 34) that

$$(3.2.12) \quad (I-C_t^{-2})Y_t \xrightarrow{D} Y \text{ on } B \text{ as } t \rightarrow \infty .$$

Let $f(t)C_t^{-1}X$ be distributed as v_t , then (3.2.6) and (3.2.12) imply ([16], Theorem 2.1, p. 58) that for any sequence $\{t_n\}$ approaching infinity, $\{v_{t_n} : n = 1, 2, \dots\}$ is conditionally compact on B . Now by Lemma 3.1 of [13], $\{v_{t_n}\}$ is a λ -family for any $\lambda \in \ell_\infty^+$ which is sufficient for $\{v_{t_n} : n = 1, 2, \dots\}$. Since $\lambda \in \ell_1^+$, it follows that $\{v_{t_n} : n = 1, 2, \dots\}$ is a λ -family, and since this is true for any sequence $\{t_n\}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ we conclude that $\{v_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$ of probability measures on $(B, \mathcal{B}(B))$. From this and (3.2.9) it follows (Lemma 3.1.10) that

$$f(t)C_t^{-1}X \xrightarrow{D} 0 \text{ on } B \text{ as } t \rightarrow \infty .$$

Hence condition (3.2.3)_(a) holds.

We now prove the converse. From (3.2.3)_(b), it follows that for any sequence $\{t_n\}$ approaching infinity, $\{\mu_{t_n} : n = 1, 2, \dots\}$ is compact. Hence it is a λ -family for any $\lambda \in \mathcal{L}_1^+$ ([13], Lemma 3.1). Thus $\{\mu_t : t \in (0, \infty)\}$ is a λ -family as $t \rightarrow \infty$, and hence condition (3.2.2)_(b) is satisfied. Furthermore, from (3.2.3)_(b) we get

$$(3.2.13) \quad \lim_{t \rightarrow \infty} a_t = 1.$$

Let G be a real valued, bounded, $\|\cdot\|_B$ -continuous function on B , and let

$$\beta_t = h(t) (\det C_t) E_{P_\lambda} \left\{ \varphi(ty) \left(\int_B G(f(t)x) \exp \{-itf(t)(y, x)\} dv(x) \right) \right\} - \int_B G(x) d\mu(s).$$

Then from (3.2.13), it follows that

$$\lim_{t \rightarrow \infty} \beta_t = \lim_{t \rightarrow \infty} \frac{\beta_t}{a_t} = \lim_{t \rightarrow \infty} \frac{1}{a_t} h(t) (\det C_t) E_{P_\lambda} \left\{ \varphi(ty) \left(\int_B G(f(t)x) \exp \{-itf(t)(y, x)\} dv(x) \right) \right\} - \int_B G(s) d\mu(s).$$

By the argument used to obtain (3.2.5) we have

$$(3.2.14) \quad \lim_{t \rightarrow \infty} \beta_t = \lim_{t \rightarrow \infty} \int_B \int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s) dv(x) d\mu_t(s) - \int_B G(s) d\mu(s).$$

From (3.2.12) and (3.2.3)_(a), it follows ([16], Lemma 1.1 and Theorem 1.1, p. 57) that

$$v \circ (f(t)C_t^{-1})^{-1} * \mu_t \circ (I - C_t^{-2})^{-1} \xrightarrow{W} \mu \quad \text{as } t \rightarrow \infty \text{ on } B.$$

Hence

$$(3.2.15) \quad \int_B \int_B G(f(t)C_t^{-1}x + (I-C_t^{-2})s) dv(x) d\mu_t(s) \rightarrow \int_B G(s) d\mu(s) \\ \text{as } t \rightarrow \infty$$

for all real valued, bounded, $\|\cdot\|_B$ -continuous functions G on B .

From (3.2.14) and (3.2.15) we get $\lim_{t \rightarrow \infty} \beta_t = 0$, and therefore the proof is completed.

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